Tropical geometry of matroids

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Abstract. Lecture notes for Current Developments in Mathematics 2016, based on joint work with Karim Adiprasito and Eric Katz. We give a gentle introduction to the main result of [AHK], the Hodge-Riemann relations for matroids, and provide a detailed description of the geometry behind from a tropical point of view.

Contents

1. PD, HL, HR

Let X be a mathematical object of "dimension" d. Often it is possible to construct from X in a natural way a graded vector space over the real numbers

$$A^*(X) = \bigoplus_{q=0}^{d} A^q(X),$$

equipped with a graded bilinear pairing $\mathrm{P} = \mathrm{P}(X)$ and a graded linear map $\mathrm{L} = \mathrm{L}(X)$:

$$\mathrm{P} : A^*(X) \times A^{d-*}(X) \longrightarrow \mathbb{R}, \qquad \mathrm{L} : A^*(X) \longrightarrow A^{*+1}(X).$$

The linear operator L usually comes in as a member of a family $\mathrm{K}(X)$, a convex cone in the space of linear operators. Here "P" is for Poincaré, "L" is for Lefschetz, and "K" is for Kähler. For example, $A^*(X)$ may be

(1) the (q, q)-part of the intersection cohomology of a complex projective variety, or
(2) the cohomology of real (q, q)-forms on a compact Kähler manifold, or
(3) algebraic cycles modulo homological equivalence on a smooth projective variety, or
(4) the combinatorial intersection cohomology of a convex polytope [**Kar04**], or
(5) the Soergel bimodule attached to an element of a Coxeter group [**EW14**], or
(6) the Chow ring of a matroid, defined below.

When X is "sufficiently smooth", $A^*(X)$ is a graded algebra, and the maps P and L respect the multiplicative structure of $A^*(X)$. In any case, we expect the following properties from the triple $\big(A^*(X), \mathrm{P}(X), \mathrm{K}(X)\big)$:

(PD) For every nonnegative integer $q \leq \frac{d}{2}$, the bilinear pairing

$$\mathrm{P} : A^q(X) \times A^{d-q}(X) \longrightarrow \mathbb{R}$$

is nondegenerate (Poincaré duality for X).

(HL) For every nonnegative integer $q \leq \frac{d}{2}$ and every $\mathrm{L} \in \mathrm{K}(X)$, the composition

$$\mathrm{L}^{d-2q} : A^q(X) \longrightarrow A^{d-q}(X)$$

is bijective (the hard Lefschetz theorem for X).

(HR) For every nonnegative integer $q \leq \frac{d}{2}$ and every $\mathrm{L} \in \mathrm{K}(X)$, the bilinear form

$$A^q(X) \times A^q(X) \longrightarrow \mathbb{R}, \qquad (x_1, x_2) \longmapsto (-1)^q \ \mathrm{P}(x_1, \mathrm{L}^{d-2q} x_2)$$

is symmetric, and is positive definite on the kernel of

$$\mathrm{L}^{d-2q+1} : A^q(X) \longrightarrow A^{d-q+1}(X)$$

(the Hodge-Riemann relations for X).

All three properties are known to hold for the objects listed above except one, which is the subject of Grothendieck's standard conjectures on algebraic cycles. The known proofs of the hard Lefschetz theorem and the Hodge-Riemann relations for different types of objects have certain structural similarities, but there is no known way of deducing one from the others. Below we describe the triple $\big(A^*(X), \mathrm{P}(X), \mathrm{K}(X)\big)$ for some of the objects mentioned above.

1.1. Polytopes. A *polytope* in $\mathbb{R}^d$ is the convex hull of a finite subset of $\mathbb{R}^d$. Let's write Π for the abelian group with generators $[P]$, one for each polytope $P \subseteq \mathbb{R}^d$, which satisfy the following relations:

(1) $[P_1 \cup P_2] + [P_1 \cap P_2] = [P_1] + [P_2]$ whenever $P_1 \cup P_2$ is a polytope,

(2) $[P+t]=[P]$ for every point t in $\mathbb{R}^d$, and
(3) $[\varnothing]=0$.

This is the *polytope algebra* of McMullen [**McM89**]. The multiplication in Π is defined by the Minkowski sum

$$[P_1]\cdot[P_2]=[P_1+P_2],$$

and this makes Π a commutative ring with $1=[\text{point}]$ and $0=[\varnothing]$.

The structure of Π can be glimpsed through some familiar translation invariant measures on the set of polytopes. For example, the Euler characteristic shows that there is a surjective ring homomorphism

$$\chi:\Pi\longrightarrow\mathbb{Z},\qquad [P]\longmapsto\chi(P),$$

and the Lebesgue measure on $\mathbb{R}^n$ shows that there is a surjective group homomorphism

$$\mathrm{Vol}:\Pi\longrightarrow\mathbb{R},\qquad [P]\longmapsto\mathrm{Vol}(P).$$

A fundamental observation is that some power of $[P]-1$ is zero in Π for every nonempty polytope P. Since every polytope can be triangulated, it is enough to check this when the polytope is a simplex. In this case, a picture drawing for $d=0,1,2$, and if necessary 3, will convince the reader that

$$([P]-1)^{d+1}=0.$$

The kernel of the Euler characteristic χ turns out to be torsion free and divisible. Thus we may speak about the logarithm of a polytope in Π, which satisfies the usual rule

$$\log[P_1+P_2]=\log[P_1]+\log[P_2].$$

The notion of logarithm leads to a remarkable identity concerning volumes of convex polytopes.

THEOREM 1.1. *Writing $\mathtt{p}$ for the logarithm of $[P]$, we have*

$$Vol(P)=\frac{1}{d!}\,Vol(\mathtt{p}^d).$$

This shows that, more generally, Minkowski's mixed volume of polytopes $P_1,\ldots,P_d$ can be expressed in terms of the product of the corresponding logarithms $\mathtt{p}_1,\ldots,\mathtt{p}_d$:

$$\mathrm{Vol}(P_1,\ldots,P_d)=\mathrm{Vol}(\mathtt{p}_1\cdots\mathtt{p}_d).$$

Let's write $P_1\preceq P_2$ to mean that P_1 is a Minkowski summand of some positive multiple of P_2. This relation is clearly transitive. We way that P_1 and P_2 are *equivalent* when

$$P_1\preceq P_2\preceq P_1.$$

Let $\mathrm{K}(P)$ be the set of all polytopes equivalent to a given polytope P. The collection $\mathrm{K}(P)$ is a convex cone in the sense that

$$P_1,P_2\in\mathrm{K}(P)\Longrightarrow\lambda_1P_1+\lambda_2P_2\in\mathrm{K}(P)\ \text{ for positive real numbers } \lambda_1,\lambda_2.$$

We will meet an analogue of this convex cone in each of the following sections.

DEFINITION 1.2. For each positive integer q, let $\Pi^q(P) \subseteq \Pi$ be the subgroup generated by all elements of the form

$$\mathtt{p}_1\mathtt{p}_2\cdots\mathtt{p}_q,$$

where $\mathtt{p}_i$ is the logarithm of a polytope in $\mathrm{K}(P)$.

Note that any two equivalent polytopes define the same set of subgroups of Π. These subgroups are related to each other in a surprising way when P is an d-dimensional *simple* polytope; this means that every vertex of the polytope is contained in exactly d edges.

THEOREM 1.3. [**McM93**] *Let $\mathtt{p}$ be the logarithm of a simple polytope in* $\mathrm{K}(P)$, *and let* $1 \leq q \leq \frac{d}{2}$.

(PD) The multiplication in Π *defines a nondegenerate bilinear pairing*

$$\Pi^q(P) \times \Pi^{d-q}(P) \longrightarrow \mathbb{R}, \qquad (x,y) \longmapsto \mathit{Vol}(xy).$$

(HL) The multiplication by $\mathtt{p}^{d-2q}$ defines an isomorphism of abelian groups

$$\Pi^q(P) \longrightarrow \Pi^{d-q}(P), \quad x \longmapsto \mathtt{p}^{d-2q}x.$$

(HR) The multiplication by $\mathtt{p}^{d-2q}$ defines a symmetric bilinear form

$$\Pi^q(P) \times \Pi^q(P) \longrightarrow \mathbb{R}, \quad (x_1,x_2) \longmapsto (-1)^q \ \mathit{Vol}\big(\mathtt{p}^{d-2q}x_1x_2\big)$$

that is positive definite on the kernel of

$$\mathtt{p}^{d-2q+1} : \Pi^q(P) \longrightarrow \Pi^{d-q+1}(P).$$

In fact, the group $\Pi^q(P)$ can be equipped with the structure of a finite dimensional real vector space in a certain natural way. The $\mathbb{Z}$-linear and $\mathbb{Z}$-bilinear maps in the above statement turns out to be $\mathbb{R}$-linear and $\mathbb{R}$-bilinear.

Here are two concrete implications of (HL) and (HR) for P.

(1) The hard Lefschetz theorem is the main ingredient in the proof of the g-conjecture for simple polytopes [**Sta80**]. This gives a numerical characterization of sequences of the form

$$f_0(P), f_1(P), \ldots, f_d(P),$$

where $f_i(P)$ is the number of i-dimensional faces of an d-dimensional simple polytope P.

(2) The Hodge-Riemann relations, in the special case $q = 1$, is essentially equivalent to the Aleksandrov-Fenchel inequality on mixed volumes of convex bodies:

$$\mathrm{Vol}(\mathtt{p}_1\mathtt{p}_1\mathtt{p}_3\cdots\mathtt{p}_d)\mathrm{Vol}(\mathtt{p}_2\mathtt{p}_2\mathtt{p}_3\cdots\mathtt{p}_d) \leq \mathrm{Vol}(\mathtt{p}_1\mathtt{p}_2\mathtt{p}_3\cdots\mathtt{p}_d)^2.$$

The inequality played a central role in the proof of the van der Waerden conjecture that the permanent of any doubly stochastic $d\times d$ nonnegative matrix is at least $d!/d^d$. An interesting account on the formulation and the solution of the conjecture can be found in [**Lin82**].

With suitable modifications, the hard Lefschetz theorem and the Hodge-Riemann relations can be extended to arbitrary polytopes [**Kar04**].

1.2. Kähler manifolds. Let ω be a *Kähler form* on an d-dimensional compact complex manifold M. This means that ω is a smooth differential 2-form on M that can be written locally in coordinate charts as

$$i\partial\overline{\partial}f$$

for some smooth real functions f whose complex Hessian matrix $\left[\frac{\partial^2 f}{\partial z_i \partial \overline{z}_j}\right]$ is positive definite; here $z_1, \ldots, z_d$ are holomorphic coordinates and ∂, $\overline{\partial}$ are the differential operators

$$\partial = \sum_{k=1}^{d} \frac{\partial}{\partial z_k} dz_k, \qquad \overline{\partial} = \sum_{k=1}^{d} \frac{\partial}{\partial \overline{z}_k} d\overline{z}_k.$$

Like all other good definitions, the Kähler condition has many other equivalent characterizations, and we have chosen the one that emphasizes the analogy with the notion of convexity.

To a Kähler form ω on M, we can associate a Riemannian metric g on M by setting

$$g(u,v) = w(u, Iv),$$

where I is the operator on tangent vectors of M that corresponds to the multiplication by i. Thus we may speak of the length, area, etc., on M with respect to ω.

THEOREM 1.4. *The volume of M is given by the integral*

$$Vol(M) = \frac{1}{d!} \int_M w^d.$$

More generally, the volume of a k-dimensional complex submanifold $N \subseteq M$ is given by

$$Vol(N) = \frac{1}{k!} \int_N w^k.$$

Compare the corresponding statement of the previous section that $\mathrm{Vol}(P) = \frac{1}{d!}\mathrm{Vol}(\mathtt{p}^d)$.

Let $\mathrm{K}(M)$ be the set of all Kähler forms on M. The collection $\mathrm{K}(M)$ is a convex cone in the sense that

$$\omega_1, \omega_2 \in \mathrm{K}(M) \Longrightarrow \lambda_1\omega_1 + \lambda_2\omega_2 \in \mathrm{K}(M) \ \text{ for positive real numbers } \lambda_1, \lambda_2.$$

This follows from the fact that the sum of two positive definite matrices is positive definite.

