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Strong Unique Continuation for the Navier–Stokes Equation with Non-Analytic Forcing

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Abstract We establish the strong unique continuation property for differences of solutions to the Navier–Stokes system with Gevrey forcing. For this purpose, we provide Carleman-type inequalities with the same singular weight for the Laplacian and the heat operator.

Keywords Navier–Stokes equation · Carleman estimates · Strong unique continuation · Gevrey class

1 Introduction

This paper is devoted to the study of the local behavior of differences of solutions to the 3D Navier–Stokes system

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= f \\ \nabla \cdot u &= 0 \end{aligned} \tag{NSE}$$

with a general non-analytic forcing. Our goal is to provide a quantitative estimate of the strong unique continuation for a difference of any two solutions (u_1, p_1) and (u_2, p_2) of the system (NSE): If the velocity vector fields u_1 and u_2 are not identically equal, then their difference $u_1 - u_2$ has finite order of vanishing at any point. We establish a polynomial estimate on the rate of vanishing, provided the forcing f lies in the Gevrey class G^σ for certain restricted range of the exponents $\sigma > 1$. The motivation for studying the strong unique continuation problem for differences of solutions comes from a result of the second

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author and Robinson [32, Theorem 1], which states that the strong uniqueness leads to a property regarding determination of solutions by a finite number of their point values.

The literature concerned with quantitative estimates of unique continuation (doubling estimates) for second order elliptic and parabolic equations, based on Carleman methods, is extensive. These type of estimates are used classically to prove weak unique continuation results (c.f. [5,10,15,19,35]). Some strong unique continuation results, under different assumptions on the coefficients, were obtained in [4,6,8,22,23,33]. For more extended historical overview of unique continuation results, see the review papers by Kenig [24,25] and Vessella [36]. On the other hand, there are very few works available on strong uniqueness for systems. Recently, in [34], Lin, Uhlmann, and Wang provided an upper bound on the order of vanishing of non-trivial solutions to the stationary Stokes system by deriving optimal three-ball inequalities. Their proof rests upon delicate Carleman-type inequalities with singular weights and interior estimates for the velocity vector field and vorticity, satisfying a coupled system of second order elliptic equations.

For applications of the quantitative estimates of the strong unique continuation in estimating the Hausdorff measure of the nodal sets of solutions to elliptic and parabolic equations, we refer to [7,14,16–18,28,33]. In [27], the second author obtained a polynomial upper bound on the size of the vorticity nodal sets for the solutions of the 2D Navier–Stokes equations written in the vorticity form. The proof relied on a modification of a unique continuation method, due to Kurata [30], for the parabolic equation

$$\partial_t u - \Delta u = w_j(x,t)\partial_j u + v(x,t)u$$

and a self-similar transformation of variables (c.f. [26]). We emphasize that this approach can not be applied to the difference of solutions of the Navier–Stokes system. For other related results on this subject see also [1,2,9,13,31].

The paper is organized as follows. In Sect. 2, we state our main results Theorem 2.1 and 2.2 for a coupled system of elliptic-parabolic type for a difference of two solutions of the system (NSE) with the same Gevrey forcing. The following section is devoted to the poof of certain Carleman estimates with singular weights for the Laplace and for the heat operator. In Sect. 4, combining these results, we provide a quantitative estimate of unique continuation (doubling estimate) for the coupled system, leading to an upper bound on the vanishing order for the difference of two solutions (c.f. Theorem 2.2).

2 Notation and the Main Result

Let $B_r = \{x \in \mathbb{R}^3 \colon |x| < r\}$ be a ball with radius r centered at the origin and let $\delta \in (0,1)$ be fixed. We consider the Navier–Stokes equation with forcing f in the Gevrey class G^σ with $\sigma \geq 1$

$$\begin{aligned}&\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = f\\ &\nabla \cdot u = 0\end{aligned}$$

for $(x,t) \in B_2 \times [t_0 - \delta^2, t_0 + \delta^2]$. More precisely, we assume that the forcing f is an infinitely smooth function in (x,t) and that there exist nonnegative constants M_0 and δ_0 such that

$$||\partial_t^m \partial_x^\alpha f(\cdot,t)||_{L^\infty(B_2)} \leq \frac{M_0 m!^\sigma |\alpha|!^\sigma}{\delta_0^{2m+|\alpha|}}, \quad t_0 - \delta^2 \leq t \leq t_0 + \delta^2$$

for all $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^3$. Let (u_1, p_1) and (u_2, p_2) be two solutions of the Navier–Stokes system which are globally in the Gevrey class G^σ. We assume that $u_1(\cdot, t_0) \neq u_2(\cdot, t_0)$ and that there exist nonnegative constants M_j and δ_0 such that

$$||\partial_t^m \partial_x^\alpha u_j(\cdot, t)||_{L^\infty(B_2)} \leq \frac{M_j m!^\sigma |\alpha|!^\sigma}{\delta_0^{2m+|\alpha|}}, \quad t_0 - \delta^2 \leq t \leq t_0 + \delta^2, \quad j = 1, 2 \tag{2.1}$$

for all $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^3$ (c.f. Remark 2.4 below). We define $v = u_1 - u_2$ and $p = p_1 - p_2$. Then (v, p) solves the coupled elliptic-parabolic system

$$\begin{aligned} &\partial_t v - \Delta v + u_1 \cdot \nabla v + v \cdot \nabla u_2 + \nabla p = 0, \\ &- \Delta p - \partial_j u_{1i} \partial_i v_j - \partial_i u_{2j} \partial_j v_i = 0 \end{aligned} \tag{2.2}$$

with coefficients in G^σ. We assume that v and p are infinitely smooth functions in (x, t) and that there exists a nonnegative constant M such that

$$\begin{aligned} &||\partial_t^m \partial_x^\alpha v(\cdot, t)||_{L^\infty(B_2)} + ||\partial_t^m \partial_x^\alpha p(\cdot, t)||_{L^\infty(B_2)} \\ &\leq \frac{M m!^\sigma |\alpha|!^\sigma}{\delta_0^{2m+|\alpha|}} \left(||v(\cdot, t)||_{L^2(B_2)} + ||p(\cdot, t)||_{L^2(B_2)}\right) \end{aligned} \tag{2.3}$$

for all $t_0 - \delta^2 \leq t \leq t_0 + \delta^2$, $m \in \mathbb{N}_0$, and $\alpha \in \mathbb{N}_0^3$. Also, we assume that v satisfies the doubling property

$$||v(\cdot, t_1)||_{L^2(B_2)} \leq K ||v(\cdot, t_2)||_{L^2(B_1)}, \quad t_0 - \delta^2 \leq t_1, t_2 \leq t_0 + \delta^2 \tag{2.4}$$

for some constant $K \geq 1$. Note that in the case of periodic boundary conditions, the constant K depends on the Dirichlet quotient $\|\nabla u(t_0 - \delta^2)\|_{L^2} / \|u(t_0 - \delta^2)\|_{L^2}$ (c.f. Remark 2.5).

