Volume polynomials

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Abstract. Volume polynomials form a distinguished class of log-concave polynomials with remarkable analytic and combinatorial properties. I will survey realization problems related to them, review fundamental inequalities they satisfy, and discuss applications to the combinatorics of algebraic matroids. These notes are based on lectures given at the 2025 Summer Research Institute in Algebraic Geometry at Colorado State University.

1. Realization problems for projection volumes and homology classes

1.1. Let π_{ij} be the coordinate projection of $\mathbb{R}^4$ onto the plane orthogonal to the standard basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$. For a convex body A in $\mathbb{R}^4$, we consider its vector of projection areas $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$, where

$$p_{ij} = \big(\text{the area of the projection } \pi_{ij}(A)\big).$$

Which tuples of six nonnegative real numbers can arise in this way? This question is the simplest nontrivial instance of the various *realization problems* for volume polynomials (Sections 2 and 3). The following answer was given in [**HHM**$^+$, Theorem 1.4].

Theorem 1.1. The following conditions are equivalent for any vector of nonnegative real numbers $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$.

(1) There is a convex body $A \subseteq \mathbb{R}^4$ that satisfies

$$p_{ij} = \big(\text{the area of the projection } \pi_{ij}(A)\big) \text{ for all } i < j.$$

(2) There is a Euclidean triangle with side lengths $\sqrt{p_{12}p_{34}}, \sqrt{p_{13}p_{24}}, \sqrt{p_{14}p_{23}}$.

In other words, $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ is realizable as the vector of projection areas of a convex body in $\mathbb{R}^4$ if and only if it satisfies the triangle inequalities

$$\sqrt{p_{12}p_{34}} \le \sqrt{p_{13}p_{24}} + \sqrt{p_{14}p_{23}} \quad \text{and}$$
$$\sqrt{p_{13}p_{24}} \le \sqrt{p_{12}p_{34}} + \sqrt{p_{14}p_{23}} \quad \text{and}$$
$$\sqrt{p_{14}p_{23}} \le \sqrt{p_{12}p_{34}} + \sqrt{p_{13}p_{24}}.$$

A triangle is said to be *nondegenerate* when all the triangle inequalities are strict.

Theorem 1.2. The following conditions are equivalent for any vector of nonnegative real numbers $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$.

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(1) There is a smooth convex body $A \subseteq \mathbb{R}^4$ that satisfies

$$p_{ij} = \big(\text{the area of the projection } \pi_{ij}(A)\big) \text{ for all } i < j.$$

(2) There is a nondegenerate triangle with side lengths $\sqrt{p_{12}p_{34}}$, $\sqrt{p_{13}p_{24}}$, $\sqrt{p_{14}p_{23}}$.

The *realization space* of $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ is the set of all convex bodies in $\mathbb{R}^4$ with the given projection areas. Theorem 1.1 characterizes the vectors with nonempty realization spaces. One can show that, in a precise sense, the realization space of a given vector only depends on the associated triangle with side lengths $\sqrt{p_{12}p_{34}}$, $\sqrt{p_{13}p_{24}}$, $\sqrt{p_{14}p_{23}}$.

EXAMPLE 1.3. The vector $(\pi, \pi, \pi, \pi, \pi, \pi)$ is realizable as the vector of projection areas of a convex body in $\mathbb{R}^4$. For example, one may take the unit ball, the hypercube with side lengths $\sqrt{\pi}$, or more generally any convex body preserved by the S_4-symmetry of $\mathbb{R}^4$, scaled appropriately.

EXAMPLE 1.4. According to Theorem 1.1, the vector $(2, 1, 1, 1, 1, 2)$ is realizable as the vector of projection areas of a convex body $A \subseteq \mathbb{R}^4$ because there is a triangle with side lengths $2, 1, 1$. By Theorem 1.2, such a convex body cannot be smooth. As a realization, one may take

$$A = \sqrt{2}\ \big(\text{the convex hull of } \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4 \text{ in } \mathbb{R}^4\big).$$

The projections $\pi_{12}(A)$ and $\pi_{34}(A)$ are squares with side lengths $\sqrt{2}$, and the remaining projections of A are triangles with side lengths 2, $\sqrt{2}$, $\sqrt{2}$.

EXAMPLE 1.5. According to Theorem 1.1, the vector $(3, 2, 1, 1, 2, 3)$ is realizable as the vector of projection areas of a convex body $A \subseteq \mathbb{R}^4$ because there is a triangle with side lengths $3, 2, 1$. For example, one may take the 4×16 matrices

$$L := \begin{bmatrix} 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \end{bmatrix},$$

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ -2 & 0 & 0 & -2 & -1 & 0 & -1 & -1 & -1 & -2 & 0 & -1 & -1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & 0 \end{bmatrix},$$

and let A be the convex hull in $\mathbb{R}^4$ of the 16 columns of $L + \frac{1}{\sqrt{2}}M$. With patience, one can check that A has the projections with given areas. For example,

$$\pi_{24}(A) = \big(\text{the convex hull of } \mathbf{e}_1, -\mathbf{e}_1, \mathbf{e}_3, -\mathbf{e}_3 \text{ in } \mathbb{R}^2\big).$$

Can you think of a simpler convex body in $\mathbb{R}^4$ that has the same six projection areas? See [**HHM**$^+$, Section 4] for a discussion of this particular case.

The condition characterizing the realizability of $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ in Theorem 1.1 is precisely the validity of the *Plücker relation*

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \quad \text{for the Grassmannian } \mathrm{Gr}(2,4),$$

interpreted over the triangular hyperfield $\mathbb{T}_2$ on the set of nonnegative real numbers [**BHKLa**]. As noted in [**HHM**$^+$, Proposition 3.1], the condition is also equivalent to the statement that the nonnegative symmetric matrix

$$\begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ p_{12} & 0 & p_{23} & p_{24} \\ p_{13} & p_{23} & 0 & p_{34} \\ p_{14} & p_{24} & p_{34} & 0 \end{pmatrix} \text{ has at most one positive eigenvalue.}$$

Theorem 1.1 is related to the fact that the set of quadratic volume polynomials $\mathbb{V}_n^2(\mathbb{R}, k)$ is equal to the set of Lorentzian polynomials (Section 3).

1.2. One quickly encounters interesting questions when trying to formulate similar realization problems in higher dimensions. The asymmetry in the following pair of conjectures points to the distinction between *volume polynomials* and *covolume polynomials* (Section 4).

Let π_{ij} be the coordinate projection of $\mathbb{R}^5$ onto the coordinate subspace orthogonal to the standard basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$.

Conjecture 1.6. The following conditions are equivalent for any vector of nonnegative real numbers $(p_{ij})_{1 \leq i < j \leq 5}$:

(1) There is a convex body $A \subseteq \mathbb{R}^5$ that satisfies

$$p_{ij} = \big(\text{the volume of the projection } \pi_{ij}(A)\big) \text{ for all } i < j.$$

(2) The nonnegative symmetric matrix

$$\begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{12} & 0 & p_{23} & p_{24} & p_{25} \\ p_{13} & p_{23} & 0 & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & 0 & p_{45} \\ p_{15} & p_{25} & p_{35} & p_{45} & 0 \end{pmatrix} \text{ has at most one positive eigenvalue.}$$

Let φ_{ij} be the coordinate projection of $\mathbb{R}^5$ onto the coordinate subspace spanned by the standard basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$.

Conjecture 1.7. The following conditions are equivalent for any vector of nonnegative real numbers $(q_{ij})_{1 \leq i < j \leq 5}$:

(1) There is a convex body $A \subseteq \mathbb{R}^5$ that satisfies

$$q_{ij} = \big(\text{the area of the projection } \varphi_{ij}(A)\big) \text{ for all } i < j.$$

(2) Every 4×4 principal submatrix of the nonnegative symmetric matrix

$$\begin{pmatrix} 0 & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{12} & 0 & q_{23} & q_{24} & q_{25} \\ q_{13} & q_{23} & 0 & q_{34} & q_{35} \\ q_{14} & q_{24} & q_{34} & 0 & q_{45} \\ q_{15} & q_{25} & q_{35} & q_{45} & 0 \end{pmatrix} \text{ has at most one positive eigenvalue.}$$

In both cases, the forward implication follows from the Alexandrov–Fenchel inequality on mixed volumes. Similar conjectures can be made more generally for projections $\pi_{ij} : \mathbb{R}^d \to \mathbb{R}^{d-2}$ and $\varphi_{ij} : \mathbb{R}^d \to \mathbb{R}^2$.