DEFINITION 1.5. For each nonnegative integer q, let $H^{q,q}(M) \subseteq H^{2q}(M,\mathbb{C})$ be the subset of all the cohomology classes of closed differential forms that can be written in local coordinate charts as

$$\sum f_{k_1,\dots,k_q,l_1,\dots,l_q} dz_{k_1} \wedge \cdots \wedge dz_{k_q} \wedge d\overline{z}_{l_1} \wedge \cdots \wedge d\overline{z}_{l_q}.$$

Note that the cohomology class of a Kähler form ω is in $H^{1,1}(M)$, and that

$$[\varphi] \in H^{q,q}(M) \Longrightarrow [\omega \wedge \varphi] \in H^{q+1,q+1}(M).$$

THEOREM 1.6 (Classical). *Let ω be an element of $\mathrm{K}(M)$, and let q be a nonnegative integer $\leq \frac{d}{2}$.*

(PD) The wedge product of differential forms defines a nondegenerate bilinear form

$$H^{q,q}(M) \times H^{d-q,d-q}(M) \longrightarrow \mathbb{C}.$$

(HL) The wedge product with ω^{d-2q} defines an isomorphism

$$H^{q,q}(M) \longrightarrow H^{d-q,d-q}(M), \quad [\varphi] \longmapsto [\omega^{d-2q} \wedge \varphi].$$

(HR) The wedge product with ω^{d-2q} defines a Hermitian form

$$H^{q,q}(M) \times H^{q,q}(M) \longrightarrow \mathbb{C}, \quad (\varphi_1, \varphi_2) \longmapsto (-1)^q \int_M \omega^{d-2q} \wedge \varphi_1 \wedge \overline{\varphi_2}$$

that is positive definite on the kernel of

$$\omega^{d-2q+1} : H^{q,q}(M) \longrightarrow H^{d-q+1,d-q+1}(M).$$

Analogous statements hold for $H^{q_1,q_2}(M)$ with $q_1 \neq q_2$, and these provide a way to show that certain compact complex manifolds cannot admit any Kähler form. For deeper applications, see [**Voi10**].

1.3. Projective varieties. Let k be an algebraically closed field, and let $\mathbb{P}^m$ be the m-dimensional projective space over k. A *projective variety* over k is a subset of the form

$$X = \{h_1 = h_2 = \ldots = h_l = 0\} \subseteq \mathbb{P}^m,$$

where h_i are homogeneous polynomials in $m+1$ variables. We can define the dimension, connectedness, and smoothness of projective varieties in a way that is compatible with our intuition when $k = \mathbb{C}$. We can also define what it means for a map between two projective varieties, each living in two possibly different ambient projective spaces, to be algebraic.

Let K be another field, not necessarily algebraically closed but of characteristic zero. A *Weil cohomology theory* with coefficients in K is an assignment

$$X \longmapsto H^*(X) = \bigoplus_k H^k(X),$$

where X is a smooth and connected projective variety over k and $H^*(X)$ is a graded-commutative algebra over K. This assignment is required to satisfy certain rules similar to those satisfied by the singular cohomology of compact complex manifolds, such as functoriality, finite dimensionality, Poincaré duality, Künneth formula, etc. For this reason the product of two elements in $H^*(X)$ will be written

$$\xi_1 \cup \xi_2 \in H^*(X).$$

One of the important rules says that every codimension q subvariety $Y \subseteq X$ defines a cohomology class

$$\mathrm{cl}(Y) \in H^{2q}(X).$$

These classes should have the property that, for example,

$$\mathrm{cl}(Y_1 \cap Y_2) = \mathrm{cl}(Y_1) \cup \mathrm{cl}(Y_2)$$

whenever Y_1 and Y_2 are subvarieties intersecting transversely, and that

$$\mathrm{cl}(H_1) = \mathrm{cl}(H_2)$$

whenever H_1 and H_2 are two hyperplane sections of $X \subseteq \mathbb{P}^m$. Though not easy, it is possible to construct a Weil cohomology theory for any k for some K. For example, when both k and K are the field of complex numbers, we can take the de Rham cohomology of smooth differential forms.

DEFINITION 1.7. For each nonnegative integer q, let $A^q(X) \subseteq H^{2q}(X)$ be the set of rational linear combinations of cohomology classes of codimension q subvarieties of X.

One of the rules for $H^*(X)$ implies that, if d is the dimension of X, there is an isomorphism

$$\deg : A^d(X) \longrightarrow \mathbb{Q}$$

determined by the property that

$$\deg(\mathrm{cl}(\mathrm{p})) = 1 \quad \text{for every} \quad \mathrm{p} \in X.$$

Writing h for the class in $A^1(X)$ of any hyperplane section of $X \subseteq \mathbb{P}^m$, the number of points in the intersection of X with a sufficiently general subspace $\mathbb{P}^{m-d} \subseteq \mathbb{P}^m$ satisfies the formula

$$\#(X \cap \mathbb{P}^{m-d}) = \deg(h^d).$$

Compare the corresponding statements of the previous sections

$$\mathrm{Vol}(P) = \frac{1}{d!}\mathrm{Vol}(\mathtt{p}^d) \quad \text{and} \quad \mathrm{Vol}(M) = \frac{1}{d!}\int_M w^d.$$

Let $\mathrm{K}(X)$ be the set of cohomology classes of hyperplane sections of X under all possible embeddings of X into projective spaces. Classical projective geometers knew that $\mathrm{K}(X)$ is a convex cone in a certain sense; keywords are "Segre embedding" and "Veronese embedding".

CONJECTURE 1.8 (Grothendieck). *Let $h \in \mathrm{K}(X)$, and let q be a nonnegative integer $\leq \frac{d}{2}$.*

(PD) The multiplication in $H^*(X)$ defines a nondegenerate bilinear pairing

$$A^q(X) \times A^{d-q}(X) \longrightarrow \mathbb{Q}, \qquad (x, y) \longmapsto \mathit{deg}(xy).$$

(HL) The multiplication by h^{d-2q} *defines an isomorphism*

$$A^q(X) \longrightarrow A^{n-q}(X), \quad \xi \longmapsto h^{d-2q} \cup \xi.$$

(HR) The multiplication by h^{d-2q} *defines a symmetric bilinear form*

$$A^q(X) \times A^q(X) \longrightarrow \mathbb{Q}, \quad (\xi_1, \xi_2) \longmapsto (-1)^q \, deg\big(h^{d-2q} \cup \xi_1 \cup \xi_2\big),$$

that is positive definite on the kernel of

$$h^{d-2q+1} : A^q(X) \longrightarrow A^{n-q+1}(X).$$

The above statements are at the heart of Grothendieck's approach to Weil's conjecture on zeta functions and other important problems in algebraic geometry [**Gro69**].

2. Log-concavity and unimodality conjectures

Logarithmic concavity is a property of a sequence of real numbers, occurring throughout algebraic geometry, convex geometry, and combinatorics. A sequence of positive real numbers $a_0, \ldots, a_d$ is *log-concave* if

$$a_i^2 \;\geq\; a_{i-1}a_{i+1} \;\text{ for all } i.$$

This means that the logarithms of a_i form a concave sequence. The condition implies unimodality of the sequence (a_i), a property easier to visualize: the sequence is *unimodal* if there is an index i such that

$$a_0 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_d.$$

We will discuss the proof of log-concavity of various combinatorial sequences in [**AHK**], such as the coefficients of the chromatic polynomial of graphs and the face numbers of matroid complexes. From a given combinatorial object M (a matroid), we construct a triple

$$\Big(A^*(\mathrm{M}),\ \mathrm{P}(\mathrm{M}),\ \mathrm{K}(\mathrm{M})\Big),$$

which satisfies Poincaré duality, the hard Lefschetz theorem, and the Hodge-Riemann relations. Log-concavity will be deduced from the Hodge-Riemann relations for M. I believe that behind any log-concave sequence that appears in nature, there is such a "Hodge structure" responsible for the log-concavity.

2.1. Coloring graphs. Generalizing earlier work of George Birkhoff, Hassler Whitney introduced in [**Whi32**] the *chromatic polynomial* of a connected graph G as the function on $\mathbb{N}$ defined by

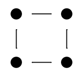
$$\chi_G(q) = \#\{\text{proper colorings of } G \text{ using } q \text{ colors}\}.$$

In other words, $\chi_G(q)$ is the number of ways to color the vertices of G using q colors so that the endpoints of every edge have different colors. Whitney noticed that the chromatic polynomial is indeed a polynomial. In fact, we can write

$$\chi_G(q)/q = a_0(G)q^d - a_1(G)q^{d-1} + \cdots + (-1)^d a_d(G)$$

for some positive integers $a_0(G), \dots, a_d(G)$, where d is one less than the number of vertices of G.

Example 2.1. Consider the square graph G with 4 vertices and 4 edges:



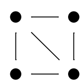
There are precisely two different ways of properly coloring the vertices using two colors:

$$\chi_G(2) = 2.$$

In fact, the graph has the chromatic polynomial

$$\chi_G(q) = 1q^4 - 4q^3 + 6q^2 - 3q.$$

Example 2.2. Let G be the graph obtained by adding a diagonal to the square:



There is no proper coloring of the vertices of this graph using two colors:

$$\chi_G(2) = 0.$$

In fact, the graph has the chromatic polynomial

$$\chi_G(q) = 1q^4 - 5q^3 + 8q^2 - 4q.$$

The chromatic polynomial was originally devised as a tool for attacking the Four Color Problem, but soon it attracted attention in its own right. Ronald Read conjectured in 1968 that the coefficients of the chromatic polynomial form a unimodal sequence for any graph [**Rea68**]. A few years later, Stuart Hoggar conjectured in [**Hog74**] that the coefficients in fact form a log-concave sequence:

$$a_i(G)^2 \geq a_{i-1}(G)a_{i+1}(G) \quad \text{for any } i \text{ and } G.$$

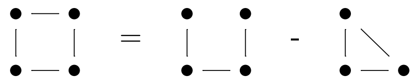
The chromatic polynomial can be computed using the *deletion-contraction relation*: if $G \backslash e$ is the deletion of an edge e from G and G/e is the contraction of the same edge, then

$$\chi_G(q) \;\;=\;\; \chi_{G\backslash e}(q) - \chi_{G/e}(q).$$

The first term counts the proper colorings of G, the second term counts the otherwise-proper colorings of G where the endpoints of e are permitted to have the same color, and the third term counts the otherwise-proper colorings of G where the endpoints of e are mandated to have the same color. Note that the sum of two log-concave sequences need not be log-concave and the sum of two unimodal sequences need not be unimodal. For example, we have

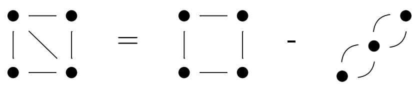
$$(1, 2, 4) + (4, 2, 1) = (5, 4, 5).$$

Example 2.3. To compute the chromatic polynomial of the graph in Example 2.1, write



and use $\chi_{G\backslash e}(q) = q(q-1)^3$ and $\chi_{G/e}(q) = q(q-1)(q-2)$.

Example 2.4. To compute the chromatic polynomial of the graph in Example 2.2, write

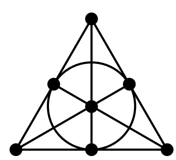


and use $\chi_{G\backslash e}(q) = 1q^4 - 4q^3 + 6q^2 - 3q$ and $\chi_{G/e}(q) = 1q^3 - 2q^2 + q$.

The Hodge-Riemann relations for the algebra $A^*(\mathrm{M})$, where M is the matroid attached to G as in Section 2.3 below, imply that the coefficients of the chromatic polynomial of G form a log-concave sequence.

2.2. Counting independent subsets. Linear independence is a fundamental notion in algebra and geometry: a collection of vectors is linearly independent if no non-trivial linear combination sums to zero. How many linearly independent collection of i vectors are there in a given configuration of vectors? Write A for a finite subset of a vector space and $f_i(A)$ for the number of independent subsets of A of size i.

Example 2.5. Let A be the set of all nonzero vectors in the three dimensional vector space over the field with two elements. Nontrivial dependencies between elements of A can be read off from the picture of the Fano plane shown below. We have $f_0 = 1$, one for the empty subset, $f_1 = 7$, seven for



the seven points, $f_2 = 21$, seven-choose-two for pairs of points, and $f_3 = 28$, seven-choose-three minus seven for triple of points not in one of the seven lines:

$$f_0 = 1, \quad f_1 = 7, \quad f_2 = 21, \quad f_3 = 28.$$

Examples suggest a pattern leading to a conjecture of John Mason and Dominic Welsh [**Mas72, Wel71**]:

$$f_i(A)^2 \geq f_{i-1}(A) f_{i+1}(A) \ \text{ for any } i \text{ and } A.$$

For any small specific case, the conjecture can be verified by computing the $f_i(A)$'s by the *deletion-contraction relation*: if $A \backslash v$ is the deletion of a nonzero vector v from A and A/v is the projection of A in the direction of v, then

$$f_i(A) \;=\; f_i(A\backslash v) + f_{i-1}(A/v).$$

The first term counts the number of independent subsets of size i, the second term counts the independent subsets of size i not containing v, and the third term counts the independent subsets of size i containing v. As in the case of graphs, we notice the apparent interference between the log-concavity conjecture and the additive nature of $f_i(A)$.

The Hodge-Riemann relations for the algebra $A^*(\mathrm{M})$, where M is the matroid attached to A as in Section 2.3 below, imply that the sequence $f_i(A)$ form a log-concave sequence.

2.3. Matroids. In the 1930s, Hassler Whitney observed that several notions in graph theory and linear algebra fit together in a common framework, that of *matroids* [**Whi35**]. This observation started a new subject with applications to a wide range of topics like characteristic classes, optimization, and moduli spaces, to name a few.

Definition 2.6. A *matroid* M on a finite set E is a collection of subsets of E, called *flats* of M, satisfying the following axioms:

(1) If F_1 and F_2 are flats of M, then their intersection is a flat of M.
(2) If F is a flat of M, then any element of $E \setminus F$ is contained in exactly one flat of M covering F.
(3) The set E is a flat of M.

