## Helpful/Important Facts In Linear Algebra

Note On Starred Statements If a statement has a $(*)$ next to it, then proving it is a good exercise.

Reading Set Notation Let's say I have a set of things $V$ (you can pretend that $V$ is a vector-space if you want). There is a notation, called propositional logic or set notation, that let's me specify some subset of $V$ using symbols and not sentences. It looks like this:

$$
U=\{v \in V \mid \text { conditions on } v \ldots\}
$$

The above expression directly translates into the following sentence: $U$ is the set of objects in $V$ that satisfy the conditions ... (the dots are standing in for whatever conditions you would actually have). Here are some examples.

Example 1: Let $A$ be an $m \times n$ matrix. Then the following expression:

$$
U=\left\{v \in \mathbb{R}^{n} \mid A v=0\right\}
$$

directly translates as " $U$ is the set of vectors $v$ in $\mathbb{R}^{n}$ such that $A v=0$. That is, $U$ is the nullspace of $A$.

Example 2: Let $A$ be an $m \times n$ matrix. Then the following expression:

$$
W=\left\{v \in \mathbb{R}^{m} \mid v=A u, u \in \mathbb{R}^{n}\right\}
$$

directly translates as " $W$ is the set of vectors $v$ in $\mathbb{R}^{m}$ such that $v=A u$ and $u \in \mathbb{R}^{n}$ ". In other words, $W$ is the column space or image of $A$. It is also common to simply write $v$ in terms of $u$ and $A$ before the line in situations like this. For example:

$$
W=\left\{A u \in \mathbb{R}^{m} \mid u \in \mathbb{R}^{n}\right\}
$$

would read " $W$ is the set of vectors $A u$ in $\mathbb{R}^{m}$ where $u$ is a vector in $\mathbb{R}^{n}$ ". These two sentences describe the same set, so the expressions do as well.

Basic Linear Algebra Let $U$ and $V$ be vector-spaces of dimension $n$ and $m$ respectively, $T: U \rightarrow V$ be a linear map, and $A$ be the matrix for $T$ in a basis $B$.

1. The following are equivalent to $T$ being one-to-one.
(a) $\operatorname{ker}(T)=\operatorname{null}(A)=\{0\}$.
(b) There is a unique solution $x$ to $A x=b$ for every $b$.
(c) The columns of $A$ are linearly independent.
2. The following are equivalent to $T$ being onto.
(a) $\operatorname{im}(T)=\operatorname{col}(A)=V$.
(b) For every $b$ there exists a solution to $A x=b$.
(c) The columns of $A$ spanning set.
3. The following are equivalent to $T$ being invertible.
(a) $T$ is one to one and onto.
(b) For every $b$ there exists a unique solution to $A x=b$.
(c) The columns of $A$ are a basis.

## Inverses, Determinants

1. The determinant has the following properties.
(a) $\operatorname{det}(A B)=\operatorname{det}(A) \operatorname{det}(B)$
(b) $\left(^{*}\right) \operatorname{det}(c A)=c^{n} \operatorname{det}(A)$
(c) $\left(^{*}\right) \operatorname{det}\left(A^{-1}\right)=\operatorname{det}(A)^{-1}$
(d) $\left(^{*}\right) \operatorname{det}\left(A^{T}\right)=\operatorname{det}(A)$
(e) $\left(^{*}\right) \operatorname{det}(A)=\lambda_{1} \lambda_{2} \ldots \lambda_{n}$ where $\lambda_{i}$ are the roots of the characteristic polynomial $p(\lambda)=\operatorname{det}(A-\lambda I)$ counted with multiplicity.
2. (Volume Property Of Determinant) If $U \subset \mathbb{R}^{n}$ is a region in $n$-dimensional space and $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is an invertible linear map with matrix $A$, then volume $(T(U))=|\operatorname{det}(A)| \cdot \operatorname{volume}(U)$.
3. (Cayley-Hamilton) A matrix satisfies its own characteristic polynomial. That is, if $M$ is a matrix and $p$ is the characteristic polynomial $p(\lambda)=\operatorname{det}(\lambda I-M)$, then the matrix $p(M)$ is 0 .
4. The following are equivalent for a square matrix $M$.
(a) $M$ is invertible.
(b) $\operatorname{det}(M) \neq 0$.
(c) All of $M$ 's eigenvalues are non-zero.
(d) 0 is not a root of the characteristic polynomial of $M$.
5. If a square matrix is $3 \times 3$, the fastest way to compute the determinant is probably via row reduction. If it is bigger, the fastest way is probably to do row reduction without scaling the rows to have 1 on the pivot. That is, during row reduction skip the part where you divide the row by the size of the entry. This will produce an upper diagonal matrix. Then to compute the determinant you just need to take the product of the diagonal entries. $\left(^{*}\right)$ Prove the last part.

## Kernel, Image, Dimension, Rank

1. (rank nullity) If $T: U \rightarrow V$ is a linear map from a vector-space $U$ to a vector-space $V$, then:

$$
\operatorname{dim}(U)=\operatorname{rank}(T)+\operatorname{dim}(\operatorname{ker}(T))
$$

For matrices, this means that if $A$ is an $m \times n$ matrix mapping $\mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$, then:

$$
\begin{aligned}
n & =\operatorname{rank}(A)+\operatorname{dim}(\operatorname{null}(A)) \\
m & =\operatorname{rank}(A)+\operatorname{dim}\left(\operatorname{null}\left(A^{T}\right)\right)
\end{aligned}
$$

2. (*) The following dimension equalities are true for a matrix $A$.

$$
\operatorname{dim}(\operatorname{col}(A))=\operatorname{dim}(\operatorname{row}(A))
$$

Bases, Change Of Basis Let $V$ be a vector-space, and let $A$ and $B$ be bases of $V$.