Example 1.8. To compare Conjectures 1.6 and 1.7, we consider the case of $(4, 1, 1, 1, 1, 1, 1, 1, 1, 1)$. Since the corresponding symmetric matrix has eigenvalues

$$3 + \sqrt{7},\ 3 - \sqrt{7},\ -1,\ -1,\ -4,$$

there is no convex body A in $\mathbb{R}^5$ such that the projection $\pi_{12}(A)$ has volume 4 while all the other projections $\pi_{ij}(A)$ have volume 1. On the other hand, every 4×4 principal submatrix of the same matrix has at most one positive eigenvalue, suggesting that there is a convex body A in $\mathbb{R}^5$ such that the projection $\varphi_{12}(A)$ has area 4 while all the other projections $\varphi_{ij}(A)$ have area 1. Indeed, there is

$$A = \big(\text{the convex hull of } 2\mathbf{e}_1, 2\mathbf{e}_2, 2\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_5, \mathbf{e}_4 + \mathbf{e}_5\big),$$

which is consistent with Conjectures 1.6 and 1.7.

1.3. We may pose analogous realization problems in the setting of projective geometry. Fix an algebraically closed field k, and consider the projective line $\mathbb{P}^1$ over k. If S is an irreducible surface in $(\mathbb{P}^1)^4$, we can uniquely express its homology class as a nonnegative integral linear combination

$$[S] = p_{12}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^0] + \cdots + p_{34}[\mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1] \in \mathrm{CH}((\mathbb{P}^1)^4).$$

Which vectors of nonnegative integers $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ can arise in this way? We call such homology classes *realizable*.[1] While the answer to this question is not known, the following partial result was given in [**HHM**$^+$, Theorem 1.6].

Theorem 1.9. The following conditions are equivalent for any vector of nonnegative rational numbers $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$.

(1) There is an irreducible surface $S \subseteq (\mathbb{P}^1)^4$ and a nonnegative $\lambda \in \mathbb{Q}$ such that

$$\lambda[S] = p_{12}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^0] + \cdots + p_{34}[\mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1].$$

(2) There is a Euclidean triangle with side lengths $\sqrt{p_{12}p_{34}}, \sqrt{p_{13}p_{24}}, \sqrt{p_{14}p_{23}}$.

The algebraic realization problem here has additional obstructions not present in the convex realization problem in Section 1.1. For example, the homology class corresponding to $(1,1,1,1,1,3)$ is not realizable by an irreducible surface in $(\mathbb{P}^1)^4$, although there is a Euclidean triangle with side lengths $\sqrt{3}, 1, 1$. To see this, note that the hypothetical surface S should satisfy

$$[\pi_3(S)] = [\pi_4(S)] = [\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0] + [\mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^1] + [\mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1],$$

where π_i is the projection $(\mathbb{P}^1)^4 \to (\mathbb{P}^1)^3$ that forgets the i-th coordinate of $(\mathbb{P}^1)^4$. Thus, the defining equations of the hypersurfaces $\pi_3^{-1}\pi_3(S)$ and $\pi_4^{-1}\pi_4(S)$ in an affine chart of $(\mathbb{P}^1)^4$ are of the form

$$\begin{aligned} *1 + *x_1 + *x_2 + *x_4 + *x_1x_2 + *x_1x_4 + *x_2x_4 + *x_1x_2x_4 = 0, \\ *1 + *x_1 + *x_2 + *x_3 + *x_1x_2 + *x_1x_3 + *x_2x_3 + *x_1x_2x_3 = 0, \end{aligned}$$

where the $*$'s are placeholders for the coefficients in k. For generic values of x_3 and x_4, the displayed system of equations reduces to

$$\begin{aligned} *1 + *x_1 + *x_2 + *x_1x_2 = 0, \\ *1 + *x_1 + *x_2 + *x_1x_2 = 0, \end{aligned}$$

which has at most 2 solutions. This contradicts the assumption that $p_{34} = 3$. The proof of Theorem 1.9 in [**HHM**$^+$] shows that, in fact, the homology class corresponding to $(2,2,2,2,2,6)$ is realizable by an irreducible surface in $(\mathbb{P}^1)^4$.

[1] Such questions are algebraic analogues of the *Steenrod problem* in topology [**Eil49**, Problem 25], which asks whether every homology class in any simplicial complex X is the image of the fundamental class of a closed oriented manifold by a map into the simplicial complex.

Modulo this subtlety involving integral coefficients, one can formulate statements in algebraic geometry parallel to Conjectures 1.6 and 1.7 in convex geometry, the first of which is a special case of [**HHM**$^+$, Theorem 1.8].

Theorem 1.10. The following conditions are equivalent for any vector of nonnegative integers $(p_{ij})_{1\leq i<j\leq 5}$:

(1) There is an irreducible surface $S \subseteq (\mathbb{P}^1)^5$ and a nonnegative $\lambda \in \mathbb{Q}$ such that

$$\lambda[S] = p_{12}[\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^0\times\mathbb{P}^0\times\mathbb{P}^0] + \cdots + p_{45}[\mathbb{P}^0\times\mathbb{P}^0\times\mathbb{P}^0\times\mathbb{P}^1\times\mathbb{P}^1].$$

(2) The nonnegative symmetric matrix

$$\begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} & p_{15} \\ p_{12} & 0 & p_{23} & p_{24} & p_{25} \\ p_{13} & p_{23} & 0 & p_{34} & p_{35} \\ p_{14} & p_{24} & p_{34} & 0 & p_{45} \\ p_{15} & p_{25} & p_{35} & p_{45} & 0 \end{pmatrix} \text{ has at most one positive eigenvalue.}$$

The following conjecture suggests the possibility that, in Conjecture 1.7, the convex body A can be chosen to be a rational convex polytope whenever all p_{ijk} are rational.

Conjecture 1.11. The following conditions are equivalent for any vector of nonnegative integers $(q_{ij})_{1\leq i<j\leq 5}$:

(1) There is an irreducible threefold $S \subseteq (\mathbb{P}^1)^5$ and a nonnegative $\lambda \in \mathbb{Q}$ such that

$$\lambda[S] = q_{12}[\mathbb{P}^0\times\mathbb{P}^0\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1] + \cdots + q_{45}[\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^0\times\mathbb{P}^0].$$

(2) Every 4×4 principal submatrix of the nonnegative symmetric matrix

$$\begin{pmatrix} 0 & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{12} & 0 & q_{23} & q_{24} & q_{25} \\ q_{13} & q_{23} & 0 & q_{34} & q_{35} \\ q_{14} & q_{24} & q_{34} & 0 & q_{45} \\ q_{15} & q_{25} & q_{35} & q_{45} & 0 \end{pmatrix} \text{ has at most one positive eigenvalue.}$$

In both cases, the Hodge index theorem for projective surfaces can be used to show the forward implication.

2. Volume polynomials in convex geometry

2.1. In [**Min03**], Minkowski made a foundational observation that has since become a cornerstone of convex geometry:

> *The volume of the Minkowski sum of convex bodies varies polynomially under scaling.*

More precisely, for positive integers n and d, and any collection of n convex bodies $C = (C_1, \ldots, C_n)$ in the d-dimensional Euclidean space $\mathbb{R}^d$, the function

$$f_C : \mathbb{R}^n_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}, \qquad (x_1, \ldots, x_n) \longmapsto \frac{1}{d!}\,\mathrm{vol}(x_1C_1 + \cdots + x_nC_n)$$

is a degree d homogeneous polynomial in $x = (x_1, \ldots, x_n)$. This polynomial, called the *volume polynomial* of C, is then used to define the *mixed volume* of convex bodies in C as its normalized coefficients

$$\mathrm{MV}(C_{i_1}, \ldots, C_{i_d}) \coloneqq \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_d}} f_C(x_1, \ldots, x_n).$$

The mixed volume is symmetric in its arguments, and it is multilinear with respect to the Minkowski sum and nonnegative scaling: For any convex bodies B_1 and B_2 in $\mathbb{R}^d$ and nonnegative real numbers λ_1 and λ_2, we have

$$\mathrm{MV}(\lambda_1 B_1 + \lambda_2 B_2, C_2, \ldots, C_d) = \lambda_1 \mathrm{MV}(B_1, C_2, \ldots, C_d) + \lambda_2 \mathrm{MV}(B_2, C_2, \ldots, C_d).$$

When all the arguments coincide, the mixed volume reduces to the usual volume: For any convex body A in $\mathbb{R}^d$, we have

$$\mathrm{MV}(\underbrace{A, \ldots, A}_{d}) = \mathrm{vol}(A).$$

More generally, if I_j is the unit interval joining the origin and $\mathbf{e}_j$ in $\mathbb{R}^d$, we have

$$\binom{d}{k} \mathrm{MV}(I_1, \ldots, I_k, \underbrace{A, \ldots, A}_{d-k}) = \frac{1}{k!}\, \mathrm{vol}(\pi_{1\cdots k} A),$$

where $\pi_{1\cdots k}$ is the projection onto the coordinate subspace orthogonal to $\mathbf{e}_1, \ldots, \mathbf{e}_k$. For a comprehensive introduction to mixed volumes, see [**Sch14**, Chapter 5].

