A mapping cone formula for Involutive Heegaard Floer homology

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Heegaard Floer homology

If $Y$ is a 3-manifold, and $s \in \text{Spin}^c(Y)$, Ozsváth and Szabó construct three $F[U]$-modules $HF^-(Y, s), HF^+(Y, s)$ and $HF^{\infty}(Y, s)$. If $Y$ is a $\mathbb{Z}_{HS}^3$ and $K \subseteq Y$, Ozsváth and Szabó constructed a graded chain complex, $CFK^{\infty}(Y, K)$, which is filtered by $\mathbb{Z} \oplus \mathbb{Z}$. 
If $Y$ is a 3-manifold, and $\mathfrak{s} \in \text{Spin}^c(Y)$, Ozsváth and Szabó construct three $\mathbb{F}[U]$-modules $HF^-(Y, \mathfrak{s})$, $HF^+(Y, \mathfrak{s})$ and $HF^\infty(Y, \mathfrak{s})$. 
Heegaard Floer homology

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- If $Y$ is a $\mathbb{Z}HS^3$ and $K \subseteq Y$, Ozsváth and Szabó constructed a graded chain complex, $CFK^\infty(Y, K)$, which is filtered by $\mathbb{Z} \oplus \mathbb{Z}$. 
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Ozsváth and Szabó’s mapping cone formula

**Theorem (Ozsváth–Szabó)**

*If* $K$ *is a knot in* $Y$, *a* $\mathbb{Z}HS^3$, *then*

$$HF^{-}(Y_n(K)) \cong H_* \left( \text{Cone} \left( \begin{array}{c} \mathbb{A} \\ D_n \rightarrow \mathbb{B} \end{array} \right) \right),$$

*where* $\mathbb{A}$ *and* $\mathbb{B}$ *are chain complexes obtained from subcomplexes of* $CFK^\infty(Y, K)$, *and* $D_n$ *is a chain map.*
Theorem (Ozsváth–Szabó)

If $K$ is a knot in $Y$, a $\mathbb{Z}HS^3$, then

$$HF^-(Y_n(K)) \cong H_* \left( \text{Cone} \left( \bigtriangleup \xrightarrow{D_n} \bigcirc \right) \right),$$

where $\bigtriangleup$ and $\bigcirc$ are chain complexes obtained from subcomplexes of $\text{CFK}^\infty(Y, K)$, and $D_n$ is a chain map.

- Bold $HF^-$ indicates coefficients in the power series $\mathbb{F}[[U]]$. 

Theorem (Ozsváth–Szabó)

*If* $K$ *is a knot in* $Y$, *a* $\mathbb{Z}HS^3$, *then*

$$\text{HF}^{-}(Y_n(K)) \cong H_\ast \left( \text{Cone} \left( A \xrightarrow{D_n} B \right) \right),$$

*where* $A$ *and* $B$ *are chain complexes obtained from subcomplexes of* $\text{CFK}^\infty(Y,K)$, *and* $D_n$ *is a chain map.*

- Bold $\text{HF}^{-}$ indicates coefficients in the power series $\mathbb{F}[[U]]$.
- When $Y = S^3$, the map $D_n$ is explicitly computable from just $\text{CFK}^\infty(S^3, K)$. 
Ozsváth and Szabó’s mapping cone formula

\[ A = \prod_{s \in \mathbb{Z}} A_s \quad \text{and} \quad B = \prod_{s \in \mathbb{Z}} B_s, \]

where \( A_s \) and \( B_s \) are subcomplexes of \( \text{CFK}_\infty(K) \), completed over \( F[[U]] \).

\( B_s \cong \text{CF}^{-}(Y) \) for all \( s \). (Actually, \( B_s \) is independent of \( s \)).

\( A_s \) is a subcomplex of \( B_s \) for all \( s \).
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Ozsváth and Szabó’s mapping cone formula

\[ \mathcal{A} = \prod_{s \in \mathbb{Z}} \mathcal{A}_s \quad \text{and} \quad \mathcal{B} = \prod_{s \in \mathbb{Z}} \mathcal{B}_s, \]

where \( \mathcal{A}_s \) and \( \mathcal{B}_s \) are subcomplexes of \( \text{CFK}^\infty(K) \), completed over \( \mathbb{F}[[U]] \).

