Introduction To Symplectic Topology
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Exercise 1.5 Carry out the inverse Legendre transform from a Hamiltonian system to a Lagrangian system.

Solution 1.5 Suppose that we are given a Hamiltonian H with $\det(\frac{\partial H}{\partial y_i \partial y_j}) \neq 0$. Then we define:

$$v_k = \frac{\partial H}{\partial y_k}; \quad L(t,x,v) = \sum_k y_k v_k - H(t,x,v)$$

Now we show that if $\gamma(t) = (x(t), y(t))$ satisfies Hamiltons equations for H, then $(x, \frac{dx}{dt})$ satisfy the Euler-Lagrange equations for L. First we see that due to Hamilton's equations, we have:

$$\frac{dx_k}{dt} = \frac{\partial H}{\partial y_k} = v_k; \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}$$

We observe that:

$$\frac{d}{dt}(\frac{\partial L}{\partial v_k}) = \frac{d}{dt}(\frac{\partial}{\partial v_k}(\sum_j y_j v_j - H(t,x,v))) = \frac{d}{dt}(y_k - \frac{\partial H}{\partial v_k} + \sum_j \frac{\partial y_j}{\partial v_k} v_j)$$

Then we see:

$$\sum_j v_j \frac{\partial y_j}{\partial v_k} = \sum_j \frac{dx_j}{dt}\frac{\partial y_j}{\partial v_k} = \sum_j \frac{\partial H}{\partial y_j}\frac{\partial y_j}{\partial v_k} = \frac{\partial H}{\partial v_k}$$

Thus:

$$\frac{d}{dt}(\frac{\partial L}{\partial v_k}) = \frac{dy_k}{dt} = -\frac{\partial H}{\partial x_k}$$

Furthermore we have:

$$\frac{\partial L}{\partial x_k} = \frac{\partial}{\partial x_k}(\sum_j y_j \frac{\partial H}{\partial v_j} - H(t,x,v)) = -\frac{\partial H}{\partial x_k} + \sum_j \frac{\partial y_j}{\partial x_k} v_j + y_j \frac{\partial v_j}{\partial x_k} - \frac{\partial H}{\partial v_j}\frac{\partial v_j}{\partial x_k} = -\frac{\partial H}{\partial x_k}$$

Here the equality of the last two terms comes from the fact that $\frac{\partial y_j}{\partial x_k} = 0$ and $y_j = \frac{\partial H}{\partial v_j}$. This proves the result

Exercise 1.12 Show that the set $\mathrm{Symp}(\mathbb{R}^{2n})$ of symplectomorphisms of $\mathbb{R}^{2n}$ form a group.

Solution 1.12 The identity map $Id : \mathbb{R}^{2n}$ is a smooth symplectomorphism since its Jacobian is the identity map $T\mathbb{R}^{2n} \to T\mathbb{R}^{2n}$, which is evidently symplectic. Furthermore given $\phi, \psi \in \mathrm{Symp}(\mathbb{R}^{2n})$ we can compose them to get a diffeomorphism $\psi \circ \phi$ and since $d(\psi \circ \phi) = d\psi \circ d\phi$, the fact that $d\psi$ and $d\phi$ are symplectic and that symplectic matrices are a group implies that $d(\psi \circ \phi)$ is symplectic. Finally, the inverse diffeomorphism ϕ^{-1} has $d(\phi^{-1}) = (d\phi)^{-1}$, thus its Jacobian is also in the linear symplectic group, so it is a symplectomorphism. Associativity follows from the same property for group composition in Diff. Thus concludes the proof.

Exercise 1.13 Consider the matrix:

$$\Phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A.B, C$ and D are real $n \times n$ matrices. Prove that Φ is symplectic if and only if its inverse is of the form

$$\Phi^{-1} = \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix}$$

Deduce that a 2×2 matrix is symplectic if and only if its determinant is equal to 1.

Solution 1.13 We simply carry out the matrix multiplication. Φ is symplectic if and only if:

$$\Phi^T J \Phi = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} \begin{pmatrix} -C & -D \\ A & B \end{pmatrix} =$$

$$\begin{pmatrix} C^T A - A^T C & C^T B - A^T D \\ D^T A - B^T C & D^T B - B^T D \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Furthermore we see that the inverse condition is true if and only if:

$$\begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D^T A - B^T C & D^T B - B^T D \\ A^T C - C^T A & A^T D - C^T B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

From these expressions it is evident that these two conditions are equivalent, since they are both true if and only if $C^T A - A^T C = D^T B - B^T D = 0$ and $D^T A - B^T C = A^T D - C^T B = 1$. In the $n = 1$ case, this is equivalent to $ad - bc = 1$ (i.e the determinant 1 condition). The other condition is trivially satisfied since 1×1 matrix commute.

Exercise 1.15 Find an element of the linear group $SL(4, \mathbb{R})$ which is not in $\mathrm{Sp}(4, \mathbb{R})$.

Solution 1.15 One cheap way of doing this is to just find a linear ϕ where $\phi^* \omega = -\omega$. Then:

$$\phi^*(\omega^2) = \phi^* \omega \wedge \phi^* \omega = (-1)^2 \omega^2 = \omega^2$$

Such a map ϕ is given, for example, by the matrix 4×4:

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Exercise 1.17 (Confirming Lemma 1.17) The Poisson bracket satisfies the Jacobi identity.

Solution 1.17 We will write this out in Einstein index notation, which will make it clear where the signs are coming from, then we will switch to a more invariant notation. Let $J = (j^{ab})$ be the co-symplectic matrix/form in coordinates. Furthermore let $f, g, h \in C^\infty(\mathbb{R}^{2n})$. Then:

$$\{f, \{g, h\}\} = \partial_c f j^{cd} \partial_d(\partial_a g j^{ab} \partial_b h) = \partial_c h j^{cd} \partial_d \partial_a g j^{ab} \partial_b h + \partial_c h j^{cd} \partial_a g j^{ab} \partial_d \partial_b h$$

$$= \partial_c h j^{cd} \partial_d \partial_a g j^{ab} \partial_b h - \partial_c h j^{cd} \partial_a g j^{ba} \partial_d \partial_b h = d^2 g(Jdf, Jdh) - d^2 h(Jdf, Jdg)$$

It is clear that if we sum over the cyclic permutations of f, g and h, the result will vanish due to term matching.

Exercise 1.19 How does the Poisson bracket behave with respect to product of functions? Prove that the Poisson bracket of two functions f and g is given by:

$$\{f, g\} = \omega_0(X_f, X_g)$$

Solution 1.19 The Poisson bracket obeys a Leibniz rule. We see that:

$$\{fg, h\} = -(\nabla(fg))^T J_0 \nabla h = f(-(\nabla g)^T J_0 \nabla h) + g(-(\nabla f)^T J_0 \nabla h) = f\{g, h\} + g\{f, h\}$$

We can use the fact that $\{fg, h\} = \{h, fg\}$ to show the analogous identity for the other entry.

For the second part, we just observe that:

$$-(\nabla f)^T J_0 \nabla g = -(\nabla f)^T(-J_0)J_0(-J_0)\nabla g = (-J_0 \nabla f)^T J_0(-J_0 \nabla g) = \omega_0(X_f, X_g)$$

Exercise 1.20 Check that in the Kepler problem (Example 1.7) the three components of the angular momentum $x \times \dot{x}$ are integrals of motion which are not in involution.

Solution 1.20 In the Kepler problem we have $p = \frac{dx}{dt}$. To show that the elements of $x \times p = x \times \frac{dx}{dt}$ are invariants of motion we just have to show that $\frac{d}{dt}(x \times \frac{dx}{dt}) = 0$. But:

$$\frac{d}{dt}(x \times \frac{dx}{dt}) = \frac{dx}{dt} \times \frac{dx}{dt} + x \times \frac{dx^2}{dt} = x \times \frac{-x}{|x|^2} = 0$$

Now observe that the whole system is symmetric under orthogonal transformations (in fact this is where these conserved quantities come from, via Noether's theorem). Thus to check that these integrals of motion are not in involution, we need only check it for one pair of components. Take $f(x, p) = (x \times p)_1 = x_2 p_3 - x_3 p_2$

and $g(x,p) = (x \times p)_2 = x_3p_1 - x_1p_3$. Then we just see that:

$$\nabla f = \begin{pmatrix} 0 \\ p_3 \\ -p_2 \\ 0 \\ -x_3 \\ x_2 \end{pmatrix}; \nabla g = \begin{pmatrix} -p_3 \\ 0 \\ p_1 \\ x_3 \\ 0 \\ -x_1 \end{pmatrix}$$

Thus it is easy to calculate $\{f, g\} = -(\nabla g)J_0\nabla f = x_2p_1 - x_2p_1$ (there may be a sign error here but this is irrelevant for showing that it's not 0).

Exercise 1.22 Consider the Hamiltonian:

$$H = \sum_{j=1}^{n} a_j(x_j^2 + y_j^2)$$

with $a_j > 0$. Find the solution of the corresponding Hamiltonian differential equation. Prove that this system is integrable. Find all periodic solutions on the energy surface $H = c$ for $c > 0$.

Solution 1.22 Consider $H_i(x,y) = x_i^2 + y_i^2$. In the coordinates $(x_1, y_1, \ldots, x_n, y_n)$ the matrix J_0 splits into blocks where on the (x_i, y_i) each block acts as the standard 90 degree rotation. Thus we evidently have $\omega_0(dH_i, dH_j) = 0$. Thus any linear combination $H = \sum_i a_iH_i$ has the property that the H_i are conserved quantities, due to the linearity of the Poisson bracket. Thus the system is integrable.

If we examine the defining ODE for the Hamiltonian flow, we see that:

$$(\frac{dx_i}{dt}, \frac{dy_i}{dt}) = -a_i(-y_i, x_i)$$

Therefore the integral curves of the Hamiltonian system are precisely the vectors:

$$(x_i(t), y_i(t)) = (r_i \cos(-a_it), r_i \sin(-a_it))$$

Here $r^2 = \sum_i r_i^2$.

Now $I \subset \{1, \ldots, n\}$. Then we make the following claim: an orbit $(r_i \cos(-a_it), r_i \sin(-a_it))$ with $r_i > 0$ if and only if $i \in I$ is periodic if and only if there exists an s such that $\frac{a_is}{2\pi} \in \mathbb{Z}$ for all $i \in I$. If this is the case, then evidently any such orbit is s-periodic. Conversely, if such an orbit $(r_i \cos(-a_it), r_i \sin(-a_it))$ is s-periodic, then $a_is \in 2\pi\mathbb{Z}$ for all $i \in I$.

To see that the system is integrable, we show that we can find n conserved quantities H_i with $\{H_i, H_j\} = 0$ and $\{H, H_i\} = 0$ for all i, j. We take:

$$H_i = \frac{1}{2}(x_i^2 + y_i^2)$$

so that $\frac{\partial H_i}{\partial x_j} = \delta_{ij} x_i$ and $\frac{\partial H_i}{\partial y_j} = \delta_{ij} y_i$. Then:

$$\{H_i, H_j\} = \sum_k \frac{\partial H_i}{\partial x_k}\frac{\partial H_j}{\partial y_k} - \frac{\partial H_i}{\partial y_k}\frac{\partial H_j}{\partial x_k} = \sum_k x_i y_j \delta_{ik}\delta_{jk} - y_i x_j \delta_{ik}\delta_{jk}$$

The above expression is 0 if $i \neq j$ since then either $\delta_{ik} = 0$ or $\delta_{jk} = 0$. If $i = j$, then the expression is 0 because $\{F, F\} = -\{F, F\}$ and thus $\{F, F\} = 0$ for any function F. Thus these are n commuting conserved quantities which commute with H since $H = \sum_i a_i H_i$ and the Poisson bracket is bilinear.

Exercise 1.23 Carry out the Legendre transformation for the geodesic flow. Prove that the g-norm of the velocity $|\dot{x}|_g = \sqrt{\langle \dot{x}, g(x)\dot{x}\rangle}$ is constant along every geodesic.

Solution 1.23 We have that the conjugate momentum is $p_i = g_{ij} y^j$ and thus that $y^j = g^{ij} p_i$. Therefore under the Legendre transform we have:

$$H(x, p) = p_i y_i - L(x, y) = g^{ij} p_i p_j - \frac{1}{2} g_{ij} y^i y^j = g^{ij} p_i p_j - \frac{1}{2} g^{ij} p_i p_j = \frac{1}{2} g^{ij} p_i p_j$$

Hamilton's equations are:

$$\frac{dx_i}{dt} = \frac{dH}{dq_i} = g^{ij} p_i \qquad \frac{dp_k}{dt} = \frac{-1}{2}\frac{\partial g^{ij}}{\partial x_k} p_i p_j$$

We see that:

$$|\frac{dx_i}{dt}|^2 = g_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt} = g_{ij} g^{ik} g^{jl} q_k q_l = g^{ij} q_i q_j = 2H(x, p)$$

So the g-norm is conserved.

Exercise 1.24 (Exponential Map) Assume $g(x) = 1$ for large x so that the solutions $x(t)$ of Equation (1.12) exist for all time. The solution with initial conditions $x(0) = x$ and $\dot{x}(0) = \xi$ is called the geodesic through (x, ξ). Define the exponential map:

$$E : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, E(x, \xi) = x(1)$$

where $x(t)$ is the geodesic through (x, ξ). Prove that this geodesic is given by $x(t) = E(x, t\xi)$. Prove that there exists a constant $c > 0$ such that:

$$|E(x, \xi) - x - \xi| \leq c|\xi|^2$$

and deduce that:

$$\frac{\partial E_j}{\partial x_k}(x, 0) = \frac{\partial E_j}{\partial \xi_k}(x, 0) = \delta_{ij}, \quad \frac{\partial^2 E_j}{\partial x_k \partial \xi_l}(x, 0) = 0$$

Solution 1.24 Given a point p and velocity ξ, let $x(t)$ be the geodesic defined for $t \in [0, \infty)$ with $x(0) = p$ and $\frac{dx}{dt}(0) = \xi$. Then observe that $x_r(t) = x(rt)$ satisfies:

$$\frac{d^2 x_r^i}{dt^2}(t) = r^2 \frac{d^2 x^i}{dt^2}(rt) = r^2 \Gamma^i_{jk}(x(rt)) \frac{dx^j}{dt}(rt) \frac{dx^k}{dt}(rt) = \Gamma^i_{jk}(x(rt)) \frac{dx^j(rt)}{dt} \frac{dx^k(rt)}{dt}$$

Thus x_r is a geodesic with initial velocity $r\xi$ and initial point p, and it follows from uniqueness of ODE solutions that this is the unique solution. It thus follows that $x(t) = x_t(1) = E(p, t\xi)$.

To show the estimate, note that the geodesic equations yield:

$$|\frac{dx}{dt^2}| \leq C|\frac{dx}{dt}|^2 \leq C|\xi|^2$$

Here $C = \sup_{x \in \mathbb{R}^n}(|\Gamma^i_{jk}|)$ (which exists because $g \equiv 1$ outside of a compact set) and we use the fact that $|\frac{dx}{dt}|^2$ is conserved. Also we may assume that the norm is just the typical Euclidean norm when writing the estimate, since on any compact set K there exists a c_k with $|v|_g^2 \leq C_K |v|^2$ where $|v|$ is the usual Euclidean norm. Again, we may use the "compact support" of g to conclude that we can pick a constant so that such an inequality holds for all $x \in \mathbb{R}^n$.

Thus we may write:

$$|\frac{dx}{dt}(t) - \frac{dx}{dt}(0)| \leq \int_0^1 |\frac{dx}{dt^2}| \leq C|\xi|^2 t$$

$$|x(1) - x(0) - \frac{dx}{dt}(0)| \leq |\int_0^1 \frac{dx}{dt}(t) - \frac{dx}{dt}(0)| \leq \int_0^1 |\frac{dx}{dt}(t) - \frac{dx}{dt}(0)| \leq C|\xi|^2$$

This is precisely our estimate.

This estimate implies the derivative identities, as it gives us the Taylor expansion:

$$E^k(x, \xi) = x^k + \xi^k + |\xi|^2 h^k(x, \xi)$$

Thus we have:

$$\frac{\partial E^k}{\partial x^j} = \delta^k_j + O(\xi)$$

$$\frac{\partial E^k}{\partial \xi^j} = \delta^k_j + O(\xi)$$

$$\frac{\partial E^k}{\partial x^i \partial \xi^j} = 0 + 2\xi^j \frac{\partial h}{\partial x^i} + |\xi|^2 \frac{\partial h}{\partial x^i \partial \xi^j}$$

Exercise 1.25 Suppose that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism and:

$$g(x) = \phi^* h(x) = d\phi(x)^T h(\phi(x)) d\phi(x)$$

Prove that every geodesic $x(t)$ for g is mapped under ϕ to a geodesic $y(t) = \phi(x(t))$ for h. Deduce that the concept of the exponential map extends to manifolds.

Solution 1.25 We calculate using Einstein notation. The geodesic equations for the metric $h = \phi^* g$ and a curve x are:

$$g_{kl}\partial_m\phi^k\partial_j\phi^l\frac{dx^j}{dt} + \frac{1}{2}(\partial_i(g_{kl}\partial_m\phi^k\partial_j\phi^l) + \partial_j(g_{kl}\partial_m\phi^k\partial_i\phi^l) - \partial_m(g_{kl}\partial_i\phi^k\partial_j\phi^l))\frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$= g_{kl}\partial_m\phi^k\partial_j\phi^l\frac{dx^j}{dt} + \frac{1}{2}(\partial_i\phi^n\partial_n g_{kl}\partial_m\phi^k\partial_j\phi^l + g_{kl}\partial_i\partial_m\phi^k\partial_j\phi^l + g_{kl}\partial_m\phi^k\partial_i\partial_j\phi^l + \partial_j\phi^n\partial_n g_{kl}\partial_m\phi^k\partial_i\phi^l + g_{kl}\partial_j\partial_m\phi^k$$

$$+g_{kl}\partial_m\phi^k\partial_j\partial_i\phi^l - \partial_m\phi^n\partial_n g_{kl}\partial_i\phi^k\partial_j\phi^l - g_{kl}\partial_m\partial_i\phi^k\partial_j\partial^l - g_{kl}\partial_i\phi^k\partial_m\partial_j\phi^l)\frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$= g_{kl}\partial_m\phi^k\partial_j\phi^l\frac{dx^j}{dt} + g_{kl}\partial_m\phi^k\partial_i\partial_j\phi^l\frac{dx^i}{dt}\frac{dx^j}{dt}$$

$$+\frac{1}{2}(\partial_n g_{kl}\partial_i\phi^n\partial_m\phi^k\partial_j\phi^l + \partial_n g_{kl}\partial_j\phi^n\partial_m\phi^k\partial_i\phi^l - \partial_n g_{kl}\partial_m\phi^n\partial_i\phi^k\partial_j\phi^l)\frac{dx^i}{dt}\frac{dx^j}{dt}$$

From the second to third line we cancel some terms in the $\frac{1}{2}(\dots)$ part and reorganize the rest of the terms into two pieces. On the other hand the geodesic equations for the metrix g and the curve $\phi(x)$ is:

$$g_{mk}\frac{d}{dt^2}(\phi(x)^j) + \frac{1}{2}(\partial_k g_{lm} + \partial_l g_{km} - \partial_m g_{kl})\partial_i\phi^k\frac{dx^i}{dt}\partial_j\phi^l\frac{dx^j}{dt}$$

$$= g_{mk}(\partial_i\partial_j\phi^k\frac{dx^i}{dt}\frac{dx^j}{dt} + \partial_i\phi^k\frac{dx^i}{dt^2}) + \frac{1}{2}(\partial_k g_{lm} + \partial_l g_{km} - \partial_m g_{kl})\partial_i\phi^k\frac{dx^i}{dt}\partial_j\phi^l\frac{dx^j}{dt}$$

These two systems of equations for x merely differ by composition with the Jacobian $(\partial\phi)$ on the m index of the latter equation. Thus the second system vanishes if and only if the first does. This shows that geodesics are coordinate independent.

Exercise 1.26 The covariant derivative of a vector field $\xi(s) \in \mathbb{R}^n$ along a curve $x(s) \in \mathbb{R}^n$ is defined by:

$$(\nabla\xi)_k = \dot{\xi}_k + \sum_{i,j=1}^{n}\Gamma^k_{ij}\dot{x}\xi_j$$

A submanifold $L \subset \mathbb{R}^n$ is called totally geodesic if $\nabla\dot{x}(s) \in T_{x(s)}L$ for every smooth curve $x(s) \in L$. Prove taht L is totally geodesic if and only if TL is invariant under the geodesic flow.

Solution 1.26 First suppose that L were closed under geodesic flow. Pick a $p \in L$ and pass to coordinates U about p where p is 0 and $L \cap U \simeq \mathbb{R}^k \cap U \subset U \subset \mathbb{R}^n$. Then any geodesic x with $x(0) = p = 0$ and $\frac{dx}{dt}(0) = \xi$ has:

$$\frac{dx^k}{dt^2}|_p = -(\sum_{i,j=1}^{n}\Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt})|_p = \eta$$

Now suppose that the the left term were not in TL_p. Then for small time ϵ we have $\frac{dx}{dt}(\epsilon) = t\eta + O(t^2)$ and thus $x(\epsilon) = 0 + \epsilon\xi + \frac{\epsilon^2}{2}\eta + O(\epsilon^3)$ (in coordinates). Now we may split η into $\eta = \eta_L + \eta_{L^\perp}$, a parallel and non-parallel component. Then we may write:

$$x(\epsilon) = \epsilon\xi + \frac{\epsilon^2}{2}\eta_{||} + \frac{\epsilon^2}{2}\eta_{\perp} + O(\epsilon^3) = v_{||}(\epsilon) + \frac{1}{2}\epsilon^2\eta_{\perp} + O(\epsilon^3)$$

Taking $\epsilon \to 0$ we see that the result must have some non-zero perpendicular component to $x(\epsilon)$. Thus it must be the case that $\eta \in TL_p$, and thus that $-(\sum_{i,j=1}^n \Gamma_{ij}^k \xi^i \xi^j)|_p \in TL_p$ for any $p \in L$. This implies that $\nabla(\frac{dx}{dt}(s)) \in T_{x(s)}L$ since $\frac{dx}{dt}(s)$ is parallel to L for any such curve.

Conversely, suppose that L is not closed under geodesic flow. Then there exists a geodesic x with $x(0) = p \in L$ and $\frac{dx}{dt}(0) \in TL_p$, but $x(t) \notin L$ for some t.

Exercise 2.1 Let (V, ω) be a symplectic vector space and $\Phi : V \to V$ be a linear map. Prove that Φ is a linear symplecticmorphism if and only if its graph

$$\Gamma_\Phi = \{(v, \Phi v) \in V \oplus V | v \in V\}$$

is Lagrangian in $V \oplus V$ with symplectic form $\tilde{\omega} = (-\omega) \oplus \omega$.

Solution 2.1 If Φ is Lagrangian then for any $v \in V$ we have:

$$\tilde{\omega}(v \oplus \Phi v, w \oplus \Phi w) = -\omega(v, w) + \omega(\Phi v, \Phi w) = \Phi^*\omega(v, w) - \omega(v, w)$$

Thus $\Phi^*\omega(v, w) = \omega(v, w)$ for all $v, w \in V$ if and only if Γ_Φ is Lagrangrian.

Exercise 2.9 Identify a matrix with its graph as in Exercise 2.1 and use a construction similar to that in Exercise 2.8 to interpret the composition of symplectic matrices in terms of symplectic reduction.

Solution 2.9 Let (V_i, ω_i), $i = 1, 2, 3$, be three symplectic vector spaces with $\phi_{12} : V_1 \to V_2$ and $\phi_{23} : V_2 \to V_3$ with graphs $\Gamma_{12} \subset V_1 \oplus V_2$, $\Gamma_{23} \subset V_2 \oplus V_3$. Then consider the symplectic vector space $V_1 \oplus V_2 \oplus V_2 \oplus V_3$ with symplectic form $(-\omega_1) \oplus \omega_2 \oplus (-\omega_2) \oplus \omega_3$. Furthermore consider the subspaces $\Gamma_{12} \oplus \Gamma_{23}$ and $W = V_1 \oplus \Delta \oplus V_3$.

The first subspace is Lagrangian and the second is coisotropic with symplectic perpendicular $W^\omega = 0 \oplus \Delta \oplus 0$. We can see that this is equal to the symplectic perpendicular because it has dimension $4n - 3n = n$ and is contained in the symplectic perpendicular by direct computation. Under symplectic reduction we have the identification $W/W^\omega = V_1 \oplus V_3$ with symplectic form $(-\omega_1) \oplus \omega_3$. Furthermore:

$$(\Gamma_{12} \oplus \Gamma_{23}) \cap W = \{v_1 \oplus \phi_{12}(v_1) \oplus v_2 \oplus \phi_{23}(v_2) | \phi_1(v_1) = v_2\}$$

and thus under the quotient the Lagrangian $\Gamma_{12} \oplus \Gamma_{23}$ goes to the Lagrangian:

$$\Gamma_{13} = \{v_1 \oplus v_2 | v_2 = \phi_{23}(\phi_{12}(v_1))\}$$

Thus we can interpret composition of symplectomorphisms in terms of taking a product of their graphs and then performing a symplectic reduction along W.

Exercise 2.10 Let (V, ω) be a symplectic vector space and $W \subset V$ be any subspace. Prove that the quotient $V' = W/(W \cap W^\omega)$ carries a natural symplectic structure.

Solution 2.10 We simply define the symplectic form $\tilde{\omega}([v],[w]) := \omega(v,w)$. To show that this is well-defined, suppose that $v' = v + a$ and $w' = w + b$ with $a, b \in W \cap W^\omega$. Then $\omega(v', w') = \omega(v,w) + \omega(a,w) + \omega(v,b) + \omega(a,b) = \omega(v,w)$. To show that $\tilde{\omega}$ is non-degenerate, suppose that we see that $\tilde{\omega}([v],[w]) = 0$ for some $[v]$ and all $[w]$. Then $\omega(v,w) = 0$ for $v \in W$ and all $w \in W$, so $v \in W^\omega \cap W$ and thus $[v] = [0]$. This proves non-degeneracy. Bilinearity and anti-symmetry follow from the definition.

Exercise 2.11 Let $A = -A^T \in \mathbb{R}^{2nn}$ be a non-degenerate skew-symmetric matrix and define $\omega(z,w) = \langle Az, w\rangle$. Prove that a symplectic basis for $(\mathbb{R}^{2n}, \omega)$ can be constructed from the eigenvectors $u_j + iv_j$ of A.

Solution 2.11 Consider the matrix iA. This matrix is Hermitian, thus it admits a diagonalization with eigenvectors $x_i = u_i + iv_i$ and real eigenvalues λ_i. This is also a diagonalization of A with eigenvalues $-i\lambda_i$. Since A is non-degenerate, $\lambda_i \neq 0$ for any i. Now observe that $iA(u_i + iv_i) = -Av_i + iAu_i = \lambda_i u_i + i\lambda_i v_i$. Since A is real, it preserves real and imaginary vectors, so it follows that $Av_i = -\lambda_i u_i$ and $Au_i = \lambda_i v_i$. This implies that $A(u_i - iv_i) = -\lambda_i(u_i - iv_i)$. Thus eigenspaces occur in conjugate pairs, and the eigenvectors are of the form $\{\pm\lambda_1, \ldots, \pm\lambda_n\}$.

Now let $e_i = \frac{1}{|u_i|}u_i$ and $f_i = \frac{-1}{\lambda_i|u_i|}v_i$ (here we take only the $\lambda_i > 0$). Then we have:

$$\omega(e_i, f_i) = \langle e_i, Af_i\rangle = \langle e_i, e_i\rangle = 1$$

Thus the subspace e_i, f_i is symplectic. Furthermore, we can choose the $u_i + iv_i$ so that $u_1 \pm iv_1, \ldots, u_n \pm iv_n$ is orthonormal. Since each span $\mathrm{span}(e_i, f_i) = \mathrm{span}(u_i, v_i)$ is a union of the $\pm\lambda_i$ eigenspaces, and since eigenspaces of a self-adjoint operator are perpendicular, it follows that the spans $\mathrm{span}(e_i, f_i)$ are mutually symplectic orthogonal. Thus $e_1, f_1, \ldots, e_n, f_n$ is a symplectic basis.

Exercise 2.12 Consider a smooth family of symplectic forms $\omega_t(z,w) = \langle z, A_t w\rangle$ on $\mathbb{R}^{2n}$. Prove Corollary 2.4 by considering the family of subspaces $E_t \subset \mathbb{C}^{2n}$ generated by the eignevectors of A_t corresponding to the eigenvalues with positive imaginary part.

Solution 2.12 This is a less general version of Exercise 2.61. See that exercise: the proof is essentially the same, except here it is over I instead of a general simply connected neighborhood $U \subset \mathbb{R}^n$.

Exercise 2.13 Show that if β is any skew-symmetric bilinear form on the vector space W, there is a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of W such that $\beta(u_j, v_k) = \delta_{jk}$ and all other pairings $\beta(b_1, b_2)$ vanish.

Solution 2.13 Let $\phi : W \to W^*$ be the map $v \mapsto \beta(v, \cdot)$ and let $B = \ker(\phi)$. Let $b_1, \ldots, b_k$ and take any complimentary subspace $V \subset W$ so that $W = V \oplus B$. Then $\beta|_V$ is non-degenerate on V since $\beta(u,v) = 0$

for some $u \in V$ and all $v \in V$ implies that $\beta(u, v+b) = 0$ for all $v \in V$ and $b \in B$ as well, thus that $u \in B \cap V = \{0\}$. Thus we can find a symplectic basis $e_1, f_1, \ldots, e_n, f_n$ on V by Theorem 2.3.

Exercise 2.14 Show that if W is an isotropic, coisotropic or symplectic subspace of (V, ω) then any standard basis for (W, ω) extends to a symplectic basis for (V, ω).

Solution 2.14 If W is symplectic, then we can take a symplectic basis $e_1, f_1, \ldots, e_k, f_k$ and a symplectic basis $e_{k+1}, f_{k+1}, \ldots, e_n, f_n$ of W^ω. The union of the bases is then a symplectic basis of V, since pairings of a basis element from W with those of W^ω are necessarily 0.

Now let W be isotropic. We prove that we can extend any basis to a symplectic basis of V inductively. If W is 1-dimensional, this is trivial. Now suppose W is $k > 1$ dimensional and let $b_1, \ldots, b_k$ be a basis. Then $W' = \text{span}(b_1, \ldots, b_{k-1})$ is an isotropic subspace and by the induction assumption we may extend its basis to a symplectic basis $a_1, b_1, \ldots, a_{k-1}, b_{k-1}, e_1, f_1, \ldots, e_{n-k-1}, f_{n-k-1}$. Let $U = \text{span}(a_1, b_1, \ldots, a_{k-1}, b_{k-1})$ and observe that $U^\omega = \text{span}(e_1, f_1, \ldots, e_{n-k-1}, f_{n-k-1})$. Now observe that there must exist an e_i or f_i such that $\omega(e_i, b_k) \neq 0$ (resp. $\omega(f_i, b_k) \neq 0$). Otherwise $b_k \in U \cap \text{span}(b_1, \ldots, b_{k-1})^\omega = \text{span}(b_1, \ldots, b_{k-1})$, contradicting that b_i is a basis. Thus we may rescale the e_i or f_i to an a_k so that $\omega(a_k, b_k) = 1$. Then the resulting $a_1, b_1, \ldots, a_k, b_k$ is a symplectic basis of its span, and we extend this to a symplectic basis of V.

Then given a standard basis of W, $e_1, f_1, \ldots, e_n, f_n, b_1, \ldots, b_k$ and let $U = \text{span}(e_1, f_1, \ldots, e_n, f_n)$. Then $b_1, \ldots, b_k$ spans an isotropic subspace of the symplectic space U^ω, so we may use the previous result to find an extension of $b_1, \ldots, b_k$ to a symplectic basis of U^ω, and then combine the bases to get an extension $e_1, f_1, \ldots, e_n, f_n, a_1, b_1, \ldots, a_k, b_k$.

Exercise 2.15 Show that any hyperplane W in a 2n-dimensional symplectic vectorspace is coisotropic. Thus $W^\omega \subset W$ and $\omega|_W$ has rank $2(n-1)$.

Solution 2.15 Simply observe that any 1-dimensional subspace is isotropic. Indeed, $\omega(v, v) = 0$ for any v. Then any hyperplane H has H^ω 1-dimensional, and thus isotropic. Then since the symplectic perpendicular to an isotropic space is coisotropic, we have $(H^\omega)^\omega = H$ is coisotropic.

Exercise 2.16 Let $\Omega(V)$ denote the space of all symplectic forms on the vector space V. By considering the action of $GL(2n, \mathbb{R})$ on $\Omega(V)$ given by $\omega \mapsto \Phi^*\omega$ show that $\Omega(V) \simeq \text{GL}(2n, \mathbb{R})/\text{Sp}(2n)$.

Solution 2.16 By Theorem 2.3 we know that the action of $\text{GL}(2n, \mathbb{R})$ is transitive. Furthermore, the stabilizer of any symplectic form is isomorphic to the symplectic group. In fact, if $\omega = \Phi^*\omega_0$ then:

$$\text{Stab}(\omega) = \{\Phi^{-1} S \Phi | S \in \text{Sp}(2n)\} = \Phi^{-1}\text{Sp}(2n)\Phi$$

Thus the map $\text{GL}(2n, \mathbb{R})/\text{Sp}(2n) \to \Omega(V)$ given by:

$$[\Phi] \mapsto \Phi^*\omega_0$$

is bijective and smooth with respect to the smooth structure on the homogeneous space $\mathrm{GL}(2n,\mathbb{R})/\mathrm{Sp}(2n)$. Note that to prove the smoothness of this map really rigorously we need to know a slice theorem for Lie group actions, which we will not develop here.

Exercise 2.17 (The Gelfand-Robbin quotient) It has been noted by physicists for a long time that symplectic structures often arise from boundary value problems. The underlying abstract principle can be formulated as follows. Let H be a Hilbert space and $D : \mathrm{dom}(D) \to H$ be a symmetric linear operator with a closed graph and a dense domain $\mathrm{dom}(D) \subset H$. Prove that the quotient:

$$V = \mathrm{dom}(D^*)/\mathrm{dom}(D)$$

is a symplectic vector space with symplectic structure:

$$\omega([x],[y]) = \langle x, D^*y\rangle - \langle D^*x, y\rangle$$

Solution 2.17 First we prove that ω is well-defined and symplectic. First suppose that $[x'] = [x]$ so that $x' = x + a$, $a \in \mathrm{dom}(D)$. Then:

$$\omega([x'],[y]) = \langle x, D^*y\rangle + \langle a, D^*y\rangle - \langle D^*x, y\rangle - \langle D^*a, y\rangle = \langle x, D^*y\rangle - \langle D^*a, y\rangle + \langle D^*(a-a), y\rangle$$

$$= \langle x, D^*y\rangle - \langle D^*x, y\rangle = \omega([x],[y])$$

And similarly $\omega([x],[y']) = \omega([x],[y])$ if $[y'] = [y]$. The form is anti-symmetric by construction. To show that it is non-degenerate, suppose that $\omega([x],[y]) = 0$ for all $[y]$ and some $[x]$. Then:

$$\langle x, D^*y\rangle - \langle D^*x, y\rangle = 0$$

for all $y \in \mathrm{dom}(D^*)$ and $x \in \mathrm{dom}(D^*)$.

To see that Λ_0 is a Lagrangian subspace, first observe for any $x, y \in \Lambda_0$ we have $D^*x = D^*y = 0$, thus $\omega([x],[y]) = 0$. Thus $\Lambda_0 \subset \Lambda_0^\omega$. Similarly, if $y \in \Lambda_0^\omega$, then $\langle D^*x, y\rangle - \langle x, D^*y\rangle = \langle x, D^*y\rangle = 0$ for every $x \in \Lambda_0$.

Exercise 2.18 Consider the linear operator:

$$D = J_0\frac{d}{dt} \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

on the Hilbert space $H = L^2([0,1],\mathbb{R}^{2n})$ with $\mathrm{dom}(D) = W_0^{1,2}([0,1],\mathbb{R}^{2n})$. Show that in this case the Gelfand-Robbin quotient is given by $V = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with symplectic form $(-\omega_0) \times \omega_0$.

Solution 2.18 The definition of $\mathrm{dom}(D^*)$ is all of the $y \in H$ such that the map $x \mapsto \langle y, Dx\rangle$ extends from $\mathrm{dom}(D)$ to H. This is the map:

$$x \mapsto \int_0^1 \langle y, J_0 \frac{dx}{dt}\rangle dt$$

Now observe that if this map extends to H then y is differentiable in the weak sense, thus in $W^{1,2} \subset L^2$. Furthermore the Sobolev inequalities imply that $W^{1,2}$ functions are continuous in dimension 1, and continuity implies absolute continuity on a compact domain. Thus $\mathrm{dom}(D^*) \subset W^{1,2}([0,1], \mathbb{R}^{2n})$, the Sobolev space with no boundary limitations. Furthermore, for any $y \in W^{1,2}([0,1], \mathbb{R}^{2n})$ and any $x \in \mathrm{dom}(D)$ we have:

$$\int_0^1 \langle y, J_0 \frac{dx}{dt}\rangle dt = \int_0^1 -\langle J_0 \frac{dy}{dt}, x\rangle dt$$

There is no boundary contribution due to the vanishing of x at the ends of $[0,1]$. Thus $W^{1,2}([0,1], \mathbb{R}^{2n}) \subset \mathrm{dom}(D^*)$ and they are therefore equal.

Continuing, we may characterize $\mathrm{dom}(D^*)/\mathrm{dom}(D)$ as $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Indeed, we have $[x] = [x']$ if and only if we have $x - x' \in W_0^{1,2}([0,1], \mathbb{R}^{2n})$, i.e if and only if $x(0) = x'(0)$ and $x(1) = x'(1)$ (the other conditions are automatically satisfied). The map to the quotient can thus be given by $x \mapsto (x(0), y(0)) \in \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Then if we consider $[x], [y] \in V = \mathrm{dom}(D^*)/\mathrm{dom}(D)$, we see that:

$$\omega([x],[y]) = \int_0^1 \langle J_0 \frac{dy}{dt}, x\rangle - \langle J_0 \frac{dx}{dt}, y\rangle dt = \int \frac{d}{dt}\langle J_0 y, x\rangle dt = \langle J_0 y(1), x(1)\rangle - \langle J_0 y(0), x(0)\rangle$$

This is precisely the symplectci from $\omega_0 \oplus -\omega_0$.

Exercise 2.24 (i) Show that if $\Phi \in \mathrm{Sp}(2n)$ is diagonalizable, then it can be diagonalized with a symplectic matrix. (ii) Deduce from Lemma 2.20 that the eigenvalues of $\Phi \in \mathrm{Sp}(2n)$ occur either in pairs $\lambda, \lambda^{-1} \in \mathbb{R}$, $\lambda, \bar{\lambda} \in S^1$, or in complex quadruplets $\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$. (iii) Work out the conjugacy classes for matrices in $\mathrm{Sp}(2)$ and $\mathrm{Sp}(4)$.

Solution 2.24 (i) Let $\Phi \in \mathrm{Sp}(2n)$ be diagonalizable by $\mathrm{GL}(2n, \mathbb{R})$. Let $e_1, \ldots, e_{2n}$ be a basis of eigenvectors. Then $\omega(e_i, e_j) = \omega(\Phi e_i, \Phi e_j) = \lambda_i \lambda_j \omega(e_i, e_j)$, so either $\omega(e_i, e_j) = 0$ or $\lambda_i \lambda_j = 1$ for any pair e_i, e_j of eigenvectors. In particular, let $V_\lambda = \mathrm{span}\{e_i | \Phi e_i = \lambda_i^{\pm 1} e_i\}$. Then $V_\lambda^\omega = \oplus_{\lambda' \in \sigma(\Phi) | \lambda' \neq \lambda} V_{\lambda'}$ (here by $\sigma(\Phi)$ we denote the set of eigenvalues with $|\lambda| \geq 1$ so that we don't double count). To see this, observe that we have $\oplus_{\lambda' \in \sigma(\Phi) | \lambda' \neq \lambda} V_{\lambda'} \subset V_\lambda^\omega$ and by dimension counting they must be equal. Thus $\omega|_{V_\lambda}$ is symplectic, and Φ splits as a direct sum of symplectic maps $\Phi = \Phi_\lambda \oplus \Phi_\lambda^\omega$ with $\Phi_\lambda : V_\lambda \to V_\lambda$ and $\Phi_\lambda^\omega : V_\lambda^\omega \to V_\lambda^\omega$.

The above discussion implies that V splits symplectically as $V = \oplus_{\lambda \in \sigma(\Phi)} V_\lambda$ with Φ splitting as $\oplus_{\lambda \in \sigma(\Phi)} \Phi_\lambda$. Each Φ_λ has only two eigenvalues, $\lambda^{\pm 1}$, or only 1 is $\lambda = \pm 1$. By the symplectic Graham-Schmidt procedure, we know that we can find a symplectic basis e_i, f_i such that for every i we have $e_i, f_i \in V_\lambda$ for some λ and so that the collection of e_i, f_i with this property form a symplectic basis for V_λ. Thus we can get Φ into the block form $\oplus_\lambda \Phi_\lambda$ via a symplectic transformation and it suffices to show that we may find a symplectic change of basis on each V_λ individually to get Φ_λ into diagonal form.

Thus we may assume that we are in one of two cases. In the first case, $\Phi : V \to V$ has two real eigenvalues, λ and λ^{-1}. In the second case, λ has only one eigenvalue, ± 1. In the second case, $\Phi = \pm I$ and is already diagonalized. Thus we may restrict to the first case.

Therefore assume that $\Phi : V \to V$ is a symplectomorphism with only two eigenvalues, λ and λ^{-1} with $|\lambda| > 1$. Thus $V = V_\lambda \oplus V_{\lambda^{-1}}$. Now let $e \in V_\lambda$. Since $V_\lambda \subset e^\omega$ we must be able to pick an $f \in V_{\lambda^{-1}}$ so that $\omega(e, f) = 1$ by non-degeneracy. Now consider $W = \text{span}(e, f)$. Then $V_\lambda \cap f^\omega \subset e^\omega \cap f^\omega = W^\omega$ and likewise $e^\omega \cap V_{\lambda^{-1}} \subset e^\omega \cap f^\omega = W^\omega$. We have $\dim(V_\lambda \cap f^\omega) = \dim(V_\lambda) - 1$ since f^ω is codimension 1 in V and it does not contain V_λ since $e \in V_\lambda$, and likewise $\dim(e^\omega \cap V_{\lambda^{-1}}) = \dim(V_{\lambda^{-1}}) - 1$. Thus $W^\omega = (V_\lambda \cap f^\omega) \oplus (V_{\lambda^{-1}} \cap e^\omega)$. We see that for any $w = u + v \in W^\omega$ with $u \in V_\lambda \cap f^\omega$ and $v \in V_{\lambda^{-1}} \cap e^\omega$, then $\Phi w = \lambda u + \lambda^{-1} v \in W^\omega$. Thus we may recurse our argument onto $V' = W^\omega$, and by repeating it acquire a symplectic basis $e_1, \ldots, e_k, f_1, \ldots, f_k$ where $\Phi e_i = \lambda e_i$ and $\Phi f_i = \lambda^{-1} f_i$. This concludes the proof.

(ii) By Lemma 2.20, for a symplectic matrix S, $\lambda \in \sigma(S)$ implies that $\lambda^{-1} \in \sigma(S)$. Furthermore $\lambda \in \sigma(S)$ implies $\bar{\lambda} \in \sigma(S)$ because S is real. Thus we have 3 cases. If λ is real, then $\bar{\lambda} = \lambda$ λ occurs in a pair λ, λ^{-1}. If λ is complex and unit norm, then $\bar{\lambda} = \lambda^{-1}$ so λ occurs in a pair $\lambda, \bar{\lambda} \in U(1)$. If λ is both complex and non-unit length, then $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$ are all distinct. So λ occurs in that group of 4.

(iii) Here we will use the fact that if two real matrices M, N are similar over $\text{GL}(n, \mathbb{C})$ if and only if they are similar over $\text{GL}(n, \mathbb{R})$.

For $\text{Sp}(2) \simeq \text{SL}(2)$, we may use the fact that SL conjugacy classes are equal to GL conjugacy classes. Thus matrices in $\text{Sp}(2)$ are classified up to conjugacy by their Jordan normal form. These are:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Here $\eta \in U(1)$ and $\lambda \in \mathbb{R}$.

In the $\text{Sp}(4)$ things get more complicated.

$$\begin{pmatrix} \xi & 0 & 0 & 0 \\ 0 & \bar{\xi} & 0 & 0 \\ 0 & 0 & \xi^{-1} & 0 \\ 0 & 0 & 0 & \bar{\xi}^{-1} \end{pmatrix} \quad \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \end{pmatrix}$$

Exercise 2.25 Use the argument of Proposition 2.22 to prove that the inclusion

$$\text{O}(2n)/\text{U}(n) \hookrightarrow \text{GL}(2n, \mathbb{R})/\text{GL}(n, \mathbb{C})$$

of homogeneous spaces is a homotopy equivalence. Prove similarly that the inclusion:

$$\text{O}(2n)/\text{U}(n) \hookrightarrow \text{GL}(2n, \mathbb{R})/\text{Sp}(2n)$$

is a homotopy equivalence.

Solution 2.25 First observe that we have the following commutative diagram.

$$\begin{array}{ccccc} \mathrm{U}(n) & \rightarrow & \mathrm{O}(2n) & \rightarrow & \mathrm{O}(2n)/\mathrm{U}(n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{GL}(2n,\mathbb{C}) & \rightarrow & \mathrm{GL}(2n,\mathbb{R}) & \rightarrow & \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}) \end{array}$$

The rows here are fibration diagrams $F \to E \to B$. This yields a commutative diagram composed of the two resulting homotopy long exact sequences.

$$\begin{array}{ccccccccccc} \dots & \rightarrow & \pi_i(\mathrm{U(n)}) & \rightarrow & \pi_i(\mathrm{O}(2n)) & \rightarrow & \pi_i(\mathrm{O}(2n)/\mathrm{U}(n)) & \rightarrow & \pi_{i-1}(\mathrm{U(n)}) & \rightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \rightarrow & \pi_i(\mathrm{GL}(2n,\mathbb{C})) & \rightarrow & \pi_i(\mathrm{GL}(2n,\mathbb{R})) & \rightarrow & \pi_i(\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})) & \rightarrow & \pi_{i-1}(\mathrm{GL}(2n,\mathbb{C})) & \rightarrow & \dots \end{array}$$

The maps of $\pi_i(\mathrm{O}(2n)) \to \pi_i(\mathrm{GL}(2n,\mathbb{R}))$ and $\pi_i(\mathrm{U}(n)) \to \pi_i(\mathrm{GL}(n,\mathbb{C}))$ are isomorphisms due to the existence of the polar decomposition. Any $M \in \mathrm{GL}(2n,\mathbb{R})$ decomposes as $M = QR$ with $Q = (MM^T)^{1/2}$ positive definite and $R \in \mathrm{O}(2n)$. We can then use the retraction $h_t(M) = (MM^T)^{-t/2}M$. This essentially relies on the fact that the space of positive definite matrices is retractable to the identity, via the same homotopy.