Mixed volumes of convex bodies satisfy a rich collection of fundamental inequalities, the most basic being *nonnegativity*:

$$0 \leq \mathrm{MV}(C_1, \ldots, C_d) \text{ for any convex bodies } C_1, \ldots, C_d \text{ in } \mathbb{R}^d.$$

More generally, mixed volumes are *monotone* in each argument: If $B_i \subseteq C_i$ are convex bodies in $\mathbb{R}^d$, we have

$$\mathrm{MV}(B_1, \ldots, B_d) \leq \mathrm{MV}(C_1, \ldots, C_d).$$

Apart from the nonnegativity, the most important inequality involving mixed volumes is the *Alexandrov–Fenchel inequality*, which generalizes classical inequalities such as the isoperimetric and Brunn–Minkowski inequalities. It states that, for any convex bodies $C_1, \ldots, C_d$ in $\mathbb{R}^d$, we have

$$\mathrm{MV}(C_1, C_1, C_3 \ldots, C_d)\, \mathrm{MV}(C_2, C_2, C_3, \ldots, C_d) \leq \mathrm{MV}(C_1, C_2, C_3 \ldots, C_d)^2.$$

This inequality is a cornerstone of modern convex geometry and underlies many structural and analytic results.

2.2. The *realization problem* for volume polynomials of convex bodies is to determine which homogeneous polynomials of degree d in n variables with nonnegative coefficients can be realized as the volume polynomials of n convex bodies in $\mathbb{R}^d$. The problem of finding the full set of inequalities for mixed volumes is sometimes referred to as *Alexandrov's problem*.

For a degree d homogeneous polynomial f in n variables $x = (x_1, \ldots, x_n)$, we write

$$f(x) = \sum_{\alpha \in \Delta_n^d} p_\alpha x^{[\alpha]}, \quad x^{[\alpha]} := \frac{x^\alpha}{\alpha!} = \frac{x_1^{\alpha_1}}{\alpha_1!} \cdots \frac{x_n^{\alpha_n}}{\alpha_n!},$$

where p_α are the normalized coefficients of f and Δ_n^d is the discrete simplex consisting of all the nonnegative vectors in $\mathbb{Z}^n$ whose coordinates sum to d. Here are the first two necessary conditions for f to be a volume polynomial.

(1) (Nonnegative change of coordinates) If $f(x)$ is a volume polynomial of n convex bodies, then $f(Ay)$ is a volume polynomial of m convex bodies, for any $n \times m$ nonnegative matrix A and variables $y = (y_1, \ldots, y_m)$.

(2) (Alexandrov–Fenchel inequality) If $f(x)$ is a volume polynomial of convex bodies, then its normalized coefficients satisfy

$$p_{\alpha+\mathrm{e}_i-\mathrm{e}_j}p_{\alpha-\mathrm{e}_i+\mathrm{e}_j} \leq p_\alpha^2 \quad \text{for any } \alpha \in \Delta_n^d \text{ and any } 1 \leq i < j \leq n.$$

The first condition is a formal consequence of the observation that the Minkowski sum of convex bodies is a convex body. The combination of the two properties leads to the conclusion that any volume polynomial must be a *Lorentzian polynomial* in the sense of [**BH20**]. Here we give an equivalent definition, following [**BL**, Section 2]. We set

$$\big\{\text{Lorentzian polynomials of degree} \leq 1\big\} = \\ \big\{\text{homogeneous polynomials of degree} \leq 1 \text{ with nonnegative coefficients}\big\}.$$

For a nonnegative vector $u = (u_1, \dots, u_n)$, we write ∂_u for the corresponding directional derivative $\sum_{i=1}^n u_i \partial_i$.

Definition 2.1. A homogeneous polynomial f of degree $d \geq 2$ in n variables with nonnegative coefficients is *Lorentzian* if, for all $v_1, \dots, v_d \in \mathbb{R}^n_{\geq 0}$, we have

$$\big(\partial_{v_1}\partial_{v_1}\partial_{v_3}\cdots\partial_{v_d} f\big)\big(\partial_{v_2}\partial_{v_2}\partial_{v_3}\cdots\partial_{v_d} f\big) \leq \big(\partial_{v_1}\partial_{v_2}\partial_{v_3}\cdots\partial_{v_d} f\big)^2.$$

Applying the Alexandrov–Fenchel inequality after a nonnegative linear change of coordinates, we see that

$$\big\{\text{volume polynomials of convex bodies}\big\} \subseteq \big\{\text{Lorentzian polynomials}\big\}.$$

Thus, Alexandrov's problem is to find inequalities between mixed volumes that identify the volume polynomials of convex bodies among Lorentzian polynomials.

Example 2.2 ($n = 2$). According to [**BH20**, Example 2.26], a bivariate polynomial with nonnegative coefficients

$$f = \sum_{a=0}^{d} p_a \, \frac{x_1^a}{a!} \frac{x_2^{d-a}}{(d-a)!}$$

is Lorentzian if and only if the sequence $p_0, \dots, p_d$ has no internal zeros and

$$p_{a-1}p_{a+1} \leq p_a^2 \quad \text{for all positive integers } a < d.$$

In [**She60**], Shephard showed that any such polynomial is the volume polynomial of two convex bodies in $\mathbb{R}^d$. This characterizes volume polynomials of convex bodies in two variables:

> *A homogeneous polynomial in two variables is the volume polynomial of two convex bodies if and only if it is Lorentzian.*

When every p_i is rational, Shephard's construction gives two rational convex polytopes. This is used in [**Huh12**, Theorem 21] to characterize realizable homology classes in $\mathbb{P}^d \times \mathbb{P}^d$ up to a multiple: Some nonnegative rational multiple of the class

$$\sum_{a=0}^{d} p_a \, [\mathbb{P}^a \times \mathbb{P}^{d-a}] \in \mathrm{CH}(\mathbb{P}^d \times \mathbb{P}^d)$$

is the class of an irreducible subvariety if and only if $p_0, \dots, p_d$ is a log-concave sequence of nonnegative rational numbers with no internal zeros.[2]

[2] As observed in [**Huh**, Section 5], there is no irreducible subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ whose homology class corresponds to the log-concave sequence $(1, 2, 3, 4, 2, 1)$.

Example 2.3 ($d = 1$). By definition, a linear form is Lorentzian if and only if all its coefficients are nonnegative. Any such linear form is a volume polynomial of convex bodies in $\mathbb{R}^1$:

$$\mathrm{vol}(x_1C_1 + \cdots + x_nC_n) = x_1 \,\mathrm{vol}(C_1) + \cdots + x_n \,\mathrm{vol}(C_n), \quad C_1, \ldots, C_n \subseteq \mathbb{R}^1.$$

Example 2.4 ($d = 2$). A quadratic form is Lorentzian if and only if all its coefficients are nonnegative and its Hessian has at most one positive eigenvalue [**BH20**, Section 2]. In [**Hei38**], Heine showed that, when there are at most three variables, any such quadratic form is the volume polynomial of three convex bodies in $\mathbb{R}^2$. This characterizes quadratic volume polynomials of convex bodies in three variables:

> *A ternary quadratic form is the volume polynomial of three convex bodies if and only if it is Lorentzian.*

The analogous statement fails when $n = 4$. For example, as observed in [**She60**, Theorem 5], there are no convex bodies C_1, C_2, C_3, C_4 in $\mathbb{R}^2$ satisfying

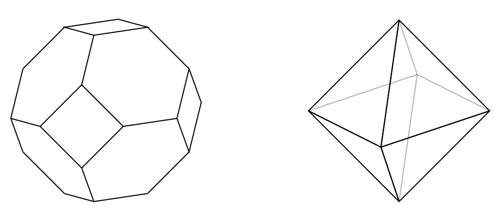
$$\mathrm{vol}(x_1C_1 + x_2C_2 + x_3C_3 + x_4C_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

even though the right-hand side is a Lorentzian polynomial. In fact, using the compactness theorem of Shephard for the affine equivalence classes of convex bodies [**She60**, Theorem 1], one can show that the displayed elementary symmetric polynomial is not even the limit of volume polynomials of convex bodies in the plane. This contrasts with the fact that there is an irreducible surface in $(\mathbb{P}^1)^4$ with class $(1,1,1,1,1,1)$ in the Chow group. For example, one may take the closure of a general two-dimensional linear subspace of an affine chart of $(\mathbb{P}^1)^4$.

2.3. The main result of [**BH20**] provides a finite description of the set of Lorentzian polynomials that generalizes Example 2.2. The central notion is that of a generalized permutohedron. Let E be a finite set with n elements, and let $\{e_i\}_{i \in E}$ be the standard basis of $\mathbb{R}^E$.

