Ribbon concordances and knot Floer homology

Ian Zemke

October 21, 2019

Ribbon concordances

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- Ribbon concordance is not symmetric.
- Gordon's notation: $K_0 \leq K_1$.

A ribbon concordance

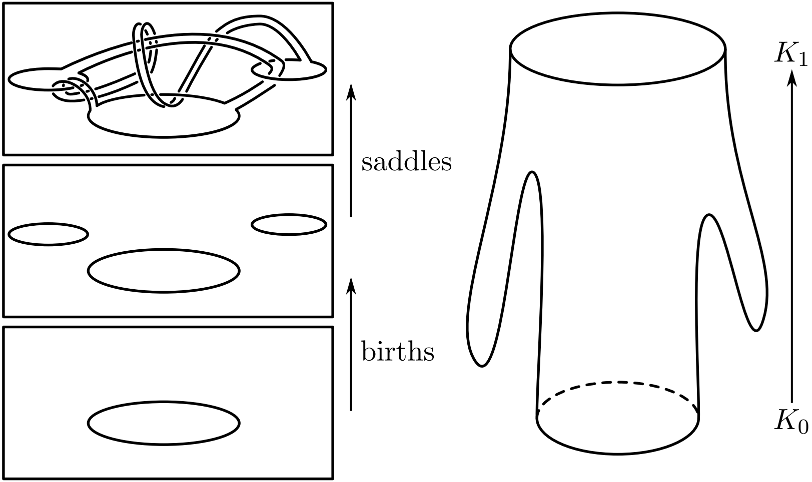

Figure: A ribbon concordance.

Gordon's work

Theorem (Gordon 1981)

If C is a ribbon concordance, then $\pi_1(K_0) \to \pi_1(C)$ is an injection, and $\pi_1(K_1) \to \pi_1(C)$ is a surjection.

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- $\pi_1(K_1) \twoheadrightarrow \pi_1(C)$: the complement of C is obtained by attaching 2-handles and 3-handles.
- $\pi_1(K_0) \hookrightarrow \pi_1(C)$: uses much harder 3-manifold topology.

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- Transfinite nilpotence: the lower central series becomes trivial at some ordinal.
- Fibered knots are transfinitely nilpotent, since $\pi_1(K)'$ is free.

Ribbon concordances and fiberedness

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Theorem (Silver 1992, Kochloukova 2006)

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- (Neuwirth 1964) If G is a knot group and G' is finitely generated, then G' is free.
- (Rapaport's conjecture 1975) If G is *knot-like* ($G/G' \cong \mathbb{Z}$ and G has deficiency 1) then G' finitely generated implies G' free.

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- Hence K_0 is fibered.

Kochloukova proved Rapaport's conjecture in 2006.

Knot Floer homology (Ozsváth and Szabó, Rasmussen)

If $K \subseteq S^3$, there is a bigraded group

$$\widehat{HFK}(K) = \bigoplus_{i,j \in \mathbb{Z}} \widehat{HFK}_i(K, j).$$

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- Categorifies the Alexander polynomial.
- Detects the Seifert genus:

$$g_3(K) = \max\{j : \widehat{HFK}(K, j) \neq \{0\}\}.$$

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- $CFK^\infty(K)$ is a graded, $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex over $\mathbb{F}_2[U, U^{-1}]$.
- $H_*(CFK^\infty(K)) \cong HF^\infty(S^3) \cong \mathbb{F}_2[U, U^{-1}]$.

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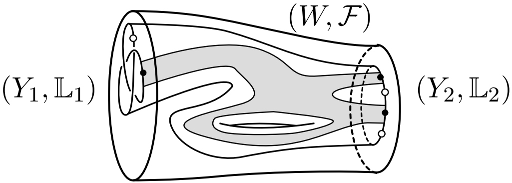

Figure: A decorated link cobordism.

Juhász's TQFT for $\widehat{HFL}$

To a decorated link cobordism

$$(W, \mathcal{F})\colon (Y_1, \mathbb{L}_1) \to (Y_2, \mathbb{L}_2),$$

Juhász associates a map

$$F_{W,\mathcal{F}}\colon \widehat{HFL}(Y_1, \mathbb{L}_1) \to \widehat{HFL}(Y_2, \mathbb{L}_2).$$

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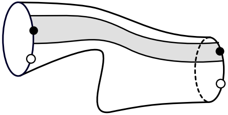

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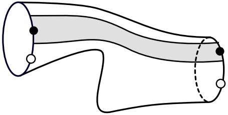

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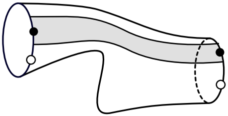

Figure: A decorated concordance.

