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On well-posedness for a free boundary fluid-structure model

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We address a fluid-structure interaction model describing the motion of an elastic body immersed in an incompressible fluid. We establish *a priori* estimates for the local existence of solutions for a class of initial data which also guarantees uniqueness.
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Dedicated to Professor Peter Constantin, on the occasion of his 60th birthday.

I. INTRODUCTION

In this paper, we derive *a priori* estimates needed for establishing the local in time well-posedness for a fluid-structure model. The model consists of the Navier-Stokes equations

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

and a wave equation

$$w_{tt} - \Delta w = 0 \tag{1.3}$$

with natural velocity and stress matching conditions imposed on the common free moving boundary.

The existence of solutions was first established in Ref. 9 by Coutand and Shkoller with initial fluid velocity u_0 belonging to H^5 and initial data for the wave equation (w_0, w_1) belonging to $H^3 \times H^2$. However, due to the divergence-free condition, the uniqueness for the model required higher regularity data, and it was proved for $(u_0, w_0, w_1) \in H^7 \times H^5 \times H^4$. In Ref. 17, the second and the fourth author of the present paper established *a priori* estimates for the existence with the data (u_0, w_0, w_1) in $H^3 \times H^{5/2+r} \times H^{3/2+r}$, where $r > 0$. The uniqueness was obtained only under the additional condition $\nabla v_{tt} \in L^2_{x,t}$ for the Lagrangian velocity v.

The main result of the present paper provides *a priori* estimates for the local existence of solutions with initial data (u_0, w_0, w_1) in $H^4 \times H^3 \times H^2$ which satisfy $\nabla v_{tt} \in L^2_{x,t}$. This, together with Ref. 17, leads to *a priori* estimates needed for the well-posedness of the system in the space $H^4 \times H^3 \times H^2$. The main difficulty in the proof is the low regularity for w_0 and w_1 which results in a substantial loss of regularity for the Lagrangian velocity—from H^4 initially to H^3 for positive time.

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We note that the presented *a priori* estimates do not require the optimal (hidden) trace regularity of solutions and thus the proof is much simpler than the ones from Refs. 16 and 17.

In our treatment of the local existence, we benefit from the coupling of the Navier-Stokes equation with a hyperbolic system even in the case where the time evolution of the domains is neglected. It is interesting to note that global solvability was proven in the case of static interface without any damping added to the wave motion (c.f. Refs. 6 and 7). However, in contrast with the present work, there are no decay rates valid for this latter model. This is due to the fact that the undamped wave motion gives rise, in a linear case, to spectrum that approaches asymptotically the imaginary axis.[2] Since the model accounting for the evolution of the domain leads to a quasilinear system, global existence of solutions should not be expected in the absence of uniform decay rates of the energy for linearized equations. We refer the reader to a large body of work that has developed in the last decade on the interaction between parabolic and hyperbolic dynamics.[1,2,6,7,11,19–22,26] Of independent interest are also, the free moving boundary problem involving the coupling of the compressible Navier-Stokes with the linear elasticity system.[4,5,18] For applications of fluid-structure interaction systems c.f. Refs. 13 and 14. For more results on hidden regularity, c.f. Refs. 23–26 and 28, and c.f. Refs. 8, 27, and 29 for applications in control theory.

The paper is organized as follows. In Sec. II, we introduce the model posed in Lagrangian coordinates and state the main result. Section III contains the main lemma for the Lagrangian coefficients a, the elliptic regularity (Stokes and Laplace) statements and the *a priori* estimate leading to the local in time well-posedness. The proof of Theorem 2.1 is presented in Sec. IV.

II. THE MAIN RESULTS

We consider the free boundary fluid-structure system which models the motion of an elastic body moving and interacting with an incompressible viscous fluid (c.f. Refs. 3, 4, 9, 10, 16, and 17). This parabolic-hyperbolic system couples the Navier-Stokes equation

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \tag{2.1}$$

$$\nabla \cdot u = 0, \tag{2.2}$$

and a wave equation

$$w_{tt} - \Delta w = 0, \tag{2.3}$$

posed in the Eulerian and the Lagrangian framework, respectively. The interaction is captured by natural velocity and stress matching conditions on the free moving interface between the fluid and the elastic body.

It is more convenient to consider the system formulated in the Lagrangian coordinates (cf. Refs. 9 and 17). With η: $\Omega_f \to \Omega_f(t)$ the position function, the incompressible Navier-Stokes equation may be written as

$$v_t^i - \partial_j(a_l^j a_l^k \partial_k v^i) + \partial_k(a_i^k q) = 0 \text{ in } \Omega_f \times (0, T), \quad i = 1, 2, 3, \tag{2.4}$$

$$a_i^k \partial_k v^i = 0 \text{ in } \Omega_f \times (0, T), \tag{2.5}$$

where $v(x, t)$ and $q(x, t)$ denote the Lagrangian velocity vector field and the pressure of the fluid over the initial domain Ω_f, i.e., $v(x, t) = \eta_t(x, t) = u(\eta(x, t), t)$ and $q(x, t) = p(\eta(x, t), t)$ in Ω_f. The matrix a with ij entry a_j^i is defined by $a(x, t) = (\nabla\eta(x, t))^{-1}$ in Ω_f, i.e., $\partial_m \eta_i a_j^m = \delta_{ij}$ for all $i, j = 1$, 2, 3. The elastic equation for the displacement function $w(x, t) = \eta(x, t) - x$ is formulated in the Lagrangian framework as

$$w_{tt}^i - \Delta w^i = 0 \text{ in } \Omega_e \times (0, T), i = 1, 2, 3 \tag{2.6}$$

over the initial domain Ω_e. We thus seek a solution (v, w, q, a, η) to the system (2.4)–(2.6), where the coefficients a_j^i for $i, j = 1, 2, 3$ and η are determined from

$$a_t = -a : \nabla v : a \text{ in } \Omega_f \times (0, T), \tag{2.7}$$

$$\eta_t = v(x,t) \text{ in } \Omega_f \times (0,T), \tag{2.8}$$

with the initial conditions $a(x, 0) = I$ and $\eta(x, 0) = x$ in Ω_f. On the interface Γ_c between Ω_f and Ω_e, we assume matching of velocities and stresses

$$v^i = w_t^i \text{ on } \Gamma_c \times (0,T), \tag{2.9}$$

$$a_l^j a_l^k \partial_k v^i N_j - a_i^k q N_k = \partial_j w^i N_j \text{ on } \Gamma_c \times (0,T), \tag{2.10}$$