Definition 2.5. A *generalized permutohedron* is a polytope in $\mathbb{R}^E$ all of whose edges are in the direction $e_i - e_j$ for some i and j in E.

A generalized permutohedron is *integral* if all its vertices belong to $\mathbb{Z}^E \subseteq \mathbb{R}^E$. For example, the *standard permutohedron* in $\mathbb{R}^n$, which is the convex hull of all permutations of $(1, 2, \ldots, n)$, and the *k-th hypersimplex* in $\mathbb{R}^n$, which is the convex hull of all permutations of $(\underbrace{1, \ldots, 1}_{k}, \underbrace{0, \ldots, 0}_{n-k})$, are integral generalized permutohedra.



The above pictures show the standard permutohedron and the second hypersimplex in $\mathbb{R}^4$. Generalized permutohedra are precisely the polytopes obtained from the standard permutohedron by moving the vertices so that all the edge directions are preserved [**Pos09**].

Definition 2.6. A subset $J \subseteq \mathbb{Z}^E_{\geq 0}$ is M-*convex* if it is the set of all lattice points of an integral generalized permutohedron. A *matroid* on E is an M-convex subset of $\mathbb{Z}^E_{\geq 0}$ consisting of zero-one vectors. The vectors in a matroid J are called *bases* of J.

The notion of M-convex sets originates in discrete convex analysis [**Mur03**]. In [**Mur03**, Chapter 4], one can find several other equivalent characterizations of M-convex sets. For example, a subset $J \subseteq \mathbb{Z}^E_{\geq 0}$ is M-convex exactly when it satisfies the *symmetric basis exchange property*:

For any $\alpha, \beta \in J$ and $i \in E$ with $\alpha_i > \beta_i$, there is $j \in E$ with

$$\alpha_j < \beta_j \quad \textit{and} \quad \alpha - e_i + e_j \in J \quad \textit{and} \quad \beta - e_j + e_i \in J.$$

For background specific to matroids, see [**Oxl11**].

Definition 2.7. A function $h : 2^E \to \mathbb{Z}_{\geq 0}$ is a *polymatroid rank function* if it satisfies the following properties:

(1) *Normalization:* $h(\varnothing) = 0$.

(2) *Monotonicity:* $h(A) \leq h(B)$ for all $A \subseteq B \subseteq E$.

(3) *Submodularity:* $h(A \cup B) + h(A \cap B) \leq h(A) + h(B)$ for all $A, B \subseteq E$.

A polymatroid rank function h is a *matroid rank function* if $h(A) \leq |A|$ for all $A \subseteq E$.

We recall the standard bijection between polymatroid rank functions on E and nonempty M-convex subsets of $\mathbb{Z}^E_{\geq 0}$ from [**Mur03**, Chapter 4]. For $A \subseteq E$ and $\alpha \in \mathbb{Z}^E_{\geq 0}$, we set $\alpha_A := \sum_{i \in A} \alpha_i$.

(1) A polymatroid rank function h defines

$$J_h := \left\{ \alpha \in \mathbb{Z}^E_{\geq 0} \;\middle|\; \alpha_E = h(E) \text{ and } \alpha_A \leq h(A) \text{ for all } A \subseteq E \right\},$$

which is an M-convex subset of $\mathbb{Z}^E_{\geq 0}$.

(2) An M-convex subset J of $\mathbb{Z}^E_{\geq 0}$ defines

$$h_J : 2^E \longrightarrow \mathbb{Z}_{\geq 0}, \quad h_J(A) := \max \left\{ \beta_A \mid \beta \leq \alpha \text{ for some } \alpha \in J \right\},$$

which is a polymatroid rank function on E.

The constructions J_h and h_J are mutually inverse, providing a polymatroid generalization of the classical cryptomorphism between the matroid rank function axioms and the symmetric basis exchange property. A *polymatroid* $\mathscr{P}$ is a pair $(h = h_J, J = J_h)$, where h is the *rank function* of $\mathscr{P}$ and J is the *set of bases* of $\mathscr{P}$. A polymatroid $\mathscr{P}$ is a *matroid* if h is a matroid rank function, or equivalently if J consists of zero-one vectors. Throughout this text, we restrict attention to integral polymatroids and do not consider nonintegral ones. Accordingly, we use the terms *polymatroid* and M-*convex set* interchangeably.

Example 2.8 (Graphic matroids). For any finite connected graph G with the edge set E, consider the set of indicator vectors

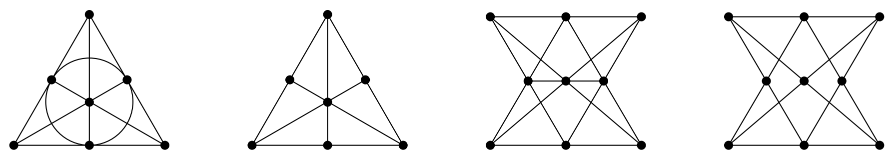
$$J(G) := \{e_B \mid B \text{ is a spanning tree of } G\} \subseteq \mathbb{Z}^E_{\geq 0}.$$

The subset $J(G)$ is M-convex for any such G. Such matroids are said to be *graphic*.

Example 2.9 (Linear matroids). For any function $\varphi : E \to W$ from a finite set E to a vector space W over a field k, consider the set of indicator vectors

$$J(\varphi) \coloneqq \{e_B \mid \varphi(B) \text{ is a basis of } W\} \subseteq \mathbb{Z}^E_{\geq 0}.$$

The subset $J(\varphi)$ is M-convex for any $\varphi : E \to W$. Such matroids are said to be *linear over* k, and the function φ is called a *linear realization over* k. One typically requires without loss of generality that the image of φ spans W. A graphic matroid is linearly realizable over every field [**Oxl11**, Section 5.1]. In general, a matroid may or may not have a linear realization over k:

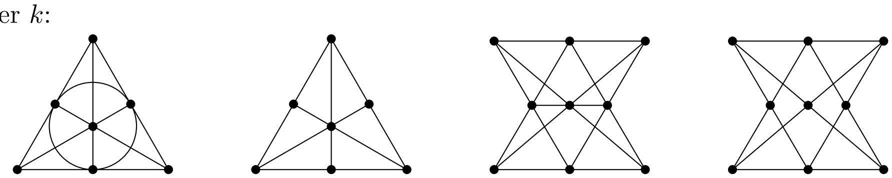


Among the four matroids pictured above, where the bases are given by all triples of points not on a line, the first is linear over k if and only if the characteristic of k is 2, the second is linear over k if and only if the characteristic of k is not 2, the third is linear over k if and only if the cardinality of k is not 2, 3, or 5, and the fourth is not linear over any field [**Oxl11**, Appendix].

Example 2.10 (Algebraic matroids). For any function $\varphi : E \to \ell$ from a finite set E to a field extension ℓ of k, consider the set of indicator vectors

$$J(\varphi) \coloneqq \{e_B \mid \varphi(B) \text{ is a transcendence basis of } \ell \text{ over } k\} \subseteq \mathbb{Z}^E_{\geq 0}.$$

The subset $J(\varphi)$ is M-convex for any $\varphi : E \to \ell$. Such matroids are said to be *algebraic over* k, and the function φ is called an *algebraic realization over* k. One typically requires without loss of generality that the image of φ contains a transcendence basis of ℓ over k. A linear matroid over k is algebraic over k [**Oxl11**, Section 6.7]. In general, a matroid may or may not have an algebraic realization over k:



Among the four matroids pictured above, where the bases are given by all triples of points not on a line, the first is algebraic over k if and only if the characteristic of k is 2, the second and the third are algebraic over any field, and the fourth is algebraic over k if and only if k has nonzero characteristic [**Oxl11**, Appendix].

Let $\mathbb{H}^d_n$ be the vector space of all homogeneous polynomials of degree d in n variables with real coefficients, and set

$$\mathbb{L}^1_n = \{\text{linear forms in } n \text{ variables with nonnegative coefficients}\}.$$

Let $\mathbb{L}^2_n \subseteq \mathbb{H}^2_n$ be the closed subset of quadratic forms with nonnegative coefficients whose Hessians have at most one positive eigenvalue. For d larger than 2, we define $\mathbb{L}^d_n \subseteq \mathbb{H}^d_n$ by setting

$$\mathbb{L}^d_n = \Big\{ f \in \mathbb{M}^d_n \mid \partial_i f \in \mathbb{L}^{d-1}_n \text{ for all } i = 1, \dots, n \Big\},$$

where $\mathbb{M}_n^d \subseteq \mathbb{H}_n^d$ is the set of polynomials with nonnegative coefficients whose supports are M-convex.[3] The following characterization in [**BH20**, Theorem 2.25] is central to the theory of Lorentzian polynomials.

THEOREM 2.11. $\mathbb{L}_n^d$ is the set of Lorentzian polynomials in $\mathbb{H}_n^d$.

