Strong unique continuation for higher order elliptic equations with Gevrey coefficients

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We address the strong unique continuation problem for higher order elliptic partial differential equations in 2D with Gevrey coefficients. We provide a quantitative estimate of unique continuation (observability estimate) and prove that the solutions satisfy the strong unique continuation property for ranges of the Gevrey exponent strictly including non-analytic Gevrey classes. As an application, we obtain a new upper bound on the Hausdorff length of the nodal sets of solutions with a polynomial dependence on the coefficients.

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1. Introduction

In this paper, we study the strong unique continuation problem for the higher order elliptic partial differential operators

\[ P(x, D)u = \sum_{|\alpha| \leq 2s} a_\alpha(x) D^\alpha u \]  (1.1)

with Gevrey coefficients in two space dimensions. Even though, in general, for higher order operators the strong unique continuation property does not hold (cf. [13,27] for counterexamples), we prove it for (1.1) with Gevrey coefficients for ranges of the Gevrey exponents strictly including non-analytic...
classes. As an application, we provide a polynomial upper bound for the Hausdorff length of the nodal sets of solutions in terms of the coefficients.

Recall that a function \( u \in C^\infty(\Omega) \) vanishes of infinite order at a point \( x_0 \) in the open set \( \Omega \subset \mathbb{R}^2 \) if \( D^\alpha u(x_0) = 0 \) for all multi-indices \( \alpha \in \mathbb{N}_0^2 \). We say that the operator (1.1) has the strong unique continuation property if every solution \( u \) of the equation \( P(x, D)u = 0 \) which vanishes of infinite order at \( x_0 \) is identically zero in a neighborhood of \( x_0 \).

By now the strong unique continuation problem for second order elliptic operators is well-understood. The results for the elliptic equation

\[
-\Delta u = W \cdot \nabla u + Vu
\]

go back to the work of Carleman [1] in the case of the space dimension \( n = 2 \) and bounded coefficients, who used as a main tool a weighted \( L^2 \) estimate. There is a large amount of work on strong continuation, achieved by different estimates of Carleman type (cf. [2,3,8,12,16,17,24,28,29] and the review papers by Kenig [18,19] and Vessella [30]).

There is a very limited number of results available regarding strong unique continuation for higher order elliptic equations with non-analytic coefficients. In fact the only result we are aware of is given by Colombini and Koch in [4] who proved the strong unique continuation for products of second order elliptic operators with Gevrey coefficients for a certain range of the Gevrey exponent (cf. [4, Theorem 1]). In this paper, we obtain a strong unique continuation result for higher order elliptic operators (1.1) with simple complex characteristics and coefficients in the Gevrey class \( G^\sigma \) provided the Gevrey exponent \( \sigma \) is less than a constant strictly greater than 1.

We note that the weak continuation for higher order equations is better understood. For instance, a result of Hörmander [14] addresses solutions of differential inequalities

\[
|P(x, D)u| \leq K \sum_{|\alpha| \leq m} |D^\alpha u|,
\]

where \( P \) is a homogeneous elliptic operator of degree \( m \) with simple characteristics; it states that if a solution \( u \) vanishes in the intersection of a neighborhood of 0 and the set \( x_1 < x_2^2 + \cdots + x_n^2 \), then \( u \) vanishes in a full neighborhood of 0.

A natural question to ask about a solution \( u \) is how large is the size of its nodal (zero, vanishing) set. In their seminal work [5–7], Donnelly and Fefferman provided upper and lower bounds on the \((n-1)\)-dimensional measure of the nodal set for an eigenfunction of the Laplacian, considered on an analytic Riemannian \( n \)-manifold (see also [11,25]). For higher order analytic elliptic equations, bounds were obtained in [23]; for estimates on nodal sets of solutions of parabolic equations cf. [10,20–22].

Most of the present paper is devoted to the proof of a Carleman-type inequality for the operator (1.1), which is needed to establish a quantitative estimate of unique continuation (observability inequality) stated in Theorem 2.1. Hörmander established in [12] a necessary and sufficient condition for a Carleman estimate to hold; thus, one of the key steps here is the choice of an appropriate radially decreasing weight function for which the pseudo-convexity condition is satisfied. Note that our choice of weight puts a restriction on the space dimension; namely Lemma 3.2 below is valid only for operators (1.1) defined on a subset of \( \mathbb{R}^2 \). In higher dimensions, we obtain the main results of this work (Theorems 2.1, 2.2, and 2.3) for classes of higher order elliptic operators satisfying a certain condition (cf. Theorem 2.4). Another obstacle we face is that the weight \( |x|^{-m} \) is singular at the origin while the operator \( P \) is too general to investigate the validity of the Carleman-type estimate directly (as it can be done for second order operators). We overcome this by establishing a uniform in \( x \) Carleman estimate for a family of operators, leading to a Carleman-type inequality on a torus \( B_1 \setminus B_\delta \) with constants which can be computed explicitly in terms of \( \delta \) (see Lemma 3.3 below).

In general, functions in the Gevrey class may not satisfy the strong unique continuation property (see for instance [24]). However, in Theorem 2.2 below, we establish strong unique continuation for the solutions of the equation \( P(x, D)u = 0 \), satisfying the Gevrey regularity condition (2.2)
(cf. [26, Section VII.1.3]) and the observability estimate from Theorem 2.1. The proof relies on [15, Theorem 2.4]. Another application of the latter theorem together with a result due to Han [9] is a polynomial upper bound on the Hausdorff length of the nodal sets (see Theorem 2.3).

The paper is organized as follows. In Section 2 we state our main results, Theorems 2.1, 2.2, 2.3, and 2.4. The next section is devoted to the Carleman-type inequality given in Theorem 3.1 and to certain auxiliary results needed for its proof. In Section 4, we establish a propagation of smallness result which leads to the proof of Theorem 2.2. In the last section, we give the proofs of Theorems 2.2, 2.3, and 2.4.

2. Notation and the main result

In this paper, we consider the elliptic partial differential operator with simple complex characteristics and coefficients in the Gevrey class with $\sigma \geq 1$

$$P(x, D)u = \sum_{|\alpha| \leq 2s} a_\alpha(x) D^\alpha u$$  \hspace{1cm} (2.1)

defined on a domain $\tilde{\Omega}$ in $\mathbb{R}^2$ where $s \in \mathbb{N}$. Let $\Omega$ be a subdomain of $\tilde{\Omega}$ such that $\text{dist}(\Omega, \partial \tilde{\Omega})$ is greater than a constant. By rescaling, we may assume that $\text{dist}(\Omega, \partial \tilde{\Omega}) \geq 4$. Let $u$ be a solution of the equation $P(x, D)u = 0$ in $\tilde{\Omega}$ which is not identically zero. We assume that $u$ is an infinitely smooth function in $\tilde{\Omega}$ and that there exist positive constants $M$ and $\delta$ such that

$$\|D^\alpha u\|_{L^2(B_2(x))} \leq M |\alpha|!^{\sigma} \delta^{|\alpha|} \|u\|_{L^2(B_4(x))}, \quad x \in \Omega$$  \hspace{1cm} (2.2)

for any $\alpha \in \mathbb{N}_0^2$ where $\sigma \geq 1$ is fixed. Also, we assume that the coefficients $a_\alpha$ are infinitely smooth functions and that there exist nonnegative constants $M_\alpha$ such that

$$\|D^\beta a_\alpha\|_{L^\infty(B_2(x))} \leq M_\alpha |\beta|!^{\sigma} \delta^{|\beta|}, \quad x \in \Omega$$  \hspace{1cm} (2.3)

for all $\alpha, \beta \in \mathbb{N}_0^2$ with $|\alpha| = 0, \ldots, 2s$. Assume $u$ satisfies the doubling property

$$\|u\|_{L^2(B_4(x))} \leq K \|u\|_{L^2(B_2(x))}, \quad x \in \Omega$$  \hspace{1cm} (2.4)

for some constant $K \geq 1$. Under the above hypotheses we establish a quantitative estimate of unique continuation (an observability estimate) for the elliptic operator (2.1) with simple complex characteristics and only Gevrey coefficients.

