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Strong unique continuation for higher order elliptic equations with Gevrey coefficients

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ABSTRACT

We address the strong unique continuation problem for higher order elliptic partial differential equations in 2D with Gevrey coefficients. We provide a quantitative estimate of unique continuation (observability estimate) and prove that the solutions satisfy the strong unique continuation property for ranges of the Gevrey exponent strictly including non-analytic Gevrey classes. As an application, we obtain a new upper bound on the Hausdorff length of the nodal sets of solutions with a polynomial dependence on the coefficients.

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1. Introduction

In this paper, we study the strong unique continuation problem for the higher order elliptic partial differential operators

$$P(x,D)u = \sum_{|\alpha|\leqslant 2s} a_\alpha(x) D^\alpha u \tag{1.1}$$

with Gevrey coefficients in two space dimensions. Even though, in general, for higher order operators the strong unique continuation property does not hold (cf. [13,27] for counterexamples), we prove it for (1.1) with Gevrey coefficients for ranges of the Gevrey exponents strictly including non-analytic

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classes. As an application, we provide a polynomial upper bound for the Hausdorff length of the nodal sets of solutions in terms of the coefficients.

Recall that a function $u \in C^\infty(\Omega)$ vanishes of infinite order at a point x_0 in the open set $\Omega \subset \mathbb{R}^2$ if $D^\alpha u(x_0) = 0$ for all multi-indices $\alpha \in \mathbb{N}_0^2$. We say that the operator (1.1) has the strong unique continuation property if every solution u of the equation $P(x, D)u = 0$ which vanishes of infinite order at x_0 is identically zero in a neighborhood of x_0.

By now the strong unique continuation problem for second order elliptic operators is well-understood. The results for the elliptic equation

$$-\Delta u = W \cdot \nabla u + V u$$

go back to the work of Carleman [1] in the case of the space dimension $n = 2$ and bounded coefficients, who used as a main tool a weighted L^2 estimate. There is a large amount of work on strong continuation, achieved by different estimates of Carleman type (cf. [2,3,8,12,16,17,24,28,29] and the review papers by Kenig [18,19] and Vessella [30]).

There is a very limited number of results available regarding strong unique continuation for higher order elliptic equations with non-analytic coefficients. In fact the only result we are aware of is given by Colombini and Koch in [4] who proved the strong unique continuation for products of second order elliptic operators with Gevrey coefficients for a certain range of the Gevrey exponent (cf. [4, Theorem 1]). In this paper, we obtain a strong unique continuation result for higher order elliptic operators (1.1) with simple complex characteristics and coefficients in the Gevrey class G^σ provided the Gevrey exponent σ is less than a constant strictly grater than 1.

We note that the weak continuation for higher order equations is better understood. For instance, a result of Hörmander [14] addresses solutions of differential inequalities

$$\left|P(x, D)u\right| \leqslant K \sum_{|\alpha|<m} \left|D^\alpha u\right|,$$

where P is a homogeneous elliptic operator of degree m with simple characteristics; it states that if a solution u vanishes in the intersection of a neighborhood of 0 and the set $x_1 < x_2^2 + \cdots + x_\nu^2$, then u vanishes in a full neighborhood of 0.

A natural question to ask about a solution u is how large is the size of its nodal (zero, vanishing) set. In their seminal work [5–7], Donnelly and Fefferman provided upper and lower bounds on the $(n-1)$-dimensional measure of the nodal set for an eigenfunction of the Laplacian, considered on an analytic Riemannian n-manifold (see also [11,25]). For higher order analytic elliptic equations, bounds were obtained in [23]; for estimates on nodal sets of solutions of parabolic equations cf. [10,20–22].

Most of the present paper is devoted to the proof of a Carleman-type inequality for the operator (1.1), which is needed to establish a quantitative estimate of unique continuation (observability inequality) stated in Theorem 2.1. Hörmander established in [12] a necessary and sufficient condition for a Carleman estimate to hold; thus, one of the key steps here is the choice of an appropriate radially decreasing weight function for which the pseudo-convexity condition is satisfied. Note that our choice of weight puts a restriction on the space dimension; namely Lemma 3.2 below is valid only for operators (1.1) defined on a subset of $\mathbb{R}^2$. In higher dimensions, we obtain the main results of this work (Theorems 2.1, 2.2, and 2.3) for classes of higher order elliptic operators satisfying a certain condition (cf. Theorem 2.4). Another obstacle we face is that the weight $|x|^{-m}$ is singular at the origin while the operator P is too general to investigate the validity of the Carleman-type estimate directly (as it can be done for second order operators). We overcome this by establishing a uniform in x Carleman estimate for a family of operators, leading to a Carleman-type inequality on a torus $B_1 \setminus B_\delta$ with constants which can be computed explicitly in terms of δ (see Lemma 3.3 below).

In general, functions in the Gevrey class may not satisfy the strong unique continuation property (see for instance [24]). However, in Theorem 2.2 below, we establish strong unique continuation for the solutions of the equation $P(x, D)u = 0$, satisfying the Gevrey regularity condition (2.2) (cf. [26, Section VII.1.3]) and the observability estimate from Theorem 2.1. The proof relies on [15, Theorem 2.4]. Another application of the latter theorem together with a result due to Han [9] is a polynomial upper bound on the Hausdorff length of the nodal sets (see Theorem 2.3).

The paper is organized as follows. In Section 2 we state our main results, Theorems 2.1, 2.2, 2.3, and 2.4. The next section is devoted to the Carleman-type inequality given in Theorem 3.1 and to certain auxiliary results needed for its proof. In Section 4, we establish a propagation of smallness result which leads to the proof of Theorem 2.2. In the last section, we give the proofs of Theorems 2.2, 2.3, and 2.4.

2. Notation and the main result

In this paper, we consider the elliptic partial differential operator with simple complex characteristics and coefficients in the Gevrey class with $\sigma \geqslant 1$

$$P(x, D)u = \sum_{|\alpha| \leqslant 2s} a_\alpha(x) D^\alpha u \tag{2.1}$$

defined on a domain $\tilde{\Omega}$ in $\mathbb{R}^2$ where $s \in \mathbb{N}$. Let Ω be a subdomain of $\tilde{\Omega}$ such that $\operatorname{dist}(\Omega, \partial\tilde{\Omega})$ is greater than a constant. By rescaling, we may assume that $\operatorname{dist}(\Omega, \partial\tilde{\Omega}) \geqslant 4$. Let u be a solution of the equation $P(x, D)u = 0$ in $\tilde{\Omega}$ which is not identically zero. We assume that u is an infinitely smooth function in $\tilde{\Omega}$ and that there exist positive constants M and δ such that

$$\left\| D^\alpha u \right\|_{L^2(B_2(x))} \leqslant \frac{M|\alpha|!^\sigma}{\delta^{|\alpha|}} \|u\|_{L^2(B_4(x))}, \quad x \in \Omega \tag{2.2}$$

for any $\alpha \in \mathbb{N}_0^2$ where $\sigma \geqslant 1$ is fixed. Also, we assume that the coefficients a_α are infinitely smooth functions and that there exist nonnegative constants M_α such that

$$\left\| D^\beta a_\alpha \right\|_{L^\infty(B_2(x))} \leqslant \frac{M_\alpha |\beta|!^\sigma}{\delta^{|\beta|}}, \quad x \in \Omega \tag{2.3}$$

for all $\alpha, \beta \in \mathbb{N}_0^2$ with $|\alpha| = 0, \dots, 2s$. Assume u satisfies the doubling property

$$\|u\|_{L^2(B_4(x))} \leqslant K \|u\|_{L^2(B_2(x))}, \quad x \in \Omega \tag{2.4}$$

for some constant $K \geqslant 1$. Under the above hypotheses we establish a quantitative estimate of unique continuation (an observability estimate) for the elliptic operator (2.1) with simple complex characteristics and only Gevrey coefficients.

Theorem 2.1. *Suppose that u is a nontrivial solution of $P(x, D)u = 0$ satisfying* (2.2)–(2.4)*. Then*

$$\|u\|_{L^2(B_2(x))} \leqslant \exp\bigl(Q_1\bigl(\delta^{-1}, K, M, \{M_\alpha\}_{|\alpha|<2s}\bigr)\bigr) \|u\|_{L^2(B_{4\delta}(x))}, \quad x \in \Omega$$

for some nonnegative polynomial Q_1 with coefficients depending on P_{2s} and Ω.

This observability estimate is one of the hypotheses needed in Lemma 5.1 (see Section 5). Thus, it enables us to establish the strong unique continuation property for the solutions of $P(x, D)u = 0$.

Theorem 2.2. *Suppose that the assumptions in Theorem* 2.1 *are satisfied and $\sigma \leqslant 1 + \eta$, where η is a constant depending on s. Then the operator* (2.1) *has the strong unique continuation property.*

Using a result due to Han [9] on the structure of the nodal sets of solutions, together with Theorem 2.1 and Lemma 5.1, we obtain an upper bound for the 1-dimensional Hausdorff measure of the zero sets of solutions of the equation $P(x,D)u=0$ with a polynomial dependence on the coefficients.

