Small data global existence for a fluid-structure model

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Thursday 30th June, 2016

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Abstract

We address the system of partial differential equations modeling motion of an elastic body inside an incompressible fluid. The fluid is modeled by the incompressible Navier-Stokes equations while the structure is represented by the damped wave equation with interior damping. The additional boundary stabilization γ, considered in our previous paper, is no longer necessary. We prove the global existence and exponential decay of solutions for small initial data in a suitable Sobolev space.

Mathematics Subject Classification: 35R35, 35Q30, 76D05
Keywords: Navier-Stokes equations, fluid-structure interaction, long time behavior, global solutions, damped wave equation

1 Introduction

In this paper, we consider a system of fluid-structure interaction with interior structural damping evolving in a domain which consists of two parts, the exterior part $\Omega_f(t)$ for the fluid and the interior part $\Omega_e(t)$ for the elastic structure. The dynamics of the fluid are described by the incompressible Navier-Stokes equations

$$\partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 \quad \text{in } \Omega_f(t) \tag{1.1}$$

$$\nabla\cdot u = 0 \quad \text{in } \Omega_f(t) \tag{1.2}$$

[Te1, Te2], while the elastic structural dynamics are governed by a damped linear wave equation

$$w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \quad \text{in } \Omega_e. \tag{1.3}$$

The interaction takes place on a common interface denoted by $\Gamma_c(t)$, and it is described by the transmission boundary conditions matching the velocities and the stress forces at the interface. Our main result is global existence of solutions and their exponential decay for small initial data, in the presence of the interior structural damping but without the boundary stabilization used in [IKLT2]. Boundary (interface) stabilization effect consists of absorbing boundary conditions, which are known to produce regularizing effect on the normal derivative of the wave component.

Fluid-structure interaction models have attracted considerable attention in both engineering and mathematical literature. This is due to a broad range of applications in various areas of applied sciences, from fluid mechanics to biochemistry and medicine. These models were originally considered within a finite element framework which is natural for numerical computations, cf. [DGHL, GGCC, GGCCL] and references therein. More recently the issues related to the mathematical theory of existence, uniqueness, and stability of solutions for such systems have become topics of major research interest. The fluid-structure interaction model considered here is an example of a quasilinear system exhibiting parabolic-hyperbolic coupling with an interface. The mismatch of regularity between hyperbolic and parabolic dynamics constitutes, as noted in [CS1, CS2], a defining feature and an obstacle in the analysis of the problem.

Earlier results in the area were mostly obtained for models where "hyperbolicity" of an elastic body was disguised by either (i) a strong damping added to the equation (thus making it parabolic) [B, DEGL] or (ii) replacing the wave component with a rigid body modeled by an ODE [F, SST]. In both cases, the analysis is more amenable due to the parabolic smoothing effects.

The interaction between parabolic and hyperbolic dynamics and induced mismatch of regularity were highlighted in a series of works [AT1, AT2, BGLT1, BGLT2, BL, DGHL, LL1, LL2, ALT, KTZ1, KTZ2, KTZ3], which considered the coupling between the Navier-Stokes equations and the wave equation under the assumption that the interface is stationary (this can be justified for small latitudinal oscillations) [L1]. The analytical tools and trace regularity results developed in some of these works have proven to be relevant for addressing more general fluid-structure interaction models and have paved the way for some of the recent works [BZ1, BZ2, KT1, KT2, IKLT1, RV].

The first results which pertain to genuine parabolic-hyperbolic problem with a dynamic interface are due to Coutand and Shkoller [CS1, CS2]. In particular, they proved local well-posedness of smooth solutions of the system [CS1]. Subsequent results pertaining to local theories but with lesser requirements imposed on the smoothness of initial data appeared in [KT1, KT2, IKLT1]. Similar results were also obtained for the case of compressible flows [BG1, BG2, KT3].

In our previous work [IKLT2], we have obtained well-posedness as well as a global existence result and exponential decay of solutions for small initial data when additional boundary damping was imposed (see the constant $\gamma > 0$ in (4.1)). The boundary damping provides regularizing effect on the normal derivative of the wave equation (the so called effect of absorbing boundary conditions). This particular feature has had beneficial effect on the analysis of the coupled structure where normal stresses provide carriers for the transport across the interface. The paper [IKLT2] is the first work to address the issue of global existence. It is well known that global existence in quasilinear dynamics is strongly tied to the asymptotic decay of solutions, which in turn requires a damping mechanism. In the case of fluid-structure interaction the least amount of damping necessary for forcing linear dynamics to decay is the boundary-interface damping [AT1, AT2]. It was shown in [IKLT2] that the presence of the boundary damping allows to prove global existence of solutions to the quasilinear model. Such result holds under certain geometric condition known as the "star shaped" domain condition. This geometric condition is natural for wave type dynamics when subject to boundary dissipation [LTr1]; however, it can be eliminated for the wave dynamics altogether using micro-local methods [LTr1]. In the case of fluid-structure interaction, it can also be eliminated, as shown in [IKLT2], by introducing an interior dissipation term αw_t. In view of the above, the following question arises. Is it possible to obtain global existence (and decay of the energy) *without any geometric restrictions* and *with one damping mechanism only*?

The present paper provides a positive answer to this question by showing that the viscous damping alone provides global and decaying solutions without any constraints on the geometry.

In the remainder of the introduction we describe the ideas behind the proof of the main global existence theorem. When proving well-posedness of the quasilinear system one is faced with a twofold task: (i) obtaining suitable a priori estimates which close at some desired level of regularity, and (ii) construction of actual solutions to which the a priori estimates can be applied.

Both tasks meet new substantial challenges which were not encountered previously. For the first task, one needs to provide a priori bounds for the integrals of potential energy, without relying on the bound on normal derivatives for the solution of the wave equation, secured by $\gamma > 0$ as was the case in [IKLT2]. (Indeed, absorbing boundary conditions provide instant L^2 regularity of the normal derivative of w.) However, in the case $\gamma = 0$, the normal wave components of finite energy solutions reside in negative order Sobolev spaces due to the failure of the Lopatinski condition [S]. We overcome this obstacle by applying suitable multipliers which cancel the effects of boundary over-spill along with developing a series of estimates on tangential derivatives by methods reminiscent to Agmon-Douglis-Nirenberg theory and elliptic theory [GS]. We note that a related multiplier estimate was successfully employed in [LL1] where static interface was considered and the Dirichlet-Stokes map served as the needed multiplier. However, the same multiplier is no longer effective for dynamic interface model and suitable modifications which exploit flow map are necessary. The second task, which is the construction of actual solutions, is even more challenging. Cancellation of the pressure term via integration by parts, typical when performing estimates on *the solutions*, no longer occurs. It is here where the mismatch of regularity comes strongly into play. Regularizing effect of parabolicity looses its momentum and strength when passing the information on the elastic system which exhibits the usual loss of regularity. Hence, the modern techniques developed in the context of hidden trace regularity theory for the wave equation (cf. [L2, LLT, LTr2, KMT]) and maximal regularity for the Stokes operator [MZ1, MZ2, PS] become critical. Maximal regularity allows us to handle the pressure term while sharp trace estimates move forward the fluid across the interface with no loss of derivatives, this latter phenomenon associated with the classical trace theory. The main ideas behind the construction of the local in time solution are described in Remark 6.3 below.

2 The main result

The system, modeling the coupling of the fluid and an elastic structure, is written in the Lagrangian variables with a precise set-up as follows. Let $\eta(\cdot,t)\colon \Omega \to \Omega$ be the flow map under which the initial domain configurations Ω_f and Ω_e evolve with time, so that $\Omega_f(t) = \eta(\Omega_f, t)$ and $\Omega_e(t) = \eta(\Omega_e, t)$. For simplicity, we assume that the domains are flat, i.e.,

$$\Omega_f = \{x = (x_1, x_2, x_3) : h_1 \le x_3 \le h_2 \text{ or } h_3 \le x_3 \le h_4\}$$

and

$$\Omega_e = \{x = (x_1, x_2, x_3) : h_2 \le x_3 \le h_3\}$$

with periodic boundary conditions with period 1 in the lateral directions.

In Lagrangian coordinates, the incompressible Navier-Stokes equation takes the form

$$v_t^i - \partial_j(a_l^j a_l^k \partial_k v^i) + \partial_k(a_i^k q) = 0 \quad \text{in } \Omega_f \times (0,T), \qquad i = 1,2,3 \tag{2.1}$$

$$a_i^k \partial_k v^i = 0 \quad \text{in } \Omega_f \times (0,T), \tag{2.2}$$

where $v(x,t) = \eta_t(x,t) = u(\eta(x,t),t)$ and $q(x,t) = p(\eta(x,t),t)$ denote the Lagrangian velocity and the pressure of the fluid over the initial domain Ω_f. For the displacement function $w(x,t) = \eta(x,t) - x$, the linear damped elasticity equation reads

$$w_{tt}^i - \Delta w^i + \alpha w_t^i + \beta w^i = 0 \quad \text{in } \Omega_e \times (0,T), \qquad i = 1,2,3 \tag{2.3}$$

over the domain Ω_e. We thus seek a solution (v, w, q, a, η) to the damped fluid-structure system (2.1)–(2.3), where the dynamics of the Lagrangian matrix a and the flow map η are described by the ODEs

$$a_t = -a : \nabla v : a \quad \text{in } \Omega_f \times (0,T) \tag{2.4}$$

and

$$\eta_t = v \quad \text{in } \Omega_f \times (0,T) \tag{2.5}$$

with the initial conditions $a(x,0) = I$ and $\eta(x,0) = x$ in Ω_f.

On the common boundary Γ_c, we assume continuity of the velocities

$$w_t^i = v^i \quad \text{on } \Gamma_c \times (0,T) \tag{2.6}$$

and the stresses

$$\partial_j w^i N_j = a_l^j a_l^k \partial_k v^i N_j - a_i^k q N_k \quad \text{on } \Gamma_c \times (0,T), \tag{2.7}$$

while on the outside fluid boundary Γ_f, we assume the non-slip boundary condition

$$v^i = 0 \quad \text{on } \Gamma_f \times (0,T) \tag{2.8}$$

for $i = 1,2,3$, where $N = (N_1, N_2, N_3)$ is the unit outward normal with respect to Ω_e. (Note that we use the strain tensor ∇v instead of the symmetric gradient matrix $\nabla v + \nabla v^T$ for the sake of simplicity.)

We use the classical spaces for the fluid velocity $H = \{v \in L^2(\Omega_f) : \operatorname{div} v = 0, v \cdot N_{|\Gamma_f} = 0\}$ and $V = \{v \in H^1(\Omega_f) : \operatorname{div} v = 0, v_{|\Gamma_f} = 0\}$.

We now state the first main result of the paper, which provides an a priori estimate for the global existence.

Theorem 2.1. *Let $\alpha, \beta > 0$. Assume that $v_0 \in V \cap H^3(\Omega_f)$, $v_t(0) \in V$, $v_{tt}(0) \in H$, $w_0 \in H^3(\Omega_e)$, and $w_1 \in H^2(\Omega_e)$ satisfying the compatibility conditions*

$$\begin{aligned}
&w_1 = v_0\\
&\Delta w_0 - \alpha w_1 - \beta w_0 = \Delta v_0 - \nabla q_0,\\
&\Delta w_1 - \alpha w_{tt}(0) - \beta w_1 = \Delta v_t(0) - \nabla q_t(0) + \partial_j(\partial_t a_j^k(0)\partial_k v^i(0)) - \partial_t a_i^k(0)\partial_k q(0)
\end{aligned} \tag{2.9}$$

on Γ_c,

$$\begin{aligned}
\frac{\partial w_0}{\partial N} \cdot \tau &= \frac{\partial v_0}{\partial N} \cdot \tau,\\
\frac{\partial w_1}{\partial N} \cdot \tau &= \frac{\partial}{\partial N}\big(\Delta v_0 - \nabla q_0\big) \cdot \tau
\end{aligned} \tag{2.10}$$

also on Γ_c, for tangential vectors τ, and

$$\begin{aligned}
&v_0 = 0,\\
&\Delta v_0 - \nabla q_0 = 0,\\
&-\partial_j(\partial_t a_j^k(0)\partial_k v^i(0)) - \Delta \partial_t v^i(0) + \partial_t a_i^k(0)\partial_k q(0) + \partial_{it} q(0) = 0
\end{aligned} \tag{2.11}$$

on Γ_f. There exists $\epsilon > 0$ such that if

$$\|v_0\|_{H^3} + \|v_t(0)\|_{H^1} + \|v_{tt}(0)\|_{L^2} + \|w_0\|_{H^3} + \|w_1\|_{H^2} \leq \epsilon \tag{2.12}$$

then any smooth solution (v, w, q, a, η) smooth solution to (2.1)–(2.5) on $[0, \infty]$ satisfies

$$\begin{aligned}
&v \in L^\infty([0,\infty); H^3(\Omega_f))\\
&v_t \in L^\infty([0,\infty); H^2(\Omega_f))\\
&v_{tt} \in L^\infty([0,\infty); L^2(\Omega_f))\\
&\nabla v_{tt} \in L^2([0,\infty); L^2(\Omega_f))\\
&\partial_t^j w \in C([0,\infty); H^{3-j}(\Omega_e)), \qquad j = 0,1,2,3
\end{aligned} \tag{2.13}$$

with $q \in L^\infty([0,\infty); H^2(\Omega_f))$, $q_t \in L^\infty([0,\infty); H^1(\Omega_f))$, $a, a_t \in L^\infty([0,\infty); H^2(\Omega_f))$, $a_{tt} \in L^\infty([0,\infty); H^1(\Omega_f))$, $a_{ttt} \in L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega_f))$, and $\eta|_{\Omega_f} \in C([0,\infty); H^3(\Omega_f))$. Also, the norm

$$\begin{aligned}
&\|v(t)\|_{H^3} + \|v_t(t)\|_{H^2} + \|v_{tt}(t)\|_{L^2} + \|q(t)\|_{H^2} + \|q_t(t)\|_{H^1}\\
&\quad + \|w(t)\|_{H^3} + \|w_t(t)\|_{H^2} + \|w_{tt}(t)\|_{H^1} + \|w_{ttt}(t)\|_{L^2}
\end{aligned} \tag{2.14}$$

decays exponentially.

Note that if the initial fluid velocity v_0 belongs to $V \cap H^4(\Omega_f)$, then it satisfies the regularity assumptions of Theorem 2.1; namely, $v_0 \in V \cap H^3(\Omega_f)$, $v_t(0) \in V$, and $v_{tt}(0) \in H$.

The proof of Theorem 2.1 is based on the energy estimates given in Section 4, which are leading to the Gronwall-type inequality (5.18) for the norm $X(t)$ defined in (5.1) below. The exponential decay of $X(t)$ is established in Lemmas 5.1 and 5.2, while the proof of Theorem 2.1 is given at the end of Section 5.

The second main result of this paper provides the global well-posedness of the system (2.1)–(2.5) for small and more regular initial data.

Theorem 2.2. *Let $\alpha, \beta > 0$. Assume that the initial data satisfy $v_0 \in V \cap H^{7/2}(\Omega_f)$, $v_t(0) \in V \cap H^{5/2}(\Omega_f)$, $v_{tt}(0) \in V$, $w_0 \in H^{15/4-\delta}(\Omega_e)$, $w_1 \in H^{11/4-\delta}(\Omega_e)$ for some $\delta \in (0, 1/4)$ and that they satisfy the compatibility conditions* (2.9), (2.10), (2.11), *and*

$$\frac{\partial w_{tt}(0)}{\partial N} \cdot \tau = \frac{\partial}{\partial N}(\Delta v_t(0) - \nabla q_t(0) + \partial_j(\partial_t a_j^k(0)\partial_k v^i(0)) - \partial_t a_i^k(0)\partial_k q(0)) \cdot \tau \tag{2.15}$$

for unit tangential vectors τ. Assume, in addition, that

$$\|v_0\|_{H^3} + \|v_t(0)\|_{H^1} + \|v_{tt}(0)\|_{L^2} + \|w_0\|_{H^3} + \|w_1\|_{H^2} \leq \epsilon, \tag{2.16}$$

where $\epsilon > 0$ is a sufficiently small constant. Then for any $T > 0$ (independent of ϵ) there exists a global in time solution (v, w, q, a, η) to (2.1)–(2.5) *obeying*

$$\begin{aligned}
&v \in L^2([0,T]; H^4(\Omega_f)) \cap H^1([0,T]; H^3(\Omega_f)) \\
&v_t \in H^1([0,T]; H^2(\Omega_f)) \\
&v_{tt} \in H^1([0,T]; L^2(\Omega_f)) \\
&\partial_t^j w \in C([0,T]; H^{15/4-\delta-j}(\Omega_e)), \qquad j = 0,1,2,3
\end{aligned} \tag{2.17}$$

with $q \in L^2([0,T]; H^3(\Omega_f)) \cap H^1([0,T]; H^2(\Omega_f))$ and $q_t \in H^1([0,T]; H^1(\Omega_f))$. Also, the norm

$$\begin{aligned}
&\|v(t)\|_{H^3} + \|v_t(t)\|_{H^2} + \|v_{tt}(t)\|_{L^2} + \|q(t)\|_{H^2} + \|q_t(t)\|_{H^1} \\
&\quad + \|w(t)\|_{H^3} + \|w_t(t)\|_{H^2} + \|w_{tt}(t)\|_{H^1} + \|w_{ttt}(t)\|_{L^2}
\end{aligned} \tag{2.18}$$

decays exponentially.

Similarly as after the statement of Theorem 2.1, we note that if $v_0 \in V \cap H^5(\Omega_f)$, then it satisfies the assumptions $v_0 \in V \cap H^{7/2}(\Omega_f)$, $v_t(0) \in H^{5/2}(\Omega_f)$, and $v_{tt}(0) \in V$. We prefer to work with the less regular assumptions because they remain invariant under the dynamics of the system.

We emphasize that the smallness of initial data is imposed only to a subset of regularity required by the initial data. The same holds with the decay rates. In order to reconcile this, we need to be careful in tracing the superlinear dependence on the higher norms, making sure that they are linear, module the topology required for decay.

We devote the last section to the proof of Theorem 2.2, which is divided into several steps. First, in Lemmas 6.2–6.8, we prove the existence and uniqueness of a solution to the linear problem with given coefficients close to the identity matrix. Here, the main tools are maximal regularity for the Stokes system used along with the sharp regularity for boundary traces of the wave operator. These allow to transfer maximal regularity of the fluid across the interface *without loosing derivatives*. In order to achieve sufficient regularity, two loops of maximal regularity are needed. The existence of a solution to the nonlinear problem then follows by the Schauder fixed point theorem, while the uniqueness is established via the contraction mapping theorem applied at the lower energy level (cf. Subsection 6.3 below).

Note that we address here the flat geometry, which we do in order to simplify the presentation. The extension to general geometry though not automatic requires introducing tangential differential operators and commutator estimates in the spirit of [BGLT2] and [KTZ3]. We suspect that the results carry over to the most general situation.

3 Preliminary results

In this section, we give formal a priori estimates on time derivatives of the unknown functions needed in the proof of Theorem 2.1. We begin with an auxiliary result from [IKLT1] providing bounds on the flow map η and the matrix a.

Lemma 3.1. *[IKLT1] Assume that $\|\nabla v\|_{L^2([0,T];H^2)} \le M$. Let $p \in [1,\infty]$ and $i,j = 1,2,3$. With $T \in [0, 1/CM]$, where C is a sufficiently large constant, the following statements hold:*
(i) $\|\nabla\eta\|_{H^2} \le C$ for $t \in [0,T]$;
(ii) $\|a\|_{H^2} \le C$ for $t \in [0,T]$;
(iii) $\|a_t\|_{L^p} \le C\|\nabla v\|_{L^p}$ for $t \in [0,T]$;
(iv) $\|\partial_i a_t\|_{L^p} \le C\|\nabla v\|_{L^{p_1}}\|\partial_i a\|_{L^{p_2}} + C\|\nabla\partial_i v\|_{L^p}$ for $i = 1,2,3$ and $t \in [0,T]$ where $1 \le p, p_1, p_2 \le \infty$ are such that $1/p = 1/p_1 + 1/p_2$;
(v) $\|\partial_{ij} a_t\|_{L^2} \le C\|\nabla v\|_{H^1}^{1/2}\|\nabla v\|_{H^2}^{1/2} + C\|\nabla v\|_{H^2}$ for $i,j = 1,2,3$ and $t \in [0,T]$;
(vi) $\|a_{tt}\|_{L^2} \le C\|\nabla v\|_{L^2}\|\nabla v\|_{L^\infty} + C\|\nabla v_t\|_{L^2}$ and $\|a_{tt}\|_{L^3} \le C\|v\|_{H^2}^2 + C\|\nabla v_t\|_{L^3}$ for $t \in [0,T]$;
(vii) $\|a_{ttt}\|_{L^2} \le C\|\nabla v\|_{H^1}^3 + C\|\nabla v_t\|_{L^2}\|\nabla v\|_{L^\infty} + C\|\nabla v_{tt}\|_{L^2}$ for $t \in [0,T]$;
(viii) for every $\epsilon \in (0, 1/2]$ and all $t \le T^* = \min\{\epsilon/CM^2, T\}$, we have

$$\|\delta_{jk} - a_l^j a_l^k\|_{H^2}^2 \le \epsilon, \qquad j,k = 1,2,3 \tag{3.1}$$

and

$$\|\delta_{jk} - a_k^j\|_{H^2}^2 \le \epsilon, \qquad j,k = 1,2,3. \tag{3.2}$$

In particular, the form $a_l^j a_l^k \xi_j^i \xi_k^i$ satisfies the ellipticity estimate

$$a_l^j a_l^k \xi_j^i \xi_k^i \ge \frac{1}{C}|\xi|^2, \qquad \xi \in \mathbb{R}^{n^2} \tag{3.3}$$

for all $t \in [0, T^*]$ and $x \in \Omega_f$, provided $\epsilon \le 1/C$ with C sufficiently large.

The statement in [IKLT1] requires $\|\nabla v\|_{L^\infty([0,T];H^2)} \le M$, but the proof only needs

$$\|\nabla v\|_{L^2([0,T];H^2)} \le M. \tag{3.4}$$

For instance, the a priori estimate for a reads

$$\|a(t)\|_{H^2} \le C + \int_0^t \|a(s)\|_{H^2}^2 \|\nabla v(s)\|_{H^2}\, ds \tag{3.5}$$

and (ii) under the condition (3.4) follows by using the Gronwall lemma, provided $T \leq 1/CM$.

We next recall the variable coefficients Stokes-type estimates for mixed Dirichlet-Neumann boundary conditions, obtained in our previous work [IKLT1].

Lemma 3.2. *[IKLT1] Assume that v and q solve the system*

$$v_t^i - \partial_j(a_l^j a_l^k \partial_k v^i) + \partial_k(a_i^k q) = 0 \quad in \quad \Omega_f \tag{3.6}$$

$$a_i^k \partial_k v^i = 0 \quad in \quad \Omega_f \tag{3.7}$$

$$v = 0 \quad on \quad \Gamma_f \tag{3.8}$$

$$a_l^j a_l^k \partial_k v^i N_j - a_i^k q N_k = \partial_j w^i N_j \quad on \quad \Gamma_c \tag{3.9}$$

for given coefficients $a_j^i \in L^\infty(\Omega_f)$ with $i, j = 1, 2, 3$ satisfying Lemma 3.1 with a sufficiently small constant $\epsilon = 1/C$. Then the estimate

$$\|v\|_{H^{s+2}} + \|q\|_{H^{s+1}} \leq C\|v_t\|_{H^s} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} \tag{3.10}$$

holds for $s = 0, 1$ and for all $t \in (0, T)$. Moreover, the time derivatives v_t and q_t satisfy

$$\begin{aligned} &\|v_t\|_{H^2} + \|q_t\|_{H^1} \\ &\quad \leq C\|v_{tt}\|_{L^2} + C\left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/2}(\Gamma_c)} + C\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\left(\|v\|_{H^2} + \|q\|_{H^1}\right) \end{aligned} \tag{3.11}$$

for all $t \in (0, T)$, where $T \leq 1/CM$ for a sufficiently large constant C.