We may thus apply the five lemma to conclude that the maps $\pi_i(\mathrm{O}(2n)/\mathrm{U}(n)) \to \pi_i(\mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}))$ are isomorphisms. Whiteheads lemma then implies that since the natural map $q : \mathrm{O}(2n)/\mathrm{U}(n) \to \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C})$ given by taking $M\mathrm{U}(n) \to M\mathrm{GL}(n,\mathbb{C})$ (as cosets) is a homotopy equivalence, since it induces an isomorphism on all homotopy groups.

An identical argument will work if we replace $\mathrm{GL}(n,\mathbb{C})$ with $\mathrm{Sp}(2n)$. The retraction in that case uses the polar decomposition described in Proposition 2.22.

Exercise 2.26 Let $\mathrm{SP}(n,\mathbb{H})$ denote the group of quaternionic matrices $W \in \mathbb{H}^{n\times n}$ such that $W^*W = 1$. Prove that $\mathrm{SP}(n,\mathbb{H})$ is a maximal compact subgroup of $\mathrm{Sp}(2n,\mathbb{C})$ and that the quotient $\mathrm{Sp}(2n,\mathbb{C})/\mathrm{SP}(n,\mathbb{H})$ is contractible.

Solution 2.26 Again we will use the polar decomposition. Any $M \in \mathrm{GL}(2n,\mathbb{C})$ decomposes as:

$$M = QR = (MM^\dagger)^{1/2}R$$

Here Q is positive definite and R is unitary.

Now we argue that $(MM^\dagger)^{1/2} \in \mathrm{Sp}(2n,\mathbb{C})$. First observe that $M \in \mathrm{Sp}(2n,\mathbb{C})$ implies $\bar{M}, M^T \in \mathrm{Sp}(2n,\mathbb{C})$ since then $M^TJM = J$ implies:

$$J = (M^T)^{-1}JM^{-1} = (MJ^{-1}M^T)^{-1} \implies J = -J^{-1} = -MJ^{-1}M^T = MJM^T$$

$$J = \bar{J} = \overline{M^TJM} = (\bar{M})^TJ\bar{M}$$

Now we prove the analogues of Lemma 2.20 and 2.21, which are the same as in the real case.

We have $M^T = JM^{-1}J^{-1}$ so M^T is conjugate to M^{-1} and thus they have the same eigenvalues. Therefore M and M^{-1} have the same eigenvalues, and thus if $\lambda \in \sigma(M)$ has $\lambda \neq \pm 1$ then $\lambda^{-1} \in \sigma(M)$. Since $\det(M) = 1$, we must therefore have an even number of -1 eigenvalues and since $\dim(M) = 2n$, we have an even number of the remaining 1 eigenvalues as well.

Now observe that if if v and w are in eigenspaces of M with eigenvalues λ, λ' and $\lambda' \neq \lambda^{-1}$ then they are symplectic orthogonal. Indeed:

$$\omega(v, w) = \omega(Mv, Mw) = \lambda\lambda'\omega(v, w)$$

So if $\lambda\lambda' \neq 1$ then $\omega(v, w) = 0$. Then we can argue again that if $P = P^\dagger$ and $P \in \mathrm{Sp}(2n, \mathbb{C})$ then $P^\alpha \in \mathrm{Sp}(2n, \mathbb{C})$ for all $\alpha \in \mathbb{R}$. We can check this by splitting $\mathbb{C}^{2n}$ into eigenspaces. If v, w in non-complimentary eigenspaces then $\omega(P^\alpha v, P^\alpha w) = \omega(v, w) = 0$ and otherwise:

$$\omega(P^\alpha v, P^\alpha w) = (\lambda\lambda^{-1})^\alpha \omega(v, w) = \omega(v, w)$$

Thus $(MM^\dagger)^{-\alpha/2} \in \mathrm{Sp}(2n, \mathbb{C})$ for $\alpha \in [0, 1]$ and thus the homotopy $h_t(M) = (MM^\dagger)^{-\alpha/2}M$ is a retraction of $\mathrm{Sp}(2n, \mathbb{C})$ to $U(n) \cap \mathrm{Sp}(2n, \mathbb{C})$.

Now we show that $\mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C}) = \mathrm{U}(2n) \cap \mathrm{GL}(n, \mathbb{H}) = \mathrm{SP}(n)$. We show this on the level of Lie algebras, i.e $\mathrm{u}(2n) \cap \mathrm{sp}(2n, \mathbb{C}) = \mathrm{u}(2n) \cap \mathrm{gl}(n, \mathbb{H})$. We see that:

$$\mathrm{u}(2n) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | A^\dagger = -A, D^\dagger = -D, C = -B^\dagger\}$$

$$\mathrm{sp}(2n, \mathbb{C}) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | C = C^T, D = -A^T, B = B^T\}$$

$$\mathrm{gl}(n, \mathbb{H}) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} | D = \bar{A}, C = -\bar{B}\}$$

Now we verify $u(2n) \cap \mathrm{sp}(2n, \mathbb{C}) \subset \mathrm{u}(2n) \cap \mathrm{gl}(n, \mathbb{H})$.

$$D = -A^T = \overline{(-A^\dagger)} = \bar{A} \qquad C = C^T = \overline{C^T} = -\bar{B}$$

Now we verify $\mathrm{u}(2n) \cap \mathrm{gl}(n, \mathbb{H}) \subset u(2n) \cap \mathrm{sp}(2n, \mathbb{C})$.

$$D = -A^T = (A^\dagger)^T = -A^T \qquad C = -\bar{B} = C^T \qquad B = -\bar{C} = B^T$$

Thus we have the equivalence of the groups as subgroups of $\mathrm{U}(2n)$. This shows that h_t retracts $\mathrm{Sp}(2n, \mathbb{C})$ to $\mathrm{SP}(n)$, this that the inclusion $\mathrm{SP}(n) \to \mathrm{Sp}(n, \mathbb{C})$ is a homotopy equivalence.

To prove that this is maximal, we show that any compact subgroup $G \subset \mathrm{SP}(n)$ is contained in a subgroup conjugate to $\mathrm{SP}(n)$. This is again the same as the real case. That is, we take the Haar measure dG associated to the Lie group G and take $A = \int_{M \in G} M^T M dG$. A is then a symmetric positive definite map which is invariant under conjugation by elements of G, and thus $G \subset U(A)$, the unitary group with respect to A.

Furthermore A is symplectic. This is a poorly elaborated point in the book! We see this as so:

$$A^T JA = \int_{M\times N\in G\times G} M^\dagger MJN^\dagger NdGdG$$

$$= \int_{M\times M\in\Delta\subset G\times G} M^\dagger MJM^\dagger MdG + \int_{M\times N\in M\times N-\Delta} \frac{1}{2}(M^T MJN^T N + N^T NJM^T M)dGdG$$

$$= J\int_{M\times M\in\Delta\subset G\times G} dG + \int_{M\times N\in M\times N-\Delta} \frac{1}{2}(M^T MJN^T N - M^T MJN^T N)dGdG = J$$

Thus we may use the conjugation map $G \to U(2n)$ given by $M \mapsto A^{-1/2}MA^{1/2}$ to see that it is conjugate to a subgroup of SP)(n).

Exercise 2.27 Let:

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$$

What is the relationship between $\det\Psi \in \mathbb{R}$ and $\det(X + iY) \in \mathbb{C}$?

Solution 2.27 We take the J matrix to be the block matrix $J = \oplus_{i=1}^n \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ instead of the block matrix $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Now observe that both determinants are invariant under $\mathrm{GL}(2n, \mathbb{C})$ conjugation, so if A is diagonalizable we can assume that Ψ is a block matrix of 2×2 of the form:

$$\Psi = \begin{pmatrix} \Lambda_1 & 0 & \dots & 0 \\ 0 & \Lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \Lambda_n \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R}) \qquad \Lambda_i = \begin{pmatrix} a_i & -b_i \\ b_i & a_i \end{pmatrix}$$

The determinant of such a matrix is the product of the determinants, so:

$$\det(\Psi) = \prod_i \det(\Lambda_i) = \prod_i a_i^2 + b_i^2$$

Furthermore, if we let A be the diagonal matrix $A_{ij} = a_i\delta_{ij}$ and similarly for B, we have $\det(A + iB) = \prod_i(a_i + ib_i)$. These are related by $|\det(A + iB)|^2 = \det(\Psi)$. Since diagonalizable matrices are dense, this formula holds for all matrices Ψ.

Exercise 2.28 The Siegel upper half space S_n is the space of complex symmetric matrices $Z = X + iY \in \mathbb{C}^{\ltimes\times\ltimes}$ with positive definite imaginary part Y. The symplectic group $\mathrm{Sp}(2n)$ acts on S_n via fractional linear transformations $\Psi_* : S_n \to S_n$ defined by:

$$\Psi_* Z = (AZ + B)(CZ + D)^{-1}, \qquad \Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Here we use the notation of Exercise 1.13. Prove that Φ_* is well-defined: if $Z \in S_n$ then the matrix is $CZ + D$ is invertible and $\Psi_* Z \in S_n$. Prove that:

$$\Psi_* \Phi_* Z = (\Psi\Phi)_* Z$$

for $\Phi, \Psi \in \mathrm{Sp}(2n)$ and $Z \in S_n$. Prove that the action is transitive. Prove that:

$$\Psi(iI) = iI \iff \Psi \in U(n)$$

Deduce that the map $\Psi \to \Psi_*(iI)$ induces a diffeomorphism from the homogeneous space $\mathrm{Sp}(2n)/U(n)$ to the Seigel upper half space S_n. Thus the quotient $\mathrm{Sp}(2n)/U(n)$ inherits the complex structure of S_n.

Solution 2.28 We prove everything except that the maps are well-defined, which we postpone until the end. First observe that if:

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \Phi = \begin{pmatrix} E & F \\ G & H \end{pmatrix} \quad \Phi\Psi = \begin{pmatrix} EA + FB & EB + FD \\ GA + HC & GB + HD \end{pmatrix}$$

Then we have:

$$\Phi_* \Psi_* Z = (E(AZ + B)(CZ + D)^{-1} + F)(G(AZ + B)(CZ + D)^{-1} + H)^{-1}$$

$$= (E(AZ + B) + F(CZ + D))(CZ + D)^{-1}((G(AZ + B) + H(CZ + D))(CZ + D)^{-1})^{-1}$$
$$= (E(AZ + B) + F(CZ + D))(G(AZ + B) + H(CZ + D))^{-1}$$
$$= ((EA + FC)Z + (EB + FD))((GA + HC)Z + (GB + HD))^{-1} = (\Phi\Psi)_* Z$$

So the map is a group homomorphism into complex automorphisms of S_n.

To see the group is transitive, it suffices to check that the map $\mathrm{sp}(2n) \to T_Z S_n$ induced by the group representation is surjective. This then implies that the group action is locally transitive in a neighborhood of any $Z \in S_n$, and then by a continuity argument and the fact that S_n is connected we may conclude that the group action is in fact globally transitive.

We thus need to check that for any $N \in T_Z S_n$ (i.e an $N = U + iV$ with U, V symmetric) there is a family of symplectic maps $\frac{d}{dt}(\Phi_t)|_{t=0} = S$ with $S \in \mathrm{sp}(2n)$ with $\frac{d}{dt}((\Phi_t)_* Z) = N$. Recall that $\mathrm{sp}(2n)$ can be described as the set of matrices splitting into blocks A, B, C, D in the obvious way with $D = -A^T$, $B = B^T$ and $C = C^T$. Now observe that:

$$\frac{d}{dt}((\Phi_t)_* Z)|_{t=0} = \frac{d}{dt}((Z + t(AZ + B) + O(t^2))(tCZ + +1 + tD + O(t^2))^{-1})|_{t=0}$$

$$= \frac{d}{dt}((Z + t(AZ + B) + O(t^2))(1 - t(CZ + D) + O(t^2)))|_{t=0}$$
$$= AZ + B - ZCZ - ZD = AZ + ZA^T + B - ZCZ$$

Thus we just need to prove that we can pick an A, B, C, D satisfying the above identities (to be in $\mathrm{sp}(2n)$) such that $AZ + ZA^T + B - ZCZ = U + iV$. We can assume $C = 0$. Then in terms of $Z = X + iY$ we want

$AX + B + XA^T = U$ and $AY + YA^T = V$. Here B is symmetric and A can be anything. But since the map $A \mapsto AY + YA^T$ is the composition of the map $A \to YA$ (which is a bijection because Y is positive definite) and the symmetrization map $P \mapsto P + P^T$ we know that it is surjective. So we can certainly pick an A. Then we may simply pick $B = U - AX - XA^T$, $D = -A^T$ and $C = 0$ to find the $S \in \mathrm{sp}(2n)$ of interest. This proves transitivity.

Now we identify the stabilizer of a point. We pick $i1$. Then we see that we want to find symplectic Ψ so that the blocks A, B, C, D satisfy:

$$(iA + B)(iC + D)^{-1} = i1; \quad iA + B = -C + iD; B = -C, A = D$$

This is in fact the condition that $\Psi J = J\Psi$. Thus Ψ is in the stabilizer of $i1$ if and only if $\Psi \in \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n) = U(n)$. Since we have in general that a space H with a transitive group action G is diffeomorphic to $G/\mathrm{Stab}(p)$.

Exercise 2.32 Prove that the orthogonal compliment of a Lagrangian subspace $\Lambda \subset \mathbb{R}^{2n}$ with respect to the standard metric is given by $\Lambda^\perp = J\Lambda$. Deduce if $u_1, \dots, u_n$ is an orthonormal basis of Λ then the vectors $u_1, \dots, u_n, Ju_1, \dots, Ju_n$ forms a basis for $\mathbb{R}^{2n}$ which is both symplectic and orthogonal. Relate this to the proof of Lemma 2.31.

Solution 2.32 Since J is orthogonal, the vectors $Ju_1, \dots, Ju_n$ are independent from each other. Now observe that since $\omega(v, w) = \langle v, Jw \rangle$, we see that $\omega(e_i, e_j) = \omega(Je_i, Je_j) = \langle e_i, Je_j \rangle = 0$ and $\langle e_i, e_j \rangle = \langle Je_i, Je_j \rangle = -\omega(Je_i, e_j) = -\delta_{ij}$. These calculations show that the set $e_1, \dots, e_n, Je_1, \dots, Je_n$ are a set of $2n$ orthonormal (thus an orthonormal basis) and standard with respect to the symplectic form.

One way to interpret this in terms of Lemma 2.31 is to note that this elucidates the relationship between the Lagrangian and its perpendicular Lagrangian, relating them via the unitary transformation J.

Exercise 2.33 State and prove the analog of Lemma 2.31 for isotropic, symplectic and coisotropic subspaces.

Solution 2.33 (i) We prove that if V is isotropic, coisotropic or symplectic then so is ΨV for any symplectic map. This is clear: $\omega(v, w) = 0$ for all $v \in V$ and some w if and only if $\omega(\Psi v, \Psi w) = 0$. Thus $(\Psi V)^\omega = \Psi V^\omega$ and $V^\omega \subset V$ (resp. $V \subset V^\omega$) if and only if $(\Psi V)^\omega \subset \Psi V$ (resp. $\Psi V \subset (\Psi V)^\omega$). Likewise if $\omega|_V$ is non-degenerate then $\omega|_{\Psi V} = \Psi^* \omega|_{\Psi V}$ is as well.

(ii) We prove that symplectic maps are transitive on isotropic, symplectic and coisotropic subspaces of the same rank. First suppose that V is isotropic of rank $k \leq n$. Then we can pick a Lagrangian L with $V \subset L$ and a matrix:

$$M = \begin{bmatrix} X \\ Y \end{bmatrix}$$

with orthogonal columns forming a basis of L, and where the first k columns form a basis of V. Then the matrix:

$$\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$$

is a map taking the isotropic space spanned by $e_1, \ldots, e_k$ to V. Note that Ψ is in fact unitary.

For the coisotropic case, where the rank of V is $k \geq n$, we can simply observe thav V^ω is isotropic and find a symplectic Ψ taking V^ω to the standard rank $2n-k$ isotropic space as above. Then since $(\Psi V^\omega)^\omega = \Psi V$ we may conclude that V goes to the symplectic perpendicular of the standard rank $2n-k$ isotropic space. Again, Ψ here is unitary.

Finally, if V is symplectic of rank $2k$ then we can take a symplectic bases $g_1, h_1, \ldots, g_k, h_k$ for V and $g_{k+1}, h_{k+1}, \ldots, g_n, h_n$ for V^ω, and then use the map Φ given by $g_i \mapsto e_i, h_i \mapsto f_i$.

(iii) Finally, we characterize these Grassmanians as homogeneous spaces. The symplectic case of $\mathrm{SGr}(n,k)$ is simple enough: the stabilizer of a symplectic subspace of rank k is isomorphic to the symplectic group $\mathrm{Sp}(2k)$ so $\mathrm{SGr}(n,k) \simeq \mathrm{Sp}(2n)/\mathrm{Sp}(2k)$. For the isotropic case, $\mathrm{IGr}(n,k)$, we observe that any choice of $X + iY \in U(n)$ yields a rank k isotropic space as the span V the first k columns of M (where M is as above). Two such M yield the same V if and only if they are related by right multiplication by an element of $O(k) \times O(n-k)$ (an orthogonal transformation preserving the span of the first k columns and their ortho-compliment). So we have $\mathrm{IGr}(n,k) \simeq \mathrm{U}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k)$. Finally, for coisotropic $\mathrm{CGr}(n,k)$ we use duality via taking the symplectic perp to see that $\mathrm{CGr}(n,k) \simeq \mathrm{IGr}(n, 2n-k) \simeq \mathrm{IGr}(n,k) \simeq \mathrm{U}(n)/\mathrm{O}(2n-k) \times \mathrm{O}(k-n)$.

Exercise 2.34 Consider the vertical Lagrangian:

$$\Lambda_{\mathrm{vert}} = \{z = (x,y) \in \mathbb{R}^{2n} | x = 0\}$$

Use Lemma 2.30 to show that $\mathcal{L}(n)$ is the disjoint union:

$$\mathcal{L}(n) = \mathcal{L}_0(n) \cup \Sigma(n)$$

where $\mathcal{L}_0(n)$ can be identified with the affine space of symmetric $n \times n$ matrices and $\Sigma(n)$ consists of all Lagrangian subspaces which do not intersect Λ_{vert} transversely. The set $\Sigma(n)$ is called the Maslov cycle.

Solution 2.34 We simply prove that a Lagrangian L can be given as a graph over Λ_{hor} if and only if it is transverse to Λ_{vert}. But observe that $\Lambda_{\mathrm{vert}} = J\Lambda_{\mathrm{hor}} = (\Lambda_{\mathrm{hor}})^\perp$. Furthermore an n-dimensional subspace V of $\mathbb{R}^{2n}$ can be described as a graph over Λ_{hor} if and only if orthogonal projection $V \to \Lambda_{\mathrm{hor}}$ is an isomorphism, i.e has no kernel. But the kernel of this map is precisely $V \cap \Lambda_{\mathrm{vert}}$. So there is no kernel if and only if $V \cap \Lambda_{\mathrm{vert}} = 0$, i.e if and only if V and Λ_{vert} are transverse.

Exercise 2.36 The Maslov index of a loop $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(V,\omega)$ of a Lagrangian subspace in a general symplectic vector space is defined as the Maslov index of the loop $t \mapsto \Psi^{-1}\Lambda(t) \in \mathcal{L}(n)$, where $\Psi : (\mathbb{R}^{2n}, \omega_0) \to (V,\omega)$ is a linear symplectomorphism. Show that this definition is independent of Ψ. Show that if one reverses the sign of ω then the sign of the Maslov index reverses also.

Solution 2.36 For the first part, simply observe that if $\Psi, \Phi : (\mathbb{R}^{2n}, \omega_0) \to (V, \omega)$ are two different symplectomorphisms then $\Psi^{-1}\Lambda(t) = \Psi^{-1}\Phi(\Phi^{-1}\Lambda(t))$. Thus if we denote the constant path $t \mapsto \Psi^{-1}\Phi$ as $\Psi^{-1}\Phi$ then we have $\mu(\Psi^{-1}\Phi) = 0$ and therefore:

$$\mu(\Phi^{-1}\Lambda) = \mu(\Psi^{-1}\Lambda) + 2\mu(\Psi^{-1}\Phi) = \mu(\Psi^{-1}\Lambda)$$

by the composition axiom. This shows that if (U, ω) and (V, ω') are two symplectic vectorspaces, $\Psi : U \to V$ is a symplectomorphism and $\Lambda : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(U, \omega)$ is a path of Lagrangians, then $\mu(\Lambda) = \mu(\Psi\Lambda)$.

For the second part, by the previous argument we may reduce to the case of $(\mathbb{R}^{2n}, \omega_0)$ and $(\mathbb{R}^{2n}, -\omega_0)$. It suffices to check that Maslov index for the generating homotopy class, $\Lambda_0(t) \oplus \mathbb{R}^{n-1} \subset \mathbb{C} \oplus \mathbb{C}^{n-1}$ with $\Lambda_0(t) = e^{2\pi i t}\mathbb{R}$ changes sign, and due to the direct sum formula it even suffices to check for $\Lambda_0(t)$. The isomorphism $c : (\mathbb{C}, \omega_0) \to (\mathbb{C}, -\omega_0)$ is just conjugation $z \mapsto \bar{z}$, so under this map the family $\Lambda_0(t) = e^{2\pi i t}\mathbb{R}$ gets sent to $c\Lambda_0(t) = e^{-2\pi i t}\mathbb{R}$. This is the same curve with the reverse parameterization, thus $\mu(c\Lambda) = \mu(\Lambda(-\cdot)) = -\mu(\Lambda)$.

Exercise 2.37 Let $\Psi : \mathbb{R}/\mathbb{Z} \to \mathrm{Sp}(V, \omega)$ be a loop of linear symplectomorphisms. Prove that the corresponding loop $\Gamma_\Psi : \mathbb{R}/\mathbb{Z} \to \mathcal{L}(V \times V, (-\omega) \times \omega)$ of Lagrangian graphs has twice the Maslov index, i.e $\mu(\Gamma_\Psi) = 2\mu(\Psi)$.

Solution 2.37 We see that $\Lambda(t) = \{v \oplus \Psi(t)v | v \in V\} = (1 \oplus \Psi(t))\Lambda_0(t)$ where $\Lambda_0(t) = \{v \oplus v | v \in V\}$ and $1 \oplus \Psi(t)$ is the family of symplectomorphisms given by $v \oplus w \mapsto v \oplus \Psi(t)w$. Thus we have:

$$\mu(\Lambda(t)) = \mu(\Lambda_0(t)) + 2\mu(1 \oplus \Psi(t)) = 0 + 2(\mu(1) + \mu(\Psi(t))) = 2\mu(\Psi(t))$$

Here we apply the product axiom, then the direct sum axiom, then the homotopy axiom.

Exercise 2.40 Prove that every anti-symplectic linear map has determinant $(-1)^n$. Prove that every anti-symplectic linear map preserves the linear symplectic width of subsets of $\mathbb{R}^{2n}$.

Solution 2.40 For the first part, suppose that Ψ is anti-symplectic. Then:

$$(-1)^n\omega^n = (-\omega)^n = (\Psi^*\omega)^n = \det(\Psi)\omega^n$$

So $\det(\Psi) = (-1)^n$. Here by ω^n we mean the nth wedge power of ω.

For the second part, consider a subset $A \subset \mathbb{R}^{2n}$. We observe that a ball $B^{2n}(r)$ can be mapped into A via a symplectic map if and only if it can be mapped in via an anti-symplectic map. Indeed, we have the standard involutive anti-symplectic map Φ given by $e_i \mapsto f_i, f_i \mapsto e_i$ which fixes any ball $B^{2n}(r)$. Thus if we have an affine symplectic (resp. anti-symplectic) $\psi : B^{2n}(r) \to A$ then $\psi \circ \Phi$ is an affine anti-symplectic (resp. symplectic) map $B^{2n}(r) \to A$.

Now for an anti-symplectic affine map ψ consider $\psi(A)$. Then if $\xi : B^{2n}(r) \to A$ were an affine symplectic map to A, then $\psi\xi\Phi$ is a symplectic map $B^{2n}(r) \to \psi(A)$. Furthermore if $\xi : B^{2n}(r) \to \psi(A)$

is symplectic, then $\psi^{-1}\xi\Phi$ is a symplectic map $B^{2n}(r) \to A$. Thus we have:

$$w(A) = \sup\{\pi r^2 | \psi(B^{2n}(r)) \subset A \text{ for some } \psi \in \mathrm{ASp}(\mathbb{R}^{2n})$$

$$= \sup\{\pi r^2 | \xi(B^{2n}(r)) \subset \psi(A) \text{ for some } \xi \in -\mathrm{ASp}(\mathbb{R}^{2n}) = w(\psi(A))$$

Exercise 2.46 Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid and define the dual ellipsoid by:

$$E^* = \{v \in \mathbb{R}^{2n} | \langle v, e \rangle \leq 1 \forall e \in E\}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. Prove that:

$$E^{**} = E, (\Psi E)^* = (\Psi^T)^{-1} E^*$$

for $\Psi \in \mathrm{Sp}(2n)$.Prove that the symplectic spectrum of E^* is given by $(1/r_n, \ldots, 1/r_1)$ where $(r_1, \ldots, r_n)$ is the symplectic spectrum of E. Deduce that the dual of a linear symplectic ball is again a linear symplectic ball.

Solution 2.46 First we show that $(\Psi E)^* = (\Psi^T)^{-1} E^*$. Indeed, we see that:

$$\langle v, e \rangle = \langle v, \Psi^{-1} \Psi e \rangle = \langle (\Psi^{-1})^T v, \Psi e \rangle$$

Thus $\langle v, e \rangle \leq 1$ for all $e \in E$ if and only if $\langle (\Psi^{-1})^T v, \Psi e \rangle \leq 1$ for all $\Psi e \in \Psi E$. This implies that E^* and $(\Psi E)^*$ are symplectomorphic. Thus it suffices to show that $E = E^{**}$ for standard ellipsoids, which will follow from the last statement.

Now suppose that E has spectrum $(r_1, \ldots, r_n)$. Then $E = \{e | \langle e, Re \rangle \leq 1\}$ where $R = \mathrm{diag}(r_1, \ldots, r_n)$. Now suppose that $v \in \bar{E}$ where $\bar{E} = \{v | \langle v, R^{-1} v \rangle \leq 1\}$. Then for any $e \in E$ we have:

$$|\langle v, e \rangle|^2 = \langle R^{-1/2} v, R^{1/2} e \rangle \leq \langle v, R^{-1} v \rangle \langle e, Re \rangle \leq 1$$

Thus $v \in E^*$. If on the other hand $\langle v, R^{-1} v \rangle = c > 1$ and $e \notin \bar{E}$ then $w = c^{-1/2} R^{-1} v$ satisfies:

$$\langle w, Rw \rangle = c^{-1} \langle R^{-1} v, R R^{-1} v \rangle = c^{-1} \langle v, R^{-1} v \rangle = 1$$

Thus $w \in E$. But then we have:

$$\langle w, v \rangle = c^{-1/2} \langle v, R^{-1} v \rangle = c^{1/2} > 1$$

so $e \notin E^*$. Thus $\bar{E} = E^*$ and we are done.

Exercise 2.49 Let (V, ω) be a symplectic vector space and J be a complex structure on V. Prove that the following are equivalent. (i) J is compatible with ω. (ii) The bilinear form $g_J(v, w) = \omega(v, Jw)$ is symmetric, positive definite and J-invariant. (iii) The form $H : V \times V \to \mathbb{C}$ given by $H(v, w) = \omega(v, Jw) + i\omega(v, w)$ is complex linear in w, complex anti-linear in v, satisfies $H(w,v) = \overline{H(v,w)}$ and has a positive definite real part.

Solution 2.49 (i) $\implies$ (ii). We have:

$$g_J(v,w) = \omega(v,Jw) = -\omega(Jw,v) = -\omega(J^2w,Jv) = \omega(w,Jv) = g_J(w,v)$$

so g_J is symmetric. Also $g_J(v,v) = \omega(Jv,Jv) > 0$ unless $v = 0$ and:

$$g_J(Jv,Jw) = \omega(Jv,J^2w) = -\omega(Jv,w) = \omega(w,Jv) = g_J(w,v) = g_J(v,w)$$

So g_J is positive definite and J-invariant.

(ii) $\implies$ (iii). We evidently have $H(u+v,w) = H(u,w) + H(v,w)$ and likewise for the other entry since H is a sum of $\mathbb{R}$-bilinear maps. Now if $c = x+iy \in \mathbb{C}$ we have:

$$H(cv,w) = g_J(cv,w) + i\omega(cv,w) = g_J(xv+yJv,w) + i\omega(xv+yJv,w)$$

$$= xg_J(v,w) + yg_J(Jv,w) + ix\omega(v,w) - iy\omega(w,Jv)$$
$$= xg_J(v,w) + yg_J(v,Jw) + ix\omega(v,w) - iyg_J(w,v)$$
$$= xg_J(v,w) - y\omega(v,w) + ix\omega(v,w) - iyg_J(v,w)$$
$$= (x-iy)(g_J(v,w) + i\omega(v,w)) = \bar{c}H(v,w)$$

Notice that we are careful to only use the compatibility between g_J and J here, which are guaranteed by (i). A nearly identical calculation shows $H(v,cw) = cH(v,w)$. We also have:

$$H(w,v) = g_J(w,v) + i\omega(w,v) = g_J(v,w) - i\omega(v,w) = \overline{H(v,w)}$$

Finally we see that $H(v,v) = g_J(v,v) + i\omega(v,v) = g_J(v,v) > 0$ unless $v = 0$.

(iii) $\implies$ (i). For any $v \neq 0$ we have $\omega_J(v,Jv) = g_J(v,v) = H(v,v) > 0$ if (iii) holds. Furthermore:

$$\omega(Jv,Jw) = \frac{1}{2}(H(iv,w) + \overline{H(iv,w)}) = \frac{1}{2}(-iH(v,w) + (i\overline{H(v,w)})) = \frac{-i}{2}(H(v,w) - \overline{H(v,w)}) = \omega(v,w)$$

Exercise 2.52 (i) Prove the continuity of the map $r : \mathfrak{Met}(V) \to \mathcal{J}(V,\omega)$ in Proposition 2.50 as follows. If $V = \mathbb{R}^{2n}$ and $\omega = \omega_0$ then an inner product $g \in \mathfrak{Met}(\mathbb{R}^{2n})$ can be written in the form $g(v,w) = w^TGv$ where $G \in \mathbb{R}^{2n\times 2n}$ is positive definite. The formula $w_0(v,w) = (J_0v)^Tw = g(Av,w)$ determines the matrix $A = G^{-1}J_0$. Prove that the g-adjoint of A represented by the matrix $A^* = G^{-1}A^TG = -A$. Prove that tghe g-square root of the matrix $P = A^*A = -A^2 = G^{-1}J_0^TG^{-1}J_0$ is given by:

$$Q = G^{-1/2}(G^{-1/2}J_0^TG^{-1}J_0G^{-1/2})^{1/2}G^{1/2}$$

Deduce that the map $G \to J_G = Q^{-1}G^{-1}J_0$ is continuous.

(ii) The algebra here is also just a reformulation of that in the proof of Lemma 2.42. Use the current method to give an alternative proof of this result.

(iii) Deduce from (ii) that a complex structure J is ω-compatible if and only if it has the form $J = \Psi^{-1}J_0\Psi$ for some $\Psi \in \mathrm{Sp}(2n)$.

Solution 2.52 (i) We observe that for any v, w we have:

$$g(Av, w) = v^T A^T G w = v^T G G^{-1} A^T G w = g(v, G^{-1}A^T G w) = g(v, A^* w)$$

Thus we must have $A^* = G^{-1}A^T G$. Furthermore:

$$g(v, Aw) = -g(w, Av) = -g(A^* w, v) = g(v, -A^* w)$$

so $A^* = -A$. Now consider $P = A^*A = G^{-1}A^T G A$. We have that $R = G^{1/2}PG^{-1/2} = G^{-1/2}A^T G A G^{-1/2}$ is a symmetric positive definite matrix, and thus $R = O^T\Lambda O$ for some orthogonal O and diagonal positive Λ. We may thus define the square root as $R^{1/2} = O^T\Lambda^{1/2}O$. Note that the map $R \to R^{1/2}$ can be defined around any multiple of the identity λI with $\lambda > 0$ using the Taylor series for $\sqrt{\lambda + x}$, which has radius of convergence λ. Thus we can see that $R \to R^{1/2}$ is a continuous (in fact, smooth!) function of the entries of R by noting that $R^{1/2} = (\lambda I + (R - \lambda I))^{1/2}$ (where the right-hand side is defined using the Taylor expansion about λ) for λ greater than any eigenvalue of R. A similar discussion holds for the map $M \to M^{-1}$ (in fact we can use the formula $M^{-1} = \det(M)^{-1} \cdot \mathrm{adj}(M)$ which show that M^{-1} can be written in terms of smooth functions in the entries of M when M isn't singular).

Thus the map $G \to Q = G^{-1/2}R^{1/2}G^{1/2} = G^{-1/2}(G^{-1/2}J_0^T G^{-1}J_0 G^{-1/2})^{1/2}G^{1/2}$ is smooth and we just need to verify that Q satisfies all of the properties we want. We certainly have $Q^2 = G^{-1/2}RG^{1/2} = P$. Furthermore we have:

$$g(v, Qw) = v^T G G^{-1/2}R^{1/2}G^{1/2}w = v^T G^{1/2}R^{1/2}G^{1/2}w = v^T(G^{1/2}R^{1/2}G^{1/2})^T w = g(Qv, w)$$

Finally we see that since R is positive, $R^{1/2}$ is positive and thus Q is because it is conjugate to $R^{1/2}$. Thus the Q given by the above formula is the g-square root, and we may conclude that the map $G \to J_G = Q^{-1}G^{-1}J_0$ is smooth due to it being a matrix product of smooth matrix-valued functions of the entries of G.

(ii) Let $z_j = u_j \pm iv_j$ and $\pm i\lambda_j$ (with $\lambda_i > 0$) be the eigenvectors and eigenvalues of $A = G^{-1}J_0$. Since A is real and anti-self-conjugate with respect to g, A must have imaginary eigenvalues coming in conjugate pairs, with corresponding eigenvectors $u_j \pm iv_j$ which are g orthonormal. In this diagonal basis $Q = (A^*A)^{1/2}$ is the simply the diagonal matrix with entries λ_i. Now observe that $A(u_j + iv_j) = \lambda_j(iu_j - v_j)$, so $Au_j = -\lambda_j v_j, Av_j = \lambda_j u_j$. Thus $J_G u_j = Q^{-1}Au_j = -v_j$ and $J_G v_j = u_j$. In other words, this is a standard basis for J_G. Furthermore we have:

$$\omega(v_i, u_j) = g(Av_i, u_j) = g(\lambda_i u_i, u_j) = \lambda_i\delta_{ij}$$

and similarly $\omega(u_i, v_j) = -\lambda_i\delta_{ij}$, $\omega(v_i, v_j) = \omega(u_i, u_j) = 0$. Thus we may set $e_i = \lambda_i^{-1}v_i, f_i = \lambda_i^{-1}u_i$ to get a standard basis for ω which is g-orthogonal. Note that in this basis J_G is still standard, since the change of basis $u_i, v_i \to e_i, f_i$ commutes with J_G (it is essentially just rescaling on the eigenspaces).

(iii) If $J = \Psi^{-1} J_0 \Psi$ for some $\Psi \in \mathrm{Sp}(2n)$, then:

$$\omega(v, Jw) = \langle v, J_0^T \Psi^{-1} J_0 \Psi w \rangle = \langle v, \Psi^T J_0^T J_0 \Psi w \rangle = \langle v, \Psi^T \Psi w \rangle$$

Thus $\omega(\cdot, J\cdot)$ is positive definite. Furthermore $\omega(Jv, Jw) = \omega(v, w)$ because J is a composition of three symplectomorphisms. Conversely, suppose J is ω-compatible. Then by the work in (ii) it is conjugate to J_0 via a symplectic transformation (given by the basis e_i and f_i).

Exercise 2.53 Here is yet another proof of the contractibility of $\mathcal{J}(V, \omega)$. This proof illustrates in a clear geometric way the relationship between Lagrangian subspaces, complex structures and inner products. Given a Lagrangian subspace $\Lambda_0 \in \mathcal{L}(V, \omega)$ there is a natural bijection:

$$\mathcal{J}(V, \omega) \to \mathcal{L}_0(V, \omega, \Lambda_0) \times S(\Lambda_0)$$

where $\mathcal{L}_0(V, \omega, \Lambda_0)$ is the space of all Lagrangian subspaces which intersect Λ_0 transversely and $S(\Lambda_0)$ is the space of all positive definite quadratic forms on Λ_0. Note that, by Lemma 2.30, the space $\mathcal{L}_0(V, \omega, \Lambda_0)$ is contractible. The above correspondence is given by the map:

$$J \mapsto (J\Lambda_0, g_J|_{\Lambda_0})$$

where $g_J(v, w) = \omega(v, Jw)$ as above. Show that this map is a bijection.

Solution 2.53 First we show injectivity. First we see that $\omega(v, Jw) = \omega(v, Iw)$ for any $v, w \in \Lambda_0$. Similarly, for any $v \in J\Lambda_0 = I\Lambda_0$ and $w \in \Lambda_0$ we have:

$$\omega(v, Jw) = \omega(Jv', Jw) = \omega(v', w) = 0 = \omega(v'', w) = \omega(Iv'', Iw) = \omega(v, Iw)$$

where $v = Jv' = Iv''$ and $v', v'' \in \Lambda_0$. But Λ_0 and $J\Lambda_0 = I\Lambda_0$ span V. So $\omega(v, Jw) = \omega(v, Iw)$ for any $v \in V$ and $w \in \Lambda_0$, and it follows that $Jw = Iw$. Furthermore, suppose that $Iv = Jw$ for some $v, w \in J\Lambda_0$. Then $v = Iv'$ and $w = Jw'$ for some $v', w' \in \Lambda_0$. Furthermore:

$$-v' = I^2 v' = Iv = Jw = J^2 w' = -w'$$

So $v' = w'$. But then $v = Iv' = Jw' = w$, so $v = w$. Since J and I carry $J\Lambda_0$ to Λ_0 bijectively, this implies that they agree on both $J\Lambda_0$, i.e $Jv = Iv$ for $v \in J\Lambda_0$. Since Λ_0 and $J\Lambda_0$ together span V, this implies that they agree on V.

Now we prove surjectivity. To see this, simply note that given a Lagrangian Λ transverse to Λ_0 and a metric g on Λ_0, we have an isomorphism induced by ω, $\Lambda \to \Lambda_0^*$, given by $w \mapsto \omega(\cdot, w)$. We may therefore define $\bar{J}$ as a map $\Lambda_0 \to \Lambda$ by the identity:

$$\omega(v, \bar{J}w) = g(v, w)$$

i.e $\bar{J} : \Lambda_0 \to \Lambda$ is the unique map such that $\omega(\cdot, \bar{J}w) = g(\cdot, w) \in \Lambda_0^*$. We may then extend this to a map $J : V \to V$ by defining $Jv = \bar{J}v$ for $v \in \Lambda_0$, $Jv = -\bar{J}^{-1}v$ for $v \in \Lambda$, and then extending by linearity. We may also extend g simply by setting $g(v,w) = g(Jv, Jw)$ for $v, w \in \Lambda$ and $g(v,w) = 0$ if $v \in \Lambda_0, w \in \Lambda$. We then have that Λ and Λ_0 are perpendicular subspaces with respect to g. Furthermore, $J^2 = -1$ and $g(v,w) = g(Jv, Jw)$ (this is easily checked on a split basis in $V = \Lambda_0 \oplus \Lambda$).

Exercise 2.54 Let ω and g be given. Show that there is a basis for V which is both g-orthogonal and ω standard if and only if there is a Lagrangian subspace Λ whose g-orthogonal compliment $\Lambda^\perp$ is also Lagrangian.

Solutuion 2.54 If there is such a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$, then we can take $\Lambda = \text{span}(e_i)$ and $\Lambda^\perp = \text{span}(f_i)$. Conversely, if two such Lagrangians exist, then we can construct such a basis via a version of the symplectic Graham-Schmidt. More specifically, we can proceed by induction: if V is 2-dimensional, we can pick the orthogonal basis $e \in \Lambda, f \in \Lambda^\perp$, picking e arbitrarily and f so that $\omega(e,f) = 1$, which we must be able to do since $\Lambda^\perp$ is transverse to Λ. If $\dim(V) = 2n$, then we pick an arbitrary non-zero $e \in \Lambda$. Then there is a unique vector

Exercise 2.55 Let $J \in \mathcal{J}(V, \omega)$. prove that a subspace $\Lambda \subset V$ is Lagrangian with respect to ω if and only if $J\Lambda$ is the orthogonal compliment of Λ with respect to the inner product g_J. Deduce that $\Lambda \in \mathcal{L}(V, \omega)$ if and only if $J\Lambda \in \mathcal{L}(V, \omega)$.

Solution 2.55 We see that:

$$\omega(v,w) = 0 \text{ for all } v, w \in \Lambda \iff g_J(v, Jw) = -\omega(v, JJw) = -\omega(v,w) = 0 \text{ for all } v \in \Lambda, Jw \in J\Lambda$$

By dimension counting, then, we must have $\Lambda^\perp = J\Lambda$. Since $J\Lambda = \Lambda^\perp$ if and only if $J^2\Lambda = \Lambda = (J\Lambda)^\perp$, we see that Λ is a Lagrangian if and only if $J\Lambda$ is.

Exercise 2.56 Suppose that J_t is a smooth family of complex structures on V depending on a parameter t. Prove that there exists a smooth family of isomorphisms $\Phi_t : \mathbb{R}^{2n} \to V$ such that $J_t \Phi_t = \Phi_t J_0$.

Solution 2.56 Let $I = (0,1)$ be the open interval. Consider $J(t) : V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ and consider the sub-bundle $E \to I$ of $I \times V \otimes \mathbb{C} \to I$ defined by $E(t) = \ker(J(t) - i1) \subset V \otimes \mathbb{C}$. This is a vector-bundle over the interval, so it is trivial. Therefore we can pick n non-vanishing, linearly independent global sections $u_j(t) + iv_j(t)$. Point-wise these u_j and v_j satisfy $J(t)(u_j(t) + iv_j(t)) = iu_j(t) - v_j(t)$, so $J(t)v_j(t) = u_j(t)$ and $J(t)u_j(t) = -v_j(t)$. Using the map $V \otimes \mathbb{C} \simeq V \oplus iV \to V$ given by $u + iv \to u + v$, we may identify the sections $u_i(t), Ju_i(t) = v_i(t)$ as $2n$ sections of V. They are point-wise linearly independent, since the vectors $u_i + iJu_i$ and $u_i - iJu_i$ were independent in the complexification. Thus the map $\Phi_t : \mathbb{R}^{2n} \to V$ given by $e_i \to u_i(t), f_i \to v_i(t)$ gives the desired family of isomorphisms.

Note that if J_t were compatible with ω_0 (the standard form, not time-dependent) then we could have chosen Φ_t to be symplectic. Indeed, in that case, we can pick $u_i + iv_i$ to be orthogonal with respect to $g_{J_t} = \omega(\cdot, J_t \cdot)$ via the argument in Paragraph 2 of Solution 2.61, and the resulting map described above would then be symplectic since then $\omega(u_i, v_i) = g_J(u_i, Jv_i) = g_J(u_i, u_i) = 1$ and $\omega(u_i, u_i) = \omega(v_i, v_i) = 0$.

Finally, this argument can be extended to a family J_t of complex structures on a trivial bundle $E = U \times \mathbb{R}^{2k}$ over U, to show that there is a family of bundle automorphisms $\Phi_t : E \to E$ such that $J\Phi_t = \Phi_t J_0$. We may or may not take J_t compatible with ω_0; in the latter case, which case we may take Φ_t to be automorphisms of E as a symplectic bundle. Again, the same argument as above will work, except this time we pick sections $u_j + iv_j$ over $I \times U$.

Exercise 2.57 Prove that the real 2×2 matrix:

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies $J^2 = -1$ if and only if $\det(J) = 1$ and $a = -d$. Deduce that J_0 and $-J_0$ lie in different components of $\mathcal{J}(\mathbb{R}^2)$. Prove that each component of $\mathcal{J}(\mathbb{R}^2)$ is contractible.

Solution 2.57 $J^2 = -1$ implies that the eigenvalues are $\pm i$, and since imaginary eigenvalues for real matrices occur in conjugate pairs, this implies that there must be 1 i eigenvalue and 1 $-i$ eigenvalue. Therefore $J^2 = -1$ if and only if $\det(J - \lambda) = \lambda^2 - \mathrm{tr}(J) + \det(J)\lambda^2 + 1$. This proves the first part.

To prove the second part, we recall that $\mathcal{J}(\mathbb{R}^2) \simeq \mathrm{GL}(2,\mathbb{R})/\mathrm{GL}(1,\mathbb{C})$ with connected components distinguished by the determinant. Thus if two complex structure are related by an orientation reversing transformation, then they are in separate components. Indeed, J_0 and $-J_0$ are related by the transformation $e_1 \to e_2, e_2 \to e_1$, which is determinant -1. So they are in different components.

To prove the third part, we observe that there exists a $G \in \mathrm{GL}^+(2,\mathbb{R})$ is connected: these matrices can be retracted via $h_t(M) = (MM^T)^{-t/2}M$ to $SO(2) = U(1)$, which is certainly connected. Thus there exists a family of maps $M(t)$ such that the path $J(t) = M(t)J_0M(t)^{-1}$ has $J_0 = J(0)$ and $J = J(1)$ for any J in the component of J_0.

Exercise 2.58 Let V be a $2n$-dimensional real vector space with complex structure J. Show that the space of all skew-forms ω which are compatible with J is convex.

Solution 2.58 Simply observe that if ω_0, ω_1 are two such forms, then $\omega(t) = t\omega_0 + (1-t)\omega_1$ is anti-symmetric. Furthermore $g_{\omega(t)} = tg_{\omega_0} + (1-t)g_{\omega_1}$ and metrics are convex, so $\omega(t)(\cdot, J\cdot)$ is certainly a metric. This also means that $\omega(t)$ is non-degenerate, since $\omega(t)(v, Jv) > 0$ for all $v \neq 0$, so $\omega(t) : V \to V^*$ is injective. This proves that any convex combination of compatible forms is compatible.

Exercise 2.59 A linear subspace $W \subset V$ is called totally real if it is of dimension n and:

$$JW \cap W = \{0\}$$

If $W \subset V$ is a totally real subspace show that the space of non-degenerate skew forms $\omega : V \times V \to \mathbb{R}$ which are compatible with J and satisfy $W \in \mathcal{L}(V,\omega)$ is naturally isomorphic to the space of inner products on W and hence is convex.

