Worksheet 10-16

Exericise 1-3 These are book problems, so you can look up the solutions.

Exercise 4 Let $\phi : U \to V$ be an invertible linear transformation from a vector space U to an inner product space V with inner product $\langle \cdot, \cdot \rangle$. Show that the pairing $\phi^* \langle \cdot, \cdot \rangle$ of vectors in U defined as:

$$\phi^* \langle u, v \rangle := \langle \phi(u), \phi(v) \rangle$$

is an inner product on U.

Solution 4 We need to verify the inner product axioms on p. 376 of the book, Ch. 6.7.

1. $\phi^* \langle u, v \rangle = \langle \phi(u), \phi(v) \rangle = \langle \phi(v), \phi(u) \rangle = \phi^* \langle v, u \rangle$. Here we use the symmetry property of $\langle \cdot, \cdot \rangle$.

2. $\phi^* \langle u + v, w \rangle = \langle \phi(u + v), \phi(w) \rangle = \langle \phi(u) + \phi(v), \phi(w) \rangle = \langle \phi(u), \phi(w) \rangle + \langle \phi(v), \phi(w) \rangle = \phi^* \langle u, w \rangle + \phi^* \langle v, w \rangle$. Here we use the linearity of ϕ and the linearity property of $\langle \cdot, \cdot \rangle$.

3. $\phi^* \langle cu, v \rangle = \langle \phi(cu), \phi(v) \rangle = \langle c\phi(u), \phi(v) \rangle = c\langle \phi(u), \phi(v) \rangle = c\phi^* \langle u, v \rangle$.

4. $\phi^* \langle u, u \rangle = \langle \phi(u), \phi(u) \rangle \geq 0$. If $\phi^* \langle u, u \rangle = 0$ then $\langle \phi(u), \phi(u) \rangle = 0$ so $\phi(u) = 0$ and thus $u = \phi^{-1}(\phi(u)) = \phi^{-1}(0) = 0$.

Exercise 5 A linear transformation $\phi : U \to V$ between inner product spaces $(U, \langle \cdot, \cdot \rangle_U)$ and $(V, \langle \cdot, \cdot \rangle_V)$ is called an **isometry** if:

$$\langle \phi u, \phi v \rangle_V = \langle u, v \rangle_U$$

That is, the inner product of u and v are the same before you apply ϕ. Another way of writing this is $\phi^* \langle \cdot, \cdot \rangle_V = \langle \cdot, \cdot \rangle_U$. Show that the matrix A of an isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis is an orthogonal matrix, i.e. $A^T A = I$.

Solution 5 Let A be the matrix of an isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with the standard (dot) inner product. Then:

$$u \cdot v = \langle u, v \rangle = \langle \phi(u), \phi(v) \rangle = (Au) \cdot (Av) = u \cdot (A^T Av)$$

Now let M be a matrix equal to $M = [m_{ij}]$ where m_{ij} is the entry in the ith column and jth row. Let e_i be the standard ith unit vector, i.e. with 1 in the ith entry and 0's elsewhere. Then:

$$m_{ij} = e_j \cdot (Me_i)$$

In these terms, the identity matrix is given by $I = (\delta_{ij})$ where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Note that $\delta_{ij} = e_i \cdot e_j$ since $e_i \cdot e_j = 1$ if $i = j$ and is 0 otherwise.

Applying this to $A^T A$, we see that:

$$e_j \cdot (A^T A e_i) = e_j \cdot e_i = \delta_{ij}$$

So the entries of $A^T A$ are the same as the entries of I, and they must be the same matrix.

Exercise 6 Let $V = C(\mathbb{R}/\mathbb{Z})$ be the space of continuous functions on the real line that are periodic with period 2π, i.e. $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$. $C(\mathbb{R}/\mathbb{Z})$ is an inner space with inner product:

$$\langle u, v \rangle = \int_{[0,2\pi]} u(t)v(t)dt$$

Show that the set $\{e_k\}$ of vectors $e_k = \cos(kt)$ and $k \in \mathbb{N}$ is an orthogonal set.

Solution 6 We note that:

$$\langle e_j, e_k \rangle = \int_0^{2\pi} \sin(jt)\sin(kt)dt = \frac{1}{2}\int_0^{2\pi} (\cos((j-k)t) - \cos((j+k)t)$$

If $j \neq k$ and $j, k > 0$, then $j - k$ and $j + k$ are both non-zero. Thus:

$$\frac{1}{2}\int_0^{2\pi} (\cos((j-k)t) - \cos((j+k)t) = \frac{1}{2}(\frac{\sin((j-k)t)}{j-k} + \frac{\sin((j+k)t)}{j+k})|_{t=0}^{t=2\pi}$$

$$= \frac{1}{2}(\frac{\sin(2\pi(j-k))}{j-k} + \frac{\sin(2\pi(j+k))}{j+k}) - \frac{1}{2}(\frac{\sin(0)}{j-k} + \frac{\sin(0)}{j+k}) = 0$$