EXERCISES FOR HEEGAARD FLOER MINICOURSE

IAN ZEMKE

1. Lecture 1: Introduction to Heegaard Floer invariants

Exercise 1.1. Let (C, ι_K) and (C', ι_K') be two ι_K-complexes. Verify that $(C \otimes C', (1 + \Phi|\Psi)(\iota_K|\iota_K'))$ is an ι_K-complex. Show that $\iota_K|\iota_K'$ does not always specify a valid knot-involution on $C \otimes C'$ (i.e. it does not always square to the Sarkar map). We write $C * C'$ for the ι_K complex so-constructed.

Exercise 1.2. Let $*$ and $*'$ be the two multiplications on ι_K-complexes. I.e., let $C * C'$ have involution $(1 + \Phi|\Psi)(\iota_K|\iota_K')$ and let $C *' C'$ have involution $(1 + \Psi|\Phi)(\iota_K|\iota_K')$. Show that $C * C'$ and $C *' C'$ are homotopy equivalent as ι_K-complexes. (See [Zem19] for similar results).

Exercise 1.3. Verify that that the set of ι_K-complexes modulo local equivalence, $\mathfrak{I}_K$, is an abelian group.

Exercise 1.4. Heegaard Floer homology as a famous surgery exact triangle

$$\to \widehat{HF}(Y) \to \widehat{HF}(Y_0(K)) \to \widehat{HF}(Y_1(K)) \to \widehat{HF}(Y) \to \cdots$$

The surgery exact triangle has a natural interpretation in terms of the Fukaya category of the torus. Recall that in the Fukaya category, objects consist of Lagrangians, and $\mathrm{Hom}(\gamma, \gamma')$ is $\widehat{CF}(\gamma, \gamma')$. Show that there there is an equivalence of twisted complexes in the Fukaya category

$$\gamma_\infty \simeq \mathrm{Cone}(\theta\colon \gamma_0 \to \gamma_1)$$

where $\gamma_0, \gamma_1, \gamma_\infty$ are as in Figure 1.1 and θ is the unique intersection of γ_0 and γ_1. For simplicity, work in the version of the Fukaya category where holomorphic curves are not allowed to cross w. (Note that technically the Fukaya category does not have *units* but rather has only *homotopy units*; that is if $\gamma \subseteq \mathbb{T}^2$ is a Lagrangian, treat $\mathrm{id}_\gamma \in \widehat{CF}(\gamma, \gamma)$ to be the top graded intersection point of $\widehat{CF}(\gamma, \gamma')$ where γ' is a small translate of γ). Hint: For this exercise, we work in the category of *twisted complexes of Lagrangians*. Therefore we think of $\mathrm{Cone}(\theta\colon \gamma_0 \to \gamma_1)$ as a "generalized" object of the Fukaya category. A homotopy equivalence between γ_∞ to $\mathrm{Cone}(\theta)$ consists of a pair of elements (x, y) where $x \in \widehat{CF}(\gamma_\infty, \gamma_0)$ and $y \in \widehat{CF}(\gamma_\infty, \gamma_1)$. We think of this as a diagram

$$\begin{array}{ccc} \gamma_\infty & & \\ \downarrow x & \searrow y & \\ \gamma_0 & \xrightarrow{\theta} & \gamma_1. \end{array}$$

For (x, y) to be a chain map, we require the above diagram to satisfy the appropriate version of $\partial^2 = 0$, which in this case means that $\mu_1(y) = \mu_2(x, \theta)$. For the homotopy equivalence in the statement, we want morphisms in both directions which compose to the identity. More about twisted complexes can be found in Seidel's book [Sei08].

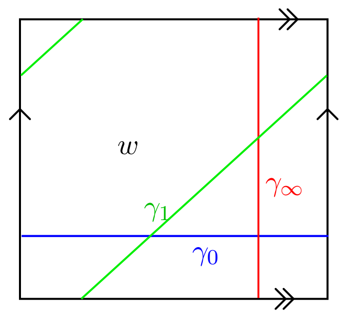


FIGURE 1.1. The Lagrangians $\gamma_\infty, \gamma_0, \gamma_1$.

2. LECTURE 2: THE MAPPING CONE FORMULA AND ITS INVOLUTIVE REFINEMENT

Exercise 2.1. Use the (involutive) mapping cone formula to compute $\mathit{CFI}(S^3_{+1}(U))$ and $\mathit{CFI}(S^3_{-1}(U))$.

