Bordered Floer homology via immersed curves in the punctured torus

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Outline

Heegaard Floer homology

Closed manifolds:

- To a closed, orientable 3-manifold Y we associate an abelian group $\widehat{HF}(Y) = H_*(\widehat{CF}(Y))$

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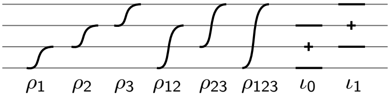
Manifolds with torus boundary:

- There is an algebra $\mathcal{A}$ associated to the torus.
- To an orientable 3-manifold M with boundary $\partial M = T^2$ and a pair of parametrizing curves (α, β) for ∂M, we associate a differential module $\widehat{CFD}(M, \alpha, \beta)$ or an $\mathcal{A}_\infty$-module $\widehat{CFA}(M, \alpha, \beta)$ over $\mathcal{A}$.

The torus algebra $\mathcal{A}$

- $\mathcal{A}$ is generated (over $\mathbb{F}_2$) by $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$ and two idempotents, ι_0 and ι_1.



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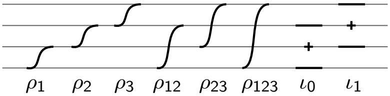

- Multiplication is concatenation, e.g.

$$\rho_1\rho_2 = \rho_{12}, \quad \rho_2\rho_1 = 0, \quad \rho_1\iota_1 = \rho_1, \quad \rho_1\iota_0 = 0$$

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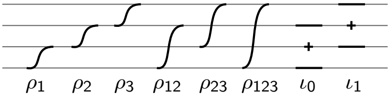

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- $\iota_0 + \iota_1 = 1 \in \mathcal{A}$. We also denote this by $\rho_\emptyset$.

$\mathcal{A}$-decorated graphs

An $\mathcal{A}$ decorated graph is a directed graph with

- vertices labeled by ι_0 or ι_1 (we depict these labels using $\bullet$ and $\circ$, respectively)
- edges labeled by ρ_I for $I \in \{1, 2, 3, 12, 23, 123, \emptyset\}$.

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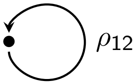
We will think of the invariants $\widehat{CFD}$ or $\widehat{CFA}$ as $\mathcal{A}$-decorated graphs
(up to appropriate equivalence)
We can always assume the graphs are *reduced* (i.e. no $\rho_\emptyset$ arrows).

Examples

$\widehat{CFD}(D^2 \times S^1, m, \ell)$



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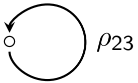
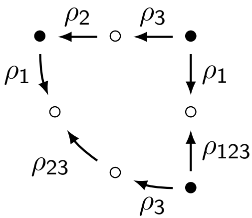
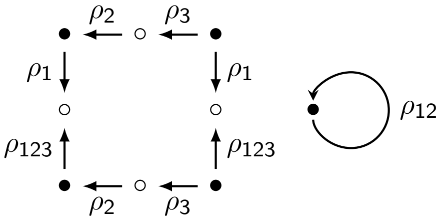


$\widehat{CFD}(\mathrm{RHT}, \mu, \lambda)$



$\widehat{CFD}(\mathrm{Fig8}, \mu, \lambda)$



Loop type manifolds

At a given vertex of a reduced $\mathcal{A}$-decorated graph, we categorize the incident edges:

$\bullet\xrightarrow{1}$	$\bullet\xrightarrow{3}$	$\circ\xrightarrow{2}$	$\circ\xleftarrow{3}$
$\bullet\xrightarrow{12}$	$\bullet\xleftarrow{2}$	$\circ\xrightarrow{23}$	$\circ\xleftarrow{23}$
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Definition

A *loop* is a connected valence two $\mathcal{A}$-decorated graph s.t. at every vertex, the two incident edges have types $\mathbf{I}_\bullet$ and $\mathbf{II}_\bullet$ or $\mathbf{I}_\circ$ and $\mathbf{II}_\circ$.

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Note: Does not depend on the choice of parametrization (α, β).

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- If M has more than one L-space filling, M is loop type
- For $K \subset S^3$, if $CFK^-(K)$ admits a horizontally and vertically simplified basis, $S^3 \setminus \nu(K)$ is loop type
- We currently do not know of any examples which are not loop type

Combinatorial description of loops

An *oriented* loop admits a well defined grading. There are four types of vertices:

$\underline{\mathbf{I}_\bullet} \bullet \underline{\mathbf{II}_\bullet}$ $\qquad$ $\underline{\mathbf{II}_\bullet} \bullet \underline{\mathbf{I}_\bullet}$ $\qquad$ $\underline{\mathbf{I}_\circ} \circ \underline{\mathbf{II}_\circ}$ $\qquad$ $\underline{\mathbf{II}_\circ} \circ \underline{\mathbf{I}_\circ}$

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An oriented loop gives a cyclic word in $\{\bullet^+, \bullet^-, \circ^+, \circ^-\}$. In fact, the converse is also true.

