Arrow's Theorem: First Visit

A voting scheme is a "digestion mechanism" that transforms the collection of individual rankings by all the voters to a societal ranking.

More precisely, if each of the voters $V_1, \ldots, V_N$ ranks the candidates $c_1, \ldots, c_N$ in a transitive ordering (ties allowed), which we could denote, e.g.,

$$S_k = \left( \sigma_1(k); \sigma_2(k); \sigma_3(k); \ldots; \sigma_N(k) \right)$$

\[ \uparrow \]

where $\{ \sigma_1(k), \sigma_2(k), \ldots, \sigma_N(k) \} = \{ c_1, c_2, \ldots, c_N \}$, then we have a map $f$ from $S = \text{set of all possible } K$-tuples $(S_1, S_2, \ldots, S_K)$ to $\hat{S} = \text{set of all possible rankings (transitive again, ties allowed)}$. This map should satisfy some requirements, of course.

Arrow proposed the following four conditions:

(As) Suppose $f(S_1, \ldots, S_K) = \hat{S}$, and that in the ordering $\hat{S}$, $A \succ B$. If in a different $K$-tuple of individual rankings $(S'_1, \ldots, S'_K)$, the only change is that for each voter, the relative ranking of $A$ over $B$ has remained the same or improved (when compared to $S_1, \ldots, S_K$) (i.e., one of the following happened:

- no change
- or, if $A$ was ranked below $B$, the $A$ and $B$ have been interchanged or tied (consecutive)
- if A was tied with B, it is now preferred)
then the relative ranking of A and B in \( S' = f(\bar{S}'_1, \ldots, \bar{S}'_k) \) must be \( A \succ B \), also

(A2) Suppose \( f(\bar{S}'_1, \ldots, \bar{S}'_k) = \bar{S} \), and that \( A \succeq B \) in \( \bar{S} \). If, insofar as A and B are concerned, their relative ranking \( f \) is the same in \( (\bar{S}'_1, \ldots, \bar{S}'_k) \) as in \( (\bar{S}_1, \ldots, \bar{S}_k) \), then it follows that \( A \succeq B \) in \( \bar{S}'_1 = f(\bar{S}'_1, \ldots, \bar{S}'_k) \).

(A3) For any pair of candidates \( c_i, c_j \) there exists some input \( (\bar{S}_1, \ldots, \bar{S}_k) \) that results in \( c_i \succeq c_j \) in \( f(\bar{S}_1, \ldots, \bar{S}_k) \) even though \( c_j \succeq c_i \) in \( \bar{S}_k \).

(A4) For all possible rankings of \( c_1, \ldots, c_N \), there exists an input \( (\bar{S}_1, \ldots, \bar{S}_k) \) so that \( f(\bar{S}_1, \ldots, \bar{S}_k) \) achieves this ranking.

He then went on to show that there exist no voting schemes that satisfy all these requirements....

Proof of the incompatibility of (A1) – (A4).

This proof requires a few lemmas first.

Lemma 1

If all voters rank A above B (i.e. \( A \succeq B \) in \( \bar{S}_1, \ldots, \bar{S}_k \)), then the societal ranking does the same (i.e. \( A \succeq B \) in \( f(\bar{S}_1, \ldots, \bar{S}_k) \)).

Proof: Take some \((\bar{S}_1, \ldots, \bar{S}_k)\) with \( A \succeq B \) for all \( k \).

There exists some \( \bar{S}'_1, \ldots, \bar{S}'_k \) so that \( A \succeq B \) in \( f(\bar{S}'_1, \ldots, \bar{S}'_k) \), by (A4).

We now want to show that there is a way to "transform"
(s_4, \ldots, s_k) \text{ into } (s_1, \ldots, s_k) \text{ that preserves the ordering } A \geq B.

First, exchange A and B in all s_j that had B \geq A, and break ties in all s_j where A = B. These "moves" lead to a new \((s_4^*, \ldots, s_k^*)\) that have \(A \geq B\) for each \(s_j^*\), and where (by (A1)) \(f(s_4^*, \ldots, s_k^*)\) still has \(A \geq B\).