while on the outside fluid boundary Γ_f we assume the non-slip condition

$$v^i = 0 \text{ on } \Gamma_f \times (0,T) \tag{2.11}$$

for $i = 1, 2, 3$, where $N = (N_1, N_2, N_3)$ is the unit outward normal with respect to Ω_e. We supplement the system (2.4)–(2.6) with the initial conditions $v(x, 0) = v_0(x)$ and $(w(x, 0), w_t(x, 0)) = (0, w_1(x))$ on Ω_f and Ω_e, respectively. We also use the classical spaces $H = \{v \in L^2(\Omega_f) : \operatorname{div} v = 0, v \cdot N|_{\Gamma_f} = 0\}$ and $V = \{v \in H^1(\Omega_f) : \operatorname{div} v = 0, v|_{\Gamma_f} = 0\}$. Based on v_0, we determine the initial pressure by solving the problem

$$\begin{aligned} &\Delta q_0 = -\partial_i v_0^k \partial_k v_0^i \text{ in } \Omega_f, \\ &\nabla q_0 \cdot N = \Delta v_0 \cdot N \text{ on } \Gamma_f, \\ &- q_0 = -\partial_j v_0^i N_j N_i + \partial_j w^i N_j N_i \text{ on } \Gamma_c. \end{aligned} \tag{2.12}$$

The following statement is our main result.

Theorem 2.1: *Assume that $v_0 \in V \cap H^4(\Omega_f)$, $w_0 \in H^3(\Omega_e)$, and $w_1 \in H^2(\Omega_e)$ with the appropriate compatibility conditions*

$$w_1 = v_0, \quad \Delta w_0 = \Delta v_0 - \nabla q_0 \text{ on } \Gamma_c, \tag{2.13}$$

$$\frac{\partial w_0}{\partial N} \cdot \tau = \frac{\partial v_0}{\partial N} \cdot \tau, \quad \frac{\partial w_1}{\partial N} \cdot \tau = \frac{\partial}{\partial N}[\Delta v_0 - \nabla q_0] \cdot \tau \text{ on } \Gamma_c, \tag{2.14}$$

$$v_0 = 0, \quad \Delta v_0 - \nabla q_0 = 0 \text{ on } \Gamma_f. \tag{2.15}$$

Assume that (v, w, q, a, η) is a smooth solution to the system (2.4)–(2.6) with the boundary conditions (2.9)–(2.11). Then the norm

$$X(t) = \|v_{tt}(t)\|^2_{L^2(\Omega_f)} + \|w_{ttt}(t)\|^2_{L^2(\Omega_e)} + \|\nabla w_{tt}(t)\|^2_{L^2(\Omega_e)} + \int_0^t \|\nabla v_{tt}(s)\|^2_{L^2(\Omega_f)}\, ds + 1 \tag{2.16}$$

remains bounded for $t \in [0, T]$, where the time $T > 0$ depends on the initial data. In particular, the solution (v, w, q, a, η) satisfies

$$v \in L^\infty([0,T]; H^3(\Omega_f)), \tag{2.17}$$

$$v_t \in L^\infty([0,T]; H^2(\Omega_f)), \tag{2.18}$$

$$\nabla v_{tt} \in L^2([0,T]; L^2(\Omega_f)), \tag{2.19}$$

$$\partial_t^j w \in C([0,T]; H^{3-j}(\Omega_e)), \quad j = 0, 1, 2, 3 \tag{2.20}$$

with $q \in L^\infty([0, T]; H^2(\Omega_f))$, $q_t \in L^\infty([0, T]; H^1(\Omega_f))$, $a, a_t \in L^\infty([0, T]; H^2(\Omega_f))$, $a_{tt} \in L^\infty([0, T]; H^1(\Omega_f))$, $a_{ttt} \in L^2([0, T]; L^2(\Omega_f))$, and $\eta|_{\Omega_f} \in C([0,T]; H^3(\Omega_f))$ and the corresponding norms and $1/T$ are bounded by a polynomial function of $\|v_0\|_{H^4(\Omega_f)\cap V}$.

Remark 2.2: The result of Theorem 2.1 depends on the existence of sufficiently smooth solutions in line with the topologies listed in (2.17)–(2.20). Existence of smooth local solutions with the initial data in $H^5 \times H^3 \times H^2$ has been shown in Ref. 9. Thus, our result shows that for the solutions established in Ref. 9 there is no finite in time blow up of $H^4 \times H^3 \times H^2$ norms. However, a full resolution of the existence problem requires construction of local solutions respecting the finiteness of $X(t)$ for the initial data in $H^4 \times H^3 \times H^2$. This construction can be carried out by taking advantage of compatibility conditions assumed in (2.13)–(2.15). These conditions are apparent when one solves coupled wave and fluid system after elimination of the pressure as in (2.12). This method is inspired by Grubb and Solonnikov[12] where solutions to Navier-Stokes equations with Neumann type of boundary conditions are shown to be equivalent to solutions of pseudo-parabolic problem with tangential boundary conditions and nonlocal pseudo-differential operators representing the pressure. The details of this procedure will be carried out in a subsequent paper.

The proof of Theorem 2.1 is given in Sec. IV.

III. PRELIMINARY RESULTS

In this section, we give the formal *a priori* estimates on the time derivatives of the unknown functions needed in the proof of Theorem 2.1. We begin with an auxiliary result providing bound on the coefficients of the matrix a.

Lemma 3.1: Assume that $\|\nabla v\|_{L^\infty([0,T];H^2(\Omega_f))} \le M$. *Let* $p \in [1, \infty]$ *and* $i, j = 1, 2, 3$. *With* $T \in [0, 1/CM]$, *where C is a sufficiently large constant, the following statements hold:*

(i) $\|\nabla\eta\|_{H^2(\Omega_f)} \le C$ *for* $t \in [0, T]$;

(ii) $\|a\|_{H^2(\Omega_f)} \le C$ *for* $t \in [0, T]$;

(iii) $\|a_t\|_{L^p(\Omega_f)} \le C\|\nabla v\|_{L^p(\Omega_f)}$ *for* $t \in [0, T]$;

(iv) $\|\partial_i a_t\|_{L^p(\Omega_f)} \le C\|\nabla v\|_{L^{p_1}(\Omega_f)}\|\partial_i a\|_{L^{p_2}(\Omega_f)} + C\|\nabla\partial_i v\|_{L^p(\Omega_f)}$ *for* $i = 1, 2, 3$ *and* $t \in [0, T]$ *where* $1 \le p, p_1, p_2 \le \infty$ *are such that* $1/p = 1/p_1 + 1/p_2$;

(v) $\|\partial_{ij} a_t\|_{L^2(\Omega_f)} \le C\|\nabla v\|^{1/2}_{H^1(\Omega_f)}\|\nabla v\|^{1/2}_{H^2(\Omega_f)} + C\|\nabla v\|_{H^2(\Omega_f)}$ *for* $i, j = 1, 2, 3$ *and* $t \in [0, T]$;