Now, let w be a solution to the wave equation (2.3). Let $D'w = (\partial_1 w, \partial_2 w)$ denote the tangential derivative of w, and let $(D')^2$ be the matrix with entries $\partial_{ij} w$ for $i, j = 1, 2$. We obtain the full regularity elliptic estimate

$$\|w\|_{H^3} \leq C\|w_{tt}\|_{H^1} + C\|w_t\|_{H^1} + C\|w\|_{H^1} + C\|(D')^2 w\|_{H^1} \tag{3.12}$$

for all $t \in (0, T)$ by applying the Agmon-Douglas-Nirenberg procedure of decomposing the elliptic (Laplace) operator into tangential and normal coordinates.

Differentiating (2.3) in time, we also have

$$\|w_t\|_{H^2} \leq C\|w_{ttt}\|_{L^2} + C\|w_{tt}\|_{L^2} + C\|w_t\|_{L^2} + C\|D'w_t\|_{H^1} \tag{3.13}$$

for all $t \in (0, T)$.

From (3.10) with $s = 1$ and (3.12), we conclude that the Stokes type estimate

$$\|v\|_{H^3} + \|q\|_{H^2} \leq C\|v_t\|_{H^1} + C\|w_{tt}\|_{H^1} + C\|w_t\|_{H^1} + C\|w\|_{H^1} + C\|(D')^2 w\|_{H^1} \tag{3.14}$$

holds for all $t \in (0, T)$, where $T \leq 1/CM$. Analogous derivation shows that

$$\|v\|_{H^2} + \|q\|_{H^1} \leq C\|v_t\|_{L^2} + C\|w_{tt}\|_{L^2} + C\|w_t\|_{L^2} + C\|w\|_{L^2} + C\|D'w\|_{H^1}. \tag{3.15}$$

By (3.11), (3.13), and (3.15), we also get

$$
\begin{aligned}
&\|v_t\|_{H^2} + \|q_t\|_{H^1} \\
&\quad \le C\|v_{tt}\|_{L^2} + C\|w_t\|_{H^2} + C\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\left(\|v\|_{H^2} + \|q\|_{H^1}\right) \\
&\quad \le C\|v_{tt}\|_{L^2} + C\|w_{ttt}\|_{L^2} + C\|w_{tt}\|_{L^2} + C\|w_t\|_{L^2} + C\|D'w_t\|_{H^1} \\
&\qquad + C\|v\|_{H^3}^{1/2}\left(C\|v_t\|_{L^2} + C\|w_{tt}\|_{L^2} + C\|w_t\|_{L^2} + C\|w\|_{L^2} + C\|D'w\|_{H^1}\right)^{3/2}
\end{aligned} \tag{3.16}
$$

for all $t \in (0,T)$, where $T \le 1/CM$.

4 Global in time solutions

In this section, we derive L^2-estimates for the fluid-structure system (2.1)–(2.5) on several different levels of energy. More precisely, as in [IKLT2], where we considered the system subjected to the transmission boundary condition

$$
w_t^i = v^i - \gamma \frac{\partial w^i}{\partial N} \quad \text{on } \Gamma_c \times (0,T), \quad i = 1,2,3 \tag{4.1}
$$

for a parameter $\gamma > 0$, we rely on a priori estimates on time derivatives of the unknown functions. Also, in order to treat the limiting case $\gamma = 0$, we further need estimates on the tangential and time-tangential derivatives. These are needed for the regularity of the wave component (cf. (3.12) and (3.13).)

4.1 First level estimates

We denote by

$$
E(t) = \frac{1}{2}\left(\|v(t)\|_{L^2}^2 + \beta\|w(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2\right), \tag{4.2}
$$

the energy of the system (2.1)–(2.3). From [IKLT2, Lemma 4.1]), recall the inequality

$$
E(t) + \int_0^t D(s)\,ds \le E(0), \tag{4.3}
$$

for all $t \in [0,T]$, where

$$
D(t) = \frac{1}{C}\|\nabla v(t)\|_{L^2}^2 + \alpha\|w_t(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial w}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2. \tag{4.4}
$$

The purpose of the next lemma is to establish estimates on the time integral of the potential energy of the wave component from the damping, secured by the kinetic energy. This entails to establishing a suitable equipartition of the wave energy in the context of the interface coupling.

Lemma 4.1. *We have*

$$
\begin{aligned}
&\int_0^t \|\nabla w\|_{L^2}^2 + \beta\int_0^t \|w\|_{L^2}^2 + \frac{\alpha}{4}\|w(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla(\eta(\cdot,t) - x)\|_{L^2}^2 + \frac{\gamma}{2}\int_{\Gamma_c}\left|\int_0^t \frac{\partial w}{\partial N}(s)\,ds\right|^2 d\sigma(x) \\
&\quad \le CE(0) + \int_0^t \|w_t\|_{L^2}^2 + C\|w_t(t)\|_{L^2}^2 + C\|v(t)\|_{L^2}^2 + \int_0^t \|v\|_{L^2}^2 + \int_0^t (\widetilde{R}(s), \eta(x,s) - x)\,ds
\end{aligned} \tag{4.5}
$$

for all $t \in [0,T]$, *where*

$$\int_0^t (\widetilde{R}(s), \eta(x,s) - x)\, ds$$
$$= \frac{1}{2}\int_0^t \int \partial_t (a_l^j a_l^k) \partial_k (\eta^i - x^i) \partial_j (\eta^i - x^i) + \int_0^t ds \int q(s) \int_0^s \partial_t a_i^k \partial_k (\eta^i - x^i)\, d\tau\, dx. \tag{4.6}$$

Remark 4.2. Note that the equipartition estimate (4.5) in Lemma 4.1 would have had much simpler form if the constants C were allowed to blow up as $\gamma \to 0$. In such case, the last term involving $\widetilde{R}$ would not be necessary. The presence of this term is due to the lack of uniform in γ control of the L^2 norm of $\partial w/\partial N$. This is the reason why the presence of only frictional damping in the wave equation (without the boundary damping) in the coupled problem leads technically to much more involved estimates.

Proof of Lemma 4.1. We take the L^2-inner product of (2.3) with w^i and sum for $i = 1, 2, 3$

$$\int w_{tt}^i w^i - \int \Delta w^i w^i + \alpha \int w_t^i w^i + \beta \int w^i w^i = 0. \tag{4.7}$$

Differentiating the first term in (4.7) by parts in time leads to

$$\frac{d}{dt}\int w_t^i w^i - \int w_t^i w_t^i + \int \partial_j w^i \partial_j w^i + \frac{\alpha}{2}\frac{d}{dt}\int |w|^2 + \beta \int |w|^2 - \int_{\Gamma_c} \partial_j w^i w^i N_j\, d\sigma(x) = 0. \tag{4.8}$$

We integrate in time

$$\int_0^t \int |\nabla w|^2 + \beta \int_0^t \int |w|^2 + \frac{\alpha}{2} \int |w|^2 \,\big|_t$$
$$= \frac{\alpha}{2}\int |w_0|^2 + \int_0^t \int |w_t|^2 - \int w_t^i w^i \,\big|_t + \int w_1^i w_0^i + \int_0^t \int_{\Gamma_c} \partial_j w^i w^i N_j\, d\sigma(x)\, ds. \tag{4.9}$$

The equality (4.9) would provide a desirable equipartition of energy relation if not for the last boundary term, which is a result of the coupling. This term In order to estimate it, we first write

$$w = \eta - x - \gamma \int_0^t \frac{\partial w}{\partial N}\, dx \quad \text{on } \Gamma_c, \tag{4.10}$$

which follows by integrating (4.1) in time. Multiplying both sides of (2.7) by $\eta^i - x^i$, summing for $i = 1, 2, 3$ and integrating the resulting equation leads to

$$\int_0^t \int_{\Gamma_c} \partial_j w^i w^i N_j\, d\sigma(x)\, ds - \int_0^t \int_{\Gamma_c} a_l^j a_l^k \partial_k v^i (\eta^i - x^i) N_j\, d\sigma(x)\, ds + \int_0^t \int_{\Gamma_c} a_i^k q (\eta^i - x^i) N_k\, d\sigma(x)\, ds$$
$$= -\gamma \int_0^t \int_{\Gamma_c} \partial_j w^i \int_0^s \frac{\partial w^i}{\partial N}\, d\tau N_j\, d\sigma(x)\, ds$$
$$= -\frac{\gamma}{2}\int_{\Gamma_c} \left| \int_0^t \frac{\partial w}{\partial N}\, ds \right|^2 d\sigma(x), \tag{4.11}$$

where we utilized (4.10) in the first equality. On the other hand, multiplying (2.1) with $\eta^i - x^i$, summing in i and integrating over Ω_f gives

$$\int v_t^i (\eta^i - x^i) - \int \partial_j \big(a_l^j a_l^k \partial_k v^i \big)(\eta^i - x^i) + \int \partial_k (a_i^k q)(\eta^i - x^i) = 0 \tag{4.12}$$

which leads to

$$\frac{d}{dt}\int v^i(\eta^i - x^i) - \int v^i\eta_t^i + \int a_l^j a_l^k \partial_k v^i \partial_j(\eta^i - x^i) - \int a_i^k q \partial_k(\eta^i - x^i)$$
$$+ \int_{\Gamma_c} a_l^j a_l^k \partial_k v^i (\eta^i - x^i) N_j \, d\sigma(x) - \int_{\Gamma_c} a_i^k q(\eta^i - x^i) N_k \, d\sigma(x) = 0. \tag{4.13}$$

Now, we integrate (4.13) in time and integrals of the last two boundary terms in (4.13) can be obtained from (4.11). Adding add the resulting equation to (4.9). Noting the cancellation of the troublesome boundary integrals due to (4.11), we arrive at

$$\begin{aligned}
&\int_0^t \int |\nabla w|^2 + \beta \int_0^t \int |w|^2 + \frac{\alpha}{2}\int |w|^2 \,\Big|_t + \frac{\gamma}{2}\int_{\Gamma_c}\left|\int_0^t \frac{\partial w}{\partial N}\, ds\right|^2 d\sigma(x) \\
&\quad = \frac{\alpha}{2}\int |w_0|^2 + \int_0^t \int |w_t|^2 - \int w_t^i w^i \,\Big|_t + \int w_1^i w_0^i - \int v^i(\eta^i - x^i)\,\Big|_t + \int_0^t \int |v|^2 \\
&\qquad - \frac{1}{2}\int a_l^j a_l^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)\,\Big|_t + \frac{1}{2}\int_0^t \int \partial_t(a_l^j a_l^k)\partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i) \\
&\qquad + \int_0^t \int a_i^k q \partial_k(\eta^i - x^i)
\end{aligned} \tag{4.14}$$

where we also utilized integration by parts in time and the relation $\eta_t = v$. The last term on the right side equals

$$\begin{aligned}
&\int_0^t \int a_i^k q \partial_k(\eta^i - x^i) \\
&\quad = \int_0^t ds \int q(s) \int_0^s \partial_t a_i^k \partial_k(\eta^i - x^i)\, d\tau\, dx + \int_0^t ds \int q(s) \int_0^s a_i^k \partial_k \partial_t(\eta^i - x^i)\, d\tau\, dx \\
&\quad = \int_0^t ds \int q(s) \int_0^s \partial_t a_i^k \partial_k(\eta^i - x^i)\, d\tau\, dx
\end{aligned} \tag{4.15}$$

since $a_i^k \partial_k \partial_t(\eta^i - x^i)\, d\tau\, dx = a_i^k \partial_k v^i = 0$ by (3.7). In conclusion, we get

$$\begin{aligned}
&\int_0^t \int |\nabla w|^2 + \beta \int_0^t \int |w|^2 + \frac{\alpha}{2}\int |w|^2 \,\Big|_t \\
&\qquad + \frac{1}{2}\int a_l^j a_l^k \partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i)\,\Big|_t + \frac{\gamma}{2}\int_{\Gamma_c}\left|\int_0^t \frac{\partial w}{\partial N}\, ds\right|^2 d\sigma(x) \\
&\quad = \frac{\alpha}{2}\int |w_0|^2 + \int_0^t \int |w_t|^2 - \int w_t^i w^i \,\Big|_t + \int w_1^i w_0^i - \int v^i(\eta^i - x^i)\,\Big|_t + \int_0^t \int |v|^2 \\
&\qquad + \frac{1}{2}\int_0^t \int \partial_t(a_l^j a_l^k)\partial_k(\eta^i - x^i)\partial_j(\eta^i - x^i) + \int_0^t ds \int q(s) \int_0^s \partial_t a_i^k \partial_k(\eta^i - x^i)\, d\tau\, dx.
\end{aligned} \tag{4.16}$$

For the fourth term on the left side we utilize the ellipticity of the matrix a, i.e., $a_l^j a_l^k \xi_j^i \xi_k^i \geq (1/C)|\xi|^2$ for all $\xi \in \mathbb{R}^3 \times \mathbb{R}^3$, while for the pointwise terms on the right, we write

$$-\int w_t^i w^i \,\Big|_t \leq \frac{\alpha}{4}\|w(t)\|_{L^2}^2 + \frac{1}{\alpha}\|w_t(t)\|_{L^2}^2 \tag{4.17}$$

and

$$-\int v^i(\eta^i - x^i)\,\Big|_t \leq \epsilon\|\eta(\cdot,t) - x\|_{L^2}^2 + C_\epsilon\|v(t)\|_{L^2}^2, \tag{4.18}$$

respectively. Since $\eta - x = 0$ on Γ_f, we may apply the Poincaré inequality in order to absorb the first term on the right side of (4.18) with the left side of (4.16). Therefore, the lemma is established. $\square$

Multiplying (4.5) with a small parameter and adding the resulting inequality to (4.3) give

$$E(t) + \int_0^t E(s)\,ds + \bar{\epsilon}\gamma \int_{\Gamma_c} \left| \int_0^t \frac{\partial w}{\partial N}\,ds \right|^2 d\sigma(x) \le CE(0) + C\int_0^t (\widetilde{R}(s), \eta(x,s) - x)\,ds, \tag{4.19}$$

where $\bar{\epsilon} > 0$ is a small parameter independent of γ. Here we utilized the Poincaré inequality for the fluid velocity v; namely, that $\|v(t)\|_{L^2} \le C\|\nabla v(t)\|_{L^2}$. The last term on the right side of (4.19) is treated in Lemma 4.13 below.

4.2 Second level estimates

For simplicity, we assume from here on that $\beta = 1$. We introduce the second level energy

$$E^{(1)}(t) = \frac{1}{2}\big(\|v_t(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2 + \|w_{tt}(t)\|_{L^2}^2 + \|\nabla w_t(t)\|_{L^2}^2\big) \tag{4.20}$$

of the system with the corresponding dissipation

$$D^{(1)}(t) = \frac{1}{C}\|\nabla v_t(t)\|_{L^2}^2 + \alpha\|w_{tt}(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial w_t}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2. \tag{4.21}$$

In order to obtain the integral inequality for $E^{(1)}(t)$, we differentiate the full system (2.1)–(2.3) in time. We obtain

$$v_{tt}^i - \partial_t\partial_j(a_l^j a_l^k \partial_k v^i) + \partial_t\partial_k(a_i^k q) = 0 \quad \text{in} \quad \Omega_f \times (0,T) \tag{4.22}$$

$$a_i^k \partial_k v_t^i + \partial_t a_i^k \partial_k v^i = 0 \quad \text{in} \quad \Omega_f \times (0,T) \tag{4.23}$$

and

$$w_{ttt}^i - \Delta w_t^i + \alpha w_{tt}^i + w_t^i = 0 \quad \text{in} \quad \Omega_e \times (0,T) \tag{4.24}$$

for $i = 1,2,3$.

We start by recalling the next statement from [IKLT2].

Lemma 4.3. *[IKLT2] The energy inequality*

$$E^{(1)}(t) + \int_0^t D^{(1)}(s)\,ds \le E^{(1)}(0) + \int_0^t (R^{(1)}(s), v_t(s))\,ds \tag{4.25}$$

holds for all $t \in [0,T]$, where

$$\int_0^t (R^{(1)}, v_t)\,ds = -\int_0^t\int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v^i \partial_j v_t^i\,dx\,ds + \int_0^t\int_{\Omega_f} \partial_t a_i^k q \partial_k v_t^i\,dx\,ds$$
$$- \int_0^t\int_{\Omega_f} \partial_t a_i^k \partial_t q \partial_k v^i\,dx\,ds. \tag{4.26}$$

For the proof of Lemma 4.3 see [IKLT2, Lemma 4.5].

We now derive estimates on the time integrals of $\|w_t(t)\|_{L^2}$ and $\|\nabla w_t(t)\|_{L^2}$.

Lemma 4.4. *We have*

$$
\begin{aligned}
&\int_0^t \|\nabla w_t\|_{L^2}^2 + \int_0^t \|w_t\|_{L^2}^2 + \frac{\alpha}{4}\|w_t(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla v(t)\|_{L^2}^2 + \frac{\gamma}{2}\left\|\frac{\partial w}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2 \\
&\quad \le CE^{(1)}(0) + C\|w_{tt}(t)\|_{L^2}^2 + C\|v_t(t)\|_{L^2}^2 + \int_0^t \|w_{tt}\|_{L^2}^2 + \int_0^t \|v_t\|_{L^2}^2 \\
&\qquad + \int_0^t (\widetilde{R}^{(1)}(s), v(s))\, ds + C\|\nabla v(0)\|_{L^2}^2 + \frac{\gamma}{2}\left\|\frac{\partial w}{\partial N}(0)\right\|_{L^2(\Gamma_c)}^2, \qquad (4.27)
\end{aligned}
$$

for all $t \in [0,T]$, *where*

$$
\int_0^t (\widetilde{R}^{(1)}(s), v(s))\, ds = -\frac{1}{2}\int_0^t \int \partial_t(a_l^j a_l^k)\partial_k v^i \partial_j v^i + \int_0^t \int \partial_t a_i^k q \partial_k v^i \qquad (4.28)
$$

and where the constant C *depends on* α.

Proof of Lemma 4.4. We take the L^2-inner product of (4.24) with w_t^i, sum for $i = 1,2,3$, and integrate in time in order to obtain

$$
\begin{aligned}
&\int_0^t \int |\nabla w_t|^2 + \int_0^t \int |w_t|^2 + \frac{\alpha}{2}\int |w_t|^2 \,\Big|_t \\
&= \frac{\alpha}{2}\int |w_1|^2 + \int_0^t \int |w_{tt}|^2 - \int w_{tt}^i w_t^i \,\Big|_t + \int w_{tt}^i w_t^i \,\Big|_{t=0} + \int_0^t \int_{\Gamma_c} \partial_j w_t^i w_t^i N_j \, d\sigma(x)\, ds. \qquad (4.29)
\end{aligned}
$$

Next, we take the L^2-inner product of (4.22) with v^i and sum for $i = 1,2,3$,

$$
\int v_{tt}^i v^i - \int \partial_j \partial_t \big(a_l^j a_l^k \partial_k v^i\big) v^i + \int \partial_k \partial_t \big(a_i^k q\big) v^i = 0 \qquad (4.30)
$$

which after differentiating by parts in time leads to

$$
\begin{aligned}
&\frac{d}{dt}\int v_t^i v^i - \int v_t^i v_t^i + \int \partial_t\big(a_l^j a_l^k \partial_k v^i\big)\partial_j v^i - \int \partial_t\big(a_i^k q\big)\partial_k v^i \\
&\quad + \int_{\Gamma_c} \partial_t\big(a_l^j a_l^k \partial_k v^i\big) v^i N_j \, d\sigma(x) - \int_{\Gamma_c} \partial_t\big(a_i^k q\big) v^i N_k \, d\sigma(x) = 0. \qquad (4.31)
\end{aligned}
$$

Now, we observe that

$$
\int \partial_t\big(a_l^j a_l^k \partial_k v^i\big)\partial_j v^i = \frac{1}{2}\int \partial_t\big(a_l^j a_l^k \partial_k v^i \partial_j v^i\big) + \frac{1}{2}\int \partial_t\big(a_l^j a_l^k\big)\partial_k v^i \partial_j v^i \qquad (4.32)
$$

and

$$
\int \partial_t\big(a_i^k q\big)\partial_k v^i = \int \partial_t a_i^k q \partial_k v^i \qquad (4.33)
$$

since $a_i^k \partial_k v^i = 0$. Therefore,

$$
\begin{aligned}
&\frac{d}{dt}\int v_t^i v^i - \int v_t^i v_t^i + \frac{1}{2}\int \partial_t\big(a_l^j a_l^k \partial_k v^i \partial_j v^i\big) + \frac{1}{2}\int \partial_t\big(a_l^j a_l^k\big)\partial_k v^i \partial_j v^i - \int \partial_t a_i^k q \partial_k v^i \\
&\quad + \int_{\Gamma_c} \partial_t\big(a_l^j a_l^k \partial_k v^i\big) v^i N_j \, d\sigma(x) - \int_{\Gamma_c} \partial_t\big(a_i^k q\big) v^i N_k \, d\sigma(x) = 0. \qquad (4.34)
\end{aligned}
$$

We integrate (4.34) in time and add the resulting equality to (4.29). Using partial cancellation of boundary terms

$$\begin{aligned}
&\int_0^t \int_{\Gamma_c} \partial_j w_t^i w_t^i N_j \, d\sigma(x)\, ds - \int_0^t \int_{\Gamma_c} \partial_t \big(a_l^j a_l^k \partial_k v^i\big) v^i N_j \, d\sigma(x)\, ds \\
&\qquad + \int_0^t \int_{\Gamma_c} \partial_t \big(a_i^k q\big) v^i N_k \, d\sigma(x)\, ds \\
&\quad = \int_0^t \int_{\Gamma_c} \partial_j w_t^i v^i N_j \, d\sigma(x)\, ds - \gamma \int_0^t \int_{\Gamma_c} \partial_j w_t^i \frac{\partial w^i}{\partial N} N_j \, d\sigma(x)\, ds - \int_0^t \int_{\Gamma_c} \partial_t \big(a_l^j a_l^k \partial_k v^i\big) v^i N_j \, d\sigma(x)\, ds \\
&\qquad + \int_0^t \int_{\Gamma_c} \partial_t \big(a_i^k q\big) v^i N_k \, d\sigma(x)\, ds \\
&\quad = -\gamma \int_0^t \int_{\Gamma_c} \partial_j w_t^i \frac{\partial w^i}{\partial N} N_j \, d\sigma(x)\, ds \\
&\quad = -\frac{\gamma}{2} \int_{\Gamma_c} \left(\frac{\partial w}{\partial N}\right)^2 d\sigma(x) \bigg|_t + \frac{\gamma}{2} \int_{\Gamma_c} \left(\frac{\partial w}{\partial N}\right)^2 d\sigma(x) \bigg|_0
\end{aligned} \tag{4.35}$$

and we thus obtain

$$\begin{aligned}
&\int_0^t \int |\nabla w_t|^2 + \int_0^t \int |w_t|^2 + \frac{\alpha}{2} \int |w_t|^2 \,\big|_t + \frac{1}{2} \int a_l^j a_l^k \partial_k v^i \partial_j v^i \,\big|_t + \frac{\gamma}{2} \int_{\Gamma_c} \left(\frac{\partial w}{\partial N}\right)^2 d\sigma(x) \bigg|_t \\
&\quad = \frac{\alpha}{2} \int |w_1|^2 + \int_0^t \int |w_{tt}|^2 - \int w_{tt}^i w_t^i \,\big|_t + \int w_{tt}^i w_t^i \,\big|_{t=0} + \int v_t^i v^i \,\big|_t - \int v_t^i v^i \,\big|_{t=0} + \int_0^t \int |v_t|^2 \\
&\qquad + \frac{1}{2} \int a_l^j a_l^k \partial_k v^i \partial_j v^i \,\big|_{t=0} - \frac{1}{2} \int_0^t \int \partial_t (a_l^j a_l^k) \partial_k v^i \partial_j v^i + \int_0^t \int \partial_t a_i^k q \partial_k v^i \\
&\qquad + \frac{\gamma}{2} \int_{\Gamma_c} \left(\frac{\partial w}{\partial N}\right)^2 d\sigma(x) \bigg|_0 .
\end{aligned} \tag{4.36}$$

For the two pointwise terms on the right side of (4.36) we utilize analogous estimates to (4.17) and (4.18); namely,

$$-\int w_{tt}^i w_t^i \,\big|_t \le \frac{\alpha}{4} \|w_t(t)\|_{L^2}^2 + \frac{1}{\alpha} \|w_{tt}(t)\|_{L^2}^2, \tag{4.37}$$

and

$$\int v_t^i v^i \,\big|_t \le \epsilon \|\nabla v(t)\|_{L^2}^2 + C_\epsilon \|v_t(t)\|_{L^2}^2 \tag{4.38}$$

respectively, where $\epsilon > 0$ is sufficiently small. This concludes the proof of the lemma. □

Multiplying (4.27) by a small parameter and adding it to (4.25) gives

$$\begin{aligned}
&E^{(1)}(t) + \int_0^t E^{(1)}(s)\, ds + \bar{\epsilon}\gamma \left\| \frac{\partial w}{\partial N}(t) \right\|_{L^2(\Gamma_c)}^2 \\
&\quad \le C E^{(1)}(0) + C \|\nabla v(0)\|_{L^2}^2 + \bar{\epsilon}\gamma \left\| \frac{\partial w}{\partial N}(0) \right\|_{L^2(\Gamma_c)}^2 \\
&\qquad + \int_0^t (R^{(1)}(s), v_t(s))\, ds + C \int_0^t (\widetilde{R}^{(1)}(s), v(s))\, ds
\end{aligned} \tag{4.39}$$

where $\bar{\epsilon}$ is a small constant independent of γ. The last two terms on the right side of (4.39) are estimated further below.

