Remarks on the inviscid limit for the Navier-Stokes equations for uniformly bounded velocity fields

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ABSTRACT. We consider the vanishing viscosity limit of the Navier-Stokes equations in a half space, with Dirichlet boundary conditions. We prove that the inviscid limit holds in the energy norm if the Navier-Stokes solutions remain bounded in $L^2_t L^\infty_x$ independently of the kinematic viscosity, and if they are equicontinuous at $x_2 = 0$.

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1. Introduction

Consider the 2D Navier-Stokes equations

$$\partial_t u^{\rm NS} + u^{\rm NS} \cdot \nabla u^{\rm NS} + \nabla p^{\rm NS} = \nu \Delta u^{\rm NS} \tag{1.1}$$

$$\nabla \cdot u^{\rm NS} = 0 \tag{1.2}$$

$$u_1^{\rm NS}|_{\partial\mathbb{H}} = u_2^{\rm NS}|_{\partial\mathbb{H}} = 0 \tag{1.3}$$

with kinematic viscosity ν, in the half space $\mathbb{H} = \{(x_1, x_2) \colon x_2 > 0\}$, and the Euler equations

$$\partial_t u^{\rm E} + u^{\rm E} \cdot \nabla u^{\rm E} + \nabla p^{\rm E} = 0 \tag{1.4}$$

$$\nabla \cdot u^{\rm E} = 0 \tag{1.5}$$

$$u_2^{\rm E}|_{\partial\mathbb{H}} = 0 \tag{1.6}$$

with asymptotically matching initial conditions

$$\lim_{\nu \to 0} \|u_0^{\rm NS} - u_0^{\rm E}\|_{L^2(\mathbb{H})} = 0. \tag{1.7}$$

We denote by

$$u_1^{\rm E}|_{\partial\mathbb{H}}(x_1, t) = U^{\rm E}(x_1, t)$$

the trace on $\partial\mathbb{H}$ of the Euler tangential flow. We omit ν in the notation for $u^{\rm NS}$. Throughout this paper we consider $0 < \nu \leq \nu_0$, and $0 \leq t \leq T$, where ν_0 is an arbitrary fixed kinematic viscosity, and T is an arbitrary fixed time. We assume that the Euler initial datum is smooth, $u_0^{\rm E} \in H^s(\mathbb{H})$ for some $s > 2$, so that there exists an unique H^s smooth solution $u^{\rm E}$ of (1.4)–(1.6) on $[0, T]$.

This paper establishes sufficient conditions for the family of Navier-Stokes solutions $\{u^{\rm NS}\}_{\nu \in (0,\nu_0]}$ to ensure that the inviscid limit holds in the energy norm:

$$\lim_{\nu \to 0} \|u^{\rm NS} - u^{\rm E}\|_{L^\infty(0,T;L^2(\mathbb{H}))} = 0. \tag{1.8}$$

Our main results are given in Theorems 1.1, 1.3, and 1.4. The main assumptions are the uniform boundedness of the Navier-Stokes solutions in $L^2(0, T; L^\infty(\mathbb{H}))$ and their equicontinuity at $x_2 = 0$.

The conditions imposed imply that the Lagrangian paths originating in a boundary layer, stay in a proportional boundary layer during the time interval considered. The physical interpretation of our result is that, as long as there is no separation of the boundary layer, the inviscid limit is possible.

1.1. Known finite time, inviscid limit results. The question of whether (1.8) holds in the case of Dirichlet boundary conditions has a rich history. Kato proved in [**Kat84**] that the inviscid limit holds in the energy norm if and only if

$$\lim_{\nu\to 0} \nu \int_0^T \int_{|x_2|\le C\nu} |\nabla u^{\mathrm{NS}}(x_1,x_2,t)|^2 dx_1 dx_2 dt = 0, \tag{1.9}$$

i.e. that the energy dissipation rate is vanishing in a thin, $\mathcal{O}(\nu)$, layer near the boundary. Kato's criterion was revisited and sharpened by many authors. For instance, in [**TW97**] and [**Wan01**] it is shown that the condition on the full gradient matrix ∇u^{NS} may be replaced by a condition on the tangential gradient of the Navier-Stokes solution alone, at the cost of considering a thicker boundary layer, of size $\delta(\nu)$, where $\lim_{\nu\to 0}\delta(\nu)/\nu = 0$. In [**Kel07**] it is shown that $\|\nabla u^{\mathrm{NS}}\|_{L^2(|x_2|\le C\nu)}$ may be replaced by $\nu^{-1}\|u^{\mathrm{NS}}\|_{L^2(|x_2|\le C\nu)}$ which has the same scaling in the Kato layer. In [**Kel08**] it is shown that (1.8) is equivalent to the weak convergence of vorticities

$$\omega^{\mathrm{NS}} \to \omega^{\mathrm{E}} - u_1^{\mathrm{E}}\,\mu_{\partial\mathbb{H}} \quad \text{in} \quad (H^1(\mathbb{H}))^* \tag{1.10}$$

where $\mu_{\partial\mathbb{H}}$ is the Dirac measure on $\partial\mathbb{H}$, and $(H^1)^*$ is the dual space to H^1 (not H^1_0). In fact, it is shown in [**BT13**] that the weak convergence of vorticity on the boundary

$$\nu\omega^{\mathrm{NS}} \to 0 \quad \text{in} \quad \mathcal{D}'([0,T]\times\partial\mathbb{H}) \tag{1.11}$$

is equivalent to (1.8) (see also [**Kel08, CKV15**] in the case of stronger convergence in (1.11)).

The idea to introduce a boundary layer corrector like Kato's, which is not based on power series expansions, and to treat the remainders with energy estimates has proven to be very fruitful. See for instance: [**Mas98**] in the case of anisotropic viscosity; [**BSJW14, BTW12**] in the context of weak-strong uniqueness; [**GN14**] for a steady flow on a moving plate; [**BN14**] for the compressible Navier-Stokes equations; [**LFNLTZ14**] for the vanishing α limit of the 2D Euler-α model.

There are three classes of functions for which there exist unconditional inviscid limit results, that is, theorems whereby conditions imposed solely on initial data guarantee that (1.8) is true for a time interval independent of viscosity (but possibly depending on initial data). The first class is that of real analytic initial data in all space variables [**SC98b**], the second is that of initial data with vorticity supported at an $\mathcal{O}(1)$ distance from the boundary [**Mae14**], and the third class is data with certain symmetries or special restrictions [**LFMNL08, LFMNLT08, MT08, Kel09**]). It is worth noting that in these three cases the Prandtl expansion of the Navier-Stokes equation is valid in a boundary layer of thickness $\sqrt{\nu}$. Moreover, in all these results, the Kato criteria also hold [**BT13, Kel14**]. However, to date, there is no robust connection between the well-posedness of the Prandtl equations, and the vanishing viscosity limit in the energy norm.