(vi) $\|a_{tt}\|_{L^2(\Omega_f)} \le C\|\nabla v\|_{L^2(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)} + C\|\nabla v_t\|_{L^2(\Omega_f)}$ *and* $\|a_{tt}\|_{L^3(\Omega_f)} \le C\|v\|^2_{H^2(\Omega_f)} + C\|\nabla v_t\|_{L^3(\Omega_f)}$ *for* $t \in [0, T]$;

(vii) $\|a_{ttt}\|_{L^2(\Omega_f)} \le C\|\nabla v\|^3_{H^1(\Omega_f)} + C\|\nabla v_t\|_{L^2(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)} + C\|\nabla v_{tt}\|_{L^2(\Omega_f)}$ *for* $t \in [0, T]$;

(viii) *for every* $\epsilon \in (0, 1/2]$ *and all* $t \le T^* = \min\{\epsilon/CM^2, T\}$, *we have*

$$\|\delta_{jk} - a^j_l a^k_l\|^2_{H^2(\Omega_f)} \le \epsilon, \qquad j, k = 1, 2, 3 \tag{3.1}$$

and

$$\|\delta_{jk} - a^j_k\|^2_{H^2(\Omega_f)} \le \epsilon, \qquad j, k = 1, 2, 3. \tag{3.2}$$

In particular, the form $a^j_l a^k_l \xi^i_j \xi^i_k$ *satisfies the ellipticity estimate*

$$a^j_l a^k_l \xi^i_j \xi^i_k \ge \frac{1}{C}|\xi|^2, \qquad \xi \in \mathbb{R}^{n^2} \tag{3.3}$$

for all $t \in [0, T^*]$ *and* $x \in \Omega_f$, *provided* $\epsilon \le 1/C$ *with C sufficiently large.*

Proof of Lemma 3.1:

(i) By (2.8), we have $\nabla\eta(x, t) = I + \int_0^t \nabla v(x, \tau)\, d\tau$ for $t \in [0, T]$. Thus, the assertion follows from $\|\nabla v\|_{L^\infty([0,T];H^2(\Omega_f))} \le M$.

(ii) We have

$$\|a(t)\|_{H^2(\Omega_f)} = \left\| a(0) + \int_0^t a_t(\tau)\, d\tau \right\|_{H^2(\Omega_f)} \le C + \int_0^t \|a_t(\tau)\|_{H^2(\Omega_f)}\, d\tau, \tag{3.4}$$

$$\le C + \int_0^t \|a(\tau)\|^2_{H^2(\Omega_f)}\|\nabla v(\tau)\|_{H^2(\Omega_f)}\, d\tau \le C + M\int_0^t \|a(\tau)\|^2_{H^2(\Omega_f)}\, d\tau, \tag{3.5}$$

for $t \in [0, T]$, where we used (2.7). Applying the Gronwall lemma, we obtain $\|a(t)\|_{H^2(\Omega_f)} \leq C$ for $t \leq 1/CM$, where C is a sufficiently large constant.

(iii) By the Sobolev inequality and (ii), we have $\|a(t)\|_{L^\infty(\Omega_f)} \leq C\|a(t)\|_{H^2(\Omega_f)} \leq C$ for $t \in [0, T]$. Using (2.7), we get

$$\|a_t(t)\|_{L^p(\Omega_f)} \leq \|a(t)\|^2_{L^\infty(\Omega_f)}\|\nabla v(t)\|_{L^p(\Omega_f)} \tag{3.6}$$

for $t \in [0, T]$, and (iii) is established.

(iv) We differentiate (2.7) to get

$$\partial_i a_t = -\partial_i a : \nabla v : a - a : \nabla \partial_i v : a - a : \nabla v : \partial_i a. \tag{3.7}$$

The desired estimate then follows by using Hölder's inequality with $1/p = 1/p_1 + 1/p_2$ and $\|a(t)\|_{L^\infty(\Omega_f)} \leq C$ for $t \in [0, T]$.

(v) Differentiating (3.7), we obtain

$$\partial_{ij} a_t = -\partial_{ij} a : \nabla v : a - \partial_i a : \nabla \partial_j v : a - \partial_i a : \nabla v : \partial_j a - \partial_j a : \nabla \partial_i v : a - a : \nabla \partial_{ij} v : a,$$

$$- a : \nabla \partial_i v : \partial_j a - \partial_j a : \nabla v : \partial_i a - a : \nabla \partial_j v : \partial_i a - a : \nabla v : \partial_{ij} a, \tag{3.8}$$

leading to

$$\|\partial_{ij} a_t\|_{L^2(\Omega_f)} \leq C\|\nabla v\|_{L^\infty(\Omega_f)} + C\|\nabla \partial_i v\|_{L^3(\Omega_f)} + C\|\nabla \partial_j v\|_{L^3(\Omega_f)} + C\|\nabla \partial_{ij} v\|_{L^2(\Omega_f)} \tag{3.9}$$

for $t \in [0, T]$, where we utilized Hölder's inequality and the part (i) of this lemma. By the interpolation inequalities $\|\nabla v\|_{L^\infty(\Omega_f)} \leq C\|\nabla v\|^{1/2}_{H^1(\Omega_f)}\|\nabla v\|^{1/2}_{H^2(\Omega_f)}$ and $\|\nabla \partial_i v\|_{L^3(\Omega_f)} \leq C\|\nabla v\|^{1/2}_{H^1(\Omega_f)}\|\nabla v\|^{1/2}_{H^2(\Omega_f)}$, we deduce the desired estimate.

(vi) Differentiating (2.7) in time gives $a_{tt} = 2a : \nabla v : a : \nabla v : a - a : \nabla v_t : a$. The assertions then follow from $\|a(t)\|_{L^\infty(\Omega_f)} \leq C$ for $t \in [0, T]$, since

$$\|a_{tt}\|_{L^2(\Omega_f)} \leq C\|a\|^3_{L^\infty(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)}\|\nabla v\|_{L^2(\Omega_f)} + C\|a\|^2_{L^\infty(\Omega_f)}\|\nabla v_t\|_{L^2(\Omega_f)} \tag{3.10}$$

and

$$\|a_{tt}\|_{L^3(\Omega_f)} \leq C\|a\|^3_{L^\infty(\Omega_f)}\|\nabla v\|^2_{L^6(\Omega_f)} + C\|a\|^2_{L^\infty(\Omega_f)}\|\nabla v_t\|_{L^3(\Omega_f)}. \tag{3.11}$$

(vii) Differentiating $a_t = -a : \nabla v : a$ twice in time, we obtain

$$a_{ttt} = 6a:\nabla v:a:\nabla v:a:\nabla v:a + 3a:\nabla v:a:\nabla v_t:a + 3a:\nabla v_t:a:\nabla v:a - a:\nabla v_{tt}:a, \tag{3.12}$$

whence

$$\|a_{ttt}\|_{L^2(\Omega_f)} \leq C\|\nabla v\|^3_{L^6(\Omega_f)} + C\|\nabla v_t\|_{L^2(\Omega_f)}\|\nabla v\|_{L^\infty(\Omega_f)} + C\|\nabla v_{tt}\|_{L^2(\Omega_f)} \tag{3.13}$$

using $\|a(t)\|_{L^\infty(\Omega_f)} \leq C$ for $t \in [0, T]$.

