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A ribbon concordance

Figure: A ribbon concordance.
Gordon’s work

Theorem (Gordon 1981)

If $C$ is a ribbon concordance, then $\pi_1(K_0) \rightarrow \pi_1(C)$ is an injection, and $\pi_1(K_1) \rightarrow \pi_1(C)$ is a surjection.
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- $\pi_1(K_1) \to \pi_1(C)$: the complement of $C$ is obtained by attaching 2-handles and 3-handles.
- $\pi_1(K_0) \hookrightarrow \pi_1(C)$: uses much harder 3-manifold topology.
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If $K_0 \leq K_1$, then $d(K_0) \leq d(K_1)$, where $d(K) = \deg \Delta_{K}(t)$.

If $K_0 \leq K_1$, $d(K_0) = d(K_1)$ and $K_1$ is transfinitely nilpotent, then $K_0 = K_1$.

Transfinite nilpotence: the lower central series becomes trivial at some ordinal.

Fibered knots are transfinitely nilpotent, since $\pi_1(K)$ is free.
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Ribbon concordances and fiberedness

Theorem (Silver 1992, Kochloukova 2006): If $K_0 \leq K_1$ and $K_1$ is fibered, then $K_0$ is fibered.

(Stallings 1965): The commutator subgroup $\pi_1(K)' \subseteq \pi_1(K)$ is free iff $K$ is fibered.

(Neuwirth 1964): If $G$ is a knot group and $G'$ is finitely generated, then $G'$ is free.

(Rapaport's conjecture 1975): If $G$ is knot-like ($G/G' \cong \mathbb{Z}$ and $G$ has deficiency 1) then $G'$ finitely generated implies $G'$ free.
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If $K \subseteq S^3$, there is a bigraded group

$$\widehat{HFK}(K) = \bigoplus_{i,j \in \mathbb{Z}} \widehat{HFK}_i(K, j).$$
Knot Floer homology (Ozsváth and Szabó, Rasmussen)

If $K \subseteq S^3$, there is a bigraded group

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- Categorifies the Alexander polynomial.
- Detects the Seifert genus:

$$g_3(K) = \max\{j : \hat{HFK}(K, j) \neq \{0\}\}.$$
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$H_*(CFK^\infty(K)) \cong HF^\infty(S^3) \cong \mathbb{F}_2[U, U^{-1}]$. 
Knot Floer homology as a TQFT

Juhász’s decorated link cobordism category
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Figure: A decorated link cobordism.
To a decorated link cobordism

\((W, \mathcal{F}): (Y_1, \mathbb{L}_1) \to (Y_2, \mathbb{L}_2)\),

Juhász associates a map

\[ F_{W,\mathcal{F}}: \widehat{HFL}(Y_1, \mathbb{L}_1) \to \widehat{HFL}(Y_2, \mathbb{L}_2). \]
Concordances

To a concordance, there is a natural choice of dividing set (with minor ambiguity).

Figure: A decorated concordance.

Studied by Juhász and Marengon. They proved the map preserves the Maslov and Alexander gradings.
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A TQFT for the full knot Floer complex

Theorem (Z. 2017)

If \((W, \mathcal{F}): (Y_1, \mathbb{L}_1) \to (Y_2, \mathbb{L}_2)\) is a decorated link cobordism and \(s \in \text{Spin}^c(W)\), there is a functorial chain map

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F^\infty_{W, \mathcal{F}, s}: \text{CFL}^\infty(Y_1, \mathbb{L}_1, s|_{Y_1}) \to \text{CFL}^\infty(Y_2, \mathbb{L}_2, s|_{Y_2}).
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For a decorated concordance \(C\), we obtain a bigraded map

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- Alishahi and Eftekharz independently gave a similar construction, in terms of a different cobordism category.
Suppose $C$ is a ribbon concordance from $K_0$ to $K_1$. Then $F_C^\infty : \hat{HFK}(K_0) \to \hat{HFK}(K_1)$ is an injection.
Theorem (Z.)

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Suppose \( C \) is a ribbon concordance from \( K_0 \) to \( K_1 \).

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is an injection.