Theorem 2.3. *Suppose that the assumptions in Theorem* 2.1 *are satisfied. Then*

$$\mathcal{H}^1\{x\in\Omega\colon u=0\}\leqslant Q_2\big(\delta^{-1},K,M,\{M_\alpha\}_{|\alpha|<2s}\big),$$

where Q_2 is a nonnegative polynomial with coefficients depending on P_{2s} and Ω.

As the next theorem shows, the restriction on the dimension $n=2$ may be removed for symbols satisfying an additional property.

Theorem 2.4. *Let $n\geqslant 3$. Suppose that the assumptions in Theorem* 2.2 *are satisfied and $\sigma\leqslant 1+\eta$, where η is a constant depending on s. Additionally, assume that for $0\neq\zeta=\xi+i\tau\nabla\psi_{x_0}(x)$, we have*

$$\big|P_{2s}(x,\zeta)\big|^2+\big|(x-x_0)\cdot\nabla_\zeta P_{2s}(x,\zeta)\big|^2>0,\quad x\in\bar{A}_{x_0},\ \xi\in\mathbb{R}^n,\ \text{and}\ \tau\in\mathbb{R},\tag{2.5}$$

for all $x_0\in\bar{\Omega}$, where $\psi_{x_0}(x)=|x-x_0|^{-m}$ and A_{x_0} is the n-dimensional unit torus centered at x_0. Then the operator (2.1) *has the strong unique continuation property. Moreover, we obtain for the Hausdorff length of the nodal sets*

$$\mathcal{H}^1\{x\in\Omega\colon u=0\}\leqslant Q_3\big(\delta^{-1},K,M,\{M_\alpha\}_{|\alpha|<2s}\big),$$

where Q_3 is a nonnegative polynomial with coefficients depending on P_{2s} and Ω.

Theorem 2.1 is proven in Section 4 and Theorems 2.2, 2.3, and 2.4 in Section 5.

3. Carleman estimate

Let $\tilde{\Omega}\subset\mathbb{R}^2$ be a domain, and let $s\in\mathbb{N}$. We consider the elliptic partial differential operator

$$P(x,D)u=\sum_{|\alpha|\leqslant 2s}a_\alpha(x)D^\alpha u$$

with coefficients in the Gevrey class G^σ where $\sigma\geqslant 1$.

In this section, we denote the principal part of the operator $P(x,D)$ by $P_{2s}(x,D)$ and the corresponding principal symbol by $P_{2s}(x,\zeta)$. Also, we adopt the notation $P_{2s}^{(j)}(x,\zeta)=(\partial P_{2s}/\partial\zeta_j)(x,\zeta)$ and $P_{2s,j}(x,\zeta)=(\partial P_{2s}/\partial x_j)(x,\zeta)$ for the first partial derivatives of $P_{2s}(x,\zeta)$.

Assume that the operator $P(x,D)$ has simple complex characteristics, i.e., the principal symbol $P_{2s}(x,\zeta)$, which is a homogeneous polynomial of degree $2s$ in the complex variable $\zeta\in\mathbb{C}^2$, has only simple (with multiplicity 1) zeros. Our goal is to derive a Carleman-type estimate for such an operator with a suitably chosen weight function ψ.

Theorem 3.1. *There exist positive constants C, τ_0, and ρ_0 such that*

$$\sum_{|\alpha|<2s}\tau^{4s-2|\alpha|-1}\int\limits_{A(x_0,\delta,\rho_0)}\big|D^\alpha v\big|^2e^{2\tau\psi_{x_0}}\leqslant C\int\limits_{A(x_0,\delta,\rho_0)}\big|P_{2s}(x,D)v\big|^2e^{2\tau\psi_{x_0}}\tag{3.1}$$

for any $v \in C_0^\infty(\tilde{\Omega})$ *with support in the annulus* $A(x_0, \delta, \rho_0) = \{x \in \tilde{\Omega}\colon\ \delta < |x - x_0| < \rho_0\}$, $x_0 \in \bar{\Omega}$ *and* $\tau \geqslant \tau_0 \delta^{-4s}$, *provided* $m \in \mathbb{N}$ *is a large enough constant and* $\psi_{x_0}(x) = |x - x_0|^{-m}$.

Above and in the sequel, the symbol C denotes a generic positive constant which is allowed to depend only on s, σ, and P_{2s}. Any additional dependence is indicated explicitly.

First, we prove a uniform Carleman estimate in the dyadic annuli $A_r(x_0) = \{x \in \tilde{\Omega}\colon\ 2^{-r-1} < |x - x_0| < 2^{-r+2}\}$ for all $x_0 \in \bar{\Omega}$ and $r \in \mathbb{N}$ such that $r \geqslant r_0$. Using a partition of unity, we then prove a corresponding Carleman-type estimate on $A(x_0, \delta, \rho_0)$.

Let $A_{x_0} = \{x \in \tilde{\Omega}\colon\ 1/2 < |x - x_0| < 4\}$ be the unit annulus centered at $x_0 \in \bar{\Omega}$. Clearly, we have $\psi_{x_0} \in C^2(\bar{A}_{x_0})$ and $\nabla\psi_{x_0}(x) = -m(x - x_0)|x - x_0|^{-(m+2)} \neq 0$ on $\bar{A}_{x_0}$. Also, we have that $P_{2s}(x, D)$ is elliptic in $\bar{A}_{x_0}$, that is, $P_{2s}(x, \xi) \neq 0$ if $x \in \bar{A}_{x_0}$ and $\xi \in \mathbb{R}^2 \setminus \{0\}$. In the next lemma we establish the pseudo-convexity condition for $P_{2s}(x, D)$ in A_{x_0} with a weight ψ_{x_0} for all $x_0 \in \bar{\Omega}$, needed in the proof of Theorem 3.1.

Lemma 3.2. *Let* $x_0 \in \bar{\Omega}$ *and* $\psi_{x_0}(x) = |x - x_0|^{-m}$. *There exists a sufficiently large number m such that the following is true. Assume that for* $0 \neq \zeta = \xi + i\tau\nabla\psi_{x_0}(x)$ *the characteristic equation* $P_{2s}(x, \zeta) = 0$ *is satisfied, where* $x \in \bar{A}_{x_0}$, $\xi \in \mathbb{R}^2$, *and* $\tau \in \mathbb{R}$. *Then*

$$\sum_{j,k=1}^{2} \partial^2\psi_{x_0}/\partial x_j \partial x_k P_{2s}^{(j)}(x, \zeta)\overline{P_{2s}^{(k)}(x, \zeta)} + \frac{1}{\tau}\operatorname{Im}\sum_{k=1}^{2} P_{2s,k}(x, \zeta)\overline{P_{2s}^{(k)}(x, \zeta)} > 0 \tag{3.2}$$

for all $x_0 \in \bar{\Omega}$.

Proof. Using that

$$\frac{\partial^2\psi_{x_0}}{\partial x_j \partial x_k} = \frac{(m+2)m(x_j - x_{0j})(x_k - x_{0k})}{|x - x_0|^{m+4}} - \frac{m\delta_{jk}}{|x - x_0|^{m+2}}$$

for $j, k = 1, 2$, we may rewrite the asserted pseudo-convexity condition (3.2) for ψ_{x_0} as

$$\begin{aligned} &\frac{(m+2)m}{|x - x_0|^{m+4}}\left|\sum_{j=1}^{2}(x_j - x_{0j})P_{2s}^{(j)}(x, \zeta)\right|^2 - \frac{m}{|x - x_0|^{m+2}}\sum_{j=1}^{2}\left|P_{2s}^{(j)}(x, \zeta)\right|^2 \\ &\quad + \frac{1}{\tau}\operatorname{Im}\sum_{k=1}^{2} P_{2s,k}(x, \zeta)\overline{P_{2s}^{(k)}(x, \zeta)} > 0. \end{aligned} \tag{3.3}$$

First, note that if $P(x, \zeta) = 0$ for $\zeta = \xi + i\eta$, then there exists a positive constant C such that

$$\frac{|\zeta|}{C} \leqslant |\xi|, |\eta| \leqslant |\zeta|. \tag{3.4}$$

The second inequality in (3.4) is trivial. In order to prove the first inequality, we consider the compact set $K = \{\zeta = \xi + i\eta\colon\ |\zeta| = 1,\ P_{2s}(x, \zeta) = 0\}$ in $\mathbb{C}^2$. Since $P_{2s}(x, \zeta)$ is elliptic and homogeneous of order $2s$ in the second variable, we have that $\xi \neq 0$ and $\eta \neq 0$ on K. Thus, there exists a constant $C > 0$ such that $|\xi| \geqslant 1/C$ and $|\eta| \geqslant 1/C$ on K. Now, let $\zeta = \xi + i\eta$ be arbitrary in $\mathbb{C}^2$. By the above property of K, we have $|\xi|/|\zeta| \geqslant 1/C$ and $|\eta|/|\zeta| \geqslant 1/C$. Hence the first inequality in (3.4) holds.

