A mapping cone formula for Involutive Heegaard Floer homology

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joint w/ K. Hendricks, J. Hom and M. Stoffregen

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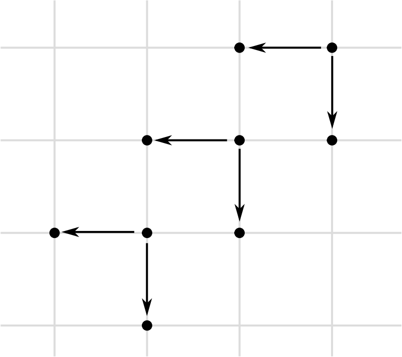

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If K is a knot in Y, a $\mathbb{Z}HS^3$, then

$$\boldsymbol{HF}^-(Y_n(K)) \cong H_*\left(\mathrm{Cone}\left(\mathbb{A} \xrightarrow{D_n} \mathbb{B}\right)\right),$$

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- Bold $\boldsymbol{HF}^-$ indicates coefficients in the power series $\mathbb{F}[[U]]$.
- When $Y = S^3$, the map D_n is explicitly computable from just $CFK^\infty(S^3,K)$.

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$$\mathbb{A} = \prod_{s \in \mathbb{Z}} \boldsymbol{A}_s \quad \text{and} \quad \mathbb{B} = \prod_{s \in \mathbb{Z}} \boldsymbol{B}_s,$$

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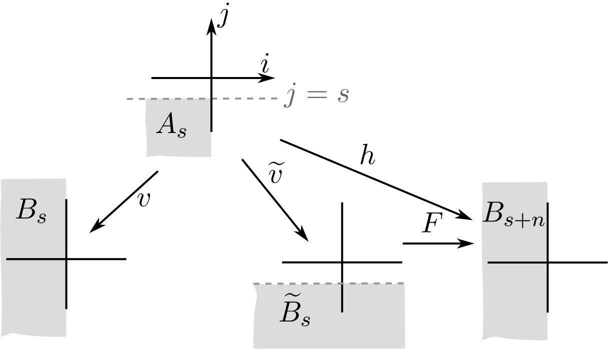

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- For knots in S^3, the homotopy type of the mapping cone is determined by just $CFK^\infty(K)$.
- Indeed, the only ambiguity is the homotopy equivalence $F\colon \widetilde{B}_s \to B_{s+n}$, but $\widetilde{B}_s$ and B_{s+n} are both homotopy equivalent to $CF^-(S^3) \simeq \mathbb{F}[U]$, so there is a unique homotopy equivalence, up to chain homotopy.

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- $\iota := \eta \circ \Psi_{\mathcal{H} \to \overline{\mathcal{H}}}$, where $\Psi_{\mathcal{H} \to \overline{\mathcal{H}}}$ is the map from naturality.

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- Applications to the homology cobordism group.
- E.g. $\exists\ \mathbb{Z}^\infty$ summand of $\Theta^3_{\mathbb{Z}}$ (Dai, Hom, Stoffregen, Truong).

Computing involutive Heegaard Floer homology

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For example, is there an analog of the mapping cone formula?

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Here, $[0]$ denotes the Spin^c structure identified with 0 under Spin$^c(Y_n(K)) \cong \mathbb{Z}_n$.

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If $K \subseteq Y$ is a framed knot there is an exact sequence

$$\cdots \widehat{HFI}(Y) \to \widehat{HFI}(Y_0) \to \widehat{HFI}(Y_1) \to \widehat{HFI}(Y) \cdots .$$

More computational tools

Theorem (Dai–Manolescu)

Involutive Heegaard Floer homology is computable for three manifolds obtained by plumbing along almost rational graphs. (This includes all Seifert fibered homology 3-spheres).

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Theorem (Hendricks–Manolescu–Z.)

If Y_1 *and* Y_2 *are homology spheres, then under the equivalence* $CF^-(Y_1 \# Y_2) \simeq CF^-(Y_1) \otimes CF^-(Y_2)$, *the involution* $\iota_{Y_1 \# Y_2}$ *is equivalent to* $\iota_{Y_1} \otimes \iota_{Y_2}$.

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Theorem (Z.)

Given $(CFK^\infty(K_1), \iota_{K_1})$ and $(CFK^\infty(K_2), \iota_{K_2})$, there is a formula for $(CFK^\infty(K_1 \# K_2), \iota_{K_1 \# K_2})$.