**Theorem 2.1.** Suppose that $u$ is a nontrivial solution of $P(x, D)u = 0$ satisfying (2.2)–(2.4). Then

$$\|u\|_{L^2(B_2(x))} \leq \exp(Q_1(\delta^{-1}, K, M, \{M_\alpha |\alpha| < 2s\})) \|u\|_{L^2(B_4(x))}, \quad x \in \Omega$$

for some nonnegative polynomial $Q_1$ with coefficients depending on $P_{2s}$ and $\Omega$.

This observability estimate is one of the hypotheses needed in Lemma 5.1 (see Section 5). Thus, it enables us to establish the strong unique continuation property for the solutions of $P(x, D)u = 0$.

**Theorem 2.2.** Suppose that the assumptions in Theorem 2.1 are satisfied and $\sigma \leq 1 + \eta$, where $\eta$ is a constant depending on $s$. Then the operator (2.1) has the strong unique continuation property.
Using a result due to Han [9] on the structure of the nodal sets of solutions, together with Theorem 2.1 and Lemma 5.1, we obtain an upper bound for the 1-dimensional Hausdorff measure of the zero sets of solutions of the equation $P(x, D)u = 0$ with a polynomial dependence on the coefficients.

Theorem 2.3. Suppose that the assumptions in Theorem 2.1 are satisfied. Then

$$\mathcal{H}^1 \{ x \in \Omega : u = 0 \} \leq Q_2(\delta^{-1}, K, M, \{ M_\alpha \} | \alpha | < 2s),$$

where $Q_2$ is a nonnegative polynomial with coefficients depending on $P_{2s}$ and $\Omega$.

As the next theorem shows, the restriction on the dimension $n = 2$ may be removed for symbols satisfying an additional property.

Theorem 2.4. Let $n \geq 3$. Suppose that the assumptions in Theorem 2.2 are satisfied and $\sigma \leq 1 + \eta$, where $\eta$ is a constant depending on $s$. Additionally, assume that for $0 \neq \xi \in \mathbb{R}^n$, we have

$$|P_{2s}(x, \xi)|^2 + |(x - x_0) \cdot \nabla x | P_{2s}(x, \xi)|^2 > 0, \quad x \in A_{x_0}, \quad \xi \in \mathbb{R}^n, \quad \text{and} \quad \tau \in \mathbb{R},$$

for all $x_0 \in \mathcal{O}$, where $\psi_{x_0}(x) = |x - x_0|^{-m}$ and $A_{x_0}$ is the $n$-dimensional unit torus centered at $x_0$. Then the operator (2.1) has the strong unique continuation property. Moreover, we obtain for the Hausdorff length of the nodal sets

$$\mathcal{H}^1 \{ x \in \Omega : u = 0 \} \leq Q_3(\delta^{-1}, K, M, \{ M_\alpha \} | \alpha | < 2s),$$

where $Q_3$ is a nonnegative polynomial with coefficients depending on $P_{2s}$ and $\Omega$.

Theorem 2.1 is proven in Section 4 and Theorems 2.2, 2.3, and 2.4 in Section 5.

3. Carleman estimate

Let $\Omega \subset \mathbb{R}^2$ be a domain, and let $s \in \mathbb{N}$. We consider the elliptic partial differential operator

$$P(x, D)u = \sum_{|\alpha| \leq 2s} a_\alpha(x)D^\alpha u$$

with coefficients in the Gevrey class $G^\sigma$ where $\sigma \geq 1$.

In this section, we denote the principal part of the operator $P(x, D)$ by $P_{2s}(x, D)$ and the corresponding principal symbol by $P_{2s}(x, \xi)$. Also, we adopt the notation $P_{2s}^{(j)}(x, \xi) = (\partial P_{2s}/\partial \xi_j)(x, \xi)$ and $P_{2s,j}(x, \xi) = (\partial P_{2s}/\partial x_j)(x, \xi)$ for the first partial derivatives of $P_{2s}(x, \xi)$.

Assume that the operator $P(x, D)$ has simple complex characteristics, i.e., the principal symbol $P_{2s}(x, \xi)$, which is a homogeneous polynomial of degree $2s$ in the complex variable $\xi \in \mathbb{C}^2$, has only simple (with multiplicity 1) zeros. Our goal is to derive a Carleman-type estimate for such an operator with a suitably chosen weight function $\psi$.

Theorem 3.1. There exist positive constants $C$, $\tau_0$, and $\rho_0$ such that

$$\sum_{|\alpha| < 2s} \tau^{4s-2|\alpha|-1} \int_{A(x_0, \delta, \rho_0)} |D^\alpha v|^2 e^{2\tau \psi_{x_0}} \leq C \int_{A(x_0, \delta, \rho_0)} |P_{2s}(x, D)v|^2 e^{2\tau \psi_{x_0}}$$

(3.1)
for any \( v \in C_0^\infty(\Omega) \) with support in the annulus \( A(x_0, \delta, \rho_0) = \{ x \in \Omega : \delta < |x - x_0| < \rho_0 \} \), \( x_0 \in \Omega \) and \( \tau \geq \tau_0 \delta^{-4s} \), provided \( m \in \mathbb{N} \) is a large enough constant and \( \psi_{x_0}(x) = |x - x_0|^{-m} \).

Above and in the sequel, the symbol \( C \) denotes a generic positive constant which is allowed to depend only on \( s, \sigma \), and \( P_{2s} \). Any additional dependence is indicated explicitly.

First, we prove a uniform Carleman estimate in the dyadic annuli \( A_r(x_0) = \{ x \in \Omega : 2^{-r-1} < |x - x_0| < 2^{-r+2} \} \) for all \( x_0 \in \Omega \) and \( r \in \mathbb{N} \) such that \( r \geq r_0 \). Using a partition of unity, we then prove a corresponding Carleman-type estimate on \( A(x_0, \delta, \rho_0) \).

Let \( A_{x_0} = \{ x \in \Omega : 1/2 < |x - x_0| < 4 \} \) be the unit annulus centered at \( x_0 \in \Omega \). Clearly, we have \( \psi_{x_0} \in C^2(A_{x_0}) \) and \( \nabla \psi_{x_0}(x) = -m(x - x_0)|x - x_0|^{-(m+2)} \neq 0 \) on \( A_{x_0} \). Also, we have that \( P_{2s}(x, D) \) is elliptic in \( \tilde{A}_{x_0} \), that is, \( P_{2s}(x, \xi) \neq 0 \) if \( x \in \tilde{A}_{x_0} \) and \( \xi \in \mathbb{R}^2 \setminus \{0\} \). In the next lemma we establish the pseudo-convexity condition for \( P_{2s}(x, D) \) in \( \tilde{A}_{x_0} \) with a weight \( \psi_{x_0} \) for all \( x_0 \in \tilde{\Omega} \), needed in the proof of Theorem 3.1.

**Lemma 3.2.** Let \( x_0 \in \tilde{\Omega} \) and \( \psi_{x_0}(x) = |x - x_0|^{-m} \). There exists a sufficiently large number \( m \) such that the following is true. Assume that for \( 0 \neq \zeta = \xi + i \tau \nabla \psi_{x_0}(x) \) the characteristic equation \( P_{2s}(x, \zeta) = 0 \) is satisfied, where \( x \in \tilde{A}_{x_0}, \xi \in \mathbb{R}^2 \), and \( \tau \in \mathbb{R} \). Then

\[
\sum_{j,k=1}^{2} \frac{\partial^2 \psi_{x_0}}{\partial x_j \partial x_k} P_{2s}^{(j)}(x, \zeta) P_{2s}^{(k)}(x, \zeta) + \frac{1}{\tau} \text{Im} \sum_{k=1}^{2} P_{2s,k}(x, \zeta) \frac{P_{2s}^{(k)}(x, \zeta)}{\psi_{x_0}(x)} > 0 \tag{3.2}
\]

for all \( x_0 \in \tilde{\Omega} \).