Next, we claim that there exists a positive constant C such that for all $x_0 \in \bar{\Omega}$

$$\left| \sum_{j=1}^{2} (x_j - x_{0j}) P_{2s}^{(j)}(x, \zeta) \right| \geqslant \frac{|\zeta|^{2s-1}}{C} \tag{3.5}$$

if $x \in \bar{A}_{x_0}$ and $\zeta = \xi + i\eta$ are satisfying $P_{2s}(x, \zeta) = 0$, where $\xi, \eta \in \mathbb{R}^2$ and $\eta = \tau y$ for y parallel to $x - x_0$ such that $|y| = 1$. It suffices to prove the claim for $\zeta \in K$, where $K = \{\zeta = \xi + i\tau y\colon\ |\zeta| = |y| = 1,\ P_{2s}(x, \zeta) = 0\}$ is a compact set. (Note that by the previous claim, for any $\zeta \in K$, we have $1/C \leqslant |\xi|,\ |\tau| \leqslant 1$.) For the sake of contradiction, suppose that there exist sequences $\{x_0^{(k)}\} \in \bar{\Omega}$, $\{x^{(k)} - x_0^{(k)}\}_{k\in\mathbb{N}} \subset \bar{A}_0$, and $\{\zeta^{(k)} = \xi^{(k)} + i\tau^{(k)} y^{(k)}\}_{k\in\mathbb{N}} \subset K$ such that

$$\left| \sum_{j=1}^{2} \bigl(x_j^{(k)} - x_{0j}^{(k)}\bigr) P_{2s}^{(j)}\bigl(x^{(k)}, \zeta^{(k)}\bigr) \right| \leqslant \frac{1}{k}, \tag{3.6}$$

where $\tau^{(k)} \in \mathbb{R}$ and $y^{(k)}$ is parallel to $x^{(k)} - x_0^{(k)}$. By passing to subsequences, we may assume that $x_0^{(k)} \to x_0$, $x^{(k)} \to x$, and $\xi^{(k)} + i\tau^{(k)} y^{(k)} \to \xi + i\tau y$ in $\bar{\Omega}$, $\bar{A}_{x_0}$, and K, respectively, where $x_0 \in \bar{\Omega}$, $x \in \bar{A}_{x_0}$, and $\xi + i\tau y \in K$. Then the limits x_0, x, and $\zeta = \xi + i\tau y$ satisfy

$$\sum_{j=1}^{2} (x_j - x_{0j}) P_{2s}^{(j)}(x, \zeta) = 0 \tag{3.7}$$

by continuity. Also, we have

$$\sum_{j=1}^{2} \zeta_j P_{2s}^{(j)}(x, \zeta) = 0, \tag{3.8}$$

which follows by the homogeneity of the polynomial P_{2s} in the second variable. Indeed, we differentiate $P_{2s}(x, \lambda\zeta) = \lambda^{2s} P_{2s}(x, \zeta)$ with respect to $\lambda \in \mathbb{R}$ and then set $\lambda = 1$. Since y is parallel to $x - x_0$, (3.7) and (3.8) imply

$$\sum_{j=1}^{2} \xi_j P_{2s}^{(j)}(x, \zeta) = 0. \tag{3.9}$$

Also, since the operator $P(x, D)$ has simple complex characteristics and the equalities (3.7) and (3.9) hold for $x, \xi \in \mathbb{R}^2$, we obtain that ξ is parallel to $x - x_0$, or equivalently that $\xi = \lambda y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then, for $\zeta = (\lambda + i\tau) y \in K$, we have $0 = P(x, \zeta) = (\lambda + i\tau)^{2s} P(x, y)$ which gives $P(x, y) = 0$ for $y \in \mathbb{R}^2 \setminus \{0\}$. The last equality contradicts the ellipticity assumption on $P(x, D)$. Therefore (3.5) is established.

Now, for all $x_0 \in \bar{\Omega}$ we prove that for a sufficiently large $m \in \mathbb{N}$ the first term on the left side of (3.3) is dominant. By the claim proven above, we obtain the upper bound

$$\frac{(m+2)m}{|x - x_0|^{m+4}} \left| \sum_{j=1}^{2} (x_j - x_{0j}) P_{2s}^{(j)}(x, \zeta) \right|^2 \geqslant \frac{(m+2)m}{C|x - x_0|^{m+4}} |\zeta|^{4s-2} \tag{3.10}$$

for $x \in \bar{A}_{x_0}$. For the second and the third term on the left side of (3.3), we have respectively

$$\frac{m}{|x-x_0|^{m+2}}\sum_{j=1}^{2}\left|P_{2s}^{(j)}(x,\zeta)\right|^2 \leqslant \frac{Cm}{|x-x_0|^{m+2}}|\zeta|^{4s-2} \tag{3.11}$$

and

$$\begin{aligned}\frac{1}{\tau}\left|\operatorname{Im}\sum_{k=1}^{2}P_{2s,k}(x,\zeta)\overline{P_{2s}^{(k)}(x,\zeta)}\right| &\leqslant \frac{C}{\tau}|\zeta|^{4s-1} \leqslant C\left|\nabla\psi_{x_0}(x)\right||\zeta|^{4s-2}\\ &\leqslant \frac{Cm}{|x-x_0|^{m+1}}|\zeta|^{4s-2}\end{aligned} \tag{3.12}$$

for $x \in \bar{A}_{x_0}$, where we used that $\tau|\nabla\psi_{x_0}(x)| \geqslant 1/C|\zeta|$ by (3.4). Choosing $m \in \mathbb{N}$ large enough, the lemma follows. □

Let $x_0 \in \bar{\Omega}$. We consider a sequence of operators for $r \in \mathbb{N}_0$ defined by

$$P_{2s}^{x_0,r}(x,D) = \sum_{|\alpha|=2s} a_\alpha\big(x_0 + 2^{-r}(x-x_0)\big)D^\alpha, \quad x \in \bar{A}_{x_0}$$

and the limiting operator with constant coefficients defined by

$$P_{2s}^{x_0}(D) = \sum_{|\alpha|=2s} a_\alpha(x_0)D^\alpha.$$

We need the following auxiliary assertion.

Lemma 3.3. *There exist constants C, $\tau_0 > 0$, and $r_0 \in \mathbb{N}$ such that for all $x_0 \in \bar{\Omega}$*

$$\sum_{|\alpha|<2s}\tau^{4s-2|\alpha|-1}\int_{A_{x_0}}\left|D^\alpha u\right|^2 e^{2\tau\psi_{x_0}} \leqslant C\int_{A_{x_0}}\left|P_{2s}^{x_0,r}(x,D)u\right|^2 e^{2\tau\psi_{x_0}} \tag{3.13}$$

holds for any $u \in C_0^\infty(A_{x_0})$, $\tau \geqslant \tau_0$, and $r \geqslant r_0$, provided that m is a large enough constant and $\psi_{x_0}(x) = |x-x_0|^{-m}$.

Proof. As a consequence of Lemma 3.2, we can find a large enough number m, so that the pseudo-convexity condition (3.2) is satisfied for all $x_0 \in \bar{\Omega}$ for the limiting operator $P_{2s}^{x_0}(D)$ with a weight function $\psi_{x_0}(x) = |x-x_0|^{-m}$ in $\bar{A}_{x_0}$. Using a nonlinear change of variables, we can straighten up the weight ψ_{x_0}, so that $\psi_{x_0}(x) = \langle x-x_0, N\rangle$ for $x \in \bar{A}_{x_0}$, where $N \neq 0$ is a constant. Now the new limiting operator, which we denote by $P_{2s}^{x_0}(x,D)$, depends on x. Also, we denote the sequence of operators after the change of variable by $P_{2s}^{x_0,r}(x,D)$. We follow the proof of [12, Theorem 8.3.1] for the limiting operator $P_{2s}^{x_0}(x,D)$ with adopting the notations therein to prove the estimate (3.13) for the sequence $P_{2s}^{x_0,r}(x,D)$ where $r \geqslant r_0$. Setting $v(x) = u(x)\exp\tau\langle x-x_0, N\rangle$ and applying the integration by parts formula from [12, Lemma 8.2.2], we obtain for all $x_0 \in \bar{\Omega}$ and $r \in \mathbb{N}_0$

$$\begin{aligned}\int\left|P_{2s}^{x_0,r}(x,D)u\right|^2 e^{2\tau\langle x-x_0,N\rangle} &= \int\left|P_{2s}^{x_0,r}(x,D+i\tau N)v\right|^2\\ &\geqslant \int\left|P_{2s}^{x_0,r}(x,D+i\tau N)v\right|^2 - \left|\bar{P}_{2s}^{x_0,r}(x,D-i\tau N)v\right|^2\\ &= \int G_\tau^{x_0,r}(x,D,\bar{D})v\bar{v},\end{aligned} \tag{3.14}$$