Solution 2.59 We define the map as so. First, let π_Λ denote projection to Λ along $J\Lambda$. Then we can define a metric $\tilde{g}$ on V from a metric g on W via:

$$\tilde{g}(v,w) = g(\pi_\Lambda v, \pi_\Lambda w) + g(\pi_\Lambda Jv, \pi_\Lambda Jw)$$

This metric is J-invariant, restricts to g on W and has $\Lambda \perp J\Lambda$, and it's easy to see that it is the unique metric satisfying these properties. Now we define the map

$$\omega_g(v,w) = -\tilde{g}(v,Jw) = -g(\pi_\Lambda v, \pi_\Lambda Jw) + g(\pi_\Lambda Jv, \pi_\Lambda w)$$

Now we have

$$\omega_g(v,w) = \tilde{g}(v,Jw) = -\tilde{g}(Jv,w) = -\tilde{g}(w,Jv) = -\omega_g(w,v)$$

Thus ω_g is an anti-symmetric. It is also non-degenerate, since $\omega_g(v,Jv) = \tilde{g}(v,v) > 0$ for any $v \neq 0$. Finally, we have:

$$\omega_g(v,w) = -g(v,0) + g(0,w) = 0$$

for $v,w \in \Lambda$ and likewise for $J\Lambda$ by J invariance. So both Λ and $J\Lambda$ are Lagrangian, and this defines a map $\Psi : \mathfrak{Met}(W) \to \mathrm{Symp}(M)$ which maps metrics on W into compatible symplectic forms on V which have W as a Lagrangian. This map is clearly injective, since if we have two metrics g,h on Λ and $v,w \in \Lambda$ with $g(v,w) \neq h(v,w)$, then $\omega_g(v,Jw) \neq \omega_h(v,Jw)$. Furthermore, the formula for ω_g is smooth in g.

Conversely, to see surjectivity, we consider any ω satisfying those properties with respect to W. We can take the metric $g_J = \omega(\cdot, J\cdot)$ and see that it is a J-invariant metric, restricting to $h = g_J|_\Lambda$ on Λ and having $\Lambda \perp J\Lambda$. Thus we have $\tilde{h} = g$ and thus $\omega_h = \omega$, so $\Psi(h) = \omega$, and the map Ψ is surjective. Note that the map $\omega \to g_J|_\Lambda$ is the inverse to Ψ, and it is smooth, so the map Ψ is in fact a diffeomorphism.

Exercise 2.61 Prove that a symplectic vector bundle as defined on p. 69 is locally symplectically trivial.

Solution 2.61 We are given a rank $2k$ vector-bundle $E \to X$ over some base X with a smooth non-degenerate section ω of $E^* \wedge E^*$. Pick a metric h. Then there exists a unique anti-self-adjoint, invertible section A of $\mathrm{End}(E)$ satisfying $h(v,Aw) = \omega(v,w)$. We may consider the section $J = (A^*A)^{-1/2}A = (-A^2)^{-1/2}A$. Here $(A^*A)^{-1/2}$ is as in Solution 2.52, see that problem for a more thorough discussion. We see that if we define $g(v,w) = h(v,(A^*A)^{1/2}w)$ then $\tilde{h}$ is a new metric satisfying $g(v,Jw) = \omega(v,w)$. Furthermore $J^2 = -1$, so J has n i eigenvalues and n $-i$ eigenvalues. Let $K \subset E \otimes \mathbb{C}$ be defined as $K = \ker(J - i1)$.

Now let $p \in X$ be any point and $U \simeq B^n \subset \mathbb{R}^n$ be any simply connected neighborhood of p. Then $K|_U$ is a line bundle over U, and thus is thus topologically trivial. Thus it possesses n independent global sections $z_j = u_j + iv_j$. In fact, we can make these orthonormal with respect to $g|_K$ (extended to $E \otimes \mathbb{C}$ as a Hermitian inner product then restricted to K). We may first pick a non-vanishing section $u_1 + iv_1$ of $K|_U$, then setting $K_1 = \mathrm{span}(u_1 + iv_1)$ pick a section $u_2 + iv_2$ in $K_1|_U^\perp$ (which is also trivial), then define $K_2 = \mathrm{span}(u_1 + iv_1, u_2 + iv_2)$ and proceed thus. Note that the real sections u_i are perpendicular in E due to this choice.

By the standard argument (see Solution 2.11) we have $Au_i = -v_i, Av_i = u_i$. Thus via the map $E \otimes \mathbb{C} \simeq$

$E \oplus iE \to E$ via $u+iv \to u+v$ we get $2n$ independent real sections u_i, Ju_i satisfying $\omega(u_i, v_j) = g(u_i, Av_j) = \delta_{ij}$ and $\omega(u_i, u_j) = \omega(v_i, v_j) = 0$. Thus the map $\psi : U \times \mathbb{R}^{2k} \to E|_U$ given by $(x, e_i) \to v_i(x), (x, f_i) \to u_i(x)$ has the property that $\psi^*\omega = \omega_0$ and constitutes a symplectic trivialization over U.

Exercise 2.64 Let $E \to M$ be a $2n$-dimensional vector-bundle with complex structure J and $F \to \partial M$ be an n-dimensional real sub-bundle. This means $J_q F_q \cap F_q = \{0\}$ for all $q \in \partial M$. Prove that there exists a symplectic bilinear form ω which is compatible with J and satisfies $F_q \in \mathcal{L}(E_q, \omega_q)$ for $q \in \partial M$. Prove that the space of such forms is contractible.

Solution 2.64 We apply an identical construction to that in Exercise 2.59. That is, define the following fiber-bundle isomorphism $\Psi : \text{Met}(F) \to \text{Symp}_{J,F}(E)$. Here $\text{Met}(F)$ denotes the bundle whose fiber is the metrics on F_p. $\text{Symp}_{J,F}(E)$ denotes the bundle whose fiber is the space of J-compatible symplectic forms on E with F as a Lagrangian. We want the isomorphism:

$$\Psi(g) = \omega_g, \quad \omega_g(v, w) := -g(\pi_F v, \pi_F Jw) + g(\pi_F Jv, \pi_F w)$$

Note that since J varies smoothly and F is a smooth sub-bundle, the section π_F of $\text{End}(E)$ to F along JF is smooth. Thus the map Ψ is a smoothly varying map which is smooth as a map of the fibers. In fact, Ψ can be extended to a section of $\text{Hom}(E^* \otimes E^*, E^* \wedge E^*)$ which is fiber-wise linear!

This map is an isomorphism on the fibers by Exercise 2.59. Furthermore, the fibers of $\text{Met}(F)$ are convex, so since Ψ extends to a section of $\text{Hom}(E^* \otimes E^*, E^* \wedge E^*)$ we may conclude that the bundle of symplectic forms also has convex fiber. A fiber-bundle with convex fiber has a contractible space of sections, so the space $\Gamma(\text{Symp}_{J,F}(E))$ is contractible. Furthermore, since $\Gamma(\text{Met}(F))$ is non-empty (we can run the usual partition of unity argument) $\Gamma(\text{Symp}_{J,F}(E))$ is also non-empty. So such an ω exists.

To prove existence once we know convexity, we could alternatively apply a partition of unity argument directly to $textSymp_{J,F}(E)$. Namely, we take locally trivially patches U_i (where $E|_{U_i}$ is a trivial complex vector bundle of rank $2n$), find ω_i on each patch by taking the standard one, and then taking convex combinations of these ω_i using a partition of unity to get a global ω_i.

Exercise 2.67 Define the notion 'symplectic trivialization.' Show that a Hermitian line bundle has a unitary trivialization if and only if its underlying symplectic bundle has a symplectic trivialization.

Solution 2.67 Let $(E_0, \omega_0, J_0, g_0) \to X$ denote the trivial Hermitian vector-bundle with $E_0 = X \times \mathbb{R}^{2k}$, $\omega_0(x) = \omega_0, J_0(x) = J_0, g_0(x) = \langle \cdot, \cdot \rangle$ and $\pi : E_0 = X \times \mathbb{R}^{2k} \to X$ the standard projection map. We will also use E_0 to denote the underlying trivial symplectic bundle.

Let $(E, \omega) \to X$ be a symplectic vector-bundle of rank $2k$. A symplectic trivialization is a bundle isomorphism $\Psi : E_0 \to E$ with $\Psi^*\omega = \omega_0$. Similary, let $(E, \omega, J, g) \to X$ be a Hermitian vector-bundle of rank $2k$. A unitary trivialization is a bundle isomorphism $\Psi : E_0 \to E$ with $\Psi^*\omega = \omega_0, J\Psi = \Psi J_0$ and $\Psi^* g = g$.

Evidently, a unitary trivialization of a unitary bundle is also a symplectic trivialization of the underlying symplectic bundle. Now suppose that $\Psi : (E_0, \omega_0, J_0, g_0) \to (E, \omega, J, g)$ is a symplectic trivialization. Then $J_1 = \Psi^{-1} J \Psi$ and J_0 are two complex structures on E_0 which are compatible with ω_0. By Solution 2.56, there exists a symplectic bundle automorphism $\Phi : E_0 \to E_0$ (connected to the identity in fact) such that $J_1 \Phi = \Phi J_0$. Thus $\Psi\Phi$ has the property that $(\Psi\Phi)^*\omega = \omega_0$ and $J\Psi\Phi = \Psi\Phi J_0$. By the compatibility condition, it follows that $(\Psi\Phi)^* g = g_0$. Thus this is a unitary trivialization of E.

Exercise 2.68 Prove that the space of paths $\Psi : [0,1] \to \mathrm{Sp}(2n)$ of symplectic matrices satisfying $\Psi(1) = \Psi(0)^{-1}$ has two components. Deduce that up to isomorphism there are precisely two symplectic vector bundles (of every given dimension) over the real projective $\mathbb{R}P^2$.

Solution 2.68 Let $\Psi : [0,1] \to \mathrm{Sp}(2n)$ be such a path. We single out two standard loops: $\Psi_0, \Psi_1 : [0,1] \to \mathrm{Sp}(2n)$ where $\Psi_0(t) \equiv 1$ and $\Psi(0) = R(\theta) \oplus 1_{2n-2} \in U(1) \oplus \mathrm{Sp}(2n-2)$. Both of these Ψ_i are evidently in our class of curves. We will show that every $\Psi(t)$ is homotopic to exactly one Ψ_i.

First observe that any Ψ is homotopic to a curve such that $\Psi(1) = \Psi(0)$. We can just take any curve $\Phi(t)$ such that $\Phi(0) = 1$ and $\Phi(1) = \Psi(1)$, letting $\Phi_s(t)$ denote the partial curve $\Phi_s(t) = \Phi((1-s) + st)$ and $\Phi_s^{-1}(t) = \Phi((1-s) + s(1-t))^{-1}$. Then $\Psi_s = \Phi_s^{-1} \circ \Psi \circ \Phi_s$ (where $\circ$ denotes path composition) is a homotopy of curves with $\Psi_s(0) = \Psi_s(1)^{-1}$ and $\Psi_1(0) = \Psi_1(1) = 1$. Thus we may consider without loss of generality that $\Psi(0) = \Psi(1) = 1$ and we may classify homotopy classes of these (we will still use homotopies of curves where $\Psi(0) \neq \Psi(1)$).

Now, if Ψ and Ψ' are two such curves, and they are homotopic as curves $S^1 \to \mathrm{Sp}(2n)$ with $0 \mapsto 1$, then they are evidently homotopic as curves with $\Psi(0) = \Psi(1)^{-1}$. Now let $\Phi(t) = \Psi_1(t)$, so that $[\Psi_1] \in \pi_1(\mathrm{Sp}(2n)) = \mathbb{Z}$ generates the group. Let $\Psi_{1,s}(t)$ and $\Psi_{1,s}^{-1}(t)$ be as Φ_s and Φ_s^{-1} above. Then $\Psi_s = \Psi_{1,s}^{-1} \circ \Psi \circ \Phi_{1,s}$ has $[\Psi] = [\Psi] + 2[\Psi_1]$ for any of our Ψ. Thus any curve is homotopic to a curve in the π_1 class of Ψ_0 or Ψ_1, and thus to Ψ_0 or Ψ_1 itself.

Conversely, suppose that Φ_s is a homotopy of curves with $\Phi_0(0) = \Phi_0(1) = \Phi_1(0) = \Phi_1(1) = 1$ and $\Phi_s(0) = \Phi_s^{-1}(1)$ for all s. Let $\Gamma(t) = \Phi_t(0)$ and let Γ_s and Γ_s^{-1} be like Φ_s and Φ_s^{-1} above. Observe that Γ itself is a closed curve, so $[\Gamma] = [\Gamma_1] = k[e]$ for some generator $[e]$ of $\pi_1(\mathrm{Sp}(2n))$. Then $\Psi_r = \Gamma_r^{-1} \circ \Phi_r \circ \Gamma_r$ is a homotopy of curves with $\Psi_r(0) = \Psi_r(1) = 1$. Thus we see that $[\Phi_1] = [\Phi_0] + 2[\Gamma_1] = [\Phi_0] + 2n[e]$. Thus the $\mod 2$ homotopy class of a curve $[\Phi]$ with $\Phi(0) = \Phi(1)^{-1}$ is invariant up to homotopy through other such curves.

Thus we has established that there are two homotopy classes of our curves. Now consider a symplectic vector bundle $E \to \mathbb{R}P^2$. We may take $\mathbb{R}P^2$ and split it along a circle into a disk D^2, where the boundary $S^1 \simeq \partial D^2$ is identified with itself in $\mathbb{R}P^2$ via the antipode map $a : \partial D^2 \to \partial D^2$. E then pulls back to the trivial bundle over D^2, since it is over a disk, coupled with a bundle map $\Psi : E_p \to E_{a(p)}$. The data of a line bundle over $\mathbb{R}P^2$ is thus the data of a bundle map $\Phi : E|_{\partial D^2} \to E|_{\partial D^2}$ identifying E_p with $E_{a(p)}$, and satisfying $\Phi(p) = \Phi(a(p))^{-1}$. Identifying $\partial D^2 = \mathbb{R}/2\mathbb{Z}$, such a map is given equivalently by a map $\Psi : [0,1] \to \mathrm{Sp}(2n)$ with $\Psi(0) = \Psi(1)^{-1}$. We can then recover the original map $\Psi : \partial D^2 \to \mathrm{Sp}(2n)$ by path composing $\Psi \circ \Psi$.

Two different trivializations of E over D^2 yield isotopic bundle maps on ∂D^2. Indeed, any two such trivializations are related by a bundle map $\Psi : E|_{D^2} \to E|_{D^2}$. Since the space of such bundle maps

If Φ and Φ' are two homotopic bundle maps on ∂D^2, they yield isomorphic bundles.

Exercise 2.75 Use the formula (2.2) (the characterization as the Euler class) to calculate the first Chern class of the normal bundle $\nu_{\mathbb{CP}^1}$ in $\mathbb{CP}^2$.

Solution 2.75 We have the trivializations $\Phi_1 : \Sigma_1 \times \mathbb{C} \to \nu_{\mathbb{C}P^1}$ and $\Phi_2 : \Sigma_2 \times \mathbb{C} \to \nu_{\mathbb{C}P^1}$ given by $\Phi_1([1 : z_1 : 0], w) = [1 : z_1 : w]$ and $\Phi_2([z_2 : 1 : 0], w) = [z_2 : 1 : w]$. Consider the section given by $([1 : z_1 : 0], 1)$ in the firsts patch and $([z_2 : 1 : 0], z_2)$ in the second patch. The transition map $\Phi_2^{-1}\Phi_1$ sends $([1 : z_1 : 0], 1) \to [1 : z_1 : 1] \to [1/z_1 : 1 : 1/z_1] \to ([1/z_1 : 1 : 0], 1/z_1) = ([z_2 : 1 : 0], z_2)$. So this is a well-defined section. Furthermore it evidently intersects the zero section (identified with $[z_2 : z_1 : 0]$) at a single point, where $z_2 = 0$ in the second patch. The orientation of $\nu_{\mathbb{CP}^1}$ is induced by the ambient space, $\mathbb{CP}^2$, via a normal neighborhood and with this orientation the intersection is positive since the section is locally the intersection of two holomorphically embedded $\mathbb{CP}^1$'s in $\mathbb{CP}^2$.

Exercise 2.76 Let $L \subset \mathbb{C}^n \times \mathbb{C}P^{n-1}$ be the incidence relation:

$$L = \{(z, l)|z \in l\} = \{(z_1, \ldots, z_n; [w_1, \ldots, w_n])|w_j z_k = w_k z_j \forall j, k\}$$

The projection $\pi : L \to \mathbb{C}P^{n-1}$ gives L the structure of a complex line-bundle over $\mathbb{C}P^{n-1}$. Show that when $n = 2$ the first Chern number of L is -1, and hence calculate $c_1(L)$ for arbitrary n.

Solution 2.76 Consider the $n = 1$ case. We have two patches for L, each over one of the disks in $\mathbb{C}P^1$: $([z, 1], \lambda) \to ([z, 1], \lambda(z, 1))$ and $([1, z], \lambda) \to ([1, z], \lambda(1, z))$. Calculating the transition map, we see that:

$$([z, 1], \lambda) \to ([z, 1], \lambda(z, 1)) = ([1, 1/z], z\lambda(1, 1/z)) \to ([1, 1/z], \lambda z) = ([1, w], w^{-1}\lambda)$$

Thus the curve $S^1 \to \mathrm{Sp}(2)$ induced by this bundle is $\theta \to e^{-2\pi i\theta}$, i.e $c_1(L) = \mu(\Psi) = -1$. For any $n > 2$, we see that the inclusion map $\mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ is covered by a bundle $L_{\mathbb{C}P^1} \to L_{\mathbb{C}P^{n-1}}$. Thus if we use $c_1(L)$ to now denote the map $H_2(\mathbb{C}P^{n-1}) \to \mathbb{Z}$ we have $\langle c_1(L)|[\mathbb{C}P^1]\rangle = -1$. But since $H_2(\mathbb{C}P^{n-1})$ is generated by $[\mathbb{C}P^1]$, this completely determines $c_1(L)$ as a map.

Exercise 2.77 Prove that every symplectic vector bundle over a Riemann surface decomposes as a direct sum of 2-dimensional vector bundles.

Solution 2.77 First observe that given a Riemann surface Σ, we can produce a plane bundle $\xi \to \Sigma$ with $c_1(\xi) = z \in \mathbb{Z}$ for any $z \in \mathbb{Z}$. To do so, we simply pick a curve $C \subset \Sigma$ such that $\Sigma - C$ has is connected and has 2 boundary components. Then we consider the trivial bundle ξ_0 over $\Sigma - C$. Given a map $\gamma : C \to \mathrm{Sp}(2)$ with $\mu(\gamma) = z$, we may produce a bundle ξ over Σ by using two trivializations: one over a cylinder/normal neighborhood $U_0 \subset \Sigma$ with $C \subset U_0$ and $U_0 \simeq (-1, 1) \times C$ and one over $U_1 = \Sigma - C \subset \Sigma$.

The transition map on $U_1 \cap U_2 = V - C \simeq (-1, 0) \times C \cup (0, 1) \times C = U_{12} \cup U_{21}$ can be the identity on U_{12}. On U_{21} we can define it using the identification $U_{21} \simeq C \cup (0, 1)$: if (x, s) are coordinates with respect to this diffeomorphism, then we want to use the transition map $\Phi(x, s) = \gamma(x)$.

Now let $C_1 = C$. Take a set of $n-1$ other splitting curves $C_2, \ldots, C_n$ so that $\Sigma - \sqcup_i C_i$ is a union of two disconnected surfaces Σ_1 and Σ_2, each homeomorphic to a disc with $n-1$ holes. Then using the trivialization of ξ on $\Sigma - C$ to induce a trivialization of ξ on $\Sigma - \sqcup_i C_i$, we see that the maps $\Psi_i : C_i \to \mathrm{Sp}(2)$ given by the transition maps at the cycles are the identity for $i > 1$ and equal to γ for $i = 1$. Thus by construction $\sum_i \mu(\Psi_i) = z$.

Now to answer the question. If we are given an arbitrary vector-bundle E of rank k, then $c_1(E) = z \in \mathbb{Z}$. Pick any set of k integers z_i so that $\sum_i z_i = z$, and let ξ_i be bundles with those Chern numbers, i.e $c_1(\xi_i) = z_i$. Then the direct sum bundle $F = \oplus_i \xi_i$ has $c_1(F) = \sum_i c_1(\xi_i) = \sum_i z_i = z$. So its rank and Chern number agree with E, and by naturality we conclude that $F \simeq E$.

Exercise 2.78 (i) Suppose that $E \to \Sigma$ is a symplectic vector bundle over an oriented Riemann surface Σ that extends over a compact oriented 3-manifold Y with boundary $\partial Y = \Sigma$. Prove that the restriction $E|_\Sigma$ has Chern class zero. (ii) Use (i) above and Exercise 2.77 to substantiate the claim made in Remark 2.70 that the Chern class $c_1(f^*E)$ depends only on the homology class of f.

Solution 2.78 (i) If $\mathrm{rank}(E) = 2k > 2$ we observe that by transversality considerations $E \to Y$ admits a global non-vanishing section. That is, if we choose any section $\sigma : Y \to E$ and then perturb it to be transverse to the 0-section, dimension counting tells us that the intersection is empty and thus that the perturbed σ is global and non-vanishing. This section restricts to a global non-vanishing section on Σ, so we may split E as $E' \oplus \mathbb{R}^2$ where $\mathrm{rank}(E') = 2k-2$. Since $E|_\Sigma = E'|_\Sigma \oplus \mathbb{R}^2|_\Sigma$, we may assume after repeating this process that $\mathrm{rank}(E) = 2$.

In this case, consider a unitary connection A on E, picked after augmenting E by some chosen compatible complex structure J. Then the curvature F_A is a closed $i\mathbb{R}$-valued 2-form on Y. By Stokes theorem we thus have $c_1(E) = \frac{i}{2\pi}\int_\Sigma F_A = \frac{i}{2\pi}\int_Y dF_A = 0$.

(ii) The easiest way to do this is to use stuff that's a little outside of the scope of the book. Given a complex vector bundle E over some X of (real) rank $2k$, we can look at $U(k)$ connections on E (for instance, the Levi-Civita connection with respect to a Hermitian inner product structure). Let P be the associated $U(k)$-principle bundle. The first Chern class can then be defined as a cohomology class via $c_1(E) = c_1(P) = \frac{i}{2\pi}\mathrm{tr}(F_A)$ where $F_A \in \Omega^2(\mathrm{Ad}P)$ is a 2-form valued in the associated bundle $P \times_{U(k)} \mathrm{Ad}(u(k))$ and $\mathrm{tr} : \Lambda^2(X) \otimes \mathrm{Ad}P \to \Lambda^2(X)$ is the map induced by $\mathrm{Ad}P \to \mathbb{R}$ given by $h \mapsto \mathrm{tr}(h) = \langle 1, h\rangle$ (and $\langle , \rangle$ is the $U(k)$-invariant inner product).

Anyway, $\mathrm{tr}(F_A)$ is closed (see Milnor-Stasheff, Appendix 3) so if $f, f' : \Sigma \to X$ are two homologous embeddings we have $f(\Sigma) \cup f'(\Sigma) = \partial C$ for some 3-cycle and thus by Stokes theorem:

$$\langle c_1(E), f_*[\Sigma] - f'_*[\Sigma]\rangle = \frac{i}{2\pi}\int_{f(\Sigma)\cup f'(\Sigma)} \mathrm{tr}(F_A) = \frac{i}{2\pi}\int_C d\mathrm{tr}(F_A) = 0$$

Exercise 2.79 Prove that every symplectic vector bundle $E \to \Sigma$ that admits a Lagrangian sub-bundle can be symplectically trivialized.

Solution 2.79 Split Σ into two Riemann surfaces with boundary Σ_0 and Σ_1 with $\Sigma = \Sigma_0 \cup_C \Sigma_1$ and $C = \sqcup_i C_i$ a disjoint union of curves. Then the symplectic $2k$-bundle E with Lagrangian sub-bundle F splits into two pairs of nested bundles $F_j \subset E_j$ over each Σ_j for $j \in \{0, 1\}$. The data of the bundle is then encoded in the transition maps $\Psi_i : C_i \to \mathrm{Sp}(2k)$, and the Chern class is defined as $\sum_i \mu(\Psi_i)$.

Now observe that $F_0|_{C_i}$ and $F_1|_{C_i}$ are yield paths of Lagrangians in $(\mathbb{R}^{2k}, \omega_0)$ via the trivialization of E_0 and E_1 over $C_i \subset \Sigma_0$ and $C_i \subset \Sigma_1$. Call these paths Λ_i^0 and Λ_i^1 respectively. Furthermore we must have $\Psi_i \Lambda_i^0 = \Lambda_i^1$ since the F_j glue together to form a sub-bundle of all of E. Thus we have $2\mu(\Psi_i) = \mu(\Lambda_i^1) - \mu(\Lambda_i^0)$ by the axioms of the Maslov index.

Now observe that the Maslov index factors as a homomorphism $\mu : H_1(\mathrm{Sp}(2k); \mathbb{Z}) \to \mathbb{Z}$ rather than $\mu : \pi_1(\mathrm{Sp}(2k)) \to \mathbb{Z}$, since $H_1(\mathrm{Sp}(2k) \simeq \mathrm{Ab}(\pi_1(\mathrm{Sp}(2k))) \simeq \pi_1(\mathrm{Sp}(2k)) \simeq \mathbb{Z}$. It is then clear that if a set of loops of Lagrangians $\Gamma_i : S^1 \to \Lambda(V, \omega)$ bound a map of a surface $\Gamma : \Sigma \to \Lambda(V, \omega)$ then the sum of the Maslov indices is 0, since the union is then null-homologous. But the maps $\Sigma_0 \to (\mathbb{R}^{2k}, \omega_0)$ and $\Sigma_1 \to (\mathbb{R}^{2k}, \omega_0)$ given by the trivialization do precisely this for the union of the curves Λ_i^0 and Λ_i^1 respectively. So:

$$c_1(E) = \sum_i \mu(\Psi_i) = \frac{1}{2}(\sum_i \mu(\Lambda_i^1) - \sum_i \mu(\Lambda_i^0)) = 0$$

Exercise 3.1 Consider cylindrical polar coordinates (θ, x_3) on the sphere minus its poles $S^2 - \{(0, 0, \pm 1)\}$ where $0 \leq \theta < 2\pi$ and $-1 < x_3 < 1$. Show that the area form induced by the Euclidean metric is precisely the form $\omega = d\theta \wedge dx_3$.

Solution 3.1 Here we use z instead of x_3. The coordinate patch (θ, z) is embedded in $\mathbb{R}^3$ via $(\theta, z) \mapsto (\sqrt{1 - z^2}\cos(\theta), \sqrt{1 - z^2}\sin(\theta), z)$. The Jacobian of this transformation Ψ is:

$$D\Psi_{\theta,z} = \begin{pmatrix} -\sqrt{1 - z^2}\sin(\theta) & -\frac{z}{\sqrt{1-z^2}}\cos(\theta) \\ \sqrt{1 - z^2}\cos(\theta) & -\frac{z}{\sqrt{1-z^2}}\sin(\theta) \\ 0 & 1 \end{pmatrix}$$

The pullback of the Euclidean metric is thus given by:

$$g_{\theta,z} = (D\Psi_{\theta,z})^T D\Psi_{\theta,z} = \begin{pmatrix} 1 - z^2 & 0 \\ 0 & \frac{1}{1-z^2} \end{pmatrix}$$

The area form induced by this metric is thus $\sqrt{\det(g_{\theta,z})} d\theta \wedge dz = d\theta \wedge dz$.

Exercise 3.5 Assume that τ is a non-degenerate 2-form on M which is not necessarily closed. In this case Hamiltonian vector fields and Poisson brackets can be defined as above. Show that:

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = d\tau(X_F, X_G, X_H)$$

for any three functions $F, G, H \in C^\infty(M)$.

Solution 3.5 We can verify this with a calculation in local coordinates. Let τ_{ij} denote the almost symplectic form and τ^{ij} denote its inverse. f, g, h will denote the functions in question. Then we have:

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\} = \partial_k(\tau^{ij}\partial_i f\partial_j g)\partial_k h + \partial_k(\tau^{ij}\partial_i g\partial_j h)\partial_f + \partial_k(\tau^{ij}\partial_i h\partial_j f)\partial_k g$$

$$= \partial_k\tau^{ij}\partial_i f\partial_j g\tau^{kl}\partial_l h + \tau^{ij}\partial_k\partial_i f\partial_j g\tau^{kl}\partial_l h + \tau^{ij}\partial_i f\partial_k\partial_j g\tau^{kl}\partial_l h$$
$$+\partial_k\tau^{ij}\partial_i g\partial_j h\tau^{kl}\partial_l f + \tau^{ij}\partial_k\partial_i g\partial_j h\tau^{kl}\partial_l f + \tau^{ij}\partial_i g\partial_k\partial_j h\tau^{kl}\partial_l f$$
$$+\partial_k\tau^{ij}\partial_i h\partial_j f\tau^{kl}\partial_l g + \tau^{ij}\partial_k\partial_i h\partial_j f\tau^{kl}\partial_l g + \tau^{ij}\partial_i h\partial_k\partial_j f\tau^{kl}\partial_l g$$

Notice that the 2nd and 9th term in the second line cancel due to the anti-symmetry of τ and the symmetry of the Hessian $\partial_i\partial_j f$. This occurs with all similar pairs of terms above, thus yielding:

$$\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\} = \partial_k\tau^{ij}\partial_i f\partial_j g\tau^{kl}\partial_l h + \partial_k\tau^{ij}\partial_i g\partial_j h\tau^{kl}\partial_l f + \partial_k\tau^{ij}\partial_i h\partial_j f\tau^{kl}\partial_l g$$

$$= \partial_k\tau_{ij}X_f^i X_g^j X_h^k + \partial_k\tau_{ij}X_g^i X_h^j X_f^k + \partial_k\tau_{ij}X_h^i X_f^j X_g^k$$

$$= \frac{1}{2}(\partial_k\tau_{ij}X_f^i X_g^j X_h^k + \partial_k\tau_{ij}X_g^i X_h^j X_f^k + \partial_k\tau_{ij}X_h^i X_f^j X_g^k - \partial_k\tau_{ij}X_g^i X_f^j X_h^k - \partial_k\tau_{ij}X_h^i X_g^j X_f^k - \partial_k\tau_{ij}X_f^i X_h^j X_g^k)$$

$$= \frac{1}{2}d\tau(X_f, X_g, X_h)$$

Here we use the formula:

$$\partial_a\tau^{ij} = \partial_a(\tau^{ik}\tau_{kl}\tau^{lj}) = \partial_a(\tau_{kl})\tau^{ik}\tau^{lj} + \tau_{kl}\partial_a\tau^{ik}\tau^{lj} + \tau_{kl}\tau^{ik}\partial_a\tau^{lj}$$

$$= \partial_a(\tau_{kl})\tau^{ik}\tau^{lj} + \delta_k^j\partial_a\tau^{ik} + \delta_l^i\partial_a\tau^{lj}$$

This implies $\partial_a(\tau_{kl})\tau^{ki}\tau^{lj} = \partial_a\tau^{ij}$, and allows us to substitute a lower index τ_{ij} and raise the indices of the $\partial_i f, \ldots$ gradient terms.

Exercise 3.7 Let S be a compact orientable hypersurface in the symplectic manifold (M, ω). Prove that there exists a smooth function $H : M \to \mathbb{R}$ such that 0 is a regular value of H and $S \subset H^{-1}(0)$. Prove that $X_H(q) \in L_q$ for $q \in S$.

Solution 3.7 Take a tubular neighborhood N of S in M, parameterized by $S \times (-1, 1)$. Since S and M are orientable, the normal bundle νS is trivial and we can pick such a parameterization. Then take any bump function $\beta : (-1, 1) \to \mathbb{R}$ which is supported on $(-1, 1)$, such that $\beta(0) \neq 0$, and such that $\beta'(x) = 0$ implies that $x \in (-\infty, -1] \cup \{a\} \cup [1, \infty)$ for some $a \neq 0$. We can take, for instance, the usual bump function:

$$\beta(x) = \begin{cases} 0 & |x+a| \geq \frac{1}{2} \\ \exp(-(1-4(x+a)^2)^{-1}) & |x+a| < \frac{1}{2} \end{cases}$$

for some small $a \neq 0$. Now define:

$$f(p) = \begin{cases} -\beta(0) & p \notin N \\ \beta(t) - \beta(0) & p = (t, x) \in N \simeq (-1, 1) \times S \end{cases}$$

This is a smooth function with 0 as a regular value and $S \subset H^{-1}(0)$, all by construction of β. At a point $q \in S$ we have $\omega(X_H, v) = dH(v) = 0$ for any $v \in T_qS$ since H is constant on S. Thus by the definition $L_q = (TS_q)^\omega$ of L as the symplectic perpendicular to TS_q we have that $X_H \in L_q$.

Exercise 3.10 Show that there is an isomorphism:

$$T_{(q,0)}T^*L \simeq T_qL \oplus T_q^*L$$

and

$$-d\lambda_{\text{can}(q,0)}(v,w) = w_1^*(v_0) - v_1^*(w_0)$$

for $v, w = (v_0, v_1^*), (w_0, w_1^*) \in T_qL \oplus T_q^*L$.

Solution 3.10 We pick coordinates x_i in a neighborhood of $q \in L$, so that (x_i, y_i) are the corresponding coordinates on T^*L with $y_i = dx_i$. Then $d\lambda_{\text{can}(q,0)} = \sum_i dy_i \wedge dx_i$. Now observe that ∂_{y_i} form a basis of $\ker(\pi_*)$ where $\pi_* : TT^*M \to TM$ is map of tangent spaces induced by the projection map $\pi : TM \to M$. We have a natural map $\phi : T^*L_q \simeq T_{(q,0)}(T^*L_q) \simeq \ker(\pi_*) \subset (TT^*M)_{(q,0)}$ given by $\sum_i a_i y_i \mapsto \sum_i a_i \partial_{y_i}$, which does not depend on our choice of x_i. Thus we may define:

$$\Psi : T_{(q,0)}T^*L \simeq T_qL \oplus T_q^*L \qquad \Psi(v) = (\pi_*(v), \phi^{-1}(v))$$

In the basis this map is simply $v = \sum_i a_i\partial_{x_i} + b_i\partial_{y_i} \mapsto (\sum_i a_i\partial_{x_i}, \sum_i b_i dx_i) = (v_0, v_1^*)$. We see that is v and w are as above, then in our basis:

$$-\omega_{\text{can}}(v,w) = (\sum_i dx_i \wedge dy_i)(v,w) = \sum_i -(v_1^*)_i(w_0)_i + (w_1^*)_i(v_0)_i$$

$$= \sum_i = (\sum_i (w_1^*)_i dx_i)(\sum_i (v_0)_i\partial_{x_i}) - (\sum_i (v_1^*)_i dx_i)(\sum_i (w_0)_i\partial_{x_i}) = w_1^*(v_0)_i - v_1^*(w_0) = \Psi^*\Omega_{\text{can}}$$

Here Ω_{can} is the usual symplectic form on $V \oplus V^*$ given to $TM_q \oplus T^*M_q \simeq TM_q \oplus (TM_q)^*$.

Exercise 3.11 Prove that there is a bundle isomorphism $\Phi : TL \oplus T^*L \to T(T^*L)$ which identifies the summand T^*L with the vertical vectors. Prove that Φ can be chosen to such that the composition $d\pi \circ \Phi$ restricts to the identity on the summand TL and $\Phi^*\omega_{\text{can}} = \Omega_{\text{can}}$.

Solution 3.11 We want to illustrate an isomorphism $\pi^*TL \oplus \pi^*T^*L \simeq T(T^*L)$. Pick an almost complex structure J on $T(T^*L)$. We still have a natural isomorphism $\ker(\pi_*)_q \simeq T^*L_{\pi(q)}$ for any $q \in T^*M$, by the same argument as in Exercise 3.10 (that argument was not dependent on q being on the 0-section). This extends to a bundle map $\ker(\pi_*) \to T^*L$ over the bases T^*L and L respectively, which is an isomorphism on the fibers (as noted on p. 92), thus a bundle isomorphism $\phi : \ker(\pi_*) \simeq \pi^*T^*L$.

Now let J be any almost complex structure on $T(T^*L)$ compatible with ω_{can}. Then $J\ker(\pi_*)$ is transverse to $\ker(\pi_*)$ itself and thus $T(T^*M) \simeq \ker(\pi_*) \oplus J\ker(\pi_*)$. Furthermore since $J\ker(\pi_*)$ is transverse to $\ker(\pi_*)$, the restriction of π_* to $J\ker(\pi_*)$ gives an isomorphism $J\ker(\pi_*) \simeq TL$ on the fibers, thus an isomorphism $J\ker(\pi_*) \simeq \pi^*TL$ given by π_* in one direction and the inverse $r_q : \pi^*TL_q \to J\ker(\pi_*)$. Thus we have a splitting:

$$\Phi : \pi^*TL \oplus \pi^*T^*L \simeq J\ker(\pi_*) \oplus \ker(\pi_*) \simeq T(T^*L) \qquad \Phi(v, v^*) = r(v) + \phi(v^*)$$

By the definition of r this has the property that $\pi_*\Phi(v, 0) = (\pi_* r(v), 0) = (v, 0)$. Furthermore, consider coordinates x_i about some $\pi(p) \in U \subset L$ for $p \in T^*L$. Let $v = \sum_i a_i\partial_{x_i} \in TL_{\pi(p)}$, $v^* = \sum_i b_i dx_i \in TL_{\pi(p)}$ and $r(v) = \sum_i a_i\partial_{x_i} + \sum_i c_i\partial_{y_i}$. Also let x_i, y_i be the corresponding coordinates on $T(T^*L)_p$. Then we have:

$$\Phi^*\omega_{\text{can}} = \omega_{\text{can}}(\Phi(v, 0), \Phi(0, v^*)) =$$

$$\omega_{\text{can}}(r(v), \phi(v^*)) = (\sum_i dx_i \wedge dy_i)(\sum_i a_i\partial_{x_i} + \sum_i c_i\partial_{y_i}, \sum_i b_i\partial_{y_i}) = \sum_i -b_i a_i = -v^*(v) = \Omega_{\text{can}}(v, v^*)$$

Since $\Phi(T^*L)$ and $\Phi(TL)$ are both Lagrangian by construction of Φ, we can conclude that $\Omega_{\text{can}}(v, w) = \Phi^*\omega_{\text{can}}(v, w)$ for all $v, w \in \pi^*TL \oplus \pi^*T^*L$.

Exercise 3.12 (i) Any diffeomorphism $\psi : L \to L$ lifts to a diffeomorphism $\Psi : T^*L \to T^*L$ by the formula:

$$\Psi(q, v^*) = (\psi(q), d\psi(q)^{-1}v^*)$$

Prove that $\Psi^*\lambda_{\text{can}} = \lambda_{\text{can}}$ and hence Ψ is a symplectomorphism of T^*L. (ii) Let $Y : L \to TL$ be a vector field on L which integrates to the parameter group ψ_t of diffeomorphisms of L. Let $X : T^*L \to TT^*L$ generate the corresponding group of symplectomorphisms Ψ_t of $(T^*L, \omega_{\text{can}})$. Show that $X = X_H$ is the Hamiltonian vector field of the function $H : T^*L \to \mathbb{R}$ given by:

$$H(q, v^*) = v^*(Y(q))$$

Solution 3.12 (i) Let q_i be coordinates on L, with corresponding coordinates q_i, p_i on T^*L. Then:

$$d\Psi_{p,q}(v, v^*) = d\psi(q)v + d\psi(q)^{-1}v^* + d(d\psi(q)^{-1})(p, v)$$

Here $d(d\psi(q)^{-1})(p, v)$ is just a makeshift expression for the term contributed by the differential of the q-depenedent part of $d\psi(q)^{-1}v^*$. It's important to note that the image of $d\psi(q)^{-1}v^*, d(d\psi(q)^{-1})(p, v) \in \ker(\pi_*) \subset T(T^*L)$ (i.e both of those vectors are in the vertical part of $T(T^*L)$). Now we see that:

$$\Psi^*\lambda_{\text{can},(p,q)} = \Psi^*_{p,q}(\sum_i p_i dq_i) = \sum_i(\sum_j d\psi(q)^{-1})^j_i p_j)(\sum_j dq_j d\psi(q)^j_i)$$

$$= \sum_{i,j} dq_j d\psi(q)^j_i d\psi(q)^{-1})^j_i p_j = \sum_{i,j} \delta^i_j p_j dq_i = \sum_i p_i dq_i = \lambda_{\text{can},(p,q)}$$

(ii) Consider the family of diffeomorphisms ψ_t generated by Y. These act on T^*L via $(q, p) \mapsto (\psi_t(q), (d\psi^{-1}(q))p)$. Thus the generating vector field X must be of the form $X = (Y, Z)$ in split coordinates (i.e its TL-coordinates agree with those of Y). Furthermore since X is symplectic, we have $0 = \mathcal{L}_X\lambda = di_X\lambda + i_X d\lambda$, i.e $-i_X d\lambda = di_X\lambda$. Since $\omega = -d\lambda$, we see that X is Hamiltonian with $H = i_X\lambda$.

But we see that:

$$i_{X(q,p)}\lambda_{p,q} = p(\pi_* X(q,p)) = p(Y(q))$$

This is the desired formula.

Exercise 3.13 (i) A Lagrangian for a variational problem on a manifold is a functional $L : TM \to \mathbb{R}$. Formulate an appropriate version of the non-degeneracy condition which permits the Legendre transformation. What is the corresponding Hamiltonian function H on $(T^*M, \omega_{\text{can}})$? Check that the equations for L on TM and the corresponding Hamiltonian equations are invariant under coordinate transformation.

Solution 3.13 (i) First we observe the following: given a manifold M, there is a natural map $\rho : T^*(TM) \to T^*M$ given in coordinates x on $U \subset M$ with corresponding coordinates (x, v, ξ_x, ξ_v) on $T^*(TM)$ by $(x, v, \xi_x, \xi_v) \to (x, \xi_v)$. We see that this is well-defined as so: given new coordinates x' with $x = \phi(x')$, we have $v = d\phi(x')v'$. Let $\Psi : TM \to TM$ be this corresponding diffeomorphism on TM. Thus given a function $L : TM \to \mathbb{R}$ we have $(\Psi^* L)(x', v') = L(\phi(x'), d\phi(x')v')$. If we denote by $d_x L, d_v L$ the x and v parts of the gradient in these coordinates (and similarly for x', v') we see that $d_{v'} L = d_v L(\Psi(x)) \circ d\phi(x')$. Thus if $(x, v, \xi_x, \xi_v) \to (x, \xi_v)$ then $(x', v', \xi'_x, \xi'_v) \to (x', \xi'_v) = (\phi^{-1}(x), d_v L \circ d\phi(\phi^{-1}(x))) = \phi^*(x, \xi_v)$. This shows that $\rho(x, v, \xi_x, \xi_v)$ is a well-defined point in T^*M.

With ρ given, we can now give an invariant form to the Legendre condition. A Lagrangian saisfies this condition if and only if the map $\rho \circ dL : TM \to T^*M$ is a diffeomorphism. This map respects the fibers of TM, i.e $\rho \circ dL(p, v) = (p, v^*)$ for any v and some v^*, and it is thus sufficient for it to be a diffeomorphism fiber to fiber, which (since the fibers are all vector-spaces) is equivalent to the non-degeneracy of the Hessian in the tangent directions (thus the coordinate description of this condition in Ch. 1).

The corresponding Hamiltonian can be given as:

$$H(x, p) = (\rho \circ dL)(v)(v) - L(x, v) = p((\rho \circ dL)^{-1}(x, p)) - L(x, (\rho \circ dL)^{-1}(x, p))$$

Here $p(\dots)$ indicates evaluating the vector $(\rho \circ dL)^{-1}(x, p) \in TM_x$ against the dual vector $p \in T^*M_x = (TM)^*_x$.

The fact that Hamilton's equations are coordinate invariant follows from its invariant formulation: A curve $\gamma : M \to T^*M$ satisfies the equations if and only if $\gamma^* i_{\dot\gamma} d\lambda = \gamma^* dH$. The fact that a diffeomorphism on M lifts to a symplectomorphsim on T^*M guarantees that this equation is fully covariant under diffeomorphisms on M. Checking this in coordinates would just involve translating this into coordinates and checking there, which is pretty uninformative so we'll skip it.

We may as well check directly for the Lagrangian case. If we change coordinates $x = \phi(x')$ and $v = d_{x'}\phi(x')v'$, then $d_x L = d_{x'} L \circ d_{x'}\phi(x') + d_{v'} L \circ d^2_{x'}\phi(x')v'$ and $d_v L = d_{v'} L \circ d_{x'}\phi(x')$. Thus:

$$\frac{d}{dt}(d_v L) = \frac{d}{dt}(d_{v'} L \circ d_{x'}\phi(x')) = \frac{d}{dt}(d_{v'} L) \circ d_{x'}\phi(x') + d_{v'} L \circ \frac{d}{dt}(d_{x'}\phi(x'))$$

$$= \frac{d}{dt}(d_{v'} L(x')) \circ d_{x'}\phi(x') + d_{v'} L \circ d^2_{x'}\phi(x')v'$$

Thus we see that:

$$d_x L - \frac{d}{dt}(d_v L) = (d_{x'} L - \frac{d}{dt}(d_{v'} L)) \circ d_{x'}\phi(x') + (d_{v'} L \circ d^2_{x'}\phi(x')v' - d_{v'} L \circ d^2_{x'}\phi(x')v')$$

Thus the left side vanishes if and only if the right-side vanishes, and the two sides are equivalent to the Euler-Lagrange equations in the x and x' coordinates respectively.

Exercise 3.18 This exercise establishes a relative form of Moser's theorem that is often useful. Let M be a compact manifold with boundary. Suppose that ω_t is a smooth family of symplectic forms that agree on $T_x M$ for every $x \in \partial M$ and satisfy, for every compact 2-manifold Σ and every smooth map $u : \Sigma \to M$ with $\partial\Sigma \subset \partial M$:

$$\frac{d}{dt}\int_\Sigma u^*\omega_t = 0$$

Prove that there exists a smooth isotopy $\psi_t : M \to M$ such that:

$$\psi_0 = \mathrm{id}, \qquad \psi_t|_{\partial M} = \mathrm{id}, \qquad \psi_t^*\omega_t = \omega_0$$

If $\omega_t = \omega_0$ in some neighborhood of ∂M, prove that ψ_t can be chosen equal to the identity in a (possibly smaller) neighborhood of ∂M.

Solution 3.18 As with the other applications of Moser's argument, we just need to show that there exists a family of 1-forms σ_t with $d\sigma_t = \frac{d}{dt}\omega_t$ and $\sigma_t = 0$ on $T_{\partial M} M$. To see this, we recall the long-exact sequence of relative cohomology for the pair $(M, \partial M)$

$$\cdots \to H^1(M;\mathbb{R}) \xrightarrow{i_*} H^1(\partial M;\mathbb{R}) \xrightarrow{\delta_*} H^2(M,\partial M;\mathbb{R}) \xrightarrow{q_*} H^2(M;\mathbb{R}) \xrightarrow{i_*} H^2(\partial M;\mathbb{R}) \to \dots$$

Now observe that by exactness of the above sequence, $i_*([\frac{d}{dt}\omega_t]) = 0$ implies that $\frac{d}{dt}\omega_t$ give a well-defined element of the relative cohomology $H^2(M,\partial M;\mathbb{R})$[1]

Furthermore we have by assumption that $\frac{d}{dt}\langle u^*[\Sigma], [\omega_t]\rangle = \langle u^*[\Sigma], \frac{d}{dt}\omega_t\rangle = 0$ for every embedding $(\Sigma, \partial\Sigma) \hookrightarrow (M, \partial M)$ and every time t. Every homology class in $H_2(M,\partial M;\mathbb{R}) \simeq (H^2(M,\partial M;\mathbb{R}))^*$ can be represented this way [2], so this implies that $[(\frac{d}{dt}\omega_t, 0)] = 0 \in H^2(M,\partial M;\mathbb{R})$ and of course that $q^*[0] = q^*[(\frac{d}{dt}\omega_t, 0)] = [\frac{d}{dt}\omega_t] = 0 \in H^2(M;\mathbb{R})$.This second condition implies that there exists a family of 1-forms $\sigma_t \in \Omega^1(M)$ such that $d\sigma_t = \frac{d}{dt}\omega_t$ (where smoothness of the family follows from similar arguments to the proof in Theorem 3.17).