It is known that for a class of initial conditions close to certain shear flows the Prandtl equations are ill-posed [**GVD10, GN11, GVN12**] and even that the Prandtl expansion is not valid [**Gre00, GGN14b, GGN14c, GGN14a**]. These results do not imply that the inviscid limit in the energy norm is invalid, but rather just that the Prandtl expansion does not describe the leading order behavior near the boundary. It would be natural to expect that working in a function space for which the local existence of the Prandtl equations holds (see, e.g. [**Ole66, MW14, AWXY14**], [**SC98a, LCS03, KV13**], [**KMVW14**], [**GVM13**]), there is a greater chance for (1.8) to be true. An instance of such a result is given in [**CKV15**], where a one-sided Kato criterion in terms of the vorticity is obtained, connecting Oleinik's monotonicity assumption and the inviscid limit: if

$$\lim_{\nu\to 0}\int_0^T \left\| \left(U^{\mathrm{E}}(x_1,t)\left(\omega^{\mathrm{NS}}(x_1,x_2,t)+\frac{\delta(\nu t)}{\nu t}\right)\right)_- \right\|^2_{L^2(|x_2|\le \nu t/\delta(\nu t))} dt = 0 \tag{1.12}$$

holds, where $\int_0^T \delta(\nu t)dt \to 0$ as $\nu \to 0$, then (1.8) holds. In particular, if there is no back-flow in the underlying Euler flow, $U^{\mathrm{E}} \geq 0$, and the Navier-Stokes vorticity ω^{NS} is larger than $-\delta(\nu t)/\nu$ (for instance if it is non-negative as in Oleinik's setting) in a boundary layer that is slightly thicker than Kato's, then the inviscid limit holds.

In contrast to the works (1.9)–(1.12) mentioned above, the goal of this paper is to establish sufficient conditions for (1.8) to hold, which do not rely on any assumptions concerning derivatives of the Navier-Stokes equations. Alternately, we establish conditions which require only L^1 uniform integrability of tangential derivatives near the boundary. Our proofs keep the idea of Kato of building an ad-hoc boundary layer corrector, but its scaling is dictated by the heat equation in x_2 (with Prandtl scaling). No explicit convergence rates are obtained with our assumptions. The main results of this paper are:

1.2. Results.

THEOREM 1.1. *Assume that there exists a constant $C_{NS} > 0$ such that*

$$\sup_{\nu\in(0,\nu_0]}\int_0^T \|u^{NS}(t)\|^2_{L^\infty(\mathbb{H})}dt \le C_{NS}\nu_0 \tag{1.13}$$

and moreover that the family

$$\{u_1^{NS}u_2^{NS}\}_{\nu\in(0,\nu_0]} \quad \textit{is equicontinuous at} \quad x_2=0. \tag{1.14}$$

Then (1.7) implies that the inviscid limit holds in the energy norm.

Specifically, in view of the Dirichlet boundary condition (1.3), by condition (1.14) we mean that there exists a function

$$0 \le \gamma(x_1,t) \in L^1_{t,x_1}([0,T]\times\mathbb{R}) \tag{1.15}$$

with the property that for any $\varepsilon > 0$, there exists $\rho = \rho(\varepsilon) > 0$ such that

$$|u_1^{\mathrm{NS}}(x_1,x_2,t)u_2^{\mathrm{NS}}(x_1,x_2,t)| \le \varepsilon\gamma(x_1,t), \quad \text{for all} \quad x_2\in(0,\rho], \tag{1.16}$$

and all $(t,x_1)\in[0,T]\times\mathbb{R}$, uniformly in $\nu\in(0,\nu_0]$.

The quantity in condition (1.13) is natural to consider: it is scale invariant under the Navier-Stokes isotropic scaling, and it appears in three dimensions as well. The same quantity was used in [**BSJW14**] to establish conditional weak-strong uniqueness of weak solutions in Hölder classes.

REMARK 1.2 (**Open problem**). Removing the equicontinuity assumption (1.14) of $u_1^{\mathrm{NS}}u_2^{\mathrm{NS}}$ at the boundary of the domain is a natural and very interesting question.

The L^1 integrability (uniform in ν) of one component of ∇u^{NS} is related to condition (1.14):

THEOREM 1.3. *Assume that (1.13) holds, and that the tangential component of the Navier-Stokes flow satisfies:*

$$\{\partial_1 u_1^{NS}\}_{\nu\in(0,\nu_0]} \quad \textit{is uniformly integrable near} \quad x_2=0, \tag{1.17}$$

meaning that for any $\varepsilon > 0$ *and any* $L > 0$*, there exists* $\rho = \rho(\varepsilon, L) > 0$ *such that*

$$\|\partial_1 u_1^{NS} \mathbf{1}_{|x_1| \le L, 0 < x_2 < \rho}\|_{L^2(0,T;L^1(\mathbb{H}))} \le \varepsilon. \tag{1.18}$$

Then (1.8) *holds.*

Condition (1.17) requires that the family of measures

$$\mu_\nu(dx_1\, dx_2) = |\partial_1 u_1^{\mathrm{NS}}(t, x_1, x_2)| dx_1\, dx_2$$

is uniformly absolutely continuous at $x_2 = 0$ with values in $L^2(0,T)$. Note that $\partial_1 u_1^{\mathrm{NS}}$ vanishes identically on $\partial\mathbb{H}$, which is not the case for the Navier-Stokes vorticity $\omega^{\mathrm{NS}} = \partial_2 u_1^{\mathrm{NS}} - \partial_1 u_2^{\mathrm{NS}}$, which is expected to develop a measure supported on the boundary of the domain in the inviscid limit [**Kel08**]. Thus, the vorticity is not expected to be uniformly integrable in $L_t^2 L_x^1$. Therefore, in (1.18) it is important that instead of a uniform integrability condition on ω^{NS} or equivalently $\partial_2 u_1^{\mathrm{NS}}$, we have only assumed a uniform integrability condition on $\partial_1 u_1^{\mathrm{NS}}$. Also, note that (uniform in ν) higher integrability of the Navier-Stokes vorticity, such as L^p for $p > 2$ cannot hold unless $U^{\mathrm{E}} \equiv 0$, as is shown in [**Kel14**].

A similar result to the one in Theorem 1.3, has been obtained independently in [**GKLF$^+$15**], where the authors prove that if ∇u^{NS} is uniformly in ν bounded in $L^\infty(0,T;L^1(\Omega))$, for a domain Ω such that the embedding $W^{1,1}(\Omega) \subset L^2$ is compact, then the vanishing viscosity limit holds in $L^\infty(0,T;L^2(\Omega))$.