- \( \mathcal{B}_s \simeq CF^{-}(Y) \) for all \( s \). (Actually, \( \mathcal{B}_s \) is independent of \( s \)).
Ozsváth and Szabó’s mapping cone formula

\[ A = \prod_{s \in \mathbb{Z}} A_s \quad \text{and} \quad B = \prod_{s \in \mathbb{Z}} B_s, \]

where \( A_s \) and \( B_s \) are subcomplexes of \( CFK^\infty(K) \), completed over \( \mathbb{F}[[U]] \).

- \( B_s \cong CF^-(Y) \) for all \( s \). (Actually, \( B_s \) is independent of \( s \)).
- \( A_s \) is a subcomplex of \( B_s \) for all \( s \).
Ozsváth and Szabó’s mapping cone formula

\[ D_n = v + h. \]

\( v \) sends \( A_s \) to \( B_s \), while \( h \) sends \( A_s \) to \( B_s + n \). 

\( v \) is the inclusion of \( A_s \) into \( B_s \).

\( h \) is the inclusion of \( A_s \) into \( \tilde{B}_s \), composed with a homotopy equivalence \( F: \tilde{B}_s \to B_s \), where \( \tilde{B}_s \) is as follows:

\[ A_s \to B_s \mapsto \ j = s \cdot v B_s + n \tilde{B}_s \tilde{v} F h. \]
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Ozsváth and Szabó’s mapping cone formula

For knots in $S^3$, the homotopy type of the mapping cone is determined by just $\text{CFK}^\infty(K)$. Indeed, the only ambiguity is the homotopy equivalence $F: \tilde{B}_s \to B_s + n$, but $\tilde{B}_s$ and $B_s + n$ are both homotopy equivalent to $\text{CF}^{-}(S^3) \cong F[\mathbb{I}]$, so there is a unique homotopy equivalence, up to chain homotopy.
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Indeed, the only ambiguity is the homotopy equivalence $F: \tilde{B}_s \to B_{s+n}$, but $\tilde{B}_s$ and $B_{s+n}$ are both homotopy equivalent to $CF^-(S^3) \cong \mathbb{F}[U]$, so there is a unique homotopy equivalence, up to chain homotopy.
Suppose $Y$ a 3-manifold, $s \in \text{Spin}^c(Y)$ and $s = s$. Hendricks and Manolescu study a homotopy involution $\iota: \text{CF}^-(Y, s) \to \text{CF}^-(Y, s)$. If $H = (\Sigma, \alpha, \beta)$ is a Heegaard diagram, there is a canonical isomorphism $\eta: \text{CF}^-(H, s) \to \text{CF}^-(H, s)$ where $H = (\Sigma, \beta, \alpha)$. $\iota := \eta \circ \Psi_{H \to H}$, where $\Psi_{H \to H}$ is the map from naturality.
Suppose $Y$ a 3-manifold, $s \in \text{Spin}^c(Y)$ and $\bar{s} = s$. 
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If \( \mathcal{H} = (\Sigma, \alpha, \beta) \) is a Heegaard diagram, there is a canonical isomorphism

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where $\bar{\mathcal{H}} = (\bar{\Sigma}, \beta, \alpha)$.

$$\iota := \eta \circ \Psi_{\mathcal{H} \to \mathcal{H}},$$

where $\Psi_{\mathcal{H} \to \mathcal{H}}$ is the map from naturality.
Involution Heegaard Floer homology

Hendricks and Manolescu define

$$\text{CFI}^-(Y, s) := \text{Cone} \left( \text{CF}^- (Y, s) \xrightarrow{Q(id + \iota)} \text{CF}^- (Y, s) \right)$$

Module over $\mathbb{F}[[U, Q]] / Q^2$.