4.3 Third level estimates

We introduce the next level of energy

$$E^{(2)}(t) = \frac{1}{2}\big(\|v_{tt}(t)\|_{L^2}^2 + \|w_{tt}(t)\|_{L^2}^2 + \|w_{ttt}(t)\|_{L^2}^2 + \|\nabla w_{tt}(t)\|_{L^2}^2\big) \tag{4.40}$$

with the corresponding dissipation

$$D^{(2)}(t) = \frac{1}{C}\|\nabla v_{tt}(t)\|_{L^2}^2 + \alpha\|w_{ttt}(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial w_{tt}}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2. \tag{4.41}$$

Differentiating the full system (2.1)–(2.3) twice in time, we obtain

$$v_{ttt}^i - \partial_{tt}\partial_j(a_l^j a_l^k \partial_k v^i) + \partial_{tt}\partial_k(a_i^k q) = 0 \ \text{ in } \ \Omega_f \times (0,T) \tag{4.42}$$

$$a_j^k \partial_k v_{tt}^j + 2\partial_t a_j^k \partial_k v_t^j + \partial_{tt} a_j^k \partial_k v^j = 0 \ \text{ in } \ \Omega_f \times (0,T) \tag{4.43}$$

$$w_{tttt}^i - \Delta w_{tt}^i + \alpha w_{ttt}^i + w_{tt}^i = 0 \ \text{ in } \ \Omega_e \times (0,T) \tag{4.44}$$

for $i = 1, 2, 3$.

We recall the next statement from [IKLT2].

Lemma 4.5. *[IKLT2] The inequality*

$$E^{(2)}(t) + \int_0^t D^{(2)}(s)ds \le E^{(2)}(0) + \int_0^t (R^{(2)}(s), v_{tt}(s))\, ds \tag{4.45}$$

holds for all $t \in [0,T]$, where

$$\int_0^t (R^{(2)}(s), v_{tt}(s))\, ds = 2\int_0^t \int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v_t^i \partial_j v_{tt}^i \, dx\, ds + \int_0^t \int_{\Omega_f} \partial_{tt}(a_l^j a_l^k)\partial_k v^i \partial_j v_{tt}^i \, dx\, ds - \int_0^t \int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_{tt}^i \, dx\, ds. \tag{4.46}$$

The proof of Lemma 4.5 is given in [IKLT2, Lemma 4.8].

Lemma 4.6. *The estimate*

$$\begin{aligned}
&\int_0^t \|\nabla w_{tt}\|_{L^2}^2 + \int_0^t \|w_{tt}\|_{L^2}^2 + \frac{\alpha}{4}\|w_{tt}(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla v_t(t)\|_{L^2}^2 + \frac{\gamma}{2}\left\|\frac{\partial w_t}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2 \\
&\le CE(0) + C\|w_{ttt}(t)\|_{L^2}^2 + C\|v_{tt}(t)\|_{L^2}^2 + \int_0^t (\widetilde{R}^{(2)}(s), v_t(s))\, ds \\
&\quad + C\|\nabla v_t(0)\|_{L^2}^2 + \frac{\gamma}{2}\left\|\frac{\partial w_t}{\partial N}(0)\right\|_{L^2(\Gamma_c)}^2,
\end{aligned} \tag{4.47}$$

holds for all $t \in [0,T]$, where

$$\int_0^t (\widetilde{R}^{(2)}(s), v_t(s))\, ds = -\frac{3}{2}\int_0^t \int \partial_t(a_l^j a_l^k)\partial_k v_t^i \partial_j v_t^i - \int_0^t \int \partial_{tt}(a_l^j a_l^k)\partial_k v^i \partial_j v_t^i + \int_0^t \int \partial_{tt}(a_i^k q)\partial_k v_t^i. \tag{4.48}$$

Proof of Lemma 4.6 (sketch). We take the L^2-inner product of (4.44) with w^i_{tt}, sum for $i = 1, 2, 3$, and integrate in time to arrive at

$$\int_0^t \int |\nabla w_{tt}|^2 + \int_0^t \int |w_{tt}|^2 + \frac{\alpha}{2} \int |w_{tt}|^2 \,\Big|_t \\ = \frac{\alpha}{2} \int |w_{tt}|^2 \,\Big|_{t=0} + \int_0^t \int |w_{ttt}|^2 - \int w^i_{ttt} w^i_{tt} \,\Big|_t + \int w^i_{ttt} w^i_{tt} \,\Big|_{t=0} + \int_0^t \int_{\Gamma_c} \partial_j w^i_{tt} w^i_{tt} N_j \, d\sigma(x) \, ds. \tag{4.49}$$

We take the L^2-inner product of (4.42) with v^i_t, sum for $i = 1, 2, 3$, and differentiate in time

$$\frac{d}{dt} \int v^i_{tt} v^i_t - \int v^i_{tt} v^i_{tt} + \int \partial_{tt} \big(a^j_l a^k_l \partial_k v^i \big) \partial_j v^i_t - \int \partial_{tt} \big(a^k_i q \big) \partial_k v^i_t \\ + \int_{\Gamma_c} \partial_{tt} \big(a^j_l a^k_l \partial_k v^i \big) v^i_t N_j \, d\sigma(x) - \int_{\Gamma_c} \partial_{tt} \big(a^k_i q \big) v^i_t N_k \, d\sigma(x) = 0. \tag{4.50}$$

Now, we use

$$\int_0^t \int \partial_{tt} (a^j_l a^k_l \partial_k v^i) \partial_j v^i_t = \frac{1}{2} \int a^j_l a^k_l \partial_k v^i_t \partial_j v^i_t - \frac{1}{2} \|\nabla v_t(0)\|^2_{L^2} \\ + \frac{3}{2} \int_0^t \int \partial_t (a^j_l a^k_l) \partial_k v^i_t \partial_j v^i_t + \int_0^t \int \partial_{tt} (a^j_l a^k_l) \partial_k v^i \partial_j v^i_t. \tag{4.51}$$

Integrating (4.50) in time, using (4.51), and adding the resulting equality to (4.49), we arrive at (4.47). □

As in (4.25), we conclude,

$$E^{(2)}(t) + \int_0^t E^{(2)}(s) \, ds + \bar{\epsilon} \gamma \left\| \frac{\partial w_t}{\partial N}(t) \right\|^2_{L^2(\Gamma_c)} \le C E^{(2)}(0) + C \|\nabla v_t(0)\|^2_{L^2} \\ + \bar{\epsilon} \gamma \left\| \frac{\partial w_t}{\partial N}(0) \right\|^2_{L^2(\Gamma_c)} + \int_0^t (R^{(2)}(s), v_{tt}(s)) \, ds + C \int_0^t (\widetilde{R}^{(2)}(s), v_t(s)) \, ds, \tag{4.52}$$

where $\bar{\epsilon} > 0$ is a small parameter.

4.4 First order tangential energy estimates

We derive estimates on the first order derivative ∂_m for a fixed $m = 1, 2$. Note that the boundary conditions are not affected by the action of tangential derivatives. Similarly as in the previous subsections we introduce the notation

$$E_m(t) = \frac{1}{2} \big(\|\partial_m v(t)\|^2_{L^2} + \|\partial_m w(t)\|^2_{L^2} + \|\partial_m w_t(t)\|^2_{L^2} + \|\nabla \partial_m w(t)\|^2_{L^2} \big) \tag{4.53}$$

for the energy and

$$D_m(t) = \frac{1}{C} \|\nabla \partial_m v(t)\|^2_{L^2} + \alpha \|\partial_m w_t(t)\|^2_{L^2} + \gamma \left\| \frac{\partial(\partial_m w)}{\partial N}(t) \right\|^2_{L^2(\Gamma_c)} \tag{4.54}$$

for the corresponding dissipation.

Lemma 4.7. *The energy inequality*

$$E_m(t) + \int_0^t D_m(s)ds \le E_m(0) + \int_0^t (R_m(s), \partial_m v(s))\, ds \tag{4.55}$$

holds for all $t \in [0,T]$, where

$$\int_0^t (R_m(s), \partial_m v(s))\, ds = -\int_0^t \int \partial_m(a_l^j a_l^k - \delta_{jk})\partial_k v^i \partial_j \partial_m v^i - \int_0^t \int \partial_m(a_i^k - \delta_{ik})\partial_m q \partial_k v^i$$
$$+ \int_0^t \int \partial_m(a_i^k - \delta_{ik}) q \partial_k \partial_m v^i. \tag{4.56}$$

Sketch of proof. The proof is obtained by applying the operator ∂_m to the system (2.1)–(2.3) and multiplying the i-th fluid equation by $\partial_m v^i$ and the i-th wave equation by $\partial_m w_t^i$, respectively. Then we use similar arguments as in [IKLT2, Lemma 4.1]. □

We now provide an estimate on the time integral of $\|\nabla \partial_m w\|_{L^2}$.

Lemma 4.8. *We have*

$$\int_0^t \|\nabla \partial_m w\|_{L^2}^2 + \int_0^t \|\partial_m w\|_{L^2}^2 + \frac{\alpha}{4}\|\partial_m w(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla \partial_m(\eta(x,t) - x)\|_{L^2}^2 + \frac{\gamma}{2}\int_{\Gamma_c} \left| \int_0^t \frac{\partial(\partial_m w)}{\partial N}\, ds \right|^2 d\sigma(x)$$
$$\le C E_m(0) + \int_0^t \|\partial_m w_t\|_{L^2}^2 + C\|\partial_m w_t(t)\|_{L^2}^2 + C\|\partial_m v(t)\|_{L^2}^2 + \int_0^t \|\partial_m v\|_{L^2}^2$$
$$+ \int_0^t (\widetilde{R}_m(s), \partial_m(\eta(x,s) - x))\, ds \tag{4.57}$$

for $t \in [0,T]$, where

$$\int_0^t (\widetilde{R}_m(s), \partial_m(\eta(x,s) - x))\, ds$$
$$= -\int_0^t \int \partial_m(a_l^j a_l^k - \delta_{jk})\partial_k v^i \partial_j \partial_m(\eta^i - x^i) + \frac{1}{2}\int_0^t \int \partial_t(a_l^j a_l^k)\partial_k \partial_m(\eta^i - x^i)\partial_j \partial_m(\eta^i - x^i)$$
$$+ \int_0^t \int \partial_m(a_i^k - \delta_{ik}) q \partial_k \partial_m(\eta^i - x^i) + \int_0^t \int a_i^k \partial_m q \partial_k \partial_m(\eta^i - x^i). \tag{4.58}$$

The proof is similar to the one of Lemma 4.1 and will be omitted.

For the last pressure term in (4.58) we use the divergence condition in order to write

$$\int_0^t \int a_i^k \partial_m q \partial_k \partial_m(\eta^i - x^i) = \int_0^t ds \int \partial_m q \int_0^s \left(\partial_t a_i^k \partial_k \partial_m(\eta^i - x^i) - \partial_m a_i^k \partial_k v^i\right) dx\, d\tau. \tag{4.59}$$

We conclude

$$E_m(t) + \int_0^t E_m(s)\, ds + \bar{\epsilon}\gamma \int_{\Gamma_c} \left| \int_0^t \frac{\partial(\partial_m w)}{\partial N}\, ds \right|^2 d\sigma(x)$$
$$\le C E_m(0) + \int_0^t (R_m(s), \partial_m v(s))\, ds + C\int_0^t (\widetilde{R}_m(s), \partial_m(\eta - x))\, ds, \tag{4.60}$$

where $\bar{\epsilon} > 0$ is a small parameter.

4.5 Second order tangential energy estimates

We next derive estimates on the derivative ∂_{mm} for fixed $m = 1, 2$. Similarly as in the previous subsections we introduce the notation

$$E_{mm}(t) = \frac{1}{2}\big(\|\partial_{mm} v(t)\|_{L^2}^2 + \|\partial_{mm} w(t)\|_{L^2}^2 + \|\partial_{mm} w_t(t)\|_{L^2}^2 + \|\nabla\partial_{mm} w(t)\|_{L^2}^2\big) \tag{4.61}$$

for the energy and

$$D_{mm}(t) = \frac{1}{C}\|\nabla\partial_{mm} v(t)\|_{L^2}^2 + \alpha\|\partial_{mm} w_t(t)\|_{L^2}^2 + \gamma\left\|\frac{\partial(\partial_{mm} w)}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2 \tag{4.62}$$

for the corresponding dissipation.

Lemma 4.9. *The energy inequality*

$$E_{mm}(t) + \int_0^t D_{mm}(s)ds \le E_{mm}(0) + \int_0^t (R_{mm}(s), \partial_{mm} v(s))\, ds \tag{4.63}$$

holds for all $t \in [0, T]$*, where*

$$\begin{aligned}\int_0^t (R_{mm}(s), \partial_{mm} v(s))\, ds &= -2\int_0^t\int \partial_m(a_l^j a_l^k - \delta_{jk})\partial_k\partial_m v^i \partial_j\partial_{mm} v^i \\ &\quad - \int_0^t\int \partial_{mm}(a_l^j a_l^k - \delta_{jk})\partial_k v^i \partial_j\partial_{mm} v^i + \int_0^t\int \partial_{mm}(a_i^k - \delta_{ik}) q \partial_k\partial_{mm} v^i \\ &\quad + 2\int_0^t\int \partial_m(a_i^k - \delta_{ik})\partial_m q \partial_k\partial_{mm} v^i + \int_0^t\int a_i^k \partial_{mm} q \partial_k\partial_{mm} v^i.\end{aligned} \tag{4.64}$$

Observe that for the last pressure term we may write

$$\int a_i^k \partial_{mm} q \partial_k\partial_{mm} v^i = -\int \partial_{mm} a_i^k \partial_{mm} q \partial_k v^i - 2\int \partial_m a_i^k \partial_{mm} q \partial_k\partial_m v^i \tag{4.65}$$

by using the divergence condition.

Lemma 4.10. *We have*

$$\begin{aligned}&\int_0^t \|\nabla\partial_{mm} w\|_{L^2}^2 + \int_0^t \|\partial_{mm} w\|_{L^2}^2 + \frac{\alpha}{4}\|\partial_{mm} w(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla\partial_{mm}\eta(t)\|_{L^2}^2 + \frac{\gamma}{2}\int_{\Gamma_c}\left|\int_0^t \frac{\partial(\partial_{mm} w)}{\partial N}\, ds\right|^2 d\sigma(x) \\ &\quad \le CE_{mm}(0) + \int_0^t \|\partial_{mm} w_t\|_{L^2}^2 + C\|\partial_{mm} w_t(t)\|_{L^2}^2 + C\|\partial_{mm} v(t)\|_{L^2}^2 + \int_0^t \|\partial_{mm} v\|_{L^2}^2 \\ &\qquad + \int_0^t (\widetilde{R}_{mm}(s), \partial_{mm}(\eta(x,s) - x))\, ds\end{aligned} \tag{4.66}$$

for $t \in [0, T]$*, where*

$$\begin{aligned}&\int_0^t (\widetilde{R}_{mm}(s), \partial_{mm}(\eta(x,s) - x))\, ds \\ &\quad = -\int_0^t\int \partial_{mm}(a_l^j a_l^k - \delta_{jk})\partial_k v^i \partial_j\partial_{mm}(\eta^i - x^i) - 2\int_0^t\int \partial_m(a_l^j a_l^k - \delta_{jk})\partial_k\partial_m v^i \partial_j\partial_{mm}(\eta^i - x^i) \\ &\qquad + \frac{1}{2}\int_0^t\int \partial_t(a_l^j a_l^k)\partial_k\partial_{mm}(\eta^i - x^i)\partial_j\partial_{mm}(\eta^i - x^i) + \int_0^t\int \partial_{mm}(a_i^k - \delta_{ik}) q \partial_k\partial_{mm}(\eta^i - x^i) \\ &\qquad + 2\int_0^t\int \partial_m(a_i^k - \delta_{ik})\partial_m q \partial_k\partial_{mm}(\eta^i - x^i) + \int_0^t\int a_i^k \partial_{mm} q \partial_k\partial_{mm}(\eta^i - x^i).\end{aligned} \tag{4.67}$$

For the last pressure term in (4.67) we use the divergence condition in order to write

$$\int_0^t \int a_i^k \partial_{mm} q \partial_k \partial_{mm}(\eta^i - x^i)$$
$$= \int_0^t ds \int \partial_{mm} q \int_0^s \big(\partial_t a_i^k \partial_k \partial_{mm}(\eta^i - x^i) - \partial_{mm} a_i^k \partial_k v^i - 2\partial_m a_i^k \partial_k \partial_m v^i\big)\, dx\, d\tau. \tag{4.68}$$

We conclude

$$E_{mm}(t) + \int_0^t E_{mm}(s)\, ds + \bar{\epsilon}\gamma \int_{\Gamma_c} \left| \int_0^t \frac{\partial(\partial_{mm} w)}{\partial N}\, ds \right|^2 d\sigma(x)$$
$$\leq C E_{mm}(0) + \int_0^t (R_{mm}(s), \partial_{mm} v(s))\, ds + C \int_0^t (\widetilde{R}_{mm}(s), \partial_{mm}(\eta - x))\, ds, \tag{4.69}$$

where $\bar{\epsilon} > 0$ is small.

4.6 Mixed time tangential energy estimates

Finally, we derive estimates on the time tangential derivative $\partial_t \partial_m$ for a fixed $m = 1, 2$. We denote by

$$E_{tm}(t) = \frac{1}{2}\big(\|\partial_t \partial_m v(t)\|_{L^2}^2 + \|\partial_t \partial_m w(t)\|_{L^2}^2 + \|\partial_t \partial_m w_t(t)\|_{L^2}^2 + \|\nabla \partial_t \partial_m w(t)\|_{L^2}^2\big) \tag{4.70}$$

the energy and

$$D_{tm}(t) = \frac{1}{C}\|\nabla \partial_t \partial_m v(t)\|_{L^2}^2 + \alpha \|\partial_t \partial_m w_t(t)\|_{L^2}^2 + \gamma \left\| \frac{\partial(\partial_m w_t)}{\partial N}(t) \right\|_{L^2(\Gamma_c)}^2 \tag{4.71}$$

the corresponding dissipation.

Lemma 4.11. *The energy inequality*

$$E_{tm}(t) + \int_0^t D_{tm}(s) ds \leq E_{tm}(0) + \int_0^t (R_{tm}(s), \partial_t \partial_m v(s))\, ds \tag{4.72}$$

holds for all $t \in [0, T]$, where

$$\int_0^t (R_{tm}(s), \partial_t \partial_m v(s))\, ds = -\int_0^t \int \partial_t (a_l^j a_l^k) \partial_m \partial_k v^i \partial_t \partial_m \partial_j v^i$$
$$- \int_0^t \int \partial_m (a_l^j a_l^k - \delta_{jk}) \partial_t \partial_k v^i \partial_t \partial_m \partial_j v^i - \int_0^t \int \partial_t \partial_m (a_l^j a_l^k) \partial_k v^i \partial_t \partial_m \partial_j v^i$$
$$+ \int_0^t \int \partial_t \partial_m a_i^k q \partial_t \partial_m \partial_k v^i + \int_0^t \int \partial_t a_i^k \partial_m q \partial_t \partial_m \partial_k v^i$$
$$+ \int_0^t \int \partial_m (a_i^k - \delta_{ik}) \partial_t q \partial_t \partial_m \partial_k v^i + \int_0^t \int a_i^k \partial_t \partial_m q \partial_t \partial_m \partial_k v^i. \tag{4.73}$$

For the last pressure term on the right side of (4.73) we use

$$a_i^k \partial_t \partial_m \partial_k v^i + \partial_t \partial_m a_i^k \partial_k v^i + \partial_t a_i^k \partial_m \partial_k v^i + \partial_m a_i^k \partial_t \partial_k v^i = 0 \tag{4.74}$$

which follows from the divergence condition.

Lemma 4.12. *We have*

$$\int_0^t \|\nabla\partial_m w_t\|_{L^2}^2 + \int_0^t \|\partial_m w_t\|_{L^2}^2 + \frac{\alpha}{4}\|\partial_m w_t(t)\|_{L^2}^2 + \frac{1}{C}\|\nabla\partial_m v(t)\|_{L^2}^2 + \frac{\gamma}{2}\left\|\frac{\partial(\partial_m w)}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2$$
$$\le CE_{tm}(0) + \frac{1}{\alpha}\|\partial_m w_{tt}(t)\|_{L^2}^2 + C\|\partial_m v_t(t)\|_{L^2}^2 + \int_0^t \|\partial_m w_{tt}\|_{L^2}^2 + \int_0^t \|\partial_m v_t\|_{L^2}^2$$
$$+ \int_0^t (\widetilde{R}_{tm}(s), \partial_m v(s))\, ds + C\|\nabla\partial_m v(0)\|_{L^2}^2 + \frac{\gamma}{2}\left\|\frac{\partial(\partial_m w)}{\partial N}(0)\right\|_{L^2(\Gamma_c)}^2, \tag{4.75}$$

for $t \in [0,T]$, where

$$\int_0^t (\widetilde{R}_{tm}(s), \partial_m v(s))\, ds = -\int_0^t \int \partial_t\partial_m(a_l^j a_l^k)\partial_k v^i \partial_m\partial_j v^i - \int_0^t \int \partial_m(a_l^j a_l^k - \delta_{jk})\partial_t\partial_k v^i\partial_m\partial_j v^i$$
$$- \frac{1}{2}\int_0^t \int \partial_t(a_l^j a_l^k)\partial_m\partial_k v^i \partial_m\partial_j v^i + \int_0^t \int \partial_t\partial_m a_i^k q \partial_m\partial_k v^i$$
$$+ \int_0^t \int \partial_t a_i^k \partial_m q \partial_m\partial_k v^i + \int_0^t \int \partial_m(a_i^k - \delta_{ik})\partial_t q\partial_m\partial_k v^i + \int_0^t \int a_i^k \partial_m\partial_t q \partial_m\partial_k v^i. \tag{4.76}$$

We also utilize

$$\int a_i^k \partial_m\partial_t q \partial_m\partial_k v^i = -\int \partial_m a_i^k \partial_m \partial_t q \partial_k v^i \tag{4.77}$$

which follows by the divergence condition. We conclude

$$E_{tm}(t) + \int_0^t E_{tm}(s)\, ds + \bar{\epsilon}\gamma\left\|\frac{\partial(\partial_m w)}{\partial N}(t)\right\|_{L^2(\Gamma_c)}^2 \le CE_{tm}(0) + C\|\nabla\partial_m v(0)\|_{L^2}^2$$
$$+ \bar{\epsilon}\gamma\left\|\frac{\partial(\partial_m w)}{\partial N}(0)\right\|_{L^2(\Gamma_c)}^2 + \int_0^t (R_{tm}(s), \partial_t\partial_m v(s))\, ds + C\int_0^t (\widetilde{R}_{tm}(s), \partial_t\partial_m(\eta - x))\, ds, \tag{4.78}$$

where $\bar{\epsilon} > 0$ is a small constant.