We conclude the introduction by noting that a similar proof to that of Theorem 1.1 yields the following:

THEOREM 1.4. *Assume that there exists a function* $M(t) \ge 0$ *such that*

$$\sup_{\nu \in (0,1]} \|u^{NS}(t)\|_{L^\infty(\mathbb{H})} \le M(t) \qquad \textit{with} \qquad \int_0^T M^2(t) dt < \infty \tag{1.19}$$

and that

$$\lim_{\nu \to 0} u_1^{NS}(x_1, \delta(\nu t)\, x_2, t) u_2^{NS}(x_1, \delta(\nu t)\, x_2, t) = 0 \tag{1.20}$$

holds for a.e. $(t, x_1, x_2) \in [0,T] \times \mathbb{H}$*, where* δ *is an increasing non-negative function such that*

$$\lim_{\nu \to 0} \nu \int_0^T \frac{1}{\delta(\nu t)} = 0. \tag{1.21}$$

Then (1.8) *holds.*

Condition (1.21) for the boundary layer thickness $\delta(\nu t)$, emerges for reasons which are similar to those in [**Wan01, Kel14, CKV15**].

1.3. Organization of the paper. In Section 2 we lay out the scheme of the proof for the above mentioned theorems, by identifying the principal error terms in the energy estimate for the corrected $u^{\mathrm{NS}} - u^{\mathrm{E}}$ flow. In Section 3 we build a caloric lift of the Euler boundary conditions, augmented by an $\mathcal{O}(1)$ correction at unit scale. In Section 4 we conclude the proof of Theorem 1.1, in Section 5 we give the proof of Theorem 1.3, while in Section 6 we show why Theorem 1.4 holds.

2. Setup of the Proof of Theorem 1.1

We consider a boundary layer corrector u^{K} (to be constructed precisely later) which for now obeys three properties

$$\nabla \cdot u^{\mathrm{K}} = 0 \tag{2.1}$$
$$u_1^{\mathrm{K}}|_{\partial\mathbb{H}} = -U^{\mathrm{E}} \tag{2.2}$$
$$u_2^{\mathrm{K}}|_{\partial\mathbb{H}} = 0. \tag{2.3}$$

The main difference between the corrector u^{K} we consider, and the one considered in [Kat84], is its characteristic length scale: we let u^{K} obey a Prandtl $\sqrt{\nu t}$ scaling. Roughly speaking, u_1^{K} is a lift of the Euler boundary condition which obeys the heat equation $(\partial_t - \nu\partial_{x_2x_2})u_1^{\mathrm{K}} = 0$ to leading order in ν. In view of (2.1)–(2.3) we then obtain u_2^{K} from u_1^{K} as

$$u_2^{\mathrm{K}}(x_1, x_2, t) = -\int_0^{x_2} \partial_1 u_1^{\mathrm{K}}(x_1, y, t)dy. \tag{2.4}$$

The function

$$v = u^{\mathrm{NS}} - u^{\mathrm{E}} - u^{\mathrm{K}}$$

is divergence free

$$\nabla \cdot v = 0$$

and obeys Dirichlet boundary conditions

$$v|_{\partial\mathbb{H}} = 0.$$

The equation obeyed by v is

$$\begin{aligned} &\partial_t v - \nu\Delta v + v \cdot \nabla u^{\mathrm{E}} + u^{\mathrm{NS}} \cdot \nabla v + \nabla q \\ &\quad = \nu\Delta u^{\mathrm{E}} - \left(\partial_t u^{\mathrm{K}} - \nu\Delta u^{\mathrm{K}} + u^{\mathrm{NS}} \cdot \nabla u^{\mathrm{K}} + u^{\mathrm{K}} \cdot \nabla u^{\mathrm{E}}\right) \end{aligned} \tag{2.5}$$

where $q = p^{\mathrm{NS}} - p^{\mathrm{E}}$. Multiplying (2.5) with v and integrating by parts, yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla v\|_{L^2}^2 \leq \|\nabla u^{\mathrm{E}}\|_{L^\infty}\|v\|_{L^2}^2 + \nu\|\Delta u^{\mathrm{E}}\|_{L^2}\|v\|_{L^2} \\ + T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \end{aligned} \tag{2.6}$$

where we have denoted

$$T_1 = -\int_{\mathbb{H}} (\partial_t u^{\mathrm{K}} - \nu\Delta u^{\mathrm{K}}) \cdot v \tag{2.7}$$
$$T_2 = -\int_{\mathbb{H}} (u^{\mathrm{NS}} \cdot \nabla u^{\mathrm{E}}) \cdot u^{\mathrm{K}} \tag{2.8}$$
$$T_3 = -\int_{\mathbb{H}} (u^{\mathrm{K}} \cdot \nabla u^{\mathrm{E}}) \cdot v \tag{2.9}$$
$$T_4 = -\int_{\mathbb{H}} u_1^{\mathrm{NS}} u_2^{\mathrm{NS}} \partial_1 u_2^{\mathrm{K}} \tag{2.10}$$
$$T_5 = -\int_{\mathbb{H}} \left((u_1^{\mathrm{NS}})^2 - (u_2^{\mathrm{NS}})^2\right) \partial_1 u_1^{\mathrm{K}} \tag{2.11}$$
$$T_6 = -\int_{\mathbb{H}} u_1^{\mathrm{NS}} u_2^{\mathrm{NS}} \partial_2 u_1^{\mathrm{K}} \tag{2.12}$$

The corrector u^{K} is designed to eliminate the contribution from T_1 to leading order in ν. In turn, this leads to $\|u^{\mathrm{K}}\|_{L^2} + \|\partial_1 u^{\mathrm{K}}\|_{L^2} \to 0$ as $\nu \to 0$, so that the terms T_2, T_3, and T_4 are harmless. Such is the case if u^{K} is localized in a layer near the boundary, which is vanishing as $\nu \to 0$. The assumptions (1.13)–(1.14) only come into play in showing that T_5 and T_6 are bounded conveniently. The next section is devoted to the construction of an u^{K} with these properties, and the conclusion of the proof is given in Section 4 below.

Throughout the text we shall denote by C_{E} any constant that depends on $\|u^{\mathrm{E}}\|_{L^\infty(0,T;H^s(\mathbb{H}))}$. Various other positive constants shall be denoted by C; these constants do not depend on ν, but they are allowed to implicitly depend on the fixed length of the time interval T, and on the largest kinematic viscosity ν_0.

