Advances in Differential Equations Volume 15, Numbers 9–10 (2010), 953–975

UNIQUE CONTINUATION AND COMPLEXITY OF SOLUTIONS TO PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH GEVREY COEFFICIENTS

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(Submitted by: Roger Temam)

Abstract. In this paper, we provide a quantitative estimate of unique continuation (doubling property) for higher-order parabolic partial differential equations with non-analytic Gevrey coefficients. Also, a new upper bound is given on the number of zeros for the solutions with a polynomial dependence on the coefficients.

1. Introduction

In this paper, we address the spatial complexity of solutions of 1D higher-order parabolic partial differential equations with Gevrey coefficients in the case of periodic boundary conditions

$$u_t + (-1)^s \partial_x^{2s} u + \sum_{k=0}^{2s-1} v_k(x,t) \partial_x^k u = 0 \tag{1.1}$$

for $(x,t) \in \mathbb{R} \times [-\delta^{2s}, \delta^{2s}]$, where $s \in \mathbb{N}$ and $\delta \in (0, 1/2]$. Even though functions from the Gevrey class may not satisfy the unique continuation property, we prove that the solutions of (1.1) do, under a very mild assumption that the Gevrey exponent is less than a universal constant. In particular, we obtain a polynomial estimate on the size of the zero sets of solutions in terms of the coefficients.

The study of complexity of solutions of elliptic and parabolic partial differential equations, through estimating the size of their nodal (zero or vanishing) sets, has been initiated by Donnelly and Fefferman ([5, 6, 7]). In the case of a real analytic compact, connected Riemannian n-manifold, they proved that the $(n-1)$-dimensional measure of the nodal set of an eigenfunction of the Laplacian with corresponding eigenvalue λ, is bounded from above and below by a constant multiple of $\sqrt{\lambda}$. For a general second-order

Accepted for publication: March 2010.
AMS Subject Classifications: 35B05, 35B60, 35K25, 35K55.

linear elliptic equation with only smooth coefficients, Hardt and Simon [14] established that the volume of the zero set of a nontrivial solution is finite in a neighborhood of any point for which the solution has finite order of vanishing. In [27], considering the Laplace and the heat equation on a compact real analytic manifold, F.-H. Lin revealed the relationship between the volume of nodal sets of solutions and their frequency (see also [13] for linear parabolic equations with non-analytic coefficients).

In the previous papers [18, 19], the second author provided estimates on the spatial complexity of solutions of the second-order parabolic equations of the type

$$\partial_t u - \Delta u = w \cdot \nabla u + vu \tag{1.2}$$

with analytic coefficients. Namely, for a complex-valued solution of the Ginzburg-Landau equation

$$\partial_t u = (1 + i\nu)u_{xx} - (1 + i\mu)|u|^2 u + au$$

with periodic boundary conditions, he obtained (in [18]) a polynomial bound on its winding number in terms of the bifurcation parameters μ and a for any fixed ν. In [19], a polynomial bound on the size of the vorticity nodal sets $\{x \in \Omega : \omega(x,t) = 0\}$, depending on the initial condition, viscosity, the size of Ω, and $1/t$, was given for the solutions of the 2D Navier-Stokes equations written in the vorticity form. The proofs employed a modification of a unique continuation method for the equation (1.2) due to Kurata [25] and a self-similar transformation of variables. Regarding the solutions u of the higher-order equations with analytic coefficients (1.1), with spatial periodicity $L > 0$, it is proven in [20] that the length of the level sets $\{x \in [0, L] : u(x,t) = \lambda\}$ can be bounded by a polynomial function on L and the coefficients for all $\lambda \in \mathbb{R}$.

There is a close relationship between the study of nodal sets and the unique continuation for solutions of elliptic and parabolic equations. To obtain a bound on the volumes of the nodal set of a solution of such a PDE, it must satisfy the strong unique continuation property; that is, if a solution vanishes at the infinite order in a point, then it is the trivial solution. There is a rich literature on this subject (cf. [8, 16, 22, 28] and the review papers by Kenig [23, 24]).

We would like to mention a recent paper of Colombini and Koch [4], in which the authors consider products of elliptic operators with coefficients in the Gevrey class G^σ and prove a strong unique continuation property provided $\sigma < 1 + 1/\alpha$ for some $\alpha > 0$. Their result relies on an estimate, obtained by iteration of a Carleman-type inequality (cf. [4, Proposition 3.1]) for second-order elliptic operators (this approach does not apply to parabolic equations).

In the present paper, we remove the restricting analyticity requirements from a previous result of the second author (cf. [20]) on the complexity of the solutions of the equation (1.1). If we follow the approach in [20], the estimate on the derivatives [20, Lemma 6.1] does not close. We overcome this difficulty by shrinking the time interval by a variable factor depending on the size of the solution (cf. Lemma 3.1 below). Another obstacle we face is that the solutions of (1.1) have only non-analytic Gevrey regularity and the Gevrey class of functions may not have the unique continuation property. Moreover, the classical approach (cf. [17, 5, 6]) relies on complex analysis methods which do not apply here. To overcome this, we use an interpolation technique (cf. Theorem 2.4). A result of independent interest is a strong unique continuation property for the equation (1.1) with coefficients in G^σ with $1 \leq \sigma \leq 1 + \eta$ for some $\eta > 0$, obtained by using a Carleman estimate, which is used classically only for weak continuation results. We emphasize that the polynomial upper bound on the number of zeros is new including for the equations of second order (the papers [18, 19] require analyticity).

The paper is organized as follows. In Section 2, we state our main results, Theorems 2.1 and 2.4. The following two sections contain auxiliary results for the proof of Theorem 2.1. In Section 3, we prove smallness of all space derivatives of the solution $u(x,t)$ in a small space-time rectangle provided the solution is small on an interval for a fixed time $t = 0$. To establish the propagation of smallness on space-time rectangles in Section 4, our main tool is a Carleman-type estimate for higher-order parabolic equations due to Isakov [15] (cf. also [12, 28]), adopted to a certain region between two parabolas. In Section 5, we give the proof of the quantitative property stated in Theorem 2.1. We develop the techniques used in [20] to the case of equations with Gevrey coefficients. Then, we prove Theorem 2.4, which is an independent result addressing quantitative uniqueness and the number of zeros for functions in a Gevrey class.

2. Notation and the main result

In this paper, we consider the 1D higher-order parabolic partial differential equation with possibly non-analytic coefficients in the Gevrey class G^σ with $\sigma \geq 1$

$$u_t + (-1)^s \partial_x^{2s} u + \sum_{k=0}^{2s-1} v_k(x,t) \partial_x^k u = 0 \qquad (2.1)$$

for $(x,t) \in \mathbb{R} \times [-\delta^{2s}, \delta^{2s}]$, where $s \in \mathbb{N}$ and $\delta \in (0, 1/2]$. Let $u(x,t)$ be a periodic solution, with period 1 in the x variable, of the equation (2.1), and denote by Ω the spatial interval of periodicity 1 which we may, without loss of generality, take to be $\Omega = (-1/2, 1/2)$. We assume that u is an infinitely smooth function of (x,t) for which $u(\cdot,0)$ is not identically zero and that there exists a constant $M > 0$ such that

$$\|\partial_t^n \partial_x^m u(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{M n!^\sigma m!^\sigma}{\delta^{2sn+m}} \|u(\cdot,t)\|_{L^2(\Omega)} \tag{2.2}$$

for $-\delta^{2s} \leq t \leq \delta^{2s}$ and $n, m \in \mathbb{N}_0$ where $\sigma \geq 1$ is fixed. Also, we assume that the coefficients are infinitely smooth functions of (x,t) and that for all $k = 0, 1, \ldots, 2s-1$ there exist constants $M_k > 0$ such that

$$\|\partial_t^n \partial_x^m v_k(\cdot,t)\|_{L^\infty(\Omega)} \leq \frac{M_k n!^\sigma m!^\sigma}{\delta^{2sn+m}} \tag{2.3}$$

for $-\delta^{2s} \leq t \leq \delta^{2s}$ and $n, m \in \mathbb{N}_0$. Assume

$$\|u(\cdot,t_1)\|_{L^2(\Omega)} \leq K \|u(\cdot,t_2)\|_{L^2(\Omega)} \tag{2.4}$$

for $-\delta^{2s} \leq t_1, t_2 \leq \delta^{2s}$ and some constant $K \geq 1$. Under the above assumptions we give the following quantitative estimate of unique continuation for the parabolic equation (2.1) with only Gevrey coefficients.

Theorem 2.1. *Suppose that u satisfies* (2.1)–(2.4). *If $\sigma \leq 1+\eta$, where $\eta > 0$ is a universal constant, then*

$$\|u(\cdot,0)\|_{L^\infty(\Omega)} \leq \exp(P(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K)) \|u(\cdot,0)\|_{L^\infty(-\delta,\delta)}$$

for some non-negative polynomial P of degree in δ^{-1} at most a constant.

