Given \( P \) in the equilateral triangle, can we characterize the set

\[ \{ P' \mid \exists P \text{ such that } P = \sum_{\sigma} P_{\sigma} T_{\sigma} (1,0,0), \]

and \( P' = \sum_{\sigma} P_{\sigma} T_{\sigma} (0,1,0) \} ? \]

**Proposition:**

This set is equal to \( \{ P' \mid \exists P \text{ such that } P' \text{ has convex coordinates } (s', t', u') \text{ such that } s' + t' + u' = 1, \text{ and } s + s' \leq 1, t + t' \leq 1, u + u' \leq 1, \text{ where } (s, t, u) \text{ are the convex coordinates of } P \} \).

**Proof:**

We first prove that these conditions are necessary. We already know that the midpoint \( P'' \) of the segment \( PP' \),

\[ P'' = \frac{1}{2} \sum_{\sigma} P_{\sigma} T_{\sigma} [(1,0,0) + (0,1,0)] \]

\[ = \sum_{\sigma} P_{\sigma} T_{\sigma} \left( \frac{1}{2}, \frac{1}{2}, 0 \right) = \sum_{\sigma} P_{\sigma} T_{\sigma} \bar{w} \frac{1}{2} \]

must lie in the convex hull of the \( \bar{w} \frac{1}{2} \), or the "inner" triangle.

The geometric interpretation of the convex coordinates \( s, t, u \) of an arbitrary point \( \tilde{P} \) shows that

\( \tilde{P} \in \) inner triangle \( \Leftrightarrow s, t, u \) are all \( \leq \frac{1}{2} \).

It follows that \( \tilde{P}'' \), the midpoint of \( PP' \), which automatically has convex coordinates \( \left( \frac{s + s'}{2}, \frac{t + t'}{2}, \frac{u + u'}{2} \right) \), must satisfy \( \frac{s + s'}{2} \leq 1, \frac{t + t'}{2} \leq 1, \frac{u + u'}{2} \leq 1 \), or

\( s + s' \leq 1, t + t' \leq 1, u + u' \leq 1 \), as claimed.

Next, we prove that the conditions are sufficient. We shall explicitly construct the \( P_{\sigma} \) such that \( P = \sum_{\sigma} P_{\sigma} T_{\sigma} (1,0,0) \) and

\( P' = \sum_{\sigma} P_{\sigma} T_{\sigma} (0,1,0) \). We do this first for the special case

[Diagram of an equilateral triangle with marked points and segments]
where \( \max (S, t, u, s', t', u') \in \{S, t, u\} \). We can, without loss of generality (if necessary, we permute the roles played by \( s, t, u \) in the argument below), assume that it is \( u \) that takes on this maximum value, i.e., \( u \geq S, t, s', t', u' \).

Set now \( \tau = \min \left( \frac{s'}{u}, \frac{t - t'}{u} \right) \); since \( s' < u, \tau < 1 \);

on the other hand \( u + s - t' = 1 - t - t' \geq 0 \), so that \( \tau \in [0, 1] \).

We also define \( \alpha = \frac{t - u + u \tau}{s} \), \( \beta = \frac{t - s' + u \tau}{t} \); by the definition of \( \tau \), \( u \tau < s' \), so that \( t - s' + u \tau < t \), and \( u \tau \leq u + s - t' \), so that \( t' + u \tau - u < s \), implying \( \alpha, \beta < 1 \).

We also have \( t' - u + u \tau = t' - u + \min (s', u + s - t') \)

\[ = \min (s' + t' - u, s) = \min (1 - u' - u, s) \geq 0 \]

as well as

\[ t - s' + u \tau = t - s' + \min (s', u + s - t') \]

\[ = \min (t, 1 - s' - t') \geq 0 \]

It follows that \( \alpha \) and \( \beta \) are both in \([0, 1]\).