We remark that the natural condition (2.3) can be derived from (2.4) for the coupled elliptic-parabolic system (2.1) with Gevrey coefficients and periodic boundary conditions.

The assumptions as stated above are local, but the aim of the theorems is to address solutions to boundary value problems (c.f. Remark 2.4 below). For instance, it is easy to show that the above assumptions hold in the case of periodic boundary conditions.

Here we state our main theorem which is proved in Section 4.

Theorem 2.1 *Suppose that v and p satisfy* (2.1)–(2.4). *Then*

$$||v(\cdot, t)||_{L^2(B_2)} \leq \exp(P(\delta^{-1}, K, M, M_1, M_2)) ||v(\cdot, t)||_{L^\infty(B_{4\delta})}, \quad t_0 - \delta^2 \leq t \leq t_0 + \delta^2$$

for a nonnegative polynomial P.

Using the above quantitative estimate of unique continuation for a fixed time $t = t_0$, we establish the following strong unique continuation result for the Navier–Stokes equation, which, however, is valid only for a certain restricted range of the Gevrey exponents σ. Recall that a difference of two solutions (u_1, p_1) and (u_2, p_2) satisfies the strong unique continuation property at a fixed time $t = t_0$ if $u_1(\cdot, t_0) \neq u_2(\cdot, t_0)$ in Ω implies that $u_1 - u_2$ has finite order of vanishing for any $x \in \Omega$ and $t = t_0$.

Theorem 2.2 *Suppose that the above hypotheses* (2.1)–(2.4) *are satisfied and $\sigma \leq 1 + \eta$, where $\eta > 0$ is a universal constant. Then the Navier–Stokes equation has the strong unique continuation property for differences of solutions at time $t = t_0$.*

This statement is a consequence of Theorem 2.1 and [20, Theorem 2.4]. The latter theorem provides an estimate on the order of vanishing and the number of zeros for Gevrey functions. We state it below for convenience.

Theorem 2.3 *[20, Theorem 2.4] Let* $f : \mathbb{R} \to \mathbb{R}$ *be an infinitely differentiable 1-periodic function which is not identically zero. Let* $a, b \geq 0$ *and* $1 \leq \sigma \leq 1 + 1/b$*. If* $\sigma = 1 + 1/b$*, we assume that* $4^{b+1}a/\delta^b \leq 1/2$*. Suppose that there exist constants* $M \geq 1$ *and* $\delta \in (0, 1/2]$ *such that*

$$\|f^{(n)}\|_{L^\infty(\Omega)} \leq \frac{Mn!^\sigma}{\delta^n}\|f\|_{L^\infty(\Omega)} \qquad n \in \mathbb{N}_0, \tag{2.5}$$

and

$$\|f\|_{L^\infty(\Omega)} \leq \exp\left(\frac{a}{\rho^b}\right)\|f\|_{L^\infty[x_0-\rho/2, x_0+\rho/2]} \tag{2.6}$$

for all $\rho \in (0, \delta]$ *and* $x_0 \in \Omega$*. Then for the number of zeros of* f *in* Ω*, we have*

$$\operatorname{card}\{x \in \Omega : f(x) = 0\} \leq CK^{1+1/b}, \tag{2.7}$$

where

$$K = \left(\frac{4^{b+1}a}{\delta^b}\right)^{1/(1+b(1-\sigma))} + \frac{4^{b+1}a}{\delta^b} + 2\log M + 2 \tag{2.8}$$

and $C = C(a, b)$*. The first term in* (2.8) *is understood to be zero if* $\sigma = 1 + 1/b$*. Moreover, we have an upper bound*

$$\operatorname{ord}_{x_0} f \leq K \tag{2.9}$$

for the order of vanishing $\operatorname{ord}_{x_0} f$ *for every* $x_0 \in \Omega$*.*

Remark 2.4 Note that the local Gevrey regularity of the solutions to (NSE), or the heat equation, is at most G^2 in time even in the case when f is analytic. However, the solutions of the boundary value problems, such as (NSE), are as regular as the forcing f; c.f. [11] for the case of forcing which is analytic in space and time variables. To illustrate this, consider the equation

$$\begin{aligned} &\partial_t u - \Delta u = f \\ &u(\cdot, 0) = 0 \end{aligned} \tag{2.10}$$

with periodic boundary conditions on [0, 1] and average zero condition for f. We assume that f is Gevrey with exponent $\sigma > 0$ in t, i.e.,

$$\|\partial_t^m f\|_{L^2} \leq \frac{M_0 m!^\sigma}{\delta_0^m} \qquad m \in \mathbb{N}_0 \tag{2.11}$$

for some $M_0 > 0$ and $\delta_0 > 0$. Then the standard energy inequality reads

$$\frac{d}{dt}\|\partial_t^m u\|_{L^2} \leq C\|\partial_t^m f\|_{L^2} \leq \frac{CM_0 m!^\sigma}{\delta_0^m} \tag{2.12}$$

which gives the Gevrey regularity in time with exponent σ for the solution. If f is jointly space-time Gevrey with exponent $\sigma > 0$, the solution is also jointly Gevrey with the same exponent as can be checked by writing the energy inequality for $\|\partial_t^m \partial_x^k u\|_{L^2}$. The similar argument extends to semilinear equations, such as the Navier–Stokes equation, using an argument from [11] (see also [12]) by considering the energy inequality for the quantity $\|\nabla e^{\alpha t(-\Delta)^{\sigma/2}} \partial_t^m u(t)\|_{L^2}$; however, the presence of a nonlinearity requires $\sigma \geq 1$.

Remark 2.5 As mentioned above, the condition (2.4) can be obtained from the bounds on the Dirichlet quotient in the case of periodic boundary conditions. In order to illustrate this, we again consider the equation

$$v_t - \Delta v + b \cdot \nabla v + cv = 0 \tag{2.13}$$

for $t \in [0, T]$ on a periodic domain $\Omega_L = [-L, L]^n$ with a period $L \leq 1/(2\sqrt{n})$ so that $\Omega_L \subseteq B_1$. The adjustments for the Navier–Stokes equation are straight-forward (c.f. [3] for instance). Denote by

$$Q(t) = \frac{\|\nabla v(t)\|_{L^2}^2}{\|v(t)\|_{L^2}^2} \tag{2.14}$$

the Dirichlet quotient, where we abbreviate $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(\Omega_L)}$ and $(\cdot,\cdot)_{L^2} = (\cdot,\cdot)_{L^2(\Omega_L)}$. Then we have (c.f. [29, Theorem 2.1])

$$\frac{1}{2}Q'(t) + \left\|(-\Delta - Q(t)I)\frac{v}{\|v\|_{L^2}}\right\|_{L^2}^2 \leq \left(-b \cdot \frac{\nabla v}{\|v\|_{L^2}} - c\frac{v}{\|v\|_{L^2}}, (-\Delta - Q(t)I)\frac{v}{\|v\|_{L^2}}\right)_{L^2} \tag{2.15}$$

which implies $Q'(t) \leq b_T^2 Q(t) + c_T^2$ where $b_T = \sup_{t\in[0,T]} \|b(t)\|_{L^\infty}$ and $c_T = \sup_{t\in[0,T]} \|c(t)\|_{L^\infty}$. Hence,

$$Q(t) \leq K(b_T, c_T, T, Q(0)) \qquad 0 \leq t \leq T, \tag{2.16}$$

where the constant K can be evaluated using the Gronwall lemma. Now, in order to obtain bounds on $\|v(t_2)\|_{L^2}^2/\|v(t_1)\|_{L^2}^2$, we multiply (2.13) by v and integrate:

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 + (b \cdot \nabla v, v)_{L^2} + (cv, v)_{L^2} = 0. \tag{2.17}$$

We observe that (2.17) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 &\geq -\|b\|_{L^\infty}\|\nabla v\|_{L^2}\|v\|_{L^2} - \|c\|_{L^\infty}\|v\|_{L^2}^2 \\ &\geq -\|\nabla v\|_{L^2}^2 - (b_T^2 + c_T)\|v\|_{L^2}^2\end{aligned} \tag{2.18}$$

or equivalently

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + (2Q + b_T^2 + c_T)\|v\|_{L^2}^2 \geq 0 \tag{2.19}$$

which, by (2.16), leads to

$$\|v(t_2)\|_{L^2}^2 \geq \exp(-2(2K + b_T^2 + c_T)T)\|v(t_1)\|_{L^2}^2 \qquad 0 \leq t_1 \leq t_2 \leq T. \tag{2.20}$$

On the other hand, using (2.17) and the Cauchy–Schwarz inequality, we obtain

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq C(\|b\|_{L^\infty}^2 + \|c\|_{L^\infty})\|v\|_{L^2}^2 \leq C(b_T^2 + c_T)\|v\|_{L^2}^2 \tag{2.21}$$

whence

$$\|v(t_2)\|_{L^2}^2 \leq \exp\left(C(b_T^2 + c_T)T\right)\|v(t_1)\|_{L^2}^2 \qquad 0 \leq t_1 \leq t_2 \leq T. \tag{2.22}$$

Finally, (2.4) follows from the bounds (2.20) and (2.22) on $\Omega_L \subseteq B_1$.

3 Carleman Estimates

Without loss of generality, we assume, from here on, that $t_0 = 0$.

First, we establish a Carleman type inequality for the Laplacian, defined in a torus centered at the origin, with a weight $\psi(x) = |x|^{-m}$, where $m \in \mathbb{N}$.

Lemma 3.1 *There exist positive constants C and τ_0 such that*

$$\tau^3 \int\limits_{A(\delta,1)} |v|^2 e^{2\tau\psi} + \tau \int\limits_{A(\delta,1)} |\nabla v|^2 e^{2\tau\psi} \leq C \int\limits_{A(\delta,1)} |\Delta v|^2 e^{2\tau\psi} \tag{3.1}$$

for any $v \in C_0^\infty(B_1)$ with support in the annulus $A(\delta, 1) = \{x \in \mathbb{R}^3 \colon \delta \leq |x| \leq 1\}$ and $\tau \geq \tau_0 \delta^{-4}$, provided m is a large enough constant.

Above and in the sequel, the symbol C denotes a generic positive constant which is allowed to depend only on σ. Any additional dependence is indicated explicitly.

Lemma 3.1 is a consequence of [21, Theorem 2.4]. The latter assertion is obtained by first establishing the pseudo-convexity condition with the singular weight $\psi = |x|^{-m}$ on the unit annulus $A(x_0, 1/2, 4)$ for principal symbols P_{2s} of elliptic operators of order $2s$ with simple complex characteristics satisfying the additional assumption: For $0 \neq \zeta = \xi + i\tau\nabla\psi_{x_0}(x)$, we have

$$|P_{2s}(x,\zeta)|^2 + |(x - x_0)\cdot\nabla_\zeta P_{2s}(x,\zeta)|^2 > 0, \quad x \in \bar{A}_{x_0}, \xi \in \mathbb{R}^n, \text{ and } \tau \in \mathbb{R}, \tag{3.2}$$

for all $x_0 \in \bar{\Omega}$, where $\psi_{x_0}(x) = |x - x_0|^{-m}$ and A_{x_0} is the n-dimensional unit torus centered at x_0. Then we derived a uniform in x_0 Carleman estimate on the dyadic annuli $A(x_0, 2^{-r-1}, 2^{-r+2})$ for $r \in \mathbb{N}$ and used a partition of unity to obtain the corresponding Carleman-type estimate on $A(x_0, \delta, 1)$ with explicit dependence of the constants C and τ on δ.

Proof of Lemma 3.1 Clearly, the Laplacian is an elliptic operator with simple complex characteristics. By [21, Theorem 2.4], it suffices to prove that for $0 \neq \zeta = (\zeta_1, \ldots, \zeta_n) = \xi + i\tau\nabla\psi(x)$ the assumption on the principal symbol $P_2(\zeta) = -|\zeta|^2$

$$|\zeta|^2 + |x \cdot 2\zeta| > 0$$

is satisfied for all $x \in \bar{A}_0$, $\xi \in \mathbb{R}^3$, and $\tau \in \mathbb{R}$, where $\psi(x) = |x|^{-m}$ and

$$A_0 = A(1/2, 4) = \left\{x \in \mathbb{R}^3 \colon \frac{1}{2} \leq |x| \leq 4\right\}$$

is the 3-dimensional unit torus centered at 0. Assume that $|\zeta|^2 = 0$, we have $|\tau| \geq 1/C$ and $|x \cdot (\xi + i\tau\nabla\psi(x))| \geq |\tau x \cdot \nabla\psi(x)| \geq 1/C$, and this implies the result. □

Using a Carleman estimates due to Isakov [19, Theorem 1.1] and a modification of arguments as in the proof of [21, Theorem 3.1], we derive a Carleman-type inequality for parabolic operators with a weight function ψ as in Lemma 3.1.