where the quadratic differential form on the right-hand side satisfies

$$\begin{aligned} G_\tau^{x_0,r}(x,\xi,\xi) &= 2\,\mathrm{Im}\sum_{k=1}^{2} P_{2s,k}^{x_0,r}(x,\xi+i\tau N)\overline{P_{2s}^{x_0,r,(k)}(x,\xi+i\tau N)} \\ &\quad + 2\,\mathrm{Im}\left(P_{2s}^{x_0,r}(x,\xi+i\tau N)\sum_{k=1}^{2}\overline{P_{2s,k}^{x_0,r,(k)}(x,\xi+i\tau N)}\right) \end{aligned} \tag{3.15}$$

for $x \in \bar{A}_{x_0}$ and $\xi \in \mathbb{R}^2$. Similar inequality as (3.14) also holds for the limiting operator $P_{2s}^{x_0}(x,D)$ (cf. the proof of [12, Theorem 8.3.1]) and the corresponding differential quadratic form reads

$$\begin{aligned} G_\tau^{x_0}(x,\xi,\xi) &= 2\,\mathrm{Im}\sum_{k=1}^{2} P_{2s,k}^{x_0}(x,\xi+i\tau N)\overline{P_{2s}^{x_0,(k)}(x,\xi+i\tau N)} \\ &\quad + 2\,\mathrm{Im}\left(P_{2s}^{x_0}(x,\xi+i\tau N)\sum_{k=1}^{2}\overline{P_{2s,k}^{x_0,(k)}(x,\xi+i\tau N)}\right). \end{aligned}$$

Moreover, there exist constants $C_1, C_2 > 0$ such that for all $x_0 \in \bar{\Omega}$

$$|\xi+i\tau N|^{4s} \leqslant \frac{C_1}{2}\tau G_\tau^{x_0}(x,\xi,\xi) + \frac{C_2}{2}\left|P_{2s}^{x_0}(x,\xi+i\tau N)\right|^2, \quad x \in \bar{A}_{x_0} \tag{3.16}$$

if $\tau \geqslant 0$ and $\xi \in \mathbb{R}^2$. (Indeed, by the homogeneity of the polynomials on both sides of (3.16), we restrict ourselves to the compact set $M = \{(\xi,\tau)\colon |\xi+i\tau N| = 1,\ \tau \geqslant 0\}$. Consider the subset of M defined by $M_0 = \{(x_0,x,\xi,\tau)\colon P_{2s}^{x_0}(x,\xi+i\tau N) = 0,\ x_0 \in \bar{\Omega},\ x \in \bar{A}_{x_0},\ (\xi,\tau) \in M\}$. Since the operator $P_{2s}^{x_0}$ satisfies the ellipticity estimate $P_{2s}^{x_0}(x,\xi) \geqslant |\xi|^{2s}/C$ and the pseudo-convexity condition (3.2) in $\bar{A}_{x_0}$, we have that τ and $G_\tau^{x_0}(x,\xi,\xi)$ are bounded from below on the compact M_0. Thus, there exists a positive constant C_1 such that $1 \leqslant (C_1/2)\tau G_\tau^{x_0}(x,\xi,\xi)$ in M_0. The last inequality depends continuously on all variables, so it holds in a small neighborhood V of M_0. Also, the polynomial $P_{2s}^{x_0}(x,\xi+i\tau N)$ has no zeros in the compact set $M \setminus V$, so it has a lower bound there. We conclude that (3.16) holds on M.)

Let $\tilde{x}_0 \in \bar{A}_{x_0}$ be arbitrary. By the continuity of the coefficients of $P_{2s}^{x_0}$ and $G_\tau^{x_0}$, we can find $r_0 \in \mathbb{N}$ such that

$$|\xi+i\tau N|^{4s} \leqslant C_1\tau G_\tau^{x_0,r}(\tilde{x}_0,\xi,\xi) + C_2\left|P_{2s}^{x_0,r}(\tilde{x}_0,\xi+i\tau N)\right|^2, \quad r \geqslant r_0 \tag{3.17}$$

for all $\tau \geqslant 0$ and $\xi \in \mathbb{R}^2$. (Indeed, on the compact set M, we have $\tau \leqslant 1/N$ and the polynomials $P_{2s}^{x_0}$ and $G_\tau^{x_0}$ are smooth functions depending only on the x variable. Thus, given $\epsilon > 0$, there exists $r_0 \in \mathbb{N}$, depending only on the first and second order derivatives of the coefficients of P_{2s}, such that for all $r \geqslant r_0$ we have $|P_{2s}^{x_0}(\tilde{x}_0) - P_{2s}^{x_0,r}(\tilde{x}_0)| < \epsilon$ and $|G_\tau^{x_0}(\tilde{x}_0) - G_\tau^{x_0,r}(\tilde{x}_0)| < \epsilon$. Choosing $\epsilon = (C_1/N + C_2)^{-1}$, we get $1 \leqslant C_1\tau G_\tau^{x_0,r}(\tilde{x}_0) + C_2|P_{2s}^{x_0,r}(\tilde{x}_0)|^2$.)

Hence, for all $r \geqslant r_0$, we obtain

$$\begin{aligned} &(2\pi)^{-2}\int \left|\hat{v}(\xi)\right|^2 |\xi+i\tau N|^{4s}\,d\xi \\ &\quad \leqslant C_1\tau\int G_\tau^{x_0,r}(\tilde{x}_0,D,\bar{D})v\bar{v}\,dx + C_2\int \left|P_{2s}^{x_0,r}(\tilde{x}_0,D+i\tau N)v\right|^2 dx \end{aligned} \tag{3.18}$$

after multiplying (3.17) by $|\hat{v}(\xi)|^2$, integrating, and applying Parseval's formula. Next, we express the polynomials $G_\tau^{x_0,r}$ in powers of τ as

$$G_\tau^{x_0,r}(x,D,\bar{D}) = \sum_{j=0}^{4s-1} \tau^j G^{x_0,r,(j)}(x,D,\bar{D}),$$

where the quadratic form $G^{x_0,r,(j)}$ is of order $(4s-j-1;2s)$ and has smooth coefficients for all $j \in \{0,\dots,4s-1\}$. Using a continuity argument, we can find $\delta_0 > 0$ such that for all $x \in A_{x_0}$ with $|x-\tilde{x}_0| < \delta_0$, we have

$$C_1\left|P_{2s}^{x_0,r}(x,D+i\tau N)v\right|^2 \leqslant C_1\left|P_{2s}^{x_0,r}(\tilde{x}_0,D+i\tau N)v\right|^2 + \frac{1}{4}\left|(D+i\tau N)^{2s}v\right|^2$$

and

$$\begin{aligned} &C_2\left|G_\tau^{x_0,r}(x,D,\bar{D})v\bar{v} - G_\tau^{x_0,r}(\tilde{x}_0,D,\bar{D})v\bar{v}\right| \\ &\quad\leqslant C_2\sum_{j=1}^{4s-1}\tau^j\left|G^{x_0,r,(j)}(x,D,\bar{D})v\bar{v} - G^{x_0,r,(j)}(\tilde{x}_0,D,\bar{D})v\bar{v}\right| \\ &\quad\leqslant \frac{1}{4}\sum_{j=1}^{4s-1}\tau^j \sum_{|\alpha|+|\beta|=4s-j-1}\left|D^\alpha v\overline{D^\beta v}\right| \end{aligned}$$

for all $r \geqslant r_0$. Thus, from (3.18) and the above inequalities

$$\begin{aligned} (2\pi)^{-2}\int\left|\hat{v}(\xi)\right|^2|\xi+i\tau N|^{4s}\,d\xi &\leqslant C_1\tau\int G_\tau^{x_0,r}(x,D,\bar{D})v\bar{v}\,dx + C_2\int\left|P_{2s}^{x_0,r}(x,D+i\tau N)v\right|^2dx \\ &\quad+\frac{1}{4}(2\pi)^{-2}\int\left|\hat{v}(\xi)\right|^2\left|(\xi+i\tau N)\right|^{4s}d\xi \\ &\quad+\frac{1}{4}\sum_{j=1}^{4s-1}\tau^{j+1}\sum_{|\alpha|+|\beta|=4s-j-1}\int\left|D^\alpha v\overline{D^\beta v}\right|dx \end{aligned}$$

for $v \in C_0^\infty(B_{\delta_0}(\tilde{x}_0)\cap A_{x_0})$ and $r \geqslant r_0$. For each term in the last sum, we use the Cauchy–Schwartz inequality and the estimate

$$\left(|N|^2\tau^2\right)^{2s-|\alpha|}\int\left|D^\alpha v\right|^2dx \leqslant (2\pi)^{-2}\int\left|\hat{v}(\xi)\right|^2|\xi+i\tau N|^{4s}\,d\xi$$

for $|\alpha| \leqslant 2s$. Also, in view of the inequality (3.14), we obtain

$$\begin{aligned} \left(|N|^2\tau^2\right)^{2s-|\alpha|}\int\left|D^\alpha u\right|^2e^{2\tau\langle x-x_0,N\rangle}\,dx &\leqslant (2\pi)^{-2}\int\left|\hat{v}(\xi)\right|^2|\xi+i\tau N|^{4s}\,d\xi \\ &\leqslant 2(C_1\tau+C_2)\int\left|P_{2s}^{x_0,r}(x,D+i\tau N)v\right|^2dx \\ &= 2(C_1\tau+C_2)\int\left|P_{2s}^{x_0,r}(x,D)u\right|^2e^{2\tau\langle x-x_0,N\rangle}\,dx \end{aligned}$$

for all $u \in C_0^\infty(B_{\delta_0}(\tilde{x}_0) \cap A_{x_0})$, $|\alpha| \leqslant 2s$, and $r \geqslant r_0$. Therefore, the estimate (3.13) holds for functions supported in a small neighborhood of $\tilde{x}_0$ in A_{x_0}. By [12, Lemma 8.3.1], the proof of Lemma 3.3 is complete. $\square$