Remark 2.2. The motivation for studying (2.1) is some pattern formation equations, that is, the Kuramoto-Sivashinsky and Cahn-Hilliard equations (cf. [29]).

Remark 2.3. The natural condition (2.4) prevents highly oscillating quickly decaying solutions. Using this condition, together with (2.1) and (2.3), we can obtain (2.2) with a certain explicit δ by first proving the Gevrey regularity on x for small δ and then using the bounds on the mixed space-time derivatives provided in [20, Lemma 4.1]. (For Gevrey class regularity and analyticity of solutions of various nonlinear PDE cf. for instance [2, 3, 9, 10, 11, 21, 26].)

The next theorem provides a new estimate for the order of vanishing and for the number of zeros for Gevrey functions which is of independent interest. Note that the Gevrey functions in general do not satisfy the unique continuation property (for instance, the function $\exp(-|x|^{1/(1-\sigma)})$ belongs to $G^\sigma(\mathbb{R})$ for $\sigma > 1$, is not identically zero, and has a zero of infinite order at $x = 0$).

For any 1-periodic function $f\colon \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, we denote by $\mathrm{ord}_x f$ the order of vanishing (i.e., order of the zero) of f at the point x.

Theorem 2.4. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable 1-periodic function which is not identically zero. Let $a, b \geq 0$ and $1 \leq \sigma \leq 1 + 1/b$. If $\sigma = 1 + 1/b$, we assume that $4^{b+1}a/\delta^b \leq 1/2$. Suppose that there exist constants $M \geq 1$ and $\delta \in (0, 1/2]$ such that*

$$\|f^{(n)}\|_{L^\infty(\Omega)} \leq \frac{M n!^\sigma}{\delta^n} \|f\|_{L^\infty(\Omega)}, \qquad n \in \mathbb{N}_0, \tag{2.5}$$

and

$$\|f\|_{L^\infty(\Omega)} \leq \exp\Big(\frac{a}{\rho^b}\Big) \|f\|_{L^\infty[x_0-\rho/2, x_0+\rho/2]} \tag{2.6}$$

for all $\rho \in (0, \delta]$ and $x_0 \in \Omega$. Then for the number of zeros of f in Ω, we have

$$\mathrm{card}\,\{x \in \Omega : f(x) = 0\} \leq C K^{1+1/b}, \tag{2.7}$$

where

$$K = \Big(\frac{4^{b+1}a}{\delta^b}\Big)^{1/(1+b(1-\sigma))} + \frac{4^{b+1}a}{\delta^b} + 2\log M + 2 \tag{2.8}$$

and $C = C(a, b)$. The first term in (2.8) is understood to be zero if $\sigma = 1 + 1/b$. Moreover, we have an upper bound

$$\mathrm{ord}_{x_0} f \leq K \tag{2.9}$$

for the order of vanishing $\mathrm{ord}_{x_0} f$ for every $x_0 \in \Omega$.

Above and in the sequel, the symbol C denotes a generic constant which may depend on s.

Theorems 2.1 and 2.4 are proven in Section 5 below.

Remark 2.5. In the case $\sigma = 1$, the function f is analytic and using only (2.5) we get

$$\mathrm{card}\,\{x \in \Omega : f(x) = 0\} \leq C \log(M + 1) \exp(C/\delta)$$

for the number of zeros of f in Ω (cf. [17]).

Combining Theorems 2.1 and 2.4, we obtain an upper bound for the number of zeros of the solutions of the equation (2.1) with a polynomial dependence on the coefficients.

Corollary 2.6. *Let u be as above and $1 \le \sigma \le 1 + \zeta$, where $\zeta > 0$ is a constant depending on s. Then for the number of zeros of $u(\cdot, 0)$ in Ω we have*

$$\operatorname{card}\{x \in \Omega : u(\cdot, 0) = 0\} \le Q(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K),$$

where Q is a nonnegative polynomial.

Proof of Corollary 2.6. By the assumption (2.2), we have

$$\|\partial_x^m u(\cdot, 0)\|_{L^\infty(\Omega)} \le \frac{M m!^\sigma}{\delta^m}\|u(\cdot, 0)\|_{L^2(\Omega)} \le \frac{M m!^\sigma}{\delta^m}\|u(\cdot, 0)\|_{L^\infty(\Omega)}.$$

Let $\rho \in (0, \delta]$ be fixed. Then Theorem 2.1 implies

$$\begin{aligned}\|u(\cdot, 0)\|_{L^\infty(\Omega)} &\le \exp\left(P(2\rho^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K)\right)\|u(\cdot, 0)\|_{L^\infty(-\rho/2, \rho/2)} \\ &\le \exp\left(\frac{a}{\rho^{Cs}}\right)\|u(\cdot, 0)\|_{L^\infty(-\rho/2, \rho/2)},\end{aligned}$$

where Cs is the highest power of the polynomial $P(\rho^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K)$ with respect to ρ^{-1} and a is a constant depending on $M, \{M_k\}_{k=0}^{2s-1}$, and K. The claim follows from Theorem 2.4. □

Remark 2.7. Note that (2.4) can be derived from (2.2), (2.3), and the equation

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2(\Omega)}^2 + \|\partial_x^s u\|_{L^2(\Omega)}^2 + \sum_{k=0}^{2s-1}(v_k(\cdot, t)\partial_x^k u, u)_{L^2(\Omega)} = 0, \tag{2.10}$$

obtained by multiplying (2.1) by u and integrating. More precisely, the lower bound on the rate of decay

$$\|u(\cdot, t)\|_{L^2(\Omega)} \ge \exp(-C_1(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1})(t - t_1))\|u(\cdot, t_1)\|_{L^2(\Omega)}, \quad t \ge t_1$$

follows directly from (2.10) and the bound

$$\begin{aligned}&\|\partial_x^s u\|_{L^2(\Omega)}^2 + \sum_{k=0}^{2s-1}(v_k(\cdot, t)\partial_x^k u, u)_{L^2(\Omega)} \\ &\quad\le \frac{M^2 s!^{2\sigma}}{\delta^{2s}}\|u\|_{L^2(\Omega)}^2 + \sum_{k=0}^{2s-1}\|v_k\|_{L^\infty(\Omega)}\|\partial_x^k u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\ &\quad\le C_1(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1})\|u\|_{L^2(\Omega)}^2.\end{aligned}$$

In order to get an estimate

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le \exp(C_2(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1})(t - t_1))\|u(\cdot, t_1)\|_{L^2(\Omega)}, \quad t \ge t_1,$$

we observe that the equation (2.10) implies

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} \le \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + \|\partial_x^s u\|^2_{L^2(\Omega)} &\le \sum_{k=0}^{2s-1} |(v_k(\cdot,t)\partial_x^k u, u)_{L^2(\Omega)}| \\ &\le C_2(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1})\|u\|^2_{L^2(\Omega)},\end{aligned}$$

where we also used the assumptions (2.2) and (2.3).

Remark 2.8. We note that, under the assumptions (2.2)–(2.4), the main results in this paper (in particular Theorem 2.1 and Corollary 2.6) also apply to the equation

$$u_t + \partial_x^{2s+1} u + \sum_{k=0}^{2s} v_k(x,t)\partial_x^k u = 0$$

which has odd highest-order derivatives in the x variable, for some fixed $s \in \mathbb{N}$.

3. Smallness of space derivatives

Under the assumptions from Section 2, we first prove smallness of all space derivatives of the solution $u(x,t)$ in a small space-time rectangle provided the solution is small on an interval for a fixed time $t = 0$. Namely, we have the following statement.

Lemma 3.1. *Assume*

$$|u(x,0)| \le \epsilon \|u(\cdot,0)\|_{L^2(\Omega)}, \qquad x \in (-\delta,\delta) \tag{3.1}$$

for some $\epsilon \in (0,1/4]$. Then for all $j_0 \in \{0,1,\ldots,2s-1\}$, we have

$$|\partial_x^{j_0} u(x,t)| \le F_{j_0}\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right) \|u(\cdot,0)\|_{L^2(\Omega)} \tag{3.2}$$

for $x \in (-\delta/2, \delta/2)$ and $t \in (-\delta^{2s}/|\log\epsilon|^\theta, \delta^{2s}/|\log\epsilon|^\theta)$, where

$$\begin{aligned}&F_{j_0}\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right) \\ &\quad = \frac{M^{1/2}}{\delta^{j_0}} \exp\Big(-\frac{1}{4}|\log\epsilon| + C\Big(\log\big(1+\sum_{k=0}^{2s-1} M_k\big)\Big)^{1/(1-\omega)}\Big) \\ &\qquad + \frac{CMK}{\delta^{j_0}} \exp\Big(-\frac{1}{C}|\log\epsilon|^\omega \log|\log\epsilon|\Big)\end{aligned}$$

and $\theta, \omega \in (0,1)$ are such that $\sigma \le 1 + \theta/16\omega$ and $\epsilon \in (0,1/C)$ with the constant C depending on θ and ω.