Lemma 3.2 *There exist constants C and τ_0 such that*

$$\tau^3 \int\limits_{-\delta^2}^{\delta^2} \int\limits_{A(\delta,1)} |u|^2 e^{2\tau\psi} + \tau \int\limits_{-\delta^2}^{\delta^2} \int\limits_{A(\delta,1)} |\nabla u|^2 e^{2\tau\psi} \leq C \int\limits_{-\delta^2}^{\delta^2} \int\limits_{A(\delta,1)} |\partial_t u - \Delta u|^2 e^{2\tau\psi} \tag{3.3}$$

holds for any $u \in C_0^\infty(A(\delta,1)\times(-\delta^2,\delta^2))$ *and* $\tau \geq \tau_0\delta^{-2}$, *provided* $m \in \mathbb{N}$ *is a large enough constant.*

Proof of Lemma 3.2 Let m be a large enough number such that the pseudo-convexity condition is satisfied for operators with the principal part $\partial_t - \Delta$ and a weight function $\psi(x) = |x|^{-m}$, defined in $\bar{A}_0 \times [-1,1]$. Then, by [19, Theorem 1.1], there exist positive constants C and τ_0 such that

$$\tau^3 \int_{-1}^{1}\int_{A_0} |u|^2 e^{2\tau\psi} + \tau \int_{-1}^{1}\int_{A_0} |\nabla u|^2 e^{2\tau\psi} \leq C \int_{-1}^{1}\int_{A_0} |\partial_t u - \Delta u|^2 e^{2\tau\psi} \tag{3.4}$$

holds for any $u \in C_0^\infty(A_0 \times (-1,1))$ and $\tau \geq \tau_0$. By rescaling the function u in the estimate (3.4), we obtain a Carleman type inequality on the dyadic cylindrical shell $A_r \times (-2^{-2r}, 2^{-2r})$ for $r \in \mathbb{N}$, where $A_r = \{x \in \mathbb{R}^3 : 2^{-r-1} \leq |x| \leq 2^{-r+2}\}$. Indeed, let $u(x,t) = v(2^{-r}x, 2^{-2r}t)$ and $\sigma = 2^{-mr}\tau$. Then we have

$$\sigma^3 2^{3rm} \int_{-2^{-2r}}^{2^{-2r}}\int_{A_r} |v|^2 e^{2\sigma\psi} + \sigma 2^{rm-2r} \int_{-2^{-2r}}^{2^{-2r}}\int_{A_r} |\nabla v|^2 e^{2\sigma\psi} \leq C 2^{-4r} \int_{-2^{-2r}}^{2^{-2r}}\int_{A_r} |\partial_t v - \Delta v|^2 e^{2\sigma\psi} \tag{3.5}$$

which leads to

$$\sigma^3 \int_{-2^{-2r}}^{2^{-2r}}\int_{A_r} |v|^2 e^{2\sigma\psi} + \sigma \int_{-2^{-2r}}^{2^{-2r}}\int_{A_r} |\nabla v|^2 e^{2\sigma\psi} \leq C \int_{-2^{-2r}}^{2^{-2r}}\int_{A_r} |\partial_t v - \Delta v|^2 e^{2\sigma\psi} \tag{3.6}$$

for all $v \in C_0^\infty(A_r \times (-2^{-2r}, 2^{-2r}))$ and $\sigma \geq \tau_0$.

Let $u \in C_0^\infty(B_1 \times (-\delta^2,\delta^2))$ be a smooth function with the support in $A(\delta,1)\times(-\delta^2,\delta^2)$. Denote by r_1 the smallest integer such that $2^{-r_1} \leq \delta$. Then $\operatorname{supp} u \subset \bigcup_{r=1}^{r_1} A_r \times (-2^{-2r}, 2^{-2r})$. Let ϕ be a smooth cut-off function with the support in $A_0 \times [-\delta^2,\delta^2]$ such that $\phi \equiv 1$ in a neighborhood of the torus $A(1,2)$ for all $t \in [-\delta^2,\delta^2]$. Define $\phi_r(x,t) = \phi(2^r x, t)$ for $r \in \mathbb{N}$. Then, we have that ϕ_r is compactly supported in $A_r \times [-\delta^2,\delta^2]$ such that $\phi_r \equiv 1$ in a neighborhood of $A(2^{-r}, 2^{-r+1})$ and $|D^\alpha \phi_r| \leq C2^{r|\alpha|}(\phi_{r+1} + \phi_{r-1})$ for all $\alpha \in \mathbb{N}^3$ with $|\alpha| = 0, 1$ and $t \in [-\delta^2,\delta^2]$.

Applying the Carleman inequality (3.6) to $u\phi_r$ and summing for $r = 1, \ldots, r_1$, we obtain

$$\sum_{r=1}^{r_1} \sigma^3 \int_{-\delta^2}^{\delta^2}\int_{A_r} |u\phi_r|^2 e^{2\sigma\psi} + \sigma \int_{-\delta^2}^{\delta^2}\int_{A_r} |\nabla(u\phi_r)|^2 e^{2\sigma\psi} \tag{3.7}$$
$$\leq C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2}\int_{A_r} |(\partial_t - \Delta)(u\phi_r)|^2 e^{2\sigma\psi}$$

for $\sigma \geq \tau_0$. First, for the left side of (3.7), we show that

$$\sum_{r=1}^{r_1} \sigma^3 \int_{-\delta^2}^{\delta^2} \int_{A_r} |u\phi_r|^2 e^{2\sigma\psi} + \sigma \int_{-\delta^2}^{\delta^2} \int_{A_r} |\nabla(u\phi_r)|^2 e^{2\sigma\psi} \geq \frac{1}{C} \sum_{r=1}^{r_1} \sigma^3 \int_{-\delta^2}^{\delta^2} \int_{A_r} |u\phi_r|^2 e^{2\sigma\psi} + \sigma \int_{-\delta^2}^{\delta^2} \int_{A_r} |\phi_r \nabla u|^2 e^{2\sigma\psi}. \tag{3.8}$$

Using the inequality $|\nabla(u\phi_r)|^2 \geq (1/2)|\phi_r \nabla u|^2 - |u \nabla \phi_r|^2$ and the assumptions on the derivatives of ϕ_r, we have

$$\sum_{r=1}^{r_1} \sum_{|\alpha| \leq 1} \sigma^{3-2|\alpha|} \int_{-\delta^2}^{\delta^2} \int_{A_r} |D^\alpha(u\phi_r)|^2 e^{2\sigma\psi} \geq \sum_{r=1}^{r_1} \left(\frac{1}{2} \sum_{|\alpha| \leq 1} \sigma^{3-2|\alpha|} \int_{-\delta^2}^{\delta^2} \int_{A_r} |\phi_r D^\alpha u|^2 e^{2\sigma\psi} - C 2^{2r} \sigma \int_{-\delta^2}^{\delta^2} \int_{A_r} \left(|\phi_{r+1} u|^2 + |\phi_{r-1} u|^2\right) e^{2\sigma\psi} \right)$$

and the two terms on the right side can be absorbed into the first term on the right side, since

$$C 2^{2r} \sigma \int_{-\delta^2}^{\delta^2} \int_{A_r} \left(|\phi_{r+1} u|^2 + |\phi_{r-1} u|^2\right) e^{2\sigma\psi} \leq \sigma^3 \int_{-\delta^2}^{\delta^2} \int_{A_{r+1}} |\phi_{r+1} u|^2 e^{2\sigma\psi} + \sigma^3 \int_{-\delta^2}^{\delta^2} \int_{A_{r-1}} |\phi_{r-1} u|^2 e^{2\sigma\psi}$$

for $r \in \{1, \ldots, r_1\}$ provided $\sigma \geq \tau_0 2^{r_1}$. Next, for the right side of (3.7), we have

$$C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2} \int_{A_r} |(\partial_t - \Delta)(u\phi_r)|^2 e^{2\sigma\psi} \leq C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2} \int_{A_r} |\phi_r (\partial_t - \Delta) u|^2 e^{2\sigma\psi} + C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2} \int_{A_r} \left(|u \Delta \phi_r|^2 + |\nabla u \nabla \phi_r|^2\right) e^{2\sigma\psi}. \tag{3.9}$$