Proof of Theorem 3.1. For simplicity of notation, we may assume that $x_0 = 0$. In order to obtain a Carleman estimate on the dyadic annuli A_r centered at 0 for $r \in \mathbb{N}$ such that $r \geqslant r_0$, we perform a change of variable $y = 2^{-r}x$ in (3.13) and write $u(x) = v(2^{-r}x)$. Denoting $\sigma = 2^{-mr}\tau$, we get

$$\sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} 2^{(4s-2|\alpha|-1)mr} 2^{-2|\alpha|r} \int_{A_r} \left|D^\alpha v\right|^2 e^{2\sigma\psi} \leqslant C 2^{-4sr} \int_{A_r} \left|P_{2s}(x, D)v\right|^2 e^{2\sigma\psi}$$

which implies

$$\sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} \int_{A_r} \left|D^\alpha v\right|^2 e^{2\sigma\psi} \leqslant C \int_{A_r} \left|P_{2s}(x, D)v\right|^2 e^{2\sigma\psi} \tag{3.19}$$

for all $v \in C_0^\infty(A_r)$ and $\sigma \geqslant \tau_0$.

Let $\varphi \in C_0^\infty(\tilde{\Omega}, [0, 1])$ be a cutoff function with support in A_0 such that $\varphi \equiv 1$ in a neighborhood of $\{x \in \tilde{\Omega}\colon\ 1 \leqslant |x| \leqslant 2\}$ and $\operatorname{supp}\varphi \subset \{x \in \tilde{\Omega}\colon\ 7/8 \leqslant |x| \leqslant 17/8\}$. Define $\varphi_r = \varphi(2^r\cdot)$. Clearly, $\varphi_r \in C_0^\infty(\tilde{\Omega}, [0, 1])$ are compactly supported in A_r, and $\varphi_r \equiv 1$ in a neighborhood of $C_r = \{x \in \tilde{\Omega}\colon\ 2^{-r} \leqslant |x| \leqslant 2^{-r+1}\}$. Also, we have that $|D^\alpha \varphi_r| \leqslant C2^{r|\alpha|}\varphi_{r+1}$ in a neighborhood of $S_r = \{x \in \tilde{\Omega}\colon\ 2^{-r-1} \leqslant |x| \leqslant 2^{-r}\}$ and $|D^\alpha \varphi_r| \leqslant C2^{r|\alpha|}\varphi_{r-1}$ in a neighborhood of $L_r = \{x \in \tilde{\Omega}\colon\ 2^{-r+1} \leqslant |x| \leqslant 2^{-r+2}\}$ for all $\alpha \in \mathbb{N}_0^2$ with $|\alpha| = 0, \dots, 2s-1$.

Let $u \in C_0^\infty(\tilde{\Omega})$ be a smooth function with support in the annulus $A(0, \delta, \rho_0)$, where $\rho_0 = 2^{-r_0}$. Let r_1 be the smallest integer such that $2^{-r_1} \leqslant \delta$. Then we have $\operatorname{supp} u \subset \bigcup_{k=r_0}^{r_1} A_r$. Applying the Carleman inequality (3.19) to $u\varphi_r$ and summing for $r = r_0, \dots, r_1$, we obtain

$$\sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \int_{A_r} \left|D^\alpha(u\varphi_r)\right|^2 e^{2\sigma\psi} \leqslant C \sum_{r=r_0}^{r_1} \int_{A_r} \left|P_{2s}(x, D)(u\varphi_r)\right|^2 e^{2\sigma\psi} \tag{3.20}$$

for $\sigma \geqslant \tau_0$. For the left side of (3.20), we claim

$$\sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \int_{A_r} \left|D^\alpha(u\varphi_r)\right|^2 e^{2\sigma\psi} \geqslant \frac{1}{C} \sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \int_{A_r} \left|\varphi_r D^\alpha u\right|^2 e^{2\sigma\psi}. \tag{3.21}$$

Indeed, we have $|D^\alpha(u\varphi_r)|^2 \geqslant (1/2)|\varphi_r D^\alpha u|^2 - C\sum_{\beta<\alpha} |D^\beta u D^{\alpha-\beta}\varphi_r|^2$ by the Leibniz rule and the triangle inequality; therefore, using the assumptions on the derivatives of φ_r

$$\begin{aligned}
&\sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \int_{A_r} \left|D^\alpha(u\varphi_r)\right|^2 e^{2\sigma\psi} \\
&\quad\geqslant \sum_{|\alpha|<2s} \sigma^{4s-2|\alpha|-1} \sum_{r=r_0}^{r_1} \Bigg(\frac{1}{2} \int_{A_r} \left|\varphi_r D^\alpha u\right|^2 e^{2\sigma\psi} \\
&\qquad - C \sum_{\beta<\alpha} 2^{2r(|\alpha|-|\beta|)} \bigg(\int_{S_r} \left|\varphi_{r+1} D^\beta u\right|^2 e^{2\sigma\psi} + \int_{L_r} \left|\varphi_{r-1} D^\beta u\right|^2 e^{2\sigma\psi} \bigg)\Bigg)
\end{aligned}$$

and the two negative terms on the right side can be absorbed into the first term on the right side, since

$$C\sigma^{4s-2|\alpha|-1}2^{2r(|\alpha|-|\beta|)}\int_{S_r}\left|\varphi_{r+1}D^\beta u\right|^2 e^{2\sigma\psi} \leqslant C\sigma^{4s-2|\alpha|-1}2^{4sr_1}\int_{S_r}\left|\varphi_{r+1}D^\beta u\right|^2 e^{2\sigma\psi}$$
$$\leqslant \sigma^{4s-2|\beta|-1}\int_{A_{r+1}}\left|\varphi_{r+1}D^\beta u\right|^2 e^{2\sigma\psi}$$

and

$$C\sigma^{4s-2|\alpha|-1}2^{2r(|\alpha|-|\beta|)}\int_{L_r}\left|\varphi_{r-1}D^\beta u\right|^2 e^{2\sigma\psi} \leqslant \sigma^{4s-2|\beta|-1}\int_{A_{r-1}}\left|\varphi_{r-1}D^\beta u\right|^2 e^{2\sigma\psi}$$

for all $\beta<\alpha$ and $r\in\{r_0,\dots,r_1\}$ provided $\sigma\geqslant\tau_0 2^{4sr_1}$. For the right side of (3.20), we have

$$C\sum_{r=r_0}^{r_1}\int_{A_r}\left|P_{2s}(x,D)(u\varphi_r)\right|^2 e^{2\sigma\psi} \leqslant C\sum_{r=r_0}^{r_1}\int_{A_r}\left|\varphi_r P_{2s}(x,D)u\right|^2 e^{2\sigma\psi}$$
$$+C\sum_{r=r_0}^{r_1}\int_{A_r}\sum_{\beta>0}\left|P_{2s}^{(\beta)}(x,D)u\right|^2\left|\frac{D^\beta\varphi_r}{\beta!}\right|^2 e^{2\sigma\psi} \tag{3.22}$$

by the Leibniz rule. The second term on the right side can be absorbed into the right side of (3.21), since

$$C\sum_{r=r_0}^{r_1}\int_{A_r}\sum_{\beta>0}\left|P_{2s}^{(\beta)}(x,D)u\right|^2\left|\frac{D^\beta\varphi_r}{\beta!}\right|^2 e^{2\sigma\psi}$$
$$\leqslant C\sum_{r=r_0}^{r_1}\sum_{|\alpha|=2s}\sum_{0<\beta\leqslant\alpha}2^{2r|\beta|}\left(\int_{S_r}\left|\varphi_{r+1}D^{\alpha-\beta}u\right|^2 e^{2\sigma\psi}+\int_{L_r}\left|\varphi_{r-1}D^{\alpha-\beta}u\right|^2 e^{2\sigma\psi}\right)$$
$$\leqslant C\sigma^{4s-2|\alpha|-1}\sum_{r=r_0}^{r_1}\int_{A_r}\left|\varphi_r D^\alpha u\right|^2 e^{2\sigma\psi}$$