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Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

(Weak version) If K is a knot in a $\mathbb{Z}HS^3$ Y, then there is a homotopy equivalence

$$\boldsymbol{CFI}^-(Y_n(K)) \simeq \begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad D_n \quad} & \mathbb{B} \\ {\scriptstyle Q(\mathrm{id}+\iota_{\mathbb{A}})}\Big\downarrow & \overset{QH_n}{\searrow} & \Big\downarrow{\scriptstyle Q(\mathrm{id}+\iota_{\mathbb{B}})} \\ Q\cdot\mathbb{A} & \xrightarrow{\quad D_n \quad} & Q\cdot\mathbb{B} \end{array}$$

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Not amenable for computations, since changing H_n could change the homotopy type.

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- *Most importantly, these conditions completely determine the homotopy type of the mapping cone.*

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- In particular, the homotopy type of $CFI^-(S^3_n(K))$ is completely determined by, and is easily computed from $(CFK^\infty(K), \iota_K)$.
- We prove similar mapping cone formulas for rational surgeries and 0-surgeries (and prove a similar computability result for knots in S^3).

Diagrams when $n = 1$

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Here is D_1 on $\mathbb{X}_1$:

$$\begin{array}{cccccccccccc}
\cdots & A_{-2} & & A_{-1} & & A_0 & & A_1 & & A_2 & & \cdots \\
\swarrow^{v} & & \searrow^{h} \;\; \swarrow^{v} & & \searrow^{h} \;\; \swarrow^{v} & & \searrow^{h} \;\; \swarrow^{v} & & \searrow^{h} \;\; \swarrow^{v} & & \searrow^{h} & \\
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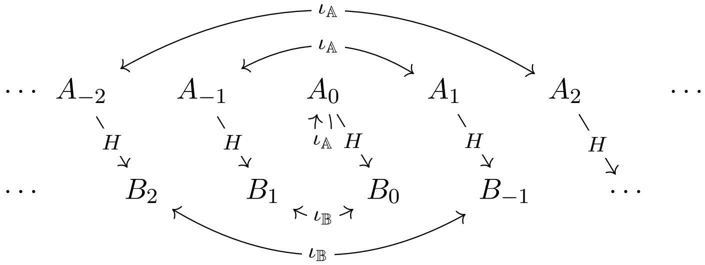
Here is $\iota_{\mathbb{X}}$ on $\mathbb{X}_1$:



(Note, B_s are shown in reverse order).

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We say an algebraic ι_K*-complex is of L-space type if* $H_*(B_s) \cong \mathbb{F}[U]$.

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For complexes arising from a knot K in Y, $H_*(B_s) \cong HF^-(Y)$.

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Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

There is a well-defined (algebraic) map

$$\mathbb{XI}_n^{\mathrm{alg}} \colon \frac{\{\iota_K\text{-complexes of } L\text{-space type}\}}{\simeq} \longrightarrow \frac{\{\iota\text{-complexes}\}}{\simeq},$$

sending an algebraic ι_K-complex to a model of the involutive mapping cone with the above factorization properties.

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The *standard complexes* approach of Dai, Hom, Stoffregen and Truong give an algebraic obstruction to being in the span of Seifert fibered spaces, and we use the cone formula to find an example.

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Definition

Two ι-complexes (C_1, ι_1) and (C_2, ι_2) are locally equivalent if there are grading preserving chain maps

$$F\colon C_1 \to C_2 \quad \textit{and} \quad G\colon C_2 \to C_1$$

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The local class of $(CF^-(Y), \iota)$ contains all the algebraic obstructions to homology cobordism coming from HFI.

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- If n is even, then $[0]$ and $[n/2]$ are the only self-conjugate Spin^c structures.

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A similar story holds for rational surgeries.

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  if n is sufficiently large.
- A *surgery exact sequence*

  $$\cdots \underline{\boldsymbol{HF}}^-(Y) \to \boldsymbol{HF}^-(Y_n) \to \boldsymbol{HF}^-(Y_{n+m}) \to \underline{\boldsymbol{HF}}^-(Y) \cdots,$$

  where $\underline{\boldsymbol{HF}}^-(Y) = \bigoplus^m \boldsymbol{HF}^-(Y)$.

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- Define a "cobordism" map

$$\boldsymbol{CFI}^-(Y_{n+1}) \to \boldsymbol{CFI}^-(Y)$$

as well as a quasi-isomorphism

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- The exact sequence for mapping cones from homological algebra gives the surgery exact sequence.