Applications to the homology cobordism group. E.g. $\exists Z^\infty$ summand of $\Theta^3 Z$ (Dai, Hom, Stoffregen, Truong).
Hendricks and Manolescu define

\[ CFI^{-}(Y, \mathfrak{s}) := \text{Cone} \left( CF^{-}(Y, \mathfrak{s}) \xrightarrow{Q(\text{id} + \iota)} Q \cdot CF^{-}(Y, \mathfrak{s}) \right). \]
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Module over \( \mathbb{F}[U, Q]/Q^2 \).

Applications to the homology cobordism group.

E.g. \( \exists \mathbb{Z}^\infty \) summand of \( \Theta^3_{\mathbb{Z}} \) (Dai, Hom, Stoffregen, Truong).
How can we compute $HFI^{-}(Y)$?
Computing involutive Heegaard Floer homology

Question

*How can we compute $HFI^-(Y)$?*

For example, is there an analog of the mapping cone formula?
Hendricks and Manolescu also defined a knot involution $\iota_K$: \[ \text{CFK}_\infty(K) \to \text{CFK}_\infty(K). \]

**Theorem (Hendricks–Manolescu)**

If $n$ is large, and $K \subseteq Y$ is a knot in a $\mathbb{Z}_{HS}^3$, $Y$, then
\[ (\text{CF}^-(Y^n(K)), [0]), \iota_K) \cong (A_0, \iota_K), \]
where \( \cong \) denotes homotopy equivalence of $\iota$-complexes.

Hence $HFI^-(Y^n(K), [0]) \cong H^*(\text{Cone}(A_0 Q \overset{id + \iota_K}{\longrightarrow} Q \cdot A_0))$.

Here, $[0]$ denotes the Spin$^c$ structure identified with 0 under Spin$^c(Y^n(K)) \cong \mathbb{Z}_n$. 
Previous results

Hendricks and Manolescu also defined a knot involution

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*where* \( \simeq \) *denotes homotopy equivalence of* \( \iota \)-*complexes. Hence*

\[
\text{HFI}^-(Y_n(K), [0]) \simeq H_* \left( \text{Cone} \left( A_0 \xrightarrow{Q(\text{id} + \iota_K)} Q \cdot A_0 \right) \right).
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Previous results

Hendricks and Manolescu also defined a knot involution

$$\iota_K : CF^K(\mathfrak{K}) \to CF^K(\mathfrak{K}).$$

**Theorem (Hendricks–Manolescu)**

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Here, \([0]\) denotes the \(\text{Spin}^c\) structure identified with 0 under \(\text{Spin}^c(Y_n(K)) \cong \mathbb{Z}_n\).
Using bordered Floer homology, \( \widehat{HFI}(Y) \) may be computed combinatorially.
Computing involutive Heegaard Floer homology

Theorem (Hendricks–Lipshitz)

Using bordered Floer homology, \( \widehat{HF}_I(Y) \) may be computed combinatorially.

If \( K \subseteq Y \) is a framed knot there is an exact sequence

\[ \cdots \rightarrow \widehat{HF}_I(Y_0) \rightarrow \widehat{HF}_I(Y_1) \rightarrow \widehat{HF}_I(Y) \rightarrow \cdots \]
Theorem (Dai–Manolescu)

*Involutive Heegaard Floer homology is computable for three manifolds obtained by plumbing along almost rational graphs. (This includes all Seifert fibered homology 3-spheres)*.
More computational tools
More computational tools

**Theorem (Hendricks–Manolescu–Z.)**

If $Y_1$ and $Y_2$ are homology spheres, then under the equivalence $CF^{-}(Y_1 \# Y_2) \simeq CF^{-}(Y_1) \otimes CF^{-}(Y_2)$, the involution $\nu_{Y_1 \# Y_2}$ is equivalent to $\nu_{Y_1} \otimes \nu_{Y_2}$. 
More computational tools

Theorem (Hendricks–Manolescu–Z.)