Note that also the lower order terms on the right side of (3.9) may be absorbed into the right side of (3.8). Indeed,

$$C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2} \int_{A_r} \left(|u \Delta \phi_r|^2 + |\nabla u \nabla \phi_r|^2\right) e^{2\sigma\psi} \leq C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2} \int_{A_r} \left(2^{4r} \left(|\phi_{r+1} u|^2 + |\phi_{r-1} u|^2\right) + 2^{2r} \left(|\phi_{r+1} \nabla u|^2 + |\phi_{r-1} \nabla u|^2\right)\right) e^{2\sigma\psi}$$

$$\leq C \sum_{r=1}^{r_1} \sum_{|\alpha|\leq 1} \sigma^{3-2|\alpha|} \int_{-\delta^2}^{\delta^2} \int_{A_r} |\phi_r D^\alpha u|^2 e^{2\sigma\psi}$$

if $\sigma \geq \tau_0 2^{2r_1}$. From (3.8) and (3.9), we conclude

$$\sum_{r=1}^{r_1} \sum_{|\alpha|\leq 1} \sigma^{3-2|\alpha|} \int_{-\delta^2}^{\delta^2} \int_{A_r} |\phi_r D^\alpha u|^2 e^{2\sigma\psi} \leq C \sum_{r=1}^{r_1} \int_{-\delta^2}^{\delta^2} \int_{A_r} |\phi_r (\partial_t - \Delta) u|^2 e^{2\sigma\psi}$$

and hence

$$\sum_{|\alpha|\leq 1} \sigma^{3-2|\alpha|} \int_{-\delta^2}^{\delta^2} \int_{A(\delta,1)} |D^\alpha u|^2 e^{2\sigma\psi} \leq C \int_{-\delta^2}^{\delta^2} \int_{A(\delta,1)} |\partial_t u - \Delta u|^2 e^{2\sigma\psi} \tag{3.10}$$

for all $u \in C_0^\infty(B_1 \times (-\delta^2, \delta^2))$ with the support in $A(\delta, 1) \times (-\delta^2, \delta^2)$ and $\sigma \geq \tau_0 \delta^{-2}$, by taking τ_0 sufficiently large. Note that the constant C does not depend on r_1, since for any $x \in A_r$, there are at most three functions from among $\{\phi_r(x, t)\}_{r=1}^{r_1}$, which are different from zero. Thus $|D^\alpha u| \leq C \left(\sum_{r=1}^{r_1} |\phi_r D^\alpha u|^2\right)^{1/2}$ by the Cauchy–Schwartz inequality. □

4 Propagation of Smallness

In this section we establish a propagation of smallness lemma, which is used in the proof of Theorem 2.1.

Let (v, p) be an infinitely smooth solution of the coupled elliptic-parabolic system

$$\begin{aligned} &v_t - \Delta v + u_1 \cdot \nabla v + v \cdot \nabla u_2 + \nabla p = 0, \\ &- \Delta p - \partial_j u_{1i} \partial_i v_j - \partial_i u_{2j} \partial_j v_i = 0 \end{aligned} \tag{4.1}$$

for $(x, t) \in B_2 \times [-\delta^2, \delta^2]$. Assume that the coefficients are infinitely smooth and that there exist nonnegative constants M_j and δ such that

$$||\partial_x^\alpha u_j(\cdot, t)||_{L^\infty(B_2)} \leq \frac{M_j |\alpha|!^\sigma}{\delta^{|\alpha|}}, \quad -\delta^2 \leq t \leq \delta^2, \quad j = 1, 2 \tag{4.2}$$

for $|\alpha| = 0, 1$. Also, assume that there exist a nonnegative constant M such that

$$||\partial_x^\alpha v(\cdot, t)||_{L^\infty(B_2)} + ||\partial_x^\alpha p(\cdot, t)||_{L^\infty(B_2)} \leq \frac{M |\alpha|!^\sigma}{\delta^{|\alpha|}}, \quad -\delta^2 \leq t \leq \delta^2 \tag{4.3}$$

for $|\alpha| = 0, 1$ and additionally that

$$||\partial_x^\alpha v(\cdot, t)||_{L^\infty(B_{2\delta})} + ||\partial_x^\alpha p(\cdot, t)||_{L^\infty(B_{2\delta})} \leq \tilde{\epsilon}, \quad -\delta^2 \leq t \leq \delta^2 \tag{4.4}$$

for $|\alpha| = 0, 1$ and some sufficiently small $\tilde{\epsilon} \in (0, 1)$ such that $\tilde{\epsilon} \leq M$.

Lemma 4.1 *Suppose that the assumptions (4.1)–(4.4) are satisfied. If*

$$\tilde{\epsilon} \leq M \exp(-P_1(\delta^{-1}, M_1, M_2))$$

then

$$||v||_{L^2(B_1\times(-\delta^3,\delta^3))} + ||p||_{L^2(B_1\times(-\delta^3,\delta^3))} \leq P_2(\delta^{-1}, M, M_1, M_2) \tilde{\epsilon}^\theta M^{1-\theta}$$

where P_1 and P_2 are nonnegative polynomials and the parameter $\theta \in (0, 1)$ is such that $\theta \leq C\delta^m$.

Proof of Lemma 4.1 We use a change of variable

$$\tilde{x} = a(t)x, \qquad a(t) = \sqrt{1 + \frac{t^2}{\delta^2\eta^2}}$$

with the parameter $\eta = \delta^2$, which transforms the region $A(\delta, 2\delta) \times (-\eta, \eta)$ into a region between the two hyperboloids $|\tilde{x}|^2 - t^2/\eta^2 = \delta^2$ and $|\tilde{x}|^2 - t^2/\eta^2 = 4\delta^2$. In the new coordinates, we denote $u_j(x,t) = \tilde{u}_j(\tilde{x}, t)$, $v(x,t) = \tilde{v}(\tilde{x}, t)$, and $p(x,t) = \tilde{p}(\tilde{x}, t)$. Then, we have that $(\tilde{v}, \tilde{p})$ is a solution of the system

$$\begin{aligned} &\tilde{v}_t - a(t)^2\Delta\tilde{v} + \frac{t\tilde{x}}{\delta^2\eta^2 a(t)^2}\cdot\nabla\tilde{v} + a(t)\tilde{u}_1\cdot\nabla\tilde{v} + a(t)\tilde{v}\cdot\nabla\tilde{u}_2 + a(t)\nabla\tilde{p} = 0, \\ &-\Delta\tilde{p} - \partial_j\tilde{u}_{1i}\partial_i\tilde{v}_j - \partial_i\tilde{u}_{2j}\partial_j\tilde{v}_i = 0. \end{aligned} \tag{4.5}$$