4.7 Superlinear estimates

Lemma 4.13. *With $\widetilde{R}$ defined in Lemma 4.1, we have*

$$|(\widetilde{R}, \eta - x)| \le C\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\|\nabla(\eta - x)\|_{L^2}^2 + C\|q\|_{H^1}\int_0^t \|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|\nabla(\eta - x)\|_{L^2}, \tag{4.79}$$

for all $t \in [0,T]$, while for the second energy level perturbation terms we have

$$|(R^{(1)}, v_t)| \le C\|v\|_{H^1}^{1/2}\|v\|_{H^2}^{1/2}\|v_t\|_{H^1}(\|v\|_{H^2} + \|q\|_{H^1}) + C\|v\|_{H^1}^{3/2}\|v\|_{H^2}^{1/2}\|q_t\|_{H^1}, \tag{4.80}$$

and

$$|(\widetilde{R}^{(1)}, v)| \le C\|v\|_{H^1}^{3/2}\|v\|_{H^2}^{3/2} + C\|v\|_{H^1}^{3/2}\|v\|_{H^2}^{1/2}\|q\|_{H^1} \tag{4.81}$$

for all $t \in [0,T]$, respectively.

Proof. The estimate (4.80) was provided in [IKLT2, Lemma 4.10], while (4.79) and (4.81) can be obtained similarly by using Hölder's and the Gagliardo-Nirenberg inequalities. Indeed, for (4.79), we have

$$
\begin{aligned}
|(\widetilde{R}, \eta - x)| \le \frac{1}{2} \|\partial_t (a_l^j a_l^k)\|_{L^\infty} \|\partial_k (\eta^i - x^i)\|_{L^2} \|\partial_j (\eta^i - x^i)\|_{L^2} \\
+ \|q\|_{L^6} \int_0^s \|\partial_t a_i^k\|_{L^3} \|\partial_k (\eta^i - x^i)\|_{L^2} \, d\tau,
\end{aligned}
\tag{4.82}
$$

where we utilized (4.6). Then we rely on Lemma 3.1 and the inequalities $\|\partial_t (a_l^j a_l^k)\|_{L^\infty} \le C \|\nabla v\|_{L^\infty} \le C \|v\|_{H^2}^{1/2} \|v\|_{H^3}^{1/2}$ and $\|\partial_t a_i^k\|_{L^3} \le C \|v\|_{H^1}^{1/2} \|v\|_{H^2}^{1/2}$ in order to arrive at (4.79). $\square$

We next recall an estimate from [IKLT2].

Lemma 4.14. *[IKLT2] For $\epsilon_0 \in (0, 1/C]$, we have*

$$
\begin{aligned}
\int_0^t (R^{(2)}, v_{tt}) \, ds \le \epsilon_0 \int_0^t \|\nabla v_{tt}\|_{L^2}^2 \, ds + C_{\epsilon_0} \int_0^t \left(\|v\|_{H^3}^2 + \|q\|_{H^2}^2 \right) \left(\|v\|_{H^1}^{5/2} \|v\|_{H^3}^{3/2} + \|v_t\|_{H^1}^2 \right) ds \\
+ C_{\epsilon_0} \int_0^t \|v\|_{H^1}^{3/2} \|v\|_{H^3}^{1/2} \|q_t\|_{H^1}^2 \, ds \\
+ \epsilon_0 \|q_t(t)\|_{H^1}^2 + \epsilon_0 \|v_t(t)\|_{H^2}^2 + \epsilon_0 \|v(t)\|_{H^3}^2 + C_{\epsilon_0} \|v(t)\|_{H^1}^6 \|v(t)\|_{H^2}^4 \\
+ C_{\epsilon_0} \|v(t)\|_{H^1}^2 \|v(t)\|_{H^2}^2 \|v_t(t)\|_{L^2}^2 + C \int_0^t \left(\|v\|_{H^2}^2 + \|v_t\|_{H^1}^{1/2} \|v_t\|_{H^2}^{1/2} \right) \|q_t\|_{H^1} \|v_t\|_{H^1} \, ds \\
+ C \int_0^t \left(\|v\|_{H^2}^3 + \|v_t\|_{H^1} \|v\|_{H^1}^{1/4} \|v\|_{H^3}^{3/4} \right) \|q_t\|_{H^1} \|v\|_{H^1}^{3/4} \|v\|_{H^3}^{1/4} \, ds \\
+ C \|v(0)\|_{H^3}^6 + C \|v_t(0)\|_{H^1}^4 + C \|q_t(0)\|_{H^1}^2
\end{aligned}
\tag{4.83}
$$

for all $t \in [0, T]$.

For the second perturbation term on the right side of (4.52), we use the following lemma.

Lemma 4.15. *The estimate*

$$
\begin{aligned}
\int_0^t (\widetilde{R}^{(2)}, v_t) \, ds \le C \int_0^t \left(\|v\|_{H^3} + \|q\|_{H^2} \right) \left(\|v\|_{H^1}^{5/4} \|v\|_{H^3}^{3/4} + \|v_t\|_{H^1} \right) \|v_t\|_{H^1} \, ds \\
+ C \int_0^t \|q_t\|_{H^1} \|v\|_{H^1}^{3/4} \|v\|_{H^3}^{1/4} \|v_t\|_{H^1} \, ds + C \|q_t(t)\|_{H^1} \|v(t)\|_{H^1}^{7/4} \|v(t)\|_{H^3}^{1/4} \\
+ C \|q_t(0)\|_{L^2} \|v(0)\|_{H^1}^{3/2} \|v(0)\|_{H^3}^{3/2} \\
+ C \int_0^t \|q_t\|_{H^1} \left(\|v\|_{H^2}^2 + \|v_t\|_{H^1}^{1/2} \|v_t\|_{H^2}^{1/2} \right) \|v\|_{H^1} \, ds
\end{aligned}
\tag{4.84}
$$

is valid for all $t \in [0, T]$.

We may bound the pointwise term on the right side of (4.84) as follows

$$
C \|q_t(t)\|_{H^1} \|v(t)\|_{H^1}^{7/4} \|v(t)\|_{H^3}^{1/4} \le \epsilon_0 \|q_t(t)\|_{H^1}^2 + \epsilon_0 \|v(t)\|_{H^3}^2 + C_{\epsilon_0} \|v(t)\|_{H^1}^{14/3}
\tag{4.85}
$$

for all $t \in [0, T]$.

Lemma 4.16. *For $m = 1, 2$, we have*

$$|(R_m, \partial_m v)| \le C\|a_l^j a_l^k - \delta_{jk}\|_{H^1}\|v\|_{H^2}^{3/2}\|v\|_{H^3}^{1/2} + C\|a_i^k - \delta_{ik}\|_{H^1}\|q\|_{H^1}\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2} \tag{4.86}$$

and

$$\begin{aligned}|(\widetilde{R}_m, \partial_m(\eta - x))| \le{}& C\|a_l^j a_l^k - \delta_{jk}\|_{H^1}\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\|\nabla\partial_m(\eta - x)\|_{L^2} \\ &+ C\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\|\nabla\partial_m(\eta - x)\|_{L^2}^2 \\ &+ C\|a_i^k - \delta_{ik}\|_{H^1}\|q\|_{H^2}\|\nabla\partial_m(\eta - x)\|_{L^2} \\ &+ C\|q\|_{H^1}\int_0^t (\|\nabla\partial_m(\eta - x)\|_{L^2} + \|a_i^k - \delta_{ik}\|_{H^1})\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}.\end{aligned} \tag{4.87}$$

Also, for the second order tangential perturbation terms we have the next statement.

Lemma 4.17. *For $m = 1, 2$, we have*

$$\begin{aligned}|(R_{mm}, \partial_{mm} v)| \le{}& C\|a_l^j a_l^k - \delta_{jk}\|_{H^2}\left(\|\nabla\partial_m v\|_{L^2}^{1/2}\|\nabla\partial_m v\|_{H^1}^{1/2}\|\nabla\partial_{mm} v\|_{L^2} + \|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\|\nabla\partial_{mm} v\|_{L^2}\right) \\ &+ C\|a_i^k - \delta_{ik}\|_{H^2}\left(\|q\|_{H^1}^{1/2}\|q\|_{H^2}^{1/2}\|\nabla\partial_{mm} v\|_{L^2} + \|q\|_{H^2}\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\right)\end{aligned} \tag{4.88}$$

and

$$\begin{aligned}|(\widetilde{R}_{mm}, \partial_{mm}(\eta - x))| \le{}& C\|a_l^j a_l^k - \delta_{jk}\|_{H^2}\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\|\nabla\partial_{mm}(\eta - x)\|_{L^2} \\ &+ C\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\|\nabla\partial_{mm}(\eta - x)\|_{L^2}^2 + C\|a_i^k - \delta_{ik}\|_{H^2}\|q\|_{H^1}^{1/2}\|q\|_{H^2}^{1/2}\|\nabla\partial_{mm}(\eta - x)\|_{L^2} \\ &+ C\|q\|_{H^2}\int_0^t \left(\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\|\nabla\partial_{mm}(\eta - x)\|_{L^2} + \|a_i^k - \delta_{ik}\|_{H^2}\|q\|_{H^2}\|\partial_m v\|_{H^1}^{1/2}\|\partial_m v\|_{H^2}^{1/2}\right)\end{aligned} \tag{4.89}$$

for all $t \in [0, T]$.

Finally, for the mixed time tangential energy level perturbation terms we have the next statement.

Lemma 4.18. *For $m = 1, 2$, we have*

$$\begin{aligned}|(R_{tm}, \partial_t\partial_m v)| \le{}& C\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}(\|v\|_{H^2} + \|q\|_{H^1})\|v_t\|_{H^2} + C\|a_l^j a_l^k - \delta_{jk}\|_{H^2}\|v_t\|_{H^1}^{1/2}\|v_t\|_{H^2}^{3/2} \\ &+ C\|a_i^k - \delta_{ik}\|_{H^2}\|q_t\|_{H^1}(\|v_t\|_{H^1}^{1/2}\|v_t\|_{H^2}^{1/2} + \|v_t\|_{H^2}) + C\|q_t\|_{H^1}\|v\|_{H^2}\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\end{aligned} \tag{4.90}$$

and

$$\begin{aligned}|(\widetilde{R}_{tm}, \partial_m v)| \le{}& C\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\|v\|_{H^2}^2 + C\|a_l^j a_l^k - \delta_{jk}\|_{H^2}\|v_t\|_{H^2}\|v\|_{H^3} \\ &+ C\|q\|_{H^1}^{1/2}\|q\|_{H^2}^{1/2}\|v\|_{H^2}^2 + C\|a_i^k - \delta_{ik}\|_{H^2}\|q_t\|_{H^1}\|v\|_{H^3}\end{aligned} \tag{4.91}$$

for all $t \in [0, T]$.

5 A priori estimates on decay

We denote

$$X(t) = E(t) + E^{(1)}(t) + E^{(2)}(t) + \sum_{m=1}^{2} E_m(t) + \sum_{m=1}^{2} E_{mm}(t) + \sum_{m=1}^{2} E_{tm}(t)$$
$$+ \widetilde{\epsilon}\|\nabla v(t)\|_{L^2}^2 + \widetilde{\epsilon}\|\nabla v_t(t)\|_{L^2}^2 + \widetilde{\epsilon}\|\nabla \partial_m v(t)\|_{L^2}^2$$
$$+ \bar{\epsilon}\gamma \left\| \frac{\partial w}{\partial N}(t) \right\|_{L^2(\Gamma_c)}^2 + \bar{\epsilon}\gamma \left\| \frac{\partial w_t}{\partial N}(t) \right\|_{L^2(\Gamma_c)}^2 + \bar{\epsilon}\gamma \left\| \frac{\partial(\partial_m w)}{\partial N}(t) \right\|_{L^2(\Gamma_c)}^2 \tag{5.1}$$

where $\widetilde{\epsilon}, \bar{\epsilon} > 0$ are small parameters. The fluid-velocity terms in the norm are controlled by the dissipation terms

$$\|\nabla v(t)\|_{L^2}^2 \le \|\nabla v(0)\|_{L^2}^2 + C\int_0^t (D(s) + D^{(1)}(s))\, ds, \tag{5.2}$$

$$\|\nabla v_t(t)\|_{L^2}^2 \le \|\nabla v_t(0)\|_{L^2}^2 + C\int_0^t (D^{(1)}(s) + D^{(2)}(s))\, ds, \tag{5.3}$$

and

$$\|\nabla \partial_m v(t)\|_{L^2}^2 \le \|\nabla \partial_m v(0)\|_{L^2}^2 + C\int_0^t (D_m(s) + D_{tm}(s))\, ds. \tag{5.4}$$

Also, observe that the dissipation terms control the last three boundary terms in (5.1) by using analogous estimates to (5.2)–(5.4).

For the flow map η and the Lagrangian matrix a, we have

$$\|\nabla(\eta - x)\|_{H^s}^2 \le t\int_0^t \|\nabla \eta_t\|_{H^s}^2\, ds \le t\int_0^t \|v\|_{H^{s+1}}^2\, ds \tag{5.5}$$

and

$$\|a_i^k - \delta_{ik}\|_{H^s}^2 \le t\int_0^t \|\partial_t a_i^k\|_{H^s}^2\, ds \le Ct\int_0^t \|v\|_{H^{s+1}}^2\, ds, \quad i,k = 1,2,3 \tag{5.6}$$

with $s = 0, 1, 2$. Similarly,

$$\|a_l^j a_l^k - \delta_{jk}\|_{H^2}^2 \le Ct\int_0^t \|v\|_{H^3}^2\, ds, \quad j,k = 1,2,3. \tag{5.7}$$

From (4.19), (4.79), and (5.5), we obtain

$$E(t) + \int_0^t E(s)\, ds + \int_0^t D(s)\, ds + \bar{\epsilon}\gamma \int_{\Gamma_c} \left| \int_0^t \frac{\partial w}{\partial N}\, ds \right|^2 d\sigma(x)$$
$$\le CE(0) + (C + Ct^2)\int_0^t P(\|v\|_{H^2}, \|q\|_{H^1})\, ds. \tag{5.8}$$

For the second and third level energy estimates, we get

$$E^{(1)}(t) + \int_0^t E^{(1)}(s)\, ds + \int_0^t D^{(1)}(s)\, ds + \bar{\epsilon}\gamma \left\| \frac{\partial w}{\partial N}(t) \right\|_{L^2(\Gamma_c)}^2 \le CE^{(1)}(0) + C\|\nabla v(0)\|_{L^2}^2$$
$$+ \bar{\epsilon}\gamma \left\| \frac{\partial w}{\partial N}(0) \right\|_{L^2(\Gamma_c)}^2 + \int_0^t P_1(\|v\|_{H^2}, \|q\|_{H^1}, \|v_t\|_{H^1}, \|q_t\|_{H^1})\, ds \tag{5.9}$$

and

$$
\begin{aligned}
&E^{(2)}(t)+\int_0^t E^{(2)}(s)\,ds+\int_0^t D^{(2)}(s)\,ds+\bar{\epsilon}\gamma\left\|\frac{\partial w_t}{\partial N}(t)\right\|^2_{L^2(\Gamma_c)}\\
&\quad\le CE^{(2)}(0)+C\|\nabla v_t(0)\|^2_{L^2}+\bar{\epsilon}\gamma\left\|\frac{\partial w_t}{\partial N}(0)\right\|^2_{L^2(\Gamma_c)}\\
&\qquad+\epsilon_0\int_0^t\|\nabla v_{tt}\|^2_{L^2}\,ds+\epsilon_0\|q_t(t)\|^2_{H^1}+\epsilon_0\|v_t(t)\|^2_{H^2}+\epsilon_0\|v(t)\|^2_{H^3}+P_2(\|v(t)\|_{H^2},\|v_t(t)\|_{L^2})\\
&\qquad+\int_0^t P_3(\|v\|_{H^3},\|q\|_{H^2},\|v_t\|_{H^2},\|q_t\|_{H^1})\,ds+P_4(\|v(0)\|_{H^3},\|v_t(0)\|_{H^1},\|q_t(0)\|_{H^1}). \qquad (5.10)
\end{aligned}
$$

Here we utilized (4.39), (4.52), and the superlinear estimates (4.80)–(4.84). Similarly, for any fixed $m\in\{1,2\}$, we have

$$
\begin{aligned}
&E_m(t)+\int_0^t E_m(s)\,ds+\int_0^t D_m(s)\,ds+\bar{\epsilon}\gamma\int_{\Gamma_c}\left|\int_0^t\frac{\partial(\partial_m w)}{\partial N}\,ds\right|^2 d\sigma(x)\\
&\quad\le CE_m(0)+(C+Ct^2)\int_0^t P_5(\|v\|_{H^3},\|q\|_{H^2})\,ds, \qquad (5.11)
\end{aligned}
$$

and

$$
\begin{aligned}
&E_{mm}(t)+\int_0^t E_{mm}(s)\,ds+\int_0^t D_{mm}(s)\,ds+\bar{\epsilon}\gamma\int_{\Gamma_c}\left|\int_0^t\frac{\partial(\partial_{mm} w)}{\partial N}\,ds\right|^2 d\sigma(x)\\
&\quad\le CE_{mm}(0)+(C+Ct^2)\int_0^t P_6(\|v\|_{H^3},\|q\|_{H^2})\,ds \qquad (5.12)
\end{aligned}
$$

with

$$
\begin{aligned}
&E_{tm}(t)+\int_0^t E_{tm}(s)\,ds+\int_0^t D_{tm}(s)\,ds+\bar{\epsilon}\gamma\left\|\frac{\partial(\partial_m w)}{\partial N}(t)\right\|^2_{L^2(\Gamma_c)}\le CE_{tm}(0)+C\|\nabla\partial_m v(0)\|^2_{L^2}\\
&\quad+\bar{\epsilon}\gamma\left\|\frac{\partial(\partial_m w)}{\partial N}(0)\right\|^2_{L^2(\Gamma_c)}+(C+Ct)\int_0^t P_7(\|v\|_{H^3},\|q\|_{H^2},\|v_t\|_{H^2},\|q_t\|_{H^1})\,ds. \qquad (5.13)
\end{aligned}
$$

We use the notation $P_1,P_2,\ldots,P_7$ to denote superlinear polynomials on all of their arguments. From (3.14) and (3.15), we have

$$\|v\|^2_{H^3}+\|q\|^2_{H^2}\le CX(t) \qquad (5.14)$$

and

$$\|v\|^2_{H^2}+\|q\|^2_{H^1}\le CX(t), \qquad (5.15)$$

while using also (3.16), we get

$$\|v_t\|^2_{H^2}+\|q_t\|^2_{H^1}\le CX(t)+CX(t)^2. \qquad (5.16)$$

Therefore,

$$X(t)+\int_0^t X(s)\,ds\le C_0X(0)+C_0(1+t^2)\sum_{j=1}^m\int_0^t X(s)^{\alpha_j}\,ds+C_0\sum_{j=1}^k X(t)^{\beta_j}+C_0\sum_{j=1}^k X(0)^{\beta_j}, \qquad (5.17)$$

for $C_0 \geq 1$, $\alpha_1, \dots, \alpha_m > 1$, and $\beta_1, \dots, \beta_k > 1$. By shifting the time, we get

$$X(t) + \int_\tau^t X(s)\,ds \leq C_0 X(\tau) + C_0(1+t^2) \sum_{j=1}^m \int_\tau^t X(s)^{\alpha_j}\,ds + C_0 \sum_{j=1}^n X(t)^{\beta_j} + C_0 \sum_{j=1}^n X(\tau)^{\beta_j}, \tag{5.18}$$

for $0 \leq \tau \leq t$.

The global well-posedness of (2.1)–(2.5) and the exponential decay of the solution given sufficiently small datum now follows from the next lemma.

Lemma 5.1. *Suppose that $X\colon [0,\infty) \to [0,\infty]$ is continuous at all t such that $X(t)$ is finite and assume that it satisfies* (5.18) *for $0 \leq \tau \leq t$ where $C_0 \geq 1$ with $\alpha_1, \dots, \alpha_m > 1$, and $\beta_1, \dots, \beta_k > 1$. If $X(0) \leq \epsilon \leq 1/C$, where the constant C depends on C_0, m, $\alpha_1, \dots, \alpha_m$ and $\beta_1, \dots, \beta_n$, then $X(t) \leq C\epsilon e^{-t/C}$.*

Proof of Lemma 5.1. By induction on $k \in \mathbb{N}_0$, we shall prove that if $\epsilon > 0$ is a sufficiently small constant (independent of k), then there exist

$$\tau_0 = 0 < \tau_1 < \dots < \tau_k \tag{5.19}$$

such that

$$\tau_j - \tau_{j-1} \leq 4C_0, \qquad j = 1, \dots, k \tag{5.20}$$

with

$$X(\tau_j) \leq \frac{\epsilon}{2^j}, \qquad j = 0, 1, \dots, k \tag{5.21}$$

and

$$X(t) \leq 2C_0 \frac{\epsilon}{2^{j-1}}, \qquad t \in [\tau_{j-1}, \tau_j], \qquad j = 0, 1, \dots, k \tag{5.22}$$

where we set $\tau_{-1} = 0$. The statement clearly holds for $k = 0$. Let now $k \in \mathbb{N}_0$, and assume that $\tau_0 = 0 < \tau_1 < \dots < \tau_k$ with the stated properties have already been constructed. First, we establish

$$X(t) \leq 2C_0 \frac{\epsilon}{2^k}, \qquad t \in [\tau_k, \tau_k + 4C_0] \tag{5.23}$$

if $\epsilon > 0$ is a sufficiently small constant (independent of k). If (5.23) does not hold, there exists $T_0 \in [\tau_k, \tau_k + 4C_0]$ such that

$$X(t) < 2C_0 \frac{\epsilon}{2^k}, \qquad t \in [\tau_k, T_0] \tag{5.24}$$

and

$$X(T_0) = 2C_0 \frac{\epsilon}{2^k}. \tag{5.25}$$

Using (5.18) with $\tau = \tau_k$ and $t = T_0$ (disregarding the second term on the left side of (5.18)), we get

$$2C_0 \frac{\epsilon}{2^k} \leq C_0 \frac{\epsilon}{2^k} + C_0(1+T_0^2) \sum_{j=1}^m 4C_0 \left(2C_0 \frac{\epsilon}{2^k}\right)^{\alpha_j} + C_0 \sum_{j=1}^n \left(2C_0 \frac{\epsilon}{2^k}\right)^{\beta_j} + C_0 \sum_{j=1}^n \left(\frac{\epsilon}{2^k}\right)^{\beta_j}. \tag{5.26}$$

Since $T_0 \leq \tau_k + 4C_0$, we have by induction $T_0 \leq 4(k+1)C_0$. Therefore, (5.26) implies

$$C_0 \leq C_0(1 + 16(k+1)^2 C_0^2) 4C_0 \sum_{j=1}^m 2^{\alpha_j} C_0^{\alpha_j} \frac{\epsilon^{\alpha_j - 1}}{2^{(\alpha_j - 1)k}} + C_0 \sum_{j=1}^n 2^{\beta_j} C_0^{\beta_j} \frac{\epsilon^{\beta_j - 1}}{2^{(\beta_j - 1)k}} + C_0 \sum_{j=1}^n \frac{\epsilon^{\beta_j - 1}}{2^{(\beta_j - 1)k}} \tag{5.27}$$

where we multiplied both sides by $2^k/\epsilon$. Since $(k+1)^2/2^{(\alpha_j-1)k}$ is uniformly bounded in $k \in \mathbb{N}_0$, the equation can not hold if $\epsilon > 0$ is a sufficiently small number.