If $Y_1$ and $Y_2$ are homology spheres, then under the equivalence $\mathcal{CF}^{-}(Y_1 \# Y_2) \simeq \mathcal{CF}^{-}(Y_1) \otimes \mathcal{CF}^{-}(Y_2)$, the involution $\iota_{Y_1 \# Y_2}$ is equivalent to $\iota_{Y_1} \otimes \iota_{Y_2}$.

Theorem (Z.)

Given $(\mathcal{CFK}^\infty(K_1), \iota_{K_1})$ and $(\mathcal{CFK}^\infty(K_2), \iota_{K_2})$, there is a formula for $(\mathcal{CFK}^\infty(K_1 \# K_2), \iota_{K_1 \# K_2})$. 
New developments: exact sequences

Theorem (In prep. Hendricks–Hom–Stoffregen–Z.)

If $K$ is a framed knot in $Y$, then there is an exact sequence

$$
\cdots \to H^{FI}_{\mathbb{Z}}(Y) \to H^{FI}_{\mathbb{Z}}(Y_0) \to H^{FI}_{\mathbb{Z}}(Y_1) \to H^{FI}_{\mathbb{Z}}(Y) \to \cdots
$$

If $K$ is a knot in a $\mathbb{Z}_{HS}^3 Y$, then there is an exact sequence

$$
\cdots \to H^{FI}_{\mathbb{Z}}(Y) \to H^{FI}_{\mathbb{Z}}(Y_n) \to H^{FI}_{\mathbb{Z}}(Y_n+m) \to H^{FI}_{\mathbb{Z}}(Y) \to \cdots
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Bold denotes coefficients in $\mathbb{Z}[[U]]$. Underline denotes twisted coefficients.
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Theorem (In prep. Hendricks–Hom–Stoffregen–Z.)

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Bold denotes coefficients in $\mathbb{F}[[U]]$. Underline denotes twisted coefficients.
The involutive mapping cone formula (weak form)

Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

(Weak version) If \( K \) is a knot in a \( \mathbb{Z} \text{HS}^3(\mathbb{Y}) \), then there is a homotopy equivalence

\[
\text{CFI}^{-}(\mathbb{Y}_n(K)) \cong A \cdot B \cdot Q \cdot (\text{id} + \iota_A) \cdot D^n \cdot QH^n \cdot Q \cdot (\text{id} + \iota_B) \cdot D^n
\]

Not amenable for computations, since changing \( H_n \) could change the homotopy type.
The involutive mapping cone formula (weak form)

**Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)**

*(Weak version)* If $K$ is a knot in a $\mathbb{Z}HS^3 Y$, then there is a homotopy equivalence

$$\text{CFI}^-(Y_n(K)) \cong Q(id + \iota_A)$$

Not amenable for computations, since changing $H_n$ could change the homotopy type.
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(Weak version) If \( K \) is a knot in a \( \mathbb{Z}HS^3 Y \), then there is a homotopy equivalence

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\cong Q \cdot A \\
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Not amenable for computations, since changing \( H_n \) could change the homotopy type.
The involutive mapping cone formula for knots in $S^3$:

**Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)**

| $\iota_A$ sends $A_s$ to $A_{s-1}$, and $\iota_K$ sends $A_s$ to $A_{s+1}$. |
| $\iota_B$ is the composition of $U_s \iota_K$, which sends $B_s$ to $\tilde{B}_{s-1}$, followed by a homotopy equivalence from $\tilde{B}_{s-1}$ to $B_{s-1} + n$. |

Most importantly, these conditions completely determine the homotopy type of the mapping cone.
The involutive mapping cone formula for knots in $S^3$:

**Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)**

(Strong version) If $K$ is a knot in a $S^3$, then we may choose the maps to satisfy the following:

\[ D_n = v + h, \text{ where } h \text{ factors through } \tilde{v} : A_s \rightarrow \tilde{B}_s. \]

\[ H_n \text{ factors as } \tilde{v}, \text{ followed by a map from } \tilde{B}_s \text{ to } B_s. \]

\[ \iota_A \text{ is } U_s \iota_K, \text{ and sends } A_s \text{ to } A_s. \]

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**Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)**

*(Strong version)* If $K$ is a knot in a $S^3$, then we may choose the maps to satisfy the following:

- $D_n = \nu + h$, where $h$ factors through $\tilde{\nu} : A_s \hookrightarrow \tilde{B}_s$.
- $H_n$ factors as $\tilde{\nu}$, followed by a map from $\tilde{B}_s$ to $B_{-s}$.
The involutive mapping cone formula for knots in $S^3$:

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- $\iota_A$ is $U^s \iota_K$, and sends $A_s$ to $A_{-s}$.
The involutive mapping cone formula for knots in $S^3$:

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- $\iota_A$ is $U^s \iota_K$, and sends $A_s$ to $A_{-s}$.
- $\iota_B$ is the composition of $U^s \iota_K$, which sends $B_s$ to $\tilde{B}_{-s}$, followed by a homotopy equivalence from $\tilde{B}_{-s}$ to $B_{-s+n}$.

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- Most importantly, these conditions completely determine the homotopy type of the mapping cone.
On the strong version of the mapping cone formula

In particular, the homotopy type of $\text{CFI}^-(S^3_n(K))$ is completely determined by, and is easily computed from $(\text{CFK}_\infty(K), \iota K)$. We prove similar mapping cone formulas for rational surgeries and 0-surgeries (and prove a similar computability result for knots in $S^3$).
In particular, the homotopy type of $\text{CFI}^-(S^3_n(K))$ is completely determined by, and is easily computed from $(\text{CFK}^\infty(K), \iota_K)$.
On the strong version of the mapping cone formula

- In particular, the homotopy type of $CFI^-(S^3_n(K))$ is completely determined by, and is easily computed from $(CFK^\infty(K), \iota_K)$.
- We prove similar mapping cone formulas for rational surgeries and 0-surgeries (and prove a similar computability result for knots in $S^3$).
Diagrams when $n = 1$
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Here is $D_1$ on $X_1$

\[
\begin{array}{cccccc}
\cdots & A_{-2} & A_{-1} & A_0 & A_1 & A_2 & \cdots \\
& v & / & h & v & / & h & v & h & v & / & \cdots \\
\vdots & B_{-1} & B_0 & B_1 & B_2 & \cdots
\end{array}
\]
Diagrams when $n = 1$
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Here is $\iota_X$ on $X_1$:

(Note, $B_s$ are shown in reverse order).
On the algebra of the involutive mapping cone

It's often useful to consider the algebraic categories of $\iota$-complexes and $\iota_K$-complexes. These are the categories consisting of $F[U]$-chain complexes, equipped with involutions, and an extra filtration structure for $\iota_K$-complexes.

Definition: We say an algebraic $\iota_K$-complex is of L-space type if $H^*(B_s) \cong F[U]$.

For complexes arising from a knot $K$ in $Y$, $H^*(B_s) \cong HF^-(Y)$. 
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On the algebra of the involutive mapping cone

Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

There is a well-defined (algebraic) map

$$XI_{alg}^n \colon \{\iota K\text{-complexes of L-space type}\} \rightarrow \{\iota\text{-complexes}\} \rightarrow \text{,}$$

sending an algebraic $$\iota K$$-complex to a model of the involutive mapping cone with the above factorization properties.
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\[ \Xi_{\text{alg}}^n : \{ \iota_K\text{-complexes of } L\text{-space type} \} \xrightarrow{\sim} \{ \iota\text{-complexes} \}, \]

sending an algebraic \( \iota_K \)-complex to a model of the involutive mapping cone with the above factorization properties.
An application to the homology cobordism group

Expanding on the work of Dai, Hom, Stoffregen and Truong, we prove the following:

Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

The homology cobordism group is not generated by Seifert fibered homology 3-spheres.

Previous work of Frøyshov (unpublished), F. Lin (2017) and Stoffregen (2020) construct classes in $\Theta^3$ which are not represented by Seifert fibered spaces. However none of these proofs imply that the classes are not connected sums of such classes.