Here, a flat of M is said to *cover* F if it is minimal among the flats of M properly containing F. For our purposes, we may and will suppose that M is *loopless*:

(4) The empty set is a flat of M.

Every maximal chain of flats of M has the same length, and this common length is called the *rank* of M.

Example 2.7. The collection of all subsets of E form a matroid, called the *Boolean matroid* on E. The rank of the Boolean matroid on E is the cardinality of E.

Example 2.8. Let E be the set of edges of a finite graph G. Call a subset F of E a flat when there is no edge in $E \setminus F$ whose endpoints are connected by a path in F. This defines a *graphic matroid* on E.

Example 2.9. A *projective space* is a set with distinguished subsets, called *lines*, satisfying:

(1) Any two distinct points are in exactly one line.
(2) Each line contains more than two points.
(3) If x, y, z, w are distinct points, no three collinear, then

$$xy \text{ intersects } zw \Longrightarrow xz \text{ intersects } yw.$$

A projective space has a structure of flats (subspaces), and this structure is inherited by any of its finite subset, defining a matroid on that finite subset. Matroids arising from subsets of projective spaces over a field k are said to be *realizable* over k (the idea of "*coordinates*"); see Example 2.12 for another, equivalent, description of realizable matroids.

We write $\mathrm{M}\backslash e$ for the matroid obtained by deleting e from the flats of M, and M/e for the matroid obtained by deleting e from the flats of M containing e. When M_1 is a matroid on E_1, M_2 is a matroid on E_2, and $E_1 \cap E_2$ is empty, the *direct sum* $\mathrm{M}_1 \oplus \mathrm{M}_2$ is defined to be the matroid on $E_1 \cup E_2$ whose flats are all sets of the form $F_1 \cup F_2$, where F_1 is a flat of M_1 and F_2 is a flat of M_2.

Matroids are determined by their *independent sets* (the idea of "*general position*"), and can be equivalently defined in terms of independent sets.

Definition 2.10. A *matroid* M on E is a collection of subsets of E, called *independent* sets, which satisfies the following properties.

(1) The empty subset of E is an independent set.
(2) Every subset of an independent set is an independent set.
(3) If I_1 and I_2 are independent sets and I_1 has more elements than I_2, then there is an element in I_1 which, when added to I_2, gives a larger independent set than I_2.

A matroid M is *loopless* if every singleton subset of E is independent.

A matroid M assigns a nonnegative integer, called *rank*, to each subset S of E:

$$\mathrm{rank}_{\mathrm{M}}(S) := (\text{the cardinality of any maximal independent subset of } S)$$

The rank of the entire set E is called the rank of M. It is the common cardinality of any one of the maximal independent subset of E. A matroid has rank 0 if and only if every element of E is a loop. If the rank of a matroid M is positive, we write

$$\mathrm{rank}_{\mathrm{M}}(E) = r + 1.$$

That Definitions 2.6 and 2.10 lead to equivalent structures is a fundamental observation in matroid theory.

Definition 2.11. A *flat* of M, in terms of independent sets of M, is a subset F of E with the following property:

> The addition of any element not in F to F increases the rank.

In other words, a flat of M is a subset of E which is maximal for its rank. We leave it as an exercise to describe independent sets in terms of flats.

Example 2.12.

(1) Let G be a finite graph, and E the set of edges. Call a subset of E independent if it does not contain a circuit. This defines a *graphic matroid* M.

(2) Let E be a finite subset of a vector space over a field k. Call a subset of E independent if it is linearly independent over k. This defines a matroid M *realizable* over k.

EXAMPLE 2.13. Write $E = \{0, 1, 2, 3\}$ for the set of edges of the square graph G in Example 2.1. The graphic matroid M on E attached to G has flats

$$\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{0, 1, 2, 3\},$$

and maximal independent sets

$$\{0, 1, 2\}, \{0, 1, 3\}, \{0, 2, 3\}, \{1, 2, 3\}.$$

EXAMPLE 2.14. Write $E = \{0, 1, 2, 3, 4, 5, 6\}$ for the configuration of vectors A in Example 2.5. The linear matroid M on E attached to A has flats

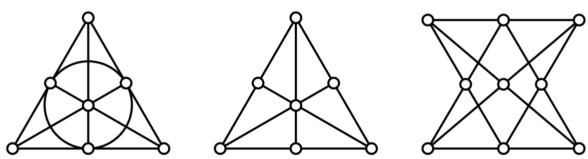
$$\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1, 2, 3\}, \{3, 4, 5\}, \{1, 5, 6\}, \\ \{0, 1, 4\}, \{0, 2, 5\}, \{0, 3, 6\}, \{2, 4, 6\}, \{0, 1, 2, 3, 4, 5, 6\}.$$

Every three element subset of E different from

$$\{1, 2, 3\}, \{3, 4, 5\}, \{1, 5, 6\}, \{0, 1, 4\}, \{0, 2, 5\}, \{0, 3, 6\}, \{2, 4, 6\},$$

is a maximal independent set of M.

Not surprisingly, the notion of realizability is sensitive to the field k. A matroid may arise from a vector configuration over one field while no such vector configuration exists over another field. A matroid may not be realizable over any field.



Among the rank 3 loopless matroids pictured above, where rank 1 flats are represented by points and rank 2 flats containing more than 2 points are represented by lines, the first is realizable over k if and only if the characteristic of k is 2, the second is realizable over k if and only if the characteristic of k is not 2, and the third is not realizable over any field.

Problems concerning the realizability of a matroid over a given field k tend to be difficult. When $k = \mathbb{Q}$, the existence of an algorithm testing the realizability is Hilbert's tenth problem over $\mathbb{Q}$ in disguise [**Stu87**], and, when $k = \mathbb{R}, \mathbb{C}$, there are universality theorems on realization spaces [**Mne88**]. It was recently shown that almost all matroids are not realizable over any field [**Nel**].

The *characteristic polynomial* $\chi_{\mathrm{M}}(q)$ of a matroid M is a generalization of the chromatic polynomial $\chi_G(q)$ of a graph G. It can be recursively defined using the following rules:

(1) If M is the direct sum $\mathrm{M}_1 \oplus \mathrm{M}_2$, then $\chi_{\mathrm{M}}(q) = \chi_{\mathrm{M}_1}(q)\ \chi_{\mathrm{M}_2}(q)$.
(2) If M is not a direct sum, then, for any e, $\chi_{\mathrm{M}}(q) = \chi_{\mathrm{M}\setminus e}(q) - \chi_{\mathrm{M}/e}(q)$.
(3) If M is the rank 1 matroid on $\{e\}$, then $\chi_{\mathrm{M}}(q) = q - 1$.
(4) If M is the rank 0 matroid on $\{e\}$, then $\chi_{\mathrm{M}}(q) = 0$.

It is a consequence of the Möbius inversion for partially ordered sets that that the characteristic polynomial of M is well-defined, see Section 3.3.

The following result from [**AHK**] confirms a conjecture of Gian-Carlo Rota and Dominic Welsh.

THEOREM 2.15. *The coefficients of the characteristic polynomial form a log-concave sequence for any matroid* M.

This implies the log-concavity of the sequence $a_i(G)$ [**Huh12**] and the log-concavity of the sequence $f_i(A)$ [**Len12**].

2.4. The Hodge-Riemann relations for matroids. Let E be a finite set, and let M be a loopless matroid on E. The vector space $A^*(\mathrm{M})$ has the structure of a graded algebra that can be described explicitly.

DEFINITION 2.16. We introduce variables x_F, one for each nonempty proper flat F of M, and set

$$S^*(\mathrm{M}) = \mathbb{R}[x_F]_{F \neq \varnothing, F \neq E}.$$

The *Chow ring* $A^*(\mathrm{M})$ of M is the quotient of $S^*(\mathrm{M})$ by the ideal generated by the linear forms

$$\sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F,$$

one for each pair of distinct elements i_1 and i_2 of E, and the quadratic monomials

$$x_{F_1} x_{F_2},$$

one for each pair of incomparable nonempty proper flats F_1 and F_2 of M.

The Chow ring of M was introduced by Eva Maria Feichtner and Sergey Yuzvinsky [**FY04**]. When M is realizable over a field k, it is the Chow ring of the "wonderful" compactification of the complement of a hyperplane arrangement defined over k as described by Corrado de Concini and Claudio Procesi [**DP95**].

Let d be the integer one less than the rank of M. It can be shown that there is a linear bijection

$$\deg\colon A^d(\mathrm{M}) \longrightarrow \mathbb{R}, \qquad x_{F_1} x_{F_2} \cdots x_{F_d} \longmapsto 1$$

for every maximal chain of nonempty proper flats $F_1 \subsetneq \cdots \subsetneq F_d$, see [**AHK**, Proposition 5.10].

DEFINITION 2.17. We define $\mathrm{P} = \mathrm{P}(\mathrm{M})$ to be the bilinear pairing

$$\mathrm{P} : A^*(\mathrm{M}) \times A^{d-*}(\mathrm{M}) \longrightarrow \mathbb{R}, \qquad (x, y) \longmapsto \deg(xy).$$

What should be the linear operator L for M? We collect all valid choices of L in a nonempty open convex cone. The cone is an analogue of the Kähler cone in complex geometry.

DEFINITION 2.18. A real-valued function c on 2^E is said to be *strictly submodular* if

$$c_\varnothing = 0, \quad c_E = 0,$$

and, for any two incomparable subsets $I_1, I_2 \subseteq E$,

$$c_{I_1} + c_{I_2} > c_{I_1 \cap I_2} + c_{I_1 \cup I_2}.$$

A strictly submodular function c defines an element

$$\mathrm{L}(c) = \sum_F c_F x_F \in A^1(\mathrm{M}),$$

that acts as a linear operator by multiplication

$$A^*(\mathrm{M}) \longrightarrow A^{*+1}(\mathrm{M}), \quad x \longmapsto \mathrm{L}(c)x.$$

The set of all such elements is a convex cone in $A^1(\mathrm{M})$.

The main result of [**AHK**] states that the triple $\big(A^*(\mathrm{M}), \mathrm{P}(\mathrm{M}), \mathrm{L}(c)\big)$ satisfies the hard Lefschetz theorem and the Hodge-Riemann relations for every strictly submodular function c:

THEOREM 2.19. *Let q be a nonnegative integer $\leq \frac{d}{2}$.*

(PD) The product in $A^*(\mathrm{M})$ defines a nondegenerate bilinear pairing

$$\mathrm{P} : A^q(\mathrm{M}) \times A^{d-q}(\mathrm{M}) \longrightarrow \mathbb{R}, \qquad (x, y) \longmapsto deg(xy).$$

(HL) The multiplication by $\mathrm{L}(c)$ defines an isomorphism

$$A^q(\mathrm{M}) \longrightarrow A^{d-q}(\mathrm{M}), \quad x \longmapsto \mathrm{L}(c)^{d-2q}\ x.$$

(HR) The symmetric bilinear form on $A^q(\mathrm{M})$ defined by

$$(x_1, x_2) \longmapsto (-1)^q\ \mathrm{P}(x_1, \mathrm{L}(c)^{d-2q} x_2)$$

is positive definite on the kernel of

$$\mathrm{L}(c)^{d-2q+1} : A^q(\mathrm{M}) \longrightarrow A^{d-q+1}(\mathrm{M}).$$

2.5. Sketch of proof of log-concavity. Why do the Hodge-Riemann relations for M imply log-concavity for $\chi_\mathrm{M}(q)$? The Hodge-Riemann relations for M, in fact, imply that the sequence (μ^i_M) in the expression

$$\chi_\mathrm{M}(q)/(q-1) = \mu^0_\mathrm{M} q^d - \mu^1_\mathrm{M} q^{d-1} + \cdots + (-1)^d \mu^d_\mathrm{M}$$

is log-concave, which is stronger.

Let's define two elements of $A^1(\mathrm{M})$: for any $j \in E$, set

$$\alpha = \sum_{j \in F} x_F, \quad \beta = \sum_{j \notin F} x_F.$$

The two elements do not depend on the choice of j, and they are limits of elements of the form $L(c)$ for strictly submodular c. A combinatorial argument shows that μ^i_{M} is a mixed degree of α and β in the ring $A^*(\mathrm{M})$.

PROPOSITION 2.20. *For every i, we have*

$$\mu^i_{\mathrm{M}} = deg(\alpha^i \ \beta^{d-i}).$$

See Section 7 for geometry behind the formula. Thus, it is enough to prove for every i that

$$\deg(\alpha^{d-i+1}\beta^{i-1})\deg(\alpha^{d-i-1}\beta^{i+1}) \leq \deg(\alpha^{d-i}\beta^{i})^2.$$

This is an analogue of the *Teisser-Khovanskii inequality* for intersection numbers in algebraic geometry, and the *Alexandrov-Fenchel inequality* for mixed volumes in convex geometry. The main case is when $i = d - 1$.