Exercise 2.2. Compute $\mathit{CFI}(S^3_{+1}(4_1))$ using the surgery formula. Verify that it is locally equivalent to $(A_0(4_1), \iota)$. Compute $\underline{d}$ and $\overline{d}$.

Exercise 2.3. Using the mapping cone formula to prove that there is a local map from $(CF^-(S^3_{+1}(K)), \iota)$ to $(A_0(K), \iota)$. Prove that there is a Spin negative definite cobordism from $S^3_{+1}(K)$ to $S^3_N(K)$ for all $N > 1$.

Exercise 2.4. Let $\mathcal{B}_n$ denote the ι_K-complex shown below (for odd n). There is a notion of local equivalence over the ring $\mathbb{F}[U, Q]/(Q^2, UQ)$. A complex over this ring is called an "almost ι-complex". Show that as an almost-ι complex, $A_0(\mathcal{B}_n \otimes \mathit{CFK}(T_{2,3}))$ is locally equivalent to the complex C_n shown in Figure 2.1. See [DHST18] and [HHSZ22].

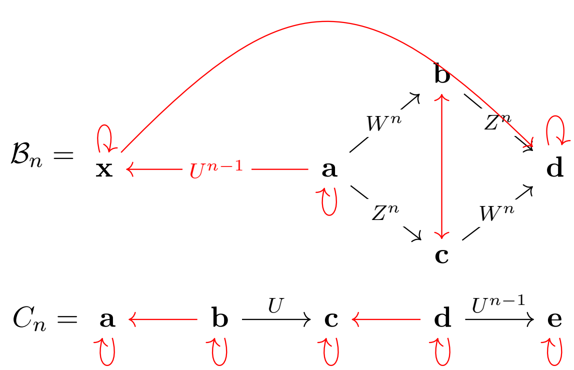


FIGURE 2.1. The complex $(\mathcal{B}_n, \iota_K)$ (top). The complex C_n (bottom). Red arrows denote the involution.

3. Lecture 3: The bordered perspective on the mapping cone formula

The main reference is [Zem21]. Note that therein, we use the basis $\mathbb{F}[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}]$ for idempotent 1, which is translated to the notation from the lecture by the change of variables $U=\mathscr{U}\mathscr{V}$ and $\mathscr{V}=T$. (We also write W in place of $\mathscr{U}$ and Z in place of $\mathscr{V}$).

Recall that $\mathcal{D}_n^{\mathcal{K}}$ is the type-D structure

$$\mathbf{x}_0 \xrightarrow{\sigma+T^n\tau} \mathbf{x}_1.$$

Recall also that ${}_{\mathcal{K}}\mathcal{D}_0$ is the type-A module as follows. Idempotent 0 consists of $\mathbb{F}[W,Z]$, and idempotent 1 consists of $\mathbb{F}[U,T,T^{-1}]$. The actions of $I_0\cdot\mathcal{K}\cdot I_0$ and $I_1\cdot\mathcal{K}\cdot I_1$ are given by polynomial multiplication on $I_0\cdot\mathcal{D}_0$ and $I_1\cdot\mathcal{D}_0$. The action of σ and τ are given by the maps

$$\sigma\cdot a=\phi^\sigma(a)\in I_1\cdot\mathcal{D}_0 \qquad \text{and} \qquad \tau\cdot a=\phi^\tau(a)\in I_1\cdot\mathcal{D}_0,$$

where $\phi^\sigma(W^iZ^j)=U^iT^{j-i}$ and $\phi^\tau(W^iZ^j)=U^jT^{j-i}$.

Exercise 3.1. Compute $HF^-(S^3_{\pm1}(U))$ by writing it as a tensor product

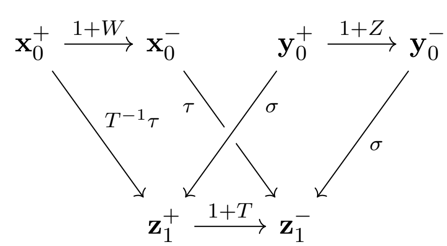
$$\mathcal{D}^{\mathcal{K}}_{\pm1}\boxtimes {}_{\mathcal{K}}\mathcal{D}_0.$$

Perform the computation in the U-adic topology. (The definition of the box tensor product $\boxtimes$ may be found in [LOT18]).