3. A pseudo-caloric lift of the boundary conditions

3.1. The tangential component of the lift u^{K}. Let

$$z = z(x_2, t) = \frac{x_2}{\sqrt{4\nu t}}$$

be the self-similar variable for the heat equation in x_2, with viscosity ν. Let η be a non-negative bump function such that

$$\operatorname{supp}(\eta) \in [1,2] \quad \text{and} \quad \int_1^2 \eta(r)dr = \frac{1}{\sqrt{\pi}} \tag{3.1}$$

which in addition obeys that $|\eta'|_{L^\infty} + |\eta''|_{L^\infty} \le C_\eta$, for some constant C_η.

We let u_1^{K} consist of a caloric lift of the Euler boundary conditions, augmented with a localization factor at large values of x_2. We define

$$u_1^{\mathrm{K}}(x_1, x_2, t) = -U^{\mathrm{E}}(x_1, t)\left(\operatorname{erfc}(z(x_2,t)) - \sqrt{4\nu t}\,\eta(x_2)\right) \tag{3.2}$$

where

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-y^2)dy.$$

The normalization of the mass of η was chosen precisely so that

$$\begin{aligned}\int_0^\infty u_1^{\mathrm{K}}(x_1,x_2,t)dx_2 &= -U^{\mathrm{E}}(x_1,t)\int_0^\infty \left(\operatorname{erfc}(z(x_2,t)) - \sqrt{4\nu t}\,\eta(x_2)\right)dx_2 \\ &= -U^{\mathrm{E}}(x_1,t)\sqrt{4\nu t}\left(\int_0^\infty \operatorname{erfc}(z)dz - \int_0^\infty \eta(x_2)dx_2\right) \\ &= 0. \end{aligned} \tag{3.3}$$

Property (3.3) of u_1^{K} allows the u_2^{K} defined in (2.4) (see also below) to decay sufficiently fast as $x_2 \to \infty$. This decay of u_2^{K} will be used essentially later on in the proof.

Note that u_1^{K} is pseudo-localized to scale $x_2 \approx \sqrt{4\nu t}$. Indeed, we have that

$$\begin{aligned}\|\operatorname{erfc}(z(x_2,t))\|_{L^p_{x_2}(0,\infty)} &= (4\nu t)^{1/(2p)}\|1-\operatorname{erf}(z)\|_{L^p_z(0,\infty)} \\ &\le C(\nu t)^{1/(2p)}\end{aligned}$$

and

$$\begin{aligned}\|\partial_{x_2}\operatorname{erfc}(z(x_2,t))\|_{L^p_{x_2}(0,\infty)} &= (4\nu t)^{1/(2p)-1/2}\|\partial_z \operatorname{erfc}(z)\|_{L^p_z(0,\infty)} \\ &\le C(\nu t)^{1/(2p)-1/2}\end{aligned}$$

for all $1 \le p \le \infty$, where $C > 0$ is a constant. The above bounds yield

$$\|u_1^{\mathrm{K}}\|_{L^p_{x_1,x_2}(\mathbb{H})} \le C\|U^{\mathrm{E}}(t)\|_{L^p_{x_1}} \left((4\nu t)^{1/(2p)} + C_\eta (4\nu t)^{1/2}\right) \le C_{\mathrm{E}}(\nu t)^{1/(2p)} \tag{3.4}$$

$$\|\partial_1 u_1^{\mathrm{K}}\|_{L^p_{x_1,x_2}(\mathbb{H})} \le C_\eta\|\partial_1 U^{\mathrm{E}}(t)\|_{L^p_{x_1}} (\nu t)^{1/(2p)} \le C_{\mathrm{E}}(\nu t)^{1/(2p)} \tag{3.5}$$

$$\|\partial_2 u_1^{\mathrm{K}}\|_{L^p_{x_1,x_2}(\mathbb{H})} \le C_\eta\|U^{\mathrm{E}}(t)\|_{L^p_{x_1}} (\nu t)^{1/(2p)-1/2} \le C_{\mathrm{E}}(\nu t)^{1/(2p)-1/2} \tag{3.6}$$

$$\|\partial_{12} u_1^{\mathrm{K}}\|_{L^p_{x_1,x_2}(\mathbb{H})} \le C_\eta\|\partial_1 U^{\mathrm{E}}(t)\|_{L^p_{x_1}} (\nu t)^{1/(2p)-1/2} \le C_{\mathrm{E}}(\nu t)^{1/(2p)-1/2} \tag{3.7}$$

for all $1 \le p \le \infty$, where $C_{\mathrm{E}} > 0$ is a constant that depends on the Euler flow, on p, the cutoff function η, through the constant C_η, on ν_0 and T. We emphasize however only the dependence on the Euler flow.

We moreover have that

$$\begin{aligned}\partial_t u_1^{\mathrm{K}} - \nu\Delta u_1^{\mathrm{K}} = &-\left(\partial_t U^{\mathrm{E}}(x_1,t) - \nu\partial_{11}U^{\mathrm{E}}(x_1,t)\right)\left(\operatorname{erfc}(z(x_2,t)) - \sqrt{4\nu t}\eta(x_2)\right)\\ &+ U^{\mathrm{E}}(x_1,t)(\partial_t - \nu\partial_{22})\left(\sqrt{4\nu t}\,\eta(x_2)\right)\end{aligned}$$

and thus

$$\begin{aligned}\|\partial_t u_1^{\mathrm{K}} - \nu\Delta u_1^{\mathrm{K}}\|_{L^2} &\le C_\eta\left(\|\partial_t U^{\mathrm{E}}\|_{L^2} + \nu\|\partial_{11}U^{\mathrm{E}}\|_{L^2}\right)(\nu t)^{1/4} + C_\eta\|U^{\mathrm{E}}\|_{L^2}\nu^{1/2}t^{-1/2}\\ &\le C_{\mathrm{E}}\left((\nu t)^{1/4} + \nu^{1/2}t^{-1/2}\right)\end{aligned} \tag{3.8}$$

where as before the dependence of all constants on ν_0 and T is ignored.