- Studied by Juhász and Marengon.
- They proved the map preserves the Maslov and Alexander gradings.

A TQFT for the full knot Floer complex

Theorem (Z. 2017)

If $(W,\mathcal{F})\colon (Y_1,\mathbb{L}_1)\to(Y_2,\mathbb{L}_2)$ *is a decorated link cobordism and* $\mathfrak{s}\in \mathrm{Spin}^c(W)$, *there is a functorial chain map*

$$F^{\infty}_{W,\mathcal{F},\mathfrak{s}}\colon \mathcal{CFL}^{\infty}(Y_1,\mathbb{L}_1,\mathfrak{s}|_{Y_1})\to \mathcal{CFL}^{\infty}(Y_2,\mathbb{L}_2,\mathfrak{s}|_{Y_2}).$$

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- For a decorated concordance C, we obtain a bigraded map

$$F_C\colon CFK^\infty(K_0) \to CFK^\infty(K_1).$$

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- Alishahi and Eftekhary independently gave a similar construction, in terms of a different cobordism category.

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- *The map F_C^∞ admits a left inverse, i.e. a filtered graded map Π such that*

$$\Pi \circ F_C^\infty \simeq \mathrm{id}_{CFK^\infty(K_0)}\,.$$

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- It suffices to show that tubing on 2-spheres does not change the cobordism maps.

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- $\widehat{HFK}(S^1 \times S^2, \mathbb{U})$ has rank 1 in the important grading.
- We can replace D with any disk D' in $N(S)$ such that $\partial D' = \partial D$.
- Replace D with $D' = S \setminus D$ to obtain $[0,1] \times K_0$.

Monotonicity of the Seifert genus

Corollary (Z.)

If there is a ribbon concordance from K_0 to K_1, then

$$g_3(K_0) \leq g_3(K_1).$$

Band sums

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- Neither proof extends for $n > 2$.

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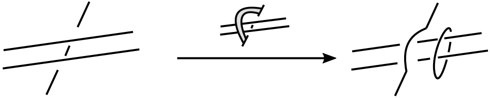

Figure: Changing a crossing of a band with a strand.

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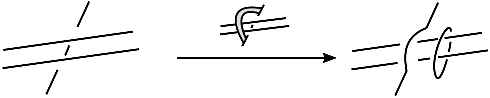

Figure: Changing a crossing of a band with a strand.

- This process terminates at $K_1 \# \cdots \# K_n$ together with some unlinked unknots, which can be capped off.

Superadditivity of the Seifert genus

Corollary (Z.)

If L is a band sum of $K_1, \ldots, K_n$ then

$$g_3(L) \geq g_3(K_1) + \cdots + g_3(K_n).$$

Strongly homotopy-ribbon concordances

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$$\begin{aligned}\{\text{ribbon concordances}\} &\subseteq \{\text{strongly homotopy-ribbon concordances}\}\\ &\subseteq \{\text{homotopy-ribbon concordances}\}\\ &\subsetneq \{\text{concordances}\}\end{aligned}$$

Strongly homotopy-ribbon concordances and knot Floer homology

Theorem (Maggie Miller, Z.)

If C is a strongly homotopy-ribbon concordance from K_0 to K_1, then

$$F_C \colon \widehat{HFK}(K_0) \to \widehat{HFK}(K_1)$$

is an injection.

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- The proof uses a similar doubling trick, and also relies on the fact that tubing in a 2-sphere does not change the cobordism map.

Khovanov homology and ribbon concordances

Theorem (Levine, Z.)

If C is a ribbon concordance, then the induced map on Khovanov homology

$$Kh(C)\colon Kh(K_0) \to Kh(K_1)$$

is an injection.

Khovanov homology and ribbon concordances

The proof follows from the previous description of the doubled concordance, as well as Bar-Natan's "dotted cobordism maps", and the tube cutting and sphere relations.

Khovanov homology and ribbon concordances

The proof follows from the previous description of the doubled concordance, as well as Bar-Natan's "dotted cobordism maps", and the tube cutting and sphere relations.

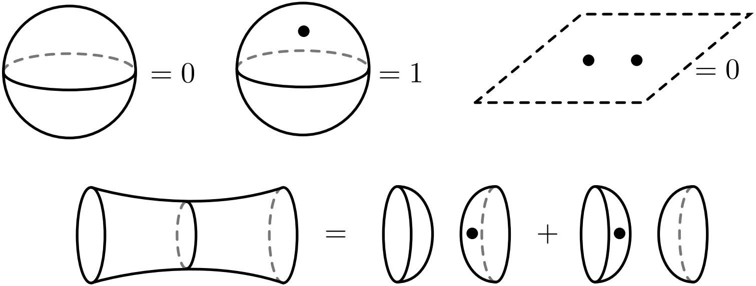

Figure: Bar-Natan's local relations.