Now we want to show that σ_t can be chosen so that $i_*\sigma_t = 0$. The fact that $[(\frac{d}{dt}\omega_t, 0)] = 0$ implies that σ_t can be chosen to be cohomologous to $i_*\delta_*\alpha_t$ for some family α_t of closed 1-forms on M. Since $i_*\delta_*\alpha_t =$

[1] Recall that the chain group $C^n(M, \partial M)$ in de Rham cohomology is defined as $\Omega^n(M) \oplus \Omega^{n-1}(M)$ with differential $d(\alpha, \beta) = (d\alpha, i_*\alpha - d\beta)$. The map $q_* : H^i(M, \partial M;\mathbb{R}) \to H^i(M, \mathbb{R})$ is given by $q_*(\alpha, \beta) = \alpha$. If $i_*\alpha = \alpha|_{\partial M} = 0$ then $(\alpha, 0)$ defines a cocycle in this model, so α is in the image of $H^2(M, \partial M;\mathbb{R}) \to H^2(M;\mathbb{R})$. Also, $\delta_* : H^i(\partial M;\mathbb{R}) \to H^{i+1}(\partial M;\mathbb{R})$ is given by $\delta_*\beta = (0, \beta)$.

[2] In dimension $2n$ with $n \geq 3$, we can represent any cycle by a smoothly embedded surface as so. Take a generic cycle representative and perturb it to a smooth immersion. Due to transverse surfaces being non-intersecting in $d \geq 5$, we the result is an embedded surface. In dimension $2n = 4$ we can perturb to have an immersed surface with only transverse double points. We can replace the double points with handles smoothly by using the model $xy = 0 \to xy = \epsilon$ in $\mathbb{C}^2$.

$(0, i_*\alpha_t)$, this implies that there exists a family of closed 1-forms α_t such that $(\frac{d}{dt}\omega_t, -i_*\alpha_t) = (\sigma_t, -i_*\alpha_t)$ is exact in $H^2(M, \partial M; \mathbb{R})$. In particular, α_t satisfies $d(\sigma_t, -i_*\alpha_t) = (d\sigma_t, i_*\sigma_t - di_*\alpha_t) = 0$. But since α_t is closed, $di_*\alpha_t = 0$ and thus $i_*\sigma_t = \sigma_t|_{\partial M} = 0$ for all t.

Thus by Moser's trick, we can set X_t such that $\sigma_t + i_{X_t}\omega_t = 0$ and take ψ_t to be the diffeomorphisms generated by X_t with initial diffeomorphism ψ_0. In particular, $i_*\sigma_t = 0$ implies that X_t will vanish on ∂M, so that $\psi_t|_{\partial M} = \mathrm{id}$.

If $\omega_t = \omega_0$ in a neighborhood V of ∂M, we can take a tubular neighborhood N of ∂M so that $U \subset V$. Any map $u : (\Sigma', \partial\Sigma') \to (M, N)$ (i.e with $u(\partial\Sigma') \subset N$) can be extended to a map $u' : (\Sigma, \partial\Sigma) \to (M, \partial M)$. We can do this by attaching a tube $[-1, 0] \times \partial\Sigma$ to Σ along $\partial\Sigma$. We can then use the fact that $N \simeq (-1, 0] \times \partial M$ is tubular to extend the map $\partial\Sigma \to N$ to a homotopy $[-1, 0] \times \partial\Sigma \to N$ with $\{1\} \times \Sigma$ agreeing with the original map and $\{0\} \times \Sigma \subset \partial M$. Then since ω_t is constant in N, we have:

$$\frac{d}{dt}\int_{\Sigma} u^*\omega_t = \frac{d}{dt}\int_{\Sigma'} u^*\omega_t = 0$$

Thus, since we never used anything specific about the boundary in the above arguments (only results about relative de Rham homology of a pair (M, A)) we can replace ∂M with N in all of the above arguments and our results carry over.

Exercise 3.20 Suppose that ω_t and τ_t are two families of symplectic forms on a closed manifold M such that $\omega_0 = \tau_0$ and ω_t is cohomologous to τ_t for all $t \in [0, 1]$. Prove that for some $\epsilon > 0$ there exists an isotopy ψ_t such that $\psi_t^*\omega_t = \tau_t$ for $0 \leq t \leq \epsilon$.

Solution 3.20 We modify the Moser argument. We want to find a family of diffeomorphisms ψ_t with $\psi_0 = \mathrm{id}$ and $\psi_t^*\omega_t = \tau_t$. Differentiating in time we see that:

$$\frac{d}{dt}\tau_t = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*(\frac{d}{dt}\omega_t + di_{X_t}\omega_t)$$

Here X_t is the family of vector fields on M satisfying $\frac{d}{dt}\psi_t = X_t \circ \psi_t$.

Exercise 3.21 Prove Darboux's theorem in the 2-dimensional case, using the fact that every non-vanishing 1-form on a surface can be written locally as $f dg$ for a suitable f and g.

Solution 3.21 We want to show that every area form ω on surface Σ is locally symplectomorphic to the standard form $dx \wedge dy$ on $\mathbb{R}^2$. For this purpose, consider a point $p \in \Sigma$ and a neighborhood U of p.

First suppose that we know that every non-zero 1-form can be written as $f dg$ for some choice of f and dg. Then look at $\omega|_U \in \Omega^2(U)$. ω is closed, so on U it is exact and there exist smooth f, g such that $\omega = d\alpha$ for a 1-form α. We can assume that α is non-vanishing at p and in U (possibly after shrinking U) by adding an exact form (perhaps a constant $adu + ydv$ in coordinates). Thus we may assume $\omega = d(f dg) = df \wedge dg$. Now observe that the map $\Psi : U \to \mathbb{R}^2$ given by $\Psi(p) = (f(p), g(p))$ is a symplectomorphism. It is certainly a diffeomorphism since in coordinates u, v we have $\det(d\psi) du \wedge dv = df \wedge dg$. Furthermore, it is a symplectomorphism since $\Psi^* \omega_0 = df \wedge dg = \omega$.

Thus we only need to know this fact that every non-vanishing 1-form over a contractible U can be written as fdg. But this is clear: given any 1-form α on U, we can look at the line bundle $\ker(\alpha)$ over U. Since U is diffeomorphic to the disk, we know that this bundle is trivial, so we can pick a global non-zero section $v \in \Gamma(\ker(\alpha)) \subset \Gamma(TU)$. We can then integrate this vector field to obtain integral curves. This foliation will be locally trivial, so we can take a smooth function $g : U \to \mathbb{R}$ such that the level sets of g are precisely the integral curves of v. Then dg is non-zero in U and $\ker(\alpha) = \ker(dg)$, so they differ by a non-zero scalar $\alpha = fdg$.

Exercise 3.22 (i) Let Σ be a closed 2-manifold. Prove that a symplectic (or area) form on Σ is determined up to strong isotopy by its cohomology class. (ii) Prove a similar result for volume forms on closed manifolds.

Solution 3.22 (i) First we prove that if two volume forms ω_0 and ω_1 on a closed n-manifold M are cohomologous, then there is a family ω_t of cohomologous forms connecting them. Let M be a closed n-manifold with symplectic forms ω_0, ω_1.

We claim that $\omega_t = (1-t)\omega_0 + t\omega_1$ is the family that we want. Evidently $[\omega_t] = [\omega_0]$. Furthermore, $\omega_1 = f\omega_0$ for some f, since $\Lambda^n(M)$ is the trivial bundle since it is a line bundle with a globals non-vanishing section. We cannot have $f = 0$ anywhere, since this would imply that ω_1 was degenerate there. Thus, either $f > 0$ or $f < 0$ everywhere, and since $\int_M \omega_0 = \int_M \omega_1$, it must be the case that $f > 0$. Thus $\omega_t = ((1-t) + tf)\omega_0$ is non-degenerate (and closed for dumb dimensional reasons).

Now consider a closed surface Σ with ω_0, ω_1. Evidently if ω_0 and ω_t are strongly isotopic then $[\omega_0] = [\omega_1]$. Conversely, suppose that $[\omega_0] = [\omega_1]$. Then as we have shown above, we have a connecting family ω_t of cohomologous symplectic forms. Thus we may apply Moser stability (Theorem 3.17) to conclude that there exists a family of diffeomorphisms ψ_t such that $\psi_1^* \omega_1 = \omega_0$.

(ii) We want to prove that if M is an n-manifold with two volume forms λ_0, λ_1 and $[\lambda_0] = [\lambda_1]$ then there exists a family of diffeomorphisms ψ_t with $\psi_0 = \mathrm{id}$ and $\psi_1^* \omega_1 = \omega_0$. By part (i), we just need to prove the analog of Moser stability: that if there exists a family of cohomologous volume forms λ_t connecting λ_0 to λ_1 then there exists a family of diffeomorphisms ψ_t with the properties above.

Suppose that such a family λ_t exists. We want to find a family of diffeomorphisms ψ_t with $\psi_t^* \lambda_t = \lambda_0$ (as in the book). Then $\lambda_t - \lambda_0$ is a family of exact forms, thus there exists a family σ_t of $n-1$-forms such that $\frac{d}{dt}\lambda_t = d\sigma_t$. Again, the fact that we can pick a smooth family σ_t of forms like this follows from de Rham theory [3]. Now we see that if X_t is the generating vector field of ψ_t, i.e satisfying $\frac{d}{dt}\psi_t = X_t \circ \psi_t$, then:

$$0 = \frac{d}{dt}(\lambda_0) = \frac{d}{dt}(\psi_t^* \lambda_t) = \psi_t^*(\mathcal{L}_{X_t}\lambda_t + \frac{d}{dt}\lambda_t) = \psi_t^*(d(i_{X_t}\lambda_t) + \frac{d}{dt}\lambda_t)$$

$$d(i_{X_t}\lambda_t) + \frac{d}{dt}\lambda_t$$

[3] The lack of detail on this point in the book is getting disturbing to me, so I'm going to discuss this in an Appendix at the end of this document.

Now observe that the map $T_pM \to \Lambda_p^{n-1}M$ given by $v \mapsto (i_v\lambda_t)_p$ is an isomorphism for any p since λ_t is a volume form. Indeed, if some $v \neq 0$ has $(i_v\lambda_t)_p = 0$, then we could complete v to a basis $v = v_1, v_2, \ldots, v_n$ and see that $\lambda_t(v_1, \ldots, v_n) = 0$, contradicting the fact that it is a volume form. Thus as in the symplectic case, if we find a σ_t with $d\sigma_t = \frac{d}{dt}\lambda_t$, we get a family of vectorfields X_t uniquely determined by $i_{X_t}\lambda_t = -\sigma_t$ which satisfy $d(i_{X_t}\lambda_t) + \frac{d}{dt}\lambda_t$. Thus we have found σ_t, X_t and (by integrating X_t) ψ_t.

Exercise 3.28 Give examples of symplectic, isotropic, coisotropic and Lagrangian submanifolds of the symplectic manifold $\mathbb{R}^4/\Gamma$ of Example 3.8.

Solution 3.28 Let $M = \mathbb{R}^4/\Gamma$ with coordinates (x_1, x_2, y_1, y_2). Let $q : \mathbb{R}^4 \to M$ denote the quotient map. We freely refer to Example 3.8 for terminology and notation.

To find a symplectic subspace, observe that (as alluded to on the bottom of p. 89) we have an embedded torus $T^2 \subset M$. If we take the plane $\mathbb{R}^2 \times 0 \subset \mathbb{R}^4$ and take its orbit O under Γ, then we get a disjoint union of planes all in the orbit of $\mathbb{R}^2$. O is by construction closed under the Γ action, and thus the image $q(O) \subset M$ is diffeomorphic to O/Γ, and it is a 2-d submanifold of M.

A fundamental domain of the action on O is given by the unit square $F = [0,1]^2 \times 0 \subset \mathbb{R}^2 \times 0$. The $g \in \Gamma$ sending points in ∂F to other points in ∂F must fix the plane $\mathbb{R}^2 \times 0$ (otherwise ∂F and $g(\partial F)$ will be in disjoint planes) so they must be in $\mathbb{Z}^2 \times 0$. Such transformations simply act by the usual $\mathbb{Z}^2$ action on $\mathbb{R}^2$, thus the resulting quotient $F/\Gamma = q(F) \simeq T^2$. Since $O \subset \mathbb{R}^4$, so is $q(O) = q(F)$.

Any isotropic sub-manifold that isn't Lagrangian will be a curve for dimensional reasons, and any curve $\gamma : \mathbb{R} \to M$ will be isotropic. Likewise any hyper-surface will be coisotropic. To find explicit ones, we could just take the curve C given by the imbedding $S^1 \to T^2 \subset M$ given by $t \mod 1 \mapsto q(t, 0, 0, 0)$.

For a hypersurface H (thus a cosiotropic manifold), we can just take the hypersurface $P = \mathbb{R}^3 \times 0 \subset \mathbb{R}^4$ spanned by the coordinates x_1, x_2 and y_1. The Γ orbit O of this hyperplane P is a disjoint union of the planes $P + 0 \times 0 \times 0 \times \mathbb{Z}$. Thus $q(O) = q(P)$ is the quotient of a sub-manifold fixed by Γ and is thus a manifold itself. It is diffeomorphic to H quotiented by the subgroup $\mathrm{Stab}(H)$ fixing H, which is the group of elements (j, k) with $k = (k_1, 0)$. This subgroup is actually isomorphic to $\mathbb{Z}^3$ since$(j,k)(j',k') = (j + j', A_{j'}k + k') = (j + j', k + k')$ when $k_2 = 0$. Likewise the action is just the typical $\mathbb{Z}^3$ action. So $H \simeq T^3$.

Finally, we want a Lagrangian. We can get this by much the same process, taking the space $L = 0 \times \mathbb{R} \times 0 \times \mathbb{R}$ spanned by x_1, y_1 and taking $q(L)$. Clearly L is a Lagrangian in $\mathbb{R}^2$. The same type of arguments as above show that $q(L) = q(O)$ where O is the orbit of L and that $q(L)$ is isomorphic to T^2.

Exercise 3.29 If Q is a coisotropic submanifold in (M, ω) show that the complimentary distribution $TQ^\omega \subset TQ$ is integrable. Since ω vanishes on TQ^ω the leaves corresponding to the foliation of Q are isotropic. This generalizes the characteristic foliation on a hypersurface discussed in Section 1.2

Solution 3.29 We have to use the Frobenius theorem. The statement we need is that if X is a manifold and $E \subset TX$ is a sub-bundle, then E is integrable if and only if it arises from a regular foliation F or M, in the sense that $E_p = TF_p$ where F_p is the leaf of F going through p. E is called integrable if it is closed under the Lie bracket, i.e if X, Y are two sections of $E \subset TM$ then $[X, Y]$ is also a section of E.

Now we apply this fundamental theorem to our situation. Suppose Q is a coisotropic sub-manifold of (M, ω), of dimension $2n - k$ with $k < n$, and let $p \in Q$ be any point. We can pick a neighborhood U of p in M and k functions H_i such that 0 is a regular value of each H_i and such that $U \cap (\cap_i H_i^{-1}(0)) = U \cap Q$ (i.e these are locally defining smooth functions).

We make several observations about the H_i. First, observe that $TQ_q = \cap_i \ker(dH_i)_q$ for any point $q \in U \cap Q$. This implies that the covectors $(dH_i)_q \in T^*M$ are independent at q. Indeed, if they weren't, then $\cap_i \ker(dH_i)_q = \cap_{i \neq j} \ker(dH_i)_q$ for some j, and thus $\dim(\cap_i \ker(dH_i)_q) = \dim(\cap_{i \neq j} \ker(dH_i)_q) \geq 2n - k - 1$, contradicting the fact that $\dim(TQ_q) = 2n - k$. Second, observe that for any $q \in U \cap Q$ and any $v \in T_q Q$ we have $\omega(X_{H_i}, v) = dH_i(v) = 0$. Thus the $X_{H_i}|_Q$ are k non-vanishing, independent sections of TQ^ω in $U \cap Q$. It follows that they are a basis of $T_q Q^\omega$ at every $q \in U \cap Q$. Thus any section X of TQ^ω over U can be expressed as $X = \sum_i a_i X_{H_i}$ for some coefficient functions a_i. Finally, observe that $[X_{H_i}, X_{H_j}] = X_{\omega(X_{H_i}, X_{H_j})} = 0$, by Proposition 3.6 and the fact that $\omega(X_{H_i}, X_{H_j}) = dH_i(X_{H_j}) = 0$.

With these comments we can prove our result. Let X, Y be two sections of TQ^ω. Consider any $p \in Q$, a neighborhood U of p and a set of local defining functions H_i as above. Then $X = \sum_i a_i X_{H_i}$ and $Y = \sum_i b_i X_{H_i}$ for some smooth a_i, b_i on U. Then we have:

$$[X, Y] = \sum_{ij} [a_i X_{H_i}, b_i X_{H_j}] = \sum_{ij} a_i b_j [X_{H_i}, X_{H_j}] + a_i X_{H_i}(b_j) X_{H_j} - b_j X_{H_j}(a_i) X_{H_i}$$

$$= \sum_j (\sum_i a_i X_{H_i}(b_j) - b_i X_{H_i}(a_j)) X_{H_j} = \sum_j c_j X_{H_j}$$

Thus $[X, Y]$ is still a section of TQ^ω and TQ^ω is an integrable distribution.

Exercise 3.31 Let Q be a 2-dimensional compact symplectic submanifold of a symplectic 4-manifold (M, ω). Prove that a neighborhood of Q is determined up to symplectomorphism by the self-intersection number $Q \cdot Q$ and the integral $\int_Q \omega$.

Solution 3.31 Suppose $Q, Q' \subset M$ are two symplectic 2-folds in M. It suffices to show that $\int_Q \omega = \int_{Q'} \omega$ and $Q \cdot Q = Q' \cdot Q'$. Let $\psi : Q \to Q'$ be any diffeomorphism. Then:

$$\langle [\psi^* \omega], [Q] \rangle = \int_Q \psi^*(\omega|_{Q'}) = \int_{Q'} \omega|_{Q'} = \int_Q \omega_Q = \langle [\omega], [Q] \rangle$$

Here $[Q]$ is the fundamental class of Q. Thus $\psi^*(\omega|_{Q'})$ and $\omega|_Q$ are cohomologous symplectic forms and by Exercise 3.22 we know that there is a diffeomorphism $\phi : Q \to Q$ such that $\phi^* \psi^* \omega|_{Q'} = \omega|_Q$. Thus $\bar{\alpha} : (Q, \omega|_Q) \to (Q', \omega|_{Q'})$ with $\bar{\alpha} = \phi\psi$ is a symplectomorphism.

Now observe that $Q \cdot Q = e(\nu Q) = c_1(Q)$ and likewise for Q'. Indeed, if σ is a generic section of νQ then we can use any diffeomorphism $\phi : \nu Q \to N(Q)$ with $\phi(0) = Q \subset N(Q)$ (where 0 is the zero-section) to get a cohomologous submanifold $Q_\sigma = \phi(\sigma(Q))$ intersecting Q itself transversely. Here $N(Q)$ is a tubular neighborhood of Q. Then the signed count of intersections $Q_\sigma \cap Q$ is clearly equal to the signed count of intersections $\sigma \cap 0$, i.e $Q \cdot Q = e(\nu Q)$.

Thus if we consider νQ and $\bar{\alpha}^* \nu Q'$, we see that $c_1(\nu Q) = c_1(\bar{\alpha}^* \nu Q')$, so the two bundles are isomorphic. Thus there is a bundle isomorphism $\alpha : \nu Q \to \nu Q'$ covering α. We can then apply Theorem 3.30. to see that there is a symplectomorphism $(N(Q), \omega) \to (N(Q'), \omega)$.

Exercise 3.32 Suppose that the normal bundles νQ_0 and νQ_1 are trivial as symplectic (or equivalently, complex) bundles, and fix a symplectic isomorphism from νQ_0 to the trivial symplectic bundle $Q_0 \times \mathbb{R}^{2k}$. Then choosing an isomorphism Φ in the preceding theorem is equivalent to choosing a symplectic framing νQ_1, and so there may well be several non-isotopic choices.

Here is an explicit example to work out. For $i = 0, 1$ let $(M_i, Q_i) = (T^2 \times \mathbb{C}, T^2 \times 0)$ with the usual product form and let $\phi = id$. Take the obvious identification $\nu Q_0 = \nu Q_1 = T^2 \times \mathbb{C}$ and define Φ by:

$$\Phi(s, t, v) = (s, t, e^{2\pi i t} v)$$

where $(s, t) \in T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Show that Φ is an isomorphism of the symplectic vector bundle νT^2 and find a formula for the symplectomorphism $\psi : N(Q_0) \to N(Q_1)$.

Solution 3.32 This question is suspiciously straight forward. We rewrite this map as a bundle map, defining the map $\bar{\phi} : Q_0 \to Q_1$ of the base spaces in the coordinates given by these trivializations by $\bar{\phi}(s, t) = (s, t)$ and the covering bundle map $\phi : \nu Q_0 \to \nu Q_1$ as $\phi_p(v) = e^{2\pi i t} v$. Using the identification $\mathbb{C} \to \mathbb{R}^2$, $z = x + iy \to (x, y)$ with the standard $\omega_0 = dx \wedge dy$, we can identify $T^2 \times \mathbb{C} = T^2 \times \mathbb{R}^2$. In this trivialization, the map $\Phi : \nu Q_0 \to \nu Q_1$ is given by $\phi_p(v) = e^{2\pi J t} v$ where:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad e^{2\pi J t} = \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix}$$

This is evidently an isomorphism on the fibers $\nu_p Q_0 \to \nu_{\phi(p)} Q_1$. To check that the map is symplectic, we just need to check that (denoting by τ_i the symplectic forms on νQ_i) $(\phi^* \tau_1)_p = (\tau_0)_p$. But in our trivialization this is equivalent to $\phi_p^*(dx \wedge dy) = dx \wedge dy$. But note that the endomorphism $J_i : \nu Q_i \to \nu Q_i$ given in our trivializations by $(J_i)_p v = Jv$ is a compatible complex structure with $\omega_i(\cdot, J_i) = \langle , \rangle$ with $\langle , \rangle$ the standard Euclidean inner product in these trivializations. Furthermore $e^{2\pi J t}$ obviously commutes with J and is orthogonal with respect to $\langle , \rangle$ (they're rotation matrices). Thus the maps are symplectic.

Using the formula:

$$\psi(s, t, x, y) = (s, t, \begin{pmatrix} \cos(2\pi t) & -\sin(2\pi t) \\ \sin(2\pi t) & \cos(2\pi t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix})$$

seems acceptable to me.

Note that this trivialization is not isotopic to the identity trivialization $\phi_0(s, t, x, y) = (s, t, x, y)$. If this were so, then $\phi \circ \phi_0^{-1}$ would be isotopic to the identity. But isotopy classes of bundle isomorphisms that fix the base are equivalent to homotopy classes of maps $[T^2, SO(2)]$, and the map $\phi \circ \phi_0^{-1}$ can't be isotopic to the identity, since it contains the map $S^1 \to SO(2)$ given by $t \to (\phi \circ \phi_0^{-1})_{0,t} = e^{2\pi J t}$, which is a map to a non-null homotopic loop. If $\phi \circ \phi_0^{-1}$ were isotopic to 0, then this map $S^1 \to SO(2)$ would be contractible, a contradiction.

Exercise 3.35 (i) Let $g : M \to T^*M$ be an embedding which is sufficiently close to the canonical embedding of the zero section in the C^1-topology. Prove that the image of g is the graph of a 1-form. (ii) Let $g : M \to M \times M$ be an embedding which is sufficiently close to the canonical embedding of the diagonal in the C^1-topology. Prove that the image of g is the graph of a diffeomorphism.

Solution 3.35 (i) Let $z : M \to T^*M$ be the zero section embedding. We just need to show that if g is C^1-close to z then $\phi = \pi \circ g : M \to M$ is a diffeomorphism. Then letting $\sigma = g \circ \phi^{-1}$ we see that σ is an embedding (as a composition of an embedding and a diffeomorphism) and $\pi \circ \sigma = \pi \circ g \circ (\pi \circ g)^{-1} = \text{id}$. Thus such a σ is a section with $\sigma(M) = g(M)$.

Assume that we have put a Riemannian metric g on M, thus inducing a metric (also g) on TM, T^*M and $T(T^*M)$ (the naturally induced metric on a TX and TX given a metric on X is easy to work out, but this is not the point of this question so we won't go into it here). Thus for two maps $\sigma, \tau : M \to T^*M$ and their corresponding differentials $d\sigma, d\tau : TM \to T(T^*M)$ we can define $\|\sigma - \tau\|_{C^0} = \max_{p \in M} \text{dist}_g(\sigma(p), \tau(p))$ and $\|d\sigma - d\tau\|_{C^0} = \max_{(p,v) \in SM} \text{dist}_g(d\sigma_p(v), d\tau_p(v))$ (here SM is the sphere bundle of TM under g), and thus $\|\sigma - \tau\|_{C^1} = \|\sigma - \tau\|_{C^0} + \|d\sigma - d\tau\|_{C^0}$.

Now consider the two maps $\phi = \pi \circ g$ and $i = \text{id} = \pi \circ z$. We will start by showing that there is an $\epsilon_1 > 0$ such that $\|g - z\|_{C^1} < \epsilon_1$ implies that $d\phi : TM \to TM$ is rank n (i.e it's a local diffeomorphism).

Start by observing that the image $di(SM) = SM$. This is a compact sub-manifold of TM which is disjoint from the zero section $Z_0 \subset TM$. So the number $d(SM, M_0) = \min_{p \in M_0, q \in SM} d(p, q)$ is non-zero (it's 1 actually, assuming that we define the metric on TM in a reasonable way). Now, there exists a constant C_1 such that $\|d(\pi g) - d(\pi z)\|_{C^0} \leq C_1 \|g - z\|_{C^1}$ (this is evident since $\pi : TM \to M$ is C^∞ bounded and $d(\pi g) = d\pi \circ dg$). Now suppose that $\|g - z\|_{C^1} < \epsilon_1 = d(SM, M_0)/C_1$ and, for the sake of contradiction, that $dg_p(v) = 0$ for some $(p, v) \in SM$. Then we see that $d(dg_p(v), di_p(v)) = d((p, 0), (p, v)) > d(SM, M_0) = C_1 \epsilon_1$. This contradicts the assumption that $\|d(\pi g) - d(\pi z)\|_{C^0} \leq C_1 \|g - z\|_{C^1} = C_1 \epsilon_1$. Thus dg_p is non-degenerate (rank n) for each p in this case.

Now assume M is connected (the not connected case is just more notationally complicated but it isn't harder). The above argument shows that assuming $\|g - z\|_{C^1} < \epsilon_1$ implies that $\phi : M \to M$ is a covering map (we can show surjectivity using a continuity argument on M if it's connected). The fiber must be finite since M is compact. But the size of the fiber $|\phi^{-1}(p)|$ is locally constant near points p where $dg(p)$ is non-degenerate, and thus it is constant on M.Then the size of the fiber of g is some integer $n \geq 1$. We see that the fiber can be expressed as $F(\phi) = \int_M \phi^* \mu$ where μ is some fixed volume form with $\int_M \mu = 1$. But the map $F : C^\infty(M, M) \to \mathbb{R}$ given by this integral is certainly continuous in the C^1 topology, so for small ϵ_2 we must have $\|\phi - i\|_{C^1} < C_1 \|g - z\|_{C^1} \leq C_1 \epsilon_2$ implies $F(\phi) = 1$ and thus that ϕ is a diffeomorphism.

Thus picking $\epsilon = \min(\epsilon_1, \epsilon_2)$ we see that $\|g - z\|_{C^1} < \epsilon$ implies that g is the graph of a section.

(ii) This admits a similar treatment to (i). Let $\delta : M \to M \times M$ denote the diagonal imbedding, and let $\pi_1, \pi_2 : M \times M \to M$ denote the two projection maps to the different factors. We want to show that if g is C^1-close enough to δ, then it is the graph of some diffeomorphism. It suffices to show that if g is close to δ

then the maps $\pi_1 g, \pi_2 g : M \to M$ are both diffeomorphisms. Then $g(M)$ is the graph of $\phi = (\pi_2 g)(\pi_1 g)^{-1}$ since $\{(\pi_1 g(x), \pi_2 g(x)) \in M \times M | x \in M\} = \{(x, (\pi_2 g)(\pi_1 g)^{-1}(x)) \in M \times M | x \in M\}$. The same argument almost verbatim as with πg in (i) should work to show that $\pi_1 g$ is a diffeomorphism for g close to δ (and likewise for $\pi_2 g$, since the problem is symmetric with respect to swapping the first and second coordinate of $M \times M$).

Exercise 3.36 (Hypersurfaces) Let ω_0 and ω_1 be symplectic forms on M which agree on a compact oriented hypersurface S. Show that the inclusion $i : S \to M$ extends to an embedding ϕ of a neighborhood of U of S into M such that $\phi^*\omega_1 = \omega_0$. Note that we only assume equality of the forms $i^*\omega_0$ and $i^*\omega_1$ on S and not $T_S M$. Deduce that a neighborhood of S is symplectomorphic to the product $S \times (-\epsilon, \epsilon)$ with the symplectic form:

$$\omega = i^*\omega_0 + d(t\alpha)$$

Here α is any 1-form on S which does not vanish in the characteristic directions TS^ω of S and t is the coordinate on $(-\epsilon, \epsilon)$.

Solution 3.36 Let νS be the normal bundle to $i(S)$ and let TS^ω be the canonical line bundle given by the symplectic perp to TS_p at each point p.

νS is trivial if S is orientable. This is true because, if we choose a metric g on M, we have the isomorphism $\Lambda^{n-1}S \to \nu S$ given by $\alpha \mapsto g^\# * \alpha$. That is, we take an element $\alpha \in \Lambda^{n-1}S_p \subset (\Lambda^{n-1}M|_S)_p$, apply the Hodge star $*$ in M to map it into $(\Lambda^1(M)|_S)_p$ and then apply the musical isomorphism to lift it to an element of $TM|_S$. The result will be perpendicular to TQ in TM, so it will be an element of the normal bundle via the identification $\nu M \simeq TS^\perp \subset TM$. S is orientable if and only if its top form bundle $\Lambda^{n-1}S$ is trivial, so this bundle isomorphism shows that νQ is trivial.

Let $p \in S$ and consider $\nu(p) \in \nu S \subset (TM|_S)_p$ (here we fixing a background metric so that $\nu M_p \simeq (TS_p)^\perp$) and some arbitrary non-zero vector $\xi(p) \in TS_p^\omega$. First observe that $\omega_i(\xi(p), \nu(p)) \neq 0$ for $i = 0, 1$. If this were that case, then $\omega_i(\xi(p), e_i(p)) = 0$ for a basis $e_i(p)$ of TS_p and thus $\omega(\xi(p), v) = 0$ for all $v \in \operatorname{span}(\nu(p), e_i(p)) = TM_p$. This contradicts non-degeneracy of ω_i.

Now we show that $\omega_t = (1-t)\omega_0 + t\omega_1$ are non-degenerate in a neighborhood of S for all t. Assume that $\omega_0(\xi(p), \nu(p))$ and $\omega_1(\xi(p), \nu(p))$ are the same sign (we will deal with this at the end of the problem). Then if $\xi(p), e_1(p), \dots, e_{2n-2}(p)$ are a basis of $T_p S$ then $\omega_t(e_i(p), \cdot)$ is non-zero for all i since there is a vector $v \in TS$ such that $\omega_t(e_i(p), v(p)) = \omega_0(e_i(p), v(p)) \neq 0$ since $\omega_t = \omega_0$ on TS and $e_i(p) \notin TS^{\omega_t} = TS^{\omega_0}$. Likewise $\omega_t(\nu(p), \cdot)$ is non-zero on $\xi(p)$ since it is the convex combination of non-zero numbers of the same sign and likewise $\omega_t(\xi(p), \cdot)$ is non-zero on $\nu(p)$. Thus the map $v(p) \mapsto \omega_t(v(p), \cdot)$ from $TM \to T^*M$ is non-degenerate. Since closedness is linear, this implies that ω_t is a family of symplectic forms.

Now we demonstrate that $\omega_1 - \omega_0 = \frac{d}{dt}\omega_t$ is exact. Then it follows from Moser's argument (or just Lemma 3.14) we have a map $\psi : N_0(S) \to N_1(S)$ between two tubular neighorhoods of S such that $\psi^*\omega_1 = \omega_0$ and $\psi|_S = \mathrm{id}$. This yields the desired result.

To see this, consider the long exact de Rham cohomology sequence for the pair (N, S) where N is any tubular neighborhood of S.

$$\cdots \to H^1(N) \to H^1(S) \to H^2(N,S) \to H^2(N) \to H^2(S) \to \dots$$

Consider $\omega_1 - \omega_0$. This form gives a well-defined class $[\omega_1 - \omega_0] \in H^2(N,S)$ since the pair $(\omega_1 - \omega_0, 0)$ is closed in the cochain complex $\Omega^*(N) \oplus \Omega^{*-1}(S)$ defining the relative cohomology of N and S, i.e $d(\omega_1 - \omega_0, 0) = (d(\omega_1 - \omega_0), (\omega_1 - \omega_0)|_S) = (0,0)$. But this relative cohomology is 0 because N retracts to S, so $(\omega_1 - \omega_0, 0) = (0, \kappa) + (d\alpha, \alpha|_S - d\beta)$ (that is, it's equal to an element in the image of $H^1(S) \to H^2(N,S)$, which is (up to an exact cocycle $(d\alpha, \alpha|_S - d\beta)$ in $\Omega^2(N) \oplus \Omega^1(S)$) something of the form $(0,\kappa)$). But this says that $\omega_1 - \omega_0$ is exact in N.

Now we cope with this sign issue. First observe that the relative sign $\omega_0(\nu(p), \xi(p))/\omega_1(\nu(p), \xi(p))$ is constant for all $p \in S$ (assuming that S is connected). To show this we use a continuity argument: fix a p_0 and define T by:

$$T = \{q \in S | \text{sign}(\frac{\omega_0(\nu(q), \xi(q))}{\omega_1(\nu(q), \xi(q))}) = \text{sign}(\frac{\omega_0(\nu(p), \xi(p))}{\omega_1(\nu(p), \xi(p))}) \forall \xi(p) \in TS^\omega_p - 0, \xi(q) \in TS^\omega_q - 0\}$$

Note that this is independent of our choice of non-zero $\xi(p)$ and $\xi(q)$. Obviously $p \in T$, so T is non-empty. It is also open: if $q \in T$, then by picking a non-zero section ξ of TS^ω in a connected neighborhood U of p we see that $\frac{\omega_0(\nu(q),\xi(q))}{\omega_1(\nu(q),\xi(q))}$ will be a continuously varying non-zero function over U and thus will not change sign. A simple argument with converging sequences of points p_i and vectors $\xi(p_i)$ also shows that the set is closed. So $T = S$.

Thus either the sign $\omega_0(\nu(p), \xi(p))/\omega_1(\nu(p), \xi(p))$ is negative everywhere or positive everywhere. We already dealt with the positive case. In the other case, we can use an automorphism $j : N \to N$ of a tubular neighborhood of S given in coordinates $S \times (-\epsilon, \epsilon)$ (induced by the trivialization of νS by ν) as $j(p,s) = (p, -s)$. This diffeomorphism restricts to the identity on S. Now if we consider $j^*\omega_1$, it satisfies $j^*\omega_1(\nu(p), \xi(p)) = -\omega_1(\nu(p), \xi(p))$. Thus $\frac{\omega_0(\nu(p),\xi(p))}{j^*\omega_1(\nu(p),\xi(p))}$ is positive. Thus applying the first case to $j^*\omega_1$ we find a $\phi : N_0(S) \to N_1(S)$ with $N_i(S) \subset N$ such that $\phi(S) = S$ and $(\phi j)^*\omega_1 = \omega_0$. Thus we still have our result, replacing ϕ with ϕj.

In the case when S is not connected, we can treat each piece separately. Deducing the last part is trivial: the symplectic manifold $S \times (-\epsilon, \epsilon)$ with form $\omega' = i^*\omega_0 + d(t\alpha)$ has the map $i : S \times 0 \to S \subset M$ which by constructioj satisfies $i^*\omega|_{i(S)} = i^*\omega = \omega'|_{S \times 0}$. So there are neighborhoods of S in M and $S \times (-\epsilon, \epsilon)$ that are isomorphic.

Exercise 3.37 State and prove analogues of Theorem 3.30 and Theorem 3.33 for isotropic and coisotropic submanifolds.

Solution 3.37 The analogue is this:

3.30/3.33 Analogue: For $j = 0, 1$ let (M_j, ω_j) be a symplectic manifold with compact submanifold Q_j. Suppose that there is a bundle map $\Phi : TQ_0^\omega \to TQ_1^\omega$ such that $\Phi^*\omega_1|_{TQ_1^\omega} = \omega_0|_{TQ_0^\omega}$ which covers a map $\phi : Q_0 \to Q_1$ such that $\phi^*\omega_1|_{TQ_1} = \omega_0|_{TQ_0}$. Then ϕ extends to a symplectomorphism $\psi : (N(Q_0), \omega_0) \to$

$(N(Q_1), \omega_1)$ of neighborhoods of Q_0 and Q_1.

Proof:

Exercise 3.38 Show that any point q of a symplectic $2k$-dimensional sub-manifold Q of M has a Darboux chart such that Q is given by the equation $x_i = 0$ for $i > 2k$. State and prove similar theorems in Lagrangian, isotropic and coisotropic cases.

Solution 3.38 This is essentially an application of Theorem 3.30, Theorem 3.33 and Exercise 3.37 (along with Exercise 3.40 which states that these results are valid for non-compact submanifolds).

First suppose $Q \subset M$ is symplectic. Take a $p \in M$ and a contractible neighborhood $U \subset M$ of p and consider $Q \cap U$. This is a symplectic manifold of dimension $2k$, so by possibly shrinking U we can find a $V \subset \mathbb{R}^{2k}$ and a symplectomorphism $\bar{\phi} : V \to Q \cap U$ with $\bar{\phi}(0) = p$. Now consider the pullback $\bar{\phi}^*\nu(Q \cap U)$ (where the normal bundle is taken by considering $Q \cap U$ as a non-compact sub-manifold of U). $\bar{\phi}^*\nu(Q \cap U)$ is a bundle over a contractible space V (diffeomorphic to the disk) so it admits a trivialization, equivalently a bundle map $\psi : \nu V = V \times \mathbb{R}^{2n-2k} \to \bar{\phi}^*\nu(Q \cap U)$. This is the same as a a bundle map $\phi : \nu V \to \nu(Q \cap U)$ covering the symplectomorphism $\phi : V \to Q \cap U$. Thus the hypotheses of Theorem 3.30 are satisfied, with $Q_0 = V$, $(M_0, \omega_0) = (W, \omega_0)$ where $W \subset \mathbb{R}^{2n}$ is a contractible open subset of $\mathbb{R}^{2n}$ with $W \cap \mathbb{R}^{2k} = V$, $Q_1 = Q$ and $(M_1, \omega_1) = (U, \omega)$. We have neighborhoods N_1 with $0 \in V \subset N_0$ in $W \subset \mathbb{R}^{2n}$, a neighborhood N_1 with $p \in Q \cap U \subset U$, and a symplectomorphism $\psi : N_0 \to N_1$ sending $\mathbb{R}^{2k} \cap W$ to V. But these are precisely Darboux coordinates about p where $Q \cap N_1 \simeq V \cap N_0 = \mathbb{R}^{2k} \cap N_0 = \{(x_i) \in N_0 | x_i = 0, i > 2k\}$.

Now suppose that $Q \subset M$ is Lagrangian. Take a $p \in M$ and a contractible neighborhood U of p such that $U \cap Q$ is also contractible. Then $U \cap Q$ is a contractible Lagrangian in the open symplectic manifold U, and is diffeomorphic via some ϕ to an open $V \subset \mathbb{R}^n \subset \mathbb{R}^{2n}$ where $\mathbb{R}^n \subset \mathbb{R}^{2n} \simeq \mathbb{C}^n$ is the usual Lagrangian (the real sub-space). We can assume that $\phi(0) = p$. Let $W \subset \mathbb{R}^{2n}$ be a simply connected open subset such that $W \cap V$. Then by Theorem 3.33, the diffeomorphism $\phi : V \to U$ is covered by a symplectomorphism $\psi : N_0 \to N_1$ of neighborhoods of N_0 of V to a neighorhood N_1 of $Q \cap U$. This is precisely a Darboux chart where $\psi(0) = p$ and $\psi^{-1}(Q \cap U) = V = W \cap \mathbb{R}^{2n} = \{(x_i) \in W | x_i = 0, i > n\}$.

Exercise 3.39 Prove Lemma 3.14 and hence Theorems 3.30 and 3.33 for non-compact sub-manifolds Q.

Solution 3.39 We will prove the following Lemma. The proof will largely be a rehashing of the proof of Lemma 3.14, with a few modifications which we will point out.

Lemma 3.14 Analogue: Let M be a $2n$-dimensional smooth manifold, and $Q \subset M$ be a closed sub-manifold whose topology is the induced topology[4]. Suppose that $\omega_0, \omega_1 \in \Omega^2(M)$ are closed 2-forms such that at each $q \in Q$ the forms ω_0 and ω_1 are equal and non-degenerate on T_qM. Then there exists open neighborhoods N_0 and N_1 of Q and a diffeomorphism $\psi : N_0 \to N_1$ such that $\psi|_Q = \text{id}$ and $\psi^*\omega_1 = \omega_0$.

Proof: First we show that there exists a neighborhood N of Q and exact 1-form $\sigma \in \Omega^1(N)$ such that

[4] So every $p \in Q$ has a neighborhood U in M and coordinates $\phi : V \to U$ such that $Q \cap U = \phi(\{(x_i) | x_1 = \ldots x_k = 0\}$ for some k. Lang takes this as the definition of a sub-manifold, but it's not always considered a necessary condition.

$\sigma|_{T_Q M} = 0$ and $d\sigma = \omega_1 - \omega_0$. As in Lemma 3.14, we prove this by considering the exponential map $\exp : TQ^\perp \to M$ from the normal bundle to Q with respect to any metric on M. By the tubular neighborhood theorem for closed sub-manifolds (see for instance Lang, Fundamentals of Differential Geometry, Theorem 5.1) there exists a smooth function $\epsilon : Q \to \mathbb{R}^+$ such that the open neighborhood $U(\epsilon) = \{(p,v) \in TQ^\perp | g(v,v) < \epsilon(p)\}$ maps diffeomorphically to $N = \exp(U(\epsilon)) \subset M$. Now define $\phi_t : N \to N$ for $0 \le t \le 1$ by:

$$\phi_t(\exp(p,v)) = \exp(p,tv)$$

ϕ_t is a diffeomorphism for $t > 0$, $\phi_0(N) = Q$, $\phi_1 = \mathrm{id}$ and $\phi_t|_Q = \mathrm{id}$. Thus letting $\tau = \omega_1 - \omega_0$, we have $\phi_0^*\tau = 0$ and $\phi_1^*\tau = \tau$. Now define the vector field X_t by:

$$X_t = (\frac{d}{dt}\phi_t) \circ \phi_t^{-1}$$

for $t > 0$. Then:

$$\frac{d}{dt}\phi_t^*\tau = \phi_t^*\mathcal{L}_{X_t}\tau = d(\phi_t^* i_{X_t}\tau) = d\sigma_t$$

where we now define $\sigma_t = \phi_t^* i_{X_t}\tau$. In particular, we have:

$$\tau = \phi_1^*\tau - \phi_0^*\tau = \int_0^1 \frac{d}{dt}\phi_t^*\tau = d\sigma \quad \sigma = \int_0^1 \sigma_t dt$$

Furthermore, $i_v\sigma_t(q) = i_{\frac{d}{dt}\phi_t(q))} i_{d\phi_t(q)v}\tau(\phi_t(q))$, so σ_t itself is smooth at 0 even though X_t is not. Furthermore this formula makes it clear that it vanishes for $q \in Q$ since then $\phi_t(q) = q$ and $\tau(\phi_t(q)) = \tau(q) = 0$.

Thus we have our σ. Now we execute Moser's argument. Consider the family of 2-forms $\omega_t = \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + td\sigma$. Since $\omega_t|_Q = \omega_0|_Q$, for every point $q \in Q$ there exists a neighborhood $U(q)$ such that ω_t is non-degenerate for all t at any $r \in U(q)$. Taking the union of these neighborhoods and intersecting the result with N, we may shrink N so that ω_t is non-degenerate in N for all N for all t. Then we can solve the equation $\sigma_t + i_{X_t}\omega_t = 0$ for a vector field X_t on N.

Now we just need to know that we can solve the equation$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \psi_0 = \mathrm{id}$ for $0 \le t \le 1$. Then $0 = \frac{d}{dt}\psi^*\omega_t = \psi_t^*(\frac{d}{dt}\omega_t + di_{X_t}\omega_t) = 0$ so in particular $\psi_1^*\omega_1 = \omega_0$. To know this, we must shrink N even further. Since X_t is C^1 (thus locally Lipchitz) and $X_t(q) = 0$ for every $q \in Q$, we know that for every q there exists a constant $C(q)$ such that in a sufficiently small ball $B(q, \epsilon(q))$ about q we have $|X_t(p)| < d(p,q)$ for all $p \in B(q,\epsilon(q))$. Furthermore we know that any flow line $p(t)$ of X_t can be continued while $p(t)$ remains in the ball: that is, if $p : [0,s] \to B(q,\epsilon(q))$ is some partially defined flow line with $p(s) \in B(q,\epsilon(q))$ then by the local existence theory of first order ODE (Picard-Lindelof) and the fact that X_t is C^∞ in t and p, we can extend $p(s)$ to a flow line $p : [0, s+\delta]$ for some small $\delta > 0$.

Now suppose that $p : [0,s) \to B(q,\epsilon(q))$ is some flow-line. Then observe that $\frac{d}{dt}(d(p(t),q)) \le \frac{d}{dt}(\mathrm{len}(p(t))) = |X_t(p(t))| \le C(q)d(p(t),q)$. Thus letting $f(t) = d(p(t),q)$ we have $\frac{df}{dt} \le C(q)f$, which implies $f(t) \le f(0)e^{C(q)t}$, i.e $d(p(t),q) \le d(p(0),q)e^{C(q)t}$. Thus if we pick $p(0) \in B(q,\eta(q))$ where $\eta(q) = \frac{1}{2}\epsilon(q)e^{-C(q)}$, $p(t)$ will stay in $B(\epsilon(q),q)$ until $t = 1$.

Thus if we let $N_0 = N \cap (\cup_{q\in Q} B(q,\eta(q)))$, then the flow along X_t is well-defined to time 1. Thus setting $\phi = \phi_1$ and $N_1 = \phi_1(N_0)$, the resulting map $\phi : N_0 \to N_1$ is the map that we desire.