3.2. The normal component of the lift u^{K}. Combining (2.4) with (3.2), we arrive at

$$\begin{aligned}u_2^{\mathrm{K}}(x_1,x_2,t) &= \partial_1 U^{\mathrm{E}}(x_1,t)\left(\int_0^{x_2}\operatorname{erfc}(z(y,t))dy - \sqrt{4\nu t}\int_0^{x_2}\eta(y)dy\right)\\ &= \sqrt{4\nu t}\,\partial_1 U^{\mathrm{E}}(x_1,t)\left(\int_0^{z(x_2,t)}\operatorname{erfc}(z)dz - \int_0^{x_2}\eta(y)dy\right)\\ &=: \sqrt{4\nu t}\,\partial_1 U^{\mathrm{E}}(x_1,t)R(x_2,t).\end{aligned} \tag{3.9}$$

An explicit calculation shows that

$$R(x_2,t) = \left(\frac{1}{\sqrt{\pi}} - \int_1^{x_2}\eta(y)dy\right) - \frac{1}{\sqrt{\pi}}\exp\left(-z(x_2,t)^2\right) + z(x_2,t)\operatorname{erfc}(z(x_2,t)).$$

Moreover, note that in view of the choice of η in (3.1), the first term on the right side of the above is identically vanishing for all $x_2 \ge 2$. It is clear that R obeys

$$R(0,t) = 0 = \lim_{x_2\to\infty} R(x_2,t),$$

and thus we may hope that R is integrable with respect to x_2, which is indeed the case. To see this, first we note that

$$\|R(t)\|_{L^\infty_{x_2}} \le \frac{1}{\sqrt{\pi}}.$$

Then, we have that

$$
\begin{aligned}
\|R(t)\|_{L^1_{x_2}} &\leq \int_0^2 \left| \frac{1}{\sqrt{\pi}} - \int_1^{x_2} \eta(y)dy \right| dx_2 + \frac{1}{\sqrt{\pi}} \int_0^\infty \exp(-z(x_2,t)^2) + z(x_2,t)\operatorname{erfc}(z(x_2,t))dx_2 \\
&\leq C_\eta + \frac{\sqrt{4\nu t}}{\sqrt{\pi}} \int_0^\infty \exp(-z^2) + z(1-\operatorname{erf}(z))dz \\
&\leq C_\eta
\end{aligned}
$$

where the dependence of all constants on ν_0 and T is ignored. By interpolation it then follows that

$$
\|R(t)\|_{L^p_{x_2}} \leq C_\eta \tag{3.10}
$$

for all $1 \leq p \leq \infty$. In view of (3.10) and (3.2), we have that the bounds

$$
\|u_2^{\mathrm{K}}\|_{L^p_{x_1,x_2}(\mathbb{H})} \leq C_\eta \sqrt{4\nu t} \|\partial_1 U^{\mathrm{E}}\|_{L^p_{x_1}} \leq C_{\mathrm{E}} (\nu t)^{1/2} \tag{3.11}
$$

$$
\|\partial_1 u_2^{\mathrm{K}}\|_{L^p_{x_1,x_2}(\mathbb{H})} \leq C_\eta \sqrt{4\nu t} \|\partial_{11} U^{\mathrm{E}}\|_{L^p_{x_1}} \leq C_{\mathrm{E}} (\nu t)^{1/2} \tag{3.12}
$$

hold for $1 \leq p \leq \infty$, where we have as before suppressed the dependence on C_η and p of the constant C_E.

Lastly, we obtain from (2.4) and (3.9) that

$$
\begin{aligned}
(\partial_t - \nu\Delta)u_2^{\mathrm{K}}(x_1,x_2,t) &= \nu\partial_{12}u_1^{\mathrm{K}}(x_1,x_2,t) - \nu\sqrt{4\nu t}\,\partial_{111}U^{\mathrm{E}}(x_1,t)R(x_2,t) \\
&\quad + \nu^{1/2}t^{-1/2}\partial_1 U^{\mathrm{E}}(x_1,t)R(x_2,t) \\
&\quad + \sqrt{4\nu t}\,\partial_1 U^{\mathrm{E}}(x_1,t)\partial_t R(x_2,t) \\
&= \nu\partial_{12}u_1^{\mathrm{K}}(x_1,x_2,t) - \nu\sqrt{4\nu t}\,\partial_{111}U^{\mathrm{E}}(x_1,t)R(x_2,t) \\
&\quad + \nu^{1/2}t^{-1/2}\partial_1 U^{\mathrm{E}}(x_1,t)R(x_2,t) \\
&\quad - \nu^{1/2}t^{-1/2}\,\partial_1 U^{\mathrm{E}}(x_1,t)z(x_2,t)\operatorname{erfc}(z(x_2,t))
\end{aligned}
$$

where we have used that

$$
\partial_t R(x_2,t) = -\frac{1}{2t} z(x_2,t)\operatorname{erfc}(z(x_2,t)).
$$

Using (3.7) and (3.10) we conclude that

$$
\begin{aligned}
\|(\partial_t - \nu\Delta)u_2^{\mathrm{K}}\|_{L^2_{x_1,x_2}(\mathbb{H})} &\leq C_\eta \nu^{1/2} t^{-1/2} (\nu t)^{1/4} \|\partial_1 U^{\mathrm{E}}(t)\|_{L^2_{x_1}} \\
&\quad + C_\eta \nu (\nu t)^{1/2} \|\partial_{111} U^{\mathrm{E}}\|_{L^2_{x_1}} + C_\eta \nu^{1/2} t^{-1/2} \|\partial_1 U^{\mathrm{E}}\|_{L^2_{x_1}} \\
&\leq C_{\mathrm{E}} \left(\nu^{1/2} t^{-1/2} + (\nu t)^{1/2} \right)
\end{aligned} \tag{3.13}
$$

holds.

4. Conclusion of the Proof of Theorem 1.1

Having constructed the corrector function u^{K}, we estimate the terms on the right side of (2.6).

4.1. Bounds for T_1, T_2, T_3, and T_4. Using (3.8) and (3.13) we arrive at

$$\begin{aligned} |T_1| &\leq \|v\|_{L^2} \|(\partial_t - \nu\Delta) u^{\mathrm{K}}\|_{L^2} \\ &\leq C_{\mathrm{E}} \|v\|_{L^2} \left(\nu^{1/2} t^{-1/2} + (\nu t)^{1/4} \right). \end{aligned} \tag{4.1}$$

In order to bound T_2 we first estimate

$$\begin{aligned} |T_2| &\leq \|\nabla u^{\mathrm{E}}\|_{L^\infty} \|u^{\mathrm{K}}\|_{L^2} \|u^{\mathrm{NS}}\|_{L^2} \\ &\leq \|\nabla u^{\mathrm{E}}\|_{L^\infty} \|u^{\mathrm{K}}\|_{L^2} \|u_0^{\mathrm{NS}}\|_{L^2} \end{aligned}$$

where we have used the L^2 energy inequality for the Navier-Stokes solution. Combining the above with (3.4) and (3.11) we arrive at

$$|T_2| \leq C_{\mathrm{E}} (\nu t)^{1/4} \tag{4.2}$$

since $\|u_0^{\mathrm{NS}}\|_{L^2} \leq C(\|u_0^{\mathrm{E}}\|_{L^2} + 1)$, for all $\nu \leq \nu_0$, as we assume $\|u_0^{\mathrm{NS}} - u_0^{\mathrm{E}}\|_{L^2} \to 0$ as $\nu \to 0$. Similarly to T_2, we may estimate

$$\begin{aligned} |T_3| &\leq \|\nabla u^{\mathrm{E}}\|_{L^\infty} \|u^{\mathrm{K}}\|_{L^2} \|v\|_{L^2} \\ &\leq C_{\mathrm{E}} (\nu t)^{1/2} \|v\|_{L^2}. \end{aligned} \tag{4.3}$$