**Proof.** Using that

\[
\frac{\partial^2 \psi_{x_0}}{\partial x_j \partial x_k} = \frac{(m+2)m(x_j - x_0j)(x_k - x_0k)}{|x - x_0|^{m+4}} - \frac{m\delta_{jk}}{|x - x_0|^{m+2}}
\]

for \( j, k = 1, 2 \), we may rewrite the asserted pseudo-convexity condition (3.2) for \( \psi_{x_0} \) as

\[
\frac{(m+2)m}{|x - x_0|^{m+4}} \left| \sum_{j=1}^{2} (x_j - x_0j) P_{2s}^{(j)}(x, \zeta) \right|^2 - \frac{m}{|x - x_0|^{m+2}} \sum_{j=1}^{2} \left| P_{2s}^{(j)}(x, \zeta) \right|^2
\]

\[
+ \frac{1}{\tau} \text{Im} \sum_{k=1}^{2} P_{2s,k}(x, \zeta) \frac{P_{2s}^{(k)}(x, \zeta)}{\psi_{x_0}(x)} > 0. \tag{3.3}
\]

First, note that if \( P(x, \zeta) = 0 \) for \( \zeta = \xi + i \eta \), then there exists a positive constant \( C \) such that

\[
\frac{\zeta}{C} \leq |\xi|, |\eta| \leq |\zeta|. \tag{3.4}
\]

The second inequality in (3.4) is trivial. In order to prove the first inequality, we consider the compact set \( K = \{ \xi = \xi + i \eta : |\xi| = 1, P_{2s}(x, \zeta) = 0 \} \) in \( \mathbb{C}^2 \). Since \( P_{2s}(x, \zeta) \) is elliptic and homogeneous of order 2s in the second variable, we have that \( \xi \neq 0 \) and \( \eta \neq 0 \) on \( K \). Thus, there exists a constant \( C > 0 \) such that \( |\xi| \geq 1/C \) and \( |\eta| \geq 1/C \) on \( K \). Now, let \( \zeta = \xi + i \eta \) be arbitrary in \( \mathbb{C}^2 \). By the above property of \( K \), we have \( |\xi|/|\zeta| \geq 1/C \) and \( |\eta|/|\zeta| \geq 1/C \). Hence the first inequality in (3.4) holds.
Next, we claim that there exists a positive constant $C$ such that for all $x_0 \in \tilde{\Omega}$

$$
\left| \sum_{j=1}^{2} (x_j - x_{0j}) P_{2s}^{(j)}(x, \xi) \right| \geq \frac{|\xi|^{2s-1}}{C} 
$$

(3.5)

if $x \in \tilde{A}_{x_0}$ and $\zeta = \xi + i\eta$ are satisfying $P_{2s}(x, \xi) = 0$, where $\xi, \eta \in \mathbb{R}^2$ and $\eta = \tau y$ for $y$ parallel to $x - x_0$ such that $|y| = 1$. It suffices to prove the claim for $\zeta \in K$, where $K = \{ \zeta = \xi + i\tau y: |\xi| = |y| = 1, P_{2s}(x, \zeta) = 0 \}$ is a compact set. (Note that by the previous claim, for any $\zeta \in K$, we have $1/C \leq |\xi|, |\tau| \leq 1$.) For the sake of contradiction, suppose that there exist sequences $\{x_0^{(k)}\} \subset \tilde{\Omega}$, $\{x^{(k)} - x_0^{(k)}\}_{k \in \mathbb{N}} \subset \tilde{A}_{x_0}$, and $\{\zeta^{(k)} = \xi^{(k)} + i\tau^{(k)} y^{(k)}\}_{k \in \mathbb{N}} \subset K$ such that

$$
\left| \sum_{j=1}^{2} (x_j^{(k)} - x_{0j}) P_{2s}^{(j)}(x^{(k)}, \zeta^{(k)}) \right| \leq \frac{1}{k} 
$$

(3.6)

where $\tau^{(k)} \in \mathbb{R}$ and $y^{(k)}$ is parallel to $x^{(k)} - x_0^{(k)}$. By passing to subsequences, we may assume that $x_0^{(k)} \to x_0$, $x^{(k)} \to x$, and $\xi^{(k)} + i\tau^{(k)} y^{(k)} \to \xi + i\tau y \in \tilde{\Omega} \setminus \tilde{\Delta}_{x_0}$, and $K$, respectively, where $x_0 \in \tilde{\Omega}$, $x \in \tilde{A}_{x_0}$, and $\xi + i\tau y \in K$. Then the limits $x_0, x$, and $\zeta = \xi + i\tau y$ satisfy

$$
\sum_{j=1}^{2} (x_j - x_{0j}) P_{2s}^{(j)}(x, \zeta) = 0 
$$

(3.7)

by continuity. Also, we have

$$
\sum_{j=1}^{2} \xi_j P_{2s}^{(j)}(x, \zeta) = 0, 
$$

(3.8)

which follows by the homogeneity of the polynomial $P_{2s}$ in the second variable. Indeed, we differentiate $P_{2s}(x, \lambda \zeta) = \lambda^{2s} P_{2s}(x, \zeta)$ with respect to $\lambda \in \mathbb{R}$ and then set $\lambda = 1$. Since $y$ is parallel to $x - x_0$, (3.7) and (3.8) imply

$$
\sum_{j=1}^{2} \xi_j P_{2s}^{(j)}(x, \zeta) = 0. 
$$

(3.9)

Also, since the operator $P(x, D)$ has simple complex characteristics and the equalities (3.7) and (3.9) hold for $x, \xi \in \mathbb{R}^2$, we obtain that $\xi$ is parallel to $x - x_0$, or equivalently that $\xi = \lambda y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then, for $\zeta = (\lambda + i\tau) y \in K$, we have $0 = P(x, \zeta) = (\lambda + i\tau)^{2s} P(x, y)$ which gives $P(x, y) = 0$ for $y \in \mathbb{R}^2 \setminus \{0\}$. The last equality contradicts the ellipticity assumption on $P(x, D)$. Therefore (3.5) is established.

Now, for all $x_0 \in \tilde{\Omega}$ we prove that for a sufficiently large $m \in \mathbb{N}$ the first term on the left side of (3.3) is dominant. By the claim proven above, we obtain the upper bound

$$
\frac{(m+2)m}{|x - x_0|^{m+4}} \left| \sum_{j=1}^{2} (x_j - x_{0j}) P_{2s}^{(j)}(x, \zeta) \right|^2 \geq \frac{(m+2)m}{C|x - x_0|^{m+4}} |\xi|^{4s-2} 
$$

(3.10)

for $x \in \tilde{A}_{x_0}$. For the second and the third term on the left side of (3.3), we have respectively
and
\[
\frac{1}{\tau} \left| \text{Im} \sum_{k=1}^{2} P_{2s,k}(x, \zeta) \bar{P}^{(k)}_{2s}(x, \zeta) \right| \leq \frac{C}{\tau} |\zeta|^{4s-2} \leq C |\nabla \psi_{x_0}(x)| |\zeta|^{4s-2}
\]
(3.12)

for \( x \in \tilde{\Omega} \), where we used that \( \tau |\nabla \psi_{x_0}(x)| \geq 1/C|\zeta| \) by (3.4). Choosing \( m \in \mathbb{N} \) large enough, the lemma follows. \( \square \)

Let \( x_0 \in \tilde{\Omega} \). We consider a sequence of operators for \( r \in \mathbb{N}_0 \) defined by
\[
P_{2s}^{x_0,r}(x, D) = \sum_{|\alpha|=2s} a_\alpha (x_0 + 2^{-r}(x - x_0)) D^\alpha, \quad x \in \tilde{\Omega} \]
and the limiting operator with constant coefficients defined by
\[
P_{2s}^{x_0}(D) = \sum_{|\alpha|=2s} a_\alpha (x_0) D^\alpha.
\]

We need the following auxiliary assertion.