Proof. Using (2.2), we have

$$\|\partial_x^j u(\cdot,0)\|_{L^2(-\delta,\delta)} \le \frac{M(2\delta)^{1/2} j!^\sigma}{\delta^j}\|u(\cdot,0)\|_{L^2(\Omega)} \le \frac{C^j M j!^\sigma}{(2\delta)^{j-1/2}}\|u(\cdot,0)\|_{L^2(\Omega)}$$

and the hypothesis (3.1) gives

$$\|u(\cdot,0)\|_{L^2(-\delta,\delta)} \le \epsilon(2\delta)^{1/2}\|u(\cdot,0)\|_{L^2(\Omega)}.$$

Then, by the proof of [20, Lemma 3.1], it follows that

$$\|\partial_x^j u(\cdot,0)\|_{L^\infty(-\delta/2,\delta/2)} \le \frac{C^{j+1}M^{1/2}\epsilon^{1/2} j!^{\sigma+1}}{\delta^j}\|u(\cdot,0)\|_{L^2(\Omega)}, \qquad j\in\mathbb{N}_0.$$

Let $n_0 \ge 8s-2$ be fixed. Then the above inequality implies

$$|\partial_x^j u(x,0)| \le \frac{CM^{1/2}\epsilon^{1/2} j!^\sigma n_0!}{(\delta/C)^j}\|u(\cdot,0)\|_{L^2(\Omega)},$$

for $x\in(-\delta/2,\delta/2)$ and $j=0,1,\ldots,n_0$. Now, by the property (2.3) and by the proof of [20, Lemma 4.1], we have

$$|\partial_t^n\partial_x^m u(x,0)| \\ \le \frac{C^{2sn+m+1}M^{1/2}\epsilon^{1/2}n_0!\left(1+\sum_{k=0}^{2s-1}M_k\right)^n(2sn+m)!^\sigma}{\delta^{2sn+m}}\|u(\cdot,0)\|_{L^2(\Omega)} \tag{3.3}$$

for $x\in(-\delta/2,\delta/2)$ and $n,m\in\mathbb{N}_0$ such that $2sn+m\le n_0$. Next, we fix $j_0\in\{0,1,\ldots,2s-1\}$. By (2.2) and (2.4), we obtain

$$\left|\partial_t^n\partial_x^{j_0}u(x,t)\right| \le \frac{CMKn!^\sigma}{\delta^{2sn+j_0}}\|u(\cdot,0)\|_{L^2(\Omega)}, \tag{3.4}$$

for $(x,t)\in\mathbb{R}\times(-\delta^{2s},\delta^{2s})$ and $n\in\mathbb{N}_0$. Also, (3.3) gives

$$\left|\partial_t^i\partial_x^{j_0}u(x,0)\right| \\ \le \frac{C^{2si+j_0+1}M^{1/2}\epsilon^{1/2}n_0!\left(1+\sum_{k=0}^{2s-1}M_k\right)^i(2si+j_0)!^\sigma}{\delta^{2si+j_0}}\|u(\cdot,0)\|_{L^2(\Omega)} \tag{3.5}$$

for $x\in(-\delta/2,\delta/2)$ provided that $2si+j_0\le n_0$. From (3.4) we have that, for any fixed x, the function $\partial_x^{j_0}u(x,\cdot)$ is in the Gevrey class of order σ for $t\in(-\delta^{2s},\delta^{2s})$. The Taylor's formula for $\partial_x^{j_0}u(x,\cdot)$ with remainder gives

$$|\partial_x^{j_0}u(x,t)| \le \sum_{i=0}^n \frac{|\partial_t^i\partial_x^{j_0}u(x,0)|}{i!}|t|^i + \frac{|\partial_t^{n+1}\partial_x^{j_0}u(x,\xi)|}{(n+1)!}|t|^{n+1} \tag{3.6}$$

for all $x \in (-\delta/2, \delta/2)$, $t \in (-\delta^{2s}/|\log \epsilon|^\theta, \delta^{2s}/|\log \epsilon|^\theta)$, and some number $\xi \in (-\delta^{2s}/|\log \epsilon|^\theta, \delta^{2s}/|\log \epsilon|^\theta)$, where $\theta \in (0,1)$ is arbitrary. Using (3.4), (3.5), and (3.6), we obtain

$$\begin{aligned} |\partial_x^{j_0} u(x,t)| \leq \sum_{i=0}^{n} & \frac{C^{2si+j_0+1} M^{1/2} \epsilon^{1/2} n_0! \left(1 + \sum_{k=0}^{2s-1} M_k\right)^i (2si+j_0)!^\sigma}{i!\, \delta^{2si+j_0}} \\ & \times \left(\frac{\delta^{2s}}{|\log \epsilon|^\theta}\right)^i \|u(\cdot,0)\|_{L^2(\Omega)} \\ & + \frac{CMK(n+1)!^\sigma}{(n+1)!\, \delta^{2s(n+1)+j_0}} \left(\frac{\delta^{2s}}{|\log \epsilon|^\theta}\right)^{n+1} \|u(\cdot,0)\|_{L^2(\Omega)} \end{aligned} \tag{3.7}$$

for $x \in (-\delta/2, \delta/2)$, $t \in (-\delta^{2s}/|\log \epsilon|^\theta, \delta^{2s}/|\log \epsilon|^\theta)$, and $n_0 \in \mathbb{N}$ such that $n_0 \geq 8s-2$. Let n_1 be the largest integer such that $2sn_1 + j_0 \leq n_0$. Since $n_0 \geq 8s-2$ and $j_0 \leq 2s-1$, we have

$$\frac{n_0}{4s} \leq n_1 \leq \frac{n_0}{2s}. \tag{3.8}$$

After simplifying the right-hand side of (3.7) and replacing n with n_1, we obtain

$$\begin{aligned} |\partial_x^{j_0} u(x,t)| \leq & \frac{C^{n_0} M^{1/2} \epsilon^{1/2} n_0!^{\sigma+1} \left(1 + \sum_{k=0}^{2s-1} M_k\right)^{n_1}}{\delta^{j_0}} \|u(\cdot,0)\|_{L^2(\Omega)} \\ & + \frac{CMK(n_1+1)!^{\sigma-1}}{|\log \epsilon|^{\theta(n_1+1)} \delta^{j_0}} \|u(\cdot,0)\|_{L^2(\Omega)} \end{aligned} \tag{3.9}$$

for $x \in (-\delta/2, \delta/2)$, $t \in (-\delta^{2s}/|\log \epsilon|^\theta, \delta^{2s}/|\log \epsilon|^\theta)$, and $n_0 \in \mathbb{N}$ such that $n_0 \geq 8s-2$. Hence, by (3.8) and (3.9),

$$\begin{aligned} |\partial_x^{j_0} u(x,t)| \leq & \frac{C^{n_0} M^{1/2}}{\delta^{j_0}} \epsilon^{1/2} n_0^{(\sigma+1)n_0} \left(1 + \sum_{k=0}^{2s-1} M_k\right)^{n_0/2s} \|u(\cdot,0)\|_{L^2(\Omega)} \\ & + \frac{CMK}{|\log \epsilon|^{\theta(n_0/4s+1)} \delta^{j_0}} \left(\frac{n_0}{2s} + 1\right)^{(\sigma-1)(n_0/2s+1)} \|u(\cdot,0)\|_{L^2(\Omega)} \end{aligned} \tag{3.10}$$

for $x \in (-\delta/2, \delta/2)$, $t \in (-\delta^{2s}/|\log \epsilon|^\theta, \delta^{2s}/|\log \epsilon|^\theta)$, and all $n_0 \in \mathbb{N}$ such that $n_0 \geq 8s-2$.