if $\sigma\geqslant\tau_0 2^{4sr_1}$. From (3.21) and (3.22), we conclude

$$\sum_{|\alpha|<2s}\sigma^{4s-2|\alpha|-1}\sum_{r=r_0}^{r_1}\int_{A_r}\left|\varphi_r D^\alpha u\right|^2 e^{2\sigma\psi} \leqslant C\sum_{r=r_0}^{r_1}\int_{A_r}\left|\varphi_r P_{2s}(x,D)u\right|^2 e^{2\sigma\psi}$$

and hence

$$\sum_{|\alpha|<2s}\sigma^{4s-2|\alpha|-1}\int_{A(0,\delta,\rho_0)}\left|D^\alpha u\right|^2 e^{2\sigma\psi} \leqslant C\int_{A(0,\delta,\rho_0)}\left|P_{2s}(x,D)u\right|^2 e^{2\sigma\psi} \tag{3.23}$$

for all $u \in C_0^\infty(\tilde{\Omega})$ with support in $A(0,\delta,\rho_0)$ and $\sigma \geqslant \tau_0\delta^{-4s}$, by taking τ_0 sufficiently large. Note that the constant C does not depend on r_1, since for any $x \in A_r$, there are at most three functions from $\{\varphi_r(x)\}_{r=r_0}^{r_1}$, which are different from zero. Thus, $|D^\alpha u| \leqslant C(\sum_{r=r_0}^{r_1}|\varphi_r D^\alpha u|^2)^{1/2}$ by the Cauchy–Schwartz inequality. □

4. Propagation of smallness

In this section we suppose that the assumptions on the operator $P_{2s}(x,D)$ from Section 3 are satisfied, so that Theorem 3.1 is applicable. Using a linear change of variable by a constant factor, we may assume without loss of generality that $\rho_0 > 2$.

Let u be an infinitely smooth solution of the equation

$$P(x,D)u = \sum_{|\alpha|\leqslant 2s} a_\alpha(x) D^\alpha u = 0 \tag{4.1}$$

for $x \in \tilde{\Omega}$. Assume that there exist nonnegative constants M and δ such that

$$\|D^\alpha u\|_{L^2(B_2(x))} \leqslant \frac{M|\alpha|!^\sigma}{\delta^{|\alpha|}} \tag{4.2}$$

for $|\alpha| = 0,\dots,2s-1$. Assume that the coefficients a_α are infinitely smooth and that there exist nonnegative constants M_α such that

$$\|a_\alpha\|_{L^\infty(B_2(x))} \leqslant M_\alpha \tag{4.3}$$

for $|\alpha| = 0,\dots,2s$. Assume additionally that

$$\|D^\alpha u\|_{L^2(B_{2\delta}(x))} \leqslant \tilde{\epsilon} \tag{4.4}$$

for $|\alpha| = 0,\dots,2s-1$ and some sufficiently small $\tilde{\epsilon} \in (0,1)$ such that $\tilde{\epsilon} \leqslant M$.

Lemma 4.1. *Suppose that the assumptions* (4.1)–(4.4) *are satisfied. If*

$$\tilde{\epsilon} \leqslant M \exp\bigl(-P_1\bigl(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\bigr)\bigr)$$

then

$$\|u\|_{L^2(B_1)} \leqslant P_2\bigl(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\bigr)\tilde{\epsilon}^\theta M^{1-\theta}$$

where P_1 and P_2 are nonnegative polynomials and the parameter $\theta \in (0,1)$ is such that $\theta \leqslant C\delta^m$.

Proof. Let $\varphi \in C_0^\infty(\tilde{\Omega},[0,1])$ be a smooth cutoff function such that $\varphi \equiv 1$ in a neighborhood of the annulus $B_{1.5} \setminus B_{2\delta}$ and $\varphi \equiv 0$ in a neighborhood of $B_\delta \cup B_2^c$. Additionally, we assume that

$$\|D^\alpha \varphi\|_{L^\infty(B_{2\delta}\setminus B_\delta)} \leqslant \frac{C}{\delta^{|\alpha|}}$$

for $|\alpha| = 0,\dots,2s-1$. Consider $u\varphi \in C_0^\infty(B_2 \setminus B_\delta)$. By Theorem 3.1, it follows

$$\sum_{|\alpha|<2s} \tau^{4s-2|\alpha|-1} \int_{B_2\setminus B_\delta} \left|D^\alpha(u\varphi)\right|^2 e^{2\tau\psi} \leqslant C \int_{B_2\setminus B_\delta} \left|P_{2s}(x,D)(u\varphi)\right|^2 e^{2\tau\psi} \tag{4.5}$$

for $\tau \geqslant \tau_0\delta^{-4s}$. We estimate the right side of (4.5) from above by $I_1 + I_2 + I_3$, where

$$I_1 = C \int_{B_{2\delta}\setminus B_\delta} \left|P_{2s}(x,D)(u\varphi)\right|^2 e^{2\tau\psi},$$

$$I_2 = C \int_{B_{1.5}\setminus B_{2\delta}} \left|P_{2s}(x,D)(u\varphi)\right|^2 e^{2\tau\psi},$$

$$I_3 = C \int_{B_2\setminus B_{1.5}} \left|P_{2s}(x,D)(u\varphi)\right|^2 e^{2\tau\psi}.$$

Using the Leibniz formula and Eq. (4.1), we have

$$\begin{aligned} P_{2s}(x,D)(u\varphi) &= \sum_{|\alpha|=2s} a_\alpha(x)\big(D^\alpha u\big)\varphi + \sum_{\beta>0}\big(P_{2s}^{(\beta)}(x,D)u\big)\frac{D^\beta\varphi}{\beta!} \\ &= -\sum_{|\alpha|<2s} a_\alpha(x)\big(D^\alpha u\big)\varphi + \sum_{\beta>0}\big(P_{2s}^{(\beta)}(x,D)u\big)\frac{D^\beta\varphi}{\beta!}. \end{aligned} \tag{4.6}$$

By the equality (4.6), the assumptions on the derivatives of φ, and the hypotheses (4.3) and (4.4), we obtain

$$\begin{aligned} I_1 &\leqslant C\sum_{|\alpha|<2s} M_\alpha^2 \int_{B_{2\delta}\setminus B_\delta} \left|D^\alpha u\right|^2 e^{2\tau\psi} + C\sum_{\beta>0}\delta^{-2|\beta|}\int_{B_{2\delta}\setminus B_\delta}\left|P_{2s}^{(\beta)}(x,D)u\right|^2 e^{2\tau\psi} \\ &\leqslant C\tilde{\epsilon}^2 e^{2\tau\psi(\delta)}\sum_{|\alpha|<2s} M_\alpha^2 + C\sum_{\beta>0}\delta^{-2|\beta|}\tilde{\epsilon}^2 e^{2\tau\psi(\delta)}\sup_{|\alpha|=2s} M_\alpha^2 \\ &\leqslant C\Big(\sup_{|\alpha|<2s} M_\alpha^2 + \delta^{-4s}\sup_{|\alpha|=2s} M_\alpha^2\Big)\tilde{\epsilon}^2 e^{2\tau\delta^{-m}} \leqslant C\delta^{-4s}\tilde{\epsilon}^2 e^{2\tau\delta^{-m}}\sup_{|\alpha|\leqslant 2s} M_\alpha^2 \end{aligned}$$

for the first integral. Above, we denoted $\psi(\delta) = \psi(x)$ where $|x| = \delta$. Next, using that $\varphi \equiv 1$ in a neighborhood of $B_{1.5}\setminus B_{2\delta}$, the estimate on the coefficients (4.3), and the equality (4.6), we have the estimate for the second integral

$$I_2 \leqslant C\sum_{|\alpha|<2s} M_\alpha^2 \int_{B_{1.5}\setminus B_{2\delta}} \left|D^\alpha u\right|^2 e^{2\tau\psi},$$

which can be absorbed in the half of the left side of (4.5) provided that

$$\tau \geqslant \max\left\{\sup_{|\alpha|<2s} M_\alpha^{2/(4s-2|\alpha|-1)}, \frac{\tau_0}{\delta^{4s}}\right\}. \tag{4.7}$$