Next, we show that there exists $\tau_{k+1} \in [\tau_k, \tau_k + 4C_0]$ such that $X(\tau_{k+1}) \leq \epsilon/2^{k+1}$. Assume, contrary to the assertion that

$$X(t) > \frac{\epsilon}{2^{k+1}}, \qquad t \in [\tau_k, \tau_k + 4C_0]. \tag{5.28}$$

Using (5.18) with $\tau = \tau_k$ and $t = \tau_k + 4C_0$ (disregarding the first term on the left side of (5.18)), we get

$$4C_0 \frac{\epsilon}{2^{k+1}} \leq C_0 \frac{\epsilon}{2^k} + C_0(1 + 16(k+1)^2 C_0^2) 4C_0 \sum_{j=1}^{m} \Big(2C_0 \frac{\epsilon}{2^k}\Big)^{\alpha_j} + C_0 \sum_{j=1}^{n} \Big(2C_0 \frac{\epsilon}{2^k}\Big)^{\beta_j} + C_0 \sum_{j=1}^{n} \Big(\frac{\epsilon}{2^k}\Big)^{\beta_j}. \tag{5.29}$$

After absorbing the first term on the right side and multiplying the resulting inequality with $2^k/\epsilon$, we obtain the contradiction using again the boundedness of $(k+1)^2/2^{(\alpha_j-1)k}$. □

This gives the following a priori bound for the solutions.

Lemma 5.2. *Assume the conditions of Theorem 2.1 imposed on the initial data, and let* (v, w, q) *be a solution of* (2.1)–(2.8). *Then there exist constants* $C > 0$ *and* $\epsilon > 0$ *such that if* $X(0) \leq \epsilon$, *then*

$$X(t) \leq C X(0) e^{-t/C} \tag{5.30}$$

where the constants C *and* ϵ *do not depend on* $\gamma > 0$.

Proof of Theorem 2.1. Let $C_0 > 0$ and let the sequence $\{\tau_j\}_{j=0}^{\infty}$ be as in the proof of Lemma 5.1. Then, we have $X(\tau_1) \leq \epsilon/2$ and $X(t) \leq 2C_0\epsilon$ for $t \in [\tau_0, \tau_1]$, which implies by (5.14) that

$$\|v\|_{H^3}^2 \leq X(t) \leq C\epsilon$$

for $t \in [\tau_0, \tau_1]$, and from here

$$\|I - a(t)\|_{H^2}^2 \leq (\tau_1 - \tau_0) \int_{\tau_0}^{\tau_1} \|\partial_t a(s)\|_{H^2}^2 \, ds \leq C(\tau_1 - \tau_0) \int_{\tau_0}^{\tau_1} \|v(s)\|_{H^3}^2 \, ds \leq C(4C_0)^2 \epsilon$$

for $t \in [\tau_0, \tau_1]$, where we utilized that $\tau_1 - \tau_0 \leq 4C_0$. By induction on $j \in \mathbb{N}_0$, we conclude that $\|v(t)\|_{H^3}^2 \leq C\epsilon e^{-t/C}$ and $\|I - a(t)\|_{H^2}^2 \leq C\epsilon$ for all $t > 0$. This concludes the proof of Theorem 2.1. □

6 Construction of solutions

In order to justify the a priori estimates provided in the previous sections, we now construct a solution to the problem (2.1)–(2.8) for data as in the statement of Theorem 2.2 which are in addition small as in the statement of Theorem 2.1. For simplicity of the presentation, we consider in detail only the case $\gamma = 0$. The construction for $\gamma > 0$ is easier due to additional a priori estimates obtainable for normal derivatives.

6.1 Linear System with Non-homogeneous Divergence Conditions

We start with a theorem from [MZ1] on existence of the Stokes system with the Neumann boundary condition (cf. [MZ2] for applications). In [MZ1, Theorem 1], the statement is worked out for L^p spaces where $p > 3$, but the same proof, with simplifications, applies also in our case where $p = 2$. The corresponding result is referred to as the *Maximal parabolic regularity.* Particular attention should be paid to the treatment of inhomogeneity in the divergence condition which is required to satisfy structural condition (6.6).

Lemma 6.1. *Consider the system*

$$v_t^k - \Delta v^k + \partial_k q = f^k \quad \text{in } \Omega_f \times (0,T) \tag{6.1}$$

$$\partial_i v^i = g \quad \text{in } \Omega_f \times (0,T) \tag{6.2}$$

subject to the mixed boundary conditions

$$\partial_j v^k N^j - q N^k = h^k \quad \text{on } \Gamma_c \times (0,T) \tag{6.3}$$

$$v^k = 0 \quad \text{on } \Gamma_f \times (0,T) \tag{6.4}$$

for $k = 1,2,3$ and subject to the compatibility conditions

$$\int_{\Omega_f} g(0)\, dx = \int_{\Gamma_c} v_0 \cdot N \, d\sigma(x) \tag{6.5}$$

and the assumption

$$g_t = \operatorname{div} A + B \tag{6.6}$$

such that $A, B \in L^2(\Omega_f \times [0,T])$. Let $v_0 \in H^1(\Omega_f)$. If the forcing terms obey $f \in L^2(\Omega_f \times [0,T])$, $g \in L^2([0,T]; H^1(\Omega_f))$, and $h \in L^2([0,T]; H^{1/2}(\Gamma_c)) \cap H^{1/4}([0,T]; L^2(\Gamma_c))$, then there exists a unique solution (v,q) on $[0,T]$ to the non-homogeneous system which satisfies

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^2(\Omega_f))} + \|v\|_{C([0,T];H^1(\Omega_f))} + \|q\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + \|q\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \|v_t\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \leq C\big(\|v_0\|_{H^1(\Omega_f)} + \|f\|_{L^2([0,T]\times\Omega_f)} + \|g\|_{L^2([0,T];H^1(\Omega_f))} + \|A\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad + \|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \|h\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\big)
\end{aligned} \tag{6.7}$$

where $C > 0$ is a constant.

Now, we apply this statement to the system

$$v_t - \Delta v + \nabla q = f \quad \text{in } \Omega_f \times (0,T) \tag{6.8}$$

$$\operatorname{div} v = g \quad \text{in } \Omega_f \times (0,T) \tag{6.9}$$

$$w_{tt} - \Delta w = -\alpha w_t - \beta w \quad \text{in } \Omega_e \times (0,T) \tag{6.10}$$

with boundary conditions

$$v = w_t \quad \text{on } \Gamma_c \times (0,T) \tag{6.11}$$

$$v = 0 \quad \text{on } \Gamma_f \times (0,T) \tag{6.12}$$

$$\frac{\partial v}{\partial N} - qN = \frac{\partial w}{\partial N} + h \quad \text{on } \Gamma_c \times (0,T) \tag{6.13}$$

where f, g, and h are given, while v, q, and w are unknown.

Lemma 6.2. *Consider the system* (6.8)–(6.10) *with the boundary conditions* (6.11)–(6.13). *Suppose that the initial data satisfy* $(v_0, w_0, w_1) \in (V \cap H^{5/2}(\Omega_f)) \times H^{11/4-\delta}(\Omega_e) \times H^{7/4-\delta}(\Omega_e)$ *for some* $\delta \in (0, 1/4)$, *and*

$$\Delta v(0) - \nabla q(0) + f(0) \in H^1(\Omega_f) \tag{6.14}$$

with $q(0)$ *determined from the elliptic system* (6.22). *In addition the quantities* f, g, *and* h *satisfy*

$$\begin{aligned} &f \in L^2([0,T]; H^1(\Omega_f)) \\ &f_t \in L^2([0,T]; L^2(\Omega_f)) \\ &g \in L^2([0,T]; H^2(\Omega_f)) \\ &g_t \in L^2([0,T]; H^1(\Omega_f)) \\ &h \in L^2([0,T]; H^{3/2}(\Gamma_c)) \\ &h_t \in H^{1/4}([0,T]; L^2(\Gamma_c)) \cap L^2([0,T]; H^{1/2}(\Gamma_c)) \\ &A, B \in L^2([0,T]; L^2(\Omega_f)) \end{aligned} \tag{6.15}$$

and

$$g_{tt} = \operatorname{div} A + B \tag{6.16}$$

for some time $T > 0$. *Assume that the compatibility conditions*

$$w_1 = v_0 \quad \text{on } \Gamma_c \tag{6.17}$$

$$\frac{\partial w_0}{\partial N} \cdot \tau = \frac{\partial v_0}{\partial N} \cdot \tau + h(0) \cdot \tau \quad \text{on } \Gamma_c \tag{6.18}$$

$$\Delta w_0 - \alpha w_1 - \beta w_0 = \Delta v_0 - \nabla q_0 + f(0) \quad \text{on } \Gamma_c \tag{6.19}$$

and

$$v_0 = 0, \quad \text{on } \Gamma_f \tag{6.20}$$

$$\Delta v_0 - \nabla q_0 + f(0) = 0 \quad \text{on } \Gamma_f \tag{6.21}$$

hold where q_0 *solves the problem*

$$\Delta q_0 = -g_t(0) + \Delta g(0) + \operatorname{div} f(0) \quad \text{in } \Omega_f \tag{6.22}$$

with the boundary conditions

$$\begin{aligned} &\frac{\partial q_0}{\partial N} = \Delta v_0 \cdot N + f(0) \cdot N \quad \text{on } \Gamma_f \\ &q_0 = \frac{\partial v_0}{\partial N} \cdot N - \frac{\partial w_0}{\partial N} \cdot N - h(0) \cdot N \quad \text{on } \Gamma_c. \end{aligned} \tag{6.23}$$

Then there exists a solution (v, q, w) *on* $(0,T)$ *which belongs to* Y *where*

$$Y = \Big\{(v,q,w) : v \in L^2([0,T]; H^3(\Omega_f)), v_t \in L^2([0,T]; H^2(\Omega_f)), v_{tt} \in L^2([0,T]; L^2(\Omega_f)),$$
$$q \in L^2([0,T]; H^2(\Omega_f)), q_t \in L^2([0,T]; H^1(\Omega_f)), q_t|_{\Gamma_c} \in H^{1/4}([0,T]; L^2(\Gamma_c)),$$
$$\partial_t^j w \in L^\infty([0,T]; H^{11/4-\delta-j}(\Omega_e)),\ j = 0,1,2\Big\} \tag{6.24}$$

and the corresponding estimates (6.39) *and* (6.44) *below hold.*

We note that as the system (6.8)–(6.10) is linear and that there is no loss of regularity in the fluid and wave variables. The time of existence, which is independent of the size of the data, can be extended up to any time $T > 0$ (by repeating the estimates below in time steps).

Remark 6.3. The main idea behind the proof below is the following. By using the maximal parabolic regularity applied to the time derivatives of the equation, we obtain sufficient regularity of the boundary data v_t which is then propagated via interface on the wave component. By resorting to standard regularity-trace estimates known for the wave dynamics, one is bound to start loosing derivatives. This results in a well-recognized mismatch between the parabolic and the hyperbolic regularity. What saves the present situation is the fact that the transfer of regularity involves only the boundary traces of the solution of the wave equation. It is now well-known (cf. [S]) that the wave solutions have better behaving traces than implied by the interior regularity and the trace theory. In fact, we shall show that there is no loss of regularity in propagating parabolic solution via the interface. This phenomenon is due to the sharp regularity of the Dirichlet-Neumann map established in [LLT] and [T]. This amounts to the following diagram of transfer:

$$v|_{\Gamma_c} = \underbrace{w_t|_{\Gamma_c} \Rightarrow \Box w \Rightarrow \frac{\partial}{\partial N} w}_{\text{D}-\text{N map}} = \underbrace{\frac{\partial}{\partial N} v - qN - h \Rightarrow \text{fluid eq} \Rightarrow v|_{\Gamma_c}}_{\text{Max}-\text{Regularity}}.$$

We would also like to point out that the above gain of regularity is due to the particular way of constructing the fixed point, where Dirichlet traces of the fluid are fed into the solid as Dirichlet data, so the normal stresses exiting the solid enter as the Neumann data into the fluid (see the diagram above).

Proof. Using Lemma 6.1 on the time differentiated system, we get

$$\begin{aligned}
&\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^1(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + \|q_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \le C\bigg(\|v_t(0)\|_{H^1(\Omega_f)} + \|f_t\|_{L^2([0,T]\times\Omega_f)} + \|g_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad\qquad + \|A\|_{L^2([0,T];L^2(\Omega_f))} + \|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h_t\|_{L^2([0,T];H^{1/2}(\Gamma_c))} \\
&\qquad\qquad + \|h_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{1/2}(\Gamma_c))}\bigg).
\end{aligned} \tag{6.25}$$

Next, we estimate the normal derivative of w_t using the sharp trace regularity result for the wave equation from [LLT, LT],

$$\left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{s-1}(\partial\Omega_e))} + \left\|\frac{\partial w}{\partial N}\right\|_{H^{s-1}([0,T];L^2(\partial\Omega_e))}$$
$$\leq C(\|w_0\|_{H^s(\Omega_e)} + \|w_1\|_{H^{s-1}(\Omega_e)} + \|w\|_{L^2([0,T];H^s(\partial\Omega_e))} + \|w\|_{H^s([0,T];L^2(\partial\Omega_e))}), \tag{6.26}$$

given the boundary data $w_t = v$ on Γ_c. We also note that similar estimate is valid when $\gamma > 0$, because in this case Lopatinski condition is satisfied and provides H^s regularity of normal stresses with H^s regularity of Dirichlet absorbing boundary data. Applying the estimate (6.26) to the time differentiated wave equation with $s = 3/2$ we have

$$\left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/4}([0,T];L^2(\Gamma_c))}$$
$$\leq C(\|w_1\|_{H^{3/2}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{1/2}(\Omega_e)} + \|v\|_{L^2([0,T];H^{3/2}(\Gamma_c))} + \|v\|_{H^{3/2}([0,T];L^2(\Gamma_c))}). \tag{6.27}$$

We next appeal to the estimates

$$\|v\|_{L^2([0,T];H^{3/2}(\Gamma_c))} \leq \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^1(\Omega_f))}, \qquad \epsilon \in (0,1] \tag{6.28}$$

and

$$\|v\|_{H^{3/2}([0,T];L^2(\Gamma_c))} \leq \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))}, \qquad \epsilon \in (0,1] \tag{6.29}$$

which follow from the trace, interpolation, and Young's inequalities. Using (6.28) and (6.29) in (6.27), we get

$$\left\|\frac{\partial w_t}{\partial N}\right\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/4}([0,T];L^2(\Gamma_c))}$$
$$\leq C(\|w_1\|_{H^{3/2}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{1/2}(\Omega_e)} + \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))}$$
$$+ \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))}) \tag{6.30}$$

and thus, substituting this inequality in (6.25),

$$\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{C([0,T];H^1(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))}$$
$$+ \|q_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))}$$
$$\leq C\bigg(\|v_t(0)\|_{H^1(\Omega_f)} + \|f_t\|_{L^2([0,T]\times\Omega_f)} + \|g_t\|_{L^2([0,T];H^1(\Omega_f))}$$
$$+ \|A\|_{L^2([0,T];L^2(\Omega_f))} + \|B\|_{L^2([0,T];L^2(\Omega_f))} + \|h_t\|_{L^2([0,T];H^{1/2}(\Gamma_c))}$$
$$+ \|h_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \|w_1\|_{H^{3/2}(\Omega_e)} + \|w_{tt}(0,\cdot)\|_{H^{1/2}(\Omega_e)} + \epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))}$$
$$+ \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))}\bigg). \tag{6.31}$$

In order to obtain the full regularity, we also use the elliptic Stokes estimate

$$\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))}$$
$$\leq C\bigg(\|f\|_{L^2([0,T];H^1(\Omega_f))} + \|v_t\|_{L^2([0,T];H^1(\Omega_f))}$$
$$+ \|g\|_{L^2([0,T];H^2(\Omega_f))} + \|h\|_{L^2([0,T];H^{3/2}(\Gamma_c))} + \left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{3/2}(\Gamma_c))}\bigg). \tag{6.32}$$

We again estimate the normal derivative of w using the sharp trace regularity estimates of the wave equation

$$\left\|\frac{\partial w}{\partial N}\right\|_{L^2([0,T];H^{3/2}(\Gamma_c))} \leq \left\|\frac{\partial w}{\partial N}\right\|_{H^{3/2}(\Sigma_c)} \leq C(\|w\|_{H^{5/2}(\Sigma_c)} + \|w_0\|_{H^{5/2}(\Omega_e)} + \|w_1\|_{H^{3/2}(\Omega_e)}) \tag{6.33}$$

where

$$\Sigma_c = \Gamma_c \times [0,T]. \tag{6.34}$$

Now, we estimate the first term on the far right side of (6.33) as

$$\begin{aligned}
\|w\|_{H^{5/2}(\Sigma_c)} &= \|\eta - x\|_{L^2([0,T];H^{5/2}(\Gamma_c))} + \|w\|_{H^{5/2}([0,T];L^2(\Gamma_c))} \\
&\leq \left\|\int_0^t v\, ds\right\|_{L^2([0,T];H^{5/2}(\Gamma_c))} + \|w\|_{H^{5/2}([0,T];L^2(\Gamma_c))} \\
&\leq CT^{1/2}\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|v\|_{H^{3/2}([0,T];L^2(\Gamma_c))} \\
&\leq CT^{1/2}\|v\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} \\
&\quad + C_\epsilon\|v\|_{L^2([0,T];H^2(\Omega_f))} \\
&\leq CT^{1/2}\|v\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} \\
&\quad + C_\epsilon T^{1/2}\|v_t\|_{L^2([0,T];H^2(\Omega_f))}
\end{aligned} \tag{6.35}$$

where we used (6.29) in the third step.

Replacing (6.33) and (6.35) in (6.32), we get

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} \\
&\quad \leq C\bigg(\|f\|_{L^2([0,T];H^1(\Omega_f))} + \|v_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + \|g\|_{L^2([0,T];H^2(\Omega_f))} + \|h\|_{L^2([0,T];H^{3/2}(\Gamma_c))} + \|w_0\|_{H^{5/2}(\Omega_e)} + \|w_1\|_{H^{3/2}(\Omega_e)} \\
&\qquad + T^{1/2}\|v\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))} \\
&\qquad + C_\epsilon T^{1/2}\|v_t\|_{L^2([0,T];H^2(\Omega_f))}\bigg).
\end{aligned} \tag{6.36}$$

Combining this inequality with (6.31) leads to

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\quad + \|v_t\|_{C([0,T];H^1(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|q_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \leq (C_\epsilon T^{1/2} + C)\|v_t(0)\|_{H^1(\Omega_f)} + C\|f\|_{L^2([0,T];H^1(\Omega_f))} + C\|f_t\|_{L^2([0,T]\times\Omega_f)} \\
&\qquad + C\|g\|_{L^2([0,T];H^2(\Omega_f))} + C\|g_t\|_{L^2([0,T];H^1(\Omega_f))} + C\|A\|_{L^2([0,T];L^2(\Omega_f))} + C\|B\|_{L^2([0,T];L^2(\Omega_f))} \\
&\qquad + C\|h\|_{L^2([0,T];H^{3/2}(\Gamma_c))} + C\|h_t\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + C\|h_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\qquad + C\|w_0\|_{H^{5/2}(\Omega_e)} + C\|w_1\|_{H^{3/2}(\Omega_e)} + C\|w_{tt}(0,\cdot)\|_{H^{1/2}(\Omega_e)} \\
&\qquad + CT^{1/2}\|v\|_{L^2([0,T];H^3(\Omega_f))} + C_\epsilon T^{1/2}\|v_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\qquad + C\epsilon\|v\|_{L^2([0,T];H^3(\Omega_f))} + C\epsilon\|v\|_{H^2([0,T];L^2(\Omega_f))},
\end{aligned} \tag{6.37}$$

where we used the inequality

$$\|v\|_{L^2([0,T];H^2(\Omega_f))} \leq CT^{1/2}\|v_0\|_{H^2(\Omega_f)} + CT^{1/2}\|v_t\|_{L^2([0,T];H^2(\Omega_f))}. \tag{6.38}$$

Choosing $\epsilon > 0$ sufficiently small and assuming T is a sufficiently small constant, the last four terms on the right side of (6.37) can be absorbed into the left side from where

$$\begin{aligned}
&\|v\|_{L^2([0,T];H^3(\Omega_f))} + \|q\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\quad + \|v_t\|_{C([0,T];H^1(\Omega_f))} + \|q_t\|_{L^2([0,T];H^1(\Omega_f))} + \|q_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \|v_{tt}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\leq (C_\epsilon T^{1/2} + C)\|v_t(0)\|_{H^1(\Omega_f)} + C\|f\|_{L^2([0,T];H^1(\Omega_f))} + C\|f_t\|_{L^2([0,T]\times\Omega_f)} \\
&\quad + C\|g\|_{L^2([0,T];H^2(\Omega_f))} + C\|g_t\|_{L^2([0,T];H^1(\Omega_f))} + C\|A\|_{L^2([0,T];L^2(\Omega_f))} + C\|B\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad + C\|h\|_{L^2([0,T];H^{3/2}(\Gamma_c))} + C\|h_t\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + C\|h_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\quad + C\|w_0\|_{H^{5/2}(\Omega_e)} + C\|w_1\|_{H^{3/2}(\Omega_e)} + C\|w_{tt}(0,\cdot)\|_{H^{1/2}(\Omega_e)}.
\end{aligned} \tag{6.39}$$

Next, using the interior regularity of the wave equation yields

$$\begin{aligned}
&\|w_{tt}\|_{L^\infty([0,T];H^{3/4-\delta}(\Omega_e))} + \|w_{ttt}\|_{L^\infty([0,T];H^{-1/4-\delta}(\Omega_e))} \\
&\quad \leq C\|v_t\|_{H^{3/4-\delta}(\Sigma_c)} + \|w_{tt}(0)\|_{H^{3/4-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-1/4-\delta}(\Omega_e)} \\
&\quad \leq C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|v_t\|_{H^{3/4-\delta}([0,T];L^2(\Gamma_c))} + \|w_{tt}(0)\|_{H^{3/4-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-1/4-\delta}(\Omega_e)} \\
&\quad \leq C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|v_t\|_{H^1([0,T];L^2(\Omega_f))} + \|w_{tt}(0)\|_{H^{3/4-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-1/4-\delta}(\Omega_e)}
\end{aligned} \tag{6.40}$$

where we used $T \leq 1$, which we may assume without loss of generality, and an interpolation trace inequality

$$\|v_t\|_{H^{3/4}([0,T];L^2(\Gamma_c))} \leq C\|v_t\|_{H^1([0,T];L^2(\Omega_f))}^{3/4}\|v_t\|_{L^2([0,T];H^2(\Omega_f))}^{1/4}. \tag{6.41}$$

By the elliptic regularity, we get (using $T \leq 1$ in particular)

$$\begin{aligned}
&\|w(t)\|_{L^\infty([0,T];H^{11/4-\delta}(\Omega_e))} \\
&\quad \leq C\|w_{tt}(t)\|_{L^\infty([0,T];H^{3/4-\delta}(\Omega_e))} + C\alpha\|w_t(t)\|_{L^\infty([0,T];H^{3/4-\delta}(\Omega_e))} + C\left\|\int_0^t v\,ds\right\|_{L^\infty([0,T];H^{9/4-\delta}(\Gamma_c))} \\
&\quad \leq C\|w_{tt}(t)\|_{L^\infty([0,T];H^{3/4-\delta}(\Omega_e))} + C\alpha\|w_t(t)\|_{L^\infty([0,T];H^{3/4-\delta}(\Omega_e))} + C\|v\|_{L^2([0,T];H^{11/4-\delta}(\Omega_f))}
\end{aligned} \tag{6.42}$$

and

$$\begin{aligned}
\|w_t(t)\|_{L^\infty([0,T];H^{7/4-\delta}(\Omega_e))} &\leq C\|w_{ttt}(t)\|_{L^\infty([0,T];H^{-1/4-\delta}(\Omega_e))} + C\alpha\|w_{tt}(t)\|_{L^\infty([0,T];H^{-1/4-\delta}(\Omega_e))} \\
&\quad + C\left\|\int_0^t v_t\,ds\right\|_{L^\infty([0,T];H^{5/4-\delta}(\Gamma_c))} + C\|w_1\|_{H^{5/4-\delta}(\Gamma_c)} \\
&\leq C\|w_{ttt}(t)\|_{L^\infty([0,T];H^{-1/4-\delta}(\Omega_e))} + C\alpha\|w_{tt}(t)\|_{L^\infty([0,T];H^{-1/4-\delta}(\Omega_e))} \\
&\quad + C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|w_1\|_{H^{7/4-\delta}(\Omega_e)}.
\end{aligned} \tag{6.43}$$

Summarizing the estimates (6.40), (6.42), and (6.43) for the wave equation, we get

$$\begin{aligned}
&\|w(t)\|_{L^\infty([0,T];H^{11/4-\delta}(\Omega_e))} + \|w_t(t)\|_{L^\infty([0,T];H^{7/4-\delta}(\Omega_e))} \\
&\quad + \|w_{tt}\|_{L^\infty([0,T];H^{3/4-\delta}(\Omega_e))} + \|w_{ttt}\|_{L^\infty([0,T];H^{-1/4-\delta}(\Omega_e))} \\
&\le C\|v\|_{L^2([0,T];H^{11/4-\delta}(\Omega_f))} + C\|v_t\|_{L^2([0,T];H^2(\Omega_f))} + C\|v_t\|_{H^1([0,T];L^2(\Omega_f))} \\
&\quad + C\|w_1\|_{H^{7/4-\delta}(\Omega_e)} + \|w_{tt}(0)\|_{H^{3/4-\delta}(\Omega_e)} + \|w_{ttt}(0)\|_{H^{-1/4-\delta}(\Omega_e)}
\end{aligned} \tag{6.44}$$

and the proof is completed. □

Recall the space Y introduced in Lemma 6.2 (cf. (6.24)), and denote

$$\begin{aligned}
Y(0) = \Big\{&(v_0, w_0, w_1) \in (V \cap H^{5/2}(\Omega_f)) \times H^{11/4-\delta}(\Omega_e) \times H^{7/4-\delta}(\Omega_e), \\
&\Delta v(0) - \nabla q(0) + f(0) \in H^1(\Omega_f) \text{ with compatibility conditions } (6.17) - (6.21)\Big\},
\end{aligned} \tag{6.45}$$

where $q(0)$ is determined from the elliptic system (6.22).