Exercise 3.40 Let $\psi_t : Q \to M$ be an isotopy of a symplectic, Lagrangian, isotropic or coisotropic submanifolds Q of M. Show that ψ_t extends symplectically over a neighborhood of Q.

Solution 3.40 First assume that Q is isotropic, Lagrangian, or symplectic. In the symplectic case, by Moser stability, we can assume that the map satisfies $\psi_t^*\omega_t = \tau$ for some fixed symplectic form τ on Q: if not, then $\psi_t^*\omega_t$ is a family of cohomologous symplectic forms on Q, thus there exists a family $\nu_t : Q \to Q$ so that $\nu_t^*\psi_t^*\omega = \psi_0^*\omega = \tau$. In the other cases we can also assume this, because the restriction of ω is 0.

Now choose a neighborhood N that retracts onto Q, so that $H^*(N, Q; \mathbb{R}) = 0$. Take any extension of ψ_t to an isotopy $\rho_t : N \to M$ of a neighborhood of Q into M, and consider $\rho_t^*\omega_t$. Then $\tau_t = \rho_t^*\omega - \rho_0^*\omega$ is a family of closed forms on N which vanish on Q. Now we examine the long exact sequence of cohomology for the pair (N, Q):

$$\dots \to H^1(N) \to H^1(Q) \to H^2(N,Q) \to H^2(N) \to H^2(Q) \to \dots$$

Since τ_t vanishes on the Q, $(\tau_t, 0)$ is a representative of an element in $H^2(N,Q)$. However, $H^2(N,Q) = 0$, so $(\tau_t, 0) = (d\alpha_t, \alpha_t|_Q - d\beta_t) + (0, \kappa_t|_Q)$ for some smooth families of 1-forms α_t on N, 0-forms β on Q and closed 1-forms κ on N. This just comes from unravelling the definition of cocycles for relative de Rham cohomology[5]. But this precisely says that $\tau_t = d\alpha_t$ where $\alpha_t|_Q = d\beta_t + \kappa_t|_Q$. We can extend β_t to a smooth function on all of U and then redefine $\alpha_t = \alpha_t - \kappa_t + d\beta_t$ to get an α_t which vanishes on the boundary and has $d\alpha_t = \tau_t$. Thus we may apply the Moser trick (solving $\alpha_t + i_{X_t}\omega_t = 0$ for X_t and then integrating X_t) to construct a family of diffeomorphisms ϕ_t which fix $Q \subset U$ and have the property that $\rho_0^*\omega = \phi_t^*\rho_t^*\omega$ and $\phi_t|_Q = \mathrm{id}$. Thus $\phi_t\rho_t : U \to M$ is a family of symplectomorphisms $(U, \rho_0^*\omega) \to (M, \omega)$ such that $\phi_t\rho_t|_Q = \psi_t$.

Exercise 3.50 Let $H = H(x_1, \dots, x_n, y_1, \dots, y_n, z)$ be a smooth function on $\mathbb{R}^{2n+1}$. Prove that the contact vector field generated by H with respet to the standard form $\alpha = dz - \sum_j y_j dx_j$ is given by the differential equation:

$$\dot{x}_j, = \frac{\partial H}{\partial y_j}, \quad \dot{y}_j = -\frac{\partial H}{\partial x_j} - y_j\frac{\partial H}{\partial z}, \quad \dot{z} = \sum_j y_j \dot{x}_j - H$$

Solution 3.50 The contact vector field X_H is characterized uniquely by $i_{X_H}\alpha = -H$ and $i_{X_H}d\alpha = dH - (i_R dH)\alpha$ where R is the Reeb vector field and α is the contact form. Consider the vector field:

$$X = \frac{\partial H}{\partial y_j}\partial_{x_j} + (-\frac{\partial H}{\partial x_j} - y_j\frac{\partial H}{\partial z})\partial_{y_j} + (\sum_j y_j \frac{\partial H}{\partial y_j} - H)\partial_z$$

Then we calculate that:

$$i_X\alpha + H = \sum_j y_j\frac{\partial H}{\partial y_j} - H - \sum_j y_j\frac{\partial H}{\partial y_j} + H = 0$$

[5]See for instance Bott & Tu, *Differential Forms In Algebraic Topology*

$$i_X d\alpha - dH + (i_R dH)\alpha = (\sum_j \frac{\partial H}{\partial y_j} dy_j + \frac{\partial H}{\partial x_j} dx_j + y_j \frac{\partial H}{\partial z} dx_j - \frac{\partial H}{\partial x_j} dx_j - \frac{\partial H}{\partial y_j} dy_j - y_j \frac{\partial H}{\partial z} dx_j) - \frac{\partial H}{\partial z} dz + \frac{\partial H}{\partial z} dz = 0$$

Thus $X = X_H$. This prove that the flow lines solving $\frac{d}{dt}\gamma = X_\gamma$ are given by the ODE written. In particular, if H is time independent then the first two equations are the Hamiltonian flow equations for (x, y) in $\mathbb{R}^{2n}$ and the last equation says that:

$$z(t) - z(0) = \int_0^t \dot{z} dt = \int_0^t \sum_j y_j \dot{x}_j - H dt = A((x, y)|_{[0,t]})$$

Here A is the symplectic action, as introduced in Ch. 1.

Exercise 3.51 Prove that the solutions of (3.11) are characteristics of the Hamilton-Jacobi equation:

$$\partial_t S + H(x, \partial_x S, S) = 0$$

for a function $S = S(t, x)$ on $\mathbb{R}^{n+1}$. More precisely, if S is a solution of (3.12) (the above equation) and $x(t)$ is a solution of the ordinary differential equation $\dot{x} = \partial_y H(x, \partial_x S, S)$, prove that:

$$x(t), \quad y(t) = \partial_x S(t, x(t)), \quad z(t) = S(t, x(t))$$

satisfy (3.11) (the contact ODE). Conversely, given an initial function $S(0, x) = S_0(x)$ use the solutions of the contact differential equation (3.11) with initial conditions of the form $x(0) = x_0, y(0) = \partial_x S_0(x_0), z(0) = S_0(x_0)$, to construct a solution to the Hamilton-Jacobi equation (3.12) for small t. Moreover, prove that a function $S = S(t, x)$ satisfies (3.12) if and only if the corresponding Legendrian submanifolds:

$$L_t = \{(x, \partial_x S(t, x), S(t, x)) | x \in \mathbb{R}^n\}$$

are related by $L_t = \psi_t(L_0)$, where $\psi_t : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is the flow of the differential equation (3.11).

Solution 3.51 Suppose that $\dot{x}_j = \partial_{y_j} H(x, \partial_x S, S)$ and we define $y(t) = \partial_x S(t, x(t))$ and $z(t) = S(t, x(t))$. We want to show that these satisfy the contact Hamilton equations. The equation $\dot{x} = \partial_y H$ is the set of equations for x. For z we have:

$$\dot{z} = \frac{d}{dt}(S(t, x(t))) = \partial_t S(t, x(t)) + \sum_j (\partial_{x_j} S) \dot{x}_j = -H(x, y, z) + \sum_j y_j \dot{x}_j$$

Here we just use chain rule, the differential equation for S and the definition of y. Likewise, we have:

$$\dot{y}_j = \frac{d}{dt}(\partial_{x_j} S(t, x)) = (\partial_t \partial_{x_j} S)(t, x) + \sum_i (\partial_{x_i} \partial_{x_j} S)(t, x) \dot{x}_i(t)$$

$$= -(\partial_{x_j}(H(x, \partial_x S, S)) + \sum_i (\partial_{x_i} \partial_{x_j} S)(t, x) \dot{x}_i(t)$$

$$= -(\partial_{x_j} H)(x, \partial_x S, S) - \sum_i (\partial_{y_i} H)(x, \partial S, S)\partial_{x_i}\partial_{x_j} S - \partial_z H \partial_{x_j} S + \sum_i (\partial_{x_i}\partial_{x_j} S)(t, x)\dot{x}_i(t)$$

$$= -(\partial_{x_j} H)(x, y, z) - \partial_z H y_j$$

This is the last equation, for $\dot{y}$.

To use solutions of the contact Hamilton equations to build a solution of the Hamilton-Jacobi equations, we essentially use these formulae backwards. For an initial function $S_0(x)$, consider the solutions to $x(w,t), y(w,t), z(w,t)$ to the contact Hamilton equations for initial conditions $x(w,0) = w$, $y(w,0) = \partial_x S_0(w)$ and $z(w,0) = S_0(w)$.

Now consider the family of smooth maps $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi_t(w) = x(w,t)$. We have $\phi_0 = \mathrm{id}$, so for each $p \in \mathbb{R}^n$ there exists a neighborhood about p, $U \subset \mathbb{R}^n \times \mathbb{R}^+$, such that for all $(q,t) \in U$ we have $d\phi_t(q)$ is full rank. In particular, for a fixed point $w_0 \in \mathbb{R}^n$ we have a neighborhood $U \times [0,t)$ of w_0 with this property. Thus, in this neighborhood, we may define $S(x,t) = z(\phi_t^{-1}(x), t)$, and if we pick $U \times [0,t)$ small enough then this map is well-defined since there $\phi_t : U \times [0,t)$ will be a diffeomorphism onto its image. In order to extend this definition to all of $\mathbb{R}^n$ we would need to make assumptions about H to make this ODE well-defined for a fixed time interval over all $\mathbb{R}^n$.

Now before we move further, we want to observe that $\partial_x S(t, x(t)) = y_{x(0)}(t)$.

With the above results, checking that $S(x,t)$ this satisfies the Hamilton-Jacobi equations in its domain of definition is simple. We just observe that:

$$\partial_t S(x,t) = \partial_t(z(\phi_t^{-1}(x), t)) = (\partial_t z)(\phi_t^{-1}(x), t) + \sum_i (\partial_{w_i} z)(\phi_t^{-1}(x), t)\frac{d\phi_{t,i}^{-1}}{dt}(x)$$

$$= \sum_i y_i(\phi_t^{-1}(x), t)\partial_{y_i} H(x, y(\phi_t^{-1}(x), t), t) - H(x, y(\phi_t^{-1}(x), t), t) - \sum_i (\partial_{w_i} z)(\phi_t^{-1}(x), t)(d\phi_t^{-1})^i_j \frac{dx_j}{dt}(t)$$

$$= \sum_i \partial_{x_j} S(x,t)\partial_{y_i} H(x, \partial_x S(x,t), t)$$

$$-H(x, \partial_x S(x,t), t) - \sum_i \partial_{x_j} S(x,t)\partial_{y_i} H(x, \partial_x S(x,t), t)$$

$$= -H(x, \partial_x S(x,t), t)$$

Exercise 3.52 Prove that the contact vector fields form a Lie algebra with $[X_F, X_G] = X_{\{F,G\}}$ for $F, G : M \to \mathbb{R}$. Deduce that the map $(F, G) \to \mathbb{R}$ determines a Lie algebra structure on $C^\infty(M)$.

Solution 3.52 Let X, Y be contact vector fields with contact Hamiltonians F, G. Consider the vector field $[X,Y]$ and the function $\{F,G\} = -\alpha([X,Y])$. We verify the formulae in Lemma 3.49 (i) for $[X,Y]$ and $\{F,G\}$, namely that:

$$i_Z\alpha = -H; \quad i_Z d\alpha = dH - (i_R dH)\alpha$$

for $Z = [X, Y]$ and $H = \{F, G\}$. Observe that this is equivalent to the condition:

$$i_Z\alpha = -H; \quad \mathcal{L}_Z\alpha = i_{[Z,R]}\alpha$$

Thus we need to prove that condition for $Z = [X, Y]$ and $H = \{F, G\}$, assuming it holds for the pairs X, F and Y, G. The first condition is trivial. For the second one we have:

$$\mathcal{L}_{[X,Y]}\alpha = \mathcal{L}_X\mathcal{L}_Y\alpha - \mathcal{L}_Y\mathcal{X}\alpha = \mathcal{L}_X(\alpha i_{[Y,R]}\alpha) - \mathcal{L}_Y(\alpha i_{[X,R]}\alpha)$$

$$= i_{[Y,R]}\mathcal{L}_X\alpha)\alpha + \alpha i_{[X,[Y,R]]}\alpha + \mathcal{L}_X\alpha i_{[Y,R]}\alpha - \alpha i_{[X,R]}\mathcal{L}_Y\alpha + \alpha i_{[X,[Y,R]]} - \mathcal{L}_Y\alpha i_{[X,R]}\alpha$$

$$= 2i_{[X,R]}\alpha i_{[Y,R]}\alpha - 2i_{[X,R]}\alpha i_{[Y,R]}\alpha + \alpha i_{[X,[Y,R]]+[Y,[R,X]]}\alpha = \alpha i_{-[R,[X,Y]]}\alpha = \alpha i_{[[X,Y],R]}\alpha$$

This confirms the two identities, and the second equation implies also that $[X, Y]$ is contact. This implies that the bracket $\{\cdot, \cdot\}$ obeys a Bianch identity. Since it is anti-symmetric and bilinear by construction, it is by definition a Lie bracket. It imbues the vector-space of C^∞ functions with a Lie algebra structure.

Exercise 3.54 Not every contact vector field is the Reeb field of some contact form. Show that X is the Reeb field of some contact form which defines ξ if and only if X is transverse to ξ, i.e $i_\xi\alpha \neq 0$ for any defining form α.

Solution 3.54 If R is the Reeb vector-field of a contact form α defining ξ as $\xi = \ker\alpha$, then for any other defining $\alpha' = f\alpha$ with $f > 0$ we have $i_R\alpha' = fi_R\alpha = f$, so X is transverse to ξ. Also $\mathcal{L}_R\alpha = 0$ so R is evidently contact.

Conversely, suppose that X is a contact vector-field transverse to ξ. Let α be any defining form for ξ. Then $i_X\alpha$ is never zero by assumption, so if take the contact Hamiltonian $H = -i_X\alpha$ then the new contact form $\frac{-\alpha}{H}$ is well-defined and smooth. Furthermore, we have:

$$i_Xd(\frac{-\alpha}{H}) = -i_X(\frac{Hd\alpha - dH \wedge \alpha}{H^2}) = -\frac{Hi_Xd\alpha - \alpha i_XdH + dHi_X\alpha}{H^2}$$

$$= -H^{-2}(Hi_Xd\alpha - HdH + \alpha\mathcal{L}_X(i_X\alpha)) = -H^{-2}(Hi_Xd\alpha - HdH + \alpha i_X\mathcal{L}_X\alpha)$$

$$= -H^{-2}(Hi_Xd\alpha - HdH - \alpha i_X\alpha i_RdH) = -H^{-2}(Hi_Xd\alpha - HdH + H\alpha i_RdH) = 0$$

This is essentially a repeated application of the 2nd defining equation for the contact Hamiltonian defined for X in Lemma 3.49.

Exercise 3.59 Let (M, ξ) be a contact manifold with contact form α and corresponding Reeb field R. If β is any 1-form such that $\beta(R) = 0$ prove that there is a unique vector field X which is tangent to $\ker\alpha$ and such that $\beta = i_Xd\alpha$.

Solution 3.59 We know by the contact condition that $\alpha \wedge (d\alpha)^n$ is a volume form, thus that $d\alpha$ is a non-degenerate symplectic form on $\ker\alpha$. Thus the map $\psi : \xi \to \xi^*$ given by $v \mapsto i_vd\alpha|_\xi$ is a bundle isomorphism. Now consider the sub-bundle $\eta \subset T^*M$ with fiber $\eta_p = \ker(R)$ where R is identified as an element of $(T^*M_p)^*$. Then we have a bundle map $\phi : \eta \to \xi$ given by $e \mapsto \psi^{-1}(i(e)|_\xi)$. Here $i : \eta \to T^*M$ is the inclusion and the map $T^*M \to \xi^*$ given by $e \mapsto e|_\xi$ is restriction.

We prove that ϕ is a bundle isomorphism, first arguing that it is surjective on the fibers. To see this, we observe that the restriction map $T * M \to \xi^*$ is certainly surjective (because $\xi \to TM$ is injective). So for any $c \in \xi^*$ there is a b so that $b|_\xi = c$. Then $a = b - \alpha b(R)$ is an element of η such that $a|_\xi = b|_\xi - b(R)\alpha|_\xi = b|_\xi = c$. So any $c \in \xi^*$ is in the image of $e \mapsto i(e)|_\xi$. Then since ψ^{-1} is an isomorphism, we know that $e \mapsto \psi^{-1}(i(e)|_\xi)$ is a composition of surjective maps on the fibers, and thus is surjective.

To prove that ϕ is injective is suffices to show that $e \mapsto i(e)|_\xi$ is injective. So suppose that $i(e)|_\xi = 0$ for $e \in \eta_p$. Then $e(v) = 0$ for any $v \in \xi$ and $e(R) = 0$ since $e \in \eta$. Thus e is zero on a basis of T_pM, thus it is identically 0. So ϕ is injective. Thus the map $\eta \to \xi$ is a bundle isomorphism. It follows that any section β of η maps to a unique section X in ξ such that $\beta = \phi^{-1}(X) = i_{i(X)}d\alpha$.

Exercise 3.55 (Darboux's theorem) Prove that every contact structure is locally diffeomorphic to the standard structure on $\mathbb{R}^{2n+1}$.

Solution 3.55 First observe that, for any vector-space V of dimension $2n + 1$, non-zero covector α on V and symplectic form β on $\ker(\alpha)$ there is a linear map $\Psi : V \to V$ such that $\Psi^*\alpha = \alpha_0|_0 = dz$ and $\Psi^*\beta = \omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. This fact is easy to see: simply do a change of coordinates to a basis $e_z, e_{x_1}, e_{y_1}, \ldots, e_{x_n}, e_{y_n}$ where e_{x_i} and e_{y_i} span $\ker(\alpha)$ and then apply the symplectic Graham-Schmidt procedure to e_{x_i} and e_{y_i} to get a symplectic basis of $\ker(\alpha)$.

Now consider a contact manifold (Y, α), a point $p \in Y$ and a neighborhood U of p. Pick coordinates $\phi : U \to \mathbb{R}^{2n+1}$ with $\phi(p) = 0$. By the above discussion we can choose this map so that $\phi^*\alpha_0 = \alpha$ and $\phi^*d\alpha_0 = d\alpha$ at p. Now consider the family of 1-forms $\alpha_t = t\phi^*\alpha_0 + (1 - t)\alpha$. At p we have $\alpha_t \equiv \alpha$ and $d\alpha_t = d\alpha$. Thus $\alpha_t \wedge d\alpha_t^n = \alpha \wedge d\alpha^n$ is a volume form at p for $t \in [0, 1]$. Since α_t and $d\alpha_t$ are smooth, and I is compact we can (after potentially shrinking U) assume that $\alpha_t \wedge d\alpha_t^n$ is a volume form on U for $t \in [0, 1]$. Then it follows from the contact Moser argument on p. 112 that (after again possibly shrinking U) there exists a family of diffeomorphisms $\psi_t : U \to M$ and a family non-vanishing of smooth functions $g_t : M \to \mathbb{R}$ such that $\psi_t^*\alpha_t = g_t\alpha_0$ for all t. In particular, $(\psi_1\phi)^*\alpha_0 = g_1\alpha$, and thus the map $\psi_1\phi : U \to \psi_1\phi(U) \subset \mathbb{R}^{2n}$ is a contactomorphism of a neighborhood of $p \in M$ to a neighborhood of $\mathbb{R}^{2n+1}$ with the standard contact form.

Exercise 3.55 (Gray's Stability Theorem) Prove that every family α_t of contact forms on a closed manifold M has the form $\psi_t^*(f_t\alpha_0)$ for some nonvanishing functions f_t.

Solution 3.55 It's equivalent to prove that $\alpha_0 = f_t\psi_t^*\alpha_t$ since then $\alpha_t = (\psi_t^*)^{-1}(\frac{1}{f_t}\alpha_0)$. This is just Moser's argument repeated. Given a compact manifold M with a family of contact forms α_t with corresponding Reeb vector field R_t, we consider the family of smooth functions $h_t = i_{R_t}\frac{d}{dt}\alpha_t$. Then $\sigma_t = \frac{d}{dt}\alpha_t - h_t\alpha_t$ is a family of 1-forms with $i_{R_t}\sigma_t = (i_{R_t}\frac{d}{dt}\alpha_t)(1 - i_{R_t}\alpha_t) = 0$. Thus there exists a unique family of vector-fields X_t tangent to $\xi = \ker(\alpha_t)$ such that $i_{X_t}d\alpha_t = \sigma_t = \frac{d}{dt}\alpha_t - h_t\alpha_t$. In particular, we have:

$$\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t = \frac{d}{dt}\alpha_t + i_{X_t}d\alpha_t + d(i_{X_t}\alpha_t) = h_t\alpha_t$$

since $i_{X_t}\alpha_t = 0$. Now since M is compact and X_t is smooth, we can solve for the flow of X_t for $t \in [0,1]$:

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = \mathrm{id}$$

Furthermore we can can set f_t to be:

$$f_t = \exp(\int_0^t h_t \circ \psi_t dt)$$

Then we have:

$$\frac{d}{dt}(f_t\psi_t^*\alpha_t) = f_t\psi_t^*(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t) - \frac{d}{dt}f_t\alpha_t = f_t\psi_t^*(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t) - f_t h_t \circ \psi_t\psi_t^*\alpha_t$$

$$= f_t\psi_t^*(\frac{d}{dt}\alpha_t + \mathcal{L}_{X_t}\alpha_t - h_t\alpha_t) = 0$$

Thus $f_t\psi_t^*\alpha_t = \alpha_0$.

Exercise 3.57 (i) Prove that $L \subset Q$ is a Legendrian sub-manifold if and only if $L \times \mathbb{R}$ is a Lagrangian sub-manifold of $Q \times \mathbb{R}$. (ii) Prove that $\psi : Q \to Q$ is a contactomorphism with $\psi^*\alpha = e^h\alpha$ if and only if the map $\tilde{\psi}(q,\theta) = (\psi(q), \theta - h(q))$ is a symplectomorphism of $Q \times \mathbb{R}$. (iii) Prove that if $X = X_H : Q \to TQ$ is the contact vector-field generated by $H : Q \to \mathbb{R}$ then the Hamiltonian vector-field $\tilde{H}(q,\theta) = e^\theta H(q)$ on $Q \times \mathbb{R}$ generates the Hamiltonian vector field $\tilde{X}(q,\theta) = (X(q), dH(Y))$. (iv) Prove that the Poisson bracket of $\tilde{F} = e^\theta F$ and $\tilde{G} = e^\theta G$ is given by $\{\tilde{F}, \tilde{G}\} = e^\theta\{F, G\}$.

Solution 3.57 (i) Pick a sub-manifold $L \subset Q$. Pick any $(p,\theta) \in L \times \mathbb{R}$, $e_\theta, e_1, \ldots, e_k$ form a basis of $T_p(L \times \mathbb{R})$, where e_θ is the basis vector in the θ direction and e_i is a basis of TL. Then L is Legendrian if and only if L is n-dimensional, with $TL \subset \xi$ and $d\alpha|_L = 0$. In the basis the last two conditions are equivalent to:

$$\omega(e_i, e_j) = e^\theta(d\alpha - \alpha \wedge d\theta) = e^\theta d\alpha(e_i, e_j) = 0; \quad \omega(e_\theta, e_j) = \alpha(e_j) = 0$$

The above equations hold if and only if $L \times \mathbb{R}$ is $n+1$-dimensional and $\omega|_{L\times\mathbb{R}} = 0$, i.e if and only if $L \times \mathbb{R} \subset Q \times \mathbb{R}$ is Lagrangian.

(ii) We see that:

$$\psi^*\alpha = e^h\alpha \iff \psi^*\alpha = e^h\alpha \text{ and } \psi^*d\alpha = e^h(dh \wedge \alpha + d\alpha)$$

$$\iff \tilde{\psi}^*\omega = \tilde{\psi}^*(e^\theta(d\alpha - \alpha \wedge d\theta)) = e^{\theta-h}(\psi^*d\alpha - \psi^*\alpha \wedge d(\theta - h))$$

$$= e^{\theta-h}(e^h(dh \wedge \alpha + d\alpha) - e^h\alpha \wedge (d\theta - dh) = e^\theta(d\alpha - \alpha \wedge d\theta)$$

The forward part of the last if and only if is part of the manipulation. The backward part comes from the fact that $e^\theta(d\alpha - \alpha \wedge d\theta) = e^{\theta-h}(\psi^*d\alpha - \psi^*\alpha \wedge d(\theta - h))$ implies that $(e^{\theta-h}\psi^*\alpha - e^\theta\alpha) \wedge d\theta = 0$. Since α only has components in the Q directions, this implies that $e^{\theta-h}\psi^*\alpha - e^\theta\alpha = 0$ identically, which implies that ψ is a contactomorphism.

(iii) We observe that if $\tilde{H} = e^\theta H$ is our Hamiltonian, then $d\tilde{H} = e^\theta H d\theta + e^\theta dH$ and defining $\tilde{X} = (X, i_R dH)$ where R is the Reeb vector field, we have:

$$i_{\tilde{X}}\omega = e^\theta(i_X d\alpha - i_X \alpha d\theta + i_R dH\alpha) = e^\theta(Hd\theta + dH) = d\tilde{H}$$

Here we use the defining equations of X, namely $i_X\alpha = -H$ and $i_X d\alpha = dH - i_R dH\alpha$.

(iv) We compute:

$$i_{X_{\tilde{F}}} i_{X_{\tilde{G}}}\omega = e^\theta i_{X_{\tilde{F}}} i_{X_{\tilde{G}}}(d\alpha - \alpha \wedge d\theta) = i_{X_{\tilde{F}}}(i_{X_G} d\alpha - i_{X_G}\alpha d\theta + \alpha i_R dG)$$

$$= i_{X_F} i_{X_G} d\alpha - i_{X_G}\alpha i_R dF + i_{X_F}\alpha i_R dG = i_{X_F} i_{X_G} d\alpha + i_{X_G} d(i_{X_F}\alpha) + i_{X_G} i_{X_F} d\alpha - i_{X_F} d(i_{X_G}\alpha) - i_{X_F} i_{X_G} d\alpha$$

$$= i_{X_G} i_{X_F} d\alpha + X_G(i_{X_F}\alpha) - X_F(i_{X_G}\alpha) = -i_{[X_F, X_G]}\alpha = \{F, G\}$$

Exercise 3.59 (i) Show that if a compact hypersurface Q has contact type, different choices of forms α such that $d\alpha = \omega|_Q$ give rise to isotopic contact structures on Q. (ii) A compact hypersurface Q in a symplectic manifold (M, ω) is said to be of restricted contact type if it is transverse to a Liouville vector field X defined on all of M. Show that every simply connected hypersurface of contact type in fact has restricted contact type provided only that ω is exact. (iii) Consider a compact Lagrangian submanifold L of Euclidean space $(\mathbb{R}^{2n}, -d\lambda_0)$. By Theorem 3.33, a neighborhood N of the zero section in T^*L embeds symplectically into $\mathbb{R}^{2n}$. For small r, the sphere bundle $S_r(T^*L)$ of radius r is contained in N and so also embeds into $\mathbb{R}^{2n}$. Show that these hypersurfaces have contact type.

Solution 3.59 (i) In order to be isotopic, it's clear that α_0 and α_1 must induce the same orientation on Q via their volume form, since isotopic volume forms induce the same orientation. Thus we may assume this. Consider two contact forms α_0 and α_1, both of which satisfy $d\alpha_i = \omega|_Q$. Then consider the family of 1-forms $\alpha_t = (1-t)\alpha_0 + t\alpha_1$. Then we have:

$$d\alpha_t = (1-t)\omega|_Q + t\omega|_Q = \omega_Q = d\alpha_0 = d\alpha_1$$

Thus we have:

$$\alpha_t \wedge d\alpha_t^n = (1-t)\alpha_0 \wedge d\alpha_t^n + t\alpha_1 \wedge d\alpha_t^n = (1-t)\alpha_0 \wedge d\alpha_0^n + t\alpha_1 \wedge d\alpha_1^n$$

As a convex combination of two volume forms inducing the same orientation, the latter expression is non-zero for all t. Thus α_t is contact for all t and thus it is an isotopy of contact structures.

(ii) Suppose that ω is exact. Then $\omega = d\alpha$ for some 1-form α. Furthermore, let X_α be the unique vector-field on M satisfying $i_{X_\alpha} d\alpha = \alpha$. Such a vector-field exists and is unique by the non-degeneracy of $\omega = d\alpha$. Then we have $\mathcal{L}_{X_\alpha}\omega = di_{X_\alpha} d\alpha = d\alpha = \omega$. Now we want to show that we can pick α so that X_α is transverse to Q.

For this, we observe the following. Since Q is of contact type, Q has a Liouville vector field X_β in a neighborhood U which is transverse to Q at every point. Let β be the corresponding 1-form $\beta = i_{X_\beta}\omega$. Then $\beta - \alpha$ is a closed 1-form on U. Since Q is simply connected, $H^1(Q;\mathbb{R}) = H^1(U;\mathbb{R}) = 0$, so $\beta - \alpha = df$ for some function f. Now let g be a function on M which is compactly supported in U, and which agrees with f in a smaller neighborhood of Q. Furthermore let $\kappa = \alpha + dg$. Then κ has $d\kappa|_Q = d\alpha|_Q = \omega|_Q$ and has $\kappa = \beta$ in a neighborhood of Q, implying that the Liouville $X_\kappa = X_\beta$ in a neighborhood of Q, and thus that it is transverse to Q. So X_κ is the globally defined Liouville that we want.

(iii) This is equivalent to the fact that the sphere bundles themselves $Q = S_r(T^*L) \subset T^*L$ are of contact-type in T^*L. To see this, let $\psi : N \to M$ be the symplectomorphism of the neighborhood of $0 \subset T^*L$ to M, and suppose X is a Liouville for $Q \subset N$ (which we can assume is defined over all of N after possibly shrinking N). Then at any point $\psi(p) \in \psi(N)$ we have:

$$\mathcal{L}_{\psi_* X}(-d\lambda_0) = \mathcal{L}_{\psi_* X}((\psi^{-1})^*(\omega_{\text{can}})) = (\psi^{-1})^*\mathcal{L}_X\omega_{\text{can}} = (\psi^{-1})^*\omega_{\text{can}} = -d\lambda_0$$

Thus the neighborhood $\psi(N)$ is a neighborhood of $\psi(Q)$ with the Liouville ψ_*X, and $\psi(Q)$ is of contact type. Now observe that $Q = S_r(T^*L)$ has the Liouville vector-field $X_{p,\xi} = -\sum_i \xi_i\partial_{\xi_i}$ (here the ξ_i are cotangent fiber coordinates in T^*L). Then we have:

$$i_X\omega_{\text{can}} = i_X(\sum_i dx_i \wedge d\xi_i) = \sum_i \xi_i d\xi_i = \alpha_{\text{can}}$$

Exercise 3.60 Show that if Q is a compact hypersurface of contact type in (M, ω) it has a preferred positive side into which any transverse Liouville vector-field points. In particular, there is no orientation reversing map $\phi : Q \to Q$ which preserves the restriction $\omega|_Q$.

Solution 3.60 Consider two Liouville vector-fields X_0, X_1 inducing contact forms $\alpha_i = i_{X_i}\omega$. We noted in Exercise 3.59 that the isotopy $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ is an isotopy of contact forms with corresponding Liouville $X_t = (1-t)X_0 + tX_1$. However, suppose that there existed a point $p \in Q$ such that $(X_0)_p \in T_pM$ were on one-side of $T_pQ \subset T_pM$ and $(X_1)_p$. We can make this more formal by picking a 1-form $\beta \in T_p^*M$ with $\ker\beta = T_pQ$, and supposing that $\beta((X_0)_p) > 0$ and $\beta((X_1)_p) < 0$. Then there must exist a s such that $\beta(X_s) = 0$, i,e $(X_s)_p \in T_pQ$. But then $[i_{X_s}(d\omega)^n]|_Q$ cannot be a volume form on T_pQ. After all, if $X_t = 0$ then $[i_{X_s}(d\omega)^n]|_Q = 0$ and if $X_t \neq 0$ then $[i_{X_s}(d\omega)^n]|_Q$ is 0 on any basis $e_1, \ldots, e_{2n-1}$ of T_pQ with $e_1 = (X_t)_p$. However, $[i_{X_s}(d\omega)^n]|_Q = n(i_{X_s}\omega) \wedge d\omega^n)|_Q = n\alpha_t \wedge d\alpha_t$, which are all volume forms because α_t is a contact form. So X_0 and X_t must have X_0 and X_1 on the same side of the hyperplane distribution $TQ \subset TM$ everywhere.

Exercise 3.63 Show that, if ω is a symplectic form, then the only functions f such that $f\omega$ is symplectic are the constant functions.

Solution 3.63 This is technically false! In dimension 2, any two-form is closed so $f\omega$ is symplectic for every 2-form. Thus we may assume that $\dim M \geq 4$.

We see that $d(f\omega) = df \wedge \omega - f \wedge d\omega = df \wedge \omega$. Suppose $df \neq 0$ at some point. Then we can pick local coordinates $x_1 = f, x_2, \ldots, x_n, y_1, \ldots, y_n$ centered at p so that $df = dx_1$. Then we can use the symplectic

Graham-Schmidt process, starting with e_{x_1} as the first, unchanged basis vector, to find a standard basis at T_pM where $df = dx_1$. We then have in this basis:

$$df_p \wedge \omega_p = dx_1 \wedge (\sum_i dx_i \wedge dy_i) = \sum_{i \neq 1} dx_1 \wedge dx_i \wedge dy_i$$

The right-hand side is evidently non-vanishing (being a simple sum of basis elements of $\Lambda^3 M_p$), so $df \wedge \omega \neq 0$.

Exercise 3.64 (i) Let D_{n+1} denote the Siegel domain:

$$D_{n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C} | \text{Im} w > |z|^2\}$$

and consider the map:

$$f : \mathbb{C}^{n+1} - \mathbb{C}^n \times \{-1\} \to \mathbb{C}^{n+1} - \mathbb{C}^{\times}\{i\}$$

defined by:

$$f(z, w) = (\frac{z}{w+1}, -i\frac{w-1}{w+1})$$

Show that f maps the interior of the unit ball holomorphically onto D_{n+1}. (ii) It follows from (i) that the boundary Q of D_{n+1} has a canonical contact structure ξ defined as in Example 3.47. Namely, at each point $q \in Q$, the contact hyper-plane ξ_q is defined to be the complex part $T_qQ \cap J_0 T_qQ$ of the tangent space T_qQ. Prove this by direct calculation, and check that the contact structure so obtained is contactomorphic to the standard structure on $\mathbb{R}^{2n+1}$. (iii) Write down an explicit contactomorphism $S^{2n+1} - \{pt\} \to \mathbb{R}^{2n+1}$.

Solution 3.64 (i) The map is evidently holomorphic, being composed of rational functions in z and w. We see that:

$$|z|^2 + |w|^2 < 1 \iff \frac{|z|^2}{|1+w|^2} < \frac{1-|w|^2}{|1+w|^2}$$

and:

$$\text{Im}(-i\frac{w-1}{w+1}) = \text{Im}(\frac{-i(w-1)(\bar{w}+1)}{|1+w|^2}) = \frac{1-|w|^2}{|1+w|^2}$$

Thus $(z, w) \in B^{2n+2} \iff f(z, w) \in D_{n+1}$.

(ii) Let $B^{2n+2} \subset \mathbb{C}^{n+1}$ have coordinates (z_j, w) and D_{n+1} have coordinates (u_j, v) where $1 \leq j \leq n$. Let $z_j = x_j + ix_j$, $w = x_0 + iy_0$, $u_j = a_j + ib_j$ and $v = a_0 + ib_0$. The map $\psi = f^{-1}$ is given in these coordinates by:

$$\psi(u_j, v) = (\frac{2i}{v+i}u_j, \frac{-v+i}{v+i}) = (z_j, w)$$

To calculate the contact structure on ∂D_{n+1} induced by the standard structure ξ on S^{2n+1}, we will characterize ξ as the kernel $\ker\alpha$ of the standard contact structure and then compute the pullback $\psi^*\alpha$. Then the induced contact structure will be $\ker(\psi^*\alpha)$.With this goal in mind, first observe that the standard contact 1-form α on B^{2n+2} can be written as:

$$\alpha = \frac{1}{2}(\sum_j x_j dy_j - y_j dx_j) = \frac{1}{2}\text{Im}(\bar{w}dw + \sum_j \bar{z}_j dz_j) = \frac{1}{2}\text{Im}(\beta)$$

Since ψ is holomorphic, we can calculate the pullback of the 1-form β (which is a product and sum of anti-holomorphic functions and holomorphic 1-forms) via ψ and then take $\psi^*\alpha = \frac{1}{2}\mathrm{Im}(\psi^*\beta)$. Calculating using the expressions for z_j, w given above, we see that:

$$dz_j = \frac{2i}{v+i}du_j + \frac{-2iu_j}{(v+i)^2}dv \quad dw = \frac{-2i}{(v+i)^2}dv$$

Thus we have:

$$\begin{aligned}\psi^*\beta &= (\overline{\frac{-v+i}{v+i}})(\frac{-2i}{(v+i)^2})dv + \sum_j (\overline{\frac{2i}{v+i}u_j})(\frac{2i}{v+i}du_j + \frac{-2iu_j}{(v+i)^2}dv) \\ &= \frac{1}{|v+i|^2}(\frac{-4|u|^2 + 2i\bar{v} - 2}{v+i}dv + \sum_j 4\bar{u}_j du_j)\end{aligned}$$

Now if we restrict to ∂D_{n+1}, we see that $|u|^2 = \mathrm{Im}(v) = \frac{-i}{2}(v - \bar{v})$. Thus simplifying the above, we have:

$$\psi^*\beta|_{\partial D_{n+1}} = \frac{1}{|v+i|^2}(\frac{2iv - 2i\bar{v} + 2i\bar{v} - 2}{v+i}dv + \sum_j 4\bar{u}_j du_j) = \frac{1}{|v+i|^2}(2idv + 4\sum_j \bar{u}du_j)$$

Thus we see that:

$$\psi^*\alpha|_{\partial D_{n+1}} = \frac{1}{2}\mathrm{Im}(\psi^*\beta) = \frac{1}{|v+i|^2}(da_0 + 2\sum_j a_j db_j - b_j da_j) = \frac{1}{a_0^2 + (1 + \sum_j a_j^2 + b_j^2)^2}(da_0 + 2\sum_j a_j db_j - b_j da_j)$$

On the other hand, ∂D_{n+1} is characterized as the set of points (u_j, v) where:

$$b_0 = \mathrm{Im}(v) = |u|^2 = \sum_j a_j^2 + b_j^2$$

i.e the zero set of the function $g(u, v) = b_0 - \sum_j a_j^2 + b_j^2$. This means that the tangent space at a point is equal to the kernel of dg, i.e the kernel of:

$$dg = db_0 - 2\sum_j a_j da_j + b_j db_j$$

Thus we have the characterization $f_*\xi = \ker(dg) \cap \ker(\psi^*\alpha)$ of the pushforward $f_*\xi$ of the contact structure ξ on S^{2n+1} through f. On the other hand, the complex sub-bundle $E \subset T(\partial D_{n+1})$ can be characterized as $E = \ker(dg) \cap \ker(J^*dg)$. So we just need to show that $\ker(J^*dg) = \ker(\psi^*\alpha)$ to see that $E = f_*\xi$. But see that $J^*da_j = -db_j$ and $J^*db_j = da_j$. Thus:

$$J^*dg|_{\partial D_{n+1}} = -db_0 - 2\sum_j -a_j db_j + b_j da_j = -|v+i|^2 \psi^*\alpha|_{\partial D_{n+1}}$$

So the 1-forms J^*dg and $\psi^*\alpha$ differ by a non-zero scalar function (recall $\mathrm{Im}(v+i) = |u|^2 + 1$) so they have the same kernels. We show that $f_*\xi$ is contactomorphic to ξ_0 in the next exercise. Note that we can abstractly observe that it is contactomorphic by using the isotopy of contact forms $(1-t)f_*\alpha + t\alpha_1$ where:

$$\alpha_1 = da_0 + \frac{1}{2}\sum_j a_j db_j - b_j da_j$$

and using the fact (Example 3.43) that α_1 is contactomorphic to α_0, the standard contact form.

(iii) We can use $a_0, a_1, b_1, \ldots, a_n, b_n$ as the $\mathbb{R}^{2n+1}$ coordinatization of $\partial D_{n+1} \simeq \mathbb{R}^{2n+1}$. Thus it suffices to find a contactomorphism taking:

$$f_*\alpha = \frac{1}{a_0^2 + (1+\sum_j a_j^2 + b_j^2)^2}(da_0 + 2\sum_j a_j db_j - b_j da_j) = h(a,b)(da_0 + 2\sum_j a_j db_j - b_j da_j)$$

to the standard contact form α. We can use $\phi : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, $(a,b) \mapsto (z,x,y)$ given by $z = a_0 + 2\sum_j a_j b_j$, $x_j = 2a_j$ and $y_j = 2b_j$. Then we have:

$$\phi^*\alpha_0 = \phi^*(dz - \sum_j y_j dx_j) = (d(a_0 + \sum_j 2a_j b_j) - 4\sum_j b_j da_j = h^{-1} f_*\alpha$$

Exercise 4.3 Calculate the local coordinate representation of the almost complex structure in Example 4.2 on S^2 using stereographic projection.

Solution 4.3 This is a pretty long calculation actually. The stereographic projection map $\psi : S^2 \subset \mathbb{R}^3 \to \mathbb{R}^2$ and its inverse $\psi^{-1} : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3$ are given by:

$$\psi(u,v) = \frac{1}{u^2+v^2+4}\begin{bmatrix} 4u \\ 4v \\ 4-u^2-v^2 \end{bmatrix} \qquad \psi^{-1}(x,y,z) = \frac{2}{1+z}\begin{bmatrix} x \\ y \end{bmatrix}$$

The almost complex structure is given on $S^2 \subset \mathbb{R}^3$ as:

$$J_p = J_{x,y,z} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

which is just the operator $v \mapsto (x,y,z) \times v$. To find the corresponding almost complex structure in coordinates given by stereographic projection, we must calculate $\psi^* J = D\psi^{-1}_{\psi(p)} J_{\psi(p)} D\psi_p$. Calculating, we see that the Jacobians for $p = (u,v)$ are:

$$D\psi_p = \frac{4}{(u^2+v^2+4)^2}\begin{bmatrix} 4+v^2-u^2 & -2uv \\ -2uv & 4+u^2-v^2 \\ -4u & -4v \end{bmatrix}; \quad D\psi^{-1}_{\psi}(p) = \frac{u^2+v^2+4}{8}\begin{bmatrix} 2 & 0 & -u \\ 0 & 2 & -v \end{bmatrix};$$

Furthermore:

$$J_{\psi(p)} = \frac{1}{u^2+v^2+4}\begin{bmatrix} 0 & u^2+v^2-4 & v \\ 4-u^2-v^2 & 0 & -u \\ -v & u & 0 \end{bmatrix}$$

Thus:

$$\psi^* J = D\psi_{\psi(p)}^{-1} J_{\psi(p)} D\psi_p$$

$$= \frac{1}{2(u^2+v^2+4)^2}\begin{bmatrix} 2 & 0 & -u \\ 0 & 2 & -v \end{bmatrix}\begin{bmatrix} 0 & u^2+v^2-4 & v \\ 4-u^2-v^2 & 0 & -u \\ -v & u & 0 \end{bmatrix}\begin{bmatrix} 4+v^2-u^2 & -2uv \\ -2uv & 4+u^2-v^2 \\ -4u & -4v \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

So the pullback is actually just the standard structure on $\mathbb{C}$. Wow.

Exercise 4.5 Prove that the 2-form $\omega_x(u,v) = \langle x, u \times v\rangle$ is non-degenerate on the orthogonal compliment of $x \in \mathbb{R}^7$.

Solution 4.5 Let $x \in \mathbb{R}^7 - 0$. We can assume $x \neq 0$ because otherwise this is trivially false (the form is 0). Take $u \in x^\perp \subset \mathbb{R}^7$. Then consider $v = x \times u$. We see that $\langle x, v\rangle = \langle x, x \times u\rangle = \langle x \times x, u\rangle = 0$ so v is in $x^\perp$. Furthermore $x \times v = x \times (x \times u) = -u \neq 0$ so $v \neq 0$. Thus we have:

$$\omega_x(u,v) = \omega_x(u, x \times u) = \langle x, u \times (x \times u)\rangle = \langle x \times u, x \times u\rangle > 0$$

So there is a $v \in x^\perp$ for every $u \neq 0$ such that $\omega_x(u,v) \neq 0$.

Exercise 4.6 Let (M,J) be an almost complex manifold of dimension $4k$. Find an identity connecting its top Pontriagin class with the Chern class c_{2k} of the complex bundle (TM,J). Deduce that none of the spheres S^{4k} admits an almost complex structure. Obtain a similar result for spheres S^{4k+2} for $k \geq 2$ using Bott's integrality theorem which asserts that for any complex vector bundle E over S^{2n}, the class $c_n(E)/(n-1)! \in H^{2n}(S^{2n})$ is integral (see for instance Husemoller [136, Chapter 18.9.8]).

Solution 4.6 Let (E,J) be a vector bundle with complex structure J. Recall that the Pontriagin classes are defined as $p_k(E) = c_{2k}(E \otimes \mathbb{C})$. We will show that $E \otimes \mathbb{C} \simeq E \oplus \bar{E}$. Thus we will have:

$$p_k(E) = (-1)^k c_{2k}(E \oplus \bar{E}) = (-1)^k \sum_{i=0}^{2k} c_i(E)c_{2k-i}(\bar{E}) = (-1)^k \sum_{i=0}^{2k} (-1)^i c_i(E) c_{2k-i}(E)$$

To see that $E \otimes \mathbb{C} \simeq E \oplus \bar{E}$, consider the map $\psi : E \otimes \bar{E} \to E \otimes \mathbb{C}$ given by $u \oplus v \mapsto u - iJu + v + iJv = \psi(u \oplus v)$. Observe that:

$$\psi(Ju \oplus 0) = Ju - iJ^2u = iu + Ju = i(u - iJu) = i\psi(Ju \oplus 0)$$

$$\psi(0 \oplus -Jv) = -Jv + iJ(-Jv) = -Jv + iv = i(v + iJv) = i\psi(0 \oplus v)$$

Thus this map is complex-linear on both factors. Since it is a bundle isomorphism (we can always pick u and v so that $u+v=x$ and $J(u-v)=y$ for any x and y so that $\psi(u\oplus v)=x+iy$) preserving the complex structure, this proves $E\otimes\mathbb{C}\simeq E\oplus\bar{E}$ as complex vector bundles.

A particular case here is the spheres S^{4n}. In this case the lower Pontriagin classes and Chern classes necessarily vanish because $H^i(S^{4n})=\mathbb{Z}$ if $i=4n,0$ and 0 otherwise. Thus we would have $p_n(TS^{4n})=2(-1)^n c_{2n}(TS^{4n})$ if S^{4n} admitted an almost complex structure. However, it is well-known that the Pontrjagin numbers $\langle\cup_{j=1}^{l}p_{i_j}|[M]\rangle=0$ for all i_j with $\sum_j i_j=n$ (see for instance Milnor Stasheff Lemma 17.3) for a manifold $M=\partial N$ for some compact manifold N with boundary. In particular, $\langle p_n(S^{4n})|[S^{4n}]\rangle=0$. But we also have $c_{2n}(E)=e(E)$ for any vector-bundle of rank $2n$, and $\langle e(S^{4n}),[S^{4n}]\rangle=\chi(S^{4n})=2$. So the formula that we derived cannot hold. It follows that a complex structure cannot exist.