Sarkar's ribbon distance and Khovanov homology

Sarkar considered the torsion order in Lee's deformation of Khovanov homology, $Kh_{Lee}(K)$, which is a finitely generated module over $R[X]$ (where R is a field).

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Theorem (Sarkar)

If K is a ribbon knot, and $2 \neq 0$ in R, then any ribbon disk for K must have at least $\mathrm{Ord}_X(Kh_{Lee}(K))$ bands.

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- Unfortunately $\mathrm{Ord}_X(Kh_{Lee}(K))$ is usually small.
- Only one example is known with $\mathrm{Ord}_X(Kh(K)) > 2$ (Marengon-Manolescu 2018).
- The proof uses a doubling trick, with a new twist.

Knot Floer homology, torsion, and the bridge index

There is an analogous version of knot Floer homology $HFK^-(K)$, which is a module over the polynomial ring $\mathbb{F}_2[v]$.

Knot Floer homology, torsion, and the bridge index

There is an analogous version of knot Floer homology $HFK^{-}(K)$, which is a module over the polynomial ring $\mathbb{F}_2[v]$. Inspired by Sarkar's work, we proved:

Theorem (Juhász, Miller, Z.)

If K is a ribbon knot, then any ribbon disk for K must have at least $\mathrm{Ord}_v(HFK^{-}(K))$ bands.

Knot Floer homology, torsion, and the bridge index

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If $K \subseteq S^3$ *then*

$$\mathrm{Ord}_v(HFK^-(K)) \leq \mathrm{br}(K) - 1.$$

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- There is a fusion disk of $K \# \overline{K}$ with $\mathrm{br}(K) - 1$ saddles.
- $\mathrm{Ord}_v(K \# \overline{K}) = \mathrm{Ord}_v(K)$, by the connected sum formula, and duality.

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- $\mathrm{br}(T_{p,q}) = \min(p, q)$ if p, q coprime (Schubert 1954).

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- $\mathrm{br}(T_{p,q}) = \min(p, q)$ if p, q coprime (Schubert 1954).
- $\mathrm{Ord}_v(HFK^-(K)) = \min(p, q) - 1$, so the bound is sharp.

Knot Floer homology, torsion, and the bridge index

Corollary (Juhász, Miller, Z.)

If J is concordant to $T_{p,q}$, then

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- $N(K) \leq \mathrm{Ord}_v(K)$, by work of DHST.
- $N(T_{p,q}) = \min(p, q) - 1$, by work of DHST.

Knot Floer homology, torsion, and the bridge index

More generally:

Theorem (Juhász, Miller, Z.)

If there is a knot cobordism from K_0 to K_1 with M local maxima, then

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Compare the effect of taking the connected sum of K_0 and another knot K to increase Ord_v

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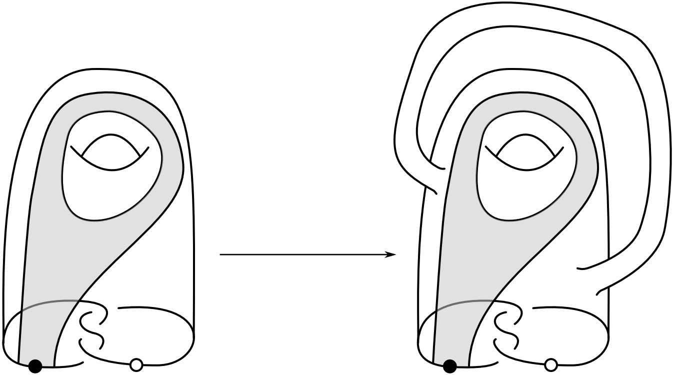

Figure: Adding a tube is multiplication by v.

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- So
$$v^b \cdot F_{\overline{D}^\circ \cup D^\circ} = v^b \cdot F_{K \times [0,1]}.$$
- Noting that $F_{\overline{D}^\circ \circ D^\circ}$ annihilates $\mathrm{Tor}_v(HFK_v^-(K))$, the proof is complete.

Gordon's conjecture

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Theorem (Z., Levine-Z.)

If $K_0 \leq K_1$ *and* $K_1 \leq K_0$*, then*

$$\widehat{HFK}(K_0) \cong \widehat{HFK}(K_1) \quad \textit{and} \quad Kh(K_0) \cong Kh(K_1),$$

as bigraded groups.

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- The Kanenobu knots are such a family (see Hedden-Watson).
- There are additional families of generalized Kanenobu knots (see Lobb).