By a continuity argument, we may replace β by $\mathrm{L} = \mathrm{L}(c)$ sufficiently close to β. The desired inequality in the main case then becomes

$$\deg(\alpha^2\mathrm{L}^{d-2})\deg(\mathrm{L}^d) \leq \deg(\alpha\mathrm{L}^{d-1})^2.$$

This follows from the fact that the signature of the bilinear form

$$A^1(\mathrm{M}) \times A^1(\mathrm{M}) \to \mathbb{R}, \quad (x_1, x_2) \mapsto \deg(x_1\mathrm{L}^{d-2}x_2)$$

restricted to the span of α and L is semi-indefinite, which, in turn, is a consequence of the Hodge-Riemann relations for M in the cases $q = 0, 1$.

This application only uses a small piece of the Hodge-Riemann relations for M. The general Hodge-Riemann relations for M may be used to extract other interesting combinatorial information about M.

EXERCISE 2.21. Let M be the rank 3 matroid on $E = \{0, 1, 2, 3\}$ whose nonempty proper flats are

$$\{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\},$$

and, in the Chow ring of M, set

$$\alpha = x_0 + x_{01} + x_{02} + x_{03}, \quad \beta = x_1 + x_2 + x_3 + x_{12} + x_{13} + x_{23}.$$

Show that the following equalities hold:

$$\alpha^2 = 1 \cdot x_0x_{01}, \quad \alpha\beta = 3 \cdot x_0x_{01}, \quad \beta^2 = 3 \cdot x_0x_{01}.$$

Compare the result with coefficients of the reduced characteristic polynomial $\chi_{\mathrm{M}}(q)/(q - 1)$.

3. Matroids from a tropical point of view

The remaining part of this note provides a detailed introduction to the geometry behind the ring $A^*(\mathrm{M})$, following [**Huh**]. The tropical point of view suggested here motivates Question 7.6, which asks whether every matroid is realizable over every field in some generalized sense.

3.1. The permutohedral variety. Let n be a nonnegative integer and let E be the set $\{0, 1, \ldots, n\}$.

DEFINITION 3.1. The n-dimensional *permutohedron* is the convex hull

$$\Xi_n = \text{conv}\Big\{(x_0, \ldots, x_n) \mid x_0, x_1, \ldots, x_n \text{ is a permutation of } 0, 1, \ldots, n\Big\} \subseteq \mathbb{R}^{n+1}.$$

The symmetric group on E acts on the permutohedron Ξ_n by permuting coordinates, and hence each one of the above $(n+1)!$ points is a vertex of Ξ_n.

The n-dimensional permutohedron is contained in the hyperplane

$$x_0 + x_1 + \cdots + x_n = \frac{n(n+1)}{2}.$$

The permutohedron has one facet for each nonempty proper subset S of E, denoted Ξ_S, is the convex hull of those vertices whose coordinates in positions in S are smaller than any coordinate in positions not in S. For example, if S is a set with one element i, then the corresponding facet is

$$\Xi_{\{i\}} = \text{conv}\Big\{(x_0, \ldots, x_n) \mid \ (x_0, \ldots, x_n) \text{ is a vertex of } \Xi_n \text{ with } x_i = 0\Big\} \subseteq \mathbb{R}^{n+1}.$$

Similarly, if S is the entire set minus one element $E \setminus \{i\}$, then the corresponding facet is

$$\Xi_{E\setminus\{i\}} = \text{conv}\Big\{(x_0, \ldots, x_n) \mid \ (x_0, \ldots, x_n) \text{ is a vertex of } \Xi_n \text{ with } x_i = n\Big\} \subseteq \mathbb{R}^{n+1}.$$

These facets can be identified with the permutohedron of one smaller dimension. In general, a facet of a permutohedron can be identified with the product of two permutohedrons of smaller dimensions:

$$\Xi_S \simeq \Xi_{|S|-1} \times \Xi_{|E\setminus S|-1}.$$

More generally, the codimension d faces of the permutohedron Ξ_n bijectively correspond to the ordered partitions of E into $d+1$ parts. Explicitly, the codimension d face corresponding to a flag of nonempty proper subsets $\big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\big)$ is the convex hull of those vertices whose coordinates in positions in $S_j \setminus S_{j-1}$ are smaller than any coordinate in positions in $S_{j+1} \setminus S_j$ for all j.

The normal fan of the n-dimensional permutohedron is a complete fan in an n-dimensional quotient of the vector space $\mathbb{R}^{n+1}$:

$$|\Delta_{A_n}| := \mathbb{R}^{n+1}/\text{span}(1, 1, \ldots, 1).$$

The quotient space $|\Delta_{A_n}|$ is generated by the vectors $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n$, where $\mathbf{u}_i$ is the primitive ray generator in the normal fan corresponding to the facet $\Xi_{\{i\}}$. In coordinates,

$$\mathbf{u}_0 = (1, 0, \ldots, 0),\ \mathbf{u}_1 = (0, 1, \ldots, 0), \\ \ldots\ \mathbf{u}_n = (0, 0, \ldots, 1) \quad \text{mod } (1, 1, \ldots, 1).$$

NOTATION. For a subset S of E, we define

$$\mathbf{u}_S := \sum_{i \in S} \mathbf{u}_i.$$

If S is a nonempty and proper subset of E, then $\mathbf{u}_S$ generates a ray in the normal fan corresponding to the facet Ξ_S.

DEFINITION 3.2. The n-dimensional *permutohedral fan* is the complete fan Δ_{A_n} whose d-dimensional cones are of the form

$$\sigma_{\mathcal{S}} = \text{cone}(\mathbf{u}_{S_1}, \mathbf{u}_{S_2}, \ldots, \mathbf{u}_{S_d}), \qquad \mathcal{S} = \Big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\Big),$$

where $\mathcal{S}$ is a flag of nonempty proper subsets of E. We call $\sigma_{\mathcal{S}}$ the cone determined by the flag $\mathcal{S}$.

The permutohedral fan Δ_{A_n} is the normal fan of the permutohedron Ξ_n, and can be identified with the fan of Weyl chambers of the root system A_n.

The geometry of the permutohedral fan is governed by the combinatorics of the Boolean lattice of all subsets of E. Let $\mathcal{T} = \big(T_1 \subsetneq T_2 \subsetneq \cdots \subsetneq T_{d-1}\big)$ be a flag of nonempty proper subsets of E. We say that a subset S of E is *strictly compatible* with $\mathcal{T}$ if $S \subsetneq T_j$ or $T_j \subsetneq S$ for each j. Then the d-dimensional cones in Δ_{A_n} containing the cone determined by $\mathcal{T}$ bijectively correspond to the nonempty proper subsets of E that are strictly compatible with $\mathcal{T}$.

The symmetric group on E acts on the Boolean lattice of subsets of E, and hence on the permutohedral fan Δ_{A_n}. In addition, the permutohedral fan has an automorphism of order 2, sometimes called the *Cremona symmetry*:

$$\text{Crem} : |\Delta_{A_n}| \longrightarrow |\Delta_{A_n}|, \qquad x \longmapsto -x.$$

This automorphism associates to a flag the flag that consists of complements:

$$\Big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d\Big) \longmapsto \Big(\hat{S}_d \subsetneq \cdots \subsetneq \hat{S}_2 \subsetneq \hat{S}_1\Big), \qquad \hat{S}_j = E \setminus S_j.$$

Let k be a field. The normal fan of the permutohedron defines a smooth projective toric variety over k. This variety is the main character of the thesis.

DEFINITION 3.3. The n-dimensional *permutohedral variety* X_{A_n} is the toric variety of the permutohedral fan Δ_{A_n} with respect to the lattice $\mathbb{Z}^{n+1}/\text{span}(1, \ldots, 1)$.

When the field k is relevant to a statement, we will say that X_{A_n} is the permutohedral variety over k. Otherwise, we do not explicitly mention the field k. Our basic reference for toric varieties is [**Ful93**].

Notation.

(1) If S is a nonempty proper subset of E, we write D_S for the torus-invariant prime divisor of X_{A_n} corresponding to the ray generated by $\mathbf{u}_S$.
(2) If $\mathcal{S}$ is a flag of nonempty proper subsets of E, we write $V(\mathcal{S})$ for the torus orbit closure in X_{A_n} corresponding to the cone determined by $\mathcal{S}$.

The codimension of $V(\mathcal{S})$ in X_{A_n} is equal to the length d of the flag

$$\mathcal{S} = \Big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_d \Big).$$

The torus orbit closure $V(\mathcal{S})$ is a transversal intersection of smooth hypersurfaces

$$V(\mathcal{S}) = D_{S_1} \cap D_{S_2} \cap \cdots \cap D_{S_d}.$$

A fundamental geometric fact is that X_{A_n} can be obtained by blowing up all the torus-invariant linear subspaces of the projective space $\mathbb{P}^n$. In fact, there are two essentially different ways of identifying X_{A_n} with the blown up projective space.

Consider the composition of blowups

$$X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \;=\; \mathbb{P}^n,$$

where $X_{d+1} \longrightarrow X_d$ is the blowup of the strict transform of the union of all the torus-invariant d-dimensional linear subspaces of $\mathbb{P}^n$. We identify the rays of the fan of $\mathbb{P}^n$ with the vectors

$$\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n,$$

and index the homogeneous coordinates of the projective space by the set E:

$$z_0, z_1, \ldots, z_n.$$

This gives one identification between X_{A_n} and X_{n-1}. We denote the above composition of blowups by $\pi_1 : X_{A_n} \longrightarrow \mathbb{P}^n$.

Definition 3.4. The map $\pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n$ is the composition of the Cremona symmetry and $\pi_1 : X_{A_n} \longrightarrow \mathbb{P}^n$.

We have the commutative diagram

$$\begin{array}{ccc} & X_{A_n} & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ \mathbb{P}^n & \overset{}{\underset{\text{Crem}}{\dashrightarrow}} & \mathbb{P}^n, \end{array}$$

where Crem is the standard Cremona transformation

$$\text{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n, \qquad (z_0 : z_1 : \cdots : z_n) \longmapsto (z_0^{-1} : z_1^{-1} : \cdots : z_n^{-1}).$$

All three maps in the diagram are torus-equivariant, and they restrict to isomorphisms between the n-dimensional tori.

Like π_1, the induced map π_2 is the blowup of all the torus-invariant linear subspaces of the target projective space. The rays in the fan of the image of π_2 are generated by the vectors

$$\mathbf{u}_{\hat{0}}, \mathbf{u}_{\hat{1}}, \ldots, \mathbf{u}_{\hat{n}},$$

where $\hat{i}$ is the complement of $\{i\}$ in E. The ray generated by $\mathbf{u}_{\hat{i}}$ correspond to the facet $\Xi_{\hat{i}}$ of the permutohedron. The homogeneous coordinates of this projective space will be written

$$z_{\hat{0}}, z_{\hat{1}}, \ldots, z_{\hat{n}}.$$

If S is a nonempty proper subset of E with $|S| \geq 2$, then D_S is the exceptional divisor of π_1 corresponding to the codimension $|S|$ linear subspace

$$\bigcap_{j \in S} \{z_j = 0\} \subseteq \mathbb{P}^n.$$

If S is a nonempty subset of E with $|E \setminus S| \geq 2$, then D_S is the exceptional divisor of π_2 corresponding to the dimension $|S|$ linear subspace

$$\bigcap_{j \notin S} \{z_{\hat{j}} = 0\} \subseteq \mathbb{P}^n.$$

The Cremona symmetry of the permutohedral fan

$$\text{Crem} : |\Delta_{A_n}| \longrightarrow |\Delta_{A_n}|, \qquad x \longmapsto -x$$

changes the role of π_1 and π_2.

The anticanonical linear system of X_{A_n} has a simple description in terms of π_1 and π_2. Choose an element i of E, and consider the corresponding hyperplanes in the two projective spaces:

$$H_i := \{z_i = 0\} \subseteq \mathbb{P}^n, \qquad H_{\hat{i}} := \{z_{\hat{i}} = 0\} \subseteq \mathbb{P}^n.$$

The pullbacks of the hyperplanes in the permutohedral variety are

$$\pi_1^{-1}(H_i) = \sum_{i \in S} D_S \quad \text{and} \quad \pi_2^{-1}(H_{\hat{i}}) = \sum_{i \notin S} D_S.$$

Since any subset of E either contains i or does not contain i, the sum of the two divisors is the union of all torus-invariant prime divisors in X_{A_n}. In other words, the sum is the torus-invariant anticanonical divisor of the permutohedral variety:

$$-K_{X_{A_n}} = \pi_1^{-1}(H_i) + \pi_2^{-1}(H_{\hat{i}}).$$

The decomposition of the anticanonical linear system gives the map

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

whose image is the closure of the graph of the Cremona transformation. This gives another proof of the result of Batyrev and Blume that $-K_{X_{A_n}}$ is nef and big [**BB11**].

PROPOSITION 3.5. *The anticanonical divisor of X_{A_n} is nef and big.*

The true anticanonical map of the linear system $|-K_{X_{A_n}}|$ fits into the commutative diagram

$$\begin{array}{ccc} X_{A_n} & \xrightarrow{\pi_1 \times \pi_2} & \mathbb{P}^n \times \mathbb{P}^n \\ \big\downarrow{\scriptstyle -K} & & \big\downarrow{\scriptstyle \mathfrak{s}} \\ \mathbb{P}^{n^2+n} & \xrightarrow{L} & \mathbb{P}^{n^2+2n}. \end{array}$$

Here $-K$ is the anticanonical map, L is a linear embedding of codimension n, and $\mathfrak{s}$ is the Segre embedding.