2. The map \( F_C^\infty \) admits a left inverse, i.e. a filtered graded map \( \Pi \) such that

\[
\Pi \circ F_C^\infty \simeq \text{id}_{CFK^\infty(K_0)}.
\]
Proof

Let $C: K_1 \rightarrow K_0$ denote the mirror of $C$. We claim $F_C \circ F_C = \text{id} \hat{\text{HFK}}(K_0)$. Each birth of $C$ has a corresponding death in $C$. Each saddle of $C$ has a corresponding saddle in $C$. The births and deaths determine 2-spheres in the complement of $[0, 1] \times K_0$. The saddles and their reverses determine tubes which connect the 2-spheres to the trivial concordance $[0, 1] \times K_0$. It suffices to show that tubing on 2-spheres does not change the cobordism maps.
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Proof

Factor through a neighborhood of the spheres. A neighborhood of each 2-sphere $N(S)$ is $D_2 \times S^2$. 

$\partial N(S) = S^1 \times S^2$. $C \cup C$ intersects $\partial N(S)$ in an unknot. 

$C \cup C$ intersects $N(S)$ in a disk $D$. 

$\hat{HFK}(S^1 \times S^2, U)$ has rank 1 in the important grading. 

We can replace $D$ with any disk $D'$ in $N(S)$ such that $\partial D' = \partial D$. 

Replace $D$ with $D' = S \setminus D$ to obtain $[0, 1] \times K_0$. 
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Corollary (Z.)

If there is a ribbon concordance from $K_0$ to $K_1$, then

$$g_3(K_0) \leq g_3(K_1).$$
Band sums

Definition

A knot $L$ is a band sum of (unlinked) knots $K_1, \ldots, K_n$ if it is obtained by attaching $n - 1$ (potentially complicated) bands to join $K_1, \ldots, K_n$ together.
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**Theorem (Gabai (1987) Scharlemann (1985))**

If \( L \) is a band sum of \( K_1 \) and \( K_2 \), then

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- Neither proof extends for $n > 2$. 
Miyazaki’s manipulation

**Theorem (Miyazaki 1998)**

*If $L$ is a band sum of $K_1, \ldots, K_n$, then*

$$K_1 \# \cdots \# K_n \leq L.$$
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![Figure: Changing a crossing of a band with a strand.](image)
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Figure: Changing a crossing of a band with a strand.

- This process terminates at $K_1 \# \cdots \# K_n$ together with some unlinked unknots, which can be capped off.
Corollary (Z.)

If $L$ is a band sum of $K_1, \ldots, K_n$ then

$$g_3(L) \geq g_3(K_1) + \cdots + g_3(K_n).$$
Strongly homotopy-ribbon concordances

Definition

A strongly homotopy-ribbon concordance is one whose complement can be built using only 1-handles and 2-handles.
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A strongly homotopy-ribbon concordance is one whose complement can be built using only 1-handles and 2-handles.

\[
\{\text{ribbon concordances}\} \subseteq \{\text{strongly homotopy-ribbon concordances}\} \\
\subseteq \{\text{homotopy-ribbon concordances}\} \\
\subsetneq \{\text{concordances}\}
\]
Strongly homotopy-ribbon concordances and knot Floer homology

Theorem (Maggie Miller, Z.)

If $C$ is a strongly homotopy-ribbon concordance from $K_0$ to $K_1$, then

$$F_C : \widehat{HFK}(K_0) \to \widehat{HFK}(K_1)$$

is an injection.
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- The proof uses a similar doubling trick, and also relies on the fact that tubing in a 2-sphere does not change the cobordism map.
Khovanov homology and ribbon concordances

Theorem (Levine, Z.)

*If C is a ribbon concordance, then the induced map on Khovanov homology

\[ Kh(C) : Kh(K_0) \rightarrow Kh(K_1) \]

is an injection.*
Khovanov homology and ribbon concordances

The proof follows from the previous description of the doubled concordance, as well as Bar-Natan’s “dotted cobordism maps”, and the tube cutting and sphere relations.
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\[ = 0 \quad \text{and} \quad = 1 \]

**Figure:** Bar-Natan’s local relations.
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**Definition**

*If $M$ is a module over $R[X]$, define $\text{Ord}_X(M)$ to be the minimum $n$ such that $X^n \cdot \text{Tor}(M) = \{0\}$.*
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**Theorem (Sarkar)**