Bott's result tells us that $c_n(E)=e(E)$ has $\frac{e(E)}{(n-1)!}$ is integral. But we see that $\frac{1}{(n-1)!}\langle e(E),[S^{2n}]\rangle=\frac{\chi(S^{2n})}{(n-1)!}=\frac{2}{(n-1)!}$. If $n>3$, this is not an integer, so the cohomology class $\frac{e(E)}{(n-1)!}$ is not integral.

Exercise 4.9 Let (ω,J,g) be a compatible triple and assume that ω is closed. Prove that:

$$(\nabla_{Jv}J)v=(\nabla_v J)Jv$$

Find an example where ω is not closed, and this equation is violated.

Solution 4.9 Using the third formula in Lemma 4.8 with $d\omega=0$, $u=u,v=v$ and $w=Jv$, we see that:

$$0=\langle(\nabla_u J)v,Jv\rangle+\langle(\nabla_v J)Jv,u\rangle+\langle(\nabla_{Jv}J)u,v\rangle=\langle(\nabla_u J)v,Jv\rangle+\langle u,(\nabla_v J)Jv-(\nabla_{Jv}J)v\rangle$$

Here we use the fact that $(\nabla_{Jv}J)$ is anti-self-adjoint (the second formula in Lemma 4.8). Thus we just need to show that $\langle(\nabla_u J)v,Jv\rangle=0$. But given any point p we can pick a vector-field $\tilde{v}$ with $\tilde{v}(p)=v(p)$ and $\nabla v=0$. Then at p we have:

$$2\langle(\nabla_u J)v,Jv\rangle=2\langle(\nabla_u J)\tilde{v},J\tilde{v}\rangle=\nabla_u\langle J\tilde{v},J\tilde{v}\rangle=\nabla_u\langle\tilde{v},\tilde{v}\rangle=2\langle\nabla_u\tilde{v},\tilde{v}\rangle=0$$

To find an example of a compatible triple that doesn't satisfy this, consider the following. Given the standard triple (ω_0,J_0,g_0), we can conformally rescale g_0 and ω_0 to get a new compatible triple $(e^f\omega_0,J_0,e^f g_0)$ on $\mathbb{R}^{2n}$. In coordinates, where g_0 is given by δ_{ij}, the new Christoffel symbols are:

$$\Gamma^i_{jk}=\frac{1}{2}(\delta^i_j\partial_k f+\delta^i_k\partial_j f-\delta_{jk}\partial^i f)$$

Observe that since, in standard $\mathbb{R}^{2n}$ coordinates $\partial_j J_0=0$, this implies that $\nabla_i J=\nabla_i J_0=[\Gamma_i,J_0]$ where by Γ_i we denote the matrix $(\Gamma^j_{ik}$ for fixed i. In particular, consider $\mathbb{R}^4$ with coordinates x_1,y_1,x_2,y_2 and the standard triple with respect to these coordinates and e_1,f_1,e_2,f_2 the corresponding basis elements in $T\mathbb{R}^4$. We will use the indices $1,2,3,4$ to denote derivatives in the respective e_1,f_1,e_2,f_2 directions. Then the formula that we want to violate can be written as:

$$[\Gamma_1,J_0]Je_1\neq[\Gamma_2,J_0]e_1$$

if we pick $v = e_1$.

Now we take a specific example. Let $f(x, y) = x_2$. Then using the formula for the Christoffel symbols given above, we have:

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Gamma_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\Gamma_1, J_0] = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad [\Gamma_2, J_0] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$[\Gamma_1, J_0]J_0 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad [\Gamma_1, J_0]J_0 e_1 = -[\Gamma_1, J_0]e_1$$

Thus, $(e^{x_2}\omega_0, J_0, e^{x_2}g_0)$ is a counter-example.

Exercise 4.10 A sub-manifold $L \subset M$ is called totally real if it is of half the dimension of M and $T_qL \cap J_qT_qL = \{0\}$ for all $q \in L$. (i) Let (ω, J, g) be a compatible triple. Show that any Lagrangian submanifold L is totally real, but not conversely. In fact, L is Lagrangian if and only if JTL is the g-orthogonal complement of TL. (ii) Prove that if L is a totally real sub-manifold of (M, J) then there exists a Riemannian metric g on M such that g is compatible with J, JTL is the orthogonal complement of TL and L is totally geodesic. (iii) Show that if L is a Lagrangian submanifold of (M, ω) then there is an ω-compatible J such that L is totally geodesic with respect to the corresponding metric g_J.

Solution 4.10 (i) Suppose that L were Langrangian and $T_qL \cap J_qT_qL \neq \{0\}$. Then we can pick non-zero $v \in T_qL \cap J_qT_qL$. $v = Jw$ for some $w \in L$, so $Jv = -w \in L$ and $Jv \neq 0$. However, $\omega(v, w) = \omega(v, Jv) > 0$, contradicting the fact that TL is Lagrangian. So L is totally real.

Counter-examples to the converse can be found in the linear theory: for instance, take (ω_0, J_0, g_0) to be the standard compatible triple on $\mathbb{R}^4$ with coordinates x_1, x_2, y_1, y_2 and corresponding tangent basis e_1, e_2, f_1, f_2. Then take $R = \text{span}(e_1, f_1 + f_2)$. Evidently this is not a Lagrangian subspace, since $\omega(e_1, f_1 + f_2) = 1$. However, $JR = \text{span}(f_1, -e_1 + -e_2)$. Since $R \oplus JR = \mathbb{R}^4$ and $\dim(R) = 2$, $JR \cap R = \{0\}$ by dimension counting. So $R \subset \mathbb{R}^4$ is an example of a totally real submanifold that is not Lagrangian.

(ii) Consider an almost complex manifold (M, J) and a totally real submanifold $L \subset M$. We begin by choosing a tubular neighborhood $N \simeq N' \subset \nu L$ of L and a projection $\pi : N \to L$ such that $\ker(d\pi)_p = JT_pL$ for each $p \in L$. We can do this as so. Let h be any metric on L. Then we may consider the metric $h \oplus J^*h$ on $TM|_L = TL \oplus JTL$. The metric will be largely fiducial, so we won't give it a better name. We can then take a covering of N by trivializations $D^n \times U_\alpha$ with $U_\alpha \subset L$ and extend the metric from $TM|_L$ to $TM|_N$ by doing so trivially on $D^n \times L$ (in coordinates) and then using a partition of unity over the U_α to add the metrics together. We can then choose an isomorphism $N \simeq N' \subset \nu L$ where M is the normal bundle with respect to our figucial metric. The projection π is then given by the pullback through $N \simeq N'$ of the standard projection $\nu L \to L$. Note that, by our construction of this metric, $TL^\perp = JTL$, so the kernel of the projection is JT_pL at any point $p \in L$, as desired.

Now we want to construct an even better metric using this projection operator. To do that, we sort of repeat the construction of $h \oplus J^*h$, but this time on all of N. Let h again be any metric on L and define $g = \pi^*h \oplus J^*\pi^*h$ on $TM|_N$. This is a metric since, by shrinking N, we can assure that $\pi^*TL \subset TM$ is a totally real sub-bundle[6] which implies that g is a well-defined metric. This metric is J invariant since $J^*g = J^*(\pi^*h \oplus J^*\pi^*h) = (-1)^*\pi^*h \oplus J^*\pi^*h = \pi^*h \oplus J^*\pi^*h$ (note that the summands switch places because J interchanges TL and JTL). Since $J^2 = -1$, we have $g(v, Jw) = g(Jv, J^2w) = -g(w, Jv)$. Thus this metric induced an almost symplectic form $\omega = g(\cdot, J\cdot)$ on $TM|_N$. We can extend this metric to the rest of M by choosing a smaller tubular neighborhood O of L, picking an arbitrary compatible metric g' on $M - O$, and then using partitions of unity α and β supported on $M - O$ and N respectively to extend g by g' to all of M. We will also refer to this extended metric as g.

We have constructed g so that g is compatible with J and so that $TL^\perp = JTL$. It remains to show that L is totally geodesic. To see this, we pass to the normal coordinates on $N \simeq N' \subset \nu L$. Take these normal coordinates to be $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Here, since we defined $g = \pi^*h \oplus J^*\pi^*$, it can be written as a block matrix:

$$g(x,y) = \begin{bmatrix} h(x) & |y|f(x,y) \\ |y|f(x,y)^T & J^T(x,y)h(x)J(x,y) \end{bmatrix}$$

This implies 3 important facts. First, $g(x,0)$ is a block matrix (i.e, along L the metric is a block matrix). Second, $\partial_{y_k} g_{x_i,x_j} = \partial_{y_k} h_{x_i,x_j} = 0$ (the subscripts on the metric denote that entry in the matrix, not derivatives). Third, $\partial_{x_k} g_{x_i,y_j} = |y|\partial_{x_k} f(x,y) = 0$ when $|y| = 0$ (i.e, along L these derivatives are 0). These 3 facts together with the formula for the Christoffel symbol $\Gamma^{y_k}_{x_ix_j}$ implies that $\Gamma^{y_k}_{x_ix_j} = 0$ for all i, jk. In particular, for any curve $\gamma \in L$ we have $\dot\gamma \in TL$ and $\ddot\gamma \in TL$ (in coordinates at least). Furthermore for any point $p = \gamma(s)$ we have $[\nabla\dot\gamma(s)]_{y_k} = [\ddot\gamma(s)]_{y_k} + \sum_{i,j} \Gamma^{y_k}_{x_ix_j}(\gamma(s))[\dot\gamma(s)]_{x_j}[\dot\gamma(s)]_{x_k}] = 0$. In other words, the covariant derivative $\nabla\dot\gamma$ is in TL for every curve $\gamma : I \to L$. Thus L is totally geodesic with respect to g.

(iii) It suffices to prove that we can pick a compatible J for some tubular neighborhood N of L with the desired properties. This is equivalent to picking a compatible metric g' on N with the desired properties. Then we can extend the metric to a global metric g on M using a partition of M into $M - O$ and N with $O \subset N$, and some partition of unity over N and $M - O$ (just as above). Afterwards, we can recover a global J agreeing with J on O by using the inverse to the map described in Proposition 2.50(i).

By the Lagrangian neighborhood theorem, we can further reduce to the case of a tubular neighborhood N of the zero section $L \subset T^*L$.

In this setting, we consider the symplectic bundle $(T(T * L), \omega)$ which is the tangent bundle of the cotangent bundle equipped with the canonical symplectic form. Note that there is a map of symplectic bundles $\pi : T(T^*L) \to T(T^*L)|_L$ which covers the projection $\pi : T^*L \to L$ and is an isomorphism on the fibers. This is literally the map $(x, y, u, v) \mapsto (x, 0, u, v) \mapsto (x, u, v)$. This restricts to a map of symplectic

[6] The totally real condition can be formulated as a determinant condition on a basis for $e_i \in \pi^*TL$ and its corresponding basis $Je_i \in JTL$, in particular as $\det([e_1, \ldots, e_n, Je_1, \ldots, Je_n]) \neq 0$, and thus is easily seen to be an open condition by picking a local trivialization.

bundles $\pi : T(T^*L)|_N \to T(T^*L)|_L$ which covers the projection $\pi : N \to L$.

Now we may pick any compatible almost complex structure J_L on the bundle $T(T^*L)|_L$ and pull it back through the bundle map π to get a compatible structure $J = \pi^* J_L$ on (TN, ω). The pullback is compatible because π is symplectic. Furthermore, the resulting metric g_J has a block matrix decomposition similar to the one described in (ii). In fact we have an even better decomposition: if we denote the restricted symplectic form on $T(T^*L)|_L$ by ω_L and the metric $\omega_L(\cdot, J_L\cdot)$ by g_L, then $g_J = \pi^* g_L$, so in the normal coordinates g has no dependence on the y variables whatsoever. In particular, in coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ we have:

$$g_J(x,y) = [\pi^* g_L](x,y) = \begin{bmatrix} h_L(\pi(x,y)) & 0 \\ 0 & h_{JL}(\pi(x,y)) \end{bmatrix} = \begin{bmatrix} h_L(x) & 0 \\ 0 & h_{JL}(x) \end{bmatrix}$$

In particular, the same arguments as above show that $\Gamma^{y_k}_{x_i,x_j} = 0$ in this metric and these coordinates, so an identical argument to the one above shows once again that L is totally geodesic.

Exercise 4.13 Check that the type $(1,0)$ vector fields on (M, J) are precisely those of the form $(1-iJ)X$ where X is a real vector field on M. Deduce that in the integrable case they have the form $\sum_j a^j \frac{\partial}{\partial z_j}$, where the a^j are complex-valued functions on M.

Solution 4.13 In coordinates, we can write any vector-field V as $V = \sum_j a_j \frac{\partial}{\partial x_j} + b_j \frac{\partial}{\partial y_j}$. Then if V is type $(1,0)$ we have:

$$\sum_j -b_j \frac{\partial}{\partial x_j} + a_j \frac{\partial}{\partial y_j} = JV = iV = \sum_j ia_j \frac{\partial}{\partial x_j} + ib_j \frac{\partial}{\partial y_j}$$

So $a_j = ib_j$. thus we have:

$$V = \sum_j a_j \frac{\partial}{\partial x_j} - ia_j \frac{\partial}{\partial y_j} = \sum_j a_j \frac{\partial}{\partial x_j} - ia_j J \frac{\partial}{\partial x_j} = \sum_j (2a_j) \frac{\partial}{\partial z_j}$$

If $a_j = b_j + ic_j$ then:

$$V = \sum_j a_j \frac{\partial}{\partial x_j} - ia_j \frac{\partial}{\partial y_j} = \sum_j b_j \frac{\partial}{\partial x_j} - ic_j J \frac{\partial}{\partial y_j} - ib_j J \frac{\partial}{\partial x_j} + c_j \frac{\partial}{\partial y_j} = (1 - iJ) \sum_j b_j \frac{\partial}{\partial x_j} + c_j \frac{\partial}{\partial y_j}$$

Exercise 4.17 Given $\tau \in \mathbb{H}$ denote by $j_\tau \in \mathbb{R}^{2\times 2}$ the complex structure associated to τ as above and define the map $\Psi_\tau : \mathbb{R}^2 \to \mathbb{C}$ by $\Psi_\tau(x,y) = x + \tau y$. Prove that:

$$\Psi_\tau \circ j_\tau = i \circ \Psi_\tau$$

or, in other words, $\Psi_\tau^* i = j_\tau$. Prove that every linear isomorphism $\Psi : \mathbb{R}^2 \to \mathbb{C}$ factors uniquely as $\Psi = \lambda \Psi_\tau$, where $\lambda \in \mathbb{C}^*$ and $\tau \in \mathbb{H}$. Deduce that the space $\mathbb{H} \simeq \mathcal{J}^+(\mathbb{R}^2)$ is diffeomorphic to the homogeneous space $\mathrm{GL}^+(2,\mathbb{R})/\mathbb{C}^* = \mathrm{SL}(2,\mathbb{R})/S^1$.

Solution 4.17 For $p = (x, y)$, we calculate that:

$$i \circ \Psi_\tau(p) = i(x + \frac{F+i}{E}y) = -\frac{1}{E}y + i(x + \frac{F}{E}y) = -Fx + \frac{F^2 - GE}{E}y + Fx + i(x + \frac{F}{E}y)$$

$$= (-Fx - Gy) + \frac{F+i}{E})(Ex + Fy) = [\Psi_\tau \circ j](p)$$

Now let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be any linear map. Then if we define $j = \Psi^{-1} \circ i \circ \Psi$ we certainly have $i \circ \Psi = \Psi \circ j$. Furthermore, $j = \Psi^{-1} \circ i \circ \Psi = \Phi^{-1} \circ i \circ \Phi$ where $\Phi v = \frac{1}{\det(\Psi)}\Psi v$, i.e j is conjugate to i via an element of $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$. Thus j remains compatible with ω_0 (since $j = \Phi^* i$ and $\omega_0 = \Phi^* \omega_0$) and is equal to j_τ for some τ. Thus $\Psi : \mathbb{R}^2 \to \mathbb{C}$ is an intertwining operator for j_τ and i for some τ.

Now consider $\Psi \circ \Psi_\tau^{-1} : \mathbb{C} \to \mathbb{C}$. We see that $\Psi \circ \Psi_\tau^{-1} \circ i = \Psi \circ j_\tau \circ \Psi_\tau = i \circ \Psi \circ \Psi_\tau^{-1}$. So $\Psi \circ \Psi_\tau^{-1} = \lambda \in \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$. Thus $\Psi = \lambda \Psi_\tau$. To see uniqueness, suppose that $\kappa \Psi_\sigma = \lambda \Psi_\tau$. Then for all $p = (x, y)$ we have:

$$\frac{\kappa}{\lambda}(x + \sigma y) = (x + \tau y)$$

In particular, setting $x = 1, y = 0$ we have $\kappa = \lambda$. Then setting $x = 0, y = 1$ we have $\sigma = \tau$.

Exercise 4.18 Two Riemannian metrics g_1 and g_2 on M are called conformally equivalent if there exists a function $\lambda : M \to \mathbb{R}$ such that $g_2 = \lambda g_1$. A diffeomorphism $f : (M_1, g_1) \to (M_2, g_2)$ of Riemannian manifolds is called conformal if $f^* g_2$ is conformally equivalent to g_1. This means that f preserves angles and orientation. A metric g is called compatible with an almost complex structure if $g(Jv, Jw) = g(v, w)$. In the case of $\dim M = 2$ prove that any two metrics g_1 and g_2 which are compatible with J are conformally equivalent.

Let (Σ_1, j_1) and (Σ_2, j_2) be 2-dimensional complex manifolds with compatible Riemannian metrics g_1 and g_2 respectively. Prove that a diffeomorphism $\phi : \Sigma_1 \to \Sigma_2$ is holomorphic if and only if it conformal.

Solution 4.18 First let g_1, g_2 be compatible with j on the Riemann surface (Σ, j). Let $p \in \Sigma$ and $v \neq 0 \in T_p\Sigma$ be arbitrary. Observe that $Jv \neq 0$, Jv is independent from v (since J has no real eigenvalues), $g_i(v, Jv) = g_i(Jv, J^2 v) = g_i(Jv, -v) = -g(v, Jv)$ (thus $g(v, Jv) = 0$) and $g(Jv, Jv) = g(v, v)$. Thus $w = av + bJv$ for any $w \in T_p\Sigma$. Now let $\lambda(p) = \frac{g_2(v,v)}{g_1(v,v)}$. Then observe that:

$$g_2(w, w) = a^2 g_2(v, v) + b^2 g_2(Jv, Jv) = (a^2 + b^2) g_2(Jv, Jv) = (a^2 + b^2)\lambda(p) g_1(v, v) = \lambda(p) g_1(w, w)$$

Note that we only have to check $g_2(v, v) = \lambda g_1(v, v)$. Then we get pairings $g_i(v, w)$ by $g_i(v, w) = \frac{1}{2}(g(v + w, v + w) - g(v, v) - g(w, w))$. We get to the last step by reversing the calculations for the first few steps with g_1 instead of g_2. Define $\lambda : M \to \mathbb{R}$. Note that by this proof, it does not matter which $v \in T_pM - 0$ we pick to define λ, any v will yield the same answer. Furthermore, we can pick a smooth section $v : U \to T\Sigma|_U$ in a neighborhood of p to define λ at each point near p to see that λ is in fact smooth. So $g_2 = \lambda g_1$ and the two metrics are conformal.

Now consider a map $\phi : (\Sigma_1, j_1, g_1) \to (\Sigma_2, j_2, g_2)$ as described above. First assume ϕ is holomorphic.

Then $\phi^* g_2$ is compatible with j_1 since:

$$\phi^* g_2(j_1 \cdot, j_1 \cdot) = g_2(d\phi j_1 \cdot, d\phi j_1 \cdot) = g_2(j_2 d\phi \cdot, j_2 d\phi \cdot) = g_2(d\phi \cdot, d\phi \cdot) = \phi^* g_2(\cdot, \cdot)$$

Thus the above theorem implies that g_1 and $\phi^* g_2$ are conformal. Conversely, if $\phi^* g_2 = \lambda g_1$ then for any non-zero $v \in T_p \Sigma_1$ we have:

$$g_2(d\phi v, j_2 d\phi v) = 0 = g_1(v, j_1 v) = \lambda(p) g_2(d\phi v, d\phi j_1 v)$$

$$g_2(j_2 d\phi v, j_2 d\phi v) = g_2(d\phi v, d\phi v) = \lambda(p)^{-1} g_1(v, v) = \lambda(p)^{-1} g_1(j_1 v, j_1 v) = g_2(d\phi j_1 v, d\phi j_1 v)$$

These two calculations show that for every $v \in T_p \Sigma_1$, $j_2 d\phi v = \pm d\phi j_1 v$. Since $d\phi$ is linear, this implies that $j_2 \circ d\phi = \pm d\phi \circ j_1$, either holomorphic or anti-holomorphic. But $g_1(\cdot, -j_1 \cdot) = -\omega_1$, so the orientation induced by $\phi^* j_2 = -j_1$ (which is represented by the non-vanishing 2-form $-\omega_1$, or indeed the 2-form $g(\cdot, -j_1)$ for any compatible metric g) is the opposite orientation to the orientation of Σ_1, represented by ω_1. Thus in order for ϕ to be orientation preserving, it must be holomorphic.

Exercise 4.20 Express the chain rule in terms of the operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$. Prove that $\phi : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if $\bar{\partial}\phi = 0$. Prove that if $\phi : \mathbb{C}^n \to \mathbb{C}$ is holomorphic then:

$$\phi^* \partial \omega = \partial \phi^* \omega, \phi^* \bar{\partial} \omega = \bar{\partial} \phi^* \omega$$

for every complex-valued differential form ω on $\mathbb{C}^n$.

Solution 4.20 The usual chain rule says that if $\phi : \mathbb{R}^{2l} \to \mathbb{R}^{2m}$ and $\psi : \mathbb{R}^{2m} \to \mathbb{R}^{2n}$ are smooth functions, then $d(\psi \circ \phi)_p = d\psi_{\psi(p)} \circ d\phi_p$. Here $d\phi_p : T_p \mathbb{R}^{2l} \to T_{\phi(p)} \mathbb{R}^{2m}$ is the map of tangent spaces given in coordinates x_k on $\mathbb{R}^{2l}$ and y_j on $\mathbb{R}^{2m}$ by:

$$d\phi_p = \sum_{j,k} \frac{\partial \phi_j}{\partial x_k}|_p (\frac{\partial}{\partial y_j} \otimes dx_k)$$

and similarly for $\psi, \psi \circ \phi$. In coordinates, this can be written (now with z_j as real coordinates on $\mathbb{R}^{2n}$) as:

$$\sum_{j,k} \frac{\partial (\psi \circ \phi)_j}{\partial x_k}|_p (\frac{\partial}{\partial z_j} \otimes dx_k) = d(\psi \circ \phi)_p = \sum_{j,k,p,q} (\frac{\partial \psi_j}{\partial y_p}|_{\phi(p)} \frac{\partial \phi_q}{\partial x_k}|_p)(\frac{\partial}{\partial z_j} \otimes dy_p) \circ (\frac{\partial}{\partial y_q} \otimes dx_k)$$

$$= \sum_{j,k,a} (\frac{\partial \psi_j}{\partial y_a}|_{\phi(p)} \frac{\partial \phi_a}{\partial x_k}|_p)(\frac{\partial}{\partial z_j} \otimes dx_k)$$

Or more simply:

$$\frac{\partial (\psi \circ \phi)_j}{\partial x_k}|_p = \sum_a \frac{\partial \psi_j}{\partial y_a}|_{\phi(p)} \frac{\partial \phi_a}{\partial x_k}|_p$$

Now define $du_j = dx_j + i dx_{j+l}$ and $du_{\bar{j}} = dx_j - i dx_{j+l}$. Dually, define $\frac{\partial}{\partial u_j} = \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial x_{j+n}}$ and $\frac{\partial}{\partial u_{\bar{j}}} = \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial x_{j+n}}$. Also define $\Phi_j = \phi_j + i\phi_j, \Phi_{\bar{j}} = \phi_j - i\phi_j$. Finally, impose similar identities for for y_j, z_j with complex variables v_j, w_j and define Ψ similarly with respect to ψ. Then by substituting the definitions above into the simple version of the chain rule identity and simplifying, we find that we may write:

$$\frac{\partial(\Psi \circ \Phi)_j}{\partial u_k}|_p = \sum_a \frac{\partial \Psi_j}{\partial v_a}|_{\phi(p)} \frac{\partial \Phi_a}{\partial u_k}|_p + \sum_{\bar{a}} \frac{\partial \Psi_j}{\partial v_{\bar{a}}}|_{\phi(p)} \frac{\partial \Phi_{\bar{a}}}{\partial u_k}|_p$$

The analogous identities hold for the pairs $\bar{j}, k$; $k, \bar{j}$; and $\bar{k}, \bar{j}$ substituted for j, k. This is the version of the chain rule for holomorphic and anti-holomorphic partial derivatives.

Now we prove that $\phi : \mathbb{C}^n \to \mathbb{C}$ is holomorphic if and only if $\bar{\partial}\phi = 0$. We take as the definition of holomorphic that ϕ satisfies the Cauchy-Riemann equations in each pair of variables x_i, y_i. We see that $\bar{\partial}\phi = \sum_j \frac{\partial \phi}{\partial \bar{z}_j} d\bar{z}_j$. The elements $d\bar{z}_j$ are independent covectors in the cotangent space at a point, so $\bar{\partial}\phi = 0$ if and only if $\frac{\partial \phi}{\partial \bar{z}_j} = 0$ for each j. But if we write $\phi = a + ib$ for real functions b, we see that this says:

$$\frac{\partial \phi}{\partial \bar{z}_j} = (\frac{\partial a}{\partial x_j} - \frac{\partial b}{\partial y_j}) + i(\frac{\partial a}{\partial y_j} + \frac{\partial b}{\partial x_j})$$

The above vanishes if and only if the real and imaginary part vanish, i.e if and only if $\frac{\partial a}{\partial y_j} + \frac{\partial b}{\partial x_j} = \frac{\partial a}{\partial x_j} - \frac{\partial b}{\partial y_j} = 0$. But this is precisely the Cauchy-Riemann equations in x_i, y_i for a and b.

Now we prove that $\phi^*\bar{\partial}\omega = \bar{\partial}\phi^*\omega$ if ϕ is holomorphic. It will follows that $\phi^*\partial\omega = \partial\phi^*\omega$ since we will then have:

$$\phi^*\partial\omega + \phi^*\bar{\partial}\omega = \phi^* d\omega = d\phi^*\omega = \partial\phi^*\omega + \bar{\partial}\phi^*\omega \implies \phi^*\partial\omega = \partial\phi^*\omega$$

First observe that, like the usual exterior derivative, we have $\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{\deg(\alpha)}\alpha \wedge \bar{\partial}\beta$. Indeed, this is typically how we define the extension of ∂ and $\bar{\partial}$ to the higher k-forms. Thus we need only prove this result for 1-forms. Then we can proceed as so. Suppose that we have proven the result for $j < k$ forms. Then we have, for any k-form ω, an expression $\sum_j \alpha_j \wedge \beta_j$ for α_j 1-forms and β_j $k-1$-forms.

$$\phi^*\partial\omega = \sum_j \phi^*\partial(\alpha_j \wedge \beta_j) = \sum_j \phi^*(\partial\alpha_j \wedge \beta_j + (-1)^{k-1}\alpha_j \wedge \partial\beta_j)$$

$$= \sum_j [\phi^*\partial\alpha_j] \wedge [\phi^*\beta_j] + (-1)^{k-1}[\phi^*\alpha_j] \wedge [\phi^*\partial\beta_j]) = \sum_j [\partial\phi^*\alpha_j] \wedge [\phi^*\beta_j] + (-1)^{k-1}[\phi^*\alpha_j] \wedge [\partial\phi^*\beta_j])$$

$$= \sum_j \partial([\phi^*\alpha_j] \wedge [\phi^*\beta_j]) = \sum_j \partial\phi^*(\alpha_j \wedge \beta_j) = \partial\phi^*\omega$$

Now suppose $\alpha = \sum_j \alpha_j dz_j + \alpha_{\bar{j}} dz_{\bar{j}}$ is a 1-form. Then:

$$\bar{\partial}\phi^*\alpha = \sum_{j,a,k} \frac{\partial}{\partial z_{\bar{j}}}(\alpha_a \circ \phi \frac{\partial \phi_a}{\partial z_k} dz_{\bar{j}} \wedge dz_k + \alpha_{\bar{a}} \circ \phi \frac{\partial \phi_{\bar{a}}}{\partial z_{\bar{k}}} dz_{\bar{j}} \wedge dz_{\bar{k}})$$

$$= \sum_{j,a,b,k} [\frac{\partial \alpha_a}{\partial z_{\bar{b}}} \circ \phi] \frac{\partial \phi_{\bar{b}}}{\partial z_{\bar{j}}} \frac{\partial \phi_a}{\partial z_k} dz_{\bar{j}} \wedge dz_k + [\frac{\partial \alpha_{\bar{a}}}{\partial z_{\bar{b}}} \circ \phi] \frac{\partial \phi_{\bar{b}}}{\partial z_{\bar{j}}} \frac{\partial \phi_{\bar{a}}}{\partial z_{\bar{k}}} dz_{\bar{j}} \wedge dz_{\bar{k}} + \alpha_{\bar{a}} \circ \phi \frac{\partial^2 \phi_{\bar{a}}}{\partial z_{\bar{j}} \partial_{\bar{k}}} dz_{\bar{j}} \wedge dz_{\bar{k}}$$

$$= \phi^*(\sum_{a,b} \frac{\partial \alpha_a}{\partial z_{\bar{b}}} dz_a \wedge dz_{\bar{b}} + \frac{\partial \alpha_{\bar{a}}}{\partial z_{\bar{b}}} dz_{\bar{a}} \wedge dz_{\bar{b}}) = \phi^*\bar{\partial}\alpha$$

Exercise 4.22 Let $f(z)$ be a real-valued function on $\mathbb{C}^n$. Find conditions under which the 2-form $\frac{1}{2}i\partial\bar{\partial}f$ is nondegenerate and compatible with J_0. Deduce that the above form τ_0 is non-degenerate and compatible with J_0.

Solution 4.22 Via the natural identification of anti-symmetric 2-tensors and 2-forms, we have:

$$\frac{i}{2}dz_i \wedge dz_{\bar{j}} = \frac{i}{2}(dz_i \otimes dz_{\bar{j}} - dz_{\bar{j}} \otimes dz_i)$$

When we compose the latter with J we get:

$$\frac{i}{2}[dz_i \wedge dz_{\bar{j}}](\cdot, J\cdot) = \frac{i}{2}(dz_i \otimes (dz_{\bar{j}} \circ J) - dz_{\bar{j}} \otimes (dz_i \circ J)) = \frac{i}{2}(-idz_i \otimes dz_{\bar{j}} - idz_{\bar{j}} \otimes dz_i) = \frac{1}{2}(dz_i \otimes dz_{\bar{j}} + dz_{\bar{j}} \otimes dz_i)$$

Thus the symmetric tensor given by $[\frac{i}{2}\partial\bar{\partial}f](\cdot, J\cdot)$ is given by:

$$\sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_{\bar{j}}} \frac{1}{2}(dz_i \otimes dz_{\bar{j}} + dz_{\bar{j}} \otimes dz_i) = \sum_{i,j} \frac{\partial^2 f}{\partial z_i \partial z_{\bar{j}}} dz_i \otimes dz_{\bar{j}}$$

This is evidently a Hermitian bilinear form, which is a symmetric form on the underlying real space. It is positive definite if and only if $\frac{\partial^2 f}{\partial z_i \partial z_{\bar{j}}}$ is positive definite Hermitian. Thus this is the condition for $\omega = \frac{i}{2}\partial\bar{\partial}f$ being compatible with J_0.

To see how this applied to τ_0, observe that in the chart U_0 where $z_0 \neq 0$ we have $\tau_0 = \frac{i}{2}\partial\bar{\partial}f_0$ where:

$$f_j(z) = \log(1 + \sum_{\nu=1}^{n} w_\nu \bar{w}_\nu)$$

We can compute that:

$$\frac{i}{2}\partial\bar{\partial}f_0 = \frac{\partial f_0}{\partial w_l \partial \bar{w}_k} dw_l \wedge d\bar{w}_k = (\frac{(1+|w|^2)\delta_{lk} - \bar{w}_l w_k}{(1+|w|^2)^2}) dw_l \wedge d\bar{w}_k$$

Then observe that if $u = (u_k)$ is a unit norm complex vector then:

$$(1+|w|^2)\frac{i}{2}\partial\bar{\partial}f_0(u, Ju) = |u|^2 - \frac{1}{1+|w|^2}\sum_{l,k} \bar{u}_k \bar{w}_l w_k u_l \geq |u|^2 - \frac{1}{1+|w|^2}|\sum_{l,k} \bar{u}_k \bar{w}_l w_k u_l|$$

$$\geq |u|^2 - \frac{1}{1+|w|^2}\sqrt{\sum_{k,l}|w_l|^2|w_k|^2}\sqrt{\sum_{k,l}|u_l|^2|u_k|^2} = |u|^2(1 - \frac{|w|^2}{1+|w|^2}) > 0$$

Thus the resulting $\tau_0(\cdot, J\cdot)$ is positive definite. Note that by symmetry this works in U_j when $j \neq 0$ as well.

Exercise 4.23 In the case $n = 1$ prove that the symplectic form τ_0 on $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ is given by:

$$\tau_0 = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}$$

in the usual coordinates $x + iy$ on $\mathbb{C}$. Use stereographic projection to prove that this form agrees up to a factor with the area form on the unit sphere $S^2 \subset \mathbb{R}^3$. Prove that the area of $(\mathbb{C}P^1, \tau_0)$ is π, while that of the unit sphere in $\mathbb{R}^3$ is 4π.

Solution 4.23 Since there is only 1 complex coordinates in $\mathbb{C}$, z say, we see that:

$$\frac{i}{2}\partial\bar{\partial} f_0 = \frac{(1 + |z|^2) - |z|^2}{(1 + |z|^2)^2})\frac{i}{2}dz \wedge d\bar{z} = \frac{1}{(1 + |z|^2)^2}\frac{i}{2}(idy \wedge dx - idx \wedge dy) = \frac{1}{(1 + x^2 + y^2)^2}dx \wedge dy$$

Now consider the stereographic projection map from the unit sphere centered at the origin to the (x, y) plane, away from the point $(0, 0, 1)$. We use cylindrical coordinates (z, θ) for the sphere, cylindrical coordinates (r, θ, z) for $\mathbb{R}^3$ and polar coordinates (ρ, ϕ) for $\mathbb{R}^2$. The map is given by:

$$(\theta, z) \mapsto (\sqrt{1 - z^2}, \theta, z) \mapsto (\sqrt{\frac{1 - z}{1 + z}}, \theta) = (\rho, \phi) = \Psi(z, \theta) \in \mathbb{R}^2$$

Thus we have:

$$d\rho = \frac{\partial\rho}{\partial z}dz = \frac{1}{(1 + z)^2} \cdot \sqrt{\frac{1 + z}{1 - z}}dz \qquad d\phi = d\theta$$

Thus:

$$\Psi^*\tau_0 = \psi^*(\frac{1}{(1 + \rho^2)^2}\rho d\rho \wedge d\phi) = \frac{1}{(1 + \frac{1-z}{1+z})^2} \cdot \frac{1 - z}{1 + z} \cdot \frac{1}{(1 + z)^2} \cdot \sqrt{\frac{1 + z}{1 - z}}dz \wedge d\theta = \frac{1}{4}dz \wedge d\theta$$

As we saw in Exercise 3.1, this is $\frac{1}{4}$ times the standard volume form. Since $\int_{S^1 \times I} dz \wedge d\theta = 2\pi \cdot (1 - (-1)) = 4\pi$, we have that the standard sphere has voluem 4π, while the sphere under the Fubini-Study metric has volume π.

Exercise 4.24 Prove that a complex submanifold of a Kähler manifold is itself a Kähler manifold.

Solution 4.24 Let $S \subset M$ be the submanifold of the Kähler manifold (M, g, ω, J) in question. Consider S with the metric $h = g|_S$ and the complex structure $j = J|_S$. Metrics can always be restricted to sub-manifolds, and by the definition of a complex sub-manifold we have $TS = JTS$, so that J also restricts to an almost complex structure (which is integrable, also part of the definition of a complex submanifold, although we will not need this in our proof). Also observe that for any $v, w \in T_pS$, we have $h(jv, jw) = g(Jv, Jw) = g(v, w) = h(v, w)$, so j is almost compatible with h and $\omega(\cdot, \cdot) = h(\cdot, j\cdot)$ is almost symplectic. Thus all we need to do is prove that $d\omega = 0$.

But observe that $\nabla^h_v j = \nabla^g_v J|_S = 0$ for any $v \in TS$. Indeed, in any neighborhood U of a point $p \in S$ we may pick an orthonormal basis e_i of $g|_S$ at T_pS, extend it to an orthonormal basis e_i on T_pM, then we can pick coordinates x_i in U so that $p = 0$, $\partial_{x_i} = e_i$ at p and $S \cap U = \{(x_i) | x_{k+1} = x_{k+2} = \cdots = x_n = 0\}$. Let Γ and $\tilde{\Gamma}$ denote the Christoffel symbols for g and h respectively. In these coordinates at p and for a, b, c denoting indices of coordinates x_i with corresponding tangent basis vectors e_i which are parallel to S, we have the following:

$$h_{ad}\tilde{\Gamma}^d_{bc} = \tilde{\Gamma}_{abc} = \frac{1}{2}(\partial_b h_{ac} + \partial_c h_{ab} - \partial_a h_{bc}) = \frac{1}{2}(\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}) = \Gamma_{abc} = g_{ad}\Gamma^d_{bc}$$

Indeed, in these coordinates at p we have $g_{ad} = h_{ad} = \delta_{ad}$ since we chose e_i to be orthonormal, so this implies that at p:

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc}$$

Now recall that J is anti-self-adjoint, which in these coordinates at p means that $J = -J^T$ or $J^a_b = -J^b_a$, and J preserves $T_pS = \mathrm{span}(e_1, \ldots, e_k)$, which in these coordinates means that $J^a_b = 0$ if $b \in \{1, \ldots, k\}$ and $a \in \{k+1, \ldots, n\}$. By the anti-symmetry, this implies that J^a_b is a block matrix, i.e $J^a_b = 0$ if $a \in \{1, \ldots, k\}$ and $b \in \{k+1, \ldots, n\}$ as well. This implies that if $a, b, c \in \{1, \ldots, k\}$ we have:

$$\tilde{\nabla}_c j^a_b = \partial_c j^a_b + \tilde{\Gamma}^a_{bd} j^d_c - \tilde{\Gamma}^d_{cb} j^a_d = \partial_c J^a_b + \Gamma^a_{bd} J^d_c - \Gamma^d_{cb} J^a_d = \nabla_c J^a_b$$

This is because we clearly have $\partial_c j^a_b = \partial_c j^a_b$ at p, and then the fact that J is a block matrix in this coordinates system implies that in the expression $\Gamma^a_{bd} J^d_c$ (which is summed over the index d) it suffices to sum only over the $d \in \{1, \ldots, k\}$ since for other values J^d_c vanishes. Then that sum is equal to $\tilde{\Gamma}^a_{bd} j^d_c$ since $\tilde{\Gamma}^a_{bd} = \Gamma^a_{bd}$ and $j^d_c = J^d_c$ for that range of d and c. The same discussion holds for the last term. This shows that $\nabla^h j = \nabla^g J|_S$ in local coordinates.

Exercise 4.30 Compute the Chern classes and Betti numbers of a complex hypersurface $M \subset \mathbb{C}P^{n+1}$ of degree d.

Solution 4.30 We can start by using the Lefchetz hyperplane theorem. Any complex hypersurface of degree d in $\mathbb{C}P^{n+1}$ can be realized as the zero set of section of the unique holomorphic line bundle L with Chern class $c_1(L) = \mathrm{PD}(d[H])$ where $[H] \in H_{2n}(\mathbb{C}P^{n+1}; \mathbb{Z})$ is the hyperplane class. This comes from the line bundle divisor correspondence in complex geometry, and also from the fact that $H^1(\mathcal{O}_{\mathbb{C}P^{n+1}}) = H^2(\mathcal{O}_{\mathbb{C}P^{n+1}}) = 0$, which implies that $\mathrm{Pic}_0(\mathbb{C}P^{n+1})$ is 1 point, and thus that line bundles are classified by degree for $\mathbb{C}P^{n+1}$.

Now we can consider a basis of the holomorphic sections of L, $\sigma_0, \ldots, \sigma_k$. We see that for any two points $p, q \in \mathbb{C}P^{n+1}$ there exists a section σ such that $\sigma(p) = 0$ and $\sigma(q) \neq 0$. If this were not the case, then every degree d hypersurface containing p would also contain q, since any section vanishing at p would also vanish at q. But this is clearly false: we can take a collection of d linear polynomials l_i which vanish at p but not q. Then we can take $p = \prod_i l_i$ and perturb the coefficients slightly. For a generic, sufficiently small perturbation the result will be a smooth degree d curve containing p but not q. The point of this is that this implies that the map $\psi : \mathbb{C}P^{n+1} \to \mathbb{C}P^k$ given by $p \mapsto [\sigma_0(p), \ldots, \sigma_k(p)]$ is injective; if $\psi(p) = \psi(q)$ for some pair, then any section that vanished on p would vanish on q.

Now if we pick σ_0 so that $M = \{p | \sigma_0(p) = 0\}$, then $M \simeq \psi(M) \simeq H \cap \psi(\mathbb{C}P^{n+1})$ where $H =$

$\{[x_0,\dots,x_k]|x_0=0\}$. Thus by the Lefchetz hyperplane theorem, we know that for $i<n$ we have $H^i(M;\mathbb{Z})=H^i(\mathbb{C}P^{n+1};\mathbb{Z})=\mathbb{Z}$ if i is even and 0 otherwise. Furthermore, by Poincare duality we know that $H^i(M;\mathbb{R})\simeq H^{2n-i}(M;\mathbb{R})$. Thus $b_i=1$ if i is even and 0 if i is odd when $i\neq n$.

To proceed further we should calculate $c(T\mathbb{C}P^{n+1})$ and $c(\nu M)$. To do this, observe that we still have $T_l\mathbb{C}P^{n+1}\simeq \mathrm{Hom}(L,L^\perp)$ and $\mathbb{C}\simeq\mathrm{Hom}(L,L)$[7], as the argument for these facts is not dimensionally dependent. Here L is the tautological line bundle on $\mathbb{C}P^{n+1}$. Thus we have $T\mathbb{C}P^{n+1}\oplus\mathbb{C}\simeq\mathrm{Hom}(L,L\oplus L^\perp)\simeq\oplus_1^{n+2}L^*$. Thus by the properties of the total Chern class with respect to Whitney sums, we have:

$$c(T\mathbb{C}P^{n+1})=c(T\mathbb{C}P^{n+1}\oplus\mathbb{C})=c(\oplus_1^{n+2}L)=c(L^*)^{n+2}=(1+h)^{n+2}$$

This is because $c_1(L^*)=-c_1(L)=-(-h)$ where h is the generator of $H^2(\mathbb{C}P^{n+1})$ corresponding to the hyperplanes via Poincare duality. this implies that $c(T_M\mathbb{C}P^{n+1})=c(i^*T\mathbb{C}P^{n+1})=i^*c(T\mathbb{C}P^{n+1})=(1+i^*h)^{n+2}$.

For νM, we observe that $c_1(\nu M)=e(\nu M)=\mathrm{PD}([\sigma=0])$ where σ is a section of νM intersecting the zero section transvsersely. But we can identify νM diffeomorphically with a tubular neighborhood of N of $M\subset\mathbb{C}P^{n+1}$ via a map $i:\nu M\to N$ sending the zero section to M itself. Under such an identification the graph of σ becomes a sub-manifold $\sigma(M)$ intersecting M transversely and homologous to M. Thus $i_*[\sigma=0]=i_*[M]\cap i_*[M]\in H_{2n-2}(M)$. In particular, if we take a surface Σ of M representing class $[\Sigma]\in H_2(M;\mathbb{Z})$ then we have:

$$\langle c_1(\nu M),[\Sigma]\rangle=[\Sigma]\cdot([M]\cap[M])=i_*[\Sigma]\cdot i_*[M]=\langle\mathrm{PD}(i_*[M]),i_*[\Sigma]\rangle=\langle dh,i_*[\Sigma]=\langle di^*h,[\Sigma]\rangle$$

So $c_1(\nu M)=di^*h$ and $c(\nu M)=1+di^*h$ because it's a line bundle. Thus by the Whitney sum property we have $c(T_M\mathbb{C}P^{n+1})=c(\nu M)c(TM)$ so that:

$$c(TM)=(1+di^*h)^{-1}(1+i^*h)^{n+2}=(\sum_{j=0}^n(-1)^jd^ji^*h^j)(\sum_{j=0}^n\binom{n+2}{j}i^*h^j)=\sum_{k=0}^n(\sum_{j=0}^k(-1)^jd^j\binom{n+2}{k-j})i^*h^k$$

For our final observation, which will give us b_n, we note that $\langle i^*h^n,[M]\rangle=\langle h^n,i_*[M]\rangle=\mathrm{PD}(h^n)\cdot i^*[M]=\mathrm{PD}(h)^n\cdot i^*[M]$. Since $\mathrm{PD}(h)$ is a hyperplane, the n-time intersection of n transversely intersecting representatives of $\mathrm{PD}(h)$ is a line, and any line intersects a degree d hyperplane d times. Thus we have:

$$\chi(M)=\langle e(M),[M]\rangle=\langle c_n(M),[M]\rangle=\sum_{j=0}^n(-1)^jd^{j+1}\binom{n+2}{n-j}$$

Thus we can use:

$$\chi(M)=(-1)^nb_n+\sum_{j\neq n}(-1)^jb_j=(-1)^nb_n+n$$

to write:

$$b_n=(-1)^n((\sum_{j=0}^n(-1)^jd^{j+1}\binom{n+2}{n-j})+n)$$

[7]This second point is obvious, the first one less so. The elaboration is poor in the book, but you can find an explanation in Ch. 3 of Milnor-Stasheff.

We can check this when $n = 2$, so that the result is:

$$b_2 = d^3 - 4d^2 + 6d - 2$$

This is correct!

Exercise 5.3 Consider the action of S^1 on $\mathbb{R}^{2n+2}$ which, under the usual identification of $\mathbb{R}^{2n+2}$ with $\mathbb{C}^{n+1}$ corresponds to multiplication by $e^{2\pi it}$. By Exercise 1.21 this action is generated by the function:

$$H(z) = -\pi|z|^2$$

Prove that the symplectic quotient at $\lambda = -\pi$ is $\mathbb{C}P^{n+1}$ with the standard symplectic form τ_0 defined in Example 4.21 above. This construction shows that τ_0 is $U(n+1)$-invariant.

