Worksheet 9-13

Exericise 1 (4.1 #5,6) Derermine if the given sets are subspaces of $\mathbb{P}_n$ for an appropriate value of n. Justify your answer. (a) All polynomials of the form $p(t) = at^2$ where $a \in \mathbb{R}$. (b) All polynomials of the form $a + t^2$ where $a \in \mathbb{R}$.

Exericise 2 (4.1 #16) Let W be the set of all vectors of the form:

$$\begin{bmatrix} 1 \\ 3a - 5b \\ 3b + 2a \end{bmatrix}$$

where a, b and c are arbitrary real numbers. Find a set S of vectors that sopan W or give an example to show that W is not a vector-space.

Exericise 3 (4.1 # 34) Suppose $u_1, \dots, u_p$ and $v_1, \dots, v_q$ are vectors in a vector space V and let $H = \text{span}(\{u_1, \dots, u_p\})$ and $K = \text{span}(\{v_1, \dots, v_q\})$. Show that $H + K = \text{span}(\{u_1, \dots, u_p, v_1, \dots, v_q\})$.

Exericise 4 (4.2 # 32) Define a linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ by:

$$T(p) = \begin{bmatrix} p(0) \\ p(0) \end{bmatrix}$$

Find polynomials p_1 and p_2 in $\mathbb{P}_2$ that span the kernel of T and describe the range of T.

Exericise 5 (4.2 # 36) Let $T : V \to W$ be a linear transformation from a vector-space V to a vector-space W, and let $Z \subset W$ be a subspace. Prove that the set S of vectors $v \in V$ such that $T(v) \in Z$ is a subspace of V.

Exericise 6 This question is extracurricular and will not be quized or tested. It's on here because the idea is interesting. However, doing it may help you come to a better understanding of the abstract concept of a vector-space.

There are many examples of infinite dimensional vector-spaces. An infinite dimensional vector-space is a set of objects that satisfies the definition of a vector-space, but is not spanned by n vectors for any $n < \infty$.

(a) Find 3 examples of infinite dimensional vector-spaces that are distinct. Check that the spaces satisfy the vector-space axioms. Try to prove that your examples are not finite dimensional.

(b) Find an example of an infinite dimensional vector-space V and a linear map $T : V \to V$ which is one-to-one but not onto. Find an example of a linear map $T : V \to V$ which is onto but not one-to-one. It's interesting to note that this can't happen with finite dimensional vector-spaces.

(c) Two vector-spaces U and V are called **isomorphic** if there is a linear map $T : U \to V$ that is onto and one-to-one. In finite dimensions, this is the same thing as $\dim(U) = \dim(V)$, but in infinite dimensions this becomes more subtle. Find an example of two infinite dimensional vector-spaces that are not isomorphic.