Remark 3.6. The permutohedral variety X_{A_n} can be viewed as the torus orbit closure of a general point in the flag variety $\mathrm{Fl}(\mathbb{C}^{n+1})$, see [**Kly85, Kly95**]. Under this identification, π_1 and π_2 are projections onto the Grassmannians

$$\begin{array}{ccc} & X_{A_n} & \\ {\scriptstyle \pi_1}\swarrow & & \searrow{\scriptstyle \pi_2} \\ \mathbb{P}^n \simeq \mathrm{Gr}(1, \mathbb{C}^{n+1}) & & \mathrm{Gr}(n, \mathbb{C}^{n+1}) \simeq \mathbb{P}^n. \end{array}$$

It may be interesting to study matroid invariants obtained by intersecting divisors coming from the intermediate Grassmannians with the Bergman fan Δ_{M} (Definition 3.9).

Recall that torus-invariant divisors on X_{A_n} may be viewed as piecewise linear functions on Δ_{A_n}. For later use, we give names to the piecewise linear functions of the divisors $\pi_1^{-1}(H_i)$ and $\pi_2^{-1}(H_{\hat{i}})$.

Definition 3.7. Let i be an element of E. We define $\alpha = \alpha(i)$ to be the piecewise linear function on Δ_{A_n} determined by the values

$$\alpha(\mathbf{u}_S) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S, \end{cases}$$

and define $\beta = \beta(i)$ to be the piecewise linear function on Δ_{A_n} determined by the values

$$\beta(\mathbf{u}_S) = \begin{cases} 0 & \text{if } i \in S, \\ -1 & \text{if } i \notin S. \end{cases}$$

The dependence of α and β on i will often be invisible from their notation. Different choices of i will give rationally equivalent divisors, and piecewise linear functions which are equal to each other modulo linear functions. The functions α and β pull-back to each other under the Cremona symmetry of Δ_{A_n}.

3.2. The Bergman fan of a matroid. We will show that a matroid on E defines an *extremal nef class* in the homology of the permutohedral variety X_{A_n}. In principle, a question on matroids on E can be translated to a question on the geometry of the permutohedral variety X_{A_n}.

DEFINITION 3.8. The *lattice of flats* of a matroid M is the poset $\mathscr{L}_{\mathrm{M}}$ of all flats of M, ordered by inclusion.

The lattice of flats has a unique minimal element, the set of all loops, and a unique maximal element, the entire set E. It is graded by the rank function, and every maximal chain in $\mathscr{L}_{\mathrm{M}} \setminus \{\min \mathscr{L}_{\mathrm{M}}, \max \mathscr{L}_{\mathrm{M}}\}$ has the same number of flats r. In what follows, we suppose that M is a loopless matroid on E with rank $r+1$.

DEFINITION 3.9. The *Bergman fan* of M, denoted Δ_{M}, is the fan in $|\Delta_{A_n}|$ consisting of cones corresponding to flags of nonempty proper flats of M. In other words, the Bergman fan of M is a collection of cones of the form

$$\sigma_{\mathcal{F}} = \mathrm{cone}(\mathbf{u}_{F_1}, \mathbf{u}_{F_2}, \ldots, \mathbf{u}_{F_d}),$$

where $\mathcal{F}$ is a flag of nonempty proper flats

$$\mathcal{F} = \Big(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_d\Big).$$

When $r = 0$, by definition, Δ_{M} is the 0-dimensional fan at the origin.

The Bergman fan Δ_{M} is an r-dimensional subfan of the permutohedral fan Δ_{A_n}. Ardila and Klivans introduced this fan in [**AK06**] and called it the fine subdivision of the Bergman fan of the matroid. Sets of the form $|\Delta_{\mathrm{M}}|$ are called *tropical linear spaces*: they are basic building blocks of *smooth tropical varieties*. Note that the permutohedral fan is the Bergman fan of the uniform matroid of full rank.

We next prove a fundamental property of the Bergman fan Δ_{M} that it satisfies the *balancing condition*. Geometrically, the condition says that, for every $(r-1)$-dimensional cone τ, the sum of the ray generators of the cones in Δ_{M} that contain τ is contained in τ. Combinatorially, the condition is a translation of the flat partition property for matroids (Definition 2.6).

PROPOSITION 3.10. *Let $F_1, \ldots, F_m$ be the nonempty proper flats of* M *that are strictly compatible with a flag of nonempty proper flats*

$$\mathcal{G} = \Big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{r-1}\Big).$$

If we set $G_0 = \varnothing$ and $G_r = E$, then there is exactly one index l such that each F_j covers G_{l-1} and is covered by G_l, and

$$\sum_{j=1}^{m} \mathbf{u}_{F_j} = \mathbf{u}_{G_l} + (m-1)\mathbf{u}_{G_{l-1}}.$$

When M has loops, the same formula holds if we replace G_0 by the set of all loops. For geometric and combinatorial reasons, we choose not to define the Bergman fan for matroids with loops.

3.3. The Möbius function of a matroid. The Möbius function of the lattice of flats will play a fundamental role in the intersection theory of matroids in the permutohedral variety. We continue to assume that M is a matroid which has no loops.

DEFINITION 3.11. Let $\mathscr{L}$ be a finite poset. The *Möbius function* of $\mathscr{L}$ is the function

$$\mu_{\mathscr{L}} : \mathscr{L} \times \mathscr{L} \longrightarrow \mathbb{Z}$$

determined by the following properties:
(1) If $x \not\leq y$, then $\mu_{\mathscr{L}}(x, y) = 0$.
(2) If $x = y$, then $\mu_{\mathscr{L}}(x, y) = 1$.
(3) If $x < y$, then

$$\mu_{\mathscr{L}}(x, y) = -\sum_{x \leq z < y} \mu_{\mathscr{L}}(x, z).$$

When $\mathscr{L}$ is the lattice of flats of M, we write μ_{M} for the Möbius function $\mu_{\mathscr{L}}$. The Möbius function of the lattice of flats of a matroid has several special properties that the Möbius function of posets in general do not have. For example, Rota's theorem says that, if F_1 is a flat contained in a flat F_2, then

$$(-1)^{\mathrm{rank}_{\mathrm{M}}(F_2) - \mathrm{rank}_{\mathrm{M}}(F_1)} \mu_{\mathrm{M}}(F_1, F_2) > 0.$$

Another basic result on matroids is the following theorem of Weisner. For proofs, see [**Rot64**, **Zas87**].

THEOREM 3.12. *Let F be a flat of* M*, and let i be an element of F. If $F_1, F_2, \ldots, F_m$ are the flats of* M *covered by F which do not contain i, then*

$$\mu_{\mathrm{M}}(\varnothing, F) = -\sum_{j=1}^{m} \mu_{\mathrm{M}}(\varnothing, F_j).$$

When M has loops, one should replace $\varnothing$ by the set of loops and choose i among elements of F which is not a loop. We will frequently use Weisner's theorem later in intersection theoretic computations in X_{A_n}. In a sense, Weisner's theorem plays a role which is Cremona symmetric to the role played by the flat partition property for matroids.

DEFINITION 3.13. The *characteristic polynomial* of M is the polynomial

$$\chi_{\mathrm{M}}(q) = \sum_{F \in \mathscr{L}_{\mathrm{M}}} \mu_{\mathrm{M}}(\varnothing, F)\ q^{\mathrm{rank}_{\mathrm{M}}(E) - \mathrm{rank}_{\mathrm{M}}(F)}.$$

By definition of the Möbius function,

$$\chi_{\mathrm{M}}(1) = \sum_{F \in \mathscr{L}_{\mathrm{M}}} \mu_{\mathrm{M}}(\varnothing, F) = 0.$$

We define the *reduced characteristic polynomial* of M by

$$\overline{\chi}_{\mathrm{M}}(q) := \chi_{\mathrm{M}}(q)/(q-1).$$

By Rota's theorem, the coefficients of the characteristic polynomial of a matroid alternate in sign. The same is true for the coefficients of the reduced characteristic polynomial.

4. Intersection theory of the permutohedral variety

Let X be an n-dimensional smooth toric variety defined from a complete fan Σ. An element of the Chow cohomology group $A^l(X)$ gives a homomorphism of Chow groups from $A_l(X)$ to $\mathbb{Z}$. The resulting homomorphism of abelian groups is the Kronecker duality homomorphism

$$A^l(X) \longrightarrow \mathrm{Hom}\big(A_l(X), \mathbb{Z}\big).$$

The Kronecker duality homomorphism for X is, in fact, an isomorphism [**FS97**]. Since $A_l(X)$ is generated by the classes of l-dimensional torus orbit closures, the isomorphism identifies Chow cohomology classes with certain integer valued functions on the set of d-dimensional cones in Σ, where $d = n - l$.

NOTATION. If σ is a d-dimensional cone containing a $(d-1)$-dimensional cone τ in the fan of X, then there is exactly one ray in σ not in τ. The primitive generator of this ray will be denoted $\mathbf{u}_{\sigma/\tau}$.

DEFINITION 4.1. A d-dimensional *Minkowski weight* on Σ is a function Δ from the set of d-dimensional cones to the integers which satisfies the *balancing condition*: For every $(d-1)$-dimensional cone τ,

$$\sum_{\tau \subset \sigma} \Delta(\sigma)\mathbf{u}_{\sigma/\tau} \quad \text{is contained in the lattice generated by } \tau,$$

where the sum is over all d-dimensional cones σ containing τ.

One may think a d-dimensional Minkowski weight as a d-dimensional subfan of Σ with integer weights on its d-dimensional cones. The balancing condition imposed on d-dimensional Minkowski weights on Σ is a translation of the rational equivalence relations between l-dimensional torus orbit closures in X [**FS97**].

THEOREM 4.2. *The Chow cohomology group $A^l(X)$ is isomorphic to the group of d-dimensional Minkowski weights on Σ:*

$$\begin{aligned} A^l(X) &\simeq Hom\big(A_l(X), \mathbb{Z}\big) \\ &\simeq \big(\textit{the group of } d\textit{-dimensional Minkowski weights}\big) = MW_l(\Sigma). \end{aligned}$$

These groups are also isomorphic to the d-dimensional homology group of X through the 'degree' map

$$A_d(X) \longrightarrow \mathrm{Hom}\big(A_l(X), \mathbb{Z}\big), \qquad \xi \longmapsto \Big(\eta \longmapsto \deg(\xi \cdot \eta)\Big).$$

Its inverse is the composition of the isomorphism $\mathrm{Hom}\big(A_l(X), \mathbb{Z}\big) \simeq A^l(X)$ and the Poincaré duality isomorphism

$$A^l(X) \longrightarrow A_d(X), \qquad \Delta \longmapsto \Delta \cap [X].$$

We say that Δ and $\Delta \cap [X]$ are *Poincaré dual* to each other.

Theorem 4.2, when applied to the permutohedral variety, says that a cohomology class of X_{A_n} is a function from the set of flags in E which satisfies the balancing condition. The balancing condition for a d-dimensional Minkowski weight Δ on the permutohedral fan can be translated as follows: For every flag of nonempty proper subsets $\mathcal{S} = \big(S_1 \subsetneq \cdots \subsetneq S_{d-1}\big)$, if $T_1, \ldots, T_m$ are the nonempty proper subsets of E that are strictly compatible with $\mathcal{S}$, then

$$\sum_{j=1}^{m} \Delta(\sigma_j)\mathbf{u}_{T_j} \quad \text{is contained in the lattice generated by } \mathbf{u}_{S_1}, \ldots, \mathbf{u}_{S_{d-1}},$$

where σ_j is the cone generated by $\mathbf{u}_{T_j}$ and $\mathbf{u}_{S_1}, \ldots, \mathbf{u}_{S_{d-1}}$.

Let M be a loopless matroid of rank $r+1$. Proposition 3.10 shows that the balancing condition is satisfied by the indicator function of the Bergman fan of M. To be more precise, we have the following.

Proposition 4.3. *The Bergman fan Δ_{M} defines an r-dimensional Minkowski weight on the permutohedral fan Δ_{A_n}, denoted by the same symbol Δ_{M}, such that*

$$\Delta_{\mathrm{M}}(\sigma_{\mathcal{S}}) = \begin{cases} 1 & \text{if } \mathcal{S} \text{ is a maximal flag of nonempty proper flats of } \mathrm{M}, \\ 0 & \text{if otherwise.} \end{cases}$$

When $r = 0$, by definition, $\Delta_{\mathrm{M}} = 1$.

The cup product of a divisor and a cohomology class of a smooth complete toric variety defines a product of a piecewise linear function and a Minkowski weight. If φ is a piecewise linear function and Δ is a d-dimensional Minkowski weight, then $\varphi \cup \Delta$ is a $(d-1)$-dimensional Minkowski weight. We will often use the following explicit formula for the cup product [**AR10**].