Solution 5.3 Consider the Fubini-Study form τ_0 as described on p. 131 and consider the projection $\pi : \mathbb{C}^{n+1} \to \mathbb{C}P^{n+1}$ given by $(x_0, \dots, x_n) \to [x_0, \dots, x_n]$. We start by showing that $\pi^*\tau_0 = \frac{i}{2}\partial\bar{\partial}f$ where:

$$f(z, \bar{z}) = \sum_{\nu=0}^{n} z_\nu \bar{z}_\nu$$

To see this, we first observe that:

$$\pi^*\tau_0 = \frac{i}{2(\sum_{\nu=0}^{n} \bar{z}_\nu z_\nu)^2} \sum_{k=0}^{n} \sum_{j\neq k} (\bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k)$$

This is the expression given for τ_0 on p. 131, but it is actually an expression for the pullback which descends to a 2-form in the patches U_j given by coordinates $(w_1, \dots, w_n) = (\frac{z_0}{z_j}, \dots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \frac{z_n}{z_j})$. It's labelled in a very misleading way. Anyway, we just need to show that this expression is $\frac{i}{2}\partial\bar{\partial}f$. But we see that:

$$\frac{i}{2}\partial\bar{\partial}f = \frac{i}{2}\partial\bar{\partial}(\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu) = \frac{i}{2}\partial(\sum_{k} \frac{z_k}{\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu} d\bar{z}_k) = \frac{i}{2}\sum_{k,j}(\frac{\delta_{jk}}{\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu} - \frac{\bar{z}_j z_k}{(\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu)^2}) dz_j \wedge d\bar{z}_k$$

$$= \frac{i}{2(\sum_{\nu=0}^{n} z_\nu \bar{z}_\nu)^2} \sum_{k,j} \bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k = \frac{i}{2(\sum_{\nu=0}^{n} \bar{z}_\nu z_\nu)^2} \sum_{k=0}^{n} \sum_{j\neq k} (\bar{z}_j z_j dz_k \wedge d\bar{z}_k - \bar{z}_j z_k dz_j \wedge d\bar{z}_k)$$

Now, both $\pi^*\tau_0$ and ω_0 restrict to 2-forms on S^{2n+1} which are equivariant under the $U(1)$ action. Furthermore, under the quotient map $q : S^{2n+1} \to \mathbb{C}P^n$ induced by restricting π, the equivariant 2-form $\pi^*\tau_0|_{S^{2n+1}}$ goes to τ_0 by construction. Thus if we show that the 2-form $\tilde{\omega}_0$ which ω_0 descends to on $\mathbb{C}P^n$ agrees with τ_0, it will follow that $(S^{2n+1}/S^1, \tilde{\omega}_0) \simeq (\mathbb{C}P^n, \tau_0)$. It suffices to show that $\pi^*\tau_0|_{S^{2n+1}} = \omega_0|_{S^{2n+1}}$. To see this, observe that on the unit sphere we have $|z|^2 = 1$ by definition, so:

$$\frac{i}{2}\partial\bar{\partial}f = \frac{i}{2}\sum_{k,j}(\delta_{jk} - \bar{z}_j z_k) dz_j \wedge d\bar{z}_k = \omega_0 + \frac{-i}{2}\sum_{j,k} \bar{z}_j z_k dz_j \wedge d\bar{z}_k = \omega_0 + \epsilon$$

Thus we just need to show that the 2-form ϵ is 0 on S^{2n+1}. But suppose that $w = (w_0, \dots, w_n)$ is a tangent vector at $z = (z_0, \dots, z_k)$. Then we have $w \cdot z = \sum_j \bar{w}_j z_j = 0$. Then we see that:

$$i_w \epsilon_z = \sum_{j,k} \bar{z}_j z_k (dz_j \bar{w}_k - w_j d\bar{z}_k) = \sum_j w \cdot z \bar{z}_j dz_j - \sum_k z \cdot w z_k d\bar{z}_k = 0$$

So the restriction of ϵ is 0.

Exercise 5.4 We saw in Exercise 4.23 that:

$$\int_{\mathbb{C}P^1} \tau_0 = \pi$$

Find yet another proof by interpreting this integral in terms of the Hopf fibration $\pi_H : S^3 \to S^2 = \mathbb{C}P^1$ and showing that it equals the integral of ω_0 over a disc in $\mathbb{R}^4$ whose boundary lies along one of the fibers of π_H.

Solution 5.4 This is actually a subtle question that involves some serious background discussion. Given a smooth fiber bundle $\pi : E \to B$ with closed fiber F of dimension k, we have an integration map (a "pushforward" if you will) on k-forms, $\pi^* : \Omega^n(E) \to \Omega^{n-k}(B)$ given by integrating over a fiber, namely:

$$(\pi^* \alpha)_p(v_1, \dots, v_{n-k}) = \int_{\pi^{-1}(p)} \alpha$$

This is largely a motivational formula since the integral above does not obviously have an invariant interpretation. We will need the following properties of this map. First, this descends to a map on cohomology. Second, for any $\alpha \in \Omega^*(E)$ and any $\beta \in \Omega^*(B)$ we have:

$$\int_E \alpha \wedge \pi^* \beta = \int_B \pi_* \alpha \wedge \beta$$

The details of this construction can be found in Bott & Tu, Ch. 6 (although the treatment there focuses on compactly supported cohomology when F is a vector space).

Now we apply these ideas to our situation. Consider the Hopf fibration $h : S^3 \to \mathbb{C}P^1$ where we consider S^3 as the unit sphere in $\mathbb{C}^2$. Also consider the 1-form $\alpha = \frac{1}{2}(\sum_i x_i dy_i - y_i dx_i)$. Notice that $d\alpha = \omega$ where ω is the standard symplectic form on $\mathbb{C}^2$. Furthermore, observe that $\alpha|_{S^3}$ is a contact form on S^3 with Reeb vector-field given in complex coordinates as $R(z) = 2iz$ for $|z|^2 = 1$, i.e $z \in S^3$. The S^1-action/Reeb flow generated by this $\psi_t(z) \mapsto e^{2it} z$ (which is just a reparameterization of the $U(1)$-action discussed in Exercise 5.3), and the quotient by this action can be identified as $\mathbb{C}P^1$ with the quotient map $q : S^3 \to S^3/S^1$ being the same as the Hopf fibration map.

Now observe the following. First, $i_R \alpha = 1$ identically (this is part of the definition of the Reeb vector-field). Thus if we pick a point $p \in \mathbb{C}P^1$ and we parameterize $h^{-1}(p)$ by an integral curve of R, γ say, then we can find $h_* \alpha \in \Omega^0(\mathbb{C}P^1)$:

$$h_* \alpha(p) = \int_{h^{-1}(p)} \alpha = \int_\gamma i_{\dot{\gamma}} \alpha = \pi$$

Thus if D is any disk bounding $\gamma = \pi^{-1}(p)$ in $\mathbb{C}^2$, we have by Stokes theorem that:

$$\pi = \int_{h^{-1}(p)} \alpha = \int_D d\alpha = \int_D \omega$$

But we can also apply our knowledge of $j_*\alpha$ to get the volume of $\mathbb{C}P^1$. Namely, we know by Exercise 5.3 that $h^*\tau = \omega|_{S^3}$. We will just denote $\omega|_{S^3}$ as ω. Thus by the integral identities for π_* above:

$$\pi \int_{\mathbb{C}P^1} \tau = \int_{\mathbb{C}P^1} \pi_*\alpha \wedge \tau = \int_{S^3} \alpha \wedge \pi^*\tau = \int_{S^3} \alpha \wedge \omega = \int_{B^4} d(\alpha \wedge \omega) = \int_{B^4} \omega^2 = 2\int_{B^4} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 = \pi^2$$

Thus we have:

$$\int_{\mathbb{C}P^1} \tau_0 = \pi = \int_D \omega$$

as desired.

Exercise 5.11 Show that Ω (defined on p. 160) has maximal rank on the odd-dimensional manifold $P \times S^2$, and that its kernel consists of all vectors tangent to the S^1 orbits. Deduce as in Lemma 5.2 that there is an induced symplectic form on the quotient $M = P \times_{S^1} S^2$. Identify this form with the one constructed in Example 5.10.

Solution 5.11 Let $P \to B$ be an S^1 principle bundle with a 1-form $\alpha \in \Omega^1(P)$ satisfying $i_X\alpha = 1$ and $d\alpha = -\pi^*\rho$ where $X : P \to TP$ is the generating vector-field of the S^1 action and ρ is a closed integral 2-form on B. Let τ_0 be a symplectic form on B such that $\tau_0 + \lambda\rho$ is also symplectic for $\lambda \in (0,1)$. Consider $P \times S^2$ with the S^1 action $a(p,s) = (a \cdot p, a^{-1} \cdot s)$ and define:

$$\Omega = \pi_B^*\tau_0 - d(H\alpha) + \pi_S^*\sigma$$

Where $H(p,s) = h(s)$ is the height function on S^2, π_B and π_S are the projections to B and S^2, and σ is an S^1 invariant volume form of unit volume on S^2. We assume through-out that we are away from the singular strata, i.e wherever the height function is 0 or 1.

Now suppose $(p,s) \in P \times S^2$ where $h(s) \neq 0, 1$ and $v \in T_{p,s}(P \times S^2)$. Then $v = w \oplus u$ where $w \in T_pP$ and $u \in T_sS^2$. Now suppose that $i_v\Omega = 0$ at p. Let $\pi_{P,B} : P \to B$ denote the projection map. Then we see that:

$$0 = i_v\Omega = i_v(\pi_B^*\tau_0 - \pi_S^* dh \wedge \alpha - Hd\alpha + \pi_S^*\sigma) = i_v(\pi_B^*(\tau_0 + H\rho) + \pi_S^*\sigma - \pi_S^* dh \wedge \pi_P^*\alpha)$$

$$= \pi_P^*(i_w \pi_{P,B}^*(\tau_0 + H\rho)) + \pi_S^*(i_u\sigma) - \pi_S^*(i_u dh)\pi_P^*\alpha - \pi_S^*(dh)\pi_P^*(i_w\alpha)$$

$$= \pi_P^*(\pi_{P,B}^*(i_{\pi_{P,B}^* w}(\tau_0 + H\rho)) - \pi_S^*(i_u dh)\alpha) + \pi_S^*(i_u\sigma - \pi_P^*(i_w\alpha)dh)$$

This mean looking set of manipulations is meant to get us to an expression with pieces that must vanish independently. In particular, in order for the above expression to vanish, both the π_P^* part and the π_S^* part must vanish, since the images of π_P^* and π_S^* are independent. Furthermore, α and the image of $\pi_{P,B}^*$ are independent by construction of α, so in order for $i_{\pi_{P,B}^* w}(\tau_0 + H\rho)) - \pi_S^*(i_u dh)\alpha = 0$, both of those terms must be zero as well.

Thus we have $i_{\pi^*_{P,B}w}(\tau_0 + H\rho)) = 0$. But $H(s) \in (0,1)$ and $\tau_0 + \lambda\rho$ is symplectic for that range of λ. So $i_{\pi^*_{P,B}w}(\tau_0 + H\rho)) = 0$ if and only if $\pi^*_{P,B}w = 0$, i.e if w is a multiple of X, the generator of the circle action on P, at each point. Since $\pi^*_S(i_u dh)\alpha = 0$ and α vanishes nowhere, we know that $i_u dh = 0$.

Thus suppose that $w(p,s) = aX(p)$ and $u(p,s) = bX_h(s)$ for some constants a and b. Then we see from the second vanishing condition that:

$$0 = i_u\sigma - \pi^*_P(i_w\alpha)dh = b[i_{X_{h(s)}}\sigma](p) - b\pi^*_P(i_X\alpha)dh(p) = bdh(p) - b\pi^*1dh(p) = (a-b)dh(p)$$

Thus $a = b$. So any v where $i_v\Omega = 0$ is of the form $v = w \oplus u = f(X \oplus -X_h)$. But the vector-field $X \oplus -X_h$ exactly generates the action on $P \times S^2$. Indeed, we see that if the action is given by $(p,s) \mapsto \psi_t(p,s) = (\psi^P_t(p), (\psi^S)^{-1}(s))$ then differentiating with respect to t in and evaluating at 0 gets us:

$$\frac{d}{dt}\psi_t(p,s)|_{t=0} = (\frac{d\psi^P_t(p)}{dt}|_{t=0}, \frac{d[\psi^S_t]^{-1}(s)}{dt}|_{t=0}) = (X(p), (d\psi^S_t)^{-1}((\psi^P_t)^{-1}(p)) \circ \frac{d\psi^1_t}{dt}(p)|_{t=0}) = (X(p), -X_h(p))$$

Thus Ω is a maximal rank 2-form with kernel equal to the tangent space of the S^1 orbits on $P \times S^2$. It is also closed and equivariant with respect to the S^1 action, since each of the terms is closed and equivariant. For instance, $\pi^*_B\tau_0$ is closed because pullback and exterior differentiation commute and it's equivariant because $\psi^*_t\pi^*_B\tau_0 = \pi^*_B(\psi^P_t)^*\tau_0 = \pi^*_B\tau_0$. The rest of the terms can be checked similarly. Thus this map descends to a well-defined symplectic form on the quotient $P \times S^2/S^1 = P \times_{S^1} S^2$.

Evidently by Proposition 5.8(ii) this form is equivariantly symplectomorphic over its domain of definition to the one constructed in Example 5.10. However, that one is constructed abstractly using Proposition 5.8(i), so a more explicit identification doesn't seem possible.

Exercise 5.12 Prove that $\tilde{\Omega}$ is symplectic and is invariant under the diagonal action of S^1. Show that V^*P is equivariantly diffeomorphic to $P \times \mathbb{R}$ and that the moment map $\mu : W = P \times \mathbb{R} \times S^2 \to \mathbb{R}$ is given by:

$$\mu(p, \eta, z) = h(z) - \eta$$

where h :^{2}: $S^2 \to \mathbb{R}$ is the height function used above. Show further that 0 is a regular value of μ and that the level sets $\mu^{-1}(0)$ can be identified with the manifold $P \times S^2$ by a map which takes $\tilde{\Omega}$ to Ω. Thus (M, ω) is the symplectic quotient of $(W, \tilde{\Omega})$.

Solution 5.12 Recall that $\tilde{\Omega}$ is defined on W, using the same information as in Exercise 5.11, as:

$$\tilde{\Omega} = \pi^*_B\tau_0 + i^*_\alpha\omega_{\text{can}} + \pi^*_S\alpha$$

Here $i_\alpha : V^*P \to T^*P$ can be written explicitly as $a \mapsto a(X)\alpha$.

Exercise 5.13 Assume that the symplectic form ω is exact (and so M is not compact). Choose a 1-form λ such that $\omega = -d\lambda$. A symplectic action of a Lie group G on M is called exact if $\psi^*_g\lambda = \lambda$ for every $g \in G$. Prove that every exact action is Hamiltonian with $H_\xi = i_{X_\xi}\lambda$ for $\xi \in \mathbf{g}$.

Solution 5.13 This is a simple computation. If we let $g : I \to G$ be a path in G with $g_0 = 1$ and $\frac{dg_t}{dt}|_{t=0} = \xi$and we let $\psi_t = \psi_{g(t)}$ be the corresponding family of diffeomorphisms, then $\frac{d\psi_t^*\lambda}{dt} = \mathcal{L}_{X_\xi}\lambda = \frac{d\lambda}{dt} = 0$ by assumption. Thus we have:

$$0 = -\mathcal{L}_{X_\xi}\lambda = -(di_{X_\xi}\lambda + i_{X_\xi}d\lambda) = -dH_\xi + i_{X_\xi}\omega$$

where $H_\xi = i_{X_\xi}\lambda$. Thus H_ξ is a Hamiltonian for the symplectic vector-field X_ξ, and G is weakly Hamiltonian. To show that it is in fact strongly Hamiltonian, we see that:

$$H_{[\xi,\eta]} = i_{X_{[\xi,\eta]}}\lambda = i_{[X_\xi,X_\eta]}\lambda = \mathcal{L}_{X_\eta}(i_{X_\xi}\lambda) = i_{X_\eta}dH_\xi = \{H_\xi, H_\eta\}$$

Exercise 5.15 Show that when G is abelian the orbits of a weakly Hamiltonian action of G on M are always isotropic submanifolds of M, i.e $\omega(X_\xi, X_\eta) = 0$ for all $\xi, \eta \in \mathbf{g}$. Give an example to show that this is not always true for symplectic actions of abelian groups.

Solution 5.15 If we do not make any compactness assumptions this is false: we can, for instance, take the action $\mathbb{R}^2 \curvearrowright \mathbb{R}^4$ given by $(x_1, y_1, x_2, y_2) \mapsto (x_1 + a, y_1 + b, x_2, y_2)$. This is clearly a Hamiltonian action given by the Hamiltonians $F(x, y) = -x_1$ and $G(x, y) = y_1$, but the group orbit is the symplectically embedded $\mathbb{R}^2 \times 0 \subset \mathbb{R}^4$.

Thus we assume that M is compact. Assume G is abelian, and that we have a weakly Hamiltonian action $G \curvearrowright (M, \omega)$. Choose a map $\mathbf{g} \to C^\infty(M)$ given by $\xi \mapsto H_\xi$ so that $X_{H_\xi} = X_\xi$ for all $\xi \in \mathbf{g}$. Since $\mathbf{g}$ is abelian, we have $[\xi, \eta] = 0$ for any $\xi, \eta \in \mathbf{g}$, and thus $H_{[\xi,\eta]} = 0$ by linearity. Then by Lemma 5.14 we have, at any point in M and for any pair $\xi, \eta \in \mathbf{g}$:

$$\{H_\xi, H_\eta\} = \tau(\xi, \eta) + H_{[\xi,\eta]} = \tau(\xi, \eta)$$

Now observe that, by compactness of M, there exists a $p \in M$ where dH_ξ vanishes (i.e a critical point of H_ξ). Thus $dH_\xi(X_\eta) = 0$ at that point, and $\tau(\xi, \eta) = 0$. Since $\tau(\xi, \eta)$ is independent of p, it follows that $\tau(\xi, \eta) = 0$ for any $\xi, \eta \in \mathbf{g}$. Thus $\{H_\xi, H_\eta\} = \omega(X_\xi, X_\eta) = 0$ for any pair ξ, η and in fact the G action is Hamiltonian.

We can easily find a counter-example to this statement if we allow $G \curvearrowright (M, \omega)$ to be only symplectic; we need only take a quotient of the $\mathbb{R}^2 \curvearrowright \mathbb{R}^4$ example. We can take the torus action $T^2 \curvearrowright T^4$ (with T^4 imbued with the quotient symplectic structure given the standard map $\mathbb{R}^4 \to \mathbb{R}^4/\mathbb{Z}^4 = T^4$) given by $(x_1, y_1, x_2, y_2) \mapsto (x_1 + a, y_1 + b, x_2, y_2)$. This action is not Hamitlonian since the 1-forms $i_{\partial_{x_1}}\omega = dy_1$ and $i_{\partial_{y_1}}\omega = -dx_1$ are not exact. This action is symplectic and the orbit of $(0, 0, 0, 0)$ is $\{(a, b, 0, 0)|(a, b) \in T^2\}$, i.e the symplectically embedded torus $T^2 \times 0 \subset T^4$.

Exercise 5.17 Prove that these definitions are consistent with the ones in Section 5.1 where G is the circle group S^1.

Solution 5.17 In Section 5.1 the moment map of an S^1 action was just defined as the Hamiltonian H corresponding to the vector-field $\frac{d\psi_t}{dt}|_{t=0}$ with $t \to \psi_t$ the group map $S^1 \to \mathrm{Symp}(M)$, with $\psi_1 = 1$. This implies that $\frac{d\psi_t}{dt}|_{t=0} = X_{2\pi i}$ where $2\pi i \in i\mathbb{R} \simeq \mathbf{u}(1)$ is the generating element of the Lie algebra given by differentiating $g(t) = \exp(2\pi i t)$ at 0. Since $\mathbf{u}(1) = \mathrm{span}(2\pi i)$, we can define a map $\mu : M \to \mathbf{u}(1)^*$ by the formula $\langle \mu(x), 2\pi i \rangle = H$ and by demanding that the map be linear. Thus any Lie algebra element $\xi = 2\pi i \lambda$ goes to $H_\xi = \lambda H$. The resulting μ is trivially a Lie algebra homomorphism because $\mathbf{u}(1)$ is 1-dimensional, thus all Lie brackets vanish, and since all H_ξ are multiples of H, all Poisson brackets vanish as well. This confirms that the terminology is consistent: all of the data of the "moment map" is carried by the Hamiltonian of $2\pi i$.

Exercise 5.19 There is a natural double cover $SU(2) \to SO(3)$. To see this identify $SU(2)$ with the unit quaternions $S^3 \subset \mathbb{R}^4 \simeq \mathbb{H}$ via the map $S^3 \to SU(2)$ defined by:

$$U_x = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

Now the unit quaternions act on the imaginary quaternions by conjugation and the map $S^3 \to SO(3)$: $x \to \Phi_x$ is defined by:

$$q(\Phi_x \xi) = q(x) q(\xi) \overline{q(x)}$$

where $q(x) = x_0 + ix_1 + jx_2 + kx_3$ and $q(\xi) = i\xi_1 + j\xi_2 + k\xi_3$ for $x \in S^3$ an $\xi \in \mathbb{R}^3$.

(i) Prove that the map $SU(2) \to SO(3) : U_x \to \Phi_x$ is a group homomorphism and a double cover. (ii) Prove that the differential of the group homomorphism $U_x \to \Phi_x$ is the map $\mathbf{su}(2) \to \mathbf{so}(3) : u_\xi \to A_\xi$ where:

$$u_\xi = \frac{1}{2} \begin{pmatrix} i\xi_1 & \xi_2 + i\xi_3 \\ -\xi_2 + i\xi_3 & -i\xi_1 \end{pmatrix}$$

for $\xi \in \mathbb{R}^3$. Prove directly that the map $u_\xi \to A_\xi$ is a Lie algebra homomorphism and identifies the two invariant inner products.

Solution 5.19 (i) First observe that the map $\xi \mapsto U_\xi$ extends to the entire quaternion algebra $\mathbb{H}$, giving an algebra embedding $U : \mathbb{H} \to \mathrm{End}(\mathbb{C}^2)$. This map is given by:

$$1 \mapsto U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i \mapsto U_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j \mapsto U_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k \mapsto U_k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Notice that the U_i, U_j, U_k form a basis of the anti-Hermitian operators, i.e the M such that $M^\dagger = -M$. Furthermore, $U_{\bar{\xi}} = U_\xi^\dagger$, as this is true if it is true for $1, i, j, k$ and it is easily checkable on the matrices above. This the action of $SU(2)$ on the imaginary quaternions can be written in terms of the U_ξ as:

$$\Phi_x U_\xi = U_{\Phi_x \xi} = U_x U_\xi U_x^\dagger$$

To check that this is a well-defined action, we need to check that the resulting matrix is in the image of the imaginary quaternions, i.e that the resulting matrix is anti-Hermitian. But:

$$(U_x U_\xi U_x^\dagger)^\dagger = (U_x^\dagger)^\dagger U_\xi^\dagger U_x^\dagger = -U_x U_\xi U_x$$

So this is true. To see that the map Φ_x is in $SO(3)$, we observe that the inner product on $\mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ can be written $\langle \xi, \eta \rangle = \frac{1}{2}\mathrm{tr}(U_\xi U_\eta^\dagger)$. This can easily be checked: $U_1 = U_1 U_1^\dagger = U_i U_i^\dagger = U_j U_j^\dagger = U_k U_k^\dagger$, so these are all norm 1 vectors and the pair-wise products are all traceless. Thus $\langle \xi, \eta \rangle$ and $\frac{1}{2}\mathrm{tr}(U^\dagger U)$ agree on a basis. So they are the same. To see that the map is a group homomorphis, we just see that:

$$\Phi_{xy} U_\xi = U_{xy} U_\xi (U_{xy})^\dagger = U_x U_y U_\xi (U_x U_y)^\dagger = U_x U_y U_\xi U_y^\dagger U_x^\dagger = \Phi_x \Phi_y U_\xi$$

To see that this is a double cover, we just need to check that the kernel of the map $\Phi : SU(2) \to SO(3)$ given by $x \to \Phi_x$ is $\{\pm 1\}$. Then we know that $d\Phi_0$ has 0-dimensional kernel (since if the kernel of $d\Phi_0$ is the tangent space of the kernel of Φ at 0) and thus by dimension counting (since $\dim(\mathbf{su}(2)) = \dim(\mathbf{so}(2)) = 3$), $d\Phi_0$ is bijective. Since Φ is a group homomorphism, this implies $d\Phi$ is bijective everywhere. So Φ is a covering map with fiber over any point isomorphic to the kernel, i.e $\{\pm 1\}$ (since $\Phi(g) = \Phi(g') \iff \Phi(gg') = \Phi(1) \iff g' = \pm g$).

To check that the kernel if ± 1, we observe that U_ξ is in the kernel if and only if $U_\xi U_a = U_a U_\xi$ for $a \in \{i, j, k\}$. But we see that U_ξ commutes with U_i if and only if U_i and U_ξ are mutually diagonalizable, i.e if and only if U_ξ is diagonal. Then:

$$U_\xi = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}$$

Since U_ξ is unitary, a is a root of unity. Furthermore, $U_j U_\xi = U_\xi U_j$ implies that $a = \bar{a}$. Thus $a = \pm 1$ and we must have $U_\xi = \pm 1$. Since ± 1 are in the center of $\mathrm{End}(\mathbb{C}^2)$ and are unitary, they are both in the kernel of Φ. So they are equal to it.

(ii) Now we examine the map of Lie algebras. First note that if U_t is a family of unitary matrices with $U_0 = 1$ and $\frac{dU_t}{dt}|_{t=0} = A \in \mathbf{su}(2)$ then, we have:

$$\frac{d}{dt}(U_t U_\xi U_t^\dagger)|_{t=0} = \frac{dU_t}{dt}|_{t=0} U_\xi + U_\xi \frac{dU_t^\dagger}{dt}|_{t=0} = AU_\xi + U_\xi A^\dagger = [A, U_\xi]$$

Here $A \in \mathbf{su}(2)$ is an anti-Hermitian matrix and so is U_ξ by the discussion in (i). Thus this is just the adjoint action of the Lie algebra on itself! To check that this map is as claimed in (ii), we just need to check on the basis i, j, k. We see that if $\xi = (a, b, c)$ then:

$$U_\xi = \begin{pmatrix} ia & b + ic \\ -b + ic & -ia \end{pmatrix}$$

Then:

$$\frac{1}{2}[U_i, U_\xi] = i \begin{pmatrix} 0 & ib - c \\ ib + c & 0 \end{pmatrix} = U_{\hat{x} \times \xi}$$

$$\frac{1}{2}[U_j, U_\xi] = i \begin{pmatrix} ic & -ia \\ -ia & ic \end{pmatrix} = U_{\hat{y}\times\xi}$$

$$\frac{1}{2}[U_k, U_\xi] = i \begin{pmatrix} i(-b) & a \\ -a & -i(-b) \end{pmatrix} = U_{\hat{z}\times\xi}$$

Thus we have checked that the map $\mathbf{su}(2) \to \mathbf{so}(3)$ is given by the map $u_\xi \to A_\xi$. To see that this is a Lie algebra homomorphism, first recall that the cross product satisfies the Jacobi identity:

$$a \times (b \times c) - b \times (a \times c) = (a \times b) \times c$$

Thus for two elements of $\mathbf{su}(2)$, ξ and η, we have:

$$[A_\xi, A_\eta]v = \xi \times (\eta \times v) - \eta \times (\xi \times v) = (\xi \times \eta) \times v = A_{\xi\times\eta}v$$

But we calculated above that $[u_\xi, u_\eta] = \frac{1}{4}[U_\xi, U_\eta] = \frac{1}{2}U_{\xi\times\eta} = u_{\xi\times\eta}$. Thus the map $u_\xi \mapsto A_\xi$ has the property that $[u_\xi, u_\eta] = u_{\xi\times\eta} \mapsto A_{\xi\times\eta} = [A_\xi, A_\eta]$. Thus we have a Lie algebra homomorphism. The fact that it preserves the inner product follows from our discussion above in (i) showing that $2\mathrm{tr}(u_\xi u_\eta^\dagger) = \frac{1}{2}\mathrm{tr}(U_\xi U_\eta^\dagger) = \langle \xi, \eta \rangle$ and the fact that:

$$\frac{1}{2}\mathrm{tr}(A_\xi A_\eta^T) = \langle \xi, \eta \rangle$$

This can be seen by noting that $\mathrm{tr}(AB^T) = \sum_{i,j} a_{ij}b_{ij}$ and thus (by examining the matrices directly) observing that $A_{\hat{x}}, A_{\hat{y}}$ and $A_{\hat{z}}$ are an orthonormal basis under $\frac{1}{2}\mathrm{tr}(A_\xi A_\eta^T)$. Thus we have:

$$2\mathrm{tr}(u_\xi u_\eta^\dagger) = \langle \xi, \eta \rangle = \frac{1}{2}\mathrm{tr}(A_\xi A^T)$$

So the two inner products are identified by $u_\xi \to A_\xi$. But these are invariant inner products with respect to the commutator, since:

$$2(\mathrm{tr}([u_\kappa, u_\xi]u_\eta^\dagger) + \mathrm{tr}(u_\xi [u_\kappa, u_\eta]^\dagger)) = 2\mathrm{tr}(u_\kappa u_\xi u_\eta^\dagger - u_\eta u_\kappa u_\eta^\dagger + u_\xi u_\eta^\dagger u_\kappa^\dagger - u_\xi u_\kappa^\dagger u_\eta^\dagger)$$

$$= 2\mathrm{tr}(u_\kappa u_\xi u_\eta^\dagger - u_\eta u_\kappa u_\eta^\dagger - u_\xi u_\eta^\dagger u_\kappa + u_\xi u_\kappa u_\eta^\dagger) = 2\mathrm{tr}(u_\kappa u_\xi u_\eta^\dagger - u_\eta u_\kappa u_\eta^\dagger - u_\kappa u_\xi u_\eta^\dagger + u_\xi u_\kappa u_\eta^\dagger) = 0$$

Here we use cyclicity of trace and the fact that $u_\kappa = -u_\kappa^\dagger$. An identical manipulation shows that $\frac{1}{2}\mathrm{tr}(A_\xi A_\eta^T)$ is invariant. Thus we have proven the last part of (ii).

Exercise 5.21 Show that the obvious action of $U(n)$ on $(\mathbb{C}P^{n-1}, \tau_0)$ is Hamiltonian and find a formula for its moment map.

Solution 5.21 It suffices to find a moment map $\mathbb{C}P^{n-1} \to \mathbf{u}(n)^*$. Then the fact that each of the vector-fields X_ξ is Hamiltonian will imply that $U(n)$ is symplectic since for any $g \in U(n)$ with $g = g(1)$ (where $g(t) = \exp(t\xi)$ for $\xi \in \mathbf{u}(n)$) we have $\phi_g^*\omega = \exp(\xi)^*\omega$ and:

$$\phi_g^*\omega - \omega = \int_0^1 \frac{d}{dt}\phi_{g(t)}^*\omega dt = \int_0^1 \phi_{g(t)}^*\mathcal{L}_{X_\xi}\omega = 0$$

So that the representation $U(n) \to \mathrm{Diff}(\mathbb{C}P^{n-1})$ is symplectic.

Now we claim that the moment map $\mu : \mathbb{C}P^{n-1} \to \mathbf{u}(n)$ is given by $\mu([z]) = \frac{izz^*}{2|z|^2}$ (where we identify $\mathbf{u}(n)$ and $\mathbf{u}(n)^*$ by the invariant inner product), so that the Hamiltonian H_ξ is $H_\xi([z]) = \frac{i}{2}\frac{\langle z^*, \xi z\rangle}{\langle z,z\rangle}$.

To show that $dH_\xi = i_{X_\xi}\tau_0$, observe the following. First, we can perform a unitary change of basis to a basis $e_0, \dots, e_n$ diagonalizing ξ, so that:

$$\xi = \sum_i 2\pi i \lambda_i e_i \otimes e_i^* = \sum_i \lambda_i \xi_i$$

is the diagonal matrix with eigenvalues $2\pi i\lambda_j$ for $j \in \{0, \dots, n\}$ and $\xi_i = 2\pi i e_i \otimes e_i^*$. In this basis the Hamiltonian becomes:

$$H_\xi([z]) = -\frac{\pi}{|z|^2}\sum_i \lambda_i |z_i|^2 = \sum_i \lambda_i H_{\xi_i}$$

We will show that the Hamiltonians H_{ξ_j} satisfy $dH_{\xi_i} = i_{X_{\xi_j}}\tau_0$, which will then imply that $dH_\xi = i_{X_\xi}\tau_0$ by linearity. First consider ξ_0. Observe that in the patch $(w_1, \dots, w_n) = \frac{1}{z_0}(z_1, \dots, z_n)$ we have:

$$H_{\xi_0} = -\frac{\pi}{1+|w|^2}; \quad dH_{\xi_0} = \frac{\pi}{(1+|w|^2)^2}\sum_i \bar{w}dw_i + wd\bar{w}_i$$

Furthermore in this patch the Fubini-Study form is given by:

$$\tau_0 = \frac{i}{2(1+|w|^2)^2}\sum_{i,j}((1+|w|^2)\delta_{ij} - \bar{w}_i w_j)dw_i \wedge d\bar{w}_j$$

Finally, to find X_{ξ_0} in this patch, we differentiate the action of $\exp(t\xi_0)$. We see that this action is:

$$\exp(t\xi_0)[z_0, \dots, z_n] = [e^{2\pi i t}z_0, \dots, z_n] \implies \exp(t\xi_0)(w_1, \dots, w_n) = (e^{-2\pi i t}w_1, \dots, e^{-2\pi i t}w_n)$$

$$\implies \frac{d}{dt}[\exp(t\xi_0)(w_1, \dots, w_n)]|_{t=0} = -2\pi i(w_1, \dots, w_n)$$

Thus the vector-field is $X_{\xi_0} = -2\pi i \sum_j w_j \partial_{w_j}$ in this patch. We then calculate that:

$$i_{X_{\xi_0}}\tau_0 = \frac{i}{2(1+|w|^2)^2}\sum_{i,j}((1+|w|^2)\delta_{ij} - \bar{w}_i w_j)(-2\pi i w_i d\bar{w}_j - 2\pi i \bar{w}_j dw_i)$$

$$= \frac{\pi}{(1+|w|^2)^2}\sum_j (1+|w|^2)(w_i d\bar{w}_i + \bar{w}_i dw_i) - \sum_{i,j}|w_i|^2 w_j d\bar{w}_j - \sum_{i,j}|w_j|^2 \bar{w}_i dw_i)$$

$$= \frac{\pi}{(1+|w|^2)^2}\sum_i w_i d\bar{w}_i + \bar{w}_i dw_i = dH_{\xi_0}$$

Thus we have proven that H_{ξ_0} is a Hamiltonian for X_{ξ_0} in the patch U_0 where $z_0 \neq 0$. Since the patch U_0 is an affine open dense set, and both H_{ξ_0}, X_{ξ_0} and τ_0 are smooth, it must be the case that the formula $i_{X_{\xi_0}}\tau_0 = dH_{\xi_0}$ holds over all of $\mathbb{C}P^{n-1}$. Thus H_{ξ_0} is a Hamiltonian for X_{ξ_0}. By symmetry, we may conclude the same for ξ_j for $j \in \{1, \dots, n\}$ as well (in this case, we can use the analogous patches U_j where $z_j \neq 0$)

and by linearity we can conclude that H_ξ is a Hamiltonian for X_ξ.

To conclude that μ is a true moment map, we must verify that the map $\xi \to H_\xi$ is a Lie algebra homomorphism. To check this, we consider the projection $\pi : \mathbb{C}^n - 0 \to \mathbb{C}P^{n-1}$. To calculate $\{H_\xi, H_\eta\} = dH_\xi(X_\eta)$ and verify that it is equal to $H_{[\xi,\eta]}$, we can take a lift of X_η through π (a $\tilde{X}_\xi$ vector-field on $\mathbb{C}^n - 0$ such that $\pi^*\tilde{X}_\xi = X_\xi$ on $\mathbb{C}P^{n-1}$) and check that $dH_\xi(\tilde{X}_\eta) = H_{[\xi,\eta]}$ (where H_ξ, H_η and $H_{[\xi,\eta]}$ are viewed as functions on $\mathbb{C}^n - 0$ via their definition $H_\xi(z) = \pi^* H_\xi([z]) = \frac{i}{2}\frac{\langle z^*, \xi z\rangle}{\langle z, z\rangle}$. We have a natural choice of $\tilde{X}_\xi$, namely the differential of the linear action $z \mapsto \exp(t\xi)z$ on $\mathbb{C}^n$. Thus we consider the vector-field $\tilde{X}_\xi = \xi z$, and observe that:

$$dH_\xi = \frac{i}{2}\frac{1}{|z|^4}(\sum_{i,j} |z|^2 \xi_{ij}(\bar{z}_i dz_j + z_j d\bar{z}_i) - \langle z^* \xi z\rangle \sum_i \bar{z}_i dz_i + z_i d\bar{z}_i)$$

$$\begin{aligned} dH_\xi(\tilde{X}_\eta) &= \frac{i}{2}\frac{1}{|z|^4}(\sum_{i,j,k} |z|^2 \xi_{ij}(\bar{z}_i \eta_{jk} z_k + z_j \bar{\eta}_{ik}\bar{z}_k) - \langle z^* \xi z\rangle \sum_{i,k} \bar{z}_i \eta_{ik} z_k + z_i \bar{\eta}_{ik}\bar{z}_k) \\ &= \frac{i}{2}\frac{1}{|z|^4}(\sum_{i,j,k} |z|^2(\bar{z}_i \xi_{ij}\eta_{jk} z_k + z_k \bar{\eta}_{ji}\xi_{jk}\bar{z}_i) - \langle z^* \xi z\rangle \sum_{i,k} \bar{z}_i \eta_{ik} z_k - \bar{z}_i \eta_{ik} z_k) \\ &= \frac{i}{2}\frac{1}{|z|^4}(\sum_{i,j,k} |z|^2(\bar{z}_i \xi_{ij}\eta_{jk} z_k - \bar{z}_i \eta_{ij}\xi_{jk} z_k)) = \frac{i}{2}\frac{\langle z^*[\xi,\eta]z\rangle}{|z|^2} = H_{[\xi,\eta]} \end{aligned}$$

Exercise 5.23 Identify the tangent space T_hG with the Lie algebra $\mathbf{g}$ by means of left translation $\mathbf{g} \to T_hG : \xi \mapsto L_h\xi$. Prove that the canonical 1-form λ_{can} on T^*G is the pull-back under the above diffeomorphism $T^*G \to G \times \mathbf{g}^*$ of the form:

$$\lambda_{(h,\eta)}(h\xi, \hat{\eta}) = \langle \eta, \xi\rangle$$

(for $g \in G, \xi \in \mathbf{g}$ and $\eta, \hat{\eta} \in \mathbf{g}^*$) on $G \times \mathbf{g}^*$. Prove the identity $H_\xi = i_{X_\xi}\lambda$ in the above example. Check that the moment map satisfies (5.6).

Solution 5.23 Let the map $\Phi : T^*G \to G \times \mathbf{g}^*$ be given by $\Phi(h, v^*) = (h, L_h^* v^*)$. Then the differential $d\Phi : T(T^*M) \to T(G \times \mathbf{g})$ is given by $d\Phi_{h,v^*}(\xi, \eta^*) = (h, L^*hv^*, \xi, L_h^*\eta^* + dL^*(\xi)v^*)$. Here $dL^*(\xi)v^*$ is ad-hoc notation denoting the term in the differential of $L_h^*v^*$ contributed by the L_h^* part. The pullback of the 1-form λ is:

$$[\Phi^*\lambda]_{h,v^*}(\xi, \eta^*) = \lambda_{h,L_h^*v^*}(\xi, L_h^*\eta^* + dL_h^*(\xi)v^*) = \langle L_h^*v^*, L_h^{-1}\xi\rangle = \langle v^*, L_h L_h^{-1}\xi\rangle = \langle v^*, \xi\rangle = \lambda_{\text{can},h,v^*}(\xi, \eta^*)$$

This makes the check of the identity $H_\xi = i_{X_\xi}\lambda$ relatively easy. We have $X_\xi(h, v^*) = (-L_h\xi, \eta^*(h, v^*))$ (i.e the G-component of the Hamiltonian vector-field on T^*G agrees with the vector-field generating the diffeomorphism $g : G \to G$). Thus we have:

$$i_{X_\xi}\lambda_{\text{can}} = \Phi^*\lambda_{h,v^*}(-L_h\xi, \eta^*(h, v^*)) = -\langle v^*, L_h\xi\rangle$$

The moment map satisfying (5.6) follows immediately from the fact that the map $T^*G \to G \times \mathbf{g}^*$ is a bundle map which is equivariant with respect to the G representations, and the fact that μ clearly satisfies (5.6) with respect to the action ψ_g on $G \times \mathbf{g}^*$. A more direct calculation is desirable though.

Exercise 5.25 Prove that the 2-form ω on $\mathcal{O}$ by (5.7) is closed. Prove that $X_\xi(\eta) = -\mathrm{ad}(\xi)^*\eta$ is the Hamiltonian vector field generated by $H_\xi(\eta) = \langle \eta, \xi \rangle$. Prove that the action of G on $\mathcal{O}$ is Hamiltonian.

Solution 5.25 It suffices to prove that the 2-form $\tau_\eta = \langle \eta, [\xi, \xi'] \rangle$ is closed on $\mathbf{g}$. Then since $\omega_\eta = (\tau_\eta)|_{\mathcal{O}}$, and closedness is preserved by restriction, we will know that ω_η is closed. Now let $\eta \in \mathcal{O}$, and take three tangent vectors $\mathrm{ad}(\alpha)^*\eta, \mathrm{ad}(\beta)^*\eta, \mathrm{ad}(\kappa)^*\eta$ at η. Then in local coordinates the gradient $\nabla\tau$ is given by:

$$\nabla_{\mathrm{ad}(\kappa)^*\eta}\tau_\eta(\mathrm{ad}(\alpha)^*\eta, \mathrm{ad}(\beta)^*\eta) = \langle \mathrm{ad}(\kappa)^*\eta, [\alpha, \beta] \rangle = \langle \eta, \mathrm{ad}(\kappa)[\alpha, \beta] \rangle = \langle \eta, [\kappa, [\alpha, \beta]] \rangle$$

Here we use the The exterior derivative is equal to the anti-symmetric form in $\mathrm{ad}(\alpha)^*\eta, \mathrm{ad}(\beta)^*\eta, \mathrm{ad}(\kappa)^*\eta$ achieved by anti-symmetrizing $\nabla\tau$ in α, β, κ. Since the Lie bracket is already anti-symmetric, this is:

$$d\tau_\eta(\mathrm{ad}(\alpha)^*\eta, \mathrm{ad}(\beta)^*\eta, \mathrm{ad}(\kappa)^*\eta) = 2\langle \eta, [\kappa, [\alpha, \beta]] + [\beta, [\kappa, \alpha]] + [\alpha, [\beta, \kappa]] \rangle = 0$$

Here we apply the Jacobi identity.

Moving on, we show that $X_\xi(\eta) = -\mathrm{ad}(\xi)^*\eta$ is generated by $H_\xi(\eta) = \langle \eta, \xi \rangle$. We calculate:

$$d[H_\xi]_\eta(\mathrm{ad}(\alpha)^*\eta) = \langle \mathrm{ad}(\alpha)^*\eta, \xi \rangle = \langle \eta, [\alpha, \xi] \rangle = i_{\mathrm{ad}(\xi)^*\eta} i_{\mathrm{ad}(\alpha)^*\eta}\omega_\eta = -i_{\mathrm{ad}(\alpha)^*\eta} i_{\mathrm{ad}(\xi)^*\eta}\omega_\eta = i_{\mathrm{ad}(\alpha)^*\eta} i_{-\mathrm{ad}(\xi)^*\eta}\omega_\eta$$

Thus H_ξ is the Hamiltonian for $X_\xi(\eta) = -\mathrm{ad}(\xi)^*\eta$. The adjoint action of G on $\mathbf{g}^*$, $\eta \to \mathrm{Ad}(g)^*\eta$, is generated by $\mathrm{ad} : \mathbf{g} \to \mathrm{Vect}(\mathbf{g}^*)$ given by $\xi \to X_\xi = \mathrm{ad}(\xi)^*\eta$. Indeed, we have for all $\nu \in \mathbf{g}$:

$$\langle \mathrm{ad}(\xi)^*\eta, \nu \rangle = \langle \eta, \mathrm{ad}(\xi)\nu \rangle = \langle \eta, \frac{d}{dt}(\mathrm{Ad}(\exp(t\xi))\nu)|_{t=0} \rangle$$

$$= \frac{d}{dt}\langle \eta, \mathrm{Ad}(\exp(t\xi))\nu \rangle|_{t=0} = \frac{d}{dt}\langle \mathrm{Ad}(\exp(t\xi))^*\eta, \nu \rangle|_{t=0} = \langle \frac{d}{dt}(\mathrm{Ad}(\exp(t\xi))^*\eta)|_{t=0}, \nu \rangle$$

Thus, the generating vector-fields of the Ad action of G on $\mathcal{O}$ are the vector-fields $X_\xi(\eta) = \mathrm{ad}(\xi)^*\eta$ and they are Hamiltonian by our previous calculations. So Ad is weakly Hamiltonian. To show that it is (strongly) Hamiltonian, we observe that:

$$H_{[\alpha,\beta]}(\eta) = \langle \eta, [\alpha, \beta] \rangle = \omega_\eta(-\mathrm{ad}(\alpha)^*\eta, -\mathrm{ad}(\beta)^*\eta) = \omega_\eta(X_\alpha, X_\beta) = \{H_\alpha, H_\beta\}$$

Exercise 5.26 For every $\eta \in \mathcal{O}$ there is a natural diffeomorphism:

$$G/G_\eta \simeq \mathcal{O} \qquad G_\eta = \{g \in G | \mathrm{Ad}(g)^*\eta = \eta\}$$

induced by the map $g \mapsto \mathrm{Ad}(g^{-1})^*\eta$. The Lie algebra of G_η is given by $\mathbf{g}_\eta = \{\xi = \mathbf{g} | \mathrm{ad}(\xi)^*\eta = 0\}$. Prove that $\mathbf{g}_\eta$ is the kernel of the skew form:

$$\mathbf{g} \times \mathbf{g} \to \mathbb{R} : (\xi, \xi') \to \langle \eta, [\xi, \xi'] \rangle$$

Give a direct proof that this form determines a symplectic structure on G/G_η.