Remark 6.4. It is important that the space of regularity imposed on the initial data (v_0, w_0, w_1) is invariant under the dynamics. Indeed, by the intermediate value theorem, we obtain that for $(v, w, w_1) \in Y$ we have $v \in C(0, T; H^{5/2}(\Omega_f))$ and $v_t \in C(0, T; H^1(\Omega_f))$. This allows us to continue the solution for all times $T > 0$.

The concluding estimate proven in Lemma 6.2 above can be summarized as follows: For any initial data $y_0 = (v_0, w_0, w_1) \in Y(0)$, for any $T > 0$, and for the forcing data $d = (f, g, h)$, there exists a unique solution $y(t) = (v(t), q(t), w(t), w_t(t))$ such that

$$\|y\|_Y \le C_T\|y_0\|_{Y(0)} + C_T\|d\|_D \tag{6.46}$$

where D denotes the space of regularity listed in (6.15).

The estimate (6.46) is the basis for constructing solutions to the non-autonomous problem with given coefficients $a(x, t)$.

In the next statement, we establish the global existence of solutions to the system (2.1)–(2.3) with given coefficients $a(x, t)$ sufficiently close to the identity matrix.

Lemma 6.5. *Consider the linear system*

$$v_t^i - \partial_j(a_l^j a_l^k \partial_k v^i) + \partial_k(a_i^k q) = F^i \quad \text{in } \Omega_f \times (0,T), \qquad i = 1,2,3 \tag{6.47}$$

$$a_i^k \partial_k v^i = G \quad \text{in } \Omega_f \times (0,T), \tag{6.48}$$

$$w_{tt}^i - \Delta w^i + \alpha w_t^i + \beta w^i = 0 \quad \text{in } \Omega_f \times (0,T), \qquad i = 1,2,3, \tag{6.49}$$

with the boundary conditions

$$v^i = w_t^i \quad \text{on } \Gamma_c \times (0,T), \qquad i = 1,2,3 \tag{6.50}$$

$$a_l^j a_l^k \partial_k v^i N_j - a_i^k q N_k = \partial_j w^i N_j + H^i \quad \text{on } \Gamma_c \times (0,T) \tag{6.51}$$

$$v^i = 0 \quad \text{on } \Gamma_c \times (0,T), \qquad i = 1,2,3, \tag{6.52}$$

where the coefficient matrix $a = a(x,t)$ is given so that it obeys

$$a(0) = I, \quad \partial_k a_j^k = 0, \qquad k, j = 1, 2, 3, \tag{6.53}$$

in addition to the conditions

$$\begin{aligned}&\|a - I\|_{L^\infty([0,T];H^2(\Omega_f))}, \|a^T : a - I\|_{L^\infty([0,T];H^2(\Omega_f))}, \|\partial_t(a^T : a)\|_{L^\infty([0,T];H^2(\Omega_f))},\\ &\|\partial_t a\|_{L^\infty([0,T];H^2(\Omega_f))}, \|\partial_{tt} a\|_{L^\infty([0,T];H^1(\Omega_f))} \le \epsilon\end{aligned} \tag{6.54}$$

for some sufficiently small ϵ, where $\epsilon < \epsilon_0 \in (0,1)$, and $T > 0$. Assume that the initial data (v_0, w_0, w_1) satisfies the assumptions and compatibility conditions in Lemma 6.2. Also, we assume that the non-homogeneous terms F, G, and H have the regularity given by (6.15). Then, there exists a unique solution (v, q, w, w_t) to the system (6.47)–(6.49) on $(0,T)$ which belongs to the space Y (cf. (6.24)).

More precisely: For any $T > 0$ and any $\epsilon < 1/2C_T$ (where C_T is determined by (6.46)), and any data $y_0 = (v_0, w_0, w_1) \in Y(0)$ and $d = (F, G, H)$, the solution $y \in Y$ satisfies the estimate

$$\|y\|_Y \le 2C_T\big(\|y_0\|_{Y(0)} + \|d\|_D\big) \tag{6.55}$$

where the topology on D is determined by the regularity of the forcing terms in (6.15).

Proof. The proof is based on a fixed point argument for the system

$$\begin{aligned}&v_t - \Delta v + \nabla q = f \quad \text{in } \Omega_f \times (0,T)\\ &\operatorname{div} v = g \quad \text{in } \Omega_f \times (0,T)\\ &w_{tt} - \Delta w + \alpha w_t + \beta w = 0 \quad \text{in } \Omega_e \times (0,T)\end{aligned}$$

with the boundary conditions

$$\begin{aligned}&w_t = u \quad \text{on } \Gamma_c \times (0,T)\\ &v = 0 \quad \text{on } \Gamma_f \times (0,T)\\ &\frac{\partial v}{\partial N} - qN = \frac{\partial w}{\partial N} + h \quad \text{on } \Gamma_c \times (0,T)\end{aligned}$$

where f, g, and h are known and given by

$$\begin{aligned}&f^i = \partial_j\big((\delta_{jk} - a_l^j a_l^k)\partial_k u^i\big) + (\delta_{ki} - a_i^k)\partial_k p + F^i\\ &g = (\delta_{kj} - a_j^k)\partial_k u^j + G\\ &h^i = (\delta_{jk} - a_l^j a_l^k)\partial_k u^i N_j + (\delta_{ki} - a_i^k) p N_k + H^i\end{aligned}$$

with $(u, p, \psi) \in Y$. From now on, we proceed by showing that f, g, and h satisfy (6.15), so that we can apply Lemma 6.2. Due to the condition $a(0) = I$, the initial data satisfy $\Delta v(0) - \nabla q(0) + f(0) = \Delta v(0) - \nabla q(0) + F(0) \in H^1(\Omega_f)$, as required by Lemma 6.2. Note that as the matrix a is given and close to the identity matrix (cf. (6.54)), the corresponding norms of f, g, and h are expected to be small, of order ϵ. Hence, one may infer that the map between the successive iterates is a contraction mapping from Y to itself.

For simplicity, we may assume that F, G and H equal zero.

We have $f \in L^2([0,T]; H^1(\Omega_f))$ by writing

$$
\begin{aligned}
&\|f\|_{L^2([0,T],H^1(\Omega_f))} \\
&\quad \le C\|a^T a - I\|_{L^\infty([0,T]\times\Omega_f)}\|u\|_{L^2([0,T];H^3(\Omega_f))} + C\|a^T a - I\|_{L^\infty([0,T];H^2(\Omega_f))}\|u\|_{L^2([0,T];H^3(\Omega_f))} \\
&\qquad + C\|a - I\|_{L^\infty([0,T]\times\Omega_f)}\|p\|_{L^2([0,T];H^2(\Omega_f))} + C\|a - I\|_{L^\infty([0,T];H^2(\Omega_f))}\|p\|_{L^2([0,T];H^2(\Omega_f))} \\
&\quad \le C\epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))} + C\epsilon\|p\|_{L^2([0,T];H^2(\Omega_f))}.
\end{aligned}
\tag{6.56}
$$

For f_t, we obtain

$$
\begin{aligned}
&\|f_t\|_{L^2([0,T]\times\Omega_f)} \\
&\quad \le C\|a^T a - I\|_{L^\infty([0,T];H^{1.5+\epsilon}(\Omega_f))}\|u_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\qquad + C\|\partial_t(a^T a - I)\|_{L^\infty([0,T];H^{1.5+\epsilon}(\Omega_f))}\|u\|_{L^2([0,T];H^2(\Omega_f))} \\
&\qquad + C\|a - I\|_{L^\infty([0,T];H^{1.5+\epsilon_0}(\Omega_f))}\|p_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + C\|\partial_t(a - I)\|_{L^\infty([0,T];H^{1.5+\epsilon}(\Omega_f))}\|p\|_{L^2([0,T];H^1(\Omega_f))} \\
&\quad \le C\epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|u\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|p_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&\qquad + C\epsilon\|p\|_{L^2([0,T];H^1(\Omega_f))}
\end{aligned}
\tag{6.57}
$$

where $\epsilon_0 \in (0, 1/2)$. Next, we estimate g and g_t as follows. First,

$$
\begin{aligned}
\|g\|_{L^2([0,T];H^2(\Omega_f))} &\le C\|a - I\|_{L^\infty([0,T];H^2(\Omega_f))}\|\nabla u\|_{L^2([0,T];H^2(\Omega_f))} \\
&\le \epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))}
\end{aligned}
\tag{6.58}
$$

and then

$$
\begin{aligned}
&\|g_t\|_{L^2([0,T];H^1(\Omega_f))} \\
&\quad \le C\|a - I\|_{L^\infty([0,T];H^{1.5+\epsilon_0}(\Omega_f))}\|u_t\|_{L^2([0,T];H^2(\Omega_f))} \\
&\qquad + C\|\partial_t a\|_{L^\infty([0,T];L^3(\Omega_f))}\|u\|_{L^2([0,T];H^3(\Omega_f))} + C\|\partial_t \nabla a\|_{L^\infty([0,T];L^2(\Omega_f))}\|u\|_{L^2([0,T];H^{2.5+\epsilon_0}(\Omega_f))} \\
&\quad \le \epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))}.
\end{aligned}
\tag{6.59}
$$

Next we write

$$
g_{tt} = \partial_{tt}\left((\delta_{kj} - a_j^k)\partial_k u^j\right) = \partial_{tt}\left(\partial_k\left((\delta_{kj} - a_j^k)u^j\right)\right) = \partial_k\left(\partial_{tt}\left((\delta_{kj} - a_j^k)u^j\right)\right) = \operatorname{div} A \tag{6.60}
$$

by setting $A_k = \partial_{tt}\left((\delta_{kj} - a_j^k)u^j\right)$ for $k = 1, 2, 3$ and $B = 0$. We then have

$$
\begin{aligned}
&\|A\|_{L^2([0,T];L^2(\Omega_f))} \\
&\quad \le C\|\partial_{tt} a\|_{L^2([0,T]\times\Omega_f)}\|u\|_{L^\infty([0,T]\times\Omega_f)} + C\|\partial_t a\|_{L^\infty([0,T]\times\Omega_f)}\|u_t\|_{L^2([0,T]\times\Omega_f)} \\
&\qquad + C\|a - I\|_{L^\infty([0,T]\times\Omega_f)}\|u_{tt}\|_{L^2([0,T]\times\Omega_f)} \\
&\quad \le C\epsilon\|u\|_{L^\infty([0,T];H^2(\Omega_f))} + C\epsilon\|u_t\|_{L^2([0,T]\times\Omega_f)} + C\epsilon\|u_{tt}\|_{L^2([0,T]\times\Omega_f)}
\end{aligned}
$$

showing that $A \in L^2([0,T]; L^2(\Omega_f))$.

Next, we estimate h and h_t. Note that

$$\|h^i\|_{L^2([0,T];H^{3/2}(\Gamma_c))} \le C\big\|(\delta_{3k} - a^3_l a^k_l)\partial_k u^i\big\|_{L^2([0,T];H^2(\Omega_f))} + C\big\|(\delta_{3i} - a^3_i)p\big\|_{L^2([0,T];H^2(\Omega_f))}$$

for $i = 1,2,3$. Thus we may proceed as in (6.56) and obtain

$$\|h\|_{L^2([0,T];H^{3/2}(\Gamma_c))} \le \epsilon\|u\|_{L^2([0,T];H^3(\Omega_f))} + \epsilon\|p\|_{L^2([0,T];H^2(\Omega_f))}. \tag{6.61}$$

Similarly, following (6.57),

$$\begin{aligned}\|h_t\|_{L^2([0,T];H^{1/2}(\Gamma_c))} &\le C\epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|u\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|p_t\|_{L^2([0,T];H^1(\Omega_f))}\\ &\quad + C\epsilon\|p\|_{L^2([0,T];H^1(\Omega_f))}.\end{aligned} \tag{6.62}$$

Also, we have

$$\begin{aligned}&\|h_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\quad \le C\|a^T : a - I\|_{L^\infty([0,T]\times\Omega_f)}\|\nabla u_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\qquad + C\|a^T : a - I\|_{W^{1/4,4}([0,T];L^\infty(\Omega_f))}\|\nabla u_t\|_{L^4([0,T];L^2(\Gamma_c))}\\ &\qquad + C\|\partial_t(a^T : a)\|_{L^\infty([0,T]\times\Omega_f)}\|\nabla u\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\qquad + C\|\partial_t(a^T : a)\|_{H^{1/4}([0,T];L^4(\Gamma_c))}\|\nabla u\|_{L^\infty([0,T];L^4(\Gamma_c))}\\ &\qquad + C\|a - I\|_{L^\infty([0,T]\times\Omega_f)}\|p_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\qquad + C\|a - I\|_{W^{1/4,4}([0,T];L^\infty(\Omega_f))}\|p_t\|_{L^4([0,T];L^2(\Gamma_c))}\\ &\qquad + C\|\partial_t a\|_{L^\infty([0,T]\times\Omega_f)}\|p\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\qquad + C\|\partial_t a\|_{H^{1/4}([0,T];L^4(\Gamma_c))}\|p\|_{L^\infty([0,T];L^4(\Gamma_c))}.\end{aligned} \tag{6.63}$$

We use the space-time interpolation inequalities for v (and similar inequalities for q)

$$\|\nabla u_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \le \epsilon_0\|u_t\|_{H^1([0,T];L^2(\Omega_f))} + C_{\epsilon_0}\|u_t\|_{L^2([0,T];H^2(\Omega_f))}, \qquad \epsilon_0 \in (0,1], \tag{6.64}$$

and

$$\|\nabla u_t\|_{L^4([0,T];L^2(\Gamma_c))} \le \epsilon_0\|u_t\|_{H^1([0,T];L^2(\Omega_f))} + C_{\epsilon_0}\|u_t\|_{L^2([0,T];H^2(\Omega_f))}, \qquad \epsilon_0 \in (0,1], \tag{6.65}$$

as well as

$$\|\nabla u\|_{L^\infty([0,T];L^4(\Gamma_c))} \le \epsilon_0\|u_t\|_{L^2([0,T];H^2(\Omega_f))} + C_{\epsilon_0}\|u\|_{L^2([0,T];H^2(\Omega_f))}, \qquad \epsilon_0 \in (0,1], \tag{6.66}$$

in order to obtain

$$\begin{aligned}&\|h_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\quad \le C\epsilon\|u_t\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|u_t\|_{L^2([0,T];H^2(\Omega_f))}\\ &\qquad + C\epsilon\|u\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|u\|_{L^2([0,T];H^2(\Omega_f))}\\ &\qquad + C\epsilon\|p_t\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + C\epsilon\|p\|_{L^2([0,T];H^1(\Omega_f))} + C\epsilon\|p_t\|_{L^2([0,T];H^1(\Omega_f))}.\end{aligned} \tag{6.67}$$

The bounds (6.56)–(6.67) show that h, f, g and their time derivatives satisfy the conditions of Lemma 6.2. As $\epsilon \in (0,1]$ is small, the estimates (6.39) and (6.44) lead to the existence and uniqueness of the solution to the linear system (2.1)–(2.3) with given coefficients. □

Remark 6.6. As in the case of Lemma 6.2, due to invariance under the dynamics of regularity of initial data, the obtained solutions can be continued indefinitely. In fact, the following quantitative estimate is obtained using the proof of the lemma: If the iterate $\hat{y} = (u, p, \psi)$ in the fixed point argument belongs to a ball in the space Y with radius M, i.e., $\hat{y} \in B_M(Y)$, and if $y_0 \in B_{r_1}(Y(0))$ and $d = (F, G, H) \in B_{r_2}(D)$ with $r = r_1 + r_2$, then

$$\|y\|_Y \leq C_T r + C_T \epsilon \|\hat{y}\|_Y \leq C_T r + C_T \epsilon M. \tag{6.68}$$

The self-mapping property requires that $r + \epsilon M \leq C_T^{-1} M$, in particular, $\epsilon \ll C_T^{-1}$.

Remark 6.7. Note that the introduced space Y provides enough regularity to accommodate topological restrictions imposed in (6.54) on the coefficient $a(x, t)$. In addition, the a priori estimates in Theorem 2.1 described by the functional $X(t)$ allow us to control the regularity of $a(x, t)$. However, the topology of the space Y is not sufficient in order to apply the a priori estimates of Theorem 2.1. It is for this reason that we need to construct solutions with higher regularity. This is done in the next section.

6.2 Data with higher regularity

The next lemma provides the global existence of the solution to the linear system (6.47)–(6.49) with more regular initial data. Here we apply Lemma 6.5 to the solution of the system as well as to its time and tangential derivatives.

Lemma 6.8. *Let the coefficient matrix $a(x, t)$ be given such that* (6.53) *and* (6.54) *hold with $\epsilon < 1/C$ for a sufficiently large constant C. Also, assume $\|a\|_{L^2([0,T];H^3(\Omega_f))} < \infty$ and $\|a_{ttt}\|_{L^2([0,T];L^2(\Omega_f))} < \infty$. Consider the linear system* (6.47)–(6.49) *with the boundary conditions* (6.50)–(6.52)*, where F, G, and H equal zero. Assume that the initial data satisfy $v_0 \in V \cap H^{7/2}(\Omega_f)$, $\partial_t v_0 \in V \cap H^{5/2}(\Omega_f)$, $\Delta v_t(0) - \nabla q_t(0) \in V$, $w_0 \in H^{15/4-\delta}(\Omega_e)$, $w_1 \in H^{11/4-\delta}(\Omega_e)$ for some $\delta \in (0, 1/4)$, with compatibility conditions* (6.17)–(6.21)*. In addition, assume that*

$$\Delta w_1 - \alpha w_{tt}(0) - \beta w_1(0) = \Delta v_t(0) - \nabla q_t(0) + f(0) \quad \text{in } \Gamma_c \tag{6.69}$$

$$\frac{\partial w_1}{\partial N} \cdot \tau = \frac{\partial}{\partial N}\big(\Delta v_0 - \nabla q_0 + f(0)\big) \cdot \tau \quad \text{on } \Gamma_c \tag{6.70}$$

and

$$-\Delta v_t(0) + \nabla q_t(0) = f(0) \quad \text{in } \Gamma_f, \tag{6.71}$$

where f, g, and h are defined in (6.74) *below. Then the unique solution $(v, q, w) \in Y$ from Lemma 6.5 obeys $(v_t, q_t, w_t) \in Y$ and $(\partial_m v, \partial_m q, \partial_m w) \in Y$ for $m = 1, 2$. The obtained solution can be extended to arbitrary $T > 0$ with the estimate given at the end of the proof.*

Proof. Consider the system

$$\begin{aligned}
&\phi^i_t - \partial_j(a^j_l a^k_l \partial_k \phi^i) + a^k_i \partial_k \chi = f^i \text{ in } \Omega_f \times (0, T)\\
&a^k_i \partial_k \phi^i = g \text{ in } \Omega_f \times (0, T)\\
&\psi_{tt} - \Delta\psi = -\alpha\psi_t - \beta\psi \text{ in } \Omega_e \times (0, T)
\end{aligned} \tag{6.72}$$

with boundary conditions

$$
\begin{aligned}
&\phi = \psi_t \quad \text{on } \Gamma_c \times (0,T)\\
&\phi = 0 \quad \text{on } \Gamma_f \times (0,T)\\
&a^j_l a^k_l \partial_k \phi^i N_j - a^i_k \chi N_k = \partial_j \psi^i N_j + h^i \quad \text{on } \Gamma_c \times (0,T)
\end{aligned}
\tag{6.73}
$$

with

$$
\begin{aligned}
f^i &= \partial_j(\partial_t(a^j_l a^k_l)\partial_k v^i) - \partial_t a^k_i \partial_k q\\
g &= -\partial_t a^k_i \partial_k v^i\\
h^i &= \partial_t(a^j_l a^k_l)\partial_k v^i N_j - \partial_t a^k_i q N_k
\end{aligned}
\tag{6.74}
$$

for $i = 1,2,3$, where $(v,q,w) \in Y$ is the unique solution to the system (6.47)–(6.49) constructed in Lemma 6.5.

Taking the initial data $\phi_0 = v_t(0) \in V \cap H^{5/2}(\Omega_f)$ $\phi_{t,0} = v_{tt}(0) \in V$, $\psi_0 = w_t(0) \in H^{11/4-\delta}(\Omega_e)$, and $\psi_t(0) = w_{tt}(0) \in H^{7/4-\delta}(\Omega_e)$, one can easily verify that f, g, and h satisfy the conditions of Lemma 6.5. Hence, we obtain $\phi \in L^2([0,T]; H^3(\Omega_f))$ and $\phi_t \in L^2([0,T]; H^2(\Omega_f))$ while $\chi \in L^2([0,T]; H^2(\Omega_f))$ and $\chi_t \in L^2([0,T]; H^1(\Omega_f))$. Similarly, we have $(\psi, \psi_t) \in L^\infty([0,T]; H^{11/4-\delta}(\Omega_e) \times H^{7/4-\delta}(\Omega_f))$. By uniqueness, it follows that $\phi = v_t$, $\chi = q_t$, and $\psi = w_t$. Therefore, $(v_t, q_t, w_t) \in Y$.

For the tangential regularity, we repeat the same procedure but now with forcing terms given by

$$
\begin{aligned}
f^i &= \partial_j(\partial_m(a^j_l a^k_l)\partial_k v^i) - \partial_m a^k_i \partial_k q\\
g &= -\partial_m a^k_i \partial_k v^i\\
h^i &= \partial_m(a^j_l a^k_l)\partial_k v^i N_j - \partial_m a^k_i q N_k
\end{aligned}
\tag{6.75}
$$

for $m = 1,2$, instead of (6.74), and initial data $\phi_0 = \partial_m v(0) \in V \cap H^{5/2}(\Omega_f)$ with $(\Delta\phi_0 - \nabla\chi_0) \in V$ while $\psi_0 = \partial_m w(0) \in H^{11/4-\delta}(\Omega_e)$, and $\psi_t(0) = \partial_m w_t(0) \in H^{7/4-\delta}(\Omega_e)$. Using Lemma 6.5 and uniqueness of solutions, we conclude $(\partial_m v, \partial_m q, \partial_m w) \in Y$ for $m = 1,2$.