We can now write

\[ P = s \; \Pi + t \; \Pi + u \; \Pi \]

\[ = d s \; \Pi \; (1, 2, 3) \; \omega_0 + (1 - d) \; s \; \Pi \; (1, 23) \; \omega_0 + \beta t \; \Pi \; (1, 32) \; \omega_0 + (1 - \beta) \; t \; \Pi \; (1, 2) \; \omega_0 + \gamma u \; \Pi \; (1, 23) \; \omega_0 + (1 - \gamma) \; u \; \Pi \; (1, 32) \; \omega_0 \]

leading to

\[ d s \; \Pi \; (0, 1, 0) + (1 - d) \; s \; \Pi \; (1, 23) \; (0, 1, 0) + \beta t \; \Pi \; (1, 32) \; (0, 1, 0) + (1 - \beta) \; t \; \Pi \; (1, 2) \; (0, 1, 0) + \gamma u \; \Pi \; (1, 23) \; (0, 1, 0) + (1 - \gamma) \; u \; \Pi \; (1, 32) \; (0, 1, 0) \]

\[ = d s \; (0, 1, 0) + (1 - d) \; s \; (0, 0, 1) + \beta t \; (0, 0, 1) + (1 - \beta) \; t \; (0, 1, 0) + \gamma u \; (1, 0, 1) + (1 - \gamma) \; u \; (0, 1, 0) \]

\[ = [\gamma u + (1 - \beta) t] \; I + [d s + (1 - \gamma) u] \; II + [\beta t + (1 - \alpha) s] III \]

By the definition of \( \alpha \) and \( \beta \), we have

\[ d s + (1 - \gamma) u = t' - u + u \tau + (1 - \gamma) u = t' \]
and \( \beta t + (1-\alpha) s = \beta t + s - (t'-u+u') \)
\[
= t - s' + u + t' + u - u' = t + u - s' - t' + s
\]
\[
= 1 - s' - t' = u',
\]
as well as \( \gamma u + (1 - \beta) t = \gamma u + t - t + s' - u x = s' \), reducing (\*\*) to: 
\[ t \quad s' \quad \Gamma + t' \quad II + u' \quad III = P'. \]
In other words, with \( \alpha, \beta, \gamma \) as defined above, and
\[
P_{(1)(2)(3)} = (1 - \alpha) s \\
P_{(1)(2)} = (1 - \alpha) s \\
P_{(1)(2)(3)} = \beta t \\
P_{(1)(2)(3)} = (1 - \beta) t \\
P_{(1)(2)} = (1 - \beta) t \\
P_{(1)(2)} = (1 - \beta) t \\
we have \[ \sum_{\sigma} \rho_\sigma \pi_\sigma \omega_\sigma = P \quad \text{and} \quad \sum_{\sigma} \rho_\sigma \pi_\sigma \hat{\omega}_1 = P'. \]
It remains to discuss the case where \( \max \{ s, t, u, s', t', u' \} \in \{ s', t', u' \} \). In this case, we can run the construction "backwards", i.e. we would write
\[
P' = \alpha' s' \pi_{(1)(2)(3)} (0,1,0) + (1 - \alpha') s' \pi_{(1)(2)(3)} (0,1,0) \\
+ \beta' t' \pi_{(1)(2)(3)} (0,1,0) + (1 - \beta') t' \pi_{(1)(2)(3)} (0,1,0) \\
+ \gamma' u' \pi_{(1)(2)(3)} (0,1,0) + (1 - \gamma') u' \pi_{(1)(2)(3)} (0,1,0),
\]
which means that we identify
\[
P_{(1)(2)(3)} = \alpha' s' \\
P_{(1)(2)(3)} = (1 - \alpha') s' \\
P_{(1)(2)(3)} = \beta' t' \\
P_{(1)(2)(3)} = (1 - \beta') t' \\
P_{(1)(2)(3)} = (1 - \gamma') u',
\]
with \( \alpha', \beta', \gamma' \) still to be set. Taking then the combination
\[
\sum_{\sigma} \rho_\sigma \pi_\sigma (1,0,0) = \left( \beta' t' + (1 - \gamma') u' \right) (1,0,0) \\
+ \left( \gamma' u' + (1 - \alpha') s' \right) (0,1,0) \\
+ \left( \alpha' s' + (1 - \beta') t' \right) (0,0,1),
\]
and requiring that this equal \( S(1,0,0) + T(0,1,0) + U(0,0,1) \),
leads to the equations
\[ \beta' t' + (1 - \gamma') w' = s \]
\[ \gamma' w' + (1 - \beta') S' = t \]
\[ d' s' + (1 - \beta') t' = u \]