Then, similarly to T_2 we estimate T_4. We appeal to the energy inequality for the Navier-Stokes solution and estimate (3.12), which is valid also for $p = \infty$, to conclude that

$$\begin{aligned} |T_4| &\leq \|u^{\mathrm{NS}}\|_{L^2}^2 \|\partial_1 u_2^{\mathrm{K}}\|_{L^\infty} \\ &\leq \|u_0^{\mathrm{NS}}\|_{L^2}^2 \|\partial_1 u_2^{\mathrm{K}}\|_{L^\infty} \\ &\leq C_{\mathrm{E}} (\nu t)^{1/2}. \end{aligned} \tag{4.4}$$

4.2. Bound for T_5. We estimate T_5 as

$$\begin{aligned} |T_5| &\leq \int_{\mathbb{H}} \left((u_1^{\mathrm{NS}})^2 + (u_2^{\mathrm{NS}})^2 \right) |\partial_1 u_1^{\mathrm{K}}| dx_1 dx_2 \\ &\leq \int_{\mathbb{H}} \left((u_1^{\mathrm{NS}})^2 + (u_2^{\mathrm{NS}})^2 \right) |\partial_1 U^{\mathrm{E}}| \left| \operatorname{erfc}(z(x_2,t)) - \sqrt{4\nu t}\, \eta(x_2) \right| dx_1 dx_2 \\ &\leq \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2 \|\partial_1 U^{\mathrm{E}}\|_{L^1_{x_1}} \| \operatorname{erfc}(z(x_2,t)) - \sqrt{4\nu t}\, \eta(x_2) \|_{L^1_{x_2}} \\ &\leq C_{\mathrm{E}} (\nu t)^{1/2} \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2. \end{aligned} \tag{4.5}$$

Using assumption (1.13), it then immediately follows that

$$\int_0^T |T_5(t)| dt \leq C_{\mathrm{E}} \nu_0 C_{\mathrm{NS}} (\nu T)^{1/2} \tag{4.6}$$

for all $\nu \in (0, \nu_0]$.

REMARK 4.1. In order to show that $\lim_{\nu \to 0} \int_0^T |T_5(t)| dt = 0$, instead of using (1.13), it would have been sufficient to assume that

$$\sup_{\nu \in (0,\nu_0]} \int_0^T \|u^{\mathrm{NS}}(t)\|_{L^2_{x_1} L^q_{x_2}(\mathbb{H})} < \infty$$

for some $q > 2$. This follows along the lines of (4.5), by using the energy inequality $\|u^{\mathrm{NS}}(t)\|_{L^2} \leq \|u_0^{\mathrm{NS}}\|_{L^2}$, and estimate (3.5) with $p = 2q/(q-2)$.

4.3. Bound for T_6. First we note that by the definition of u_1^{K} in (3.2) we have

$$\begin{aligned}
|T_6| &\leq (4\nu t)^{1/2} \left| \int_{\mathbb{H}} u_1^{\mathrm{NS}}(x_1,x_2,t) u_2^{\mathrm{NS}}(x_1,x_2,t) U^{\mathrm{E}}(x_1,t) \eta'(x_2) dx_1 dx_2 \right| \\
&\qquad + \left| \int_{\mathbb{H}} u_1^{\mathrm{NS}}(x_1,x_2,t) u_2^{\mathrm{NS}}(x_1,x_2,t) U^{\mathrm{E}}(x_1,t) \partial_{x_2} \operatorname{erfc}(z(x_2,t)) dx_1 dx_2 \right| \\
&\leq C_{\mathrm{E}} (\nu t)^{1/2} \|u^{\mathrm{NS}}\|_{L^2}^2 + |T_{6,\nu}| \\
&\leq C_{\mathrm{E}} (\nu t)^{1/2} + |T_{6,\nu}| \qquad (4.7)
\end{aligned}$$

where we have used the energy inequality $\|u^{\mathrm{NS}}\|_{L^2} \leq \|u_0^{\mathrm{NS}}\|_{L^2} \leq C(1+\|u_0^{\mathrm{E}}\|_{L^2})$, and have denoted

$$\begin{aligned}
T_{6,\nu} &= \int_{\mathbb{H}} u_1^{\mathrm{NS}}(x_1,x_2,t) u_2^{\mathrm{NS}}(x_1,x_2,t) U^{\mathrm{E}}(x_1,t) \partial_{x_2} \operatorname{erfc}(z(x_2,t)) dx_1 dx_2 \\
&= -\frac{1}{\sqrt{\pi \nu t}} \int_{\mathbb{H}} u_1^{\mathrm{NS}}(x_1,x_2,t) u_2^{\mathrm{NS}}(x_1,x_2,t) U^{\mathrm{E}}(x_1,t) \exp(-z(x_2,t)^2) dx_1 dx_2 \\
&= -\frac{2}{\sqrt{\pi}} \int_{\mathbb{H}} u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t} y,t) u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t} y,t) U^{\mathrm{E}}(x_1,t) \exp(-y^2) dx_1 dy. \qquad (4.8)
\end{aligned}$$

The goal is now to show that assumptions (1.13)–(1.14) imply

$$\lim_{\nu \to 0} \int_0^T |T_{6,\nu}(t)| dt = 0 \qquad (4.9)$$

which yields the desired T_6 estimate.