**Lemma 3.3.** There exist constants \( C, \tau_0 > 0 \), and \( r_0 \in \mathbb{N} \) such that for all \( x_0 \in \tilde{\Omega} \)
\[
\sum_{|\alpha|<2s} \tau^{4s-2|\alpha|-1} \int_{\tilde{\Omega}} |D^\alpha u|^{2} e^{2\tau |\psi_{x_0}|} \leq C \int_{\tilde{\Omega}} |P_{2s}^{x_0,r}(x, D)u|^{2} e^{2\tau |\psi_{x_0}|}
\]
(3.13)

holds for any \( u \in C^{\infty}_c(A_{x_0}), \tau \geq \tau_0, \) and \( r \geq r_0, \) provided that \( m \) is a large enough constant and \( \psi_{x_0}(x) = |x - x_0|^{-m} \).

**Proof.** As a consequence of Lemma 3.2, we can find a large enough number \( m \), so that the pseudo-convexity condition (3.2) is satisfied for all \( x_0 \in \tilde{\Omega} \) for the limiting operator \( P_{2s}^{x_0}(D) \) with a weight function \( \psi_{x_0}(x) = |x - x_0|^{-m} \) in \( \tilde{\Omega} \). Using a nonlinear change of variables, we can straighten up the weight \( \psi_{x_0} \), so that \( \psi_{x_0}(x) = |x - x_0, N| \) for \( x \in \tilde{\Omega} \), where \( N \neq 0 \) is a constant. Now the new limiting operator, which we denote by \( P_{2s}^{x_0}(x, D) \), depends on \( x \). Also, we denote the sequence of operators after the change of variable by \( P_{2s}^{x_0,r}(x, D) \). We follow the proof of [12, Theorem 8.3.1] for the limiting operator \( P_{2s}^{x_0}(x, D) \) with adopting the notations therein to prove the estimate (3.13) for the sequence \( P_{2s}^{x_0,r}(x, D) \) where \( r \geq r_0 \). Setting \( \psi(x) = u(x) \exp \tau (x - x_0, N) \) and applying the integration by parts formula from [12, Lemma 8.2.2], we obtain for all \( x_0 \in \tilde{\Omega} \) and \( r \in \mathbb{N}_0 \)

\[
\int |P_{2s}^{x_0,r}(x, D)u|^{2} e^{2\tau (x-x_0,N)} = \int |P_{2s}^{x_0,r}(x, D + i\tau N)v|^{2}
\]
\[
\geq \int |P_{2s}^{x_0,r}(x, D + i\tau N)v|^{2} - \left| \tilde{P}_{2s}^{x_0,r}(x, D - i\tau N)v \right|^{2}
\]
\[
= C^{x_0,r}(x, D, \tilde{D})v \bar{v},
\]
(3.14)
where the quadratic differential form on the right-hand side satisfies

\[
G_{\tau}^{x_0, r}(x, \xi, \xi) = 2 \text{Im} \sum_{k=1}^{2} p_{2s,k}^{x_0, r}(x, \xi + i\tau N) \overline{p_{2s,k}^{x_0, r,(k)}}(x, \xi + i\tau N)
+ 2 \text{Im} \left( p_{2s}^{x_0, r}(x, \xi + i\tau N) \sum_{k=1}^{2} p_{2s,k}^{x_0, r,(k)}(x, \xi + i\tau N) \right)
\]  

(3.15)

for \( x \in \tilde{A}_{x_0} \) and \( \xi \in \mathbb{R}^2 \). Similar inequality as (3.14) also holds for the limiting operator \( P_{2s}^{x_0}(x, D) \) (cf. the proof of [12, Theorem 8.3.1]) and the corresponding differential quadratic form reads

\[
G_{\tau}^{x_0}(x, \xi, \xi) = 2 \text{Im} \sum_{k=1}^{2} p_{2s,k}^{x_0}(x, \xi + i\tau N) \overline{p_{2s,k}^{x_0,(k)}}(x, \xi + i\tau N)
+ 2 \text{Im} \left( p_{2s}^{x_0}(x, \xi + i\tau N) \sum_{k=1}^{2} p_{2s,k}^{x_0,(k)}(x, \xi + i\tau N) \right).
\]

Moreover, there exist constants \( C_1, C_2 > 0 \) such that for all \( x_0 \in \tilde{\Omega} \)

\[
|\xi + i\tau N|^{4s} \leq C_1 \tau G_{\tau}^{x_0}(x, \xi, \xi) + C_2 \left| p_{2s}^{x_0}(x, \xi + i\tau N) \right|^2, \quad x \in \tilde{A}_{x_0}
\]  

(3.16)

if \( \tau \geq 0 \) and \( \xi \in \mathbb{R}^2 \). (Indeed, by the homogeneity of the polynomials on both sides of (3.16), we restrict ourselves to the compact set \( M = \{(\xi, \tau): |\xi + i\tau N| = 1, \ \tau \geq 0\} \). Consider the subset of \( M \) defined by \( M_0 = \{(x_0, x, \xi, \tau): P_{2s}^{x_0}(x, \xi + i\tau N) = 0, x_0 \in \tilde{\Omega}, \ x \in \tilde{A}_{x_0}, \ (\xi, \tau) \in M\} \). Since the operator \( P_{2s}^{x_0} \) satisfies the ellipticity estimate \( P_{2s}^{x_0}(x, \xi) \geq |\xi|^{2s}/C \) and the pseudo-convexity condition (3.2) in \( \tilde{A}_{x_0} \), we have that \( \tau \) and \( G_{\tau}^{x_0}(x, \xi, \xi) \) are bounded from below on the compact \( M_0 \). Thus, there exists a positive constant \( C_1 \) such that \( 1 \leq (C_1/2) \tau G_{\tau}^{x_0}(x, \xi, \xi) \) in \( M_0 \). The last inequality depends continuously on all variables, so it holds in a small neighborhood \( V \) of \( M_0 \). Also, the polynomial \( P_{2s}^{x_0}(x, \xi + i\tau N) \) has no zeros in the compact set \( M \setminus V \), so it has a lower bound there. We conclude that (3.16) holds on \( M \).

Let \( \tilde{x}_0 \in \tilde{A}_{x_0} \) be arbitrary. By the continuity of the coefficients of \( P_{2s}^{x_0} \) and \( G_{\tau}^{x_0} \), we can find \( r_0 \in \mathbb{N} \) such that

\[
|\xi + i\tau N|^{4s} \leq C_1 \tau G_{\tau}^{x_0}(\tilde{x}_0, \xi, \xi) + C_2 \left| p_{2s}^{x_0}(\tilde{x}_0, \xi + i\tau N) \right|^2, \quad r \geq r_0
\]  

(3.17)

for all \( \tau \geq 0 \) and \( \xi \in \mathbb{R}^2 \). (Indeed, on the compact set \( M \), we have \( \tau \leq 1/N \) and the polynomials \( P_{2s}^{x_0} \) and \( G_{\tau}^{x_0} \) are smooth functions depending only on the \( x \) variable. Thus, given \( \epsilon > 0 \), there exists \( r_0 \in \mathbb{N} \), depending only on the first and second order derivatives of the coefficients of \( P_{2s} \), such that for all \( r \geq r_0 \) we have \( |P_{2s}^{x_0}(\tilde{x}_0) - P_{2s}^{x_0}(\tilde{x}_0)| < \epsilon \) and \( |G_{\tau}^{x_0}(\tilde{x}_0) - G_{\tau}^{x_0}(\tilde{x}_0)| < \epsilon \). Choosing \( \epsilon = (C_1/N + C_2)^{-1} \), we get \( 1 \leq C_1 \tau G_{\tau}^{x_0}(\tilde{x}_0) + C_2 |P_{2s}^{x_0}(\tilde{x}_0)|^2 \).)