Now, choose $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2}|\log \epsilon|^\omega \leq n_0 \leq |\log \epsilon|^\omega \tag{3.11}$$

for some $\omega \in (0,1)$ to be determined. For the first term on the right-hand side of (3.10), we obtain

$$
\begin{aligned}
&\frac{C^{n_0} M^{1/2}}{\delta^{j_0}}\, \epsilon^{1/2} n_0^{(\sigma+1)n_0} \Big(1+\sum_{k=0}^{2s-1} M_k\Big)^{n_0/2s} \qquad (3.12)\\
&\quad = \frac{M^{1/2}}{\delta^{j_0}} \exp\Big(-\frac{1}{2}|\log\epsilon| + (\sigma+1) n_0 \log n_0 + n_0 \log C \\
&\qquad\qquad + \frac{n_0}{2s} \log\Big(1+\sum_{k=0}^{2s-1} M_k\Big)\Big)\\
&\quad \le \frac{M^{1/2}}{\delta^{j_0}} \exp\Big(-\frac{1}{2}|\log\epsilon| + \omega(\sigma+1)|\log\epsilon|^{\omega} \log|\log\epsilon| + |\log\epsilon|^{\omega}\log C \\
&\qquad\qquad + \frac{1}{2s}|\log\epsilon|^{\omega} \log\Big(1+\sum_{k=0}^{2s-1} M_k\Big)\Big).
\end{aligned}
$$

Note that for $\epsilon \in (0,1/4]$ all the terms in the exponent can be controlled by $-1/4|\log\epsilon|$. Indeed, using the ϵ-Cauchy inequality, we have

$$
\omega(\sigma+1)|\log\epsilon|^{\omega}\log|\log\epsilon| \le \frac{1}{12}|\log\epsilon| + C
$$

for a constant C depending on ω. Also

$$
|\log\epsilon|^{\omega}\log C \le \frac{1}{12}|\log\epsilon| + C,
$$

and

$$
\frac{1}{2s}|\log\epsilon|^{\omega}\log\Big(1+\sum_{k=0}^{2s-1} M_k\Big) \le \frac{1}{12}|\log\epsilon| + C\Big(\log\Big(1+\sum_{k=0}^{2s-1} M_k\Big)\Big)^{1/(1-\omega)}.
$$

Thus, using the above inequalities in (3.12), we get

$$
\begin{aligned}
&\frac{C^{n_0} M^{1/2}}{\delta^{j_0}}\, \epsilon^{1/2} n_0^{(\sigma+1)n_0} \Big(1+\sum_{k=0}^{2s-1} M_k\Big)^{n_0/2s} \qquad (3.13)\\
&\quad \le \frac{M^{1/2}}{\delta^{j_0}} \exp\Big(-\frac{1}{4}|\log\epsilon| + C\Big(\log\Big(1+\sum_{k=0}^{2s-1} M_k\Big)\Big)^{1/(1-\omega)}\Big).
\end{aligned}
$$

For the second term on the right-hand side of (3.10), we have

$$
\frac{CMK}{\delta^{j_0}}|\log\epsilon|^{-\theta(n_0/4s+1)}\Big(\frac{n_0}{2s}+1\Big)^{(\sigma-1)(n_0/2s+1)}
$$

$$\begin{aligned}
&\leq \frac{CMK}{\delta^{j_0}} \exp\Big(-\theta \left(\frac{n_0}{4s}+1\right) \log|\log\epsilon| \\
&\qquad\qquad + (\sigma-1)\left(\frac{n_0}{2s}+1\right)\log\left(\frac{n_0}{2s}+1\right)\Big) \\
&\leq \frac{CMK}{\delta^{j_0}} \exp\Big(-\frac{\theta n_0}{4s}\log|\log\epsilon| + (\sigma-1)\left(\frac{n_0}{2s}+1\right)\log n_0\Big).
\end{aligned}$$

Now, by (3.11), we obtain

$$\begin{aligned}
&\frac{CMK}{\delta^{j_0}}|\log\epsilon|^{-\theta(n_0/4s+1)}\left(\frac{n_0}{2s}+1\right)^{(\sigma-1)(n_0/2s+1)} \qquad (3.14)\\
&\quad\leq \frac{CMK}{\delta^{j_0}} \exp\Big(-\frac{\theta}{8s}|\log\epsilon|^{\omega}\log|\log\epsilon| \\
&\qquad\qquad + \omega(\sigma-1)\Big(\frac{1}{2s}|\log\epsilon|^{\omega}+1\Big)\log|\log\epsilon|\Big) \\
&\quad\leq \frac{CMK}{\delta^{j_0}} \exp\Big(-\frac{1}{C}|\log\epsilon|^{\omega}\log|\log\epsilon|\Big)
\end{aligned}$$

provided $\sigma \leq 1+\theta/16\omega$. Therefore, for all $j_0 \in \{0,1,\dots,2s-1\}$, we conclude from (3.13) and (3.14) that

$$\begin{aligned}
|\partial_x^{j_0}u(x,t)| \leq \Big(&\frac{M^{1/2}}{\delta^{j_0}}\exp\Big(-\frac{1}{4}|\log\epsilon| + C\Big(\log\Big(1+\sum_{k=0}^{2s-1}M_k\Big)\Big)^{1/(1-\omega)}\Big) \\
&+ \frac{CMK}{\delta^{j_0}}\exp\Big(-\frac{1}{C}|\log\epsilon|^{\omega}\log|\log\epsilon|\Big)\Big)\|u(\cdot,0)\|_{L^2(\Omega)}
\end{aligned}$$

for $x \in (-\delta/2, \delta/2)$ and $t \in (-\delta^{2s}/|\log\epsilon|^{\theta}, \delta^{2s}/|\log\epsilon|^{\theta})$ provided that $\sigma \leq 1+\theta/16\omega$ for arbitrary $\theta,\omega \in (0,1)$ and $\epsilon \in (0,1/4]$.

4. Propagation of smallness on space-time rectangles

Let $u(x,t)$ be a periodic solution, with period 1 in the x variable, of the equation

$$u_t + (-1)^s\partial_x^{2s}u + \sum_{k=0}^{2s-1} v_k(x,t)\partial_x^k u = 0 \qquad (4.1)$$

for $(x,t) \in \mathbb{R}\times(-\delta^{2s}/|\log\epsilon|^{\theta}, \delta^{2s}/|\log\epsilon|^{\theta})$, where $\epsilon,\theta \in (0,1)$ and $\delta \in (0,1/4]$. As in Section 2, we suppose that u and v_k are infinitely smooth functions in (x,t). Also, assume there exist non-negative constants M and M_j for $j \in \{0,1,\dots,2s-1\}$ such that

$$|\partial_x^j u(x,t)| \leq \frac{Mj!^{\sigma}}{\delta^j} \qquad (4.2)$$

and

$$|v_j(x,t)| \le M_j \tag{4.3}$$

for $(x,t) \in \mathbb{R} \times (-\delta^{2s}/|\log\epsilon|^\theta, \delta^{2s}/|\log\epsilon|^\theta)$. Assume additionally that

$$|\partial_x^j u(x,t)| \le \widetilde{\epsilon} \tag{4.4}$$

for $(x,t) \in (-\delta,\delta) \times (-\delta^{2s}/|\log\epsilon|^\theta, \delta^{2s}/|\log\epsilon|^\theta)$ and some $\widetilde{\epsilon} \in (0,1)$.

Lemma 4.1. *Suppose that the assumptions* (4.1)–(4.4) *are satisfied. If* $\widetilde{\epsilon} \le \exp\left(-P_1\left(|\log\epsilon|, \delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)\right)$, *then*

$$\int_{-\delta^{2s}/4|\log\epsilon|^\theta}^{\delta^{2s}/4|\log\epsilon|^\theta} \int_0^1 u(x,t)^2 \, dx \, dt \le P_2\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right) \widetilde{\epsilon}^{2/3},$$

where P_1 and P_2 are non-negative polynomials.

Proof. We begin by performing the change of variables

$$(x,t) \to \Big(x - \frac{4|\log\epsilon|^{2\theta}}{\delta^{4s}} t^2, t\Big),$$

which transforms the rectangle $(-\delta,\delta) \times (-\delta^{2s}/|\log\epsilon|^\theta, \delta^{2s}/|\log\epsilon|^\theta)$ into a region between the two parabolas $x + \delta = 4|\log\epsilon|^{2\theta} t^2/\delta^{4s}$ and $x - \delta = 4|\log\epsilon|^{2\theta} t^2/\delta^{4s}$. Define

$$\widetilde{u}(x,t) = u\Big(x - \frac{4|\log\epsilon|^{2\theta}}{\delta^{4s}} t^2, t\Big). \tag{4.5}$$

Then $\widetilde{u}$ is a periodic solution, with period 1 in x variable, of the equation

$$\widetilde{u}_t + (-1)^s \partial_x^{2s} \widetilde{u} + \sum_{k=0}^{2s-1} v_k(x,t) \partial_x^k \widetilde{u} + \frac{8|\log\epsilon|^{2\theta} t}{\delta^{4s}} \widetilde{u}_x = 0. \tag{4.6}$$

Using hypotheses (4.3) and (4.4), we have

$$\left|v_1 + \frac{8|\log\epsilon|^{2\theta} t}{\delta^{4s}}\right| \le M_1 + \frac{8|\log\epsilon|^\theta}{\delta^{2s}} \tag{4.7}$$

for $(x,t) \in \mathbb{R} \times (-\delta^{2s}/|\log\epsilon|^\theta, \delta^{2s}/|\log\epsilon|^\theta)$ and for all $j \in \{0,1,\dots,2s-1\}$

$$|\partial_x^j \widetilde{u}(x,t)| \le \widetilde{\epsilon} \tag{4.8}$$

for $(x,t) \in S$, where with S we denote the region

$$S = \Big\{(x,t) : -\delta \le x - \frac{4|\log\epsilon|^{2\theta}}{\delta^{4s}} t^2 \le \delta, \, -\delta \le x \le 4 - \delta\Big\}.$$