(viii) The first inequality follows from $\delta_{jk} - a^j_l a^k_l = -\int_0^t \partial_t (a^j_l a^k_l)(s)\, ds$ and the multiplicative Sobolev inequalities, while the second one follows from $\delta_{jk} - a^j_k = -\int_0^t \partial_t a^j_k(s)\, ds$. □

Lemma 3.2: Assume that v and q are solutions to the system

$$v^i_t - \partial_j(a^j_l a^k_l \partial_k v^i) + \partial_k(a^k_i q) = 0 \text{ in } \Omega_f, \tag{3.14}$$

$$a^k_i \partial_k v^i = 0 \text{ in } \Omega_f, \tag{3.15}$$

$$v = 0 \text{ on } \Gamma_f, \tag{3.16}$$

$$a^j_l a^k_l \partial_k v^i N_j - a^k_i q N_k = \partial_j w^i N_j \text{ on } \Gamma_c, \tag{3.17}$$

for given coefficients $a^i_j \in L^\infty(\Omega_f)$ with $i, j = 1, 2, 3$ satisfying Lemma 3.1 with a sufficiently small constant $\epsilon = 1/C$. Then the estimate

$$\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)} \le C\|v_t\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} \tag{3.18}$$

holds for $s = 0, 1$ and for all $t \in (0, T)$. Moreover, the time derivatives v_t and q_t satisfy

$$\begin{aligned}&\|v_t\|_{H^2(\Omega_f)} + \|q_t\|_{H^1(\Omega_f)}\\ &\quad\le C\|v_{tt}\|_{L^2(\Omega_f)} + C\left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/2}(\Gamma_c)} + C\|v\|^{1/2}_{H^2(\Omega_f)}\|v\|^{1/2}_{H^3(\Omega_f)}\left(\|v\|_{H^2(\Omega_f)} + \|q\|_{H^1(\Omega_f)}\right)\end{aligned} \tag{3.19}$$

for all $t \in (0, T)$, where $T \le 1/CM$ for a sufficiently large constant C.

Proof of Lemma 3.2: Let ϕ be a solution to the elliptic equation

$$\Delta\phi = -(\delta_{jk} - a^k_j)\partial_k v^j \text{ in } \Omega_f \tag{3.20}$$

with the Dirichlet boundary condition $\phi = 0$ on $\Gamma_c \cup \Gamma_f$. Then the function $u = v + \nabla\phi$ satisfies the stationary Stokes problem

$$\begin{aligned}&-\Delta u^i + \partial_i q = -\Delta\partial_i\phi - \partial_j((\delta_{jk} - a^j_l a^k_l)\partial_k v^i) + \partial_k((\delta_{ik} - a^k_i)q) - v^i_t \text{ in } \Omega_f \qquad (3.21)\\ &\partial_j u^j = 0 \text{ in } \Omega_f\\ &u = \nabla\phi \text{ on } \Gamma_f\\ &\partial_j u^i N_j - qN_i = \partial_j w^i N_j + \partial_{ij}\phi N_j + (\delta_{jk} - a^j_l a^k_l)\partial_k v^i N_j - (\delta_{ik} - a^k_i)qN_k \text{ on } \Gamma_c.\end{aligned}$$

Thus, we have (c.f. Ref. 29 and 32, for instance)

$$\begin{aligned}&\|u\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)}\\ &\quad\le C\|\Delta\nabla\phi\|_{H^s(\Omega_f)} + C\|\partial_j((\delta_{jk} - a^j_l a^k_l)\partial_k v)\|_{H^s(\Omega_f)} + C\sum_i \|\partial_k((\delta_{ik} - a^k_i)q)\|_{H^s(\Omega_f)}\\ &\qquad + C\|v_t\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)} + C\left\|\frac{\partial(\nabla\phi)}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)}\\ &\qquad + C\|(\delta_{jk} - a^j_l a^k_l)\partial_k v N_j\|_{H^{s+1/2}(\Gamma_c)} + C\sum_i \|(\delta_{ik} - a^k_i)qN_k\|_{H^{s+1/2}(\Gamma_c)} + C\|\nabla\phi\|_{H^{s+3/2}(\Gamma_f)}.\end{aligned} \tag{3.22}$$

Using the trace theorem and $\|\nabla\Delta\phi\|_{H^s(\Omega_f)} \le C\|(\delta_{jk} - a^k_j)\partial_k v^j\|_{H^{s+1}(\Omega_f)}$ for the sixth and the ninth term on the right side, we obtain

$$\begin{aligned}&\|v\|_{H^{s+2}(\Omega_f)} + \|q\|_{H^{s+1}(\Omega_f)}\\ &\quad\le C\|(\delta_{jk} - a^k_j)\partial_k v^j\|_{H^{s+1}(\Omega_f)} + C\sum_j \|(\delta_{jk} - a^j_l a^k_l)\partial_k v\|_{H^{s+1}(\Omega_f)}\\ &\qquad + C\sum_{i,k} \|(\delta_{ik} - a^k_i)q\|_{H^{s+1}(\Omega_f)} + C\|v_t\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)}\\ &\quad\le C\epsilon\|v\|_{H^{s+2}(\Omega_f)} + C\epsilon\|q\|_{H^{s+1}(\Omega_f)} + C\|v_t\|_{H^s(\Omega_f)} + C\left\|\frac{\partial w}{\partial N}\right\|_{H^{s+1/2}(\Gamma_c)},\end{aligned} \tag{3.23}$$

where we also utilized the multiplicative Sobolev inequalities (namely, $\|uv\|_{H^s} \le \|u\|_{H^s}\|v\|_{H^2}$ for $0 \le s \le 2$ and $\|uv\|_{H^s} \le \|u\|_{H^s}\|v\|_{H^s}$ for $s \ge 2$) and the part $(viii)$ of Lemma 3.1. The inequality (3.18) now follows by choosing ϵ sufficiently small so that the first and the second term on the far right side are absorbed by the terms on the left side.

