Worksheet 10-16

Exericise 1 (6.7 # 3-7) Let $\mathbb{P}_2$ be the inner product space of polynomials of degree ≤ 2 with inner product:

$$\langle p, q\rangle = p(-1)q(-1) + p(0)q(0) + p(1)q(1)$$

Compute $\langle p, q\rangle$, $\|p\|$ and $\|q\|$ when $p = 4+t, q(t) = 5-4t^2$. Compute the orthogonal projection of p onto the subspace spanned by q.

Exercise 2 (6.7 # 15,17) Use the inner product axioms in the beginning of 6.7 to verify that $\langle u, cv\rangle = c\langle u, v\rangle$ for all scalars c, and that $\langle u, v\rangle = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$.

Exercise 3 (6.7 # 22,24) Let $V = C[0,1]$ be the space of continuous functions on $[0,1]$ with inner product:

$$\langle u, v\rangle = \int_{[0,1]} u(t)v(t)dt$$

Compute $\langle f, g\rangle$, $\|f\|$ and $\|g\|$ where $f = 1 = 3t^2$ and $g(t) = t - t^3$.

Exercise 4 Let $\phi : U \to V$ be an invertible linear transformation from a vector space U to an inner product space V with inner product $\langle\cdot,\cdot\rangle$. Show that the pairing $\phi^*\langle\cdot,\cdot\rangle$ of vectors in U defined as:

$$\phi^*\langle u, v\rangle := \langle\phi(u), \phi(v)\rangle$$

is an inner product on U.

Exercise 5 A linear transformation $\phi : U \to V$ between inner product spaces $(U, \langle\cdot,\cdot\rangle_U)$ and $(V, \langle\cdot,\cdot\rangle_V)$ is called an **isometry** if:

$$\langle\phi u, \phi v\rangle_V = \langle u, v\rangle_U$$

That is, the inner product of u and v are the same before you apply ϕ. Another way of writing this is $\phi^*\langle\cdot,\cdot\rangle_V = \langle\cdot,\cdot\rangle_U$. Show that the matrix A of an isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with respect to the standard basis is an orthogonal matrix, i.e. $A^TA = I$.

Exercise 6 Let $V = C(\mathbb{R}/\mathbb{Z})$ be the space of continuous functions on the real line that are periodic with period 2π, i.e. $f(x) = f(x+2\pi)$ for all $x \in \mathbb{R}$. $C(\mathbb{R}/\mathbb{Z})$ is an inner space with inner product:

$$\langle u, v \rangle = \int_{[0,2\pi]} u(t)v(t)dt$$

Show that the set $\{e_k\}$ of vectors $e_k = \cos(kt)$ and $k \in \mathbb{N}$ is an orthogonal set.