Also, for $r \in (0,3]$ we use the notation

$$O_l(r) = \Big\{(x,t) : -\delta < x < r, -\frac{\delta^{2s}(x+\delta)^{1/2}}{2|\log \epsilon|^\theta} < t < \frac{\delta^{2s}(x+\delta)^{1/2}}{2|\log \epsilon|^\theta}\Big\}$$

and

$$O_s(r) = \Big\{(x,t) : \delta < x < r, -\frac{\delta^{2s}(x-\delta)^{1/2}}{2|\log \epsilon|^\theta} < t < \frac{\delta^{2s}(x-\delta)^{1/2}}{2|\log \epsilon|^\theta}\Big\}$$

for the regions inside the parabolas $x+\delta = 4|\log\epsilon|^{2\theta}t^2/\delta^{4s}$ and $x-\delta = 4|\log\epsilon|^{2\theta}t^2/\delta^{4s}$, respectively, and to the left of $x = r$. Define a smooth cut-off function $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $0 \le \phi \le 1$ in $\mathbb{R}^2$, $\phi = 0$ in a neighborhood of $\mathbb{R}^2 \backslash O_l(3)$, and $\phi = 1$ in a neighborhood of $\overline{O_s(11/4)}$. Additionally, we impose

$$|\partial_x^j \phi(x,t)| \le \frac{C}{\delta^j}, \qquad j \in \{1,\dots,2s-1\}$$

and

$$|\partial_t \phi(x,t)| \le \frac{C|\log\epsilon|^\theta}{\delta^{2s+1}}.$$

for $(x,t) \in S$. Next, we use the Carleman estimate for $\widetilde{u}\phi \in C_0^\infty(\mathbb{R}^2)$

$$\sum_{k=0}^{2s-1} \tau^{4s-2k-1} \int_O \left(\partial_x^k(\widetilde{u}\phi)\right)^2 e^{2\tau\psi} \le C \int_O \left((\widetilde{u}\phi)_t + (-1)^s \partial_x^{2s}(\widetilde{u}\phi)\right)^2 e^{2\tau\psi} \tag{4.9}$$

for $\tau \ge \tau_0$, where $O = (-4,4)^2$ and the weight function ψ is given by $\psi(x) = -x + x^2/100$. We estimate the right-hand side of (4.9) from above by $I_1 + I_2 + I_3$, where

$$I_1 = C \int_{O_s(11/4)} \left((\widetilde{u}\phi)_t + (-1)^s \partial_x^{2s}(\widetilde{u}\phi)\right)^2 e^{2\tau\psi},$$

$$I_2 = C \int_S \left((\widetilde{u}\phi)_t + (-1)^s \partial_x^{2s}(\widetilde{u}\phi)\right)^2 e^{2\tau\psi},$$

$$I_3 = C \int_{O_s(3)\backslash O_s(11/4)} \left((\widetilde{u}\phi)_t + (-1)^s \partial_x^{2s}(\widetilde{u}\phi)\right)^2 e^{2\tau\psi}.$$

Using the equation (4.6) and the bounds on the coefficients (4.3) and (4.7), we obtain the estimate

$$I_1 \le C \sum_{k=0}^{2s-1} M_k^2 \int_{O_s(11/4)} \left(\partial_x^k \widetilde{u}\right)^2 e^{2\tau\psi} + \frac{C|\log\epsilon|^{2\theta}}{\delta^{4s}} \int_{O_s(11/4)} \widetilde{u}_x^2 e^{2\tau\psi}$$

for the first integral. We can absorb this estimate for I_1 in the half of the left-hand side of (4.9) under the condition

$$\tau \geq \max\Big\{ \max_{k=0,\dots,2s-1} CM_k^{2/(4s-2k-1)},\ \frac{C|\log\epsilon|^{2\theta/(4s-3)}}{\delta^{4s/(4s-3)}},\ \tau_0 \Big\}. \tag{4.10}$$

For the second integral, we use also the hypotheses on the derivatives of ϕ

$$\begin{aligned}
I_2 &\leq C\int_S \left(\widetilde{u}_t + (-1)^s\partial_x^{2s}\widetilde{u}\right)^2 e^{2\tau\psi} \\
&\qquad + C\sum_{k=0}^{2s-1}\int_S \left(\partial_x^{2s-k}\phi\right)^2\left(\partial_x^k\widetilde{u}\right)^2 e^{2\tau\psi} + C\int_S \phi_t^2\widetilde{u}^2 e^{2\tau\psi} \\
&\leq C\sum_{k=0}^{2s-1}\left(M_k^2 + \frac{1}{\delta^{4s-2k}}\right)\int_S \left(\partial_x^k\widetilde{u}\right)^2 e^{2\tau\psi} \\
&\qquad + \frac{C|\log\epsilon|^{2\theta}}{\delta^{4s}}\int_S \widetilde{u}_x^2 e^{2\tau\psi} + \frac{C|\log\epsilon|^{2\theta}}{\delta^{4s+2}}\int_S \widetilde{u}^2 e^{2\tau\psi}.
\end{aligned}$$

Now, by (4.8) and since $|S| \leq C\delta^{2s+1}/|\log\epsilon|^\theta$ and $\psi < 2\delta$ on S, we obtain

$$I_2 \leq C\widetilde{\epsilon}^2 e^{4\tau\delta}\Big(\sum_{k=0}^{2s-1} M_k^2 + \frac{|\log\epsilon|^{2\theta}}{\delta^{4s+2}}\Big)|S| \leq C\widetilde{\epsilon}^2 e^{4\tau\delta}\Big(\sum_{k=0}^{2s-1}\frac{M_k^2\delta^{2s+1}}{|\log\epsilon|^\theta} + \frac{|\log\epsilon|^\theta}{\delta^{2s+1}}\Big).$$

Finally, for the third integral, the assumptions (4.2) and (4.3) imply

$$\begin{aligned}
I_3 &\leq C\int_{O_s(3)\setminus O_s(11/4)} \left(\widetilde{u}_t + (-1)^s\partial_x^{2s}\widetilde{u}\right)^2 e^{2\tau\psi} \\
&\qquad + C\sum_{k=0}^{2s-1}\int_{O_s(3)\setminus O_s(11/4)} \left(\partial_x^{2s-k}\phi\right)^2\left(\partial_x^k\widetilde{u}\right)^2 e^{2\tau\psi} \\
&\qquad\qquad + C\int_{O_s(3)\setminus O_s(11/4)} \phi_t^2\widetilde{u}^2 e^{2\tau\psi} \\
&\leq Ce^{-5\tau}\left(\sum_{k=0}^{2s-1}\left(M_k^2 + \frac{1}{\delta^{4s-2k}}\right)\frac{M^2 k!^{2\sigma}}{\delta^{2k}} + \frac{|\log\epsilon|^{2\theta}}{\delta^{4s}}\frac{M^2}{\delta^2} + \frac{|\log\epsilon|^{2\theta}}{\delta^{4s+2}}M^2\right) \\
&\qquad\qquad \times |O_s(3)\setminus O_s(11/4)| \\
&\leq CM^2 e^{-5\tau}\left(\sum_{k=0}^{2s-1}\left(\frac{M_k^2\delta^{2s-2k}}{|\log\epsilon|^\theta} + \frac{1}{|\log\epsilon|^\theta\delta^{2s}}\right)k!^{2\sigma} + \frac{2|\log\epsilon|^\theta}{\delta^{2s+2}}\right),
\end{aligned}$$

where we used that $|O_s(3)\backslash O_s(11/4)| \leq C\delta^{2s}/|\log\epsilon|^\theta$ and $\psi < -5/2$ on $O_s(3)\backslash O_s(11/4)$. Therefore,

$$\begin{aligned}
\tau^{4s-1}&\int_O (\widetilde{u}\phi)^2 e^{2\tau\psi} \\
&\leq C\widetilde{\epsilon}^2 e^{4\tau\delta}\left(\sum_{k=0}^{2s-1}\frac{M_k^2\delta^{2s+1}}{|\log\epsilon|^\theta}+\frac{|\log\epsilon|^\theta}{\delta^{2s+1}}\right) \\
&\qquad + CM^2 e^{-5\tau}\left(\sum_{k=0}^{2s-1}\left(\frac{M_k^2\delta^{2s-2k}}{|\log\epsilon|^\theta}+\frac{1}{|\log\epsilon|^\theta\delta^{2s}}\right)k!^{2\sigma}+\frac{2|\log\epsilon|^\theta}{\delta^{2s+2}}\right).
\end{aligned}$$