In order to prove the second part of the lemma, we differentiate the stationary Stokes problem (3.21) in time. By a similar argument as above, we have

$$\begin{aligned}
&\|v_t\|_{H^2(\Omega_f)} + \|q_t\|_{H^1(\Omega_f)} \\
&\quad \le C\|v_{tt}\|_{L^2(\Omega_f)} + C\|\partial_j(\partial_t(a_l^j a_l^k)\partial_k v)\|_{L^2(\Omega_f)} + C\sum_i \|\partial_k(\partial_t a_i^k q)\|_{L^2(\Omega_f)} \\
&\quad\quad + C\left\|\frac{\partial w_t}{\partial N}\right\|_{H^{1/2}(\Gamma_c)} + C\|\partial_t a_i^k \partial_k v^i\|_{H^1(\Omega_f)}.
\end{aligned} \tag{3.24}$$

Using Lemma 3.1, we bound the terms on the right side of (3.24) involving the entries of a. By the Poincaré inequality, we then obtain

$$\begin{aligned}
\|\partial_j(\partial_t(a_l^j a_l^k)\partial_k v^i)\|_{L^2(\Omega_f)} &\le C\left(\|a_t\|_{L^6(\Omega_f)}\|\nabla a\|_{L^6(\Omega_f)} + \|\nabla a_t\|_{L^3(\Omega_f)}\|a\|_{L^\infty(\Omega_f)}\right)\|\nabla v\|_{L^6} \\
&\quad + C\|a_t\|_{L^\infty(\Omega_f)}\|a\|_{L^\infty(\Omega_f)}\|v\|_{H^2(\Omega_f)} \\
&\le C\|v\|_{H^2(\Omega_f)}^{1/2}\|v\|_{H^3(\Omega_f)}^{1/2}\|v\|_{H^2(\Omega_f)}
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
\sum_i \|\partial_k(\partial_t a_i^k q)\|_{L^2(\Omega_f)} &\le C\|\nabla a_t\|_{L^3(\Omega_f)}\|q\|_{L^6(\Omega_f)} + \|a_t\|_{L^\infty(\Omega_f)}\|\nabla q\|_{L^2(\Omega_f)} \\
&\le C\|v\|_{H^2(\Omega_f)}^{1/2}\|v\|_{H^3(\Omega_f)}^{1/2}\|q\|_{H^1(\Omega_f)},
\end{aligned} \tag{3.26}$$

since $v = 0$ on Γ_f. Similarly, we may estimate $\|\partial_t a_i^k \partial_k v^i\|_{H^1}$ by the same right side as in (3.25), and the proof of (3.19) is established. □

Now, let w be a solution to the wave equation (2.6) satisfying the velocity matching condition (2.9) on the common boundary Γ_c. Note that we have $w(t) = w_0 + \int_0^t v(s)\,ds$ on Γ_c. Hence, we obtain the elliptic estimate

$$\|w\|_{H^3(\Omega_e)} \le C\|w_{tt}\|_{H^1(\Omega_e)} + C\int_0^t \|v(s)\|_{H^3(\Omega_f)}\,ds + C\|w_0\|_{H^3(\Omega_e)} \tag{3.27}$$

for all $t \in (0, T)$. Differentiating (2.6) in time, we also have by the ellipticity

$$\|w_t\|_{H^2(\Omega_e)} \le C\|w_{ttt}\|_{L^2(\Omega_e)} + C\|v\|_{H^2(\Omega_f)} \tag{3.28}$$

for all $t \in (0, T)$.

From (3.18) with $s = 1$ and (3.27), we conclude that the Stokes type estimate

$$\|v\|_{H^3} + \|q\|_{H^2} \le C\|v_t\|_{H^1} + C\|w_{tt}\|_{H^1} + C\int_0^t \|v\|_{H^3}\,ds + C\|w_0\|_{H^3} \tag{3.29}$$

holds for all $t \in (0, T)$, where $T \le 1/CM$. Now, using the Gronwall inequality, we obtain

$$\begin{aligned}
\|v\|_{H^3} + \|q\|_{H^2} &\le C\|v_t\|_{H^1} + C\|w_{tt}\|_{H^1} + C\|w_0\|_{H^3} \\
&\quad + Ce^{Ct}\int_0^t \left(\|v_t\|_{H^1} + \|w_{tt}\|_{H^1} + \|w_0\|_{H^3}\right) ds.
\end{aligned} \tag{3.30}$$

Analogous derivation shows that

$$\begin{aligned}
\|v\|_{H^2} + \|q\|_{H^1} &\le C\|v_t\|_{L^2} + C\|w_{tt}\|_{L^2} + C\|w_0\|_{H^2} \\
&\quad + Ce^{Ct}\int_0^t \left(\|v_t\|_{L^2} + \|w_{tt}\|_{L^2} + \|w_0\|_{H^2}\right) ds.
\end{aligned} \tag{3.31}$$

By (3.19), (3.28), and (3.31) with $s = 0$, we also get

$$\begin{aligned}
&\|v_t\|_{H^2} + \|q_t\|_{H^1} \\
&\quad \le C\|v_{tt}\|_{L^2} + C\|w_t\|_{H^2} + C\|v\|_{H^2}^{1/2}\|v\|_{H^3}^{1/2}\big(\|v\|_{H^2} + \|q\|_{H^1}\big) \\
&\quad \le C\|v_{tt}\|_{L^2} + C\|w_{ttt}\|_{L^2} + C\|v\|_{H^2} \\
&\qquad + C\|v\|_{H^3}^{1/2}\bigg(\|v_t\|_{L^2} + \|w_{tt}\|_{L^2} + \|w_0\|_{H^2} \\
&\qquad\qquad + e^{Ct}\int_0^t \Big(\|v_t\|_{L^2} + \|w_{tt}\|_{L^2} + \|w_0\|_{H^2}\Big)\,ds\bigg)^{3/2}
\end{aligned} \tag{3.32}$$

for all $t \in (0, T)$, where $T \le 1/CM$.