Theorem 2.11 makes it possible to decide whether a given polynomial is Lorentzian or not. For example, the following polynomials are not Lorentzian because their supports are not M-convex:

$$x_1^3 + x_2^3, \quad x_1^2 x_3 + x_2^3.$$

Note that, in each case, all the partial derivatives $\partial_i f$ are Lorentzian.

One can also use Theorem 2.11 to show that a given polynomial is Lorentzian. For example, the elementary symmetric polynomial of degree d in n variables is Lorentzian because its support is M-convex and all its quadratic partial derivatives have the Hessian

$$\begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix},$$

which have exactly one positive eigenvalue $n - d + 1$. When $n = 2$, Theorem 2.11 specializes to the explicit description of bivariate Lorentzian polynomials given in Example 2.2.

3. Volume polynomials in projective geometry

3.1. The analogous volume polynomial in algebraic geometry is defined as follows: Let $D = (D_1, \ldots, D_n)$ be a collection of semiample divisors on a d-dimensional projective variety Y over a field k.[4] The *volume polynomial* of D is

$$f_D(x) \coloneqq \frac{1}{d!} \int_Y \left(\sum_{i=1}^n x_i D_i \right)^d,$$

which is a homogeneous polynomial of degree d in $x = (x_1, \ldots, x_n)$. When the base is the field of complex numbers and each D_i is ample, the restriction of f_D to the nonnegative orthant measures the volume of Y with respect to the Kähler class determined by x.

DEFINITION 3.1. Let k be a field.

(1) A homogeneous polynomial f is a *realizable volume polynomial over* k if $f = \lambda f_D$ for some $\lambda \in \mathbb{Q}_{\geq 0}$ and a collection of semiample divisors D on a projective variety Y over k.

(2) A homogeneous polynomial f is a *volume polynomial over* k if it is a limit of realizable volume polynomials over k.

---

[3]By definition, the *support* of a polynomial f is the set of all monomials appearing in f with nonzero coefficients.

[4]Throughout this paper, a variety over k is by definition a reduced and irreducible scheme of finite type over k. A Cartier divisor on a complete variety is *semiample* if some positive multiple moves in a basepoint-free linear system. For background and any undefined terms concerning divisors on varieties, we refer to [**Laz04**].

We write $\mathbb{V}_n^d(\mathbb{Q},k)$ for the set of realizable volume polynomials over k of degree d in n variables, and $\mathbb{V}_n^d(\mathbb{R},k)$ for the set of all volume polynomials over k of degree d in n variables.

Recall that a quadratic form is Lorentzian if and only if all its coefficients are nonnegative and its Hessian has at most one positive eigenvalue. It is easy to see that, for any n and any k, we have

$$\mathbb{V}_n^1(\mathbb{Q},k) = \big\{\textit{linear forms in } n \textit{ variables with nonnegative rational coefficients}\big\},$$
$$\mathbb{V}_n^1(\mathbb{R},k) = \big\{\textit{linear forms in } n \textit{ variables with nonnegative coefficients}\big\}.$$

By [**HHM**$^+$, Theorem 1.8], for any n and any k, we have

$$\mathbb{V}_n^2(\mathbb{Q},k) = \big\{\textit{Lorentzian quadratic forms in } n \textit{ variables with rational coefficients}\big\},$$
$$\mathbb{V}_n^2(\mathbb{R},k) = \big\{\textit{Lorentzian quadratic forms in } n \textit{ variables}\big\}.$$

Also, by [**Huh12**, Theorem 21], for any d and any k, we have

$$\mathbb{V}_2^d(\mathbb{Q},k) = \big\{\textit{Lorentzian bivariate forms of degree } d \textit{ with rational coefficients}\big\},$$
$$\mathbb{V}_2^d(\mathbb{R},k) = \Big\{\textit{Lorentzian bivariate forms of degree } d\Big\}.$$

In general, $\mathbb{V}_n^d(\mathbb{R},k)$ is preserved under a nonnegative linear change of coordinates, and the normalized coefficients p_α of its members satisfy the *Khovanskii–Teissier inequality*:

$$p_{\alpha+\mathrm{e}_i-\mathrm{e}_j}\,p_{\alpha-\mathrm{e}_i+\mathrm{e}_j} \le p_\alpha^2 \ \text{ for any } \alpha\in\Delta_n^d \text{ and any } 1\le i<j\le n.$$

As observed in [**BH20**, Section 4.2], it follows that

$$\big\{\text{volume polynomials over } k\big\} \subseteq \big\{\text{Lorentzian polynomials}\big\} \text{ for any } k.$$

The *realization problem* for volume polynomials over k is to identify the volume polynomials over k among Lorentzian polynomials. The distinction between the notions of realizable volume polynomials and volume polynomials will be relevant in applications to algebraic matroids in Section 5.

Example 3.2. A standard construction in toric geometry shows that any volume polynomial of rational convex polytopes arises as the volume polynomial of semiample divisors on projective varieties [**Ful93**, Section 5.4]. Thus, we have

$$\big\{\text{volume polynomials of } n \text{ rational polytopes in } \mathbb{R}^d\big\} \subseteq \mathbb{V}_n^d(\mathbb{Q},k) \text{ for any } k.$$

Since any convex body is a limit of a sequence of rational convex polytopes [**Sch14**, Section 1.8], we have

$$\big\{\text{volume polynomials of } n \text{ convex bodies in } \mathbb{R}^d\big\} \subseteq \mathbb{V}_n^d(\mathbb{R},k) \text{ for any } k.$$

As noted in Example 2.4, the elementary symmetric polynomial $x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4$ is not in the left-hand side. On the other hand, any Lorentzian quadratic form is a volume polynomial over k for any k, so the inclusion is strict.

Example 3.3. Let $P=(P_1,\dots,P_n)$ be a collection of $d\times d$ positive semidefinite Hermitian matrices. The *volume polynomial* of P is

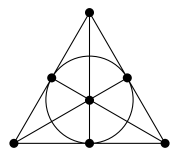
$$f_P(x) \coloneqq \det(x_1P_1+\cdots+x_nP_n),$$

which is a homogeneous polynomial of degree d in $x = (x_1, \dots, x_n)$. Using the abelian variety $\mathbb{C}^d/(\mathbb{Z}^d + \mathbb{Z}^d\sqrt{-1})$, one can show that

$$\big\{\text{volume polynomials of } n \text{ positive semidefinite } d \times d \text{ matrices}\big\} \subseteq \mathbb{V}_n^d(\mathbb{R}, \mathbb{C}).$$

Since volume polynomials of positive semidefinite Hermitian matrices are *stable* [**Wag11**, Proposition 2.1], the bivariate Lorentzian polynomial $x_1^3+6x_1^2x_2+6x_1x_2^2+2x_2^3$ is not in the left-hand side. On the other hand, any Lorentzian bivariate form is a volume polynomial over k for any k, so the inclusion is strict. For a detailed discussion of volume polynomials of positive semidefinite Hermitian matrices, see [**HMWX**].

Example 3.4. The *Fano matroid* F_7 is the rank 3 matroid on 7 elements whose bases are all the triples that are not colinear in the following configuration:



Its *basis generating polynomial* is the cubic polynomial in seven variables

$$b_{F_7}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \sum_{ijk \in F_7} x_i x_j x_k,$$

where the sum is over all the 28 bases of the Fano matroid. According to [**GHM**$^+$, Example 5.5], we have

$$b_{F_7} \in \mathbb{V}_7^3(\mathbb{Q}, k) \text{ if and only if } \operatorname{char}(k) = 2.$$

Is b_{F_7} a volume polynomial over k when the characteristic of k is not 2?

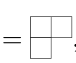
Example 3.5. If a matroid J of rank d on n elements is linear over k, then its basis generating polynomial f_J is a homogeneous polynomial of degree d in n variables. The *arrangement Schubert variety* of any linear realization of J over k witnesses the fact that f_J is a realizable volume polynomial over k [**BHM**$^+$, Section 1.3].

Example 3.6. The *Schur module* $\mathrm{V}(\lambda)$ of a Young diagram λ is the irreducible representation of the general linear group $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ. It has the weight space decomposition

$$\mathrm{V}(\lambda) = \bigoplus_{\mu} \mathrm{V}(\lambda)_\mu \quad \text{with} \quad \dim \mathrm{V}(\lambda)_\mu = K_{\lambda\mu},$$

where $K_{\lambda\mu}$ is the *Kostka number* counting semistandard Young tableaux of given shape λ and weight μ [**Ful97**, Section 8.3]. The *normalized Schur polynomial* is the generating polynomial

$$f_\lambda(x_1, \dots, x_n) \coloneqq \sum_{\mu} K_{\lambda\mu} x^{[\mu]}, \quad \text{where} \quad x^{[\mu]} \coloneqq \frac{x^\mu}{\mu!} = \frac{x_1^{\mu_1}}{\mu_1!} \cdots \frac{x_n^{\mu_n}}{\mu_n!},$$

For example, for the Young diagram $\lambda =$ , we have

$$f_\lambda(x_1, x_2, x_3) = \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_1^2x_3 + \frac{1}{2}x_1x_2^2 + \frac{1}{2}x_2^2x_3 + \frac{1}{2}x_1x_3^2 + \frac{1}{2}x_2x_3^2 + 2x_1x_2x_3.$$

The proof of [**HMMSD22**, Theorem 3] shows that any normalized Schur polynomial is a realizable volume polynomial over k for any k.