If $K$ is a ribbon knot, and $2 \neq 0$ in $R$, then any ribbon disk for $K$ must have at least $\text{Ord}_X(Kh_{Lee}(K))$ bands.
Unfortunately $\text{Ord}_X(Kh_{\text{Lee}}(K))$ is usually small.
Sarkar’s ribbon distance and Khovanov homology

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- Unfortunately $\text{Ord}_X(\text{Kh}_{\text{Lee}}(K))$ is usually small.
- Only one example is known with $\text{Ord}_X(\text{Kh}(K)) > 2$ (Marengon-Manolescu 2018).
- The proof uses a doubling trick, with a new twist.
There is an analogous version of knot Floer homology $HFK^{-}(K)$, which is a module over the polynomial ring $\mathbb{F}_2[v]$. 

Inspired by Sarkar’s work, we proved:

Theorem (Juhász, Miller, Z.)

If $K$ is a ribbon knot, then any ribbon disk for $K$ must have at least $\text{Ord}_v(HFK^{-}(K))$ bands.
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Definition

If $K \subseteq S^3$, then the bridge number $\text{br}(K)$ is the smallest number of local maxima in any diagram of $K$. 

Corollary (Miller, Juhász, Z.)

If $K \subseteq S^3$, then $\text{Ord}_v(HFK^- (K)) \leq \text{br}(K) - 1$.

There is a fusion disk of $K \# K$ with $\text{br}(K) - 1$ saddles.

$\text{Ord}_v(K \# K) = \text{Ord}_v(K)$, by the connected sum formula, and duality.
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Knot Floer homology, torsion, and the bridge index

\[ \text{br}(T_{p,q}) = \min(p, q) \text{ if } p, q \text{ coprime (Schubert 1954)}. \]
Knot Floer homology, torsion, and the bridge index

- $\text{br}(T_{p,q}) = \min(p, q)$ if $p, q$ coprime (Schubert 1954).
- $\text{Ord}_v(\text{HFK}^{-}(K)) = \min(p, q) - 1$, so the bound is sharp.
Corollary (Juhász, Miller, Z.)

If $J$ is concordant to $T_{p,q}$, then

$$\text{br}(J) \geq \text{br}(T_{p,q}).$$
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- $N(T_{p,q}) = \min(p, q) - 1$, by work of DHST.
More generally:

**Theorem (Juhász, Miller, Z.)**

If there is a knot cobordism from $K_0$ to $K_1$ with $M$ local maxima, then

$$\text{Ord}_v\{HFK^-(K_0)\} \leq \max\{\text{Ord}_v(HFK^-(K_1)), M\} + 2g(S).$$
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**Example:** if there is a ribbon concordance from $K_0$ to $K_1$ with $b$ bands, then

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Outline of the proof of the fusion number bound

Adding a tube to the unshaded subregion of a decorated surface induces multiplication by $v$. 
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Figure: Adding a tube is multiplication by $v$. 
Consider a ribbon disk $D$ for $K$, with $b$ bands and $b + 1$ maxima.
Outline of the proof of the fusion number bound

- Consider a ribbon disk $D$ for $K$, with $b$ bands and $b + 1$ maxima.
- Write $D^\circ$ for the induced cobordism from $K$ to $U$ with $b$ bands and $b$ maxima.

$D^\circ \cup D^\circ$ is a concordance from $K$ to itself.
Tube the maxima of $D^\circ$ to the minima of $D^\circ$ with $b$ tubes.
Upon inspection, we arrive at a copy of $K \times [0,1]$ with $b$ tubes added.

So $v_b \cdot F D^\circ \cup D^\circ = v_b \cdot F K \times [0,1]$.
Noting that $F D^\circ \cup D^\circ$ annihilates Tor $v_b(HFK - v_b(K))$, the proof is complete.
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Gordon’s conjecture

Conjecture (Gordon 1981)

If \( K_0 \leq K_1 \) and \( K_1 \leq K_0 \), then \( K_0 = K_1 \).
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Theorem (Z., Levine-Z.)

If $K_0 \leq K_1$ and $K_1 \leq K_0$, then

$$\widehat{HFK}(K_0) \cong \widehat{HFK}(K_1) \quad \text{and} \quad \text{Kh}(K_0) \cong \text{Kh}(K_1),$$

as bigraded groups.
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