Using the hypothesis (4.4), we obtain

$$|\partial_x^\alpha\tilde{v}(\tilde{x}, t)| + |\partial_x^\alpha\tilde{p}(\tilde{x}, t)| \le \frac{\tilde{\epsilon}}{a(t)^{|\alpha|}}, \quad (\tilde{x}, t) \in S \tag{4.6}$$

for $|\alpha| = 0, 1$, where the region S is defined by

$$S = \left\{(\tilde{x}, t)\colon \delta \le \frac{|\tilde{x}|}{a(t)} \le 2\delta, \delta \le |\tilde{x}| \le 2\right\}.$$

For the regions enclosed by the two hyperboloids and the cylinder $B_r \times (-\delta^2, \delta^2)$ with radius $r \in (2\delta, 3]$, we use, respectively, the notation

$$O_l(r) = \left\{(\tilde{x}, t)\colon \delta \le |\tilde{x}| \le r, -\eta\sqrt{|\tilde{x}|^2 - \delta^2} \le t \le \eta\sqrt{|\tilde{x}|^2 - \delta^2}\right\}$$

and

$$O_s(r) = \left\{(\tilde{x}, t)\colon 2\delta \le |\tilde{x}| \le r, -\eta\sqrt{|\tilde{x}|^2 - 4\delta^2} \le t \le \eta\sqrt{|\tilde{x}|^2 - 4\delta^2}\right\}.$$

Now, using Lemma 3.2, we establish a Carleman estimate for parabolic operators with the principal part $\partial_t - a(t)^2\Delta$. We change the time variable $\tilde{t} = A(t)$, where $A(t)$ is the solution of the equation $A'(t) = a(A(t))^{-2}$ with an initial condition $A(0) = 0$; the solution is given implicitly by $A(t) + (1/3)\delta^{-2}\eta^{-2}A(t)^3 = t$. Then, for $v(x,t) = \tilde{v}(x, \tilde{t})$, we obtain

$$\partial_t v - \Delta v = \frac{1}{a(\tilde{t})^2}\left(\partial_{\tilde{t}}\tilde{v} - a(\tilde{t})^2\Delta\tilde{v}\right)$$

provided $-(1/C)t \le A(t) \le Ct$ for $t \le \delta^3$ and $-(1/C)\delta^{2/3}\eta^{2/3}t^{1/3} \le A(t) \le C\delta^{2/3}\eta^{2/3}$ $t^{1/3}$ for $t \ge \delta^3$. By Lemma 3.2, we have

$$\begin{aligned} &\tau^3\int_{-A(\delta^2)}^{A(\delta^2)}\int_{A(\delta,1)} |\tilde{v}|^2 e^{2\tau\psi} + \tau\int_{-A(\delta^2)}^{A(\delta^2)}\int_{A(\delta,1)} |\nabla\tilde{v}|^2 e^{2\tau\psi} \\ &\qquad \le C\int_{-A(\delta^2)}^{A(\delta^2)}\int_{A(\delta,1)} |\partial_{\tilde{t}}\tilde{v} - a(\tilde{t})^2\Delta\tilde{v}|^2 e^{2\tau\psi} \end{aligned} \tag{4.7}$$

for all $\tilde{v} \in C_0^\infty(A(\delta, 1) \times (-A(\delta^2), A(\delta^2)))$ and $\tau \ge \tau_0\delta^{-2}$.

Next, we determine η so that $\eta \le A(\delta^2)$. More precisely, we choose the critical value $\eta = C\delta^2$. Let $\phi \in C_0^\infty(B_2 \times (-\delta^2, \delta^2))$ be a smooth cut-off function such that $\phi \equiv 1$ in a neighborhood of $O_s(1.5)$ and $\phi \equiv 0$ in a neighborhood of $O_l(2)^c$. Additionally, for $(\tilde{x}, t) \in S$ we assume that $|D_x^\alpha \phi(\tilde{x}, t)| \le C\delta^{-|\alpha|}$ if $|\alpha| = 0, 1$ and $|\partial_t \phi(\tilde{x}, t)| \le C\delta^{-3}$. Then $\tilde{v}\phi$ and $\tilde{p}\phi$ are infinitely smooth functions with compact supports in $A(\delta, 2) \times (-\delta^2, \delta^2)$. Using a linear change of variable, we have that (3.1) and (4.7) are valid for functions in $C_0^\infty(B_2 \times (-\delta^2, \delta^2))$. Adding the two Carleman-type estimates (3.1) and (4.7) for $\tilde{p}\phi$ and $\tilde{v}\phi$, respectively, we obtain

$$\begin{aligned}\tau^3 \int_{-\delta^2}^{\delta^2} \int_{A(\delta,2\delta)} (|\tilde{v}\phi|^2 + |\tilde{p}\phi|^2) e^{2\tau\psi} + \tau \int_{-\delta^2}^{\delta^2} \int_{A(\delta,2\delta)} (|\nabla(\tilde{v}\phi)|^2 + |\nabla(\tilde{p}\phi)|^2) e^{2\tau\psi} \\ \le C \int_{-\delta^2}^{\delta^2} \int_{A(\delta,2\delta)} (|\partial_t(\tilde{v}\phi) - a(t)^2 \Delta(\tilde{v}\phi)|^2 + |\Delta(\tilde{p}\phi)|^2) e^{2\tau\psi}\end{aligned} \tag{4.8}$$

for $\tau \ge \tau_0 \delta^{-4}$. We estimate the right side of (4.8) from above by $I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &= C \int_{O_s(1.5)} \left(|\partial_t(\tilde{v}\phi) - a(t)^2 \Delta(\tilde{v}\phi)|^2 + |\Delta(\tilde{p}\phi)|^2\right) e^{2\tau\psi}, \\ I_2 &= C \int_S \left(|\partial_t(\tilde{v}\phi) - a(t)^2 \Delta(\tilde{v}\phi)|^2 + |\Delta(\tilde{p}\phi)|^2\right) e^{2\tau\psi}, \\ I_3 &= C \int_{O_s(2)\setminus\overline{O_s(1.5)}} \left(|\partial_t(\tilde{v}\phi) - a(t)^2 \Delta(\tilde{v}\phi)|^2 + |\Delta(\tilde{p}\phi)|^2\right) e^{2\tau\psi}. \end{aligned}$$