Example 3.7. The product of two realizable volume polynomials over k is a realizable volume polynomial over k: If $f_1(x)$ is the realizable volume polynomial obtained from a collection of semiample divisors D_1 on Y_1 and $f_2(x)$ is the realizable volume polynomial obtained from a collection of semiample divisors D_2 on Y_2, then $f_1(x)f_2(x)$ is the realizable volume polynomial obtained from the collection of semiample divisors $\pi_1^* D_1 + \pi_2^* D_2$ on $Y_1 \times Y_2$, where the addition is defined componentwise.

It is known that $\mathbb{V}_n^d(\mathbb{Q}, k)$, and hence $\mathbb{V}_n^d(\mathbb{R}, k)$, only depends on the characteristic of k [**GHM**$^+$, Proposition 2.10]. It is not known whether $\mathbb{V}_n^d(\mathbb{R}, k)$ depends on k when $d \geq 3$ and $n \geq 3$.

Conjecture 3.8. The set of volume polynomials over k is independent of the choice of k.

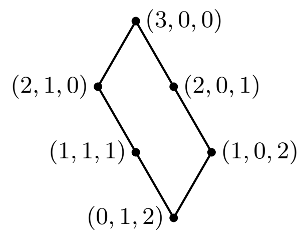
3.2. As observed in [**Huh23**, Example 14], we have the proper inclusion

$$\mathbb{V}_n^d(\mathbb{R}, k) \subsetneq \mathbb{L}_n^d \quad \text{for any field } k \text{ when } d \geq 3 \text{ and } n \geq 3.$$

Thus, the realization problem for volume polynomials over k has a nontrivial answer in these cases. For example, consider the cubic polynomial

$$f = 14x_1^3 + 6x_1^2x_2 + 24x_1^2x_3 + 12x_1x_2x_3 + 6x_1x_3^2 + 3x_2x_3^2.$$

The support of f is M-convex, as it is the set of all lattice points of the following integral generalized permutohedron:



The Hessians of the partial derivatives $\partial_1 f, \partial_2 f, \partial_3 f$ are

$$\begin{pmatrix} 84 & 12 & 48 \\ 12 & 0 & 12 \\ 48 & 12 & 12 \end{pmatrix}, \quad \begin{pmatrix} 12 & 0 & 12 \\ 0 & 0 & 0 \\ 12 & 0 & 6 \end{pmatrix}, \quad \begin{pmatrix} 48 & 12 & 12 \\ 12 & 0 & 6 \\ 12 & 6 & 0 \end{pmatrix},$$

each of which has exactly one positive eigenvalue. Then, by Theorem 2.11, f is a Lorentzian polynomial. The fact that f is not the volume polynomial over k follows from the *reverse Khovanskii–Teissier inequality* [**LX17**, Theorem 5.7]: For any nef divisors D_1, D_2, D_3 on a d-dimensional projective variety Y and any $e \leq d$,

$$\binom{d}{e} \, (D_2^e \cdot D_1^{d-e})_Y \; (D_1^e \cdot D_3^{d-e})_Y \geq (D_1^d)_Y \; (D_2^e \cdot D_3^{d-e})_Y.$$

The complex analytic proof of the inequality in [**LX17**] relies on the Calabi–Yau theorem [**Yau78**]. The algebraic proof of the inequality in [**JL23**] using Okounkov bodies works over any algebraically closed field. As mentioned before,

$$\{\text{volume polynomials of } n \text{ convex bodies in } \mathbb{R}^d\} \subseteq \mathbb{V}_n^d(\mathbb{R}, k) \text{ for any } k.$$

Since Lorentzian polynomials are strongly log-concave [**BH20**, Theorem 2.31], the Lorentzian cubic f provides a counterexample to Gurvits' conjecture that a strongly log-concave homogeneous polynomial in three variables with nonnegative coefficients is the volume polynomial of three convex bodies [**Gur09**, Conjecture 4.1].

In [**HMWX**], the authors introduce a new family of inequalities for volume polynomials that subsumes both the Khovanskii–Teissier and the reverse Khovanskii–Teissier inequalities as special cases.

3.3. A basic property of volume polynomials over k is that it is preserved under any nonnegative linear change of coordinates, as in the case of volume polynomials of convex bodies (Section 2). More precisely, for any $n \times m$ matrix A with nonnegative rational entries and sets of variables $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, we have the implication

$$f(x) \in \mathbb{V}_n^d(\mathbb{Q}, k) \implies f(Ay) \in \mathbb{V}_m^d(\mathbb{Q}, k).$$

It follows that, for any $n \times m$ matrix A with nonnegative real entries, we have

$$f(x) \in \mathbb{V}_n^d(\mathbb{R}, k) \implies f(Ay) \in \mathbb{V}_m^d(\mathbb{R}, k).$$

Here are two additional basic operations on the set of realizable volume polynomials over k, and hence on the set of volume polynomials over k. Let E be a finite set indexing the variables in a polynomial ring with real coefficients.

Definition 3.9. For a nonzero degree d homogeneous polynomial $f \in \mathbb{R}[x_i]_{i \in E}$ and an element $j \in E$, we write

$$f = \sum_{e=e_{\min}}^{e_{\max}} f_{d-e} \frac{x_j^e}{e!},$$

where f_{d-e} are polynomials in $\mathbb{R}[x_i]_{i \neq j}$ that are nonzero for $e = e_{\min}, e_{\max}$.

(1) The *deletion* of f by j is the degree d homogeneous polynomial

$$f \setminus j = \sum_{e=e_{\min}}^{e_{\max}-1} f_{d-e} \frac{x_j^e}{e!},$$

(2) The *contraction* of f by j is the degree $d-1$ homogeneous polynomial

$$f/j = \sum_{e=e_{\min}+1}^{e_{\max}} f_{d-e} \frac{x_j^{e-1}}{(e-1)!},$$

A *minor* of f is a polynomial obtained from f by a sequence of deletion and contraction operations.

When applied to the spanning tree polynomials of graphs and the basis generating polynomials of matroids, Definition 3.9 recovers the corresponding notions of contraction, deletion, and minor in the context of graph theory and matroid theory [**Oxl11**, Chapter 3].[5] For a discussion of minors in the more general framework of polymatroids over tracts, see [**BHK$^+$**, Section 2.2].

A special case of [**GHM$^+$**, Corollary 3.3] implies that deletions and contractions of realizable volume polynomials over k are realizable volume polynomials over k. For a systematic study of linear operators preserving the set of realizable volume polynomials over k, see Section 4.

[5] In [**Oxl11**, Section 1.2], a matroid is required to have at least one basis. This leads to minor differences in the definition of $f \setminus j$ and f/j when j is a loop (when j is contained in no basis of the matroid) or a coloop (when j is contained in every basis of the matroid).

PROPOSITION 3.10. Any minor of a realizable volume polynomial over k is a realizable volume polynomial over k.

It follows that any minor of a volume polynomial over k is a volume polynomial over k. This contrasts with the fact that the set of volume polynomials of convex bodies is not closed under minors.

EXAMPLE 3.11. Let C_1, C_2, C_3, C_4 be four equiangular unit segments in $\mathbb{R}^3$, and let C_5 be the unit ball in $\mathbb{R}^3$. The volume polynomial for $C = (C_1, C_2, C_3, C_4, C_5)$ is the cubic in five variables

$$f_C = \frac{4\pi}{3} x_5^3 + \pi \Bigg[\sum_{1 \le i \le 4} x_i \Bigg] x_5^2 + \frac{4\sqrt{2}}{3} \Bigg[\sum_{1 \le i < j \le 4} x_i x_j \Bigg] x_5 + \frac{4\sqrt{3}}{9} \Bigg[\sum_{1 \le i < j < k \le 4} x_i x_j x_k \Bigg].$$

Recall from Example 2.4 that the quadratic elementary symmetric polynomial in x_1, x_2, x_3, x_4 is not a volume polynomial of convex bodies. However, it is a minor of the volume polynomial f_C.

To what extent do the volume polynomials arising in algebraic geometry coincide with those arising in convex geometry? Proposition 3.10 shows that any minor of a volume polynomial of convex bodies is a volume polynomial over k for any k. The following strengthening of Conjecture 3.8 was suggested during a discussion with Shouda Wang.