Lemma 3.3: For $\epsilon_0 \in (0, 1/C]$, we have

$$\begin{aligned}
&\|v_{tt}(t)\|_{L^2}^2 + \|w_{ttt}(t)\|_{L^2}^2 + \|\nabla w_{tt}(t)\|_{L^2}^2 + \int_0^t \|\nabla v_{tt}(s)\|_{L^2}^2\,ds \\
&\quad \le C E(0)^3 + \epsilon_0 \int_0^t \|\nabla v_{tt}\|_{L^2}^2\,ds + C_{\epsilon_0}\int_0^t \|q_t\|_{H^1}^2\|v\|_{H^1}^{3/2}\|v\|_{H^3}^{1/2}\,ds \\
&\qquad + C_{\epsilon_0}\int_0^t \big(\|v\|_{H^3}^2 + \|q\|_{H^2}^2\big)\Big(\|v\|_{H^1}^{5/2}\|v\|_{H^3}^{3/2} + \|v_t\|_{H^1}^2\Big)\,ds + \epsilon_0\|q_t(t)\|_{H^1}^2 + \epsilon_0\|v_t(t)\|_{H^2}^2 \\
&\qquad + \epsilon_0\|v(t)\|_{H^3}^2 + C_{\epsilon_0}\left(\|v(0)\|_{H^1}^2 + t\int_0^t \|v_t(s)\|_{H^1}^2\,ds\right)^3\left(\|v(0)\|_{H^2} + t\int_0^t \|v_t(s)\|_{H^2}^2\,ds\right)^2 \\
&\qquad + C_{\epsilon_0}\left(\|v_t(0)\|_{L^2}^2 + t\int_0^t \|v_{tt}(s)\|_{L^2}^2\,ds\right)^2\left(\|v(0)\|_{H^1}^2 + t\int_0^t \|v_t(s)\|_{H^1}^2\,ds\right)^3 \\
&\qquad + C\int_0^t \|q_t\|_{H^1}\left(\|v\|_{H^2}^2 + \|v_t\|_{H^1}^{1/2}\|v_t\|_{H^2}^{1/2}\right)\|v_t\|_{H^1}\,ds \\
&\qquad + C\int_0^t \|q_t\|_{H^1}\left(\|v\|_{H^2}^3 + \|v_t\|_{H^1}\|v\|_{H^1}^{1/4}\|v\|_{H^3}^{3/4}\right)\|v\|_{H^1}^{3/4}\|v\|_{H^3}^{1/4}\,ds
\end{aligned} \tag{3.33}$$

for all $t \in [0, T]$, where $E(0) = \|v_0\|_{H^3(\Omega_f)}^2 + \|v_t(0)\|_{H^1(\Omega_f)}^2 + \|v_{tt}(0)\|_{L^2(\Omega_f)}^2 + \|w_0\|_{H^3(\Omega_e)}^2 + \|w_1\|_{H^2(\Omega_e)}^2 + 1$.

Proof of Lemma 3.3: We first differentiate the system (2.4)–(2.6) twice in time. We obtain

$$v_{ttt}^i - \partial_{tt}\partial_j(a_l^j a_l^k \partial_k v^i) + \partial_{tt}\partial_k(a_i^k q) = 0 \text{ in } \Omega_f \times (0, T), \tag{3.34}$$

$$a_j^k \partial_k v_{tt}^j + 2\partial_t a_j^k \partial_k v_t^j + \partial_{tt} a_j^k \partial_k v^j = 0 \text{ in } \Omega_f \times (0, T), \tag{3.35}$$

$$w_{tttt}^i - \Delta w_{tt}^i = 0 \text{ in } \Omega_e \times (0, T), \tag{3.36}$$

with the boundary conditions

$$v_{tt}^i = w_{ttt}^i \text{ on } \Gamma_c \times (0, T), \tag{3.37}$$

$$\partial_{tt}(a_l^j a_l^k \partial_k v^i) N_j - \partial_{tt}(a_i^k q) N_k = \partial_j w_{tt}^i N_j \text{ on } \Gamma_c \times (0, T), \tag{3.38}$$

and

$$v_{tt}^i = 0 \text{ on } \Gamma_f \times (0, T), \tag{3.39}$$

for $i = 1, 2, 3$. Multiplying (3.34) by v_{tt}^i, integrating over Ω_f, and summing for $i = 1, 2, 3$, we have

$$\frac{1}{2}\frac{d}{dt}\|v_{tt}\|_{L^2}^2 + \int_{\Omega_f} \partial_{tt}(a_l^j a_l^k \partial_k v^i)\partial_j v_{tt}^i\,dx + \int_{\Gamma_c} \partial_{tt}(a_l^j a_l^k \partial_k v^i) v_{tt}^i N_j\,d\sigma(x) \tag{3.40}$$
$$- \int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_{tt}^i\,dx - \int_{\Gamma_c} \partial_{tt}(a_i^k q) v_{tt}^i N_k\,d\sigma(x) = 0,$$

after integrating by parts. Similarly, we multiply (3.36) by w_{ttt}^i, sum for $i = 1, 2, 3$, and integrate over Ω_e to obtain

$$\frac{1}{2}\frac{d}{dt}\|w_{ttt}\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla w_{tt}\|_{L^2}^2 - \int_{\Gamma_c} \partial_k w_{tt}^i w_{ttt}^i N_k\,d\sigma(x) = 0. \tag{3.41}$$

Adding (3.40) and (3.41) and applying the boundary conditions (3.37) and (3.38) leads to

$$\frac{1}{2}\frac{d}{dt}\left(\|v_{tt}\|_{L^2}^2 + \|w_{ttt}\|_{L^2}^2 + \|\nabla w_{tt}\|_{L^2}^2\right) + \int_{\Omega_f} a_l^j a_l^k \partial_k v_{tt}^i \partial_j v_{tt}^i\,dx \tag{3.42}$$
$$+ 2\int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v_t^i \partial_j v_{tt}^i\,dx + \int_{\Omega_f} \partial_{tt}(a_l^j a_l^k)\partial_k v^i \partial_j v_{tt}^i\,dx - \int_{\Omega_f} \partial_{tt}(a_i^k q)\partial_k v_{tt}^i\,dx = 0.$$

Using the ellipticity of the form $a_l^j a_l^k \xi_j \xi_k$ and integrating in time, we get

$$\|v_{tt}(t)\|_{L^2}^2 + \|w_{ttt}(t)\|_{L^2}^2 + \|\nabla w_{tt}(t)\|_{L^2}^2 + \frac{1}{C}\int_0^t \|\nabla v_{tt}(s)\|_{L^2}^2\,ds \tag{3.43}$$
$$\le C\left|\int_0^t \int_{\Omega_f} \partial_t(a_l^j a_l^k)\partial_k v_t^i \partial_j v_{tt}^i\,dx\,ds\right| + C\left|\int_0^t \int_{\Omega_f} \partial_{tt}(a_l^j a_l^k)\partial_k v^i \partial_j v_{tt}^i\,dx\,ds\right|$$
$$+ C\left|\int_0^t \int_{\Omega_f} q\,\partial_{tt} a_i^k \partial_k v_{tt}^i\,dx\,ds\right| + C\left|\int_0^t \int_{\Omega_f} q_t \partial_t a_i^k \partial_k v_{tt}^i\,dx\,ds\right|$$
$$+ C\left|\int_0^t \int_{\Omega_f} q_{tt} a_i^k \partial_k v_{tt}^i\,dx\,ds\right| + C\|v_{tt}(0)\|_{L^2}^2 + C\|w_{ttt}(0)\|_{L^2}^2 + C\|\nabla w_{tt}(0)\|_{L^2}^2$$
$$\le A_1 + A_2 + A_3 + A_4 + A_5 + CE(0).$$

