MAT 566: FUKAYA CATEGORIES AND FLOER HOMOLOGY
PROBLEM SET 1

Please turn in at least **6** of the following 9 problems.

(1) Complete the argument from the end of Lecture 1 to prove $\dim(\widehat{HF}(Y)) > 1$ when Y contains an incompressible torus. That is, prove that any two immersed curves in the punctured torus intersect at least twice provided:

- neither curve is nullhomotopic; and
- neither curve is *loose* (i.e. neither curve is homotopic to a straight line in the cover $\mathbb{R}^2 \setminus \mathbb{Z}^2$);

[Hint: you may assume the curves are "pulled tight", meaning that they consist of arcs following a ball of some radius ϵ around the punctures (call these arcs corners*) and straight segments connecting the corners. To ensure transverse intersection, one should use a different ϵ for each corner. If some care is taken when choosing the relative sizes of the epsilon, curves in this position will always intersect minimally. What does not being loose tell you about the number of corners on a curve? What happens locally near a corner?]*

Optional: Give an example where the intersection number is 2. Can you improve the bound by further assuming that the two curves do not cobound an immersed annulus and that neither curve does a full wrap around the puncture when pulled tight?

(2) Given an A_∞-category $\mathcal{A}$ (as defined in Lecture 2), show that the cohomological category is a well defined ordinary category. In particular, show that the map $[\mu^2]$ induced by μ^2 is defines an associative composition.

(3) Let (M, ω) be a symplectic manifold with almost complex structure J compatible with ω. Let Σ be a Riemann surface with a standard complex structure j. A map $u : (\Sigma, j) \to (M, J)$ has energy $E(u) = \int_\Sigma |du|^2$. Show that the energy can be written as

$$E(u) = \int_\Sigma u^*\omega + \int_\Sigma |\bar{\partial}_J(u)|^2.$$

(4) Let M be a symplectic 2-manifold. A mod 2 grading on a Lagrangian is a lift of $L \to \mathcal{G}M$ to E where the fibers of E are the double cover the fibers of $\mathcal{G}M$ (which are $\mathcal{G}(1) \cong \mathbb{R}P^1$). Show that this is equivalent to an orientation on L. Given graded Lagrangians L_0 and L_1, show that intersection sign defines a mod 2 grading on $CF^*(L_0, L_1)$.

(5) Let M be a symplectic 2-manifold, and assume that L_0, L_1 are graded Lagrangians so that $\deg(p)$ is defined for $p \in L_0 \cap L_1$ as in Lecture 4. For a strip u from p to q, show that

$$\mathrm{ind}(u) = \deg(q) - \deg(p).$$

Show that a strip u corresponding to a bigon with acute corners has $\mathrm{ind}(u) = 1$. Draw an example of (the image of) an index 2 strip.

(6) Consider the cotangent bundle T^*L with zero-section L. Show that $HF^*(L, L) \cong H^*(L)$.

[Hint: For help, refer to Example 1.12 in Auroux, "A beginner's introduction to Fukaya categories"]

(7) Let $\mathbb{D}$ denote the unit disk in the complex plane. Fix any three distinct points p_1, p_2, p_3 in $\partial\mathbb{D}$. Show that there is unique complex automorphism $\mathbb{D} \to \mathbb{D}$ taking p_1 to 1, p_2 to i, and p_3 to -1.

(8) Show that the Deligne-Mumford compactification of the moduli space of disks with $k + 1$ punctures on the boundary (described in Lecture 6) is equivalent to the kth Stasheff associahedron (described in Lecture 2).

(9) Let M be the punctured torus (viewed as $(\mathbb{R}/\mathbb{Z})^2 \setminus \{(0, 0)\}$ with symplectic form $dx \wedge dy$). Let L_0 and L_2 be parallel copies of the vertical simple closed curve, separated by some distance a, and let L_1 and L_3 be copies of the horizontal simple closed curve, separated by some distance b. Let q be the unique point in $L_0 \cap L_3$ and let p_i be the unique point in $L_{i-1} \cap L_i$ for $1 \le i \le 3$. Compute, up to sign, the coefficient of q in $\mu^3(p_3, p_2, p_1)$.