Solution 5.26 The first part is simple enough. We see that for a fixed $\eta \in \mathcal{O}$ and $\xi \in \mathbf{g}$ we have:

$$\langle \eta, [\xi, \xi'] \rangle = 0 \text{ for all } \xi' \in \mathbf{g} \iff \langle \mathrm{ad}(\xi)^*\eta, \xi' \rangle = 0 \text{ for all } \xi' \in \mathbf{g} \iff \mathrm{ad}(\xi)^*\eta = 0 \iff \xi \in \mathbf{g}_\eta$$

To prove that the above bilinear form induces a symplectic form on G/G_η, we argue as so. Define the 2-form ω_g for any $g \in G$ and $L_g\xi, L_g\xi' \in T_gG$ and $\xi, \xi' \in \mathbf{g} = T_0G$ by:

$$\omega_g(L_g\xi, L_g\xi') = \langle \eta, [\xi, \xi'] \rangle = \langle L_g^*\eta, [L_g\xi, L_g\xi'] \rangle$$

We observed that $\omega_g(L_g\xi, L_g\xi') = 0$ for some ξ and all ξ' if and only if $\xi \in \mathbf{g}_\eta$, i.e if and only if $L_g\xi$ is in the tangent space of the G_η orbit of g. Furthermore, ω_g itself is G invariant in the sense that $\omega_{L_hg}(L_{L_hg}\xi, L_{L_hg}\xi') = L_g(L_g\xi, L_g\xi')$. Thus ω_g descends to a well-defined, non-degenerate 2-form on G/G_η via $\tilde{\omega}_{[g]}([L_g\xi], [L_g\xi']) = \omega_g(L_g\xi, L_g\xi')$.

To show that $\tilde{\omega}_g$ is closed, it suffices to show that ω_g is closed. This is because if we consider the quotient map $q : G \to G/G_\eta$, the pullback map $q^* : \Omega^*(G/G_\eta) \to \Omega^*(G)$ is injective and $dq^*\alpha = q^*d\alpha$ for any $\alpha \in \Omega^*(G/G_\eta$. Thus $d\tilde{\omega}_\eta = 0$ if and only if $q^*d\tilde{\omega}_\eta = dq^*\tilde{\omega}_\eta = d\omega_\eta = 0$.

To see that ω_η is closed, observe that:

$$d\omega_g(L_g\alpha, L_g\beta, L_g\gamma)$$

$$= d[\omega(L_g\beta, L_g\gamma)](L_g\alpha) + (-1)d[\omega(L_g\alpha, L_g\gamma)](L_g\beta) + d[\omega(L_g\alpha, L_g\beta)](L_g\gamma)$$
$$+(-1)\omega_g([L_g\alpha, L_g\beta], L_g\kappa) + \omega([L_g\alpha, L_g\kappa], L_g\beta) + (-1)\omega([L_g\beta, L_g\kappa], L_g\alpha)$$

Here we are looking at $L_g\alpha, L_g\beta, L_g\gamma$ as vector-fields on G, and we are using a well-known invariant formula for the exterior derivative. Note that the above formula would hold for any choice of $X_\alpha, X_\beta, X_\kappa$ with $X_\alpha(g) = L_g\alpha$ and similarly for β, κ, but our choice of $X_\alpha = L_g\alpha$ and so on makes things particularly easy.

By $d[\omega(L_g\beta, L_g\gamma)]$ we mean df where f is the function $f = \omega(L_g\beta, L_g\gamma)$. Then by $df(X)$ we mean the usual i_Xdf. The first thing to notice about the above calculation is that $d[\omega(L_g\beta, L_g\gamma)] = d[\langle \eta, [\beta, \gamma] \rangle] = 0$ because it is constant with respect to g. The same statement holds for the other 2 terms like this, so the whole second line above vanishes. The second thing to note is that $[L_g\alpha, L_g\beta] = L_g[\alpha, \beta]$ (i.e the map $\mathbf{g} \to \mathrm{Vect}(G)$ from the Lie algebra to the invariant vector-fields is a Lie algebra homomorphism). With these two facts we may continue with only the third line, writing:

$$= \omega_g(L_g[\alpha, \beta], L_g\kappa) + \omega(L_g[\alpha, \kappa], L_g\beta) + (-1)\omega(L_g[\beta, \kappa], L_g\alpha)$$

$$= \langle \eta, -[[\alpha,\beta],\kappa] + [[\alpha,\kappa],\beta] - [[\beta,\kappa],\alpha]\rangle = \langle \eta, [[\beta,\alpha],\kappa] + [[\alpha,\kappa],\beta] + [[\kappa,\beta],\alpha]) = 0$$

The last step is an application of the Jacobi identity.

Exercise 5.27 Show that the symplectic action of a connected semi-simple group is always Hamiltonian.

Solution 5.27 Let G be a semi-simple Lie group with a symplectic action $\phi : G \times M \to M$ on symplectic manifold (M,ω). Let the associated Lie algebra map $\mathbf{g} \to \mathrm{Vect}(M)$ be denoted by $\xi \mapsto X_\xi$.

We begin by proving that this action is weakly Hamiltonian. Fix a $\xi \in \mathbf{g}$. Since $\mathbf{g}$ is semi-simple, we have $\xi = [\eta,\nu]$ for some $\eta,\nu \in \mathbf{g}$ (since $\mathbf{g} = [\mathbf{g},\mathbf{g}]$). Define the smooth function H_ξ as $H_\xi = \omega(X_\eta, X_\nu)$. Observe then that:

$$i_{X_\xi}\omega = i_{[X_\eta,X_\nu]}\omega = \mathcal{L}_{X_\eta}(i_{X_\nu}\omega) = (di_{X_\nu} + i_{X_\nu}d)(i_{X_\nu}\omega)$$
$$= d(i_{X_\eta}i_{X_\nu}\omega) + i_{X_\eta}\mathcal{L}_{X_\nu}\omega = \omega(X_\eta,X_\nu) = H_\xi$$

Here we use (in order) the Leibniz rule for the Lie derivative and the fact that $\mathcal{L}_{X_\eta}\omega = 0$, then Cartan's magic formula, then the fact that $\mathcal{L}_{X_\nu}\omega = 0$.

Thus we can choose a linear map $\mathbf{g} \to C^\infty(M)$ given by $\xi \mapsto H_\xi$ where H_ξ is a Hamiltonian for X_ξ for all ξ. By Lemma 5.14 we know that $\{H_\eta, H_\nu\} - H_{[\xi,\nu]} = \tau(\xi,\eta)$ where τ is a 2-cocycle in the Lie algebra chain groups composed of anti-symmetric 2-forms on $\mathbf{g}$. By the hint (the vanishing of the second Lie algebra cohomology $H^2(\mathbf{g})$) we know that $\tau(\xi,\eta) = \sigma([\xi,\eta])$ is a coboundary. Thus we may redefine $\xi \mapsto H_\xi$ to $\xi \mapsto H_\xi + \sigma(\xi)$ to get a map which yields a map of Lie algebras with $C^\infty(M)$ given a Lie algebra structure via the Poisson bracket. The action is thus (strongly) Hamiltonian with moment map $\mu : M \to \mathbf{g}^*$ defined by $\langle \mu(p), \xi\rangle = H_\xi$.

Exercise 5.28 Suppose that G acts in a Hamiltonian way on the symplectic manifolds (M_j,ω_j) for $j = 1,2$ with moment maps $\mu_j : M_j \to \mathbf{g}^*$. Prove that the obvious diagonal action $G \to \mathrm{Symp}(M_1 \times M_2)$ is Hamiltonian with moment map $\mu : M_1 \times M_2 \to \mathbf{g}^*$ given by $\mu(p_1,p_2) = \mu_1(p_1) + \mu_2(p_2)$ for $p_j \in M_j$.

Solution 5.28 Consider a $\xi \in \mathbf{g}$. Observe that the vector-field X_ξ generating the diagonal action on $M_1 \times M_2$ splits as $X_\xi(p_1,p_2) = X^1_\xi(p_1) + X^2_\xi(p_2) \in \pi_1^* T_{p_1}M_1 \oplus \pi_2^* T_{p_2}M_2 = T_{p_1,p_2}(M_1 \times M_2)$. The sub-bundle $\pi_1^*TM_1 \subset T(M_1 \times M_2)$ is the sub-bundle tangent to the $M_1 \times p_2$ sub-manifolds and can be picked out as the kernel of the projection map $d\pi_2 : T(M_1 \times M_2) \to TM_2$. We can analogously define $\pi_2^*TM_2$. Likewise, a splitting $T^*(M_1 \oplus M_2) = T^*M_1 \oplus T^*M_2$ is induced by the splitting of the cotangent bundle. The vector-field X^1_ξ is then defined as the unique vector-field in the sub-bundle $T_{p_1}M_1$ whose image $d\pi_1(X^1_\xi)$ under the bundle map $d\pi_1 : \pi_1^*T_{p_1}M_1 \to TM_1$ is the Hamiltonian vector-field on M_1 corresponding to ξ. This is well-defined because the map $d\pi_1$ is an isomorphism on the fibers. We define X^2_ξ analogously.

Thus, letting $\omega = \omega_1 \oplus \omega_2$ we have:

$$i_{X_\xi}\omega = \pi_1^* i_{X^1_\xi}\omega_1 + \pi_2^* i_{X^2_\xi}\omega_2 = \pi_1^* d\langle \mu_1,\xi\rangle + \pi_2^* d\langle \mu_2,\xi\rangle = d\langle \pi_1^*\mu_1 + \pi_2^*\mu_2, \xi\rangle \in T^*_{p_1}M_1 \oplus T^*_{p_2}M_2 = T^*_{p_1,p_2}(M_1 \times M_2)$$

But $[\pi_1^*\mu_1 + \pi_2^*\mu_2](p_1,p_2) = \mu_1(p_1) + \mu_2(p_2) = \mu(p_1,p_2)$. Thus this precisely says that $H_\xi = \langle \mu, \xi\rangle$ is a

Hamiltonian for X_ξ. A similar computation shows that we have a Lie algebra homomorphism, i.e:

$$\langle \mu, [\xi, \eta] \rangle = \pi_1^* \langle \mu_1, \xi \rangle + \pi_2^* \langle \mu_2, \xi \rangle = \pi_1^* (\omega_1 (d\pi_1 X_\xi^1, d\pi_1 X_\eta^1)) + \pi_2^* (\omega_2 (d\pi_2 X_\xi^2, d\pi_2 X_\eta^2))$$

$$= (\pi_1^* \omega_1)(X_\xi^1, X_\eta^2) + (\pi_2^* \omega_2)(X_\xi^2, X_\eta^2) = (\pi_1^* \omega_1 + \pi_2^* \omega_2)(X_\xi^1 + X_\xi^2, X_\eta^1 + X_\eta^2) = \omega(X_\xi, X_\eta) = \{H_\xi, H_\eta\}$$

Exercise 5.29 Use the previous exercise to calculate the moment map $\mu_n : \mathbb{C}^n \to \mathbb{R}^n$ of the action of the n-torus $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ on $\mathbb{C}^n$ given by:

$$(\theta_1, \dots, \theta_n) \cdot (z_1, \dots, z_n) = (e^{2\pi i \theta_1} z_1, \dots, e^{2\pi i \theta_n} z_n)$$

If $i : \mathbb{T}^k \to \mathbb{T}^n$ is a linear embedding and $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the dual projection show that:

$$\mu_k = \pi \circ \mu_n : \mathbb{C}^n \to \mathbb{R}^k$$

is the moment map for the induced action of $\mathbb{T}^k$.

Solution 5.29 The moment map must be $\mu(z) = -\pi(|z_1|^2, |z_2|^2, \dots, |z_n|^2)$. To see this, consider any $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{u}(1)^n \simeq \mathbb{R}^n$. We have the $\mathbb{R}$ action on $\mathbb{C}^n$ generated by X_θ, i.e the action $t \cdot (z_1, \dots, z_n) = (e^{2\pi i \theta_1 t} z_1, \dots, e^{2\pi i \theta_n t} z_n)$. This is the "diagonal" $\mathbb{R}$ action on $\mathbb{C}^n$ induced by the n $\mathbb{R}$ actions on $\mathbb{C}$ given by $t \cdot z = e^{2\pi i \theta_j t}$ for $j \in \{1, \dots, n\}$. By the previous result in the previous Exercise 5.28, the Hamiltonian for this action is the sum of the Hamiltonians for each of the $\mathbb{R}$ actions pulled back along the n projection maps. More simply:

$$H_\theta(z) = -\pi \sum_j \theta_j |z_i|^2 = \langle \mu, \theta \rangle$$

Since our θ was arbitrary, this shows that $\langle \mu, \theta \rangle$ is a Hamiltonian for all $\theta \in \mathbf{u}(1)^n$, and the action is weakly Hamiltonian. Then the fact that $U(1)^n$ is abelian implies trivially that $\langle \mu, [\xi, \eta] \rangle = \{\langle \mu, \xi \rangle, \langle \mu, \eta \rangle\}$, since everything commutes, so both expressions vanish. More directly, we see that any combination of $|z_1|^2, \dots, |z_n|^2$ will be constant along $U(1)^n$ orbits, so $\{X_\xi, X_\eta\} = dH_\xi(X_\eta) = \mathcal{L}_{X_\eta} H_\xi = 0$ for any ξ, η, sicne X_η is an infinitesimal rotation of this form and H_ξ is a combination of $|z_i|^2$ terms.

To see that $\mu_k = \pi \circ \mu_n$, denote the Lie alegebras of $\mathbb{T}^k$ and $\mathbb{T}^n$ as $\mathbf{t}^k$ and $\mathbf{t}^n$ respectively. Then observe that the map $\mathbf{t}^k \to \mathrm{Vect}(M)$ factors as $\mathbf{t}^k \to \mathbf{t}^n \to \mathrm{Vect}(M)$ where the first map is the map $di_0 : \mathbf{t}^k \to \mathbf{t}^n$ induced by the Jacobian of i at 0. Thus $\xi \mapsto X_{di_0(\xi)}$. In particular, the Hamiltonian is given by $H_\xi = H_{di_0(\xi)} = \langle \mu, di_0 \xi \rangle = \langle (di_0)^* \mu, \xi \rangle$. But since i is given as the quoitent of a linear map $i : \mathbb{R}^k \to \mathbb{R}^n$, $di_0 = i$ via the identifications $\mathbb{R}^n \simeq T_0 \mathbb{R}^n \simeq T_0 \mathbb{T}^n = \mathbf{t}^n$ (with the analogous identifcation for $\mathbb{R}^k$). So $H_\xi = \langle i^* \mu, \xi \rangle = \langle \pi \mu, \xi \rangle$ (since the dual projection π is precisely the adjoint of i with respect to the dual pairing on $\mathbf{g}^* \times \mathbf{g}$. Furthermore the fact that $\xi \to \langle \mu, \xi \rangle$ was a Lie algebra homomorphism ensures that the same is true for $\xi \to \langle \pi \mu, \xi \rangle$, since in particular:

$$\langle \pi \mu, [\xi, \eta] \rangle = 0 = \{\langle \mu, di_0 \xi \rangle, \langle \mu, di_0 \eta \rangle\} = \{\langle \pi \mu, \xi \rangle, \langle \pi \mu, \eta \rangle\}$$

Exercise 5.39 Use a construction similar to that in Example 5.38 to interpret the composition of symplectomorphisms in terms of symplectic quotients.

Solution 5.39 Let $M_1, \omega_i)$ for $i \in \{1,2,3\}$ be 3 symplectic manifolds, $\phi_{12} : M_1 \to M_2$ and $\phi_{23} : M_2 \to M_3$ be two symplectomorphisms. Consider the manifold $X = M_1 \times \bar{M}_2 \times M_2 \times \bar{M}_3$ with form $\omega_1 \times (-\omega_2) \times \omega_2 \times (-\omega_3)$. We have a coisotropic subspace $C = M_1 \times \Delta \times \bar{M}_3$ with isotropic leaves $p \times \Delta \times q$ and a Lagrangian subspace $L = \Gamma_{12} \times \Gamma_{23}$, where the two components are the graphs of ϕ_{12} and ϕ_{23} respectively.

Let M be the symplectic quotient of C by the foliation $p \times \Delta \times q$. The map $[p \times \Delta \times q] \mapsto (p, q)$ is smooth, since it is induced by a smooth map $M_1 \times \Delta \times \bar{M}_3 \to M_1 \times \bar{M}_3$ which is constant on leaves, and its trivial to check that the map is in fact a symplectomorphism. Furthermore, the Lagrangian $\Gamma_{12} \times \Gamma_{23}$ intersects $M_1 \times \Delta \times \bar{M}_3$ transversely.

To see that the intersection is good, look at a point $x = (p, q, q, r) = (p, \phi_{12}(p), \phi_{12}(p), \phi_{23}(\phi_{12}(q)))$. Since $\dim(C) = 3n$ and $\dim(L) = 2n$ it suffices to show that $T_xC + T_xL = T_xX$ to show that the intersection is transverse. But we see that the tangent vectors to T_xC at this point are all vectors of the form $u \oplus v \oplus v \oplus w \in T_xX$. Meanwhile, tangent vectors to L are of the form $a \oplus D\phi_{12}a \oplus b \oplus D\phi_{23}b$. But here we can pick b to be anything and a to be 0. Thus for any $a \oplus b \oplus c \oplus d \in TX$ we have:

$$a \oplus b \oplus c \oplus d = [a \oplus b \oplus b \oplus (d - D\phi_{23}(c-b))] + [0 \oplus 0 \oplus \oplus (c-b) \oplus (D\phi_{23}(c-b))] \in T_xC \oplus T_xL$$

Thus the intersection is transverse. It is clear that for a fixed p and q, the leaf $p \times \Delta \times q$ intersects L at most at one point, in which case that point is $p \times \phi_{12}(p) \times \phi_{12}(p) \times \phi_{23}(\phi_{12}(q))$. Thus the image under the map $C \to M$ of $L \cap C$ is precisely Γ_{13}, the graph of $\phi_{23} \circ \phi_{12}$.

Exercise 5.42 Let $\mu : M \to \mathbf{g}^*$ be the moment map of a Hamiltonian group action and $\mathcal{O} \subset \mathbf{g}^*$ be a coadjoint orbit. Prove that $\mathcal{O}$ contains a regular value of μ if and only if every point in $\mathcal{O}$ is a regular value of μ. In view of (5.8) this is equivalent to the condition:

$$\mathbf{g}^* = T_{\mu(p)}\mathcal{O} + \{d\mu(p)v | v \in T_pM\}$$

For every $p \in \mu^{-1}(\mathcal{O})$. But this means that μ is transverse to $\mathcal{O}$.

Solution 5.42 One direction of implication is trivial. Thus assume $\mathcal{O}$ contains a regular value, i.e a point η where $d\mu_p$ is full rank for all p with $\mu(p) = \eta$. Then if $\eta' = \mathrm{Ad}(g^{-1})^*\eta$. Then for any q with $\mu(q) = \eta'$, the point $p = \psi_{g^{-1}}(q)$ satisfies $\mu(q) = \mu(\psi_g(p))$, so $p \in \mu^{-1}(\eta)$, and $\eta' = \mathrm{Ad}(g^{-1})^*\mu(p)$. Thus:

$$d[\mathrm{Ad}(g^{-1})^*]_{\mu(p)} \circ d\mu_p = d(\mathrm{Ad}(g^{-1})^*\mu_p) = d(\mu \circ \psi_g(p)) = d\mu_{\psi_g(p)} \circ d\psi_{g,p} = d\mu_q \circ d\psi_{g,p}$$

$$d\mu_q = d[\mathrm{Ad}(g^{-1})^*]_{\mu(p)} \circ d\mu_p \circ d\psi_{g,p}^{-1}$$

Thus, since the rank of $d\mu_p$ is maximal and the maps $d[\mathrm{Ad}(g^{-1})^*]_{\mu(p)}, d\psi_{g,p}^{-1}$ are isomorphisms, we may conclude that $d\mu_q$ is of maximal rank.

Exercise 5.43 Consider the obvious action of $U(k)$ on the space $\mathbb{C}^{n\times k}$ of complex $n\times k$-matrices with the standard symplectic structure. Identify the Lie algebra $\mathbf{u}(k)$ with its dual as above and prove that the moment map of the action is given by:

$$\mu(B) = \frac{1}{2i}B^*B$$

for $B\in\mathbb{C}^{n\times k}$. Deduce that $\mu^{-1}(1/2i)$ is the space of unitary k-frames $B\in\mathbb{C}^{n\times k}$ with $B^*B=1$ and the quotient:

$$\mu^{-1}(1/2i)/U(k) = G(k,n)$$

is the Grassmanian.

Solution 5.43 Let z_{ab} with $a\in\{1,\dots,n\}$ and $b\in\{1,\dots,k\}$ be the complex coordinates on $\mathbb{C}^{n\times k}$. Let $A=(A_{bc})\in\mathbf{u}(k)$ be an anti-Hermitian matrix, $U(t)=e^{At}$ and $Z=(z_{ab})\in\mathbb{C}^{n\times k}$. Then $X_A(Z)=\frac{d}{dt}(ZU(t))|_{t=0}=ZA\in\mathbb{C}^{n\times k}$. If we denote the $z,\bar z$ basis of the tangents space as $\partial_{z_{ab}},\partial_{\bar z_{ab}}=\partial_{ab},\partial_{\overline{ab}}$ coordinates we thus have:

$$X_A(Z)=\sum_{a,b}(\sum_c z_{ac}A_{cb})\partial_{ab}+(\sum_c \bar z_{ac}\bar A_{cb})\partial_{\overline{ab}}$$

Thus we must find a Hamiltonian H_A with Hamiltonian vector-field equal to this. The standard form is:

$$\omega=\frac{i}{2}\sum_{a,b}dz_{ab}\wedge d\bar z_{ab}$$

Thus we have:

$$i_{X_A}\omega=\frac{i}{2}\sum_{a,b,c}(z_{ac}A_{cb})d\bar z_{ab}-(\bar z_{ac}\bar A_{cb})dz_{ab}=\frac{i}{2}\sum_{a,b,c}(z_{ac}A_{cb})d\bar z_{ab}-(\bar z_{ab}\bar A_{bc})dz_{ac}$$

$$=\frac{i}{2}\sum_{a,b,c}(z_{ac}A_{cb})d\bar z_{ab}-(\bar z_{ab}A^*_{cb})dz_{ac}=\frac{i}{2}\sum_{a,b,c}(z_{ac}A_{cb})d\bar z_{ab}+(\bar z_{ab}A_{cb})dz_{ac}=d\left(\frac{i}{2}\sum_{a,b,c}z_{ac}A_{cb}\bar z_{ab})\right)$$

$$=d(\frac{i}{2}\mathrm{tr}(ZAZ^*))=d(\mathrm{tr}(\frac{1}{2i}Z^*ZA^*))=d\langle\frac{1}{2i}Z^*Z,A\rangle$$

Observe above that we use the fact that $Y^\dagger=-Y$ and the fact that the invariant inner product is given by $\langle A,B\rangle=\mathrm{tr}(AB^*)$. Thus we may define $\mu:\mathbb{C}^{n\times k}\to\mathbf{u}(k)$ by $\mu(B)=\frac{1}{2i}B^*B$ and $\langle\mu(B),A\rangle=H_A$ is a Hamiltonian for X_A. It remains to check that this moment map induces a Lie algebra homomorphism. But we see that:

$$\{H_X,H_Y\}=dH_X(X_Y)=(\frac{i}{2}\sum_{a,b,c}(z_{ac}X_{cb})d\bar z_{ab}-(\bar z_{ac}\bar X_{cb})dz_{ab})(\sum_{a,b}(\sum_d z_{ad}Y_{db})\partial_{ab}+(\sum_d \bar z_{ad}\bar Y_{db})\partial_{\overline{ab}})$$

$$=\frac{i}{2}\sum_{a,b,c,d}z_{ac}X_{cb}\bar z_{ad}\bar Y_{db}-\bar z_{ac}\bar X_{cb}z_{ad}Y_{db}=\frac{i}{2}\sum_{a,b,c,d}z_{ab}X_{bc}\bar z_{ad}\bar Y_{dc}-\bar z_{ad}\bar X_{dc}z_{ab}Y_{bc}$$

$$=\frac{i}{2}\sum_{a,b,c,d}z_{ab}[X_{bc}Y^*_{cd}-Y_{bc}X^*_{cd}]\bar z_{ad}=\sum_{a,b,c,d}\frac{1}{2i}z_{ab}[X_{bc}Y_{cd}-Y_{bc}X_{cd}]\bar z_{ad}=H_{[X,Y]}$$

Thus we have that $\mu^{-1}(1/2i)$ is the space of unitary frames, since it is exactly the matrices such that $B^*B = 1$. Thus $\mu^{-1}(1/2i)/U(n) = \{\text{unitary}k - \text{frames}\}/U(n)$, which is one of the homogeneous space realtization of $\mathrm{Gr}(n, k, \mathbb{C})$.

Exercise 5.44 (Toric Manifolds) Consider the action of the k-torus $\mathbb{T}^k$ on $\mathbb{C}^n$ which is induced by the inclusion $\mathbb{T}^k \to \mathbb{T}^n$ as in Exercise 5.29. A symplectic manifold is said to be toric if it is a symplectic quotient $M_{\mathcal{O}}$ formed from this action, where $\mathcal{O} \subset \mathbb{R}^n$ s a coadjoint orbit of $\mathbb{T}^k$. Of course, since $\mathbb{T}^k$ is abelian, $\mathcal{O}$ is simply a point. Show that any product of the form:

$$(M, \omega) = (S^2 \times \cdots \times S^2, \lambda_1\sigma \times \cdots \times \lambda_m\sigma)$$

is toric, where σ is an area from on S^2 and $\lambda_i > 0$. More generally, any product of projective spaces is toric. Show that any symplectic toric manifold of dimension $2m$ supports a Hamiltonian action of the torus $\mathbb{T}^m$ and calculate its moment map.

Solution 5.44 To see that the M above is toric, consider the standard $\mathbb{T}^{2n} = \mathbb{T}^n \times \mathbb{T}^n$ action on $\mathbb{C}^n \times \mathbb{C}^n$ with coordinates $z_1, \dots, z_n, w_1, \dots, w_n$, and consider the diagonal embedding $\mathbb{T}^n \to \mathbb{T}^{2n}$ given by $g \mapsto g \times g$. Then this $\mathbb{T}^n$ action is just the product of the diagonal $\mathbb{T}^1$ actions on the (z_j, w_j)-planes and thus the moment map is simply:

$$(z_1, \dots, z_n, w_1, \dots, w_n) \mapsto -\pi \cdot (|z_1|^2 + |w_1|^2, |z_2|^2 + |w_2|^2, \dots, |z_n|^2 + |w_n|^2) \in \mathbb{R}^n = \mathbf{t}^n$$

i.e the product of the moment maps of the individual $\mathbb{T}^1$ actions. Now consider the point $\lambda = -\pi \cdot (\lambda_1, \dots, \lambda_n) \in \mathbf{t}^n$ and Then:

$$\mu^{-1}(\lambda) = \mu_1^{-1}(\lambda_1) \times \mu_2^{-1}(\lambda_2) \times \cdots \times \mu_n^{-1}(\lambda_n) = \lambda_1 S^3 \times \lambda_2 S^3 \times \cdots \times \lambda_n S^3$$

Here $\lambda_j S^3$ is the radius $\lambda_j^{1/2}$ sphere in the (z_j, w_j)-plane in $\mathbb{C}^n \times \mathbb{C}^n$.

Now observe that $\omega_0|_{\lambda_j S^3} = \lambda_j \pi^* \tau_0|_{\lambda_j S^3}$ where τ_0 is the standard Fubini-Study form on $\mathbb{C}P^1 = S^2$ and $\pi : \mathbb{C}^2 - 0 \to \mathbb{C}P^1$ is the quotient map $(x_1, x_2) \mapsto [x_1, x_2]$. In fact we see that, considering the (z_j, w_j)-plane as $\mathbb{C}^2$ and looking at $\pi : \mathbb{C}^2 \to \mathbb{C}P^1$ we have:

$$\pi^*\tau_0 = \frac{i}{2(|z|^2 + |w|^2)}(dz_j \wedge d\bar{z}_j + dw_j \wedge d\bar{w}_j)$$

$$-\frac{i}{2(|z_j|^2 + |w_j|^2)^2}(\bar{z}_j z_j dz_j \wedge d\bar{z}_j + \bar{w}_j z_j dw_j \wedge d\bar{z}_j + \bar{z}_j w_j dz_j \wedge d\bar{w}_j + \bar{w}_j w_j dw_j \wedge d\bar{w}_j)$$

Restricted to $\lambda_j S^3$ we have:

$$\pi^*\tau_0|_{S^3} = \frac{i}{2\lambda_j}(dz_j \wedge d\bar{z}_j + dw_j \wedge d\bar{w}_j)$$

$$-\frac{i}{2\lambda_j^2}(\bar{z}_j z_j dz_j \wedge d\bar{z}_j + \bar{w}_j z_j dw_j \wedge d\bar{z}_j + \bar{z}_j w_j dz_j \wedge d\bar{w}_j + \bar{w}_j w_j dw_j \wedge d\bar{w}_j)$$

And the same calculation as in Solution 5.3 shows that the latter part vanishes identically on $\lambda_j S^3$, so:

$$\pi^*\tau_0|_{\lambda_j S^3} = \frac{1}{\lambda_j}\frac{i}{2}(dz_j \wedge d\bar{z}_j + dw_j \wedge d\bar{w}_j) = \frac{1}{\lambda_j}\omega_0|_{\lambda_j S^3}$$

In particular, if we take the group quotient $\lambda_j S^3/U(1) = \pi(\lambda_j S^3) = \mathbb{C}P^1$ then the equivariant 2-form $\omega_0|_{\lambda_j S^3}$ descends to the 2-form $\lambda_j \tau_0$. In other words:

$$(\mu_j^{-1}(\lambda_j)/\mathbb{T}^1, \omega_0/\mathbb{T}^1) \simeq (\lambda_j S^3/U(1), \lambda_j \tau_0)$$

Thus we have:

$$\mu^{-1}(\lambda)/\mathbb{T}^n = \times_j(\mu_j^{-1}(\lambda_j)/\mathbb{T}^n, \omega_0/\mathbb{T}^1) \simeq (\mathbb{C}P^1, \lambda_j\tau_0) = (M, \omega)$$

and we have realized (M, ω) as a toric manifold.

To see the next part, consider a linear torus embedding $\mathbb{T}^k \to \mathbb{T}^n$, the dual projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ and an arbitrary $p \in \mathbb{R}^k$. Let $M = \mathbb{C}^n//\mathbb{T}^k = [\pi\mu]^{-1}(p)/\mathbb{T}^k$ be the toric manifold associated to this data. Assuming that p is a regular value, we know that the dimension of $2n - 2k = 2(n-k) =: 2m$.

Now observe that M inherits a symplectic action of $\mathbb{T}^m = \mathbb{T}^n/\mathbb{T}^k$ defined for $[g] \in \mathbb{T}^n/\mathbb{T}^k$ and $[x] \in [\pi\mu]^{-1}(p)$ as:

$$[g] \cdot [x] \mapsto [gx]$$

We verify that this is well-defined. First observe that if $x \in [\pi\mu]^{-1}(p)$ then $g \in [\pi\mu]^{-1}(p)$ since $\mu(p) = \mu(gp)$ and thus $\pi\mu(p) = \pi\mu(gp)$. Thus we must show that our choice of x and g doesn't matter. To see this, observe that if $[h] = [g]$ so that $g = fh$ for $f \in \mathbb{T}^k$, then gx and hx differ by multiplication by f, and thus $[gx] = [hx]$. Likewise if $[x] = [y]$ so that $x = fy$ for $f \in \mathbb{T}^k$ then $gx = fgy$ so $[gx] = [gy]$. Thus the action $\mathbb{T}^m$ on M is well-defined.

The fact that this action is symplectic follows from the fact that $\omega|_{[\pi\mu]^{-1}(p)}$ is equivariant under the full $\mathbb{T}^n$ action. Indeed, if we denote the quotient symplectic form as $\tilde{\omega}$, and let $[v], [w] \in T_{[q]}M$ for $[q] \in M$ then $d[g]_{[q]}[v] = [dg_q v]$, so:

$$[g]^*\tilde{\omega}_{[p]}([v], [w]) = \tilde{\omega}_{[gp]}(d[g]_{[q]}[v], d[g]_{[q]}[w]) = \omega_{gp}(dg_q v, dg_q w) = \omega_p(v, w) = \tilde{\omega}_{[p]}([v], [w])$$

Now we argue that this action is Hamiltonian. Given $\xi \in \mathbf{t}^n$ let X_ξ be the symplectic vector-field generating the infinitesimal action on $\mathbb{C}$. Then observe that $g_* X_\xi = X_{L_g \xi}$ since (COMING BACK TO THIS ONE).

Exercise 5.45 Examine the manifold $M_{\mathcal{O}} = \mu^{-1}(\mathcal{O})/G$ in the case where $M = T^*G \simeq G \times \mathbf{g}^*$ with the action in Exercise 5.22.

Solution 5.45 Consider $M = G \times \mathbf{g}^*$ with the G action $g \cdot (h, \xi) = (hg^{-1}, \mathrm{Ad}(g^{-1})^*\xi)$. The moment map $\mu : G \times \mathbf{g}^* \to \mathbf{g}^*$ was observed in Exercise 5.22 and the associated example to be given by $\mu(h, \xi) = -\xi$.

Thus, if $\mathcal{O} = \mathcal{O}(-\xi)$ is an orbit in $\mathbf{g}^*$ under the adjoint action $g \cdot \eta = \mathrm{Ad}(g^{-1})^*\eta$ of $-\xi \in \mathcal{O}$ then:

$$\mu^{-1}(\mathcal{O}) = \{(h,\eta) | \eta = \mathrm{Ad}(g^{-1})^*\xi\} = G \times \mathcal{O}(\xi)$$

Now consider the map $\Phi : \mu^{-1}(\mathcal{O})/G \to G/\mathrm{Stab}(\xi)$ given by:

$$[h,\eta] \mapsto [hg^{-1}] \in G/\mathrm{Stab}(\xi) \text{ with } g \text{ s.t } \mathrm{Ad}(g^{-1})^*\eta = \xi$$

We claim that this is a homeomorphism (probably a diffeomorphism as well). We show that it is well defined and a bijection. First, suppose that $(a,\eta) = g \cdot (b,\nu) = (bg^{-1}, \mathrm{Ad}(g^{-1})^*\nu)$ and suppose that $e \cdot (a,\eta) = (ae^{-1}, \xi)$ and $f \cdot (b,\nu) = (bf^{-1}, \nu)$. Then $(e^{-1}gf) \cdot (ae^{-1}, \xi) = (bf^{-1}, \xi)$, so that ae^{-1} and bf^{-1} differ by an element of the stabilizer.

The map is obviously surjective, since $[g,\eta] \mapsto [g]$. Thus we show that the map is injective. If $\Phi([a,\eta]) = \Phi([b,\nu])$ then for some $e, f \in G$ and some $h^{-1} \in \mathrm{Stab}(\xi)$ we have:

$$e \cdot (a,\eta) = (ae^{-1}, \mathrm{Ad}(e^{-1})^*\eta) = (c, \xi)$$

$$f \cdot (b,\nu) = (bf^{-1}, \mathrm{Ad}(f^{-1})^*\nu) = (ch^{-1}, \xi)$$

But this implies that $(a,\eta) = e^{-1} \cdot (c \cdot (f \cdot (b,\nu))) = (fce^{-1}) \cdot (b,\nu)$ (note how we use that c is in the stabilizer here so that $\mathrm{Ad}(c^{-1})^*\xi = \xi$), and thus that $[b,\eta] = [a,\nu] \in \mu^{-1}(\mathcal{O})/G$.

Continuity comes because the inverse of Φ is given by $[g] \mapsto [g,\xi]$. This is continuous since it is induced by the continuous map $G \to \mu^{-1}(\mathcal{O})/G$ given by $g \mapsto [g,\xi]$ which is the composition of the embedding $G \to \mu^{-1}(\mathcal{O})$ given by $g \mapsto (g,\xi)$ with the quotient map $\mu^{-1}(\mathcal{O}) \to \mu^{-1}(\mathcal{O})/G$. Since the maps that are continuous $G/\mathrm{Stab}(\xi) \to \mu^{-1}(\mathcal{O})/G$ are precisely those induced by continuous maps $G \to \mu^{-1}(\mathcal{O})/G$ which are constant on $\mathrm{Stab}(\xi)$ orbits, this shows that the map Φ is continuous.

Exercise 5.46 Suppose that G acts in a Hamiltonian way on a symplectic manifold (M,ω) with moment map $\mu : M \to \mathbf{g}^*$. Consider the action of G on the product $M' = M \times T^*G$ with symplectic form $\omega' = \omega \times \omega_{\mathrm{can}}$. By Exercise 5.28 this action is Hamiltonian. If we identify T^*G with $G \times \mathbf{g}^*$ as in Example 5.22 then the moment map is given by:

$$\mu'(p,h,\eta) = \mu(p) - \eta$$

for $p \in M, h \in G$ and $\eta \in \mathbf{g}^*$. Prove that the Marsden-Weinstein quotient can be identified with (M,ω).

Solution 5.46 We see that $[\mu']^{-1}(0) = \{(p,h,\mu(p)) | (p,h) \in M \times G\} \simeq M \times G$. Now we postulate the map $\Phi : [\mu']^{-1}(0)/G \to M$ given by:

$$[p,g,\mu(p)] \mapsto g \cdot p$$

We show that this map is a diffeomorphism. First note that $\Phi(p,g,\mu(p)) = g \cdot p = h \cdot q = \Phi(q,h,\mu(q))$ if and only if:

$$(h^{-1}g) \cdot (p,g,\mu(p)) = (h^{-1}(g(p)), g(h^{-1}g)^{-1}, \mathrm{Ad}((h^{-1}g)^{-1})^*\mu(p)) = (q,h,\mu(h^{-1}g(p))) = (q,h,\mu(q))$$

Thus if $[p, g, \mu(p)] = [q, h, \mu(q)] \in [\mu']^{-1}(0)/G$ then $\Phi([p, g, \mu(p)]) = \Phi([q, h, \mu(q)])$ (so Φ is well-defined), and conversely if $\Phi([p, g, \mu(p)]) = \Phi([q, h, \mu(q)])$ then $[p, g, \mu(p)] = [q, h, \mu(q)]$ (so it is injective). It is evidently surjective, since every point $p \in M$ is in the image of $[p, 1, \mu(p)]$. Thus Φ is a bijection.

We can see it is a diffeomorphism by observing that the map $\Phi' : [\mu']^{-1}(0) \to M$ given by $(p, g, \mu(p)) \mapsto g \cdot p$ is smooth (this is, in fact, equivalent to the smoothness of the representation map $M \times G \to M$). Thus Φ is smooth because it is induced by a smooth function on $\Phi' : [\mu']^{-1}(0)$ which is constant on group orbits. Conversely, Φ^{-1} is given by $p \mapsto [p, 1, \mu(p)]$, and we can see that this map is smooth by noting that it is the composition of the smooth map $M \to \Phi' : [\mu']^{-1}(0)$ given by $p \mapsto (p, 1, \mu(p))$ with the smooth quotient map $\Phi' : [\mu']^{-1}(0) \to \Phi' : [\mu']^{-1}(0)/G$.

Finally we must show that Φ is a symplectomorphism. Let us first examine $\Omega = \omega \times \omega_{\text{can}}$ on $[\mu']^{-1}(0)$. The tangent space of a point $(p, h, \eta) \in [\mu']^{-1}(0)$ is all the vectors $v \oplus \alpha \oplus \xi \in T_pM \oplus \mathbf{g} \oplus \mathbf{g}^*$ such that $d\mu_p v = \xi$. The tangent space to the group orbit is all vectors of the form $X_\xi(p) \oplus -h\xi \oplus \mathrm{ad}(-\xi)^*\eta$.

Now observe that $d\Phi_{p,h,\eta}(v \oplus \alpha \oplus \xi) = dg_p v + X_\alpha(gp) = dg_p v + dg_p X_\alpha(p) \in T_{g\cdot p}M$. Thus:

$$\Phi^*\omega(v \oplus \alpha \oplus \xi, w \oplus \beta \oplus \eta) = \omega_{gp}(dg_p v + dg_p X_\alpha(p), dg_p w + dg_p X_\beta(p))$$

$$= g^*\omega(v + X_\alpha(p), w + X_\beta(p)) = \omega(v + X_\alpha(p), w + X_\beta(p)) = \omega(v, w) + \omega(X_\alpha(p), w) + \omega(v, X_\beta(p))$$
$$= \omega(v, w) + dH_\alpha(w) - dH_\beta(v) = \omega(v, w) + \langle d\mu_p w, \alpha\rangle - \langle d\mu_p v, \beta\rangle$$

But observe that if $\xi = d\mu_p v$ and $\eta = d\mu_p w$ then:

$$\Omega(v \oplus \alpha \oplus d\mu_p v, w \oplus \beta \oplus d\mu_p w) = \omega(v, w) - \langle d\mu_p v, \beta\rangle + \langle d\mu_p w, \alpha\rangle$$

so Φ is a symplectomorphism.

Exercise 5.49 Consider the case of $n = 2$ in Example 5.48. Show that the inverse image of any vertex P_i in Δ is a single point, of any point o the edge is S^1, and of any point in the interior is T^2. What is the inverse image of an edge? Of a line segment such that AB, AB' as in Fig. 5.3? Of the triangle ABP_0?

Solution 5.49 We recall that the moment map on $\mathbb{C}P^2$ is:

$$\mu([z_0, z_1, z_2]) = \pi(\frac{|z_1|^2}{|z|^2}, \frac{|z_2|^2}{|z|^2})$$

The image of μ is the set of points $\{(\pi x, \pi y) | x + y \leq 1; x, y \geq 0\}$. There are 3 vertices, $(\pi, 0), (0, \pi)$ and $(0, 0)$. These correspond to points where $|z_i| = |z|$ for $i = 0, 1, 2$, i.e points of the form $[z, 0, 0], [0, z, 0]$ and $[0, 0, z]$. Each of these represents a single point in projective space, so the inverse image is one point.

Now examine the points on a side, say where $|z_2| = 0$. The fiber of a point here has $|z_2| = 0$ and $|z_0|^2 = a^2|z|^2, |z_1|^2 = b^2|z|^2$ for fixed $a, b \neq 0$ with $a^2 + b^2 = 1$. In particular, $z_1 = ce^{i\theta}z_0$ for some constant c. Thus the points in the inverse image are:

$$[z_0, ce^{i\theta}z_0, 0] = [1, ce^{i\theta}, 0]$$

Every such point has a unique representative $[1, ce^{i\theta}, 0]$ where the first coordinate is 1, so the image is diffeomorphic to the circle $\{(1, e^{i\theta}, 0) | \theta \in \mathbb{R}\}$. A nearly identical argument takes care of the other sides. For a point in the interior, the condition is that none of the z_i are zero and $|z_1| = a|z_0|, |z_2| = b|z_1|$ for non-zero a, b, so that $(z_0, z_1, z_2) = (z_0, ae^{i\theta}z_0, be^{i\phi}z_0)$. Each such point has a unique representative with $z_0 = 1$, so that the map $[z_0, e^{i\theta}z_0, be^{i\phi}z_0] \to (e^{i\theta}, e^{i\phi})$ is a diffeomorphism to the flat Clifford torus in $\mathbb{C}^2$.

The inverse image of an entire edge (including the end-points) is a sphere. We can see this by noting that the inverse image of the edge without the end-points is an open cylinder (being a circle bundle over a line-segment, which is always trivial). The two ends of this cylinder are then each glued along the inverse image of the two vertices at the end-points of the edge, which are points, so the inverse image of the closed line-segment can be identied with $S^1 \times I/\sim$ with the equivalence relation that identified the two circles at either end-point with two points respectively. This is a sphere.

The inverse images of the sides AB, AB' are diffeomorphic to S^3, and the inverse image of the triangle ABP_0 is 4-ball B^4. The easiest way to argue this is to use projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ onto the perpendicular line to AB (resp. $\mathbb{R}^4$) and look at $f = \pi \circ \mu : \mathbb{C}P^2 \to \mathbb{R}$. f is then Morse with critical points corresponding to the vertices of $\mu(\mathbb{C}P^2)$ on the interval $f^{-1}([f(P_0), f(A)])$ with $f^{-1}(ABP_0)$, and this interval contains only one critical point which is the minimum at P_0. Thus by standard Morse theory, we have that $f^{-1}([f(P_0), f(P_0) + \epsilon]) \simeq B^4$ for all ϵ such that there are no critical points in $[f(P_0), f(P_0) + \epsilon]$.

Appendix 1: De Rham Theory

Appendix 2: Tidbits Here I'm throwing some things that I proved that didn't end up being useful for the problem I was trying to do. Enjoy!

Lemma 3.14 (Parameterized Version) Let M be a $2n$-dimensional smooth manifold and $\psi_s : Q \to M$ be an isotopy of a compact sub-manifold Q through M. Suppose that $\omega_0, \omega_1 \in \Omega^2(M)$ are closed 2-forms that are equal and non-degenerate on T_qM for any $q \in \psi(I \times Q)$. Then there exists smooth isotopies $\psi_s^i : U \to M$ $(i = 0, 1)$ for some U containing Q which is diffeomorphic to tubular neighborhoods of Q along with a family of diffeomorphisms $\phi_s : U_0 \to U_1$ so that $\phi_s^*(\psi_s^1)_1^*\omega_1 = (\psi_s^0)^*\omega_0$. Thus u is a multiple of X_h at p.

Thus suppose that $u = aX_h$ and $w = bX$ for some constants a, b at p.

Proof: Fix an extension of of ψ to an isotopy $\psi_s^0 : U \to M$ for some U containing Q (we can do this using the usual smooth isotopy extension theorem). Then we can use a version of Moser's argument to prove our result. It suffices to find a smooth family of 1-forms $\sigma_s \in \Omega^1(U)$ such that $(\psi_s^0)^*\sigma_s = 0$ and $d\sigma_s = (\psi_s^0)^*(\omega_1 - \omega_0)$. Then we can consider the family of closed forms:

$$\omega_{t,s} = (\psi_s^0)^*(\omega_0 + t(\omega_1 - \omega_0)) = (\psi_s^0)^*\omega_0 + td\sigma_s$$

Since $(\psi_s)^*\omega_0 = (\psi_s)^*\omega_1$ and thus $(\psi_s^0)^*(\omega_0 + t(\omega_1 - \omega_0))$ is non-degenerate all s, t and all $p \in Q$, we may assume that $\omega_{t,s}$ is symplectic on all of U for all t, s possibly after shrinking U. Then we may solve the equation:

$$\sigma_s + i_{X_{t,s}}\omega_{t,s} = 0$$

for $X_{t,s}$. The resulting family of vector fields is smooth and vanishes on Q for all t, s. Now we can solve the system of ODE:

$$\frac{d}{dt}\phi_{t,s} = X_{t,s} \circ \phi_{t,s}$$

Since $X_{t,s}$ vanishes on Q and Q is compact, we pick a U_0 such that this isotopy is well-defined for $t, s \in I$ and $p \in U_0$. The resulting map is a smooth family of maps $\phi_{t,s} : U_0 \to \psi_{t,s}(U_0) \subset U$. The resulting family of diffeomorphisms will satisfy:

$$0 = \frac{d}{dt}\phi^*_{t,s}\omega_{t,s} = \phi^*_{t,s}(\frac{d}{dt}\omega_{t,s} + di_{X_{t,s}}\omega_{t,s})$$

Picking some family of diffeomorphisms $\xi_t : U \to U$ so that $\xi_s(\psi_{1,s}(U_0)) = U_0$, setting $\phi_s = \xi\phi_{1,s}$ and $\psi^1_s = \psi^0_s\xi^{-1}$ we have:

$$(\phi_s\psi^1_s)^*\omega_1 = (\phi_{1,s}\psi^0_s)^*\omega_1 = \phi^*_{1,s}\omega_{1,s} = \phi^*_{0,s}\omega_{0,s} = (\psi^0_s)^*\omega_0$$

Thus we have found our desired family. $\square$