Next, leaving A and B alone, reshuffle the other candidates in each \(s_j^*\) so that they end up in the same places as in \(s_j\). This leads to \((s_4^*, \ldots, s_k^*)\) through "moves" that, by (A2), do not affect the societal ranking of A, B.

It follows that \(A \geq B\) if \(f(s_4, \ldots, s_k)\).

To prove incompatibility of (A1) \(\rightarrow\) (A4), we need only give 1 situation that is contradictory. We shall give this for the case \(K=2\) (two voters only!) and \(N=3\) (3 candidates). For the next lemma, we already specialize to this situation.

**Lemma 2.**

If \(A \geq B\) in \(s_1\), and \(B \geq A\) in \(s_2\), then the societal ranking \(f(s_1, s_2)\) must have \(A \geq B\).

**Proof.** Suppose \(A \geq B\) in \(f(s_1, s_2)\). Take now arbitrary \(s_1', s_2'\) with \(A \geq B\) in \(s_1'\). If \(A \geq B\) in \(s_2'\), then we have \(A \geq B\) in \(f(s_1', s_2')\), by Lemma 1. If \(B \geq A\) in \(s_2'\), then we can, by
just moving the third candidate in both $S'_1$, $S'_2$, make $S'_1$ identical to $S_1$, $S'_2$ identical to $S_2$. It follows that the relative ranking of $A$ and $B$ must be the same in $f(S'_1, S'_2)$ as in $f(S_1, S_2)$, i.e. that $A \geq B$ in $f(S'_1, S'_2)$ in this case as well.

There remains only the case when $A \not\geq B$ in $f(S'_2)$. In this case we can start from $S_2$, and consider $S''_2$, obtained by moving $A$ "up" in $S_2$ until it ties with $B$. Then, since $A \not\geq B$ in $f(S_1, S_2)$, we also have (by (A1)) $A \not\geq B$ in $f(S_2, S''_2)$. The difference between $(S_2, S''_2)$ and $(S'_1, S'_2)$ concerns only the position of the third candidate $C$, so that the relative ranking of $A, B$ in $S'_1, S'_2$ must also be $A \not\geq B$, by (A2).

Consequently, if $A \not\geq B$ in $f(S_1, S_2)$, we would have $A \not\geq B$ in all $f(S'_1, S'_2)$ with $A \not\geq B$ in $S'_1$, regardless of how vote 2 ranks $A$ and $B$. This is not allowed, however, by (A3). It follows that $A \not\geq B$ in $f(S_1, S_2)$ is not possible.

Similarly, $A \not\geq B$ in $f(S_1, S_2)$ is impossible.

It follows that $A \not\geq B$ if $f(S_1, S_2)$.

We are now ready to derive our contradiction.

Suppose $S_1, S_2$ are as follows:

$S_1 : A \not\geq C \not\geq B$

$S_2 : B \not\geq A \not\geq C$

Then, because $A \not\geq B$ in $S_1$, $B \not\geq A$ in $S_2$, we must have $A \not\geq B$ in $f(S_1, S_2)$.
Likewise, because \( B \succ C \) in \( S_2 \), \( C \succ B \) in \( S_1 \), we must have \( B \succ C \) in \( f(S_1, S_2) \).

Because \( f(S_1, S_2) \) is transitive, it follows from \( A \sim B \) and \( B \succ C \) that \( A \succ C \) in \( f(S_1, S_2) \).

Yet \( A \preceq C \) in \( S_2 \) as well as in \( S_2 \Rightarrow A \preceq C \) in \( f(S_1, S_2) \)

by Lemma 1.

\[ \Rightarrow \text{contradiction!} \]

Of course one can try to make Arrow's hypotheses weaker, and find arguments that are not quite as restrictive. The basic problem remains however, in all these incarnations. There simply isn't a map from individual transitive rankings to societal transitive rankings that is always fair. Given that unfairness is thus a fact of voting life, it is all the more important to study carefully what goes on in different voting schemes.

This will be the topic of the next few lectures. We'll first see an introduction to convex sets, so that we can confidently apply notions involving convex hulls and such, and then go on to the "geometry of voting" (based in good part on the book by D. Saari of the same name).