Hence, for all \( r \geq r_0 \), we obtain

\[
(2\pi)^{-2} \int |\hat{\nu}(\xi)|^2 |\xi + i\tau N|^{4s} d\xi
\leq C_1 \tau \int G_{\tau}^{x_0}(\tilde{x}_0, D, \tilde{D}) v \hat{\nu} dx + C_2 \int |P_{2s}^{x_0}(\tilde{x}_0, D + i\tau N) v|^2 dx
\]  

(3.18)
after multiplying (3.17) by $|\hat{v}(\xi)|^2$, integrating, and applying Parseval’s formula. Next, we express the polynomials $G_{\tau}^{0,r}$ in powers of $\tau$ as

$$G_{\tau}^{0,r}(x, D, \bar{D}) = \sum_{j=0}^{4s-1} \tau^j G_{\tau}^{x_0,r,(j)}(x, D, \bar{D}),$$

where the quadratic form $G_{\tau}^{x_0,r,(j)}$ is of order $(4s - j - 1; 2s)$ and has smooth coefficients for all $j \in \{0, \ldots, 4s - 1\}$. Using a continuity argument, we can find $\delta_0 > 0$ such that for all $x \in A_0$ with $|x - \tilde{x}_0| < \delta_0$, we have

$$C_1 |P_{2s}^{x_0,r}(x, D + i\tau N)v|^2 \leq C_1 |P_{2s}^{x_0,r}(\tilde{x}_0, D + i\tau N)v|^2 + \frac{1}{4} |(D + i\tau N)^{2s}v|^2$$

and

$$C_2 |G_{\tau}^{x_0,r}(x, D, \bar{D})v\tilde{v} - G_{\tau}^{x_0,r}(\tilde{x}_0, D, \bar{D})v\tilde{v}|$$

$$\leq C_2 \sum_{j=1}^{4s-1} \tau^j |G_{\tau}^{x_0,r,(j)}(x, D, \bar{D})v\tilde{v} - G_{\tau}^{x_0,r,(j)}(\tilde{x}_0, D, \bar{D})v\tilde{v}|$$

$$\leq \frac{1}{4} \sum_{j=1}^{4s-1} \tau^j \sum_{|\alpha|+|\beta|=4s-j-1} |D^\alpha v D^\beta \tilde{v}|$$

for all $r \geq r_0$. Thus, from (3.18) and the above inequalities

$$(2\pi)^{-2} \int |\hat{v}(\xi)|^2 |\xi| + i\tau N|^{4s} d\xi \leq C_1 \int G_{\tau}^{x_0,r}(x, D, \bar{D})v\tilde{v} \, dx + C_2 \int |P_{2s}^{x_0,r}(x, D + i\tau N)v|^2 \, dx$$

$$+ \frac{1}{4} (2\pi)^{-2} \int |\hat{v}(\xi)|^2 |(\xi + i\tau N)|^{4s} d\xi$$

$$+ \frac{1}{4} \sum_{j=1}^{4s-1} \tau^{j+1} \sum_{|\alpha|+|\beta|=4s-j-1} \int |D^\alpha v D^\beta \tilde{v}| \, dx$$

for $v \in C^\infty_0(B_{\delta_0}(\tilde{x}_0) \cap A_0)$ and $r \geq r_0$. For each term in the last sum, we use the Cauchy–Schwarz inequality and the estimate

$$(|N|^2 \tau^2)^{2s-|\alpha|} \int |D^\alpha v|^2 \, dx \leq (2\pi)^{-2} \int |\hat{v}(\xi)|^2 |\xi| + i\tau N|^{4s} d\xi$$

for $|\alpha| \leq 2s$. Also, in view of the inequality (3.14), we obtain

$$(|N|^2 \tau^2)^{2s-|\alpha|} \int |D^\alpha u|^2 e^{2\tau(x-x_0,N)} \, dx \leq (2\pi)^{-2} \int |\hat{v}(\xi)|^2 |\xi| + i\tau N|^{4s} d\xi$$

$$\leq 2(C_1 \tau + C_2) \int |P_{2s}^{x_0,r}(x, D + i\tau N)v|^2 \, dx$$

$$= 2(C_1 \tau + C_2) \int |P_{2s}^{x_0,r}(x, D)u|^2 e^{2\tau(x-x_0,N)} \, dx$$
for all $u \in \mathcal{C}^\infty_0(B_{b_0}({\bar{x}_0}) \cap A_{b_0})$, $|\alpha| \leq 2s$, and $r \geq r_0$. Therefore, the estimate (3.13) holds for functions supported in a small neighborhood of $\bar{x}_0$ in $A_{b_0}$. By [12, Lemma 8.3.1], the proof of Lemma 3.3 is complete. □

**Proof of Theorem 3.1.** For simplicity of notation, we may assume that $\bar{x}_0 = 0$. In order to obtain a Carleman estimate on the dyadic annuli $A_r$ centered at 0 for $r \in \mathbb{N}$ such that $r \geq r_0$, we perform a change of variable $y = 2^{-r}x$ in (3.13) and write $u(x) = v(2^{-r}x)$. Denoting $\sigma = 2^{-mr}\tau$, we get

$$\sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} |D^\alpha v|^2 e^{2\sigma \psi} \leq C 2^{-4sr} \int_{A_r} |P_{2s}(x, D)v|^2 e^{2\sigma \psi}$$

which implies

$$\sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} |D^\alpha v|^2 e^{2\sigma \psi} \leq C \sum_{r=0}^{r_1} \int_{A_r} |P_{2s}(x, D)v|^2 e^{2\sigma \psi}$$

for all $v \in \mathcal{C}^\infty_0(A_r)$ and $\sigma \geq r_0$.

Let $\varphi \in \mathcal{C}^\infty_0(\tilde{\Omega}, [0, 1])$ be a cutoff function with support in $A_0$ such that $\varphi \equiv 1$ in a neighborhood of $\{x \in \tilde{\Omega} : 1 \leq |x| \leq 2\}$ and supp $\varphi \subset \{x \in \tilde{\Omega} : 7/8 \leq |x| \leq 17/8\}$. Define $\varphi_\alpha = \varphi(2^\alpha \cdot)$. Clearly, $\varphi_\alpha \in \mathcal{C}^\infty_0(\Omega, [0, 1])$ is compactly supported in $A_r$, and $\varphi_\alpha \equiv 1$ in a neighborhood of $C_r = \{x \in \tilde{\Omega} : 2^{-r} \leq |x| \leq 2^{-r+1}\}$. Also, we have that $|D^\alpha \varphi_\alpha| \leq C 2^{|\alpha|} |\varphi_{\alpha+1}|$ in a neighborhood of $S_r = \{x \in \tilde{\Omega} : 2^{-r-1} \leq |x| \leq 2^{-r}\}$ and $|D^\alpha \varphi_\alpha| \leq C 2^{|\alpha|} |\varphi_{\alpha-1}|$ in a neighborhood of $L_r = \{x \in \tilde{\Omega} : 2^{-r+1} \leq |x| \leq 2^{-r+2}\}$ for all $\alpha \in \mathbb{N}^2_0$ with $|\alpha| = 0, \ldots, 2s - 1$.