As a consequence, we also obtain higher regularity of $w \in L^\infty([0,T]; H^{15/4-\delta}(\Omega_e))$ from the elliptic estimate

$$
\|w\|_{H^{15/4-\delta}(\Omega_e)} \le C\|w_{tt}\|_{H^{7/4-\delta}(\Omega_e)} + C\|w_t\|_{H^{7/4-\delta}(\Omega_e)} + C\|w\|_{H^{7/4-\delta}(\Omega_e)} + C\|D'w\|_{H^{11/4-\delta}(\Omega_e)} \tag{6.76}
$$

for all $t \in (0,T)$, where D' denotes the tangential derivative. Similarly, we may also conclude that $v \in L^2([0,T]; H^4(\Omega_f))$ and $q \in L^2([0,T]; H^3(\Omega_f))$ by using elliptic estimates for the stationary Stokes operator. □

We denote by $\widetilde{Y}$ the solution space, obtained in Lemma 6.8, for the system (6.47)–(6.49) with the higher regularity data; namely,

$$
\begin{aligned}
\widetilde{Y} = \Big\{(v,q,w) : v \in L^2([0,T]; H^4(\Omega_f)) \cap H^1([0,T]; H^3(\Omega_f)), v_t \in H^1([0,T]; H^2(\Omega_f)),\\
v_{tt} \in H^1([0,T]; L^2(\Omega_f)), q \in L^2([0,T]; H^3(\Omega_e)) \cap H^1([0,T]; H^2(\Omega_f)),\\
q_t \in H^1([0,T]; H^1(\Omega_f)), w \in L^\infty([0,T]; H^{15/4-\delta-j}(\Omega_e)), j = 0,1,2,3\Big\}.
\end{aligned}
\tag{6.77}
$$

Remark 6.9. Using the 1D Agmon inequality in time, we have $\widetilde{Y} \subseteq X$, where X consists of the norms in (2.13) from Theorem 2.1. Moreover, as in the case of the previous lemma, there is no loss of regularity with respect to the initial data. Thus solutions can be contracted for an arbitrary long time interval.

Remark 6.10. Here we state the quantitative estimate on the norm in $\widetilde{Y}$ of the solution. In analogy to $Y(0)$, define the space of initial data $\widetilde{Y}(0)$ as

$$\begin{aligned}\widetilde{Y}(0) = \Big\{&(v_0, w_0, w_1) \in (V \cap H^{7/2}(\Omega_f)) \times H^{15/4-\delta}(\Omega_e) \times H^{11/4-\delta}(\Omega_e),\\ &\partial_t v_0 \in V \cap H^{5/2}(\Omega_f), \Delta v_t(0) - \nabla q_t(0) \in H^1(\Omega_f),\\ &\text{with the compatibility conditions (6.17)–(6.21) and (6.69)–(6.71)}\Big\}.\end{aligned} \tag{6.78}$$

Then, for any $T > 0$, there exists $\epsilon > 0$ sufficiently small so that we have

$$\|y(t)\|_{\widetilde{Y}} \le C\|y_0\|_{\widetilde{Y}(0)} + C\Big(\|a(t)\|_{L^2H^3} + \|a_t(t)\|_{L^2H^2} + C\|a_{tt}(t)\|_{L^2H^1} + C\|a_{ttt}(t)\|_{L^2L^2}\Big)\|y(t)\|_X \tag{6.79}$$

for $t \in [0, T]$, which is obtained from the respective estimates for the terms f, g, h in (6.74). It is important that the space $\widetilde{Y}(0)$ is invariant under the dynamics.

Now, consider the norm $X(t)$ defined in (5.1) with $\gamma = 0$. Assume that the coefficient matrix a is given and satisfies (6.53) and (6.54) for all $T > 0$. Then we have a Gronwall-type inequality

$$X(t) + \int_\tau^t X(s)\,ds \le C_0 X(\tau) + C_0\epsilon(t-\tau)^2 \int_\tau^t X(s)\,ds \tag{6.80}$$

for all $0 \le \tau \le t$. We omit the detailed derivation of this inequality as it is similar to the one given in Section 5, with the only difference being that we use the smallness of the coefficients a instead of the superlinear estimates from Subsection 4.7. Next, we prove that, similarly to (5.18), this inequality also implies the exponential decay of $X(t)$.

Lemma 6.11. *Suppose that $X\colon [0,\infty) \to [0,\infty]$ is continuous at all t such that $X(t)$ is finite and assume that it satisfies (6.80) for $0 \le \tau \le t$ where $C_0 \ge 1$. There exists $\epsilon > 0$ depending on C_0 such that if $X(0) \le M$, where $M > 0$, then $X(t) \le CMe^{-t/C}$.*

Proof of Lemma 6.11. Choose $\epsilon > 0$ such that

$$C_0\epsilon(8C_0)^2 \le \frac{1}{2}. \tag{6.81}$$

This implies $C_0\epsilon(t-\tau)^2 \le 1/2$ provided $t - \tau \le 8C_0$. Therefore, we have

$$X(t) + \int_\tau^t X(s)\,ds \le 2C_0X(\tau), \qquad 0 \le \tau \le t \le \tau + 8C_0. \tag{6.82}$$

We shall prove by induction in $k \in \mathbb{N}_0$, that there exist

$$\tau_0 = 0 < \tau_1 < \cdots < \tau_k \tag{6.83}$$

such that

$$\tau_j - \tau_{j-1} \le 8C_0, \qquad j = 1, \ldots, k \tag{6.84}$$

with

$$X(\tau_j) \leq \frac{M}{2^j}, \qquad j = 0, 1, \ldots, k \tag{6.85}$$

and

$$X(t) \leq 2C_0 \frac{M}{2^{j-1}}, \qquad t \in [\tau_{j-1}, \tau_j], \qquad j = 0, 1, \ldots, k \tag{6.86}$$

where we set $\tau_{-1} = 0$. The statement clearly holds for $k = 0$. Let $k \in \mathbb{N}_0$, and assume that $\tau_0 = 0 < \tau_1 < \cdots < \tau_k$ with the stated properties have been constructed. First, by (6.82), we have

$$X(t) \leq 2C_0 \frac{M}{2^k}, \qquad t \in [\tau_k, \tau_k + 8C_0]. \tag{6.87}$$

It remains to be shown that there exists $\tau_{k+1} \in [\tau_k, \tau_k + 8C_0]$ such that $X(\tau_{k+1}) \leq M/2^{k+1}$. If this is not true, then we have

$$X(t) > \frac{M}{2^{k+1}}, \qquad t \in [\tau_k, \tau_k + 8C_0]. \tag{6.88}$$

Using (6.82) with $\tau = \tau_k$ and $t = \tau_k + 8C_0$, disregarding the first term on the left side, we get

$$8C_0 \frac{M}{2^{k+1}} \leq 2C_0 \frac{M}{2^k} \tag{6.89}$$

which is a contradiction. $\square$

6.3 A construction for the nonlinear problem

We now proceed with the proof of Theorem 2.2 by constructing a solution as a limit of the sequence of iterates; the main idea is to show the contractive property in a lower regularity topology than where solutions belong, however still strong enough to control the cofactor matrix a.

Proof of Theorem 2.2. We assume that the initial data $y_0 = (v_0, w_0, w_1)$ belongs to $\widetilde{Y}(0)$ for some $\delta \in (0, 1/4)$, $\|y_0\|_{X(0)} \leq \epsilon$, and that it satisfies the compatibility conditions (6.17)–(6.21) and (6.69)–(6.71). Here $\|\cdot\|_{X(0)}$ denotes the norm corresponding to (2.12).

We start the iteration with $(u^{(0)}, p^{(0)}, \psi^{(0)})$ given by the unique solution (cf. Lemma 6.5) to the linear homogeneous system corresponding to (6.47)–(6.52) with coefficients $a^{(0)} = a^{(0)}(x, t)$, forcing terms F, G and H equal to zero, and initial data (v_0, w_0, w_1). Here we assume that the coefficient matrix $a^{(0)}$ satisfying $a^{(0)}(x, 0) = I$, $\partial_t a^{(0)}(x, 0) = -\nabla u^{(0)}(x, 0)$, (6.53), and it is close to the identity matrix, i.e., it obeys (6.54).

Now, given the iterates $y^{(j)} = (u^{(j)}, p^{(j)}, \psi^{(j)})$ for $j = 0, 1, \ldots, n-1$, we construct $a^{(n)}$ by solving

$$\partial_t a^{(n)} = -a^{(n)} : \nabla u^{(n-1)} : a^{(n)}. \tag{6.90}$$

Given $a^{(n)}$, we then solve the linear system for the new iterate

$$y^{(n)} = (u^{(n)}, p^{(n)}, \psi^{(n)}); \tag{6.91}$$

denoting $(v,q,w) = (u^{(n)}, p^{(n)}, \psi^{(n)})$, the system reads

$$v_t^i - \partial_j((a^{(n)})_l^j (a^{(n)})_l^k \partial_k v^i) + \partial_k((a^{(n)})_i^k q) = 0 \ \text{ in } \Omega_f \times (0,T), \qquad i = 1,2,3 \tag{6.92}$$

$$(a^{(n)})_i^k \partial_k v^i = 0 \ \text{ in } \Omega_f \times (0,T), \tag{6.93}$$

$$w_{tt}^i - \Delta w^i + \alpha w_t^i + \beta w^i = 0 \ \text{ in } \Omega_f \times (0,T), \qquad i = 1,2,3, \tag{6.94}$$

with the boundary conditions

$$v^i = w_t^i \ \text{ on } \Gamma_c \times (0,T), \qquad i = 1,2,3 \tag{6.95}$$

$$(a^{(n)})_l^j (a^{(n)})_l^k \partial_k v^i N_j - (a^{(n)})_i^k q N_k = \partial_j w^i N_j + H^i \ \text{ on } \Gamma_c \times (0,T) \tag{6.96}$$

$$v^i = 0 \ \text{ on } \Gamma_c \times (0,T), \qquad i = 1,2,3. \tag{6.97}$$

Note that Lemma 6.11 with $\|y_0\|_{X(0)} \le \epsilon$, where $\|\cdot\|_{X(0)}$ denotes the norm as in (2.12), applies inductively to the iterates. Therefore, if $\epsilon > 0$ is sufficiently small, then

$$\|y^{(n)}\|_X \le C\epsilon e^{-t/C}, \qquad t \ge 0 \tag{6.98}$$

where $\|\cdot\|_X$ denotes the norm corresponding to (5.1). Based on the definition of the norm $\|\cdot\|_X$, we obtain easily that all $a = a^{(n)}$ satisfy (6.53) and (6.54) for all $T > 0$. Here we check the first inequality in (6.53), omitting the details for other inequalities since the proofs are similar. First, by (2.4), we have

$$\partial_t(a - I) = -(a-I) : \nabla u : (a-I) - \nabla u : (a-I) - (a-I) : \nabla u - \nabla u \tag{6.99}$$

where we abbreviate $a = a^{(n)}$ and $u = u^{(n)}$. Therefore,

$$\begin{aligned}(a-I)(t) = &-\int_0^t (a-I) : \nabla u : (a-I)\,ds - \int_0^t \nabla u : (a-I)\,ds \\ &- \int_0^t (a-I) : \nabla u\,ds - \int_0^t \nabla u\,ds\end{aligned} \tag{6.100}$$

and thus, using (6.98),

$$\|(a-I)(t)\|_{H^2} \le C\epsilon \int_0^t \big(\|(a-I)(s)\|_{H^2}^2 + 1\big) e^{-s/C}\,ds \tag{6.101}$$

and $\|(a-I)(t)\|_{H^2} \le \epsilon$ for all $t > 0$ follows from the Gronwall lemma.

Using the estimate (6.79), the iterates are defined for all $t > 0$ and satisfy

$$\|y^{(n)}\|_{\widetilde{Y}} \le C\|y_0\|_{\widetilde{Y}(0)} + C\epsilon\|a^{(n)}\|_{L^2H^3(\Omega_f)} + C\epsilon\|a_t^{(n)}\|_{L^2H^2(\Omega_f)} + C\epsilon\|a_{tt}^{(n)}\|_{L^2H^1(\Omega_f)} + C\epsilon\|a_{ttt}^{(n)}\|_{L^2L^2(\Omega_f)} \tag{6.102}$$

on $[0,T]$, where C depends on T. Reducing $\epsilon > 0$ if necessary, we obtain

$$\|y^{(n)}\|_{\widetilde{Y}} \le C\|y_0\|_{\widetilde{Y}(0)} + \frac{1}{2}\|y^{(n-1)}\|_{\widetilde{Y}}. \tag{6.103}$$

Thus all the iterates belong to the ball B_M in $\widetilde{Y}$ on the interval $[0,T_0]$, where T_0 is a fixed constant; for instance, we may take $T_0 = 1$. Clearly, M is a constant multiple of $\|y_0\|_{\widetilde{Y}(0)}$.

We first construct the solution on the interval $[0, T_0]$ and then outline the details for continuing the construction for $t \geq T_0$. The argument is based on a generalized fixed point technique, where the iterates are bounded uniformly in the space $\widetilde{Y}$ while we have a contraction in a lower order norm.

Consider the map $\Lambda \colon Y \to Y$ between the successive iterates, i.e., $\Lambda(u^{(j)}, p^{(j)}, \psi^{(j)}) = (u^{(j+1)}, p^{(j+1)}, \psi^{(j+1)})$. In order to avoid superscripts, we denote

$$(u, p, \psi) = (u^{(n-1)}, p^{(n-1)}, \psi^{(n-1)}) \tag{6.104}$$

and

$$(v, q, w) = (u^{(n)}, p^{(n)}, \psi^{(n)}) \tag{6.105}$$

and thus assume that

$$(v, q, w) = \Lambda(u, p, \psi) \tag{6.106}$$

is the solution to the system

$$v_t - \Delta v + \nabla q = f \quad \text{in } \Omega_f \times (0, T) \tag{6.107}$$

$$\operatorname{div} v = g \quad \text{in } \Omega_f \times (0, T) \tag{6.108}$$

$$w_{tt} - \Delta w = -\alpha w_t - \beta w \quad \text{in } \Omega_e \times (0, T) \tag{6.109}$$

with the boundary conditions

$$w_t = u \quad \text{on } \Gamma_c \times (0, T) \tag{6.110}$$

$$v = 0 \quad \text{on } \Gamma_f \times (0, T) \tag{6.111}$$

$$\frac{\partial v}{\partial N} - qN = \frac{\partial w}{\partial N} + h \quad \text{on } \Gamma_c \times (0, T) \tag{6.112}$$

where

$$\begin{aligned}
f^i &= \partial_j((\delta_{jk} - a^j_l a^k_l)\partial_k v^i) + (\delta_{ki} - a^k_i)\partial_k q \\
g &= (\delta_{kj} - a^k_j)\partial_k v^j \\
h^i &= (\delta_{jk} - a^j_l a^k_l)\partial_k v^i N_j + (\delta_{ki} - a^k_i) q N_k
\end{aligned} \tag{6.113}$$

for $i = 1, 2, 3$. The matrix a is the coefficient matrix corresponding to u, and is obtained by solving the system

$$\begin{aligned}
a_t &= -a : \nabla u : a \quad \text{in } \Omega_f \times (0, T) \\
a(x, 0) &= I \quad \text{in } \Omega_f.
\end{aligned}$$

We next show that Λ is a contraction on the lower topological space

$$\begin{aligned}
Y_0 = \big\{&(v, q, w) : v \in L^2([0, T]; H^2(\Omega_f)), v_t \in L^2([0, T]; L^2(\Omega_f)), q \in L^2([0, T]; H^1(\Omega_f)), \\
&q \in H^{1/4}([0, T]; L^2(\Gamma_c)), w \in L^\infty([0, T]; H^1(\Omega_f)), w_t \in L^\infty([0, T]; L^2(\Omega_f))\big\}
\end{aligned} \tag{6.114}$$

with $T = T_0$ with the norm

$$\begin{aligned}
\|(v, q, w)\|_{Y_0} = {}& \|v\|_{L^2([0,T];H^2(\Omega_f))} + \|v_t\|_{L^2([0,T];L^2(\Omega_f))} + \|q\|_{L^2([0,T];H^1(\Omega_f))} \\
&+ \|q\|_{H^{1/4}([0,T];L^2(\Gamma_c))} + \epsilon_0 \|w\|_{L^\infty([0,T];H^1(\Omega_f))} + \epsilon_0 \|w_t\|_{L^\infty([0,T];L^2(\Omega_f))}
\end{aligned} \tag{6.115}$$

where ϵ_0 is a sufficiently small constant which is determined below. Let (u,p,ψ) and $(\widetilde{u},\widetilde{p},\widetilde{\psi})$ be two elements in the ball B_M in Y. We estimate the difference between two solutions (v,q,w) and $(\widetilde{v},\widetilde{q},\widetilde{w})$ arising from (u,p,ψ) and $(\widetilde{u},\widetilde{p},\widetilde{\psi})$ respectively in the topology of Y_0. Denote the differences of the old variables

$$U = u - \widetilde{u}, \qquad P = p - \widetilde{p}, \qquad \Psi = \psi - \widetilde{\psi} \tag{6.116}$$

and the differences of the new variables

$$V = v - \widetilde{v}, \qquad Q = q - \widetilde{q}, \qquad W = w - \widetilde{w}, \tag{6.117}$$

respectively. They obey the two coupled equations

$$V_t - \Delta V + \nabla Q = F \quad \text{in } \Omega_f \times (0,T) \tag{6.118}$$

$$\operatorname{div} V = G \quad \text{in } \Omega_f \times (0,T) \tag{6.119}$$

$$\frac{\partial V}{\partial N} - QN = \frac{\partial W}{\partial N} + H \quad \text{on } \Gamma_c \times (0,T) \tag{6.120}$$

$$V = 0 \quad \text{on } \Gamma_f \times (0,T) \tag{6.121}$$

and

$$W_{tt} - \Delta W + \alpha W_t + \beta W = 0 \quad \text{in } \Omega_e \times (0,T) \tag{6.122}$$

$$W_t = V \quad \text{on } \Gamma_c \times (0,T), \tag{6.123}$$

where

$$\begin{aligned}
F^i &= \partial_j((\delta_{jk} - \widetilde{a}_l^j \widetilde{a}_l^k)\partial_k \widetilde{v}^i) - \partial_j((\delta_{jk} - a_l^j a_l^k)\partial_k v^i) + (\delta_{ki} - \widetilde{a}_i^k)\partial_k \widetilde{q} - (\delta_{ki} - a_i^k)\partial_k q \\
G &= (\delta_{kj} - \widetilde{a}_j^k)\partial_k \widetilde{v}^j - (\delta_{kj} - a_j^k)\partial_k v^j = \partial_k\Big((\delta_{kj} - \widetilde{a}_j^k)\widetilde{v}^j - (\delta_{kj} - a_j^k)v^j\Big) \\
H^i &= (\delta_{jk} - \widetilde{a}_l^j \widetilde{a}_l^k)\partial_k \widetilde{v}^i N_j - (\delta_{jk} - a_l^j a_l^k)\partial_k v^i N_j + (\delta_{ki} - \widetilde{a}_i^k)\widetilde{q} N_k - (\delta_{ki} - a_i^k) q N_k
\end{aligned} \tag{6.124}$$

for $i = 1,2,3$, and

$$G_t = \operatorname{div} \bar{A} + \bar{B} \tag{6.125}$$

with $\bar{B} = 0$ and

$$\bar{A}_k = \partial_t\Big((\delta_{kj} - \widetilde{a}_j^k)\widetilde{v}^j - (\delta_{kj} - a_j^k)v^j\Big), \qquad k = 1,2,3. \tag{6.126}$$

Here we denoted by a and $\widetilde{a}$ the coefficients associated with the flow maps induced by u and $\widetilde{u}$ respectively. We then appeal to Lemma 6.1 associated with the linear system (6.118)–(6.121) satisfied by the difference of solutions (V,Q,W) with zero initial data, and obtain

$$\begin{aligned}
&\|V\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{L^2([0,T];L^2(\Omega_f))} + \|Q\|_{L^2([0,T];H^1(\Omega_f))} + \|Q\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\quad \le C\Big(\|F\|_{L^2([0,T];L^2(\Omega_f))} + \|G\|_{L^2([0,T];H^1(\Omega_f))} + \|\bar{A}\|_{L^2([0,T];L^2(\Omega_f))} \\
&\qquad + \|\bar{B}\|_{L^2([0,T];L^2(\Omega_f))} + \|H\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \|H\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\qquad + \left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \left\|\frac{\partial W}{\partial N}\right\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\Big).
\end{aligned} \tag{6.127}$$

We next estimate the norms on the right side of (6.127), starting with F, which we write as

$$F^i = \partial_j((\widetilde{a}_l^j \widetilde{a}_l^k - \delta_{jk})\partial_k V^i) - \partial_j((a_l^j a_l^k - \widetilde{a}_l^j \widetilde{a}_l^k)\partial_k v^i) + (\widetilde{a}_i^k - \delta_{ki})\partial_k Q - (a_i^k - \widetilde{a}_i^k)\partial_k q.$$

Using the triangle inequality, we have

$$\begin{aligned}\|F\|_{L^2([0,T];L^2(\Omega_f))} \le C \sum_{i,j} &\|(\widetilde{a}_l^j \widetilde{a}_l^k - \delta_{jk})\partial_k V^i\|_{L^2([0,T];H^1(\Omega_f))} + \|(a_l^j a_l^k - \widetilde{a}_l^j \widetilde{a}_l^k)\partial_k v^i\|_{L^2([0,T];H^1(\Omega_f))} \\ &+ \|(\widetilde{a}_i^k - \delta_{ki})\partial_k Q\|_{L^2([0,T];L^2(\Omega_f))} + \|(a_i^k - \widetilde{a}_i^k)\partial_k q\|_{L^2([0,T];L^2(\Omega_f))}.\end{aligned}$$

Applying Hölder and Sobolev inequalities we have

$$\begin{aligned}\|F(t)\|_{L^2(\Omega_f)} \le C\Big(&\|\widetilde{a}_l^j \widetilde{a}_l^k - \delta_{jk}\|_{H^{1.5+\epsilon}(\Omega_f)} \|V\|_{H^2(\Omega_f)} + \|a_l^j a_l^k - \widetilde{a}_l^j \widetilde{a}_l^k\|_{H^1(\Omega_f)} \|\nabla v\|_{L^\infty(\Omega_f)} \\ &+ \|\widetilde{a}_i^k - \delta_{ki}\|_{H^{1.5+\epsilon}(\Omega_f)} \|Q\|_{H^1(\Omega_f)} + \|a_i^k - \widetilde{a}_i^k\|_{L^6(\Omega_f)} \|\nabla q\|_{L^3(\Omega_f)}\Big).\end{aligned} \tag{6.128}$$

Here we use that the coefficient matrices a and $\widetilde{a}$ obey (6.53) and (6.54). Also, in order to estimate $a - \widetilde{a}$, we write

$$\begin{aligned}\|a - \widetilde{a}\|_{H^1} \le C \int_0^t \|a - \widetilde{a}\|_{H^1} \|\nabla u\|_{H^2} \|a\|_{H^2}\, ds + C \int_0^t \|\widetilde{a}\|_{H^2} \|\nabla U\|_{H^1} \|a\|_{H^2}\, ds \\ + C \int_0^t \|\widetilde{a}\|_{H^2} \|\nabla \widetilde{u}\|_{H^2} \|a - \widetilde{a}\|_{H^1}\, ds\end{aligned} \tag{6.129}$$

whence using Gronwall's inequality and assuming that $\epsilon > 0$ is sufficiently small, we get

$$\|a - \widetilde{a}\|_{H^1} \le C \int_0^t \|U\|_{H^2}\, ds. \tag{6.130}$$

Similarly,

$$\|a_l^j a_l^k - \widetilde{a}_l^j \widetilde{a}_l^k\|_{H^1} \le C \int_0^t \|U\|_{H^2}\, ds. \tag{6.131}$$