We now estimate the terms on the far right side of (3.43). Using Hölder's inequality and Lemma 3.1, we have

$$A_1 + A_2 + A_3 \le C\int_0^t (\|\nabla v\|_{L^\infty} + \|q\|_{L^\infty})(\|\nabla v\|_{L^2}\|\nabla v\|_{L^\infty} + \|\nabla v_t\|_{L^2})\|\nabla v_{tt}\|_{L^2}\,ds \tag{3.44}$$
$$\le C\int_0^t (\|v\|_{H^3} + \|q\|_{H^2})\left(\|v\|_{H^1}^{5/4}\|v\|_{H^3}^{3/4} + \|v_t\|_{H^1}\right)\|\nabla v_{tt}\|_{L^2}\,ds$$

and

$$A_4 \le C\int_0^t \|q_t\|_{L^6}\|\nabla v\|_{L^3}\|\nabla v_{tt}\|_{L^2}\,ds \le C\int_0^t \|q_t\|_{H^1}\|v\|_{H^1}^{3/4}\|v\|_{H^3}^{1/4}\|\nabla v_{tt}\|_{L^2}\,ds, \tag{3.45}$$

where we also utilized the Sobolev and the interpolation inequalities. Regarding the term A_5, using (3.35) and integrating by parts in time, we obtain

$$
\begin{aligned}
A_5 &= \left| 2\int_0^t \int_{\Omega_f} q_{tt}\partial_t a_i^k \partial_k v_t^i \,dx\,ds + \int_0^t \int_{\Omega_f} q_{tt}\partial_{tt} a_i^k \partial_k v^i \,dx\,ds \right| \\
&\le C\left| \int_{\Omega_f} q_t(t)\partial_t a_i^k(t)\partial_k v_t^i(t)\,dx \right| + C\left| \int_{\Omega_f} q_t(t)\partial_{tt} a_i^k(t)\partial_k v^i(t)\,dx \right| \\
&\quad + C\left| \int_{\Omega_f} q_t(0)\partial_t a_i^k(0)\partial_k v_t^i(0)\,dx \right| + C\left| \int_{\Omega_f} q_t(0)\partial_{tt} a_i^k(0)\partial_k v^i(0)\,dx \right| \\
&\quad + C\left| \int_0^t \int_{\Omega_f} q_t\partial_{tt} a_i^k \partial_k v_t^i \,dx\,ds \right| + C\left| \int_0^t \int_{\Omega_f} q_t\partial_t a_i^k \partial_k v_{tt}^i \,dx\,ds \right| \\
&\quad + C\left| \int_0^t \int_{\Omega_f} q_t\partial_{ttt} a_i^k \partial_k v^i \,dx\,ds \right| \\
&\le C\|q_t(t)\|_{L^6}\|\nabla v(t)\|_{L^3}\|\nabla v_t(t)\|_{L^2} + C\|q_t(t)\|_{L^6}\|a_{tt}(t)\|_{L^2}\|\nabla v(t)\|_{L^3} \\
&\quad + C\|q_t(0)\|_{L^6}\|\nabla v(0)\|_{L^3}\|\nabla v_t(0)\|_{L^2} + C\|q_t(0)\|_{L^6}\|a_{tt}(0)\|_{L^2}\|\nabla v(0)\|_{L^3} \\
&\quad + C\int_0^t \|q_t\|_{L^6}\|a_{tt}\|_{L^3}\|\nabla v_t\|_{L^2}\,ds + C\int_0^t \|q_t\|_{L^6}\|\nabla v\|_{L^3}\|\nabla v_{tt}\|_{L^2}\,ds \\
&\quad + C\int_0^t \|q_t\|_{L^6}\|a_{ttt}\|_{L^2}\|\nabla v\|_{L^3}\,ds,
\end{aligned} \tag{3.46}
$$

which by Lemma 3.1 leads to

$$
\begin{aligned}
A_5 &\le \epsilon_0\|q_t(t)\|_{H^1}^2 + C_{\epsilon_0}\left(\|\nabla v(t)\|_{L^2}^2\|\nabla v(t)\|_{L^\infty}^2 + \|\nabla v_t(t)\|_{L^2}^2\right)\|\nabla v(t)\|_{L^3}^2 \\
&\quad + C\|v(0)\|_{H^3}^6 + C\|v_t(0)\|_{H^1}^4 + C\|q_t(0)\|_{H^1}^2 \\
&\quad + C\int_0^t \|q_t\|_{H^1}\left(\|v\|_{H^2}^2 + \|\nabla v_t\|_{L^2}^{1/2}\|\nabla v_t\|_{H^1}^{1/2}\right)\|\nabla v_t\|_{L^2}\,ds \\
&\quad + C\int_0^t \|q_t\|_{H^1}\|\nabla v\|_{L^2}^{3/4}\|\nabla v\|_{H^2}^{1/4}\|\nabla v_{tt}\|_{L^2}\,ds \\
&\quad + C\int_0^t \|q_t\|_{H^1}\left(\|v\|_{H^2}^3 + \|\nabla v_t\|_{L^2}\|\nabla v\|_{L^\infty}\right)\|\nabla v\|_{L^3}\,ds.
\end{aligned} \tag{3.47}
$$

Observe that for the second term on the right side of (3.47) we have

$$
\begin{aligned}
&C_{\epsilon_0}\|\nabla v(t)\|_{L^2}^2\|\nabla v(t)\|_{L^\infty}^2\|\nabla v(t)\|_{L^3}^2 \le C_{\epsilon_0}\|\nabla v(t)\|_{L^2}^3\|\nabla v(t)\|_{H^1}^2\|\nabla v(t)\|_{H^2} \\
&\quad \le \epsilon_0\|v(t)\|_{H^3}^2 + C_{\epsilon_0}\|v(t)\|_{H^1}^6\|v(t)\|_{H^2}^4 \\
&\quad \le \epsilon_0\|v(t)\|_{H^3}^2 + C_{\epsilon_0}\left(\|v(0)\|_{H^1}^2 + t\int_0^t \|v_t(s)\|_{H^1}^2\,ds\right)^3\left(\|v(0)\|_{H^2}^2 + t\int_0^t \|v_t(s)\|_{H^2}^2\,ds\right)^2
\end{aligned} \tag{3.48}
$$