Finally, the assumptions (4.2), (4.3), and the equality (4.6) imply

$$I_3 \leqslant C \sum_{|\alpha|<2s} M_\alpha^2 \int\limits_{B_2\setminus B_{1.5}} \left|D^\alpha u\right|^2 e^{2\tau\psi} + C \sum_{|\alpha|=2s} M_\alpha^2 \sum_{\beta>0} \int\limits_{B_2\setminus B_{1.5}} \left|D^{\alpha-\beta} u\right|^2 \left|D^\beta \varphi\right|^2 e^{2\tau\psi}$$

$$\leqslant CM^2 \sum_{|\alpha|<2s} M_\alpha^2 \frac{|\alpha|!^{2\sigma}}{\delta^{2|\alpha|}} e^{2\tau\psi(1.5)} + CM^2 \sum_{|\alpha|=2s} M_\alpha^2 \sum_{\beta>0} \frac{|\alpha-\beta|!^{2\sigma}}{\delta^{2|\alpha-\beta|}} \delta^{-2|\beta|} e^{2\tau\psi(1.5)}$$

$$\leqslant CM^2 \sum_{|\alpha|\leqslant 2s} \frac{M_\alpha^2}{\delta^{2|\alpha|}} e^{2\tau 1.5^{-m}} \leqslant \frac{CM^2}{\delta^{-4s}} e^{2\tau 1.5^{-m}} \sup_{|\alpha|\leqslant 2s} M_\alpha^2$$

for the third integral. Hence,

$$\tau^{4s-1} \int\limits_{B_1\setminus B_{2\delta}} |u|^2 e^{2\tau\psi} \leqslant \frac{C\tilde{\epsilon}^2}{\delta^{4s}} e^{2\tau\delta^{-m}} \sup_{|\alpha|\leqslant 2s} M_\alpha^2 + \frac{CM^2}{\delta^{4s}} e^{2\tau 1.5^{-m}} \sup_{|\alpha|\leqslant 2s} M_\alpha^2$$

provided (4.7) holds. Note that $\psi \geqslant 1$ on $B_1 \setminus B_{2\delta}$. Dividing both sides of the above inequality by $\tau^{4s-1}\exp(2\tau)$, we get

$$\int\limits_{B_1\setminus B_{2\delta}} |u|^2 \leqslant \frac{C}{\tau^{4s-1}\delta^{4s}} \left(\tilde{\epsilon}^2 e^{2\tau(\delta^{-m}-1)} + M^2 e^{2\tau(1.5^{-m}-1)}\right) \sup_{|\alpha|\leqslant 2s} M_\alpha^2.$$

Now, we choose τ such that $\tilde{\epsilon} = M\exp(\tau(1.5^{-m} - \delta^{-m}))$ and this τ satisfies (4.7) if $\tilde{\epsilon}$ is sufficiently small. Then

$$\frac{1}{\tilde{\epsilon}^{2\theta}} \int\limits_{B_1\setminus B_{2\delta}} |u|^2 \leqslant \frac{CM^{2-2\theta}}{\delta^{-4s}} \sup_{|\alpha|\leqslant 2s} M_\alpha^2 = P_2\left(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\right) M^{2-2\theta}$$

for $\theta = (1 - 1.5^{-m})/(\delta^{-m} - 1.5^{-m})$ provided

$$\log\frac{M}{\tilde{\epsilon}} \geqslant \max\left\{\sup_{|\alpha|<2s} M_\alpha^{2/(4s-2|\alpha|-1)}, \frac{\tau_0}{\delta^{4s}}\right\}\left(\delta^{-m} - 1.5^{-m}\right) = P_1\left(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\right).$$

Therefore, the lemma is proven. □

We shall need the following result on the smallness of the derivatives of u.

Lemma 4.2. *Suppose*

$$\left\|D^\alpha u\right\|_{L^2(B_{4\delta})} \leqslant \frac{C^{|\alpha|} M |\alpha|!^\sigma}{\delta^{|\alpha|-1}}$$

for $\alpha \in \mathbb{N}_0^n$ *and*

$$\|u\|_{L^2(B_{4\delta})} \leqslant \epsilon\delta.$$

Then

$$\left\|D^\alpha u\right\|_{L^2(B_{2\delta})} \leqslant \frac{C^{|\alpha|+1}\epsilon^{1/2} M^{1/2} |\alpha|!^\sigma}{\delta^{|\alpha|-1}}$$

for all $\alpha \in \mathbb{N}_0^n$.

This interpolation-type lemma is proven in [22] for real analytic functions when $\sigma = 1$. Using analogous arguments for functions u in the Gevrey class G^σ with $\sigma > 1$, one obtains the more general assertion stated above.

Proof of Theorem 2.1. Denote

$$\epsilon = \frac{\|u\|_{L^2(B_{4\delta})}}{\|u\|_{L^2(B_2)}}.$$

We shall prove that $\epsilon \geqslant \exp(-Q_1(\delta^{-1}, K, M, \{M_\alpha\}_{|\alpha|<2s}))$ for some nonnegative polynomial Q_1. By the definition of ϵ, in particular, we have

$$\|u\|_{L^2(B_{4\delta})} \leqslant \epsilon \|u\|_{L^2(B_2)}.$$

Then Lemma 4.2 implies

$$\left\| D^\alpha u \right\|_{L^2(B_{2\delta})} \leqslant \frac{C^{|\alpha|+1} \epsilon^{1/2} M^{1/2} |\alpha|!^\sigma}{\delta^{|\alpha|}} \|u\|_{L^2(B_2)}$$

for $|\alpha| = 0, \ldots, 2s-1$. Denote

$$K_0 = \max_{|\alpha| \leqslant 2s-1} \frac{\delta^{|\alpha|} \| D^\alpha u \|_{L^2(B_2)}}{|\alpha|!^\sigma}$$

and set $\tilde{u}(x) = K_0^{-1} u(x)$. Clearly, the function $\tilde{u}$ also solves Eq. (4.1) and

$$\left\| D^\alpha \tilde{u} \right\|_{L^2(B_2)} \leqslant \frac{|\alpha|!^\sigma}{\delta^{|\alpha|}}$$

for $|\alpha| = 0, \ldots, 2s-1$. Next, we denote

$$\tilde{\epsilon} = \max_{|\alpha|<2s} \frac{C^{|\alpha|+1} \epsilon^{1/2} M^{1/2} |\alpha|!^\sigma}{\delta^{|\alpha|}}. \tag{4.8}$$

Then

$$\left\| D^\alpha \tilde{u} \right\|_{L^2(B_{2\delta})} \leqslant \frac{1}{K_0} \left\| D^\alpha u \right\|_{L^2(B_{2\delta})} \leqslant \tilde{\epsilon}$$

for $|\alpha| = 0, \ldots, 2s-1$. Since the hypotheses of Lemma 3.1 are satisfied, we have

$$\|\tilde{u}\|_{L^2(B_1)} \leqslant P_2\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big) \tilde{\epsilon}^\theta$$

provided that

$$\tilde{\epsilon} \leqslant \exp\big(-P_1\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big)\big), \tag{4.9}$$

where P_1 and P_2 are the nonnegative polynomials obtained explicitly in the proof of the lemma.

Using the hypotheses (2.2) and (2.4), we get

$$K_0 \leqslant M\|u\|_{L^2(B_4)} \leqslant MK\|u\|_{L^2(B_2)}.$$

Thus, we obtain the estimate

$$\begin{aligned}\|u\|_{L^2(B_2)} &\leqslant K\|u\|_{L^2(B_1)} \leqslant KK_0\|\tilde{u}\|_{L^2(B_1)} \leqslant KK_0 P_2\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big)\tilde{\epsilon}^\theta \\ &\leqslant MK^2 P_2\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big)\tilde{\epsilon}^\theta\|u\|_{L^2(B_2)},\end{aligned}$$

which holds only if

$$\tilde{\epsilon} \geqslant \frac{1}{M^{1/\theta}K^{2/\theta}} P_2\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big)^{-1/\theta}. \tag{4.10}$$

Therefore, by (4.9) and (4.10)

$$\tilde{\epsilon} \geqslant \min\left\{\exp\big(-P_1\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big)\big), \frac{1}{M^{1/\theta}K^{2/\theta}} P_2\big(\delta^{-1}, \{M_\alpha\}_{|\alpha|<2s}\big)^{-1/\theta}\right\}.$$

Using (4.8), we solve the last inequality for ϵ. We conclude that $\epsilon \geqslant \exp(-Q_1(\delta^{-1}, K, M, \{M_\alpha\}_{|\alpha|<2s}))$ for a nonnegative polynomial Q_1 of degree in δ^{-1} depending only on s and m. □

5. Nodal sets

First, we recall a result from our paper [15] which addresses the order of vanishing and the size of the nodal sets for any 1-periodic Gevrey function in one real variable.

Lemma 5.1. *(See [15].) Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable 1-periodic function which is not identically zero. Let $a, b \geqslant 0$ and $1 \leqslant \sigma \leqslant 1 + 1/b$. If $\sigma = 1 + 1/b$, we assume that $4^{b+1}a/\delta^b \leqslant 1/2$. Suppose that there exist constants $M \geqslant 1$ and $\delta \in (0, 1/2]$ such that*

$$\big\|f^{(n)}\big\|_{L^\infty(\Omega)} \leqslant \frac{Mn!^\sigma}{\delta^n}\|f\|_{L^\infty(\Omega)}, \quad n \in \mathbb{N}_0, \tag{5.1}$$

and

$$\|f\|_{L^\infty(\Omega)} \leqslant \exp\left(\frac{a}{\rho^b}\right)\|f\|_{L^\infty[x_0-\rho/2, x_0+\rho/2]} \tag{5.2}$$

for all $\rho \in (0, \delta]$ and $x_0 \in \Omega$. Then for the number of zeros of f in Ω, we have

$$\operatorname{card}\big\{x \in \Omega\colon\ f(x) = 0\big\} \leqslant CK^{1+1/b}, \tag{5.3}$$

where

$$K = \left(\frac{4^{b+1}a}{\delta^b}\right)^{1/(1+b(1-\sigma))} + \frac{4^{b+1}a}{\delta^b} + 2\log M + 2 \tag{5.4}$$

and $C = C(a, b)$. The first term in (5.4) is understood to be zero if $\sigma = 1 + 1/b$. Moreover, we have an upper bound

$$\mathrm{ord}_{x_0} f \leqslant K \tag{5.5}$$

for the order of vanishing $\mathrm{ord}_{x_0} f$ *for every* $x_0 \in \Omega$.