Remark 3.2. We remark that there is another natural linear topology to put on the surgery algebra, which is called the *chiral topology.* (See [Zem21, Section 6]) The definition for finitely generated type-D modules can be simplified to avoid linear topologies. In this case, one can just ignore the topologies, and work with ordinary type-D modules over the "chirally" completed algebra

$$I_0\cdot\boldsymbol{\mathcal{K}}\cdot I_0=\mathbb{F}[\![W,Z]\!],\quad I_1\cdot\boldsymbol{\mathcal{K}}\cdot I_0=\mathbb{F}[\![T,T^{-1}]\!][\![U]\!]\langle\sigma\rangle\oplus\mathbb{F}[T,T^{-1}]\!][\![U]\!]\langle\tau\rangle$$
$$I_1\cdot\boldsymbol{\mathcal{K}}\cdot I_1=\mathbb{F}[T,T^{-1}][\![U]\!].$$

We now recall the type-D module of the ∞-framed solid torus:

$\mathcal{D}^{\mathcal{K}}_\infty =$  (3.1)

Exercise 3.3. Verify the surgery exact triangle for type-D modules in the *chiral topology.* Show that there is a chain map $f^1\colon\mathcal{D}^{\mathcal{K}}_0\to\mathcal{D}^{\mathcal{K}}_1$ and a homotopy equivalence

$$\mathcal{D}^{\mathcal{K}}_\infty\simeq \operatorname{Cone}(f^1\colon\mathcal{D}^{\mathcal{K}}_0\to\mathcal{D}^{\mathcal{K}}_1).$$

It is also possible to show this in the U-adic topology, though it is a bit more involved [Zem23, Section 12]. Hint: $1+W$ and $1+Z$ are units in the chirally completed algebra.

Remark 3.4. The morphisms in the above homotopy equivalence can by derived by performing curve counts on the torus. For example, we can define f^1 by counting curves on the torus.

The next exercises use the dual knot Floer of Eftekhary-Hedden-Levine [Eft06] [HL19]. This corresponds to the DA-bimodule of the Hopf link ${}_{\mathcal{K}}\mathcal{H}^{\mathcal{K}}_{(0,0)}$, which is computed in [Zem21]. We recall the computation here:

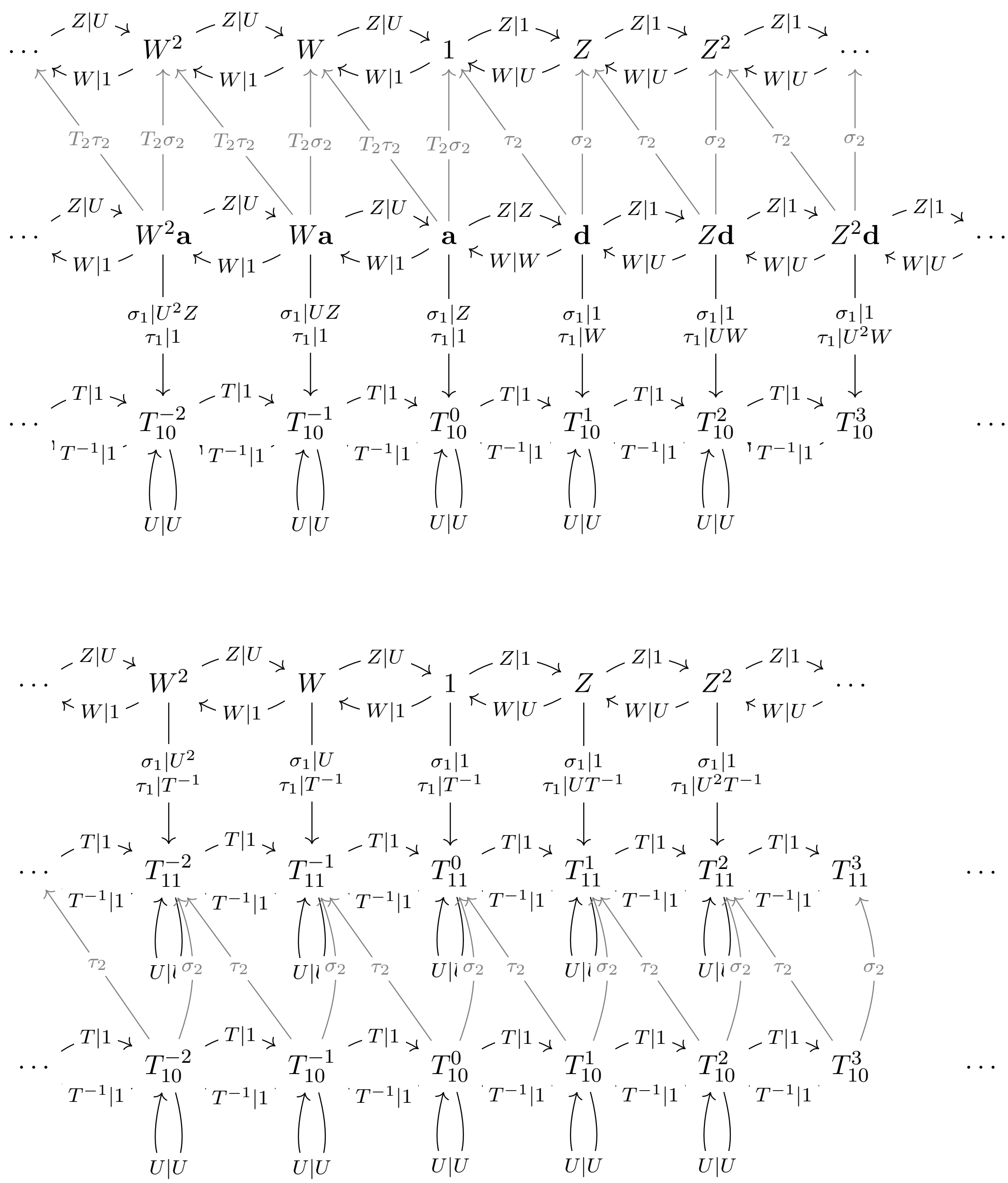


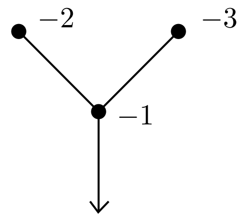
FIGURE 3.1. The DA-bimodule of the Hopf link. Generators $W^i\mathbf{a}$, $Z^i\mathbf{d}$ are idempotent $(0,0)$. Generators W^i and Z^j are idempotent $(0,1)$. Generators T^i_ε are idempotent ε. Gray arrows are δ^1_1 and black arrows are δ^1_2.

Exercise 3.5. Derive the invariant for the ∞-framed solid torus by using the Hedden-Levine-Eftekhary dual knot formula. I.e. show that

$$\mathcal{D}_{\infty}^{\mathcal{K}} = \mathcal{D}_0^{\mathcal{K}} \boxtimes {}_{\mathcal{K}}\mathcal{H}_{(0,0)}^{\mathcal{K}}$$

Exercise 3.6. Compute $\mathcal{D}_{p/q}^{\mathcal{K}}$ by induction using the dual knot formula (i.e. write p/q as a continued fraction). Show that the answer recovers the rational surgery formula of Ozsváth and Szabó [OS11]. (See [Zem21, Section 18.2]).

Exercise 3.7. Compute the invariant for the trefoil in the following inefficient manner: View the trefoil as a plumbed link for the following plumbing, and compute using the invariants for a solid torus, the dual knot formula, and the connected sum formula.



Exercise 3.8. Consider the two "twist" bimodules ${}_{\mathcal{K}}\mathcal{T}_{\tau}^{\mathcal{K}}$ and ${}_{\mathcal{K}}\mathcal{T}_{\sigma}^{\mathcal{K}}$, defined as follows. As a vector space, both are isomorphic to the idempotent ring $\mathbf{I}$. They both have a δ_2^1 given by $\delta_2^1(a,1) = 1 \otimes a$. Additionally, $\mathcal{T}_\tau$ has a δ_4^1 given by

$$\delta_4^1(a, b, c\tau, i_0) = i_1 \otimes \partial_U(a) T \partial_T(b) c\tau.$$

The bimodule $\mathcal{T}_\sigma$ is similar, except we replace τ with σ, above. Show that there is a homotopy equivalence

$${}_{\mathcal{K}}\mathcal{T}_{\sigma}^{\mathcal{K}} \simeq {}_{\mathcal{K}}\mathcal{T}_{\tau}^{\mathcal{K}}.$$

Show that ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}} \boxtimes {}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}} \simeq {}_{\mathcal{K}}\mathbb{I}^{\mathcal{K}}$. See [Zem21, Section 14.3 and 14.4] for more on similar bimodules.