The standard complexes approach of Dai, Hom, Stoffregen and Truong give an algebraic obstruction to being in the span of Seifert fibered spaces, and we use the cone formula to find an example.
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The *standard complexes* approach of Dai, Hom, Stoffregen and Truong give an algebraic obstruction to being in the span of Seifert fibered spaces, and we use the cone formula to find an example.
More applications: local equivalence classes

Definition

Two \( \iota \)-complexes \((C_1,\iota_1)\) and \((C_2,\iota_2)\) are locally equivalent if there are grading preserving chain maps \(F: C_1 \to C_2\) and \(G: C_2 \to C_1\) such that \(F\iota_1 + \iota_2 F \simeq 0\) and \(G\iota_2 + \iota_1 G \simeq 0\), such that \(F\) and \(G\) become isomorphisms on homology after inverting \(U\).

The local class of \((C^F - (Y),\iota)\) contains all the algebraic obstructions to homology cobordism coming from \(HFI\).
More applications: local equivalence classes

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Two ω-complexes \((C_1, \omega_1)\) and \((C_2, \omega_2)\) are locally equivalent if there are grading preserving chain maps

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such that \(F \omega_1 + \omega_2 F \simeq 0\) and \(G \omega_2 + \omega_1 G \simeq 0\), such that \(F\) and \(G\) become isomorphisms on homology after inverting \(U\).
More applications: local equivalence classes

**Definition**

Two $\nu$-complexes $(C_1, \nu_1)$ and $(C_2, \nu_2)$ are locally equivalent if there are grading preserving chain maps

$$F: C_1 \to C_2 \quad \text{and} \quad G: C_2 \to C_1$$

such that $F\nu_1 + \nu_2 F \simeq 0$ and $G\nu_2 + \nu_1 G \simeq 0$, such that $F$ and $G$ become isomorphisms on homology after inverting $U$.

The local class of $(CF^-(Y), \nu)$ contains all the algebraic obstructions to homology cobordism coming from $HFI$. 
More applications: local equivalence classes

Recall:

If $K \subseteq S^3$, then $\text{Spin}^c(S^3 \#_{n} (K)) \sim = \mathbb{Z}/n$. If $n$ is odd, then $[0]$ is the only self-conjugate Spin$^c$ structure. If $n$ is even, then $[0]$ and $[n/2]$ are the only self-conjugate Spin$^c$ structures.
Recall:

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- If $n$ is even, then $[0]$ and $[n/2]$ are the only self-conjugate $\text{Spin}^c$ structures.
Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

Suppose $K \subseteq S^3$, and $n > 0$.

Then \( \text{CFI}^{-}(S^3_n(K), [0]) \) is locally equivalent to \( (A_0, \iota_K) \).

Also, \( \text{CFI}^{-}(S^3_2^n(K), [n]) \) is locally equivalent to

\[ A_n \times A_n \times B_n \varepsilon \]

with the involution which swaps the two copies of \( A_n \) via the identity map.

A similar story holds for rational surgeries.
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$$
\begin{array}{c}
A_n \\
\Downarrow \nu \\
B_n
\end{array}
\quad
\begin{array}{c}
A_n \\
\Downarrow \nu'
\end{array}
$$

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More applications: correction terms

Theorem (In prep., Hendricks–Hom–Stoffregen–Z.)

Suppose $K \subseteq S^3$, and $n > 0$.

$$d(S^3_n(K), [0]) = d(L(n, 1), [0]) - 2V_0(K).$$

$$d(S^3_2^n(K), [n]) = d(S^3_2^n(K), [n]).$$

$$d(S^3_2^n(K), [n]) = d(L(2n, 1), [n]).$$

This is an analog of a result by Ni and Wu, concerning the ordinary $d$-invariants of surgeries.
More applications: correction terms

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This is an analog of a result by Ni and Wu, concerning the ordinary $d$-invariants of surgeries.
Idea of proof

Recall the main steps in Ozsváth and Szabó’s proof of the ordinary mapping cone formula:

A large surgeries formula, which states that $\tilde{HF}^{-\left(\frac{Y}{n}\right)}(K)\sim H^\ast\left(A_i(K)\right)$, if $n$ is sufficiently large.