CONJECTURE 3.12. Every volume polynomial over k is a limit of minors of volume polynomials of convex bodies.

3.4. A real $(1,1)$-class $[\omega]$ on a compact Kähler manifold Y is *semipositive* if it contains a smooth semipositive representative, that is, if there is a smooth function φ on Y such that

$$\omega + i\partial\overline{\partial}\varphi \ge 0.$$

DEFINITION 3.13. A degree d homogeneous polynomial f in n variables is a *realizable analytic volume polynomial* if there is a d-dimensional compact Kähler manifold Y and semipositive classes $[\omega_1], \ldots, [\omega_n]$ such that

$$f(x_1, \ldots, x_n) = \frac{1}{d!} \int_Y (x_1\omega_1 + \cdots + x_n\omega_n)^{\wedge d}.$$

A homogeneous polynomial f is an *analytic volume polynomial* if it is a limit of realizable analytic volume polynomials.

By [**Gro90**], the $(1,1)$-part of the cohomology of Y satisfies the *mixed Hodge–Riemann relations*, and hence

$$\big\{\text{analytic volume polynomials}\big\} \subseteq \big\{\text{Lorentzian polynomials}\big\}.$$

It follows that the support of an analytic volume polynomial is M-convex, defining the class of *analytic polymatroids* [**GHM**$^+$, Section 5]. On the other hand, the resolution of singularities for complex projective varieties implies that any realizable volume polynomial over $\mathbb{C}$ is a realizable analytic volume polynomial, and hence

$$\big\{\text{volume polynomials over } \mathbb{C}\big\} \subseteq \big\{\text{analytic volume polynomials}\big\}.$$

The answers to the following basic questions regarding analytic volume polynomials remain unknown.

QUESTION 3.14. Is there an analytic volume polynomial that is not a volume polynomial over $\mathbb{C}$?

QUESTION 3.15. Is the class of analytic volume polynomials closed under taking minors?

4. Linear operators preserving volume polynomials

4.1. The set of volume polynomials over k is, in a precise sense, dual to the set of *covolume polynomials* over k. To define covolume polynomials and state their main properties, it will be convenient to work with the dual pair of polynomial rings

$$\mathbb{R}[\partial] = \mathbb{R}[\partial_i]_{i \in E} \quad \text{and} \quad \mathbb{R}[x] = \mathbb{R}[x_i]_{i \in E}.$$

We write $\mathbb{Z}^E_{\geq 0}$ for the set of exponent vectors of the monomials in the two polynomial rings, and set

$$\partial^\alpha := \prod_{i \in E} \partial_i^{\alpha_i} \quad \text{and} \quad x^{[\alpha]} := \prod_{i \in E} \frac{x_i^{\alpha_i}}{\alpha_i!} \quad \text{for } \alpha \in \mathbb{Z}^E_{\geq 0}.$$

The polynomial ring $\mathbb{R}[\partial]$ acts on $\mathbb{R}[x]$ as differential operators by the usual rule

$$\partial^\alpha \circ x^{[\beta]} := \begin{cases} x^{[\beta-\alpha]} & \text{if } \alpha \leq \beta, \\ 0 & \text{if otherwise,} \end{cases}$$

where $\alpha \leq \beta$ means that their components satisfy $\alpha_i \leq \beta_i$ for all $i \in E$. For any further conventions for multivariate polynomials, we refer to [**BH20**, Section 2]. For $\mu \in \mathbb{Z}^n_{\geq 0}$, we consider

$$\mathbb{R}[\partial]_{\leq \mu} := \text{span}(\partial^\alpha)_{\alpha \leq \mu} \quad \text{and} \quad \mathbb{R}[x]_{\leq \mu} := \text{span}(x^\alpha)_{\alpha \leq \mu}.$$

Then $\mathbb{R}[x]_{\leq \mu}$ is an $\mathbb{R}[\partial]$-submodule of $\mathbb{R}[x]$ generated by $x^{[\mu]}$, and the linear map

$$\mathbb{R}[\partial]_{\leq \mu} \longrightarrow \mathbb{R}[x]_{\leq \mu}, \qquad \partial^\alpha \longmapsto \partial^\alpha \circ x^{[\mu]} = x^{[\mu-\alpha]}$$

is an isomorphism of finite-dimensional vector spaces.

DEFINITION 4.1. Let g be a homogeneous polynomial in $\mathbb{R}[\partial]_{\leq \mu}$.

(1) We say that g is a *realizable covolume polynomial over* k if $g(\partial) \circ x^{[\mu]}$ is a realizable volume polynomial over k.

(2) We say that g is a *covolume polynomial over* k if it is a limit of realizable covolume polynomials over k.

As observed in [**Alu24**, Remark 2.2], the property of being a realizable covolume polynomial over k does not depend on the choice of μ. This follows from the translation invariance

$$\Big(\sum_\alpha c_\alpha x^{[\alpha]} \text{ is a realizable volume polynomial over } k\Big) \Longleftrightarrow$$
$$\Big(\sum_\alpha c_\alpha x^{[\alpha+\beta]} \text{ is a realizable volume polynomial over } k\Big), \text{ for any } \beta \in \mathbb{Z}^E_{\geq 0}.$$

For the cone construction that justifies this, see [**GHM$^+$**, Section 2].

REMARK 4.2. According to [**BHK**$^+$, Section 2.1], the *dual* of an M-convex set $J \subseteq \mathbb{Z}^E_{\geq 0}$, defined up to translation in $\mathbb{Z}^E_{\geq 0}$, is the M-convex subset

$$\mu - J \coloneqq \{\mu - \alpha \mid \alpha \in J\} \subseteq \mathbb{Z}^E_{\geq 0},$$

where μ is any nonnegative integral vector satisfying $\alpha \leq \mu$ for all $\alpha \in J$.[6] Since the support of a covolume polynomial over k is the dual of the support of a volume polynomial over k, the support of a covolume polynomial is an M-convex set.

REMARK 4.3. The class of (realizable) volume polynomials over k is closed under nonnegative (rational) linear changes of coordinates, as well as under taking products and minors. Similarly, the class of (realizable) covolume polynomials over k is closed under nonnegative (rational) linear changes of coordinates [**GHM**$^+$, Theorem 2.7], as well as taking products and minors [**GHM**$^+$, Theorem 1.5]. It is interesting to note that the corresponding statements for volume polynomials and covolume polynomials sometimes have substantially different proofs.

EXAMPLE 4.4. Let f_J be the basis generating polynomial of a matroid J linear over k. By Example 3.5, f_J is a realizable volume polynomial over k. Since the dual of a linear matroid over k is linear over the same field [**Oxl11**, Corollary 2.2.9], f_J is a realizable covolume polynomial over k as well.

EXAMPLE 4.5. The proof of [**HMMSD22**, Theorem 6] shows that the *Schubert polynomial* $s_w(\partial)$ is a realizable covolume polynomial over k for any permutation w and any k. In particular, any *Schur polynomial* is a realizable covolume polynomial over k for any field k. By Example 3.6, any *normalized Schur polynomial* is a realizable volume polynomial over k for any k. It is not known whether *normalized Schubert polynomials* are realizable volume polynomials over k for any k. See [**HMMSD22**, Conjecture 15] for a weaker statement.

EXAMPLE 4.6. The convex polytope in Example 1.8 shows that, for any k,

$$x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + x_1x_3x_4 + x_1x_3x_5 \\ + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 + x_2x_4x_5 + 4x_3x_4x_5$$

is a realizable volume polynomial over k. It follows that, for any k,

$$\partial_4\partial_5 + \partial_3\partial_5 + \partial_3\partial_4 + \partial_2\partial_5 + \partial_2\partial_4 + \partial_2\partial_3 + \partial_1\partial_5 + \partial_1\partial_4 + \partial_1\partial_3 + 4\partial_1\partial_2$$

is a realizable covolume polynomial over k. This covolume polynomial over k is not a Lorentzian polynomial, and hence it is not a volume polynomial over k for any k.

Conjecture 1.11 suggests that a quadratic polynomial $\sum_{1\leq i<j\leq n} q_{ij}\partial_i\partial_j$ is a covolume polynomial over k if and only if its coefficients are nonnegative and satisfy the *Plücker relations* over the triangular hyperfield $\mathbb{T}_2$:

$$\sqrt{q_{ij}q_{kl}} \leq \sqrt{q_{ik}q_{jl}} + \sqrt{q_{il}q_{jk}} \quad \text{for any } 1 \leq i < j < k < l \leq n.$$

For general discussions of Grassmannians over triangular hyperfields, see [**BHKLa, BHKLb**].

[6] A standard choice used in [**BHK**$^+$, Section 2.1] is to take μ to be the *duality vector* $\delta_J = \delta_J^+ + \delta_J^-$, where $\delta_J^+ = \sup J$ and $\delta_J^- = \inf J$.