Denote $R=[1/2,3/2]\times[-\delta^{2s}/4|\log\epsilon|^\theta,\delta^{2s}/4|\log\epsilon|^\theta]$. Then $R\subseteq O_s(11/4)$ and $\psi\geq -3/2$ on R. We get

$$\tau^{4s-1}\int_O(\widetilde{u}\phi)^2e^{2\tau\psi}\geq \tau^{4s-1}\int_R \widetilde{u}^2 e^{2\tau\psi}\geq \tau^{4s-1}e^{-3\tau}\int_R\widetilde{u}^2$$

and thus

$$\begin{aligned}
\int_R \widetilde{u}^2 \leq \frac{C\widetilde{\epsilon}^2 e^{\tau(4\delta+3)}}{\tau^{4s-1}}&\left(\sum_{k=0}^{2s-1}\frac{M_k^2\delta^{2s+1}}{|\log\epsilon|^\theta}+\frac{|\log\epsilon|^\theta}{\delta^{2s+1}}\right) \\
&+\frac{CM^2e^{-2\tau}}{\tau^{4s-1}}\left(\sum_{k=0}^{2s-1}\left(\frac{M_k^2\delta^{2s-2k}}{|\log\epsilon|^\theta}+\frac{1}{|\log\epsilon|^\theta\delta^{2s}}\right)k!^{2\sigma}+\frac{2|\log\epsilon|^\theta}{\delta^{2s+2}}\right).
\end{aligned}$$

The above inequality is true for any τ satisfying (4.10). We choose τ such that $e^{-3\tau}=\widetilde{\epsilon}$. Then

$$\begin{aligned}
\frac{1}{\widetilde{\epsilon}^{2/3}}\int_R\widetilde{u}^2 \leq C&\left(\sum_{k=0}^{2s-1}\frac{M_k^2\delta^{2s+1}}{|\log\epsilon|^\theta}+\frac{|\log\epsilon|^\theta}{\delta^{2s+1}}\right) \\
&+CM^2\left(\sum_{k=0}^{2s-1}\left(\frac{M_k^2\delta^{2s-2k}}{|\log\epsilon|^\theta}+\frac{1}{|\log\epsilon|^\theta\delta^{2s}}\right)k!^{2\sigma}+\frac{2|\log\epsilon|^\theta}{\delta^{2s+2}}\right) \\
=P_2&\left(|\log\epsilon|,\delta^{-1},M,\{M_k\}_{k=0}^{2s-1}\right)
\end{aligned}$$

provided $\widetilde{\epsilon}$ satisfies

$$\begin{aligned}
\log\frac{1}{\widetilde{\epsilon}}&\geq\max\left\{3\max_{k=0,\dots,2s-1}CM_k^{2/(4s-2k-1)},\ \frac{3C|\log\epsilon|^{2\theta/(4s-3)}}{\delta^{4s/(4s-3)}},\ 3\tau_0\right\} \\
&=P_1\left(|\log\epsilon|,\delta^{-1},\{M_k\}_{k=0}^{2s-1}\right).
\end{aligned}$$

Thus, the proof of the lemma is complete.

5. Proofs of the main results

In this section we prove Theorems 2.1 and 2.4.

Proof of Theorem 2.1. Denote

$$\epsilon = \frac{\sup_{x\in(-\delta,\delta)} |u(x,0)|}{\|u(\cdot,0)\|_{L^2(\Omega)}}. \tag{5.1}$$

Our goal is to show that there exists a universal constant $\eta > 0$ such that for $\sigma \le 1+\eta$, we have either $\epsilon > 1/4$ or $\epsilon \ge \exp(-P(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K))$ for some nonnegative polynomial P. Let $\epsilon \in (0, 1/4]$. Then by (5.1), we have that the hypothesis of Lemma 3.1

$$|u(x,0)| \le \epsilon \|u(\cdot,0)\|_{L^2(\Omega)}, \qquad x \in (-\delta,\delta)$$

is satisfied. Hence,

$$|\partial_x^j u(x,t)| \le F_j\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right) \|u(\cdot,0)\|_{L^2(\Omega)}$$

for $x \in (-\delta/2, \delta/2)$ and $t \in (-\delta^{2s}/2^{2s}|\log\epsilon|^\theta, \delta^{2s}/2^{2s}|\log\epsilon|^\theta)$ with $\theta \in (0,1)$, and $j \in \{0,1,\ldots,2s-1\}$. Next, we denote

$$K_0 = \max_{j\in\{0,\ldots,2s-1\}} \sup_{|t|<\delta^{2s}/2^{2s}|\log\epsilon|^\theta} \frac{\delta^j \|\partial_x^j u(\cdot,t)\|_{L^\infty(\Omega)}}{j!^\sigma}$$

and let

$$\widetilde{u}(x,t) = \frac{1}{K_0} u(x,t).$$

Clearly, the dilated function $\widetilde{u}$ satisfies the equation

$$\widetilde{u}_t + (-1)^s \partial_x^{2s}\widetilde{u} + \sum_{k=0}^{2s-1} v_k(x,t)\partial_x^k \widetilde{u} = 0. \tag{5.2}$$

Also, for all $j \in \{0,1,\ldots,2s-1\}$, we have

$$|\partial_x^j \widetilde{u}(x,t)| \le \frac{j!^\sigma}{\delta^j}, \qquad (x,t) \in \mathbb{R} \times \left(-\frac{\delta^{2s}}{2^{2s}|\log\epsilon|^\theta}, \frac{\delta^{2s}}{2^{2s}|\log\epsilon|^\theta}\right) \tag{5.3}$$

and by the assumption (2.3)

$$|v_j(x,t)| \le M_j, \qquad (x,t) \in \mathbb{R} \times \left(-\frac{\delta^{2s}}{2^{2s}|\log\epsilon|^\theta}, \frac{\delta^{2s}}{2^{2s}|\log\epsilon|^\theta}\right). \tag{5.4}$$

Denote

$$\epsilon_j = F_j\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right), \qquad j \in \{0,1,\ldots,2s-1\}$$

and

$$\widetilde{\epsilon} = \max_{j\in\{0,\dots,2s-1\}} \epsilon_j.$$

Then we obtain

$$|\partial_x^j \widetilde{u}(x,t)| \le \widetilde{\epsilon}, \qquad (x,t) \in \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \times \left(-\frac{\delta^{2s}}{2^{2s}|\log\epsilon|^\theta}, \frac{\delta^{2s}}{2^{2s}|\log\epsilon|^\theta}\right) \tag{5.5}$$

for $j \in \{0,1,\dots,2s-1\}$ since

$$|\partial_x^j \widetilde{u}(x,t)| \le \frac{1}{K_0} F_j\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right) \|u(\cdot,0)\|_{L^2(\Omega)} \le \epsilon_j \le \widetilde{\epsilon}.$$

Now, we can apply Lemma 4.1 with $M=1$. If

$$\widetilde{\epsilon} \le \exp\left(-P_1\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)\right)$$

then

$$\int_{-\delta^{2s}/4^{s+1}|\log\epsilon|^\theta}^{\delta^{2s}/4^{s+1}|\log\epsilon|^\theta} \int_0^1 \widetilde{u}(x,t)^2\,dx\,dt \le P_2\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right) \widetilde{\epsilon}^{2/3}.$$

Note that the assumptions (2.2) and (2.4) imply

$$K_0 \le MK\|u(\cdot,0)\|_{L^2(\Omega)}.$$

We have

$$\begin{aligned}\|u(\cdot,0)\|_{L^2(\Omega)}^2 &= \frac{2\cdot 4^{s+1}|\log\epsilon|^\theta}{\delta^{2s}} \int_{-\delta^{2s}/4^{s+1}|\log\epsilon|^\theta}^{\delta^{2s}/4^{s+1}|\log\epsilon|^\theta} \|u(\cdot,0)\|_{L^2(\Omega)}^2\,dt \\ &\le \frac{2\cdot 4^{s+1}K^2K_0^2|\log\epsilon|^\theta}{\delta^{2s}} \int_{-\delta^{2s}/4^{s+1}|\log\epsilon|^\theta}^{\delta^{2s}/4^{s+1}|\log\epsilon|^\theta} \|\widetilde{u}(\cdot,t)\|_{L^2(\Omega)}^2\,dt.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\|u(\cdot,0)\|_{L^2(\Omega)} \\ &\quad\le \frac{2^{s+2}MK^2|\log\epsilon|^{\theta/2}}{\delta^s} P_2\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)^{1/2} \widetilde{\epsilon}^{1/3}\|u(\cdot,0)\|_{L^2(\Omega)}.\end{aligned}$$