Let $u \in \mathcal{C}^\infty_0(\tilde{\Omega})$ be a smooth function with support in the annulus $A(0, \delta, \rho_0)$, where $\rho_0 = 2^{-r_0}$. Let $r_1$ be the smallest integer such that $2^{-r_1} \leq \delta$. Then we have supp $u \subset \bigcup_{k=0}^{r_1} A_r$. Applying the Carleman inequality (3.19) to $u \varphi_\tau$ and summing for $r = r_0, \ldots, r_1$, we obtain

$$\sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} |D^\alpha (u \varphi_\tau)|^2 e^{2\sigma \psi} \leq C \sum_{r=0}^{r_1} \int_{A_r} |P_{2s}(x, D)(u \varphi_\tau)|^2 e^{2\sigma \psi}$$

for $\sigma \geq r_0$. For the left side of (3.20), we claim

$$\sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} |D^\alpha (u \varphi_\tau)|^2 e^{2\sigma \psi} \geq 1 \sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} |\varphi_\tau D^\alpha u|^2 e^{2\sigma \psi}. \quad (3.21)$$

Indeed, we have $|D^\alpha (u \varphi_\tau)|^2 \geq (1/2) |\varphi_\tau D^\alpha u|^2 - C \sum_{\beta < \alpha} |D^\beta u D^{\alpha-\beta} \varphi_\tau|^2$ by the Leibniz rule and the triangle inequality; therefore, using the assumptions on the derivatives of $\varphi_\tau$

$$\sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} |D^\alpha (u \varphi_\tau)|^2 e^{2\sigma \psi}$$

$$\geq \sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} \left( \frac{1}{2} |\varphi_\tau D^\alpha u|^2 e^{2\sigma \psi} - C \sum_{\beta < \alpha} 2^{2r(|\alpha|-|\beta|)} \left( \int_{S_r} |\varphi_{\tau+1} D^\beta u|^2 e^{2\sigma \psi} + \int_{L_r} |\varphi_{\tau-1} D^\beta u|^2 e^{2\sigma \psi} \right) \right)$$
and the two negative terms on the right side can be absorbed into the first term on the right side, since

\[ C \sigma^{4s-2|\alpha|-1} 2^{2r(|\alpha|-|\beta|)} \int_{S_r} \left| \varphi_{r+1} D^\beta u \right|^2 e^{2\sigma \psi} \leq C \sigma^{4s-2|\alpha|-1} 2^{4s} \int_{S_r} \left| \varphi_{r+1} D^\beta u \right|^2 e^{2\sigma \psi} \]

\[ \leq \sigma^{4s-2|\beta|-1} \int_{A_{r+1}} \left| \varphi_{r+1} D^\beta u \right|^2 e^{2\sigma \psi} \]

and

\[ C \sigma^{4s-2|\alpha|-1} 2^{2r(|\alpha|-|\beta|)} \int_{L_r} \left| \varphi_{r-1} D^\beta u \right|^2 e^{2\sigma \psi} \leq C \sigma^{4s-2|\alpha|-1} \int_{A_{r-1}} \left| \varphi_{r-1} D^\beta u \right|^2 e^{2\sigma \psi} \]

for all \( \beta < \alpha \) and \( r \in \{r_0, \ldots, r_1\} \) provided \( \sigma \geq \tau_0 2^{4s r_1} \). For the right side of (3.20), we have

\[ C \sum_{r=r_0}^{r_1} \int P_{2s}(x, D)(\varphi_{r}) \left| D^{2\sigma \psi} \right| \leq C \sum_{r=r_0}^{r_1} \int \left| \varphi_{r} P_{2s}(x, D) u \right|^2 e^{2\sigma \psi} \]

\[ + C \sum_{r=r_0}^{r_1} \int \sum_{\beta > 0} \left| P_{2s}(x, D) u \right| \left| \frac{D^\beta \varphi_{r}}{\beta!} \right|^2 e^{2\sigma \psi} \]

(3.22)

by the Leibniz rule. The second term on the right side can be absorbed into the right side of (3.21), since

\[ C \sum_{r=r_0}^{r_1} \int \sum_{\beta > 0} \left| P_{2s}(x, D) u \right| \left| \frac{D^\beta \varphi_{r}}{\beta!} \right|^2 e^{2\sigma \psi} \]

\[ \leq C \sum_{r=r_0}^{r_1} \sum_{|\alpha|=2s \atop 0 < |\alpha| \leq \alpha} \sum_{|\beta| \leq \alpha} 2^{2r|\beta|} \left( \int_{S_r} \left| \varphi_{r+1} D^{\alpha-\beta} u \right|^2 e^{2\sigma \psi} + \int_{L_r} \left| \varphi_{r-1} D^{\alpha-\beta} u \right|^2 e^{2\sigma \psi} \right) \]

\[ \leq C \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \int \left| \varphi_{r} D^{\alpha} u \right|^2 e^{2\sigma \psi} \]

if \( \sigma \geq \tau_0 2^{4s r_1} \). From (3.21) and (3.22), we conclude

\[ \sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \int \left| \varphi_{r} D^{\alpha} u \right|^2 e^{2\sigma \psi} \leq C \sum_{r=r_0}^{r_1} \int \left| \varphi_{r} P_{2s}(x, D) u \right|^2 e^{2\sigma \psi} \]

and hence

\[ \sum_{|\alpha| < 2s} \sigma^{4s-2|\alpha|-1} \int_{A(0, \delta, \rho_0)} \left| D^{\alpha} u \right|^2 e^{2\sigma \psi} \leq C \int_{A(0, \delta, \rho_0)} \left| P_{2s}(x, D) u \right|^2 e^{2\sigma \psi} \]

(3.23)
for all $u \in C_0^\infty(\tilde{\Omega})$ with support in $A(0, \delta, \rho_0)$ and $\sigma \geq \tau_0 \delta^{-4s}$, by taking $\tau_0$ sufficiently large. Note that the constant $C$ does not depend on $r_1$, since for any $x \in A_r$, there are at most three functions from $\{\phi_r(x)\}_{r_1=r}^{r}$, which are different from zero. Thus, $|D^\alpha u| \leq C(\sum_{r=r_0}^{r_1} |\phi_r D^\alpha u|^2)^{1/2}$ by the Cauchy–Schwartz inequality. 

4. Propagation of smallness

In this section we suppose that the assumptions on the operator $P_{2s}(x, D)$ from Section 3 are satisfied, so that Theorem 3.1 is applicable. Using a linear change of variable by a constant factor, we may assume without loss of generality that $\rho_0 > 2$.

Let $u$ be an infinitely smooth solution of the equation

$$P(x, D)u = \sum_{|\alpha| \leq 2s} a_\alpha(x) D^\alpha u = 0 \quad (4.1)$$

for $x \in \tilde{\Omega}$. Assume that there exist nonnegative constants $M$ and $\delta$ such that

$$\|D^\alpha u\|_{L^2(B_2(x))} \leq \frac{M|\alpha|^\sigma}{\delta|\alpha|} \quad (4.2)$$

for $|\alpha| = 0, \ldots, 2s - 1$. Assume that the coefficients $a_\alpha$ are infinitely smooth and that there exist nonnegative constants $M_\alpha$ such that

$$\|a_\alpha\|_{L^\infty(B_2(x))} \leq M_\alpha \quad (4.3)$$

for $|\alpha| = 0, \ldots, 2s$. Assume additionally that

$$\|D^\alpha u\|_{L^2(B_\delta(x))} \leq \tilde{\epsilon} \quad (4.4)$$

for $|\alpha| = 0, \ldots, 2s - 1$ and some sufficiently small $\tilde{\epsilon} \in (0, 1)$ such that $\tilde{\epsilon} \leq M$.

**Lemma 4.1.** Suppose that the assumptions (4.1)–(4.4) are satisfied. If

$$\tilde{\epsilon} \leq M \exp\left(-P_1(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s})\right)$$

then

$$\|u\|_{L^2(B_1)} \leq P_2(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}) \tilde{\epsilon}^\theta M^{1-\theta}$$

where $P_1$ and $P_2$ are nonnegative polynomials and the parameter $\theta \in (0, 1)$ is such that $\theta \leq C \delta^m$.