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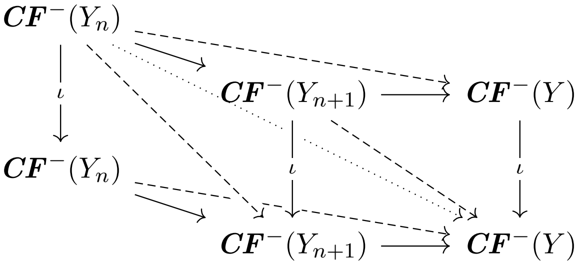
$$\Phi\colon \boldsymbol{CFI}^-(Y_n) \to \mathrm{Cone}(\boldsymbol{CFI}^-(Y_{n+1}) \to \boldsymbol{CFI}^-(Y))$$

we start by building a hypercube (i.e. a cubical diagram whose total complex is a chain complex)



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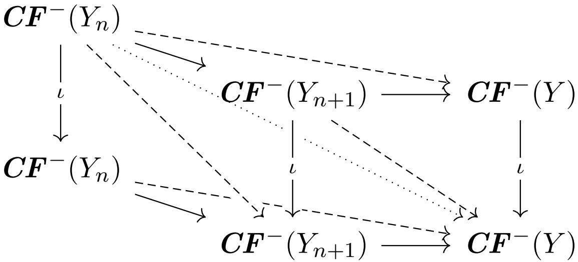

- Furthermore, the maps along top coincide with the maps along the bottom.

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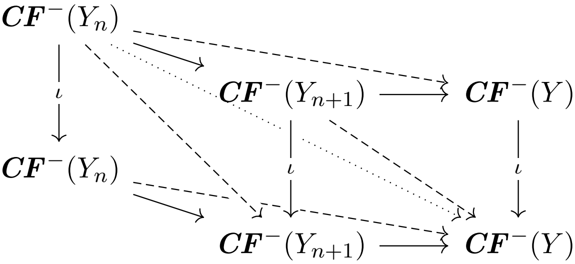

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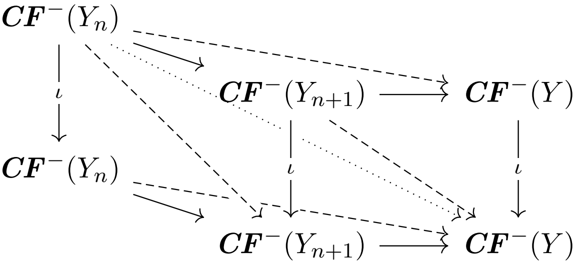

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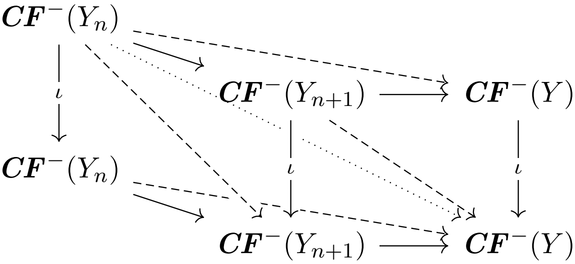

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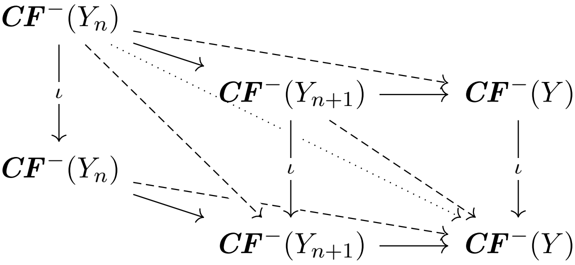

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- The challenging part which is new to our work is the length 3 dotted arrow.

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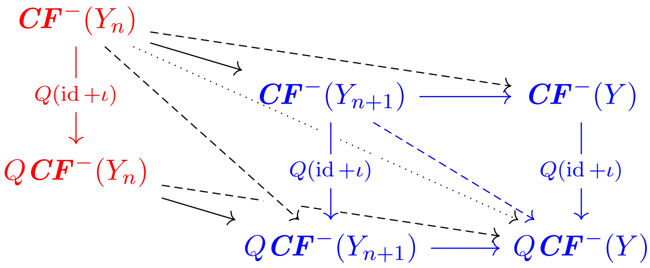

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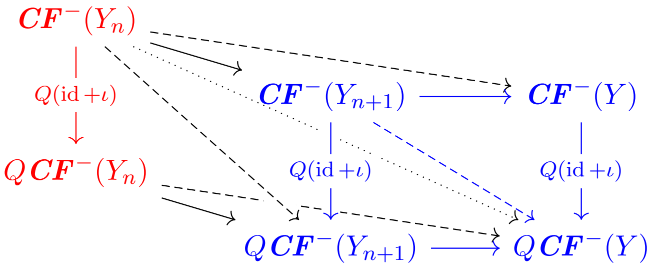

This is the same as a chain map from $\boldsymbol{CFI}^-(Y_n)$ to $\mathrm{Cone}(\boldsymbol{CFI}^-(Y_{n+1}) \to \boldsymbol{CFI}^-(Y))$.

Thanks for listening!