The last inequality holds if

$$\frac{2^{s+2}MK^2|\log\epsilon|^{\theta/2}}{\delta^s} P_2\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)^{1/2} \widetilde{\epsilon}^{1/3} \ge 1$$

or equivalently if

$$\widetilde{\epsilon} \ge \frac{\delta^{3s}}{CM^3K^6|\log\epsilon|^{3\theta/2}} P_2\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)^{-3/2}.$$

We conclude

$$\widetilde{\epsilon} \geq \min\Big\{ \exp\big(- P_1\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)\big), \quad \frac{\delta^{3s}}{CM^3K^6|\log\epsilon|^{3\theta/2}} P_2\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big)^{-3/2}\Big\}. \tag{5.6}$$

We need to solve the inequality (5.6) for ϵ. Using the definition of $\widetilde{\epsilon}$ and the expression for $F_j\left(|\log\epsilon|, \delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}\right)$ from Lemma 3.1, we estimate from above the left-hand side of (5.6) in the following way:

$$\begin{aligned}\widetilde{\epsilon} &\leq \frac{M^{1/2}}{\delta^{2s-1}} \exp\Big(- \frac{1}{4}|\log\epsilon| + C\Big(\log\Big(1 + \sum_{k=0}^{2s-1} M_k\Big)\Big)^{1/(1-\omega)}\Big) \\ &\qquad + \frac{CMK}{\delta^{2s-1}} \exp\Big(- \frac{1}{C}|\log\epsilon|^{\omega} \log|\log\epsilon|\Big) \\ &\leq \frac{2CMK}{\delta^{2s-1}} \exp\Big(- \frac{1}{C}|\log\epsilon|^{\omega} \log|\log\epsilon|\Big) \\ &\leq \exp\Big(\log\frac{2CMK}{\delta^{2s-1}} - \frac{1}{C}|\log\epsilon|^{\omega}\log|\log\epsilon|\Big).\end{aligned} \tag{5.7}$$

Next, we proceed by estimating from below the right-hand side of (5.6). We have

$$\begin{aligned}&\min\Big\{ \exp\big(- P_1\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right)\big), \\ &\qquad\qquad \frac{\delta^{3s}}{CM^3K^6|\log\epsilon|^{3\theta/2}} P_2\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big)^{-3/2}\Big\} \\ &\qquad \geq \exp\Big(- P_1\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big) - \log\frac{CM^3K^6}{\delta^{3s}} \\ &\qquad\qquad - \log|\log\epsilon|^{3\theta/2} - \frac{3}{2}\log P_2\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big)\Big).\end{aligned} \tag{5.8}$$

We shall find ϵ from the inequality

$$\begin{aligned}&\log\frac{2CMK}{\delta^{2s-1}} - \frac{1}{C}|\log\epsilon|^{\omega}\log|\log\epsilon| \\ &\qquad \geq -P_1\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big) - \log\frac{CM^3K^6}{\delta^{3s}} - \log|\log\epsilon|^{3\theta/2} \\ &\qquad\qquad - \frac{3}{2}\log P_2\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big),\end{aligned} \tag{5.9}$$

which we get from (5.6) by using the estimates (5.7) and (5.8). Regarding the polynomials P_1 and P_2, obtained in Lemma 4.1, we have respectively

$$P_1\left(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\right) \tag{5.10}$$

$$= \max\left\{3 \max_{k=0,\dots,2s-1} C\{M_k\}^{2/(4s-2k-1)}, \frac{3C|\log\epsilon|^{2\theta/(4s-3)}}{\delta^{4s/(4s-3)}}, 3\tau_0\right\}$$
$$\leq C\sum_{k=0}^{2s-1} M_k^2 + \frac{3C|\log\epsilon|^{2\theta/(4s-3)}}{\delta^2}$$

and

$$\begin{aligned} &P_2\big(|\log\epsilon|, 2\delta^{-1}, \{M_k\}_{k=0}^{2s-1}\big) \\ &\quad = C\sum_{k=0}^{2s-1}\frac{M_k^2\delta^{2s+1}}{|\log\epsilon|^\theta} + C\frac{|\log\epsilon|^\theta}{\delta^{2s+1}} \\ &\qquad + C\sum_{k=0}^{2s-1}\left(\frac{M_k^2\delta^{2s-2k}}{|\log\epsilon|^\theta} + \frac{1}{|\log\epsilon|^\theta\delta^{2s}}\right)k!^{2\sigma} + 2C\frac{|\log\epsilon|^\theta}{\delta^{2s+2}} \\ &\quad \leq C\sum_{k=0}^{2s}\frac{(M_k^2+1)k!^{2\sigma}}{\delta^{2s}} + 3C\frac{|\log\epsilon|^\theta}{\delta^{2s+2}} \leq C\sum_{k=0}^{2s-1}\frac{(M_k^2+1)k!^{2\sigma}}{\delta^{2s+2}}|\log\epsilon|^\theta. \end{aligned} \tag{5.11}$$

Now, by (5.10) and (5.11), we rewrite the inequality (5.9) as

$$\begin{aligned} &\log\frac{2CMK}{\delta^{2s-1}} - \frac{1}{C}|\log\epsilon|^\omega\log|\log\epsilon| \\ &\quad \geq -\frac{3C|\log\epsilon|^{2\theta/(4s-3)}}{\delta^2} - \log\frac{CM^3K^6}{\delta^{3s}} - \log|\log\epsilon|^{3\theta/2} \\ &\qquad - \frac{3}{2}\log\left(C\sum_{k=0}^{2s-1}\frac{(M_k^2+1)k!^{2\sigma}}{\delta^{2s+2}}|\log\epsilon|^\theta\right) - C\sum_{k=0}^{2s-1}M_k^2. \end{aligned} \tag{5.12}$$

Finally, (5.12) gives

$$\begin{aligned} &-\frac{1}{C}|\log\epsilon|^\omega\log|\log\epsilon| + \frac{3C|\log\epsilon|^{2\theta/(4s-3)}}{\delta^2} + 2|\log\epsilon|^{\theta/2} \\ &\quad \geq -\frac{3}{2}C\sum_{k=0}^{2s-1}\frac{(M_k^2+1)k!^{2\sigma}}{\delta^{2s+2}} - \frac{CM^3K^6}{\delta^{3s}} - \frac{2CMK}{\delta^{2s-1}} - C\sum_{k=0}^{2s-1}M_k^2, \end{aligned} \tag{5.13}$$

by using the well-known estimates $\log x \leq Cx^{1/3}$ and $\log x \leq x$. Note that all the terms on the left-hand side of (5.13) may be absorbed in the first one, provided $\omega \geq 2\theta/(4s-3)$ and $\omega \geq \theta/2$. We choose $\omega = 2\theta$. Since $\sigma \leq 1 + \theta/16\omega$, we get $\sigma \leq 33/32$. Then $\epsilon \geq \exp(-P(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K))$ for some polynomial P and thus

$$\|u(\cdot,0)\|_{L^2(\Omega)} \leq \exp(P(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K))\|u(\cdot,0)\|_{L^\infty(-\delta,\delta)}. \tag{5.14}$$

By Agmon's inequality, (5.14), and (2.2), we get

$$\begin{aligned}\|u(\cdot,0)\|_{L^\infty(\Omega)} &\le C\|u(\cdot,0)\|_{L^2(\Omega)}^{1/2}\|\partial_x u(\cdot,0)\|_{L^2(\Omega)}^{1/2} + C\|u(\cdot,0)\|_{L^2(\Omega)} \qquad (5.15)\\ &\le \frac{CM^{1/2}}{\delta^{1/2}}\|u(\cdot,0)\|_{L^2(\Omega)}\\ &\le \exp(P(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K))\|u(\cdot,0)\|_{L^\infty(-\delta,\delta)},\end{aligned}$$

where we denote again by $P(\delta^{-1}, M, \{M_k\}_{k=0}^{2s-1}, K)$ the new polynomial, obtained in the last line of (5.15), with the highest power with respect to δ^{-1} a constant depending on s.