Remark 3.9. The above bimodule ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$ is related to the *half-identity bimodule* of Lipshitz-Ozsváth-Thurston. It turns out that there is a fundamental asymmetry in the construction of the surgery formula, which produces *a-priori* two modules $\mathcal{X}_n(K)^{\mathcal{K}}$ and $\mathcal{X}_n'(K)^{\mathcal{K}}$. Tensoring with ${}_{\mathcal{K}}\mathcal{T}^{\mathcal{K}}$ switches between these two models. (*A-posteri* it turns out that the distinction is not important for knots, but becomes non-trivial when considering links and surgery *bi*modules).

4. Lecture 4: Formality and algebraic links

Exercise 4.1. Let $R = \mathbb{F}[X_1, \dots, X_n]$. Let $\Lambda = \Lambda(\theta_1, \dots, \theta_n)$ be the exterior algebra on n generators (over $\mathbb{F} = \mathbb{Z}/2$). There is a type-$DD$ module ${}^R\Lambda^R$ where $\delta^{1,1}(\theta_i) = X_i|1|1 + 1|1|X_i$, extended via the Leibniz rule. Verify that

$${}_R R_R \boxtimes {}^R\Lambda^R \simeq {}_R[\mathbb{I}]^R.$$

(Hint: verify the $n = 1$ case, and then use this to prove the general case).

Exercise 4.2. Let $R = \mathbb{F}[W, Z]$ and let ${}_R M$ be an A_∞-module which is $(\mathrm{gr}_w, \mathrm{gr}_z)$-bigraded such that M is supported in $2\mathbb{Z} \times 2\mathbb{Z}$. Let ${}_R H(M)$ be the module obtained by taking the homology of M, and then using multiplication induced by $m_2 \colon R \otimes M \to M$. Show that if $H(M)$ is supported in bigradings $2\mathbb{Z} \times 2\mathbb{Z}$, then there is a homotopy equivalence of A_∞-modules $M \simeq H(M)$. Verify that Ozsváth and Szabó's result follows: if K is an L-space knot, then $\mathit{CFK}(K)$ is a staircase complex. Hint: Consider ${}^R\Lambda^R \boxtimes {}_R M$ and some related tensor products.

Exercise 4.3. Let (X, ∂_X) be a free chain complex over $\mathbb{F}[W,Z]$ whose homology $H(X)$ admits a 2-step free resolution. I.e. such that we have an exact sequence

$$0 \to C_0 \xrightarrow{f} C_1 \to H(X) \to 0,$$

where C_i are free $\mathbb{F}[W,Z]$-modules (with trivial differential). Show that X is quasi-isomorphic to $\mathcal{C} := \mathrm{Cone}(f\colon C_0 \to C_1)$. Explain why this gives another proof of Ozsváth and Szabó's structure theorem for L-space knot complexes.

Exercise 4.4. Compute the link Floer homology modules for the Whitehead link and $T(2,2)$ using the h-function. (See [BLZ22, Section 7.1]). Compute free resolutions of these modules, and compare with the complexes obtained from genus 0 Heegaard link diagrams. Hint: the complexes obtained from Heegaard diagrams are typically only free-resolutions after a basis change.

Remark 4.5. The h-function for the Whitehead link and $T(2,2)$ are shown below:

$$h(T(2,2)) = \begin{array}{cccccc} & & & {\scriptstyle s_1=\frac{1}{2}} & & \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & 1 \\ 4 & 3 & 2 & 2 & 2 & 2 \\ 5 & 4 & 3 & 3 & 3 & 3 \end{array} \begin{array}{c} \\ \\ \\ {\scriptstyle s_2=\frac{1}{2}} \\ \\ \\ \\ \end{array} \qquad h(Wh) = \begin{array}{ccccc} & & {\scriptstyle s_1=0} & & \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 2 & 2 & 2 \end{array} \begin{array}{c} \\ \\ \\ {\scriptstyle s_2=0} \\ \\ \\ \end{array}$$