We then determine \( \alpha' \), \( \beta' \), \( \gamma' \) so that these equations are satisfied, similarly to the earlier case. For instance, if \( t' = \max (s', t', u', s, t, u) \), we would set
\[ \beta' = \min \left( \frac{s}{t'}, 1 + \frac{s' - u}{t'} \right) \]
and \( d' = \frac{u - t' + \beta' t'}{s'} \), \( \gamma' = \frac{u' + \beta' t' - s}{u} \); it would follow again that \( d', \beta', \gamma' \in [0, 1] \), and that the three equations at the top of the page are satisfied.

What does this mean geometrically? We already saw, in the proof, that the necessary and sufficient condition \( s + s' \leq 1, t + t' \leq 1 \) and \( u + u' \leq 1 \) is equivalent to requiring that the midpoint of the segment PP' lies in the small "inner triangle". To find all the P', starting from a given P, we can thus just look at the endpoints of segments that have one end at P, and their midpoint in the inner triangle, and their other endpoint still within the (convex) big triangle.

the lightly shaded "upside down" large triangle is the collection of all candidate endpoints P' of segments PP' with a midpoint in the little triangle; only those in the heavier shaded region fall inside the original convex hull, however.

large upright triangle.
Now that we have understood this geometry, we can revisit our earlier assertions, and prove them.

In instance let's prove that:

Theorem. For all \( s_1, s_2 \in [0, \frac{1}{2}] \) with \( s_1 \neq s_2 \), and for any two

societal rankings \( \beta_1, \beta_2 \), there exists a profile \( p \) so that,

for \( i = 1 \) or \( 2 \), \( \sum_{o} p_o \pi_o \omega_i \) gives rise to ranking \( \beta_i \).

Proof.

- First of all, note that, by the preceding theorem, we can find, \( V, P, P' \) in the
  inner triangle, an appropriate

\[
P = (p_o)_{o \in S_3}
\]

so that

\[
\mathcal{P} = \sum_{o} p_o \pi_o \omega_0
\]

\[
P' = \sum_{o} p_o \pi_o \omega_2
\]

Set \( P'' = \text{midpoint of the segment } PP' \); then

\[
\overrightarrow{PP''} = \frac{1}{2} PP' = (1 - \mu) P' \quad \mu \in [0, 1]
\]

\[
= \frac{1}{2} \sum_{o} p_o \pi_o \omega_0 \quad s \in [0, \frac{1}{2}]
\]

- In particular, this is true if \( P, P' \) are both within the circle
  centered at the midpoint of the triangle, with radius

\[
r = \frac{1}{\sqrt{2}}
\]

(where the length of the side of the large triangle
is equal to 1). (in dotted lines on figure).

- In the region corresponding to ranking \( \beta_2 \), pick a point
  \( P \) closed to the center \( C \) of the triangle than \( \frac{1}{\sqrt{2}} |s_1 - s_2| r \).

This point will correspond to \( \sum_{o} p_o \pi_o \omega_2 \). With this
point \( A \) as its center, define \( C \) to be the circle with radius

\[
R = \frac{1}{2} |s_1 - s_2| r = \frac{1}{2} |s_1 - s_2| \frac{1}{\sqrt{2}} \frac{1}{2} |s_1 - s_2| r
\]

The center \( C \) of the triangle will automatically lie inside this circle. Moreover, for each point

\[
\text{since } d(C, A) \leq \frac{1}{4} |s_1 - s_2| r
\]
Q inside this circle, we have
\[ d(Q, C) \leq d(Q, A) + d(A, C) \]
\[ \leq R + \frac{1}{4} |s_1 - s_2| r \]
\[ \leq R \leq r \quad \text{(since } s_1, s_2 \in [0, \frac{1}{2}] \text{)} \]
\[ \Rightarrow \] \( Q \in \text{insidetriangle inner triangle} \).