Using the hypotheses (4.2) and (4.4), and that $\phi \equiv 1$ in a neighborhood of $O_s(1.5)$, we obtain

$$\begin{aligned} I_1 &= C \int_{O_s(1.5)} (|\tilde{v}_t - a(t)^2 \Delta\tilde{v}|^2 + |\Delta\tilde{p}|^2) e^{2\tau\psi} \\ &\le C \int_{O_s(1.5)} \Bigg(\left|\frac{t\tilde{x}}{\delta^4 a(t)^2}\right|^2 |\nabla\tilde{v}|^2 + a(t)^2 |\tilde{u}_1|^2 |\nabla\tilde{v}|^2 + a(t)^2 |\nabla\tilde{u}_2|^2 |\tilde{v}|^2 + a(t)^2 |\nabla\tilde{p}|^2 \\ &\qquad + |\nabla\tilde{u}_1|^2 |\nabla\tilde{v}|^2 + |\nabla\tilde{u}_2|^2 |\nabla\tilde{v}|^2\Bigg) e^{2\tau\psi} \\ &\le (C\delta^{-4} + C(M_1^2 + M_2^2)\delta^{-2}) \int_{O_s(1.5)} |\nabla\tilde{v}|^2 e^{2\tau\psi} + CM_2^2 \delta^{-2} \int_{O_s(1.5)} |\tilde{v}|^2 e^{2\tau\psi} \\ &\qquad + C \int_{O_s(1.5)} |\nabla\tilde{p}|^2 e^{2\tau\psi} \end{aligned}$$

which can be absorbed in the half of the left side of (4.8) provided

$$\tau \ge \max\left\{C(M_1^2 + M_2^2)\delta^{-2}, \tau_0 \delta^{-4}\right\}. \tag{4.9}$$

For the second integral, we have

$$
\begin{aligned}
I_2 \le C \int_S & \left(|\tilde{v}_t - a(t)^2 \Delta \tilde{v}|^2 + |\partial_t \phi|^2 |\tilde{v}|^2 + |a(t)^2 \Delta \phi|^2 |\tilde{v}|^2 + |(a(t)^2 \nabla \phi)|^2 |\nabla \tilde{v}|^2 \right) e^{2\tau\psi} \\
& + C \int_S \left(|\Delta \tilde{p}|^2 + |\nabla \phi|^2 |\nabla \tilde{p}|^2 + |\Delta \phi|^2 |\tilde{p}|^2 \right) e^{2\tau\psi} \\
\le C \int_S & \left(\left| \frac{t\tilde{x}}{\delta^4 a(t)^2} \right|^2 |\nabla \tilde{v}|^2 + a(t)^2 |\tilde{u}_1|^2 |\nabla \tilde{v}|^2 + a(t)^2 |\nabla \tilde{u}_2|^2 |\tilde{v}|^2 + a(t)^2 |\nabla \tilde{p}|^2 \right) e^{2\tau\psi} \\
& + C\delta^{-6} \tilde{\epsilon}^2 e^{2\tau\delta^{-m}} |S| + C \int_S \left(|\nabla \tilde{u}_1|^2 |\nabla \tilde{v}|^2 + |\nabla \tilde{u}_2|^2 |\nabla \tilde{v}|^2 \right) e^{2\tau\psi} \\
\le C (\delta^{-4} & + M_1^2 + M_2^2) \tilde{\epsilon}^2 e^{2\tau\delta^{-m}},
\end{aligned}
$$

where we used the estimates on the coefficients (4.2) and (4.6), $\psi(\tilde{x}) \le \delta^{-m}$ for $(\tilde{x}, t) \in S$, and $|S| \le \delta^2$. Finally, the assumptions (4.2), (4.3), $\psi(\tilde{x}) \le 1.5^{-m}$ for $(\tilde{x}, t) \in O_s(2) \setminus \overline{O_s(1.5)}$, and $|O_s(2) \setminus \overline{O_s(1.5)}| \le \delta^2$ imply

$$
\begin{aligned}
I_3 \le C & \int_{O_s(2)\setminus\overline{O_s(1.5)}} \left(|\tilde{v}_t - a(t)^2 \Delta \tilde{v}|^2 + |\partial_t \phi|^2 |\tilde{v}|^2 + |a(t)^2 \Delta \phi|^2 |\tilde{v}|^2 + |(a(t)^2 \nabla \phi)|^2 |\nabla \tilde{v}|^2 \right) e^{2\tau\psi} \\
& + C \int_{O_s(2)\setminus\overline{O_s(1.5)}} \left(|\Delta \tilde{p}|^2 + |\nabla \phi|^2 |\nabla \tilde{p}|^2 + |\Delta \phi|^2 |\tilde{p}|^2 \right) e^{2\tau\psi} \\
\le C & \left(\delta^{-4} + M_1^2 + M_2^2 \right) M^2 e^{2\tau 1.5^{-m}}
\end{aligned}
$$

for the third integral. Hence,

$$
\tau^3 \int_{-\delta^3}^{\delta^3} \int_{A(2\delta,1)} \left(|\tilde{v}|^2 + |\tilde{p}|^2 \right) e^{2\tau\psi} \le P_2(\delta^{-1}, M_1, M_2) \left(\tilde{\epsilon}^2 e^{2\tau\delta^{-m}} + M^2 e^{2\tau 1.5^{-m}} \right)
$$

provided (4.9) holds. Note that $\psi \ge 1$ on $A(2\delta, 1)$. Dividing both sides of the above inequality by $\tau^3 \exp(2\tau)$, we get

$$
\int_{-\delta^3}^{\delta^3} \int_{A(2\delta,1)} \left(|\tilde{v}|^2 + |\tilde{p}|^2 \right) \le P_2(\delta^{-1}, M_1, M_2) \left(\tilde{\epsilon}^2 e^{2\tau(\delta^{-m}-1)} + M^2 e^{2\tau(1.5^{-m}-1)} \right).
$$

Now, we choose τ such that $\tilde{\epsilon} = M \exp(\tau(1.5^{-m} - \delta^{-m}))$ and this τ satisfies (4.9) if $\tilde{\epsilon}$ is sufficiently small. Then

$$
\int_{-\delta^3}^{\delta^3} \int_{A(2\delta,1)} \left(|\tilde{v}|^2 + |\tilde{p}|^2 \right) \le P_2(\delta^{-1}, M_1, M_2) \tilde{\epsilon}^{2\theta} M^{2-2\theta}
$$

for $\theta = (1 - 1.5^{-m})/(\delta^{-m} - 1.5^{-m})$ provided

$$
\log \frac{M}{\tilde{\epsilon}} \ge \max \left\{ C(M_1^2 + M_2^2)\delta^{-2}, \tau_0 \delta^{-4} \right\} (\delta^{-m} - 1.5^{-m}) = P_1(\delta^{-1}, M_1, M_2).
$$

Therefore, the lemma is proven. □

Proof of Theorem 2.1 Suppose that the hypotheses of Theorem 2.1 are satisfied. Define

$$\epsilon = \sup_{t\in[-\delta^2,\delta^2]} \frac{||v(\cdot,t)||_{L^\infty(B_{4\delta})}}{||v(\cdot,t)||_{L^2(B_2)}}. \tag{4.10}$$