4.2. The main result of [**GHM**$^+$] is the following characterization of realizable covolume polynomials. This parallels the characterization of dually Lorentzian polynomials in [**RSW25**, Theorem 1.2].

THEOREM 4.7. The following conditions are equivalent for any $g \in \mathbb{Q}[\partial]$.

(1) The polynomial g is a realizable covolume polynomial over k.

(2) For any realizable volume polynomial f over k, the polynomial $g(\partial) \circ f(x)$ is a realizable volume polynomial over k.

The corresponding characterization of covolume polynomials is obtained by taking limits.

COROLLARY 4.8. The following conditions are equivalent for any $g \in \mathbb{R}[\partial]$.

(1) The polynomial g is a covolume polynomial over k.

(2) For any volume polynomial f over k, the polynomial $g(\partial) \circ f(x)$ is a volume polynomial over k.

Example 3.11 shows that the set of volume polynomials of convex bodies does not satisfy the analogous statement.[7] For a parallel statement characterizing volume polynomials as linear operators preserving covolume polynomials, see [**GHM**$^+$, Theorem 1.9].

Corollary 4.8 can be used to deduce new inequalities for mixed volumes of convex bodies, or more generally, for intersection numbers of nef divisors on a projective variety. For instance, given a Schubert polynomial $s_w(\partial)$ and a volume polynomial $f_C(x)$, any known inequality for the coefficients of a volume polynomial can be applied to $s_w(\partial) \circ f_C(x)$ to produce another inequality for the coefficients of $f_C(x)$. For an overview of known inequalities for the coefficients of the volume polynomial, such as the Khovanskii–Teissier inequality or the reverse Khovanskii–Teissier inequality, see [**HMWX**].

REMARK 4.9. One can define *(realizable) analytic covolume polynomials* as the duals of (realizable) analytic volume polynomials as in Definition 4.1. Do they satisfy the analogues of Theorem 4.7 and Corollary 4.8?

4.3. The *symbol theorem* for Lorentzian polynomials states that, if the symbol of a linear operator T is a Lorentzian polynomial, then T sends Lorentzian polynomials to Lorentzian polynomials [**BH20**, Theorem 3.2]. Theorem 4.7 can be used to derive a volume polynomial analogue of the symbol theorem for Lorentzian polynomials.[8]

Let $x = (x_i)_{i \in E}$ and $y = (y_j)_{j \in F}$ be two finite sets of variables. Let T be a homogeneous linear operator[9]

$$T : \mathbb{R}[x]_{\leq \mu} \longrightarrow \mathbb{R}[y]_{\leq \nu}, \quad \text{where } \mu \in \mathbb{Z}^E_{\geq 0} \text{ and } \nu \in \mathbb{Z}^F_{\geq 0}.$$

The *symbol* of T is the homogeneous polynomial in variables (x, y) given by

$$\mathrm{sym}_T(x, y) = \sum_{0 \leq \alpha \leq \mu} T(x^{[\alpha]}) x^{[\mu - \alpha]}.$$

[7] For example, $\partial_5 f_C$ is not a volume polynomial of convex bodies.

[8] The study of symbols of linear operators dates back to Gårding [**Gar51**] and appears prominently in the work of Borcea and Brändén on the Pólya–Schur program for stable polynomials [**BB09a, BB09b**].

[9] This means that T is linear over $\mathbb{R}$ and deg $T(x^\alpha)$ − deg $x^\alpha \in \mathbb{Z}$ does not depend on $\alpha \leq \mu$.

THEOREM 4.10. If the symbol of T is a realizable volume polynomial over k, then T sends realizable volume polynomials over k to realizable volume polynomials over k.

COROLLARY 4.11. If the symbol of T is a volume polynomial over k, then T sends volume polynomials over k to volume polynomials over k.

Geometrically, one may view T as a graded linear map between Chow groups

$$\varphi_T : \mathrm{CH}(\mathbb{P}^\mu) \otimes \mathbb{R} \to \mathrm{CH}(\mathbb{P}^\nu) \otimes \mathbb{R}.$$

If this map is induced by an irreducible correspondence $\Gamma \subseteq \mathbb{P}^\mu \times \mathbb{P}^\nu$ so that

$$\varphi_T(\Lambda) = p_{2*}\big(\Gamma \cap p_1^*(\Lambda)\big),$$

then, by [**GHM**$^+$, Lemma 2.1], it preserves the classes of irreducible cycles up to a rational multiple.

The symbol theorem for realizable volume polynomials shows that many familiar operators from the theory of Lorentzian polynomials preserve realizable volume polynomials over k for any k:

(1) The *upper truncation operators* and the *lower truncation operators* preserve realizable volume polynomials over k [**GHM**$^+$, Corollary 3.3].

(2) The *polarization operator* $\Pi^\uparrow$ preserves realizable volume polynomials over k [**GHM**$^+$, Proposition 4.1].

(3) The *normalization operator* N preserves realizable volume polynomials over k [**GHM**$^+$, Proposition 4.2].

(4) For any nonnegative rational number t, the *interlacing operator* $1 + tx_i\partial_j$ preserves realizable volume polynomials over k [**GHM**$^+$, Proposition 4.3].

(5) For any nonnegative rational number t, the *symmetric exclusion process* $\Phi_t^{i,j}$ preserves realizable multiaffine volume polynomials over k [**GHM**$^+$, Proposition 4.4].

The corresponding statements for volume polynomials over k follow from taking limits.

5. Realization problems for polymatroids

Recall that the *support* of a polynomial f in $\mathbb{R}[x_i]_{i\in E}$ is the set of all exponent vectors $\alpha \in \mathbb{Z}^E_{\geq 0}$ such that the monomial x^α appears in f with nonzero coefficient.

QUESTION 5.1. If f is a volume polynomial over k (Definition 3.1), then the support of f is the set of bases of a polymatroid. Which polymatroids arise in this way?

QUESTION 5.2. If f is a covolume polynomial over k (Definition 4.1), then the support of f is the set of bases of a polymatroid. Which polymatroids arise in this way?

QUESTION 5.3. If f is an analytic volume polynomial (Definition 3.13), then the support of f is the set of bases of a polymatroid. Which polymatroids arise in this way?

QUESTION 5.4. If f is an analytic covolume polynomial (Remark 4.9), then the support of f is the set of bases of a polymatroid. Which polymatroids arise in this way?

At present, the author is not aware of any obstructions for any of the above cases.

The following connection between algebraic matroids and the support of *realizable* volume polynomials is known [**GHM$^+$**, Proposition 5.4]. A polymatroid on E is *algebraic over k* if there are field extensions $k \subseteq \ell_i \subseteq \ell$ for $i \in E$ such that

$$h(A) = \mathrm{trdeg}_k\Big(\vee_{i \in A} \, \ell_i \Big) \ \text{ for all } A \subseteq E,$$

where h is the rank function of the polymatroid.

PROPOSITION 5.5. A polymatroid is algebraic over k if and only if it is the support of a realizable volume polynomial over k.

Proposition 5.5 implies that every minor of an algebraic polymatroid over k is an algebraic polymatroid over k [**GHM$^+$**, Section 5].[10] In the classical case of matroids, this statement is typically deduced from a theorem of Lindström [**Lin89**], who proved Piff's conjecture that M is algebraic over k if M is algebraic over an extension of k, see [**Oxl11**, Corollary 6.7.14]. Since the set of realizable volume polynomials over k depends only on the characteristic of k [**GHM$^+$**, Proposition 2.10], Proposition 5.5 gives the following version of Lindström's theorem for polymatroids.

COROLLARY 5.6. A polymatroid is algebraic over some field of characteristic p if and only if it is algebraic over all fields of characteristic p.

Another consequence of Proposition 5.5 is that the intersection $M_1 \wedge M_2$ of algebraic matroids over k is an algebraic matroid over k [**GHM$^+$**, Theorem 5.11]. This generalizes Piff's theorem that the truncation of an algebraic matroid is algebraic [**Wel76**, Section 11.3].[11]

Is the support of a realizable covolume polynomial over k an algebraic polymatroid over k? This question extends the following long-standing open problem in matroid theory. For up-to-date discussions, see [**BFP25, Hoc**].

QUESTION 5.7. Is the dual of an algebraic matroid over k algebraic over k?

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[10] Definition 3.9, applied to the basis generating polynomials of polymatroids, define the notion of *minor* of polymatroids [**BHK$^+$**, Section 2.1].

[11] In [**Wel76**, Section 11.3], Welsh proves the dual statement that the union $M_1 \vee M_2$ of algebraic matroids over k is an algebraic matroid over k. It is interesting to note that, as in Remark 4.3, the proof of the statement for $M_1 \wedge M_2$ is substantially different from that for $M_1 \vee M_2$.

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