In order to prove (4.9), we fix an $\varepsilon > 0$, arbitrary, which in turn fixes a $\rho = \rho(\varepsilon) > 0$ such that (1.16) holds. We then have

$$\begin{aligned}
&\int_0^T |T_{6,\nu}(t)| dt \\
&\leq \frac{2}{\sqrt{\pi}} \int_0^T \int_{y \geq \frac{\rho}{\sqrt{4\nu t}}} \left| u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t} y,t) u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t} y,t) U^{\mathrm{E}}(x_1,t) \right| \exp(-y^2) dx_1 dy dt \\
&\quad + \frac{2}{\sqrt{\pi}} \int_0^T \int_{y \leq \frac{\rho}{\sqrt{4\nu t}}} \left| u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t} y,t) u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t} y,t) U^{\mathrm{E}}(x_1,t) \right| \exp(-y^2) dx_1 dy dt \\
&\leq \frac{2}{\sqrt{\pi}} \|U^{\mathrm{E}}\|_{L^\infty(0,T;L^1_{x_1}(\mathbb{R}))} \int_0^T \|u^{\mathrm{NS}}(t)\|^2_{L^\infty_{x_1,x_2}(\mathbb{H})} \left(\int_{y \geq \frac{\rho}{\sqrt{4\nu t}}} \exp(-y^2) dy \right) dt \\
&\quad + \frac{2}{\sqrt{\pi}} \|U^{\mathrm{E}}\|_{L^\infty(0,T;L^\infty_{x_1}(\mathbb{R}))} \int_0^T \int_{y \leq \frac{\rho}{\sqrt{4\nu t}}} \varepsilon\, \gamma(x_1,t) \exp(-y^2) dx_1 dy dt \\
&\leq \|U^{\mathrm{E}}\|_{L^\infty(0,T;L^1_{x_1}(\mathbb{R}))} C_{\mathrm{NS}} \nu_0 \operatorname{erfc}\left(\frac{\rho}{\sqrt{4\nu T}} \right) \\
&\quad + \varepsilon \|U^{\mathrm{E}}\|_{L^\infty(0,T;L^\infty_{x_1}(\mathbb{R}))} \|\gamma\|_{L^1(0,T;L^1_{x_1}(\mathbb{R}_+))} \qquad (4.10)
\end{aligned}$$

where we have also appealed to (1.13). By passing $\nu \to 0$ in (4.10), since ρ and T are fixed, and $\operatorname{erfc}(z) \to 0$ as $z \to \infty$, we arrive at

$$\lim_{\nu \to 0} \int_0^T |T_{6,\nu}(t)| dt \leq \varepsilon \|U^{\mathrm{E}}\|_{L^\infty(0,T;L^\infty_{x_1}(\mathbb{R}))} \|\gamma\|_{L^1(0,T;L^1_{x_1}(\mathbb{R}_+))}. \qquad (4.11)$$

Since γ is independent of ε, and $\varepsilon > 0$ is arbitrary, (4.11) implies (4.9) as desired.

4.4. Proof of Theorem 1.1. From (2.6), (4.1), (4.2), (4.3), (4.4), (4.5), and (4.7) we conclude that

$$\frac{d}{dt}\|v\|_{L^2}^2 \leq C_{\mathrm{E}}\|v\|_{L^2}^2 + C_{\mathrm{E}}\nu^{1/2}t^{-1/2}\|v\|_{L^2} + C_{\mathrm{E}}(\nu t)^{1/4} + C_{\mathrm{E}}(\nu t)^{1/2}\|u^{\mathrm{NS}}(t)\|_{L^\infty}^2 + T_{6,\nu} \tag{4.12}$$

where as usual C_{E} implicitly depends on ν_0 and T. Upon integrating (4.12) in time, using (1.7), (4.6), and (4.9) we arrive at

$$\lim_{\nu\to 0}\|v\|_{L^\infty(0,T;L^2(\mathbb{H}))} = 0.$$

The above yields the proof of (1.8) once we recall that $u^{\mathrm{NS}} - u^{\mathrm{E}} = v + u^{\mathrm{K}}$, and that cf. (3.4) and (3.11) we have $\lim_{\nu\to 0}\|u^{\mathrm{K}}\|_{L^\infty(0,T;L^2(\mathbb{H}))} = 0$.

5. Proof of Theorem 1.3

The proof follows from the proof of Theorem 1.1, as soon as we manage to establish the limit (4.9) for the T_6 term. Recall that

$$\frac{\sqrt{\pi}}{2}|T_{6,\nu}(t)| = \left|\int_{\mathbb{H}} u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)U^{\mathrm{E}}(x_1,t)\exp(-y^2)dx_1dy\right|.$$

Since (1.13) holds, and

$$\|U^{\mathrm{E}}(x_1,\cdot)\|_{L^\infty([0,T])}dx_1 \quad \text{and} \quad \exp(-y^2)dy$$

are finite measures on $\mathbb{R}$ respectively $\mathbb{R}_+$, by Chebyshev's inequality we have that for any $L>0$:

$$\begin{aligned}
&\left|\int_{|y|\geq L} u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)U^{\mathrm{E}}(x_1,t)\exp(-y^2)dx_1dy\right| \\
&\qquad \leq \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2 \int_{|y|\geq L}|U^{\mathrm{E}}(x_1,t)|\exp(-y^2)dx_1dy \\
&\qquad \leq \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2\|U^{\mathrm{E}}\|_{L^1_{x_1}L^\infty_t}\frac{\sqrt{\pi}}{2}\operatorname{erfc}(L)
\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}
&\left|\int_{|x_1|\geq L} u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)U^{\mathrm{E}}(x_1,t)\exp(-y^2)dx_1dy\right| \\
&\qquad \leq \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2 \int_{|x_1|\geq L}\sup_{t\in[0,T]}|U^{\mathrm{E}}(x_1,t)|dx_1 \\
&\qquad \leq \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2\|U^{\mathrm{E}}\|_{L^1_{x_1}L^\infty_t}\frac{1}{L}.
\end{aligned} \tag{5.2}$$

Combining (5.1)–(5.2) with (1.13) it follows that for a given $\varepsilon>0$, there exists a sufficiently large $L = L(\varepsilon, C_{\mathrm{E}}, C_{\mathrm{NS}}, \nu_0) > 0$ such that

$$\int_0^T\left|\int_{|y|\geq L \text{ or } |x_1|\geq L} u_1^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)u_2^{\mathrm{NS}}(x_1,\sqrt{4\nu t}y,t)U^{\mathrm{E}}(x_1,t)\exp(-y^2)dx_1dy\right|dt \leq \varepsilon. \tag{5.3}$$