**Proof.** Let $\phi \in C_0^\infty(\tilde{\Omega}, [0, 1])$ be a smooth cutoff function such that $\phi \equiv 1$ in a neighborhood of the annulus $B_{1.5} \setminus B_{2.8}$ and $\phi \equiv 0$ in a neighborhood of $B_\delta \cup B_2^c$. Additionally, we assume that

$$\|D^\alpha \phi\|_{L^\infty(B_{2\delta} \setminus B_\delta)} \leq \frac{C}{\delta|\alpha|}$$

for $|\alpha| = 0, \ldots, 2s - 1$. Consider $u\phi \in C_0^\infty(B_2 \setminus B_\delta)$. By Theorem 3.1, it follows
\[
\sum_{|\alpha|<2s} \tau^{4s-2|\alpha|-1} \int_{B_2 \setminus B_\delta} |D^\alpha (u\varphi)|^2 e^{2\tau \psi} \leq C \int_{B_2 \setminus B_\delta} |P_{2s}(x, D) (u\varphi)|^2 e^{2\tau \psi} \tag{4.5}
\]

for \(\tau \geq \tau_0 \delta^{-4s}\). We estimate the right side of (4.5) from above by \(I_1 + I_2 + I_3\), where

\[
I_1 = C \int_{B_{2\delta} \setminus B_\delta} |P_{2s}(x, D) (u\varphi)|^2 e^{2\tau \psi},
\]

\[
I_2 = C \int_{B_{1.5} \setminus B_{2\delta}} |P_{2s}(x, D) (u\varphi)|^2 e^{2\tau \psi},
\]

\[
I_3 = C \int_{B_2 \setminus B_{1.5}} |P_{2s}(x, D) (u\varphi)|^2 e^{2\tau \psi}.
\]

Using the Leibniz formula and Eq. (4.1), we have

\[
P_{2s}(x, D) (u\varphi) = \sum_{|\alpha|=2s} a_\alpha (x) (D^\alpha u) \varphi + \sum_{\beta > 0} (P_{2s}^{(\beta)} (x, D) u) \frac{D^\beta \varphi}{\beta!}
\]

\[
= - \sum_{|\alpha|<2s} a_\alpha (x) (D^\alpha u) \varphi + \sum_{\beta > 0} (P_{2s}^{(\beta)} (x, D) u) \frac{D^\beta \varphi}{\beta!}. \tag{4.6}
\]

By the equality (4.6), the assumptions on the derivatives of \(\varphi\), and the hypotheses (4.3) and (4.4), we obtain

\[
I_1 \leq C \sum_{|\alpha|<2s} M_\alpha^2 \int_{B_{2\delta} \setminus B_\delta} |D^\alpha u|^2 e^{2\tau \psi} + C \sum_{\beta > 0} |\delta^{-2|\beta|}| \int_{B_{2\delta} \setminus B_\delta} |P_{2s}^{(\beta)} (x, D) u|^2 e^{2\tau \psi}
\]

\[
\leq C \epsilon^2 e^{2\tau \psi (\delta)} \sum_{|\alpha|<2s} M_\alpha^2 + C \sum_{\beta > 0} \delta^{-2|\beta|} \epsilon^2 e^{2\tau \psi (\delta)} \sup_{|\alpha|=2s} M_\alpha^2
\]

\[
\leq C \left( \sup_{|\alpha|<2s} M_\alpha^2 + \delta^{-4s} \sup_{|\alpha|=2s} M_\alpha^2 \right) \epsilon^2 e^{2\tau \delta^{-m}} \leq C \delta^{-4s} \epsilon^2 e^{2\tau \delta^{-m}} \sup_{|\alpha|\leq 2s} M_\alpha^2
\]

for the first integral. Above, we denoted \(\psi (\delta) = \psi (x)\) where \(|x| = \delta\). Next, using that \(\varphi \equiv 1\) in a neighborhood of \(B_{1.5} \setminus B_{2\delta}\), the estimate on the coefficients (4.3), and the equality (4.6), we have the estimate for the second integral

\[
I_2 \leq C \sum_{|\alpha|<2s} M_\alpha^2 \int_{B_{1.5} \setminus B_{2\delta}} |D^\alpha u|^2 e^{2\tau \psi},
\]

which can be absorbed in the half of the left side of (4.5) provided that

\[
\tau \geq \max \left\{ \sup_{|\alpha|<2s} M_\alpha^{2/(4s-2|\alpha|-1)}, \frac{\tau_0}{\delta^{4s}} \right\}. \tag{4.7}
\]

Finally, the assumptions (4.2), (4.3), and the equality (4.6) imply
Then for all

\[ \frac{2}{3} \alpha \in N \]

Therefore, the lemma is proven. \( \square \)

**Lemma 4.2.** Suppose

\[ \| D^\alpha u \|_{L^2(B_{2\delta})} \leq \frac{C|\alpha| |M| |\alpha|^{\sigma}}{\delta |\alpha|^{-1}} \]

for \( \alpha \in \mathbb{N}_0 \) and

\[ \| u \|_{L^2(B_{2\delta})} \leq \epsilon \delta. \]

Then

\[ \| D^\alpha u \|_{L^2(B_{2\delta})} \leq \frac{C|\alpha|^{1+1/2}M^{1/2}|\alpha|^{\sigma}}{\delta |\alpha|^{-1}} \]

for all \( \alpha \in \mathbb{N}_0 \).
This interpolation-type lemma is proven in [22] for real analytic functions when $\sigma = 1$. Using analogous arguments for functions $u$ in the Gevrey class $G^\sigma$ with $\sigma > 1$, one obtains the more general assertion stated above.

**Proof of Theorem 2.1.** Denote
\[
\epsilon = \frac{\|u\|_{L^2(B_{4\delta})}}{\|u\|_{L^2(B_2)}}.
\]

We shall prove that $\epsilon \geq \exp(-Q_1(\delta^{-1}, K, M, \{M_\alpha\}_{\|\alpha\| < 2s}))$ for some nonnegative polynomial $Q_1$. By the definition of $\epsilon$, in particular, we have
\[
\|u\|_{L^2(B_{4\delta})} \leq \epsilon \|u\|_{L^2(B_2)}.
\]

Then Lemma 4.2 implies
\[
\left\| D^\alpha u \right\|_{L^2(B_{2\delta})} \leq \frac{C^{\|\alpha\|+1} \epsilon^{1/2} M^{1/2} \|\alpha\|^{\sigma}}{\delta^{\|\alpha\|}} \|u\|_{L^2(B_{2\delta})}
\]
for $|\alpha| = 0, \ldots, 2s - 1$. Denote
\[
K_0 = \max_{|\alpha| \leq 2s - 1} \frac{\delta^{\|\alpha\|} \|D^\alpha u\|_{L^2(B_{2\delta})}}{|\alpha|^{\sigma}}
\]
and set $\tilde{u}(x) = K_0^{-1} u(x)$. Clearly, the function $\tilde{u}$ also solves Eq. (4.1) and
\[
\left\| D^\alpha \tilde{u} \right\|_{L^2(B_{2\delta})} \leq \frac{|\alpha|^{\sigma}}{\delta^{\|\alpha\|}}
\]
for $|\alpha| = 0, \ldots, 2s - 1$. Next, we denote
\[
\tilde{\epsilon} = \max_{|\alpha| < 2s} \frac{C^{\|\alpha\|+1} \epsilon^{1/2} M^{1/2} |\alpha|^{\sigma}}{\delta^{\|\alpha\|}}. \tag{4.8}
\]

Then
\[
\left\| D^\alpha \tilde{u} \right\|_{L^2(B_{2\delta})} \leq \frac{1}{K_0} \left\| D^\alpha u \right\|_{L^2(B_{2\delta})} \leq \tilde{\epsilon}
\]
for $|\alpha| = 0, \ldots, 2s - 1$. Since the hypotheses of Lemma 3.1 are satisfied, we have
\[
\|\tilde{u}\|_{L^2(B_1)} \leq P_2(\delta^{-1}, \{M_\alpha\}_{|\alpha| < 2s}) \tilde{\epsilon}^\theta
\]
provided that
\[
\tilde{\epsilon} \leq \exp(-P_1(\delta^{-1}, \{M_\alpha\}_{|\alpha| < 2s})). \tag{4.9}
\]

where $P_1$ and $P_2$ are the nonnegative polynomials obtained explicitly in the proof of the lemma.
Using the hypotheses (2.2) and (2.4), we get

\[ K_0 \leq M \|u\|_{L^2(B_4)} \leq MK\|u\|_{L^2(B_2)}. \]