Proof of Theorem 2.4. We only prove the theorem when $\sigma < 1 + 1/b$. The modifications for the case $\sigma = 1 + 1/b$ are straightforward. Without loss of generality, let $x_0 = 0$. Assume that f has n zeros in the interval $[-\rho/2, \rho/2]$, counting the multiplicity. Let $x_1, x_2, \dots, x_k \in [-\rho/2, \rho/2]$ be the k distinct zeros of f with multiplicities $m_1, m_2, \dots, m_k$, respectively, such that $m_1 + m_2 + \cdots + m_k = n$. By the Hermite interpolation theorem (cf. [1, p. 878]), there exists a unique interpolation polynomial p_{n-1} of f, of degree less than or equal to $n-1$, satisfying

$$p_{n-1}^{(j)}(x_l) = f^{(j)}(x_l), \qquad 1 \le l \le k, \qquad 0 \le j \le m_l - 1.$$

Moreover, for all $x \in [-\rho/2, \rho/2]$ there exists $\xi \in [-\rho/2, \rho/2]$ depending on x such that

$$f(x) - p_{n-1}(x) = \frac{(x-x_1)^{m_1}\dots(x-x_k)^{m_k}}{n!} f^{(n)}(\xi)$$

and consequently

$$\begin{aligned}&\|f - p_{n-1}\|_{L^\infty[-\rho/2,\rho/2]}\\ &\qquad \le \frac{\sup_{x\in[-\rho/2,\rho/2]} |(x-x_1)^{m_1}\dots(x-x_k)^{m_k}|}{n!}\|f^{(n)}\|_{L^\infty[-\rho/2,\rho/2]}.\end{aligned}$$

Clearly, $p_{n-1} \equiv 0$, and from the above inequality, we obtain

$$\|f\|_{L^\infty[-\rho/2,\rho/2]} \le \frac{\rho^n}{n!}\|f^{(n)}\|_{L^\infty[-\rho/2,\rho/2]}.$$

Now, the hypotheses (2.5) and (2.6) give

$$\|f\|_{L^\infty[-\rho/2,\rho/2]} \le Mn!^{\sigma-1}\frac{\rho^n}{\delta^n}\exp\left(\frac{a}{\rho^b}\right)\|f\|_{L^\infty[-\rho/2,\rho/2]}. \qquad (5.16)$$

In order for the inequality (5.16) to hold, we need

$$Mn!^{\sigma-1}\frac{\rho^n}{\delta^n}\exp\left(\frac{a}{\rho^b}\right) \ge 1. \qquad (5.17)$$

Using $n^n e^{-n} \le n! \le n^n$, (5.17) implies

$$M\frac{(n^{\sigma-1}\rho)^n}{\delta^n}\exp\left(\frac{a}{\rho^b}\right) \ge 1,$$

which is the same as

$$\exp\left(n\log(n^{\sigma-1}\rho) - n\log\delta + \frac{a}{\rho^b} + \log M\right) \ge 1,$$

or equivalently

$$n\log(n^{\sigma-1}\rho) - n\log\delta + \frac{a}{\rho^b} + \log M \ge 0. \tag{5.18}$$

Let n_0 be the largest integer such that $n_0 \le K$, where K is defined in (2.8), and let

$$\rho = \frac{(2a)^{1/b}}{n_0^{1/b}}. \tag{5.19}$$

Then

$$\rho = \frac{(2a)^{1/b}}{n_0^{1/b}} \le \frac{(2a)^{1/b}}{(4^{b+1}a/\delta^b)^{1/b}} = \frac{2^{1/b}a^{1/b}\delta}{4^{1+1/b}a^{1/b}} \le \frac{\delta}{4}.$$

Also,

$$n_0^{\sigma-1}\rho \le \frac{(2a)^{1/b}\delta}{((4^{b+1}a)^{1/(1+b(1-\sigma))})^{1/b-\sigma+1}} = \frac{(2a)^{1/b}\delta}{4^{1+1/b}a^{1/b}} \le \frac{\delta}{4}. \tag{5.20}$$

Therefore, by (5.18) with n replaced by n_0 and (5.20), we get

$$\begin{aligned} n_0\log(n_0^{\sigma-1}\rho) - n_0\log\delta + \frac{a}{\rho^b} + \log M &= n_0\log\frac{n_0^{\sigma-1}\rho}{\delta} + \frac{a}{\rho^b} + \log M \\ \le n_0\log\frac{1}{4} + \frac{a}{\rho^b} + \log M &= \left(-\frac{n_0}{2} + \frac{a}{\rho^b}\right) + \left(-\frac{n_0}{2} + \log M\right). \end{aligned}$$

The first term on the far right vanishes by (5.19), while the second is less than zero by the definition of n_0. Therefore the number $n = n_0$ does not satisfy (5.18) and thus also does not satisfy (5.17). Hence, f has less than n_0 zeros in $[-\rho/2, \rho/2]$, where ρ is defined in (5.19). Therefore, f has at most $C(a,b)n_0^{1/b+1}$ zeros in $[-1/2, 1/2]$.

Acknowledgments. Both authors were supported in part by the NSF grants DMS-0604886 and DMS-1009769.

References

[1] M. Abramowitz and I.A. Stegun, "Handbook of Mathematical Functions," Dover Publications, Inc., New York, 1972.

[2] C. Bardos, S. Benachour, and M. Zerner, *Analycité des solutions périodiques de l'équation d'Euler en deux dimensions*, C. R. Acad. Sci. Paris Sér. A-B, 282 (1976), A995–A998.

[3] J.L. Bona, Z. Grujić, and H. Kalisch, *Algebraic lower bounds for the uniform radius of spatial analyticity for the generalized KdV equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), 783–797.

[4] F. Colombini and H. Koch, *Strong unique continuation for products of elliptic operators of second order*, Trans. Amer. Math. Soc., 362 (2009), 345–355.

[5] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions on Riemannian manifolds*, Inv. Math., 93 (1988), 161–183.

[6] H. Donnelly and C. Fefferman, *Nodal sets for eigenfunctions of the Laplacian on surfaces*, J. Amer. Math. Soc., 3 (1990), 333–353.

[7] H. Donnelly and C. Fefferman, *Nodal sets of eigenfunctions: Riemannian manifolds with boundary*, Moser volume, Analysis, Academic Press, 1990, 251–262.

[8] L. Escauriaza, F.J. Fernandez, and S. Vessella, *Doubling properties of caloric functions*, Appl. Anal., 85 (2006), 205–223.

[9] A.B. Ferrari and E.S. Titi, *Gevrey regularity for nonlinear analytic parabolic equations*, Commun. PDE, 23 (1998), 1–16.

[10] C. Foias and R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal., 87 (1989), 359–369.

[11] Z. Grujić and I. Kukavica, *Space analyticity for the Navier-Stokes and related equations with initial data in L^p*, J. Funct. Anal., 152 (1998), 447–466.

[12] L. Hörmander, "The Analysis of Linear Partial Differential Operators III: Pseudo-Differential Operators," Springer-Verlag, 1985.

[13] Q. Han and F.-H. Lin, *Nodal sets of solutions of parabolic equations*: II, Comm. Pure Appl. Math., 47 (1994), 1219–1238.

[14] R. Hardt and L. Simon, *Nodal sets for solutions of elliptic equations*, J. Differential Geom., 30 (1989), 505–522.

[15] V. Isakov, *Carleman type estimates in an anisotropic case and applications*, J. Diff. Equations, 105 (1993), 217–238.

[16] D. Jerison and C. Kenig, *Unique continuation and absence of positive eigenvalues for Schrödinger operators*, Ann. of Math., 121 (1985), 463–494.

[17] I. Kukavica, *An upper bound for the winding number for solutions of the Ginzburg-Landau equation*, Indiana Univ. Math. J., 41 (1992), 825–836.

[18] I. Kukavica, *Self-similar variables and the complex Ginzburg-Landau equation*, Commun. PDE, 24 (1999), 545–62.

[19] I. Kukavica, *Length of vorticity nodal sets for solutions of the 2D Navier-Stokes system*, Commun. PDE, 28 (2003), 771–793.

[20] I. Kukavica, *Spatial complexity of solutions of higher order partial differential equations*, Nonlinearity, 17 (2004), 459–476.

[21] I. Kukavica and V. Vicol, *On the radius of analyticity of solutions to the three-dimensional Euler equations*, Proc. Amer. Math. Soc., 137 (2009), 669–677.

[22] C. Kenig, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation*, Lecture Notes in Math., 1384 (1989), 69–90.

[23] C. Kenig, *Some recent applications of unique continuation*, Contemp. Math., 439 (2007), 25–56.

[24] C. Kenig, *Quantitative unique continuation, logarithmic convexity of Gaussian means and Hardy's uncertainty principle*, Proc. Sympos. Pure Math., 79 (2008), 207–227.

[25] K. Kurata, *On a backward estimate for solutions of parabolic differential equations and its application to unique continuation*, Adv. Stud. Pure Math., 23 (1994), 247–257.

[26] C.D. Levermore and M. Oliver, *Analyticity of solutions for a generalized Euler equation*, J. Diff. Equations, 133 (1997), 321–339.

[27] F. Lin, *Nodal sets of solutions of elliptic and parabolic equations*, Comm. Pure Appl. Math., 44 (1991), 287–308.

[28] J.-C. Saut and B. Scheurer, *Unique continuation for some evolution equations*, J. Diff. Equations, 66 (1987), 118–139.

[29] R. Temam, "Infinite-Dimensional Dynamical Systems in Mechanics and Physics," Appl. Math. Science, 2nd ed., New York, Springer, 1997.