A key step in the proof of formality of HFL for algebraic links [BLZ22] is the following algebraic lemma, which we use as an exercise:

Exercise 4.6. Suppose that R is an algebra over the field of two elements. Let $\mathscr{C}$ be the bounded chain complex of R-modules

$$0 \longrightarrow C_n \xrightarrow{\ \partial\ } C_{n-1} \xrightarrow{\ \partial\ } \cdots \xrightarrow{\ \partial\ } C_0 \longrightarrow 0$$

be a bounded chain complex of R-modules. Each C_i is a chain complex of R-modules and each ∂ commutes with the R-action. Suppose that $\mathscr{C}$ is supported in a single grading. Show that $\mathscr{C}$ is formal. Hint: consider first the case that $H_*(\mathscr{C})$ is supported in $H_0(\mathscr{C})$. In this case $\mathscr{C}$ is just a free-resolution of its homology. For the general case, use the homological perturbation lemma.

Remark 4.7. In the context of the proof of formality, the chain complex $\mathscr{C}$ is a version of lattice link Floer homology. The subscript n is the "lattice grading". There are a number of gradings on this complex. In addition to the lattice grading, there are Maslov gradings $(\mathrm{gr}_w, \mathrm{gr}_z)$, as well as Alexander gradings. The differential has Maslov bigrading $(-1,-1)$, and preserves the Alexander grading. Note that with respect to either of the Maslov gradings, we cannot typically write knot or link Floer homology as a chain complex with the above form. Instead, it is a very special implication of the equivalence of lattice homology and Heegaard Floer homology [Zem22] as well as its refinement for lattice link Floer homology [BLZ22] .

Remark 4.8. Write C for the negative trefoil complex

$$C = \begin{array}{ccccc} \mathbf{x} & & & & \mathbf{y} \\ & \searrow^{W} & & \swarrow^{Z} & \\ & & \mathbf{z} & & \end{array}$$

Use the homological perturbation lemma to equip $H_*(C)$ with the structure of an A_∞-module. Compare this to the complex for the figure-8, which takes the form

$$C = \mathbf{z} \oplus \begin{array}{ccccc} & & \mathbf{a} & & \\ & \swarrow^{W} & & \searrow^{Z} & \\ \mathbf{b} & & & & \mathbf{c} \\ & \searrow^{Z} & & \swarrow^{W} & \\ & & \mathbf{d} & & \end{array}$$

References

[BLZ22] Maciej Borodzik, Beibei Liu, and Ian Zemke, *Lattice homology, formality, and plumbed L-space links*, 2022. e-print, arXiv 2210.15792 [math.GT].

[DHST18] Irving Dai, Jennifer Hom, Matthew Stoffregen, and Linh Truong, *An infinite-rank summand of the homology cobordism group*, 2018. preprint, arXiv:1810.06145.

[Eft06] Eaman Eftekhary, *Heegaard Floer homology and Morse surgery*, 2006. e-print, arXiv:0603171.

[HHSZ22] Kristen Hendricks, Jennifer Hom, Matthew Stoffregen, and Ian Zemke, *On the quotient of the homology cobordism group by seifert spaces*, 2022.

[HL19] Matthew Hedden and Adam Simon Levine, *A surgery formula for knot Floer homology*, 2019. e-print, arXiv:1901.02488.

[LOT18] Robert Lipshitz, Peter S. Ozsvath, and Dylan P. Thurston, *Bordered Heegaard Floer homology*, Mem. Amer. Math. Soc. **254** (2018), no. 1216, viii+279.

[OS11] Peter Ozsváth and Zoltán Szabó, *Knot Floer homology and rational surgeries*, Algebr. Geom. Topol. **11** (2011), no. 1, 1–68.

[Sei08] Paul Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008.

[Zem19] Ian Zemke, *Connected sums and involutive knot Floer homology*, Proc. Lond. Math. Soc. (3) **119** (2019), no. 1, 214–265.

[Zem21] ———, *Bordered manifolds with torus boundary and the link surgery formula*, 2021. e-print, arXiv:2109.11520.

[Zem22] ———, *The equivalence of lattice and heegaard floer homology*, 2022.

[Zem23] ———, *A general heegaard floer surgery formula*, 2023.