On the other hand, because the circle \( C \), it intersects with the 6 ranking regions that involve only strict inequalities, and with the 6 dividing lines. It follows that the deck \( \{ \{ Q ; d(A, Q) \leq R \} \} = \{ \} \) contains points of all 13 ranking regions. (The only one that was still missing is \( C \) itself, the 1-point regime where the societal ranking ties all candidates.)

Pick a point \( B \) in the deck \( D \) that lies in the ranking regime corresponding to \( \beta_2 \). This point will correspond to \( \sum \frac{p_\sigma}{\Pi_\sigma} \mathbf{w}_\sigma \). \( \mathbf{w}_\sigma \) (Note that we have not determined \( p \) yet at this point!)

Introduce coordinates on the line \( AB \), and label \( A \) by \( s_1 \), \( B \) by \( s_2 \), denote by \( P \) the point corresponding to \( 0 \), and by \( P' \) the point corresponding to \( \beta_1 \).

Then \( d(P, C) \leq d(P, A) + d(A, C) \)

\[ \frac{s_1}{1s_1 - s_2} \mathbf{d}(B, A) + \mathbf{d}(A, C) \leq \frac{s_1}{1s_1 - s_2} R + \frac{1}{4} |s_1 - s_2| r \]
\[ \leq \frac{1 + \frac{3}{2}}{2} s_1 r + \frac{1}{4} r \leq \frac{2}{3} r < r \]
and \[ d(P', C) \leq d(P', A) + d(A, C) \]
\[ = \frac{\frac{1}{2} - S_1}{1S_1 - S_2} d(B, A) + d(A, C) \]
\[ \leq \left( \frac{\frac{1}{2} - S_1}{1S_1 - S_2} \right) r + \frac{1}{4} r \leq \frac{3}{4} \] so that both \( P \) and \( P' \) lie within the circle inscribed in the inner triangle, and thus \( P, P' \in \text{inner triangle} \). Now we can apply the preceding theorem, and use the coordinates of \( P, P' \) to identify the profile \( p \) so that
\[
\begin{align*}
P &= \sum_{\sigma} p_\sigma T_{\sigma} \omega_0^3, \\
P' &= \sum_{\sigma} p_\sigma T_{\sigma} \omega_1^3.
\end{align*}
\]
By the construction of \( P, P' \) from \( A \) and \( B \), it follows that
\[
\begin{align*}
A &= \sum_{\sigma} p_\sigma T_{\sigma} \omega_0^3 + (1 - S_1) \sum_{\sigma} p_\sigma T_{\sigma} \omega_1^3, \\
B &= \sum_{\sigma} p_\sigma T_{\sigma} \omega_2^3,
\end{align*}
\]
and likewise
\[
B = \sum_{\sigma} p_\sigma T_{\sigma} \omega_2^3.
\]
\[ \] In fact, we can conclude even more than this. If \( \beta_1, \beta_2 \) are both strict rankings (no ties), then we can pick \( A, B \) in this proof so that there will be small disks around \( A, B \) so that \( VA' \in \mathcal{D}_A, \ YB' \in \mathcal{D}_B \) there still exists a profile \( p' \) so that
\[ A' = \sum_{\sigma} p_\sigma \Pi_{\sigma} \tilde{\omega}_{\phi} \quad \text{and} \quad B' = \sum_{\sigma} p_\sigma \Pi_{\sigma} \tilde{\omega}_{\phi^{1/2}}, \]

and \( A' \leftrightarrow \beta_1, B' \leftrightarrow \beta_2 \).

By continuity, this means that there exists a small ball in \( \mathbb{R}^6 \), centered at \( p \), with some strictly positive radius \( \rho \), so that

\[
\text{for all } \tilde{p} \in \tilde{B} = \{ p'' \mid \| \tilde{p}'' - p \| < \rho \}, \sum_{\sigma} p'' = 1
\]

we have \( \sum_{\sigma} \tilde{p} \Pi_{\sigma} \tilde{\omega}_{\phi_1} \leftrightarrow \beta_1 \), \( \sum_{\sigma} \tilde{p} \Pi_{\sigma} \tilde{\omega}_{\phi_2} \leftrightarrow \beta_2 \).

This means the result of the theorem is robust. (It doesn't change under small perturbations.)