Proof of Theorem 2.2. Let u be a solution of the elliptic equation with Gevrey coefficients $P(x, D)u = 0$ for $x \in \Omega$, which satisfies the condition (2.2) and the quantitative estimate of unique continuation from Theorem 2.1. Applying the 1-dimensional result from Lemma 5.1 on rays through x_0, we obtain a polynomial upper bound on the order of vanishing of u at any point x_0 in the 2-dimensional unit ball B_1 provided the Gevrey exponent $\sigma \leqslant 1 + \eta$, where η is a constant multiple of s. □

To give an upper bound on the 1-dimensional Hausdorff measure of the nodal sets of solutions, we rely on an argument depending on their geometric structure. We recall a result, due to Han [9], on the structure of the nodal and singular sets of solutions to higher order elliptic equations in $\mathbb{R}^n$ with Hölder continuous leading coefficients. It turns out that under the assumption that the solutions vanish at finite order, the singular and nodal sets are countable unions of subsets of $(n-2)$-dimensional and $(n-1)$-dimensional submanifolds, respectively. Define

$$\mathcal{N}(u) = \{x \in B_1\colon\ u(x) = 0\}$$

and

$$\mathcal{S}(u) = \{x \in B_1\colon\ D^\beta u = 0,\ |\beta| = 0, \ldots, 2s-1\},$$

where $2s$ is the order of the elliptic equation.

Theorem 5.2. *(See [9].) The set* $\mathcal{N}(u)$ *is countably* $(n-1)$*-rectifiable and the set* $\mathcal{S}(u)$ *is countably* $(n-2)$*-rectifiable. In fact there exist decompositions*

$$\mathcal{N}(u) = \bigcup_{j=0}^{n-1} \mathcal{N}^j(u),$$

$$\mathcal{S}(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^j(u)$$

where (i) *each* $\mathcal{N}^j(u)$ *is on a countable union of* j*-dimensional* C^1 *graphs for* $0 \leqslant j \leqslant n-2$ *and* $\mathcal{N}^{n-1}(u)$ *is on a countable union of* $(n-1)$*-dimensional* $C^{1,\alpha}$ *manifolds and* (ii) *each* $S^j(u)$ *is on a countable union of* j*-dimensional* C^1 *graphs for* $0 \leqslant j \leqslant n-3$ *and* $\mathcal{S}^{n-2}(u)$ *is on a countable union of* $(n-2)$*-dimensional* $C^{1,\alpha}$ *manifolds for some* $0 < \alpha < 1$.

In our case the space dimension is $n = 2$ and the coefficients of the elliptic equation $P(x, D)u = 0$ are infinitely smooth.

Proof of Theorem 2.3. By Theorem 5.2, we conclude that the nodal set $\mathcal{N}(u)$ of a solution is a union of a set with Hausdorff length zero and a countable union of 1-dimensional manifolds. Let Γ be a 1-dimensional manifold. Since, at any point z of Γ the corresponding tangent line makes an angle of at most 45° with one of the coordinate axes (cf. [5]), we have

$$\mathcal{H}^1\big\{\Gamma \cap \big\{|z| < \rho\big\}\big\} \leqslant C\bigg(\int\limits_{|s|<\rho} \mathcal{H}^0\big\{\Gamma \cap \{x = s\} \cap \big\{|y| < \rho\big\}\big\}\,ds + \int\limits_{|s|<\rho} \mathcal{H}^0\big\{\Gamma \cap \{y = s\} \cap \big\{|x| < \rho\big\}\big\}\,ds\bigg)$$

for $\rho > 0$ sufficiently small. An application of Lemma 5.1 gives the result. □

Proof of Theorem 2.4. Note that by the proof of Lemma 3.2, the pseudo-convexity condition (3.2) for the operator (2.1) is satisfied provided (2.5) holds. The assertion follows as in the 2-dimensional case treated above. □

References

[1] T. Carleman, Sur un problème d'unicité pur les systèmes d'équations aux dérivées partielles à deux variables indépendantes, Ark. Mat., Astr. Fys. 26 (1939) 1–9.
[2] F. Colombini, C. Grammatico, A result on strong uniqueness in Gevrey spaces for some elliptic operators, Comm. Partial Differential Equations 30 (2005) 39–57.
[3] F. Colombini, C. Grammatico, D. Tataru, Strong uniqueness for second order elliptic operators with Gevrey coefficients, Math. Res. Lett. 13 (2006) 15–27.
[4] F. Colombini, H. Koch, Strong unique continuation for products of elliptic operators of second order, Trans. Amer. Math. Soc. 362 (2009) 345–355.
[5] H. Donnelly, C. Fefferman, Nodal sets of eigenfunctions on Riemannian manifolds, Invent. Math. 93 (1988) 161–183.
[6] H. Donnelly, C. Fefferman, Nodal sets for eigenfunctions of the Laplacian on surfaces, J. Amer. Math. Soc. 3 (1990) 333–353.
[7] H. Donnelly, C. Fefferman, Nodal sets of eigenfunctions: Riemannian manifolds with boundary, in: Moser Volume, Analysis, Academic Press, 1990, pp. 251–262.
[8] L. Escauriaza, F.J. Fernandez, S. Vessella, Doubling properties of caloric functions, Appl. Anal. 85 (2006) 205–223.
[9] Q. Han, Schauder estimates for elliptic operators with application to nodal sets, J. Geom. Anal. 10 (2000) 455–480.
[10] Q. Han, F.-H. Lin, Nodal sets of solutions of parabolic equations: II, Comm. Pure Appl. Math. 47 (1994) 1219–1238.
[11] R. Hardt, L. Simon, Nodal sets for solutions of elliptic equations, J. Differential Geom. 30 (1989) 505–522.
[12] L. Hörmander, The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators, Springer-Verlag, 1985.
[13] L. Hörmander, On the uniqueness of the Cauchy problem, Math. Scand. 6 (1958) 213–225.
[14] L. Hörmander, On the uniqueness of the Cauchy problem II, Math. Scand. 7 (1959) 177–190.
[15] M. Ignatova, I. Kukavica, Unique continuation and complexity of solutions to parabolic partial differential equations with Gevrey coefficients, Adv. Differential Equations 15 (2010) 953–975.
[16] D. Jerison, C. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math. 121 (1985) 463–494.
[17] C. Kenig, Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation, Lecture Notes in Math. 1384 (1989) 69–90.
[18] C. Kenig, Some recent applications of unique continuation, Contemp. Math. 439 (2007) 25–56.
[19] C. Kenig, Quantitative unique continuation, logarithmic convexity of Gaussian means and Hardy's uncertainty principle, Proc. Sympos. Pure Math. 79 (2008) 207–227.
[20] I. Kukavica, Self-similar variables and the complex Ginzburg–Landau equation, Comm. Partial Differential Equations 24 (1999) 545–562.
[21] I. Kukavica, Length of vorticity nodal sets for Solutions of the 2D Navier-Stokes system, Comm. Partial Differential Equations 28 (2003) 771–793.
[22] I. Kukavica, Spatial complexity of solutions of higher order partial differential equations, Nonlinearity 17 (2004) 459–476.
[23] I. Kukavica, Nodal volumes for eigenfunctions of analytic regular elliptic problems, J. Anal. Math. 67 (1995) 269–280.
[24] N. Lerner, Résultats d'unicité forte pour des opérateurs elliptiques à coefficients Gevrey, Comm. Partial Differential Equations 6 (1981) 1163–1177.
[25] F. Lin, Nodal sets of solutions of elliptic and parabolic equations, Comm. Pure Appl. Math. 44 (1991) 287–308.
[26] J.L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. III, Springer-Verlag, New York, 1973.
[27] A. Plis, Unique continuation theorems for the solutions of partial differential equations, in: Proceedings of the International Congress of Mathematicians, 15–20 August, 1962.
[28] J.-C. Saut, B. Scheurer, Unique continuation for some evolution equations, J. Differential Equations 66 (1987) 118–139.
[29] H. Koch, D. Tataru, Recent results on unique continuation for second order elliptic equations. Carleman estimates and applications to uniqueness and control theory, Progr. Nonlinear Differential Equations Appl. 46 (2001) 73–84.
[30] S. Vessella, Carleman estimates, optimal three cylinder inequalities and unique continuation properties for parabolic operators, in: Progress in Analysis, vols. I, II, Berlin, 2001, pp. 485–492.