On the other hand, since $\partial_2 u_2^{\rm NS} = -\partial_1 u_1^{\rm NS}$, by (1.18) we have that

$$
\begin{aligned}
&\int_0^T \left| \int_{|y|\le L \text{ and } |x_1|\le L} u_1^{\rm NS}(x_1,\sqrt{4\nu t}y,t) u_2^{\rm NS}(x_1,\sqrt{4\nu t}y,t) U^{\rm E}(x_1,t)\exp(-y^2) dx_1 dy \right| dt \le \\
&\le \|U^{\rm E}\|_{L^\infty_t L^\infty_{x_1}} \int_0^T \|u_1^{\rm NS}(t)\|_{L^\infty} \int_0^L \exp(-y^2) \int_{|x_1|\le L} \int_0^{\sqrt{4\nu t}L} |\partial_1 u_1^{\rm NS}(x_1,x_2,t)| dx_2 dx_1 dy dt \\
&\le C_{\rm E} \int_0^T \|u_1^{\rm NS}(t)\|_{L^\infty} \|\partial_1 u_1^{\rm NS}(t) \mathbf{1}_{|x_1|\le L, 0<x_2<\sqrt{4\nu t}L}\|_{L^1(\mathbb{H})} dt \\
&\le C_{\rm E} C_{\rm NS} \nu_0 \|\partial_1 u_1^{\rm NS}(t) \mathbf{1}_{|x_1|\le L, 0<x_2<\sqrt{4\nu t}L}\|_{L^2(0,T;L^1(\mathbb{H}))} \\
&\le \varepsilon
\end{aligned}
\tag{5.4}
$$

assuming ν is sufficiently small so that $\sqrt{4\nu t}L \le \rho(\varepsilon, L)$. Therefore, by adding (5.3) and (5.4) we have that for a fixed $y>0$

$$
\lim_{\nu\to 0} \int_0^T |T_{6,\nu}(t)| dt \le C\varepsilon
$$

for any $\varepsilon > 0$, as desired.

6. Proof of Theorem 1.4

The proof follows closely that of Theorem 1.1. To avoid redundancy, here we only point out the main differences. Moreover, for the sake of simplicity, we first consider the case $\delta(\nu t) = 2\sqrt{\nu t}$, which clearly obeys condition (1.21).

Condition (1.19) implies that (1.13) holds. Therefore, it remains to show that (1.19) and (1.20) imply

$$
\lim_{\nu\to 0} \int_0^T \int_{\mathbb{H}} \left| u_2^{\rm NS}(x_1, \sqrt{4\nu t}\, x_2, t) u_1^{\rm NS}(x_1, \sqrt{4\nu t}\, x_2, t) U^{\rm E}(x_1,t) \exp\left(-x_2^2\right) \right| dx_1 dx_2 dt = 0, \tag{6.1}
$$

i.e. that (4.9) holds. Once (6.1) is proven, the theorem follows with the same proof as Theorem 1.1.

For this purpose, notice that the function

$$
A(x_1,x_2,t) = M^2(t) |U^{\rm E}(x_1,t)| \exp(-x_2^2)
$$

is independent of ν, obeys

$$
A \in L^1(dt dx_1 dx_2),
$$

since the Euler trace $U^{\rm E}$ is bounded in $L^\infty(0,T;L^1_{x_1}(\mathbb{R}))$, and we have that

$$
\left| u_2^{\rm NS}(x_1, \sqrt{4\nu t}\, x_2, t) u_1^{\rm NS}(x_1, \sqrt{4\nu t}\, x_2, t) U^{\rm E}(x_1,t) \exp\left(-x_2^2\right) \right| \le A(x_1,x_2,t)
$$

for a.e. (x_1,x_2,t), and all $\nu \in (0,\nu_0]$. Thus, in view of (1.20), which guarantees that

$$
\lim_{\nu\to 0} \left| u_1^{\rm NS}(x_1, \delta(\nu t)\, x_2, t) u_2^{\rm NS}(x_1, \delta(\nu t)\, x_2, t) U^{\rm E}(x_1,t) \exp\left(-x_2^2\right) \right| = 0
$$

we may apply the Dominated Convergence Theorem and conclude that (6.1) holds. This concludes the proof of the theorem when $\delta(\nu t) = 2\sqrt{\nu t}$.

To treat the more general case $\delta(\nu)$ which obeys (1.21), we need to define a different corrector. For this purpose, we recall cf. [CKV15] that the function

$$
\varphi(x_1,x_2,t) = (\varphi_1(x_1,x_2,t), \varphi_2(x_1,x_2,t))
$$

where

$$\varphi_1(x_1,x_2,t) = -U^{\rm E}(x_1,t)\left(e^{-\frac{x_2}{\delta(\nu t)}} - \delta(\nu t)\psi(x_2)\right) \tag{6.2}$$

$$\varphi_2(x_1,x_2,t) = \delta(\nu t)\partial_1 U^{\rm E}(x_1,t)\left(\left(1-\int_0^{x_2}\psi(y)dy\right) - e^{-\frac{x_2}{\delta(\nu t)}}\right) \tag{6.3}$$

where $\psi\colon [0,\infty)\to[0,\infty)$ is a C_0^∞ function supported in $[1/2,4]$, which is non-negative and has mass $\int\psi(z)dz = 1$, is divergence free and obeys the boundary conditions (2.2)–(2.3)

$$\varphi_1(x_1,0,t) = -U^{\rm E}(x_1,t)$$
$$\varphi_2(x_1,0,t) = 0.$$

We then consider the same argument as in the proof of Theorem 1.1, except that $u^{\rm K}$ is replaced by φ. In [**CKV15**], the bounds

$$\|\varphi\|_{L^p(\mathbb{H})} + \|\partial_t\varphi\|_{L^p(\mathbb{H})} + \|\partial_1\varphi\|_{L^p(\mathbb{H})} + \|\partial_{11}\varphi\|_{L^p(\mathbb{H})} \le C_{\rm E}\delta(\nu t)^{1/p}$$
$$\|\partial_2\varphi_1\|_{L^p(\mathbb{H})} \le C_{\rm E}\delta(\nu t)^{-1+1/p}$$
$$\|\partial_1\varphi_2\|_{L^p(\mathbb{H})} \le C_{\rm E}\delta(\nu t)$$

were established. It then follows that the terms $T_1,\ldots,T_5$ defined in (2.7)–(2.11) obey the estimates

$$|T_1| \le C_{\rm E}\delta(\nu t)^{1/2}\|v\|_{L^2} + C_{\rm E}\nu\delta(\nu t)^{1/2}\|v\|_{L^2} + \frac{\nu}{2}\|\partial_2 v\|_{L^2}^2 + \nu\|v\|_{L^2}^2 + C_{\rm E}\frac{\nu}{\delta(\nu t)} \tag{6.4}$$

$$|T_2| \le C_{\rm E}\delta(\nu t)^{1/2} \tag{6.5}$$

$$|T_3| \le C_{\rm E}\delta(\nu t)^{1/2}\|v\|_{L^2} \tag{6.6}$$

$$|T_4| \le C_{\rm E}\delta(\nu t) \tag{6.7}$$

$$|T_5| \le M^2(t)\delta(\nu t)^{1/2} \tag{6.8}$$

where $M(t)$ is as given by condition (1.19). For the term T_6 we proceed as above, by appealing to the Dominated convergence theorem. Condition (1.21) is necessary in order to ensure that the time integral of the last term on the right side of (6.4) vanishes as $\nu\to 0$. The proof now follows. We omit further details.

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