and

$$
\begin{aligned}
&C_{\epsilon_0}\|\nabla v_t(t)\|_{L^2}^2\|\nabla v(t)\|_{L^3}^2 \\
&\quad\le C_{\epsilon_0}\|v_t(t)\|_{L^2}\|v_t(t)\|_{H^2}\|\nabla v(t)\|_{L^2}^{3/2}\|\nabla v(t)\|_{H^2}^{1/2} \\
&\quad\le \epsilon_0\|v_t(t)\|_{H^2}^2+\epsilon_0\|v(t)\|_{H^3}^2+C_{\epsilon_0}\|v_t(t)\|_{L^2}^4\|v(t)\|_{H^1}^6 \\
&\quad\le \epsilon_0\|v_t(t)\|_{H^2}^2+\epsilon_0\|v(t)\|_{H^3}^2 \\
&\qquad\quad + C_{\epsilon_0}\left(\|v_t(0)\|_{L^2}^2+t\int_0^t\|v_{tt}(s)\|_{L^2}^2\,ds\right)^2\left(\|v(0)\|_{H^1}^2+t\int_0^t\|v_t(s)\|_{H^1}^2\,ds\right)^3,
\end{aligned}
\tag{3.49}
$$

where we utilized

$$
\|v(t)\|_{H^s}^2\le C\|v(0)\|_{H^s}^2+Ct\int_0^t\|v_t(s)\|_{H^s}^2\,ds \tag{3.50}
$$

for $s = 1, 2$ and

$$
\|v_t(t)\|_{L^2}^2\le C\|v_t(0)\|_{L^2}^2+Ct\int_0^t\|v_{tt}(s)\|_{L^2}^2\,ds \tag{3.51}
$$

for all $t \in (0, T]$. By (3.43)–(3.47), we then deduce that the estimate (3.33) holds, and the proof of Lemma 3.3 is complete. □

IV. PROOF OF THE MAIN RESULT

This section is devoted to the proof of Theorem 2.1.

Proof of Theorem 2.1: Denote

$$
X(t)=\|v_{tt}(t)\|_{L^2(\Omega_f)}^2+\|w_{ttt}(t)\|_{L^2(\Omega_e)}^2+\|\nabla w_{tt}(t)\|_{L^2(\Omega_e)}^2+\int_0^t\|\nabla v_{tt}(s)\|_{L^2(\Omega_f)}^2\,ds+1. \tag{4.1}
$$

We assume that C is large enough so that $\alpha \le C$ and

$$
\|v_0\|_{H^3},\ \|v_{tt}(0)\|_{L^2},\ \|w_0\|_{H^3},\ \|w_1\|_{H^2}\le C. \tag{4.2}
$$

By the Poincaré inequality, we obtain

$$
\|v_t(t)\|_{H^1}^2\le C\|\nabla v_t(t)\|_{L^2}^2\le C\|\nabla v_t(0)\|_{L^2}^2+Ct\int_0^t\|\nabla v_{tt}(s)\|_{L^2}^2\,ds \tag{4.3}
$$

which leads to

$$
\|v_t(t)\|_{H^1}^2\le C+CtX(t). \tag{4.4}
$$

We also have

$$
\|w_{tt}(t)\|_{L^2}^2\le\|w_{tt}(0)\|_{L^2}^2+Ct\int_0^t\|w_{ttt}(s)\|_{L^2}^2\,ds\le C+Ct\int_0^t X(s)\,ds. \tag{4.5}
$$

Using (3.30) and (4.5), we obtain

$$
\|v(t)\|_{H^3}^2+\|q(t)\|_{H^2}^2\le C(t+1)X(t)+Ce^{Ct}\int_0^t X(s)\,ds. \tag{4.6}
$$

In particular, we used $t \le e^{Ct}$ and

$$
\begin{aligned}
\|w_{tt}\|_{H^1}^2 &= \|w_{tt}\|_{L^2}^2+\|\nabla w_{tt}\|_{L^2}^2 \\
&\le C\|w_{tt}(0)\|_{L^2}^2+Ct\int_0^t\|w_{ttt}\|_{L^2}^2\,ds+\|\nabla w_{tt}(t)\|_{L^2}^2\le CX(t)+Ct\int_0^t X(s)\,ds.
\end{aligned}
\tag{4.7}
$$

Similarly, we have by (3.32) and by $\|v\|_{H^2}^2 + \|q\|_{H^1}^2 \le C + Ce^{Ct}\int_0^t X(s)\,ds$, which results from (3.31),

$$\begin{aligned}\|v_t(t)\|_{H^2}^2 + \|q_t(t)\|_{H^1}^2 &\le C + CX(t)\\ &\quad + Ce^{Ct}\int_0^t X(s)\,ds + Ce^{Ct}X(t)^{1/2}\int_0^t X(s)^{3/2}\,ds + Ce^{Ct}\int_0^t X(s)^2\,ds\\ &\le C + CX(t) + Ce^{Ct}\int_0^t X(s)^3\,ds. \end{aligned} \tag{4.8}$$

Now, we consider (3.33). The sum of the fifth, sixth, seventh, eight, and ninth term on the right side of (3.33) is estimated from above by

$$\begin{aligned}&C\epsilon_0\left(1 + X(t) + e^{Ct}\int_0^t X(s)^3\,ds\right)\\ &\quad + C_{\epsilon_0}\left(1 + t\int_0^t X(s)\,ds\right)^3\left(1 + e^{Ct}\int_0^t X(s)^3\,ds\right)^2 + C_{\epsilon_0}\left(1 + t\int_0^t X(s)\,ds\right)^5.\end{aligned} \tag{4.9}$$

Let $C_0 > 0$ be a large enough constant. We collect all the estimates and choose $\epsilon_0 > 0$ sufficiently small. We obtain

$$X(t) \le C_0 e^{C_0 t}\sum_{j=1}^m \int_0^t X(s)^{\alpha_j}\,ds, \tag{4.10}$$

where $\alpha_1, \ldots \alpha_m \ge 1$. We assume $X(0) \le C_0$. Let T_* be such that $X(t) < 2C_0$ for all $t \in (0, T_*)$ and $X(T_*) = 2C_0$. We now show that there is a lower bound on T_* in terms of C_0; thus, the Gronwall lemma is applicable for the inequality (4.10). Indeed, for $t \in (0, T_*]$, we have

$$X(t) \le C_0 e^{C_0 t}\sum_{j=1}^m t(2C_0)^{\alpha_j} + C_0, \tag{4.11}$$

which for $t = T_*$ implies

$$X(T_*) - C_0 \le C_0 e^{C_0 T}\sum_{j=1}^m T_*(2C_0)^{\alpha_j}. \tag{4.12}$$

Note that the right side of the inequality (4.12) tends to 0 as $T_* \to 0$ while the left side equals C_0. Therefore, we deduce that there is a lower bound for T_* and the necessary *a priori* estimate is thus established. □

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