Therefore, we get

$$\begin{aligned}\|F\|_{L^2([0,T];L^2(\Omega_f))} &\le C\epsilon \|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|P\|_{L^2([0,T];H^1(\Omega_f))} \\ &\quad + C\epsilon \|V\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|Q\|_{L^2([0,T];H^1(\Omega_f))}\end{aligned} \tag{6.132}$$

where we used $T \le T_0$. For G, by Hölder's inequality we have

$$\|G\|_{H^1(\Omega_f)} \le \|\widetilde{a}_j^k - \delta_{kj}\|_{L^\infty(\Omega_f)} \|\nabla V\|_{H^1(\Omega_f)} + \|a_j^k - \widetilde{a}_j^k\|_{H^1(\Omega_f)} \|\nabla v\|_{L^\infty(\Omega_f)}.$$

We appeal to the properties of the coefficients a and $\widetilde{a}$, (6.54) and (6.130), to get

$$\|G\|_{L^2([0,T];H^1(\Omega_f))} \le C\epsilon \|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|V\|_{L^2([0,T];H^2(\Omega_f))}. \tag{6.133}$$

An estimate for the boundary term H in $L^2([0,T];H^{1/2}(\Gamma_c))$ is similar to that for F in $L^2([0,T] \times \Omega_f)$, hence

$$\begin{aligned}\|H\|_{L^2([0,T];H^{1/2}(\Gamma_c))} &\le C\epsilon \|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|P\|_{L^2([0,T];H^1(\Omega_f))} \\ &\quad + C\epsilon \|V\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon \|Q\|_{L^2([0,T];H^1(\Omega_f))}.\end{aligned} \tag{6.134}$$

Next, we estimate $\|H\|_{H^{1/4}([0,T];L^2(\Gamma_c))}$, where we write

$$H^i = (\widetilde{a}^j_l \widetilde{a}^k_l - \delta_{jk})\partial_k V^i N_j - (a^j_l a^k_l - \widetilde{a}^j_l \widetilde{a}^k_l)\partial_k v^i N_j + (\widetilde{a}^k_i - \delta_{ki}) Q N_k - (a^k_i - \widetilde{a}^k_i) q N_k. \tag{6.135}$$

Using Kato-Ponce type estimates as in (6.63), we have

$$\begin{aligned}
&\|H\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\quad \le C\|\widetilde{a}^T : \widetilde{a} - I\|_{L^\infty([0,T]\times\Omega_f)} \|\nabla V\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\qquad + C\|\widetilde{a}^T : \widetilde{a} - I\|_{W^{1/4,4}([0,T];L^\infty(\Omega_f))} \|\nabla V\|_{L^4([0,T];L^2(\Gamma_c))} \\
&\qquad + C\|a^T : a - \widetilde{a}^T : \widetilde{a}\|_{L^\infty([0,T];L^4(\Gamma_c))} \|\nabla v\|_{H^{1/4}([0,T];L^4(\Gamma_c))} \\
&\qquad + C\|a^T : a - \widetilde{a}^T : \widetilde{a}\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \|\nabla v\|_{L^\infty([0,T]\times\Omega_f)} \\
&\qquad + C\|\widetilde{a} - I\|_{L^\infty([0,T]\times\Omega_f)} \|Q\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\qquad + C\|\widetilde{a} - I\|_{W^{1/4,4}([0,T];L^\infty(\Omega_f))} \|Q\|_{L^4([0,T];L^2(\Gamma_c))} \\
&\qquad + C\|a - \widetilde{a}\|_{L^\infty([0,T];L^4(\Gamma_c))} \|q\|_{H^{1/4}([0,T];L^4(\Gamma_c))} \\
&\qquad + C\|a - \widetilde{a}\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \|q\|_{L^\infty([0,T];L^\infty(\Gamma_c))}.
\end{aligned} \tag{6.136}$$

Now we rely on the space-time interpolation inequality (6.64) for V,

$$\|\nabla V\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \le \epsilon_0 \|V\|_{H^1([0,T];L^2(\Omega_f))} + C_{\epsilon_0}\|V\|_{L^2([0,T];H^2(\Omega_f))}, \qquad \epsilon_0 \in (0,1], \tag{6.137}$$

together with

$$\|Q\|_{L^4([0,T];L^2(\Gamma_c))} \le C\|Q\|_{H^{1/4}([0,T];L^2(\Gamma_c))}. \tag{6.138}$$

For the differences of coefficients $a^T : a - \widetilde{a}^T : \widetilde{a}$ and $a - \widetilde{a}$ (which by (6.130) and (6.131) are bounded in $L^\infty([0,T];H^1(\Omega_f)))$, we also have

$$\|a - \widetilde{a}\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \le C\|\nabla U\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \tag{6.139}$$

and

$$\|a^T : a - \widetilde{a}^T : \widetilde{a}\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \le C\|\nabla U\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \tag{6.140}$$

for $T \le T_0$. Therefore, we conclude

$$\begin{aligned}
\|H\|_{H^{1/4}([0,T];L^2(\Gamma_c))} &\le C\epsilon\|U\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|U\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|P\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\quad + C\epsilon\|V\|_{H^1([0,T];L^2(\Omega_f))} + C\epsilon\|V\|_{L^2([0,T];H^2(\Omega_f))} + C\epsilon\|Q\|_{H^{1/4}([0,T];L^2(\Gamma_c))}.
\end{aligned} \tag{6.141}$$

We next use the optimal trace regularity for the wave equation (6.122)–(6.123) to estimate

$$\begin{aligned}
&\left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \left\|\frac{\partial W}{\partial N}\right\|_{H^{1/4}([0,T];L^2(\Gamma_c))} \\
&\quad \le C\left\|\int_0^t V\right\|_{L^2([0,T];H^{3/2}(\Gamma_c))} + C\,\|W\|_{H^{3/2}([0,T];L^2(\Gamma_c))} \\
&\quad \le CT^{1/2}\|V\|_{L^2([0,T];H^2(\Omega_f))} + C\|V\|_{H^{1/2}([0,T];L^2(\Gamma_c))},
\end{aligned} \tag{6.142}$$

where we utilized $W = \int_0^t V(s)\,ds$ on Γ_c. Observe that for the last term on the right side we may apply the interpolation inequality

$$\|V\|_{H^{1/2}([0,T];L^2(\Gamma_c))} \le \epsilon_0 \|V_t\|_{L^2([0,T];L^2(\Omega_f))} + C_{\epsilon_0}\|V\|_{L^2([0,T];H^1(\Omega_f))}. \tag{6.143}$$

Thus we conclude

$$\begin{aligned}&\left\|\frac{\partial W}{\partial N}\right\|_{L^2([0,T];H^{1/2}(\Gamma_c))} + \left\|\frac{\partial W}{\partial N}\right\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\quad\le C\|V\|_{L^2([0,T];H^2(\Omega_f))} + C\|V_t\|_{L^2([0,T];L^2(\Omega_f))}.\end{aligned} \tag{6.144}$$

In order to estimate $G_t = \operatorname{div}\bar{A}$ where $\bar{A}$ is given in (6.126), we write

$$\bar{A}_k = (\widetilde{a}_j^k - \delta_{kj})\partial_t \widetilde{V}^j - (\widetilde{a}_j^k - a_j^k)\partial_t v^j + \partial_t \widetilde{a}_j^k V^j - (\partial_t \widetilde{a}_j^k - \partial_t a_j^k)v^j. \tag{6.145}$$

We have

$$\begin{aligned}\|\bar{A}(t)\|_{L^2(\Omega_f)} &\le C\|a - I\|_{L^\infty(\Omega_f)}\|V_t\|_{L^2(\Omega_f)} + C\|a - \widetilde{a}\|_{L^6(\Omega_f)}\|\partial_t v\|_{L^3(\Omega_f)}\\ &\quad + C\|\partial_t \widetilde{a}\|_{L^3(\Omega_f)}\|V\|_{L^6(\Omega_f)} + C\|\partial_t(a - \widetilde{a})\|_{L^2(\Omega_f)}\|v\|_{L^\infty(\Omega_f)}.\end{aligned} \tag{6.146}$$

Using (6.129) and

$$\|\partial_t(a - \widetilde{a})\|_{L^2([0,T];L^2(\Omega_f))} \le C\|\nabla V\|_{L^2([0,T];L^2(\Omega_f))}, \tag{6.147}$$

we conclude

$$\|\bar{A}\|_{L^2([0,T];L^2(\Omega_f))} \le C\epsilon\|V\|_{L^2([0,T];H^2(\Omega_f)} + C\epsilon\|V_t\|_{L^2([0,T];L^2(\Omega_f)}. \tag{6.148}$$

Regarding W we have

$$\|W\|_{L^\infty([0,T];H^1(\Omega_e))} + \|W_t\|_{L^\infty([0,T];L^2(\Omega_e))} \le C\|W\|_{H^1(\Sigma_c)} \tag{6.149}$$

where

$$\begin{aligned}\|W\|_{H^1(\Sigma_c)} &= \left\|\int_0^t V\right\|_{L^2([0,T];H^1(\Gamma_c))} + \|W\|_{H^1([0,T];L^2(\Gamma_c))}\\ &\le CT^{1/2}\|V\|_{L^2([0,T];H^{3/2}(\Omega_f))} + \|V\|_{L^2([0,T];L^2(\Gamma_c))} \le C\|V\|_{L^2([0,T];H^2(\Omega_f))}\end{aligned} \tag{6.150}$$

for $T \le T_0$.

From the above estimates (applied on time steps), we conclude

$$\begin{aligned}&\|V\|_{L^2([0,T];H^2(\Omega_f))} + \|V_t\|_{L^2([0,T];L^2(\Omega_f))} + \|Q\|_{L^2([0,T];H^1(\Omega_f))} + \|Q\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\\ &\quad\le C\epsilon\left(\|U\|_{L^2([0,T];H^2(\Omega_f))} + \|U_t\|_{L^2([0,T];L^2(\Omega_f))} + \|P\|_{L^2([0,T];H^1(\Omega_f))} + \|P\|_{H^{1/4}([0,T];L^2(\Gamma_c))}\right)\end{aligned} \tag{6.151}$$

and

$$\|W\|_{L^\infty([0,T];H^1(\Omega_e))} + \|W_t\|_{L^\infty([0,T];L^2(\Omega_e))} \le C\|V\|_{L^2([0,T];H^2(\Omega_f))}. \tag{6.152}$$

If $\epsilon > 0$ and $\epsilon_0 > 0$ (in (6.115)) are sufficiently small, we get

$$\|(V,Q,W)\|_{Y_0} \leq \frac{1}{4}\|(U,P,\Psi)\|_{Y_0}. \tag{6.153}$$

Therefore, the solution map Λ is a contraction on the space Y_0 and thus there exists a unique fixed point solution in (v,q,w) in Y_0 which belongs to $\widetilde{Y}$. The solution satisfies the bound

$$\|v(t)\|_X \leq C\epsilon e^{-t/C}, \qquad 0 \leq t \leq T_0. \tag{6.154}$$

Now, we show how to continue the argument on the intervals $[kT_0,(k+1)T_0]$, where $k = 1,2,\ldots$. The main difficulty is that the inequalities (6.132), (6.140), and (6.150) require T_0 to be bounded. The main idea of the proof is to use the fact that, using induction, the iterates have already been shown to converge to the solution on $[0,kT_0]$ at an exponential rate and only the size of the interval $[kT_0,(k+1)T_0]$ needs to be taken into account. More precisely, let $Y_0(T_1,T_2)$ denote the space (6.114) with $[0,T]$ replaced by $[T_1,T_2]$ with the norm $\|(v,q,w)\|_{Y_0(T_1,T_2)}$ where in (6.115) the interval $[0,T]$ is replaced by $[T_1,T_2]$. Note that (6.153) shows that

$$\|(V,Q,W)\|_{Y_0[0,T_0]} \leq \frac{1}{4}\|(U,P,\Psi)\|_{Y_0[0,T_0]} \tag{6.155}$$

where we continue with using the notation (6.116)–(6.117) with (6.104)–(6.105).

Now, we show the modification of the inequality (6.155) when we work inductively on the interval $[kT_0,(k+1)T_0]$, where $k \in \mathbb{N}$.

For $t \in [kT_0,(k+1)T_0]$, the inequality (6.130) is replaced by

$$\begin{aligned}\|a-\widetilde{a}\|_{H^1} &\leq C\int_0^{kT_0}\|U\|_{H^2}\,ds + C\int_{kT_0}^{t}\|U\|_{H^2}\,ds\\ &\leq C_k\|U\|_{L^2([0,kT_0];H^2(\Omega_f))} + C\|U\|_{L^2([kT_0,(k+1)T_0];H^2(\Omega_f))}.\end{aligned} \tag{6.156}$$

We split the integral in (4.37) similarly and thus (4.27) is replaced by

$$\begin{aligned}&\|F\|_{L^2([kT_0,(k+1)T_0];L^2(\Omega_f))}\\ &\quad\leq C_k\epsilon\|U\|_{L^2([0,kT_0];H^2(\Omega_f))} + C_k\epsilon\|P\|_{L^2([0,kT_0];H^1(\Omega_f))}\\ &\qquad + C\epsilon\|U\|_{L^2([kT_0,(k+1)T_0];H^2(\Omega_f))} + C\epsilon\|P\|_{L^2([kT_0,(k+1)T_0];H^1(\Omega_f))}.\end{aligned} \tag{6.157}$$

Analogous replacements are obtained for the inequalities (6.141) and (6.150). Therefore, (6.153) becomes

$$\|(V,Q,W)\|_{Y_0(kT_0,(k+1)T_0)} \leq C_k\|(U,P,\Psi)\|_{Y_0(0,kT_0)} + \frac{1}{4}\|(U,P,\Psi)\|_{Y_0(kT_0,(k+1)T_0)}, \tag{6.158}$$

i.e.,

$$\|y^{(n+1)}-y^{(n)}\|_{Y_0(kT_0,(k+1)T_0)} \leq C_k\|y^{(n)}-y^{(n-1)}\|_{Y_0(0,kT_0)} + \frac{1}{4}\|y^{(n)}-y^{(n-1)}\|_{Y_0(kT_0,(k+1)T_0)} \tag{6.159}$$

for $n \in \mathbb{N}$. Now, using inductively the fact that as $n \to \infty$, the first term on the right side converges to zero exponentially fast, we get that, as $n \to \infty$, the iterates converge exponentially fast also on the interval $[kT_0,(k+1)T_0]$. Note that we also have

$$\|y^{(n)}\|_{\widetilde{Y}(kT_0,(k+1)T_0)} \leq C\|y^{(n)}(kT_0)\|_{\widetilde{Y}(0)} + \frac{1}{2}\|y^{(n-1)}\|_{\widetilde{Y}(kT_0,(k+1)T_0)}. \tag{6.160}$$

where $\widetilde{Y}(T_1, T_2)$ denotes the analog of $\widetilde{Y}$ but on the time interval $[T_1, T_2]$. This, again, implies that all the iterates belong to a ball B_M in $\widetilde{Y}$ on the interval $[kT_0, (k+1)T_0]$, where M is large enough, independent of n. The proof of the theorem is thus concluded. □

Acknowledgments

We thank Piotr Mucha for useful discussions. We also thank the two referees for their suggestions. M.I. was supported in part by the NSF FRG grant DMS-115893, I.K. was supported in part by the NSF grants DMS-1311943 and DMS-1615239, I.L. was supported in part by the NSF grant DMS-0104305 and by the Air Force grant OSR FA9550-09-1-0459.

References

[ALT] G. Avalos, I. Lasiecka, and R. Triggiani, *Higher regularity of a coupled parabolic-hyperbolic fluid-structure interactive system*, Georgian Math. J. **15** (2008), no. 3, 403–437.

[AT1] G. Avalos and R. Triggiani, *The coupled PDE system arising in fluid/structure interaction. I. Explicit semigroup generator and its spectral properties*, Fluids and waves, Contemp. Math., vol. 440, Amer. Math. Soc., Providence, RI, 2007, pp. 15–54.

[AT2] G. Avalos and R. Triggiani, *Fluid-structure interaction with and without internal dissipation of the structure: a contrast study in stability*, Evol. Equ. Control Theory **2** (2013), no. 4, 563–598.

[B] M. Boulakia, *Existence of weak solutions for the three-dimensional motion of an elastic structure in an incompressible fluid*, J. Math. Fluid Mech. **9** (2007), no. 2, 262–294.

[BG1] M. Boulakia and S. Guerrero, *Regular solutions of a problem coupling a compressible fluid and an elastic structure*, J. Math. Pures Appl. (9) **94** (2010), no. 4, 341–365.

[BG2] M. Boulakia and S. Guerrero, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **26** (2009), no. 3, 777–813.

[BGLT1] V. Barbu, Z. Grujić, I. Lasiecka, and A. Tuffaha, *Existence of the energy-level weak solutions for a nonlinear fluid-structure interaction model*, Fluids and waves, Contemp. Math., vol. 440, Amer. Math. Soc., Providence, RI, (2007), 55–82.

[BGLT2] V. Barbu, Z. Grujić, I. Lasiecka, and A. Tuffaha, *Smoothness of weak solutions to a nonlinear fluid-structure interaction model*, Indiana Univ. Math. J. **57** (2008), no. 3, 1173–1207.

[BL] F. Bucci and I. Lasiecka, *Optimal boundary control with critical penalization for a PDE model of fluid-solid interactions*, Calc. Var. Partial Differential Equations **37** (2010), no. 1-2, 217–235.

[BZ1] L. Bociu and J.-P. Zolésio, *Sensitivity analysis for a free boundary fluid-elasticity interaction*, Evol. Equ. Control Theory **2** (2013), no. 1, 55–79.

[BZ2] L. Bociu and J.-P. Zolésio, *Existence for the linearization of a steady state fluid/nonlinear elasticity interaction*, Discrete Contin. Dyn. Syst. (2011), no. Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. I, 184–197.

[CS1] D. Coutand and S. Shkoller, *Motion of an elastic solid inside an incompressible viscous fluid*, Arch. Ration. Mech. Anal. **176** (2005), no. 1, 25–102.

[CS2] D. Coutand and S. Shkoller, *The interaction between quasilinear elastodynamics and the Navier-Stokes equations*, Arch. Ration. Mech. Anal. **179** (2006), no. 3, 303–352.

[DEGL] B. Desjardins, M.J. Esteban, C. Grandmont, and P. Le Tallec, *Weak solutions for a fluid-elastic structure interaction model*, Rev. Mat. Complut. **14** (2001), no. 2, 523–538.

[DGHL] Q. Du, M.D. Gunzburger, L.S. Hou, and J. Lee, *Analysis of a linear fluid-structure interaction problem*, Discrete Contin. Dyn. Syst. **9** (2003), no. 3, 633–650.

[F] E. Feireisl, *On the motion of rigid bodies in a viscous incompressible fluid*, J. Evol. Equ. **3** (2003), no. 3, 419–441, Dedicated to Philippe Bénilan.

[GS] G. Grubb and V.A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand. **69** (1991), no. 2, 217–290 (1992).

[GGCC] G. Guidoboni, R. Glowinski, N. Cavallini, and S. Canic, *Stable loosely-coupled-type algorithm for fluid-structure interaction in blood flow*, J. Comput. Phys. **228** (2009), no. 18, 6916–6937.

[GGCCL] G. Guidoboni, R. Glowinski, N. Cavallini, S. Canic, and Sergey Lapin, *A kinematically coupled time-splitting scheme for fluid-structure interaction in blood flow*, Appl. Math. Lett. **22** (2009), no. 5, 684–688.

[IKLT1] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha, *On well-posedness for a free boundary fluid-structure model*, J. Math. Phys. **53** (2012), no. 11, 115624, 13 pp.

[IKLT2] M. Ignatova, I. Kukavica, I. Lasiecka, and A. Tuffaha, *On well-posedness and small data global existence for an interface damped free boundary fluid-structure model*, Nonlinearity **27** (2014), no. 3, 467–499.

[KMT] I. Kukavica, A.L. Mazzucato, and A. Tuffaha, *Sharp trace regularity for an anisotropic elasticity system*, Proc. Amer. Math. Soc. **141** (2013), no. 8, 2673–2682.

[KT1] I. Kukavica and A. Tuffaha, *Solutions to a fluid-structure interaction free boundary problem*, Discrete Contin. Dyn. Syst. **32** (2012), no. 4, 1355–1389.

[KT2] I. Kukavica and A. Tuffaha, *Regularity of solutions to a free boundary problem of fluid-structure interaction*, Indiana Univ. Math. J. **61** (2012), no. 5, 1817–1859.

[KT3] I. Kukavica and A. Tuffaha, *Well-posedness for the compressible Navier-Stokes-Lamé system with a free interface*, Nonlinearity **25** (2012), no. 11, 3111–3137.

[KTZ1] I. Kukavica, A. Tuffaha, and M. Ziane, *Strong solutions to a nonlinear fluid structure interaction system*, J. Differential Equations **247** (2009), no. 5, 1452–1478.

[KTZ2] I. Kukavica, A. Tuffaha, and M. Ziane, *Strong solutions for a fluid structure interaction system*, Adv. Differential Equations **15** (2010), no. 3-4, 231–254.

[KTZ3] I. Kukavica, A. Tuffaha, and M. Ziane, *Strong solutions to a Navier-Stokes-Lamé system on a domain with a non-flat boundary*, Nonlinearity **24** (2011), no. 1, 159–176.

[L1] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[L2] J.-L. Lions, *Hidden regularity in some nonlinear hyperbolic equations*, Mat. Apl. Comput. **6** (1987), no. 1, 7–15.

[LL1] I. Lasiecka and Y. Lu, *Asymptotic stability of finite energy in Navier Stokes-elastic wave interaction*, Semigroup Forum **82** (2011), no. 1, 61–82.

[LL2] I. Lasiecka and Y. Lu, *Interface feedback control stabilization of a nonlinear fluid-structure interaction*, Nonlinear Anal. **75** (2012), no. 3, 1449–1460.

[LLT] I. Lasiecka, J.-L. Lions, and R. Triggiani, *Nonhomogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pures Appl. (9) **65** (1986), no. 2, 149–192.

[LT] I. Lasiecka and D. Toundykov, *Semigroup generation and "hidden" trace regularity of a dynamic plate with non-monotone boundary feedbacks*, Commun. Math. Anal. **8** (2010), no. 1, 109–144.

[LTr1] I. Lasiecka and R. Triggiani, *Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim. **25** (1992), no. 2, 189–224.

[LTr2] I. Lasiecka and R. Triggiani, *Sharp regularity theory for elastic and thermoelastic Kirchoff equations with free boundary conditions*, Rocky Mountain J. Math. **30** (2000), no. 3, 981–1024.

[MZ1] P.B. Mucha and W.M. Zajaczkowski, *On the existence for the Cauchy–Neumann problem for the Stokes system in the L_p-framework*, Studia Math. **143** (2000), 75–101.

[MZ2] P.B. Mucha and W.M. Zajaczkowski, *On local existence of solutions of free boundary problem for incompressible viscous self-gravitating fluid motion*, Applicationes Mathematicae **27** (2000), no. 3, 319–333.

[PS] J. Prüss and G. Simonett, *On the two-phase Navier-Stokes equations with surface tension*, Interfaces Free Bound. **12** (2010), no. 3, 311–345.

[RV] J.-P. Raymond and M. Vanninathan, *A fluid-structure model coupling the Navier-Stokes equations and the Lamé system*, J. Math. Pures Appl. (9) **102** (2014), no. 3, 546–596.

[S] R. Sakamoto, *Hyperbolic boundary value problems*, Cambridge University Press, Cambridge-New York, 1982, Translated from the Japanese by Katsumi Miyahara.

[SST] J.A. San Martín, V. Starovoitov, and M. Tucsnak, *Global weak solutions for the two-dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Ration. Mech. Anal. **161** (2002), no. 2, 113–147.

[T] D. Tataru, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 1, 185–206.

[Te1] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*, second ed., Applied Mathematical Sciences, vol. 68, Springer-Verlag, New York, 1997.

[Te2] R. Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, 2001, Theory and numerical analysis, Reprint of the 1984 edition.