Theorem 4.4. *Let φ be a piecewise linear function and let Δ be a d-dimensional Minkowski weight on Σ. If τ is a $(d-1)$-dimensional cone in Σ, then*

$$\Big(\varphi \cup \Delta\Big)(\tau) = \varphi\left(\sum_{\tau \subset \sigma} \Delta(\sigma)\mathbf{u}_{\sigma/\tau}\right) - \sum_{\tau \subset \sigma} \varphi\Big(\Delta(\sigma)\mathbf{u}_{\sigma/\tau}\Big),$$

where the sums are over all d-dimensional cones σ containing τ.

In particular, if Δ is nonnegative and φ is linear on the cone generated by the cones containing τ, then

$$\Big(\varphi \cup \Delta\Big)(\tau) = 0.$$

Similarly, if Δ is nonnegative and φ is concave on the cone generated by the cones containing τ, then

$$\Big(\varphi \cup \Delta\Big)(\tau) \geq 0.$$

Corollary 4.5. *If φ is concave and Δ is nonnegative, then $\varphi \cup \Delta$ is nonnegative.*

Theorem 4.4 can be used to compute the cup product of the piecewise linear function α and the Bergman fan Δ_{M}. Proposition 4.7 below shows that the result of the cup product is the Bergman fan of another matroid. Recall that $\alpha = \alpha(i)$ is the piecewise linear function on the permutohedral fan Δ_{A_n} determined by its values

$$\alpha(\mathbf{u}_S) = \begin{cases} -1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

For any element i of E and any nonempty proper subset G of E, the function α is linear when restricted to the cone

$$\mathrm{cone}(\mathbf{u}_j \mid j \in G) = \bigcup_{\mathcal{F}} \sigma_{\mathcal{F}},$$

where the union is over all flag of nonempty proper subsets contained in G.

Definition 4.6. When $r \geq 1$, we define the *truncation* of M to be the matroid $\overline{\mathrm{M}}$ on the same set E defined by the following condition:

> A subset I is independent for $\overline{\mathrm{M}}$ if and only if I is independent for M and $|I| \leq r$.

We do not define the truncation for rank 1 matroids.

A proper subset F of E is a flat for $\overline{\mathrm{M}}$ if and only if F is a flat for M and $\mathrm{rank}_{\mathrm{M}}(F) < r$. In other words, $\mathscr{L}_{\overline{\mathrm{M}}} \setminus \{E\}$ is obtained from $\mathscr{L}_{\mathrm{M}} \setminus \{E\}$ by deleting all flats of rank r.

Proposition 4.7. *If $r \geq 1$ and $\overline{\mathrm{M}}$ is the truncation of* M, *then*

$$\alpha \cup \Delta_{\mathrm{M}} = \Delta_{\overline{\mathrm{M}}}.$$

A repeated application of Proposition 4.7 gives the equality between the 0-dimensional Minkowski weights

$$\Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r}\Big) \cup \Delta_{\mathrm{M}} = 1.$$

Exercise 4.8. Let S_Σ be the polynomial ring with variables x_ρ indexed by the rays of a (not necessarily complete) fan Σ, and define

$$A^*(\Sigma) := S_\Sigma/(I_\Sigma + J_\Sigma),$$

where I_Σ is the ideal generated by the squarefree monomials

$$x_{\rho_1}x_{\rho_2}\cdots x_{\rho_k}, \quad \{\rho_1, \rho_2, \ldots, \rho_k\} \text{ does not generate a cone in } \Sigma,$$

and J_Σ is the ideal generated by the linear forms

$$\sum_\rho \langle \mathbf{u}_\rho, m\rangle\, x_\rho, \quad m \text{ is an element of the dual space } N^\vee.$$

Show that the graded component $A^k(\Sigma)$ is spanned by degree k squarefree monomials in S_Σ.

Exercise 4.9. The graded ring $A^*(\Sigma)$ defined in the previous exercise is isomorphic to the Chow cohomology ring of the toric variety of Σ; see [**Bri96**]. Show that the graded ring $A^*(\mathrm{M})$ in Definition 2.16 is isomorphic to the Chow cohomology ring of the toric variety of the Bergman fan Δ_M.

Exercise 4.10. Define balancing condition for a k-dimensional weight on a rational fan. How should one define the balancing condition for weights on fans with irrational rays?

Exercise 4.11. Let P be a rational polytope containing 0 in its interior, and let Σ be the fan obtained by taking the cone over the edges of P. Show that Σ supports a positive 2-dimensional balanced weight.

Exercise 4.12. Let Q be the standard octahedron with vertices

$$(\pm 1, 0, 0),\ (0, \pm 1, 0),\ (0, 0, \pm 1),$$

and let Σ be the fan obtained by taking the cone over the proper faces of Q. Compute the dimensions of the spaces of Minkowski weights

$$\mathrm{MW}_0(\Sigma), \quad \mathrm{MW}_1(\Sigma), \quad \mathrm{MW}_2(\Sigma), \quad \mathrm{MW}_3(\Sigma).$$

Exercise 4.13. Let M be a uniform matroid of rank 4 on $E = \{0, 1, 2, 3, 4\}$. Compute the dimensions of the spaces of Minkowski weights

$$\mathrm{MW}_0(\Delta_\mathrm{M}), \quad \mathrm{MW}_1(\Delta_\mathrm{M}), \quad \mathrm{MW}_2(\Delta_\mathrm{M}), \quad \mathrm{MW}_3(\Delta_\mathrm{M}).$$

5. Every matroid is nef, effective, and extremal

We have seen that the Bergman fan of a matroid on E defines a cohomology class of the permutohedral variety X_{A_n}. The goal of this section is to show that its Poincaré dual is an effective homology class which generates an extremal ray of the nef cone of X_{A_n}.

Definition 5.1. Let X be an n-dimensional smooth complete variety over a field k, and $d = n - l$.

(1) A d-dimensional Chow homology class of X is *nef* if it intersects all l-dimensional effective cycles nonnegatively.
(2) A d-dimensional Chow homology class of X is *effective* if it is the class of an d-dimensional effective cycle.

If X is a toric variety, then every effective cycle is rationally equivalent to a torus-invariant effective cycle [**FMSS95**]. Therefore, in this case, a d-dimensional Chow homology class ξ is nef if and only if

$$\xi \cdot \big[V(\sigma)\big] \geq 0$$

for every l-dimensional torus orbit closure $V(\sigma)$ of X. In other words, ξ is nef if and only if its Poincaré dual is a nonnegative function when viewed as a d-dimensional Minkowski weight. For example, the Bergman fan of a matroid

M on E defines a nef homology class $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ of the permutohedral variety X_{A_n}.

If X is a toric variety, then every nef class of X is effective. This is a special case of the result of Li on spherical varieties [**Li13**].

THEOREM 5.2. *If X is a toric variety, then every nef class of X is effective.*

PROOF. The main observation is that every effective cycle in a toric variety is rationally equivalent to a torus-invariant effective cycle. Applying this to the diagonal embedding

$$\iota : X \longrightarrow X \times X, \qquad x \longmapsto (x, x),$$

we have an expression

$$\big[\iota(X)\big] = \sum_{\sigma,\tau} m_{\sigma,\tau} \big[V(\sigma) \times V(\tau)\big] \in A_n(X \times X), \qquad m_{\sigma,\tau} \geq 0,$$

where the sum is over all cones σ, τ in the fan of X such that $\dim \sigma + \dim \tau = n$. The choice of the integers $m_{\sigma,\tau}$ is in general not unique, but the knowledge of such constants characterizes both the cap product and the cup product of (co)homology classes on X [**FMSS95**].

Let ξ be a d-dimensional nef class of X. We show that ξ is the class of a torus-invariant effective cycle. If Δ is the Poincaré dual of ξ, viewed as a d-dimensional Minkowski weight, then

$$\xi = \Delta \cap [X] = \sum_{\sigma,\tau} m_{\sigma,\tau}\ \Delta(\sigma)\ \big[V(\tau)\big],$$

where the sum is over all d-dimensional cones σ and l-dimensional cones τ. Since ξ is nef, for all σ, we have

$$\Delta(\sigma) = \deg\Big(\xi \cdot \big[V(\sigma)\big]\Big) \geq 0.$$

Therefore ξ is the class of a torus-invariant effective cycle in X. □

An application of Theorem 5.2 to the permutohedral variety X_{A_n} gives the following.

COROLLARY 5.3. *If* M *is a loopless matroid on E, then $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ is effective.*

We stress that the statement does not involve the field k which is used to define the permutohedral variety X_{A_n}. The proof of Theorem 5.2 shows that an explicit effective cycle with the matroid homology class $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ can be found by degenerating the diagonal of the permutohedral variety in $X_{A_n} \times X_{A_n}$.

DEFINITION 5.4. Let $N_d(X)$ be the real vector space of d-dimensional algebraic cycles with real coefficients modulo numerical equivalence on a smooth complete variety X.

(1) The *nef cone* of X in dimension d, denoted $\text{Nef}_d(X)$, is the cone in $N_d(X)$ generated by d-dimensional nef classes.
(2) The *pseudoeffective cone* of X in dimension d, denoted $\text{Peff}_d(X)$, is the closure in $N_d(X)$ of the cone generated by the d-dimensional effective classes.

The nef cone in dimension d and the pseudoeffective cone in dimension l are dual to each other under the intersection pairing

$$N_d(X) \times N_l(X) \longrightarrow \mathbb{R}.$$

If X is the toric variety of a complete fan Σ, then $N_d(X) \simeq A_d(X) \otimes \mathbb{R}$ can be identified with the set of real valued functions Δ from the set of d-dimensional cones in Σ which satisfy the balancing condition over $\mathbb{R}$: For every $(d-1)$-dimensional cone τ,

$$\sum_{\tau \subset \sigma} \Delta(\sigma)\mathbf{u}_{\sigma/\tau} \text{ is contained in the subspace generated by } \tau,$$

where the sum is over all d-dimensional cones σ containing τ.

In the toric case, any integral effective cycle is rationally equivalent to an effective torus-invariant cycle, and hence there is no need to take the closure when defining the pseudoeffective cone. Furthermore, the pseudoeffective cone and the nef cone of X are polyhedral cones. These polyhedral cones depend only on the fan Σ and not on the field k used to define X. Theorem 5.2 shows that one is contained in the other.

THEOREM 5.5. *If X is a toric variety, then the nef cone of X in dimension d is contained in the pseudoeffective cone of X in dimension d, for every d.*

We remark that there is a 4-dimensional complex abelian variety whose nef cone in dimension 2 is not contained in the pseudoeffective cone in dimension 2 [**DELV11**].

We now show that a loopless matroid on E gives an *extremal* nef class of X_{A_n}. The main combinatorial ingredient is the following theorem of Björner [**Bjo92**]. Recall that the *order complex* of a finite poset $\mathscr{L}$ is a simplicial complex which has the underlying set of $\mathscr{L}$ as vertices and the finite chains of $\mathscr{L}$ as faces.

THEOREM 5.6. *The order complex of the lattice of flats of a matroid is shellable.*

The shellability of the order complex of the lattice of flats $\mathscr{L}_{\mathrm{M}}$ implies, among many other things, that the Bergman fan Δ_{M} is *connected in codimension* 1: If σ and $\tilde{\sigma}$ are r-dimensional cones in Δ_{M}, then there are r-dimensional cones $\sigma_0, \sigma_1, \ldots, \sigma_l$ and $(r-1)$-dimensional cones $\tau_1, \ldots, \tau_l$ in Δ_{M} such that

$$\sigma = \sigma_0 \supset \tau_1 \subset \sigma_1 \supset \tau_2 \subset \cdots \supset \tau_{l-1} \subset \sigma_{l-1} \supset \tau_l \subset \sigma_l = \tilde{\sigma}.$$

THEOREM 5.7. *If* M *is a loopless matroid on E of rank $r+1$, then the class $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ generates an extremal ray of the nef cone of X_{A_n} in dimension r.*

PROOF. The claim is that Δ_{M} cannot be written as a sum of two nonnegative real Minkowski weights in a nontrivial way. Suppose Δ is an r-dimensional Minkowski weight with the following property:

> If $\mathcal{S} = \big(S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_r\big)$ is a flag of nonempty proper subsets of E and one of the S_j is not a flat of M, then $\Delta(\sigma_{\mathcal{S}}) = 0$.

In short, we suppose that Δ is a Minkowski weight whose support is contained in the support of Δ_{M}. Note that any nonnegative summand of Δ_{M} should have this property. We show that there is a constant c such that $\Delta = c\, \Delta_{\mathrm{M}}$.