1. The coordinate vector $[x]_{B}$ of any $x \in V$ is the unique column vector:

$$
[x]_{B}=\left[\begin{array}{c}
x_{1} \\
x_{2} \\
\ldots \\
x_{n}
\end{array}\right]
$$

such that $x=\sum_{i=1}^{n} x_{i} b_{i}$ where $B=\left\{b_{i}\right\}$ is the basis $B$. There is only one set of numbers $x_{i}$ that satisfy this property, because $B$ is a basis.
2. The coordinate map $C_{B}: V \rightarrow \mathbb{R}^{n}$, given by $C_{B}(x)=[x]_{B}$, is the map that takes a vector $x$ to its coordinate vector.
3. The inverse of the coordinate map, $P_{B}: \mathbb{R}^{n} \rightarrow V$, is the map:

$$
P_{B}\left(\left[\begin{array}{l}
x_{1} \\
x_{2} \\
\ldots \\
x_{n}
\end{array}\right]\right)=\sum_{i=1}^{n} x_{i} b_{i}
$$

Since it is the inverse, it satisfies $P_{B}\left(C_{B}(x)\right)=P_{B}\left([x]_{B}\right)=x$.
4. The change of basis matrix $P_{A \rightarrow B}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is the map $P_{A \rightarrow B}=C_{B} \circ P_{A}$. It takes the coordinate vector for $x$ in basis $A$ and spits out the coordinate vector for $x$ in basis $B$.

## Eigenvectors, Eigenvalues, Diagonalization

1. Let $A$ be an $n \times n$ matrix. $\lambda$ is an eigenvalue of $A$ if and only if one of the following is true.
(a) $\lambda$ is a root of the characteristic polynomial $p(\lambda)=\operatorname{det}(A-\lambda I)$.
(b) There exists a vector $v$ such that $A v=\lambda v$.
(c) The matrix $A-\lambda I$ has a non-zero nullspace.
2. Let $A$ be an $n \times n$ matrix. If any of the following properties is true, then $A$ is diagonalizable.
(a) The characteristic polynomial of $A$ has no repeated roots.
(b) $\left(^{*}\right) A$ has a basis of eigenvectors.
(c) $A$ is symmetric.

## Inner Products, Orthonormal Sets

1. Any orthogonal set is independent.
2. If $A$ is any matrix (not necessarily square) then $y \cdot(A x)=\left(A^{T} y\right) \cdot x$ for every pair of vectors $x$ and $y$.
3. Let $V$ be a vector-space with inner product $\langle\cdot, \cdot\rangle$. Let $W$ be a subspace. Then the following are equivalent definitions of orthogonal projection $\operatorname{proj}_{W}: V \rightarrow$ $W \subset V$.
(a) $\operatorname{proj}_{W}(v)$ is the unique vector such that $v=\operatorname{proj}_{W}(v)+u$ for some $u \in W^{\perp}$.
(b) $\operatorname{proj}_{W}(v)$ is equal to:

$$
\operatorname{proj}_{W}(v)=\sum_{i=1}^{k} u_{i}\left\langle u_{i}, v\right\rangle
$$

for any orthonormal basis $\left\{u_{i}\right\}$ of $W$.
4. Let $V$ be a vector-space with inner product $\langle\cdot, \cdot\rangle$. Let $W$ be a subspace. Then:

$$
\operatorname{dim}(V)=\operatorname{dim}(W)+\operatorname{dim}\left(W^{\perp}\right)
$$

## Quadratic Forms, Symmetric Matrices

1. Every quadratic form $q$ on a vector-space $V$ with inner product $\langle\cdot, \cdot\rangle$ is given by:

$$
q(x)=\sum_{i=1}^{n} \lambda_{i}\left\langle u_{i}, x\right\rangle^{2}
$$

for some orthonormal basis $u$.
2. (Spectral Theorem) If $A$ is a symmetric matrix, then:

$$
A=U D U^{T}
$$

for a diagonal matrix $D$ and an orthogonal matrix $U$.
3. Let $I_{m, n}$ be the block matrix:

$$
I_{m, n, p}=\left[\begin{array}{ccc}
I_{m} & 0 & 0 \\
0 & -I_{n} & 0 \\
0 & 0 & 0_{p}
\end{array}\right]
$$

where $I_{m}$ and $I_{n}$ are the $m$-dimensional and $n$-dimensional identity matrices, respectively, and $0_{p}$ is the $p \times p$ square matrix with all 0 's. Then any symmetric matrix $A$ can be written as:

$$
A=B^{T} I_{m, n, p} B
$$

for some invertible $B$ and a unique $m, n$ and $p$. $A$ is positive definite if and only if $n=p=0$, in which case $A$ can be written:

$$
A=B^{T} B
$$

for an invertible $B$.

## Extra Exercises

1. A matrix is anti-symmetric if $A=-A^{T}$. Show that an anti-symmetric matrix has no non-zero real eigenvalues.
2. Find an example of a matrix that has no real eigenvalues.