Thus, we obtain the estimate

\[ \|u\|_{L^2(B_2)} \leq K\|u\|_{L^2(B_1)} \leq KK_0\|\tilde{u}\|_{L^2(B_1)} \leq KK_0 P_2(\delta^{-1}, \{M\alpha\}_{|\alpha|<2s})\bar{\epsilon}^\theta \leq MK^2 P_2(\delta^{-1}, \{M\alpha\}_{|\alpha|<2s})\|u\|_{L^2(B_2)}, \]

which holds only if

\[ \bar{\epsilon} \geq \frac{1}{M^{1/\theta} K^{2/\theta}} P_2(\delta^{-1}, \{M\alpha\}_{|\alpha|<2s})^{-1/\theta}. \quad (4.10) \]

Therefore, by (4.9) and (4.10)

\[ \bar{\epsilon} \geq \min\left\{ \exp\left(-P_1(\delta^{-1}, \{M\alpha\}_{|\alpha|<2s})\right), \frac{1}{M^{1/\theta} K^{2/\theta}} P_2(\delta^{-1}, \{M\alpha\}_{|\alpha|<2s})^{-1/\theta} \right\}. \]

Using (4.8), we solve the last inequality for \( \epsilon \). We conclude that \( \epsilon \geq \exp(-Q_1(\delta^{-1}, K, M, \{M\alpha\}_{|\alpha|<2s})) \) for a nonnegative polynomial \( Q_1 \) of degree in \( \delta^{-1} \) depending only on \( s \) and \( m \). \( \square \)

5. Nodal sets

First, we recall a result from our paper [15] which addresses the order of vanishing and the size of the nodal sets for any 1-periodic Gevrey function in one real variable.

Lemma 5.1. (See [15].) Let \( f : \mathbb{R} \to \mathbb{R} \) be an infinitely differentiable 1-periodic function which is not identically zero. Let \( a, b \geq 0 \) and \( 1 \leq \sigma \leq 1 + 1/b \). If \( \sigma = 1 + 1/b \), we assume that \( 4^{b+1}a/\delta^b \leq 1/2 \). Suppose that there exist constants \( M \geq 1 \) and \( \delta \in (0, 1/2] \) such that

\[ \|f^{(n)}\|_{L^\infty(\Omega)} \leq \frac{Mn!\sigma}{\delta^n} \|f\|_{L^\infty(\Omega)}, \quad n \in \mathbb{N}_0, \quad (5.1) \]

and

\[ \|f\|_{L^\infty(\Omega)} \leq \exp\left(\frac{a}{\rho^b}\right) \|f\|_{L^\infty[\{x_0-\rho/2, x_0+\rho/2] \quad (5.2) \}

for all \( \rho \in (0, \delta] \) and \( x_0 \in \Omega \). Then for the number of zeros of \( f \) in \( \Omega \), we have

\[ \text{card}\{x \in \Omega: f(x) = 0\} \leq CK^{1+1/b}, \quad (5.3) \]

where

\[ K = \left(\frac{4^{b+1}a}{\delta^b}\right)^{1/(1+b(1-\sigma))} + \frac{4^{b+1}a}{\delta^b} + 2\log M + 2 \quad (5.4) \]

and \( C = C(a, b) \). The first term in (5.4) is understood to be zero if \( \sigma = 1 + 1/b \). Moreover, we have an upper bound
ord\(x_0 f\) \(\leq K\) \hspace{1cm} (5.5)

for the order of vanishing \(\text{ord}_{x_0} f\) for every \(x_0 \in \Omega\).

**Proof of Theorem 2.2.** Let \(u\) be a solution of the elliptic equation with Gevrey coefficients 
\(P(x,D)u = 0\) for \(x \in \Omega\), which satisfies the condition (2.2) and the quantitative estimate of 
unique continuation from Theorem 2.1. Applying the 1-dimensional result from Lemma 5.1 on rays 
through \(x_0\), we obtain a polynomial upper bound on the order of vanishing of \(u\) at any point \(x_0\) in the 
2-dimensional unit ball \(B_1\) provided the Gevrey exponent \(\sigma \leq 1 + \eta\), where \(\eta\) is a constant multiple 
of \(s\).

To give an upper bound on the 1-dimensional Hausdorff measure of the nodal sets of solutions, we 
rely on an argument depending on their geometric structure. We recall a result, due to Han [9], on 
the structure of the nodal and singular sets of solutions to higher order elliptic equations in \(\mathbb{R}^n\) with 
Hölder continuous leading coefficients. It turns out that under the assumption that the solutions van-
ish at finite order, the singular and nodal sets are countable unions of subsets of \((n-2)\)-dimensional 
and \((n-1)\)-dimensional submanifolds, respectively. Define

\[ N(u) = \left\{ x \in B_1 : u(x) = 0 \right\} \]

and

\[ S(u) = \left\{ x \in B_1 : D^\beta u = 0, \ |\beta| = 0, \ldots, 2s - 1 \right\}, \]

where \(2s\) is the order of the elliptic equation.

**Theorem 5.2.** (See [9].) The set \(N(u)\) is countably \((n-1)\)-rectifiable and the set \(S(u)\) is countably \((n-2)\)-
rectifiable. In fact there exist decompositions

\[ N(u) = \bigcup_{j=0}^{n-1} N^j(u), \]

\[ S(u) = \bigcup_{j=0}^{n-2} S^j(u) \]

where (i) each \(N^j(u)\) is on a countable union of \(j\)-dimensional \(C^1\) graphs for \(0 \leq j \leq n-2\) and \(N^{n-1}(u)\) is 
on a countable union of \((n-1)\)-dimensional \(C^{1,\alpha}\) manifolds and (ii) each \(S^j(u)\) is on a countable union of 
\(j\)-dimensional \(C^1\) graphs for \(0 \leq j \leq n-3\) and \(S^{n-2}(u)\) is on a countable union of \((n-2)\)-dimensional \(C^{1,\alpha}\) 
manifolds for some \(0 < \alpha < 1\).

In our case the space dimension is \(n = 2\) and the coefficients of the elliptic equation \(P(x,D)u = 0\) 
are infinitely smooth.

**Proof of Theorem 2.3.** By Theorem 5.2, we conclude that the nodal set \(N(u)\) of a solution is a union 
of a set with Hausdorff length zero and a countable union of 1-dimensional manifolds. Let \(\Gamma\) be a 
1-dimensional manifold. Since, at any point \(z\) of \(\Gamma\) the corresponding tangent line makes an angle of 
at most 45° with one of the coordinate axes (cf. [5]), we have
\[ \mathcal{H}^1 \left\{ \Gamma \cap \{|z| < \rho\} \right\} \leq C \left( \int_{|s| < \rho} \mathcal{H}^0 \left\{ \Gamma \cap \{|x| = s\} \cap \{|y| < \rho\} \right\} ds + \int_{|s| < \rho} \mathcal{H}^0 \left\{ \Gamma \cap \{|y| = s\} \cap \{|x| < \rho\} \right\} ds \right) \]

for \( \rho > 0 \) sufficiently small. An application of Lemma 5.1 gives the result. \( \square \)

**Proof of Theorem 2.4.** Note that by the proof of Lemma 3.2, the pseudo-convexity condition (3.2) for the operator (2.1) is satisfied provided (2.5) holds. The assertion follows as in the 2-dimensional case treated above. \( \square \)

**References**