Bordered perspectives on the link surgery formula

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- If $K \subseteq S^3$ is a knot, there is a relative version $\mathcal{CFK}(K)$, which takes the form of a chain complex over $\mathbb{F}[\mathscr{U}, \mathscr{V}]$, defined using a doubly pointed Heegaard diagram.

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- We think of U as acting by $\mathscr{U}\mathscr{V}$.
- Φ^K and Φ^{-K} are not $\mathbb{F}[\mathscr{U}, \mathscr{V}]$-equivariant, though they are $\mathbb{F}[U]$-equivariant. Homotopy equivalence in mapping cone formula is of chain complexes over $\mathbb{F}[[U]]$.

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3. To see this, $S^3 \setminus \nu(K_1 \# K_2)$ is obtained by gluing an annulus to $S^3 \setminus \nu(K_1)$ and $S^3 \setminus \nu(K_2)$, so that $\mu_1 \mapsto \mu_2$.

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4. This is the same as gluing complements together along a 1-handle, then gluing 2-handles along $\mu_1 * -\mu_2$ and $\lambda_1 * \lambda_2$, and then gluing a 3-handle.

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3. Goal: Construct a similar theory for $\boldsymbol{CF}^-$ using the link surgery formula.

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6. $\sigma \mathscr{U} = \mathscr{U} \sigma \quad \sigma \mathscr{V} = \mathscr{V} \sigma \quad \tau \mathscr{U} = \mathscr{V}^{-1} \tau \quad \text{and} \quad \tau \mathscr{V} = \mathscr{U} \mathscr{V}^2 \tau$.

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7. More symmetric description: write $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1 \cong \mathbb{F}[U, T, T^{-1}]$ where $U = \mathscr{U}\mathscr{V}$ and $T = \mathscr{V}$.

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1. If L is an n-component link with framing Λ, the link surgery formula determines a type-D module

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5. Φ^K gives terms weighted by σ. Φ^{-K} gives terms weighted by τ.

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3. Φ^K and Φ^{-K} satisfy the relations

$$\Phi^K \circ \mathscr{U} = \mathscr{U} \circ \Phi^K \qquad \Phi^K \circ \mathscr{V} = \mathscr{V} \circ \Phi^K$$

$$\Phi^{-K} \circ \mathscr{U} = \mathscr{V}^{-1} \circ \Phi^{-K} \qquad \Phi^{-K} \circ \mathscr{V} = \mathscr{U}\mathscr{V}^2 \circ \Phi^{-K}.$$

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$$\begin{tikzcd}[column sep=small] & \mathbf{x}^0 \arrow[dl, "\mathscr{V}"'] \arrow[dr, "\mathscr{U}"] & \\ \mathbf{y}^0 \arrow[dr, "\mathscr{U}\mathscr{V}^{-1}\sigma+\mathscr{V}^{-2}\tau"'] & & \mathbf{z}^0 \arrow[dl, "\sigma+\mathscr{U}\mathscr{V}\tau"] \\ & \mathbf{z}^1 & \end{tikzcd}$$

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6. Can view $\mathcal{D}_\lambda$ as an AA-bimodule

$$ {}_{\mathcal{K}}[\mathcal{D}_\lambda]_{\mathbb{F}[U]}, $$

where U acts by $\mathscr{U}\mathscr{V}$. (Type-A modules for other knots and links are similar).

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2. The right hand side has an action of $\mathbb{F}[U_1, \dots, U_n]$ (one U_i for each $\mathcal{D}_0$). This reflects the fact that the Manolescu-Ozsváth complex is a module over $\mathbb{F}[U_1, \dots, U_n]$.

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2. The Hopf link has complement $\mathbb{T}^2 \times [0,1]$. We may view the Hopf link complement as the mapping cylinder of a diffeomorphism which sends $\mu \mapsto \lambda$ and $\lambda \mapsto -\mu$.
3. The Hopf link gives a DA-bimodule ${}_{\mathcal{K}}\mathcal{H}^{\mathcal{K}}$ which has the effect of changing the boundary parametrization.

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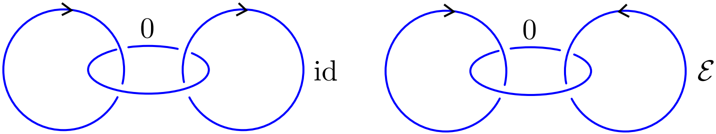

(Remove neighborhoods of arrow labeled components to get $\mathbb{T}^2 \times [0, 1]$).

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5. Known to be isomorphic to Heegaard Floer homology for many families of plumbed 3-manifolds. (Némethi, Ozsváth, Stipsicz, Szabó). The isomorphism in general was a conjecture.

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Gradings is in progress, but is (slightly technical) bookkeeping.

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Definition

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3. *An algebraic link is the intersection of a the boundary of a small ball centered at an isolated complex curve singularity in* $\mathbb{C}^2$.

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Theorem (Gorsky-Némethi (links), Hedden (knots))

Algebraic links are L-space link.

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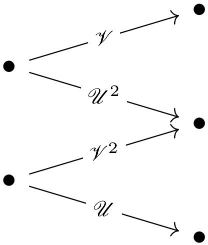

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1. Important since these complexes are computable from their Alexander polynomials.
2. An open question is how to properly generalize this result to links.

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4. $\mathcal{CFL}(T(3,3))$ and $\mathcal{CFL}(T(4,4))$ have 18 and 68, generators, resp.

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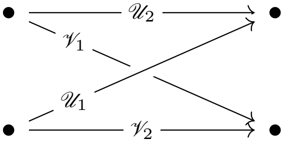
Examples:

$T(2,2)$



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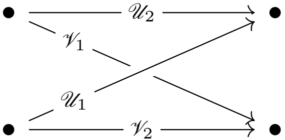

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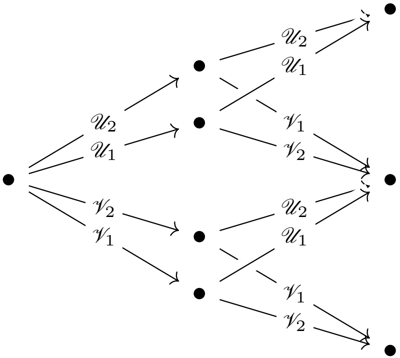

Thanks for listening!