An exact sequence

$\cdots \tilde{HF}^{-\left(\frac{Y}{n}\right)} \rightarrow \tilde{HF}^{-\left(\frac{Y}{n}+m\right)} \rightarrow \tilde{HF}^{-\left(\frac{Y}{n}\right)} \cdots$,

where $\tilde{HF}^{-\left(\frac{Y}{n}\right)} = \bigoplus m \tilde{HF}^{-\left(\frac{Y}{n}\right)}$. 


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HF^-(Y_n(K), [i]) \cong H_*(A_i(K)),
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if \( n \) is sufficiently large.
Idea of proof

Recall the main steps in Ozsváth and Szabó’s proof of the ordinary mapping cone formula:

- A large surgeries formula, which states that

\[ HF^{-}(Y_{n}(K), [i]) \cong H_{*}(A_{i}(K)), \]

if \( n \) is sufficiently large.

- A surgery exact sequence

\[ \cdots \xrightarrow{HF^{-}} (Y) \rightarrow HF^{-}(Y_{n}) \rightarrow HF^{-}(Y_{n+m}) \rightarrow HF^{-}(Y) \cdots , \]

where \( HF^{-}(Y) = \bigoplus^{m} HF^{-}(Y) \).
Idea of proof: exact sequence, $m = 1$
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- Consider the $m = 1$ case

\[ \cdots \; \mathcal{H}F^-(Y) \rightarrow \mathcal{H}F^-(Y_n) \rightarrow \mathcal{H}F^-(Y_{n+1}) \rightarrow \mathcal{H}F^-(Y) \cdots , \]
Idea of proof: exact sequence, $m = 1$

- Consider the $m = 1$ case

$$\cdots \longrightarrow HF^{-}(Y) \longrightarrow HF^{-}(Y_{n}) \longrightarrow HF^{-}(Y_{n+1}) \longrightarrow HF^{-}(Y) \longrightarrow \cdots$$

- Define a “cobordism” map

$$CFI^{-}(Y_{n+1}) \rightarrow CFI^{-}(Y)$$

as well as a quasi-isomorphism

$$\Phi: CFI^{-}(Y_{n}) \rightarrow \text{Cone} \left( CFI^{-}(Y_{n+1}) \rightarrow CFI^{-}(Y) \right).$$
Idea of proof: exact sequence, $m = 1$

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- The exact sequence for mapping cones from homological algebra gives the surgery exact sequence.
Idea of proof: exact sequence, $m = 1$
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To build

$$\Phi: CFI^{-}(Y_n) \rightarrow \text{Cone}(CFI^{-}(Y_{n+1}) \rightarrow CFI^{-}(Y))$$

we start by building a hypercube (i.e. a cubical diagram whose total complex is a chain complex)
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we start by building a hypercube (i.e. a cubical diagram whose total complex is a chain complex)

Furthermore, the maps along top coincide with the maps along the bottom.
Idea of proof: exact sequence, $m = 1$

The maps along the top and bottom were constructed by Ozsváth and Szabó. The maps along the left and front face were constructed by Hendricks and Manolescu. The challenging part which is new to our work is the length 3 dotted arrow.
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Idea of proof: exact sequence, $m = 1$
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Since the maps along the top and bottom, agree, we can add id to each vertical map, and total complex will still be a chain complex:
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\[
\begin{align*}
&CF^{-}(Y_n) \\
| & Q(id + \iota) \\
\downarrow & \quad \downarrow \\
QCF^{-}(Y_n) & \quad \quad \quad \quad QCF^{-}(Y_{n+1}) & \xrightarrow{\quad Q(id + \iota) \quad \quad \quad \quad} & QCF^{-}(Y)
\end{align*}
\]
Idea of proof: exact sequence, $m = 1$

Since the maps along the top and bottom, agree, we can add id to each vertical map, and total complex will still be a chain complex:

This is the same as a chain map from $CFI^{-}(Y_n)$ to $\text{Cone}(CFI^{-}(Y_{n+1}) \to CFI^{-}(Y))$.
Thanks for listening!