Let τ be an $(r-1)$-dimensional cone determined by a flag of nonempty proper flats $\mathcal{G} = \big(G_1 \subsetneq G_2 \subsetneq \cdots \subsetneq G_{r-1}\big)$. If $F_1, \ldots, F_m$ are the flats which are strictly compatible with $\mathcal{G}$, then the balancing condition for Δ at τ says that

$$\sum_{j=1}^{m} \Delta(\sigma_j)\mathbf{u}_{F_j} \quad \text{is in the subspace generated by } \tau,$$

where σ_j is the r-dimensional cone generated by $\mathbf{u}_{F_j}$ and $\mathbf{u}_{G_1}, \ldots, \mathbf{u}_{G_{r-1}}$. Writing G_{l-1} for the flat in $\mathcal{G}$ which is covered by (any) one of the F_j, we have

$$\sum_{j=1}^{m} \Delta(\sigma_j)\mathbf{u}_{F_j \setminus G_{l-1}} = c_1 \mathbf{u}_{G_1} + c_2 \mathbf{u}_{G_2 \setminus G_1} + \cdots + c_{r-1}\mathbf{u}_{G_{r-1}\setminus G_{r-2}}$$

for some real numbers $c_1, c_2, \ldots, c_{r-1}$. One can solve this equation explicitly using the fact that $G_l \setminus G_{l-1}$ is a disjoint union of the nonempty sets $F_j \setminus G_{l-1}$:

(i) If $l \neq r$, then $\Delta(\sigma_1) = \cdots = \Delta(\sigma_m) = c_l$ and all the other c_k are zero.
(ii) If $l = r$, then $\Delta(\sigma_1) = \cdots = \Delta(\sigma_m) = c_1 = \cdots = c_{r-1}$.

In any case, we write c for the common value of $\Delta(\sigma_j)$ and repeat the above argument for all $(r-1)$-dimensional cones τ. Since Δ_{M} is connected in codimension 1, we have $\Delta = c\, \Delta_{\mathrm{M}}$. □

EXAMPLE 5.8. The permutohedral surface X_{A_2} is the blowup of the three torus invariant points of $\mathbb{P}^2$. Let $\pi_1 : X_{A_2} \longrightarrow \mathbb{P}^2$ be the blowup map, D_0, D_1, D_2 be the exceptional curves, and H be the pull-back of a general line. The nef cone of curves in X_{A_2} is a four-dimensional polyhedral cone with five rays. The rays are generated by the classes of

$$H, \quad H - D_0, \quad H - D_1, \quad H - D_2, \quad \text{and} \quad 2H - D_0 - D_1 - D_2.$$

The first four classes come from matroids on $E = \{0, 1, 2\}$. The matroid corresponding to H has five flats

$$\varnothing,\ \{0\},\ \{1\},\ \{2\},\ E,$$

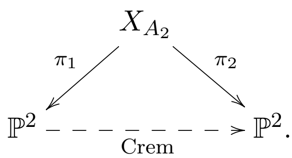
and the matroid corresponding to $H - D_i$ has four flats

$$\varnothing,\ \{i\},\ E\setminus\{i\},\ E.$$

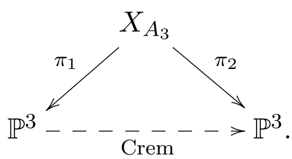
The remaining class is the class of the strict transform under π_1 of a general conic passing through the three torus-invariant points of $\mathbb{P}^2$. It is Cremona symmetric to the class of H, and comes from the matroid on $\hat{E} = \{\hat{0}, \hat{1}, \hat{2}\}$, where $\hat{i} = E \setminus \{i\}$, whose flats are

$$\varnothing,\ \{\hat{0}\},\ \{\hat{1}\},\ \{\hat{2}\},\ \hat{E}.$$

It is the class of the pull-back of a general line through the map π_2 in the diagram



Example 5.9. The fan displacement rule of [**FS97**] shows that the product of two nef classes in a toric variety is a nef class. However, one should not expect that the product of two extremal nef classes in a toric variety is an extremal nef class. For example, consider the diagram



If H_1, H_2 are hyperplanes in $\mathbb{P}^3$, then the classes of their pullbacks $\pi_1^{-1}(H_1)$, $\pi_2^{-1}(H_2)$ are extremal nef classes in X_{A_3}. We note that the class of the product $\pi_1^{-1}(H_1) \cdot \pi_1^{-1}(H_2)$ is a sum of three different extremal nef curve classes in X_{A_3}. One may show this by directly computing the cup product of the piecewise linear functions α and β using Theorem 4.4. Alternatively, one may see this geometrically as follows. Let H_1 be the plane

$$z_0 + z_1 + z_2 + z_3 = 0,$$

and choose H_2 so that $\pi_2^{-1}(H_2)$ is the strict transform under π_1 of the cubic surface

$$z_0z_1z_2 + z_0z_1z_3 + z_0z_2z_3 + z_1z_2z_3 = 0.$$

Then the intersection in $\mathbb{P}^3$ is the union of three lines

$$\{z_0 + z_1 = z_2 + z_3 = 0\} \cup \{z_0 + z_2 = z_1 + z_3 = 0\} \cup \{z_0 + z_3 = z_1 + z_2 = 0\}.$$

The strict transform of any one of the three lines under π_1 generates an extremal ray of the nef cone of X_{A_3}. Their classes correspond to, respectively, to rank 2 matroids whose nonempty proper flats are

$$\big\{\{0,1\},\{2,3\}\big\} \quad \text{and} \quad \big\{\{0,3\},\{1,3\}\big\} \quad \text{and} \quad \big\{\{0,3\},\{1,2\}\big\}.$$

In this case, the intersection of the strict transforms is the strict transform of the intersection. It follows that the sum of the three nef curve classes in X_{A_3} is the product $\pi_1^{-1}(H_1) \cdot \pi_1^{-1}(H_2)$.

In general, finding all extremal rays of the nef cone is difficult, even for relatively simple toric varieties. Here is a sample question: How many extremal rays are there in the nef cone of X_{A_n} in dimension two for some small values of n?

Let Δ be a two-dimensional nonnegative Minkowski weight on the fan of X. It is convenient to think the support of Δ as a geometric graph G_Δ, whose vertices are the primitive generators of the rays of the cones in the support of Δ. Two vertices of G_Δ are connected by an edge if and only if they generate a cone on which Δ is nonzero.

The main idea used in the proof of Theorem 5.7 gives a simple condition on G_Δ which guarantees that the class $\Delta \cap [X]$ generates an extremal ray of the nef cone of X. A few graphs G_Δ, including those of the Bergman fans of rank 3 matroids, satisfy this condition.

Proposition 5.10. *If G_Δ is connected and the set of neighbors of any vertex is linearly independent, then $\Delta \cap [X]$ generates an extremal ray of the nef cone of X in dimension two.*

The condition on G_Δ is not necessary for $\Delta \cap [X]$ to be extremal.

Example 5.11. Consider the two-dimensional Minkowski weight Δ on Δ_{A_4} which has value 1 on the cones corresponding to flags of the form

$$\{i\} \subsetneq \{i, j, k\}, \qquad i \neq j \neq k,$$

and has value 0 on all other two-dimensional cones of Δ_{A_4}. A direct computation shows that $\Delta \cap [X_{A_4}]$ generates an extremal ray of the nef cone of X_{A_4}. The graph of Δ has 10 vertices at which the set of neighbors is linearly dependent.

We note that $\Delta \cap [X_{A_4}]$ is an intersection of two extremal nef divisor classes. In fact, there is a single irreducible family of surfaces in X_{A_4} whose members have the class $\Delta \cap [X_{A_4}]$. The family consists of strict transforms under π_1 of the cubic surfaces in $\mathbb{P}^4$ defined by the equations

$$c_0z_0 + c_1z_1 + c_2z_2 + c_3z_3 + c_4z_4 = 0,$$

$$\begin{aligned} &c_{234}z_2z_3z_4 + c_{c134}z_1z_3z_4 + c_{124}z_1z_2z_4 + c_{123}z_1z_2z_3 + c_{034}z_0z_3z_4 \\ &\quad + c_{024}z_0z_2z_4 + c_{023}z_0z_2z_3 + c_{014}z_0z_1z_4 + c_{013}z_0z_1z_3 + c_{012}z_0z_1z_2 = 0, \end{aligned}$$

where c_i and c_{ijk} are parameters. Each one of the above two equations defines a basepoint free linear system on X_{A_4} whose class generates an extremal ray of the nef cone of divisors.

Example 5.12. Consider the two-dimensional Minkowski weight Δ on Δ_{A_n} which has value 1 on the cones corresponding to flags of the form

$$\{i\} \subsetneq E \setminus \{j\}, \qquad i \neq j,$$

and has value 0 on all other two-dimensional cones of Δ_{A_n}. Then the graph G_Δ satisfies the condition of Proposition 5.10, and hence $\Delta \cap [X_{A_n}]$ generates an extremal ray of the nef cone of X_{A_n}. This homology class is invariant under the Cremona symmetry of X_{A_n} and the action of the symmetric group on E.

When $n = 3$, the graph is that of a three-dimensional cube. Since the codimension of Δ is 1, it is not difficult to describe families of surfaces in X_{A_3} whose members have the class $\Delta \cap [X_{A_3}]$. There is a single irreducible family, and it consists of strict transforms under π_1 of the quadric surfaces in $\mathbb{P}^3$ defined by

$$c_{01}z_0z_1 + c_{02}z_0z_2 + c_{03}z_0z_3 + c_{12}z_1z_2 + c_{13}z_1z_3 + c_{23}z_2z_3 = 0,$$

where c_{ij} are parameters. The equation defines a basepoint free linear system on X_{A_3} which is invariant under the Cremona symmetry of X_{A_3} and the action of the symmetric group on E.

When $n = 4$, the extremal nef class $\Delta \cap [X_{A_4}]$ has the interesting property that

(1) $\Delta \cap [X_{A_4}]$ is not a product of two nef (integral) divisor classes, and
(2) $2\Delta \cap [X_{A_4}]$ is a product of two nef (integral) divisor classes.

To see that $\Delta \cap [X_{A_4}]$ is not an intersection of two nef divisor classes, one notes that any surface S in X_{A_4} which has the class $\Delta \cap [X_{A_4}]$ should map to a cubic surface in $\mathbb{P}^4$ under the maps π_1 and π_2. The cubic surfaces $\pi_1(S)$ and $\pi_2(S)$ are obtained by intersecting general members of torus-invariant linear systems on $\mathbb{P}^4$ outside their common base locus. If the common base locus has dimension less than 2, then the degrees of the linear systems are 1 and 3, and one can check directly that the class of S is not invariant either under the Cremona symmetry of X_{A_4} or under the action of the symmetric group on E. If the common base locus has dimension 2, then, since complete intersections are connected in codimension 1, $\pi_1(S)$ and $\pi_2(S)$ intersect some 2-dimensional torus orbits in $\mathbb{P}^4$ in curves. This contradicts that the Minkowski weight Δ has value zero on all flags involving two element subsets of E.

On the other hand, $2\Delta \cap [X_{A_4}]$ is an intersection of two nef divisor classes. The corresponding family comes from sextic surfaces in $\mathbb{P}^4$ defined by the equations

$$\begin{aligned} &c_{01}z_0z_1 + c_{02}z_0z_2 + c_{03}z_0z_3 + c_{04}z_0z_4 + c_{12}z_1z_2 \\ &\qquad + c_{13}z_1z_3 + c_{14}z_1z_4 + c_{23}z_2z_3 + c_{24}z_2z_4 + c_{34}z_3z_4 = 0, \end{aligned}$$

$$\begin{aligned} &c_{234}z_2z_3z_4 + c_{134}z_1z_3z_4 + c_{124}z_1z_2z_4 + c_{123}z_1z_2z_3 + c_{034}z_0z_3z_4 \\ &\qquad + c_{024}z_0z_2z_4 + c_{023}z_0z_2z_3 + c_{014}z_0z_1z_4 + c_{013}z_0z_1z_3 + c_{012}z_0z_1z_2 = 0, \end{aligned}$$

where c_{ij} and c_{ijk} are parameters. The family of strict transforms under π_1 is invariant under the Cremona symmetry of X_{A_4} and the action of the symmetric group on E. Each of its members has the class $2\Delta \cap [X_{A_4}]$. Each one of the above two equations defines a basepoint free linear system on X_{A_4} whose class generates an extremal ray of the nef cone of divisors.

It is more difficult to describe families of surfaces in X_{A_4} whose members have the homology class $\Delta\cap[X_{A_4}]$. In fact, there is a single irreducible family, and it is the family of strict transforms of cubic surfaces in $\mathbb{P}^4$ defined by the 2×2 minors of the matrix

$$\begin{bmatrix} c_{110}z_0 + c_{111}z_1 + c_{112}z_2 + c_{113}z_3 + c_{114}z_4 & c_{120}z_0 + c_{121}z_1 + c_{122}z_2 + c_{123}z_3 + c_{124}z_4 \\ c_{210}z_0 + c_{211}z_1 + c_{212}z_2 + c_{213}z_3 + c_{214}z_4 & c_{220}z_0 + c_{221}z_1 + c_{222}z_2 + c_{223}z_3 + c_{224}z_4 \\ c_{310}z_0 + c_{311}z_1 + c_{312}z_2 + c_{313}z_3 + c_{314}z_4 & c_{320}z_0 + c_{321}z_1 + c_{322}z_2 + c_{323}z_3 + c_{324}z_4 \end{bmatrix}$$

which is given by five sufficiently general rank 1 matrices

$$\begin{bmatrix} c_{110} & c_{120} \\ c_{210} & c_{220} \\ c_{310} & c_{320} \end{bmatrix}, \begin{bmatrix} c_{111} & c_{121} \\ c_{211} & c_{221} \\ c_{311} & c_{321} \end{bmatrix}, \begin{bmatrix} c_{112} & c_{122} \\ c_{212} & c_{222} \\ c_{312} & c_{322} \end{bmatrix}, \begin{bmatrix} c_{113} & c_{123} \\ c_{213} & c_{223} \\ c_{313} & c_{323} \end{bmatrix}, \begin{bmatrix} c_{114} & c_{124} \\ c_{214} & c_{224} \\ c_{314} & c_{324} \end{bmatrix}.$$

The family of strict transforms under π_1 is invariant under the Cremona symmetry of X_{A_4} and the action of the symmetric group on E.