To establish the statement, it suffices to prove that $\epsilon \geq P(\delta^{-1}, K, M, M_1, M_2)$ for a non-negative polynomial P. By (4.10), we have

$$||v(\cdot,t)||_{L^\infty(B_{4\delta})} \leq \epsilon ||v(\cdot,t)||_{L^2(B_2)}, \quad -\delta^2 \leq t \leq \delta^2$$

for some $\epsilon \in (0, 1]$. Then, as in the proof of [28, Lemma 3.1], we obtain

$$|\partial_x^\alpha v(x,t)| \leq \frac{C^{|\alpha|+1}\epsilon^{1/2}M^{1/2}|\alpha|!^\sigma}{\delta^{|\alpha|}} ||v(\cdot,t)||_{L^2(B_2)}, \quad x \in B_{2\delta},\ -\delta^2 \leq t \leq \delta^2$$

and

$$|\partial_t^m v(x,t)| \leq \frac{C^{m+1}\epsilon^{1/2}M^{1/2}m!^\sigma}{\delta^{2m}} ||v(\cdot,t)||_{L^2(B_2)}, \quad x \in B_{4\delta},\ -\delta^2/2 \leq t \leq \delta^2/2.$$

Without loss of generality, we assume that $p(0,t) = 0$ for all $t \in [-\delta^2, \delta^2]$. Now, using the parabolic equation from the system (2.2), we get

$$\begin{aligned}
&||\nabla p(\cdot,t)||_{L^\infty(B_{2\delta})} \\
&\quad \leq ||\partial_t v(\cdot,t)||_{L^\infty(B_{2\delta})} + ||\Delta v(\cdot,t)||_{L^\infty(B_{2\delta})} + ||u_1\cdot\nabla v(\cdot,t)||_{L^\infty(B_{2\delta})} + ||v\cdot\nabla u_2(\cdot,t)||_{L^\infty(B_{2\delta})} \\
&\quad \leq \frac{C\epsilon^{1/2}M^{1/2}}{\delta^2} \max\{1, M_1, M_2\} ||v(\cdot,t)||_{L^2(B_2)}
\end{aligned}$$

for $t \in [-\delta^2/2, \delta^2/2]$. Thus, we obtain

$$||\partial_x^\alpha v(\cdot,t)||_{L^\infty(B_{2\delta})} + ||\partial_x^\alpha p(\cdot,t)||_{L^\infty(B_{2\delta})} \leq \tilde{\epsilon} ||v(\cdot,t)||_{L^2(B_2)}$$

for all $t \in [-\delta^2/2, \delta^2/2]$, where

$$\tilde{\epsilon} = \max_{|\alpha|=0,1} \left\{ \epsilon, \frac{C^{|\alpha|+1}\epsilon^{1/2}M^{1/2}(|\alpha|+1)!^\sigma}{\delta^{|\alpha|+1}} \max\{1, M_1, M_2\} \right\} \tag{4.11}$$

Denote

$$K_0 = \max_{|\alpha|=0,1} \max_{t\in[-\delta^2/2,\delta^2/2]} \frac{\delta^{|\alpha|}}{|\alpha|!^\sigma} \left(||\partial_x^\alpha v(\cdot,t)||_{L^\infty(B_2)} + ||\partial_x^\alpha p(\cdot,t)||_{L^\infty(B_2)} \right)$$

and set $(\tilde{v}, \tilde{p}) = K_0^{-1}(v, p)$. Clearly, $(\tilde{v}, \tilde{p})$ also solves the system (2.1) and

$$||\partial_x^\alpha \tilde{v}(\cdot,t)||_{L^\infty(B_2)} + ||\partial_x^\alpha \tilde{p}(\cdot,t)||_{L^\infty(B_2)} \leq \frac{|\alpha|!^\sigma}{\delta^{|\alpha|}}, \quad -\delta^2/2 \leq t \leq \delta^2/2, \quad |\alpha| = 0, 1.$$

Then

$$\begin{aligned}
&||\partial_x^\alpha \tilde{v}(\cdot,t)||_{L^\infty(B_{2\delta})} + ||\partial_x^\alpha \tilde{p}(\cdot,t)||_{L^\infty(B_{2\delta})} \\
&\quad \leq \frac{1}{K_0} \left(||\partial_x^\alpha v(\cdot,t)||_{L^\infty(B_{2\delta})} + ||\partial_x^\alpha p(\cdot,t)||_{L^\infty(B_{2\delta})} \right) \leq \tilde{\epsilon}
\end{aligned}$$

for $t \in [-\delta^2/2, \delta^2/2]$ and $|\alpha| = 0, 1$. Now, Lemma 4.1 implies

$$||\tilde{v}||_{L^2(B_1)\times(-\delta^3,\delta^3)} \leq P_2(\delta^{-1}, M_1, M_2)\tilde{\epsilon}^\theta$$

provided

$$\tilde{\epsilon} \leq \exp(-P_1(\delta^{-1}, M_1, M_2)). \tag{4.12}$$

Also, by the hypothesis (2.3), we get

$$K_0 \leq M \left(||v(\cdot, t)||_{L^2(B_2)} + ||p(\cdot, t)||_{L^2(B_2)} \right), \quad -\delta^2/2 \leq t \leq \delta^2/2.$$

Hence, we obtain the estimate

$$\begin{aligned} ||v(\cdot, t)||_{L^2(B_2)} &\leq K||v(\cdot, t)||_{L^2(B_1)} \leq K K_0 ||\tilde{v}(\cdot, t)||_{L^2(B_1)} \leq \frac{C K K_0}{\delta^{3/2}} ||\tilde{v}||_{L^2(B_1 \times (-\delta^3, \delta^3))} \\ &\leq \frac{C K K_0}{\delta^{3/2}} P_2(\delta^{-1}, M_1, M_2) \tilde{\epsilon}^\theta \leq \frac{C K M}{\delta^{3/2}} P_2(\delta^{-1}, M_1, M_2) \tilde{\epsilon}^\theta \left(||v(\cdot, t)||_{L^2(B_2)} + ||p(\cdot, t)||_{L^2(B_2)} \right) \end{aligned}$$

for all $-\delta^2/2 \leq t \leq \delta^2/2$, which holds only if

$$\tilde{\epsilon} \geq \frac{\delta^{3/2\theta}}{(C M K)^{1/\theta}} P_2(\delta^{-1}, M_1, M_2)^{-1/\theta}. \tag{4.13}$$

Therefore, by (4.12) and (4.13)

$$\tilde{\epsilon} \geq \min \left\{ \exp(-P_1(\delta^{-1}, M_1, M_2)), \frac{\delta^{3/2\theta}}{(C M K)^{1/\theta}} P_2(\delta^{-1}, M_1, M_2)^{-1/\theta} \right\}.$$

Using (4.11), we solve the last inequality for ϵ. We conclude that $\epsilon \geq \exp(-P(\delta^{-1}, K, M, M_1, M_2))$ for a non-negative polynomial P of degree in δ^{-1} depending only on m. □

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