We will replace $\circ^{\pm}$ with $\alpha^{\pm 1}$ and $\bullet^{\pm}$ with $\beta^{\pm 1}$. We have:

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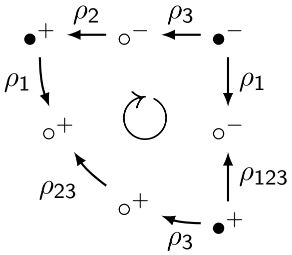
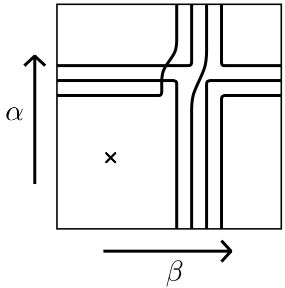


$\beta\alpha\alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}$



Bordered invariants as curves

- Given a loop type manifold M with parametrizing curves α and β, $\widehat{CFD}(M, \alpha, \beta)$ is represented by a collection of loops.
- These correspond to a collection of immersed curves in the punctured torus.
- We think of this as a collection $\gamma(M, \alpha, \beta)$ in $\partial M \setminus \{z\}$, where z is a fixed basepoint.

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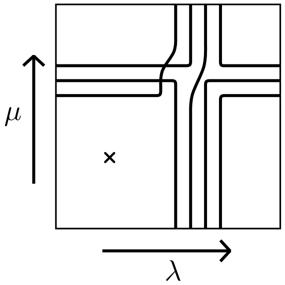
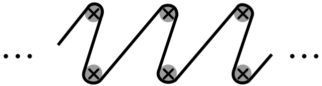
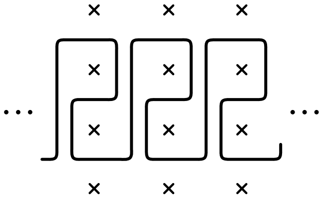
Theorem 1 (H-Rasmussen-Watson)

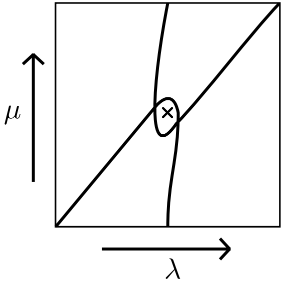
The curves $\gamma(M) := \gamma(M, \alpha, \beta)$ do not depend on the parametrizing curves α and β.

Example: $\widehat{CFD}(\mathrm{RHT}, \mu, \lambda)$









Pairing: What happens when we glue?

Bordered Floer homology has a pairing theorem:

$$\widehat{CFA}(M_1, \alpha_1, \beta_1) \boxtimes \widehat{CFD}(M_2, \alpha_2, \beta_2) \simeq \widehat{CF}(M_1 \cup M_2)$$

Suppose M_1 and M_2 are loop type manifolds. Then we have collections of immersed curves $\gamma_1 \subset \partial M_1$ and $\gamma_2 \subset \partial M_2$.

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Theorem 2 (H.-Rasmussen-Watson)

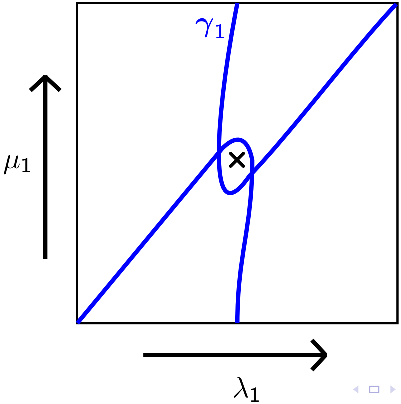
Let $Y = M_1 \cup_h M_2$, where $h : \partial M_2 \to \partial M_1$ is a diffeomorphism. Then

$$\widehat{HF}(Y) \cong HF(\gamma_1, h(\gamma_2)),$$

Where right side denotes the intersection Floer homology of the two sets of curves in the punctured torus $\partial M_1 \setminus \{z\}$.

Example

Let Y be the 3-manifold obtained by splicing two RHT complements, that is, by gluing them with a map taking μ_1 to λ_2 and λ_1 to μ_2.



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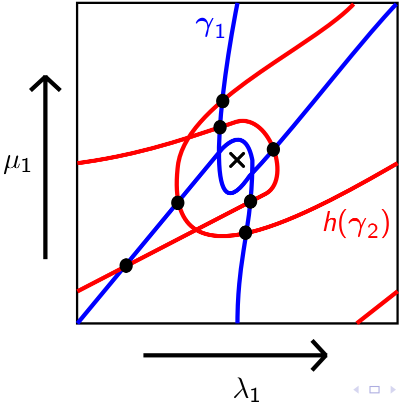

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Let $\mathcal{L}_{M_i}$ denote the set of L-space slopes on ∂M_i.

Theorem 3 (H.-Rasmussen-Watson)

If M_1 and M_2 are loop type and neither is the solid torus, then $M_1 \cup M_2$ is an L-space iff every slope on $\partial M_1 = \partial M_2$ is in either $\mathcal{L}^\circ_{M_1}$ or $\mathcal{L}^\circ_{M_2}$

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- This was the key remaining step in confirming a conjecture of Boyer-Gordon-Watson for graph manifolds.
- Using curves, the proof is essentially an application of the Mean Value Theorem.

Other applications

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- Connections to Seiberg-Witten theory?
- Recovering HF^+?

Thank you!