Math 54 Section 4: Quiz 4

Problem 1 True Or False?

(a) If a system $Ax = b$ has more than one solution, then so does the system $Ax = 0$. **True.** If $Ax = Ay = b$ and $x \neq y$, then $A(x - y) = 0$.

(b) If the equation $Ax = 0$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix. **False.** This is the same as being one-to-one, but to be row equivalent to I_n, A must also be onto.

(c) If the columns of an $m \times n$ matrix A are linearly independent, then the columns of A span $\mathbb{R}^n$. **False.** You can have a set of linearly independent vectors that do not form a basis (for instance if the number of vectors is smaller than n).

(d) If $BC = BD$, then $C = D$. **False.** B could be non-invertible.

(e) If A and B are $n \times n$ matrices, then $(A + B)(A - B) = A^2 - B^2$. **False.** This is only true if $AB = BA$.

(f) If A is invertible and $r \neq 0$, then $(rA)^{-1} = rA^{-1}$. **False**. The real formula is $(rA)^{-1} = r^{-1}A^{-1}$.

(g) $\det(AB - BA) = 0$ for any A and B. **False**. There are 2×2 counter-examples for A and B.

(h) Any onto linear transformation is invertible. **Very False.**

(i) Row operations on a matrix can change the nullspace. **False.**

Problem 2 Suppose that v_1, v_2, v_3 are distinct points on a line L in $\mathbb{R}^3$. Here L does **not** necessarily pass through the origin. Show that $\{v_1, v_2, v_3\}$ is linearly dependent.

Solution 2 This is obviously true if L does go through the origin, since then v_1, v_2, v_3 are 3 vectors in a 1-dimensional vector-space L.

If L doesn't, then the set of points in L is given by $\{ut + v | t \in \mathbb{R}\}$ where u and v are non-zero vectors. In particular, every point on L is a linear combination of u and v, and L lies inside of the plane $H = \text{span}(u, v)$. H is 2-dimensional, so any 3 vectors in H are linearly dependent. Thus if v_1, v_2, v_3 are 3 points in L, then they are also in H and they must be linearly dependent.

Problem 3 Suppose A is invertible. Explain why $A^T A$ is also invertible. Then show that $A^{-1} = (A^T A)^{-1} A^T$.

Solution 3 The determinant $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2$ is not zero since $\det(A)$ is not zero. Thus $A^T A$ is invertible. Furthermore:

$$((A^T A)^{-1} A^T)A = (A^T A)^{-1}(A^T A) = A^{-1}(A^T)^{-1} A^T A = I$$

So $(A^T A)^{-1} A^T$ must be the inverse of A, since multiplying it by A produces 1.

Problem 4 Suppose that $A^n = 0$ for some $n > 1$. Find an inverse for $I - A$.

Solution 4 Let $B = 1 + A + \cdots + A^{n-1}$. Then we see that:

$$B(I-A) = (1+A+\cdots+A^{n-1})(I-A) = 1+A+\cdots+A^{n-1}-A-A^2-\cdots-A^{n-1}-A^n$$

We see on the right that every power of A cancels except for A^n and I. But since $A^n = 0$, that means that $BA = I$. So $B = A^{-1}$.

Problem 5 Compute the following determinants.

$$\begin{vmatrix} -1 & 5 & 2 \\ 5 & 6 & 3 \\ 1 & 3 & 1 \end{vmatrix} \qquad \begin{vmatrix} 9 & 5 & 2 \\ 1 & 0 & 0 \\ 4 & 4 & 1 \end{vmatrix}$$

Solution 5 For the first matrix A, we have $\det(A) = -1(6-9) - 5(5-3) + 2(15-6) = 3 - 10 + 18 = 11$. For the second matrix B, we have $\det(B) = 9(0-0) - 5(1-0) + 2(4) = 3$.

Problem 6 Let A, B, C, D be $n \times n$ matrices with A invertible. Find matrices X and Y to produce the block factorization:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & Y \end{bmatrix}$$

and use this to show that:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D - CA^{-1}B)$$

Solution 6 Multiplying out the block matrices, we see that:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ XA & XB+Y \end{bmatrix}$$

Therefore $XA = C$, so $X = CA^{-1}$, and $XB + Y = CA^{-1}B + Y = D$ so $Y = D - CA^{-1}B$. Therefore we have:

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det \begin{bmatrix} I & 0 \\ CZ^{-1} & I \end{bmatrix} \cdot \det \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} = \det M \cdot \det N$$

Here we define:

$$M = \begin{bmatrix} I & 0 \\ CZ^{-1} & I \end{bmatrix} \quad N = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

By subtracting multiples of the top n rows of M from the bottom rows of N, we can turn M into the $2n \times 2n$ identity. These operationes don't change the determinant, so $\det M = 1$. Likewise, we can subtract the first n columns of N from the second n columns of N to make N into the matrix:

$$\begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix}$$

This doesn't change the determinant and the resulting matrix has determinant $\det A \cdot \det(D - CA^{-1}B)$ due to the block structure. Thus $\det M \cdot \det N = 1 \cdot \det(A) \cdot \det(D - CA^{-1}B)$.

Problem 7 Let V be a vector-space. We say that a subspace G is smaller than a subspace H of V if $G \subset H$, that is if G is a subspace of H. Let $v_1, \ldots, v_p$ be vectors in a vector-space V. What is the smallest subspace containing $v_1, \ldots, v_p$? Prove it.

Solution 7 The smallest subspace containing $v_1, \ldots, v_p$ is the span $S = \text{span}(v_1, \ldots, v_p)$.

We can prove that S is the smallest subspace containing $v_1, \ldots, v_p$ as so. Let H be such a subspace. Then since H is closed under addition and scalar multiplication (by definition) and since $v_1, \ldots, v_p$ are in H, we have that any linear combination $c_1v_1 + c_2v_2 + \cdots + c_pv_p$ is in H also. In other words, every vector in the span S is in H. Thus $S \subset H$, and S is smaller than any subspace H containing $v_1, \ldots, v_p$.

Problem 8 The rank of a matrix A, denoted by $\text{rank}(A)$, is the number of pivots of A after row reduction. Equivalently, the rank is the dimension of the range/image of the linear map T_A of A, and the rank is also the dimension of the column space of A.

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix, such that $AB = 0$. Show that $\text{rank}(A) + \text{rank}(B) \leq n$.

Solution 8 The dimension $\mathrm{null}(A) = \dim(\mathrm{Null}(A))$ of the null space of A and the dimension $\mathrm{rank}(A) = \dim(\mathrm{Im}(A))$ of the column space of A satisfy:

$$\mathrm{null}(A) + \mathrm{rank}(A) = n$$

This is true for A and B. Furthermore, if $AB = 0$, then the column space of A must be a subspace of the nullspace of B. Thus means that $\mathrm{rank}(A) = \dim(\mathrm{Im}(A)) \leq \dim(\mathrm{Null}(B)) = \mathrm{null}(B)$. So:

$$n = \mathrm{null}(A) + \mathrm{rank}(A) \geq \mathrm{rank}(B) + \mathrm{rank}(A)$$