In general, for $n \geq 3$, there is a single irreducible family whose members have the extremal nef class

$$\Delta \cap [X_{A_n}].$$

The family consists of strict transforms of rational scrolls in $\mathbb{P}^n$ which contain all the torus-invariant points and intersect no other torus orbits of codimension ≥ 2. The homology class is not an intersection of nef divisor classes. On the other hand,

$$(n-1)! \cdot \Delta \cap [X_{A_n}]$$

is an intersection of nef divisor classes. The corresponding family is the family of strict transforms of complete intersections in $\mathbb{P}^n$ defined by general linear combinations of square-free monomials in $z_0, z_1, \ldots, z_n$ with degrees $2, 3, \ldots, n-1$.

EXERCISE 5.13. Let Δ be a fan in $\mathbb{R}^n$ with fixed ray generators $\mathbf{u}_i$. A 2-dimensional weight w on Δ is said to be *geometrically balanced* if, for each ray i of Δ, there is a real number d_i satisfying

$$d_i \mathbf{u}_i = \sum_{i \sim j} w_{ij} \mathbf{u}_j,$$

where the sum is over all the neighbors j of i in Δ. The *tropical Laplacian* of such w is the square matrix L_w defined by

$$(L_w)_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -w_{ij} & \text{if } i \sim j, \\ 0 & \text{if } i \nsim j. \end{cases}$$

How is the corank of L_w related to the dimension of $\mathbb{R}^n$?

EXERCISE 5.14. Let w be the unique 2-dimensional positive weight on the Bergman fan of a rank 3 matroid. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

EXERCISE 5.15. Let w be the unique 2-dimensional positive weight on the fan over the edges of the standard cube. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

EXERCISE 5.16. Let w be any 2-dimensional positive weight on the fan over the edges of the standard octahedron. Show that the tropical Laplacian of w has exactly one negative eigenvalue.

EXERCISE 5.17. Let Δ_D be the 2-dimensional "Desargues" fan on $N_{\{0,1,2,3,4\}}$ with ray generators

$$\mathbf{e}_{01}, \ \mathbf{e}_{02}, \ \mathbf{e}_{03}, \ \mathbf{e}_{04}, \ \mathbf{e}_{12}, \ \mathbf{e}_{13}, \ \mathbf{e}_{14}, \ \mathbf{e}_{23}, \ \mathbf{e}_{24}, \ \mathbf{e}_{34},$$
$$\mathbf{e}_{234}, \ \mathbf{e}_{134}, \ \mathbf{e}_{124}, \ \mathbf{e}_{123}, \ \mathbf{e}_{034}, \ \mathbf{e}_{024}, \ \mathbf{e}_{023}, \ \mathbf{e}_{014}, \ \mathbf{e}_{013}, \ \mathbf{e}_{012},$$

whose 2-dimensional cones correspond to inclusions between the subsets of $\{0, 1, 2, 3, 4\}$.

(1) Show that there is, up to a multiple, unique 2-dimensional positive balanced weight w on Δ_D.
(2) Show that the tropical Laplacian of w has exactly one negative eigenvalue.

EXERCISE 5.18. Is there a 2-dimensional positive balanced weight w in $\mathbb{R}^3$ whose tropical Laplacian has more than one negative eigenvalue? For an example in $\mathbb{R}^4$, see [**BH17**].

6. Realizing matroids in the permutohedral variety

Now the field k comes into play. We will show that a matroid M is realizable over k if and only if the corresponding homology class $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ in the permutohedral variety is the class of a subvariety over k. This sharply contrasts Corollary 5.3, which says that the class $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ is the class of an effective cycle over k for any matroid M and any field k.

Let X_{A_n} be the permutohedral variety over k, and let M be a loopless matroid on $E = \{0, 1, \ldots, n\}$. By a subvariety of X_{A_n}, we mean a geometrically reduced and geometrically irreducible closed subscheme of finite type over k. As before, the rank of M is $r + 1$.

DEFINITION 6.1. A *realization* $\mathscr{R}$ of M over k is a collection of vectors $f_0, f_1, \ldots, f_n$ in an $(r + 1)$-dimensional vector space V over k with the following property:

> A subset I of E is independent for M if and only if $\{f_i \mid i \in I\}$ is linearly independent in V.

Since M has no loops, all the f_j are nonzero. The *arrangement* associated to $\mathscr{R}$, denoted $\mathcal{A}_{\mathscr{R}}$, is the hyperplane arrangement

$$\mathcal{A}_{\mathscr{R}} := \{f_0 f_1 \cdots f_n = 0\} \subseteq \mathbb{P}(V^\vee),$$

where $\mathbb{P}(V^\vee)$ is the projective space of hyperplanes in V. We say that a linear subspace of $\mathbb{P}(V^\vee)$ is a *flat* of $\mathcal{A}_{\mathscr{R}}$ if it is an intersection of hyperplanes in

$\mathcal{A}_{\mathscr{R}}$. There is an inclusion reversing bijection between the flats of M and the flats of $\mathcal{A}_{\mathscr{R}}$:

$$F \longmapsto \bigcap_{j \in F} \{f_j = 0\}.$$

The *embedding* associated to $\mathscr{R}$, denoted $L_{\mathscr{R}}$, is the map from the projectivized dual

$$L_{\mathscr{R}} : \mathbb{P}(V^\vee) \simeq \mathbb{P}^r \longrightarrow \mathbb{P}^n, \qquad L_{\mathscr{R}} = [f_0 : f_1 : \cdots : f_n].$$

Since $f_0, f_1, \ldots, f_n$ generate V, the linear map $L_{\mathscr{R}}$ is well-defined and is an embedding. Furthermore, since M has no loops, the generic point of $\mathbb{P}(V^\vee)$ maps to the open torus orbit of $\mathbb{P}^n$. If k is infinite, then a general point of $\mathbb{P}(V^\vee)$ maps to the open torus orbit of $\mathbb{P}^n$. Under the embedding $L_{\mathscr{R}}$, the union of the torus-invariant hyperplanes in $\mathbb{P}^n$ pullbacks to the arrangement $\mathcal{A}_{\mathscr{R}}$.

DEFINITION 6.2. The variety of $\mathscr{R}$, denoted $Y_{\mathscr{R}}$, is the strict transform of the image of $L_{\mathscr{R}}$ under the composition of blowups $\pi_1 : X_{A_n} \longrightarrow \mathbb{P}^n$. By definition, there is a commutative diagram

$$\begin{array}{ccc} Y_{\mathscr{R}} & \xrightarrow{\iota_{\mathscr{R}}} & X_{A_n} \\ \downarrow{\scriptstyle \pi_{\mathscr{R}}} & & \downarrow{\scriptstyle \pi_1} \\ \mathbb{P}(V^\vee) & \xrightarrow[L_{\mathscr{R}}]{} & \mathbb{P}^n, \end{array}$$

where $\iota_{\mathscr{R}}$ is the inclusion and $\pi_{\mathscr{R}}$ is the induced blowup.

Recall that π_1 can be factored into

$$X_{A_n} \;=\; X_{n-1} \longrightarrow X_{n-2} \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 \;=\; \mathbb{P}^n,$$

where $X_{d+1} \longrightarrow X_d$ is the blowup of the strict transform of the union of all the torus-invariant d-dimensional linear subspaces of $\mathbb{P}^n$. If S is a proper subset of E with $|S| \geq 2$, then D_S is the exceptional divisor of π_1 corresponding to the codimension $|S|$ linear subspace

$$\bigcap_{i \in S} \big\{z_i = 0\big\} \subseteq \mathbb{P}^n.$$

If $S = \{i\}$, then D_S is the strict transform of the hyperplane $\{z_i = 0\}$. The union of all the D_S is a simple normal crossings divisor whose complement in X_{A_n} is the open torus orbit.

Similarly, $\pi_{\mathscr{R}}$ is the blowup of all the flats of the hyperplane arrangement $\mathcal{A}_{\mathscr{R}}$. It is the composition of maps

$$Y_{\mathscr{R}} \;=\; Y_{r-1} \longrightarrow Y_{r-2} \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y_0 \;=\; \mathbb{P}(V^\vee),$$

where $Y_{d+1} \longrightarrow Y_d$ is the blowup of the strict transform of the union of all the d-dimensional flats of $\mathcal{A}_{\mathscr{R}}$. Exceptional divisors of $\pi_{\mathscr{R}} : Y_{\mathscr{R}} \longrightarrow \mathbb{P}(V^\vee)$ are indexed by flats with rank at least 2.

NOTATION. If F is a flat of rank at least 2, then we write E_F for the exceptional divisor of $\pi_{\mathscr{R}}$ corresponding to the codimension $\operatorname{rank}_M(F)$ linear subspace

$$\bigcap_{j \in F} \{f_j = 0\} \subseteq \mathbb{P}(V^\vee).$$

When F is a flat of rank 1, we define E_F to be the strict transform of the hyperplane of $\mathbb{P}(V^\vee)$ corresponding to F.

The union of all the E_F is a simple normal crossings divisor whose complement in $Y_{\mathscr{R}}$ is the intersection of $Y_{\mathscr{R}}$ with the open torus orbit of X_{A_n}. In the language of De Concini and Procesi [**DP95**], the variety $Y_{\mathscr{R}}$ is the wonderful compactification of the arrangement complement $\mathbb{P}(V^\vee) \setminus \mathcal{A}_{\mathscr{R}}$ corresponding to the maximal building set.

The statement below is a classical variant of the tropical statement of Katz and Payne [**KP11**].

THEOREM 6.3. *Let X_{A_n} be the n-dimensional permutohedral variety over k.*

(i) If $\mathscr{R}$ is a realization of M over k, then

$$[Y_{\mathscr{R}}] = \Delta_{\mathrm{M}} \cap [X_{A_n}] \in A_r(X_{A_n}).$$

(ii) If Y is an r-dimensional subvariety of X_{A_n} such that

$$[Y] = \Delta_{\mathrm{M}} \cap [X_{A_n}] \in A_r(X_{A_n}),$$

then $Y = Y_{\mathscr{R}}$ for some realization $\mathscr{A}$ of M over k.

In particular, M is realizable over k if and only if $\Delta_{\mathrm{M}} \cap [X_{A_n}]$ is the class of a subvariety over k.

PROOF. For a nonempty proper subset F of E, the subvariety $Y_{\mathscr{R}}$ intersects the torus-invariant divisor D_F if and only if F is a flat of M. In this case,

$$Y_{\mathscr{R}} \cap D_F = E_F.$$

Let $\mathcal{F} = \big(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\big)$ be a flag of nonempty proper subsets of E. If one of the F_j is not a flat of M, then

$$Y_{\mathscr{R}} \cap V(\mathcal{F}) = Y_{\mathscr{R}} \cap D_{F_1} \cap \cdots \cap D_{F_r} = \varnothing.$$

If all the F_j are flats of M, then

$$Y_{\mathscr{R}} \cap V(\mathcal{F}) = Y_{\mathscr{R}} \cap D_{F_1} \cap \cdots \cap D_{F_r} = E_{F_1} \cap \cdots \cap E_{F_r},$$

and this intersection is a reduced point. Therefore, the class of $Y_{\mathscr{R}}$ in X_{A_n} is Poincaré dual to the Bergman fan of M. In other words,

$$[Y_{\mathscr{R}}] = \Delta_M \cap [X_{A_n}] \in A_r(X_{A_n}).$$

This proves the first assertion.

Conversely, suppose that Y is an r-dimensional subvariety of X_{A_n} defined over k such that

$$[Y] = \Delta_{\mathrm{M}} \cap [X_{A_n}] \in A_r(X_{A_n}).$$

As an intermediate step, we prove that Y is not contained in any torus-invariant hypersurface of X_{A_n}.

Consider a torus-invariant prime divisor of X_{A_n}. It is of the form D_S for some nonempty proper subset S of E. We show that Y is not contained in D_S. Choose a rank 1 flat F_1 which is not comparable to S. This is possible because M has no loops and E is a disjoint union of the rank 1 flats of M. We extend F_1 to a maximal flag of proper flats

$$\mathcal{F} = \Big(F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_r\Big).$$

By definition of the Bergman fan Δ_{M}, we have

$$D_{F_1} \cdot D_{F_2} \cdot \ldots \cdot D_{F_r} \cdot [Y] = 1.$$

On the other hand, if Y is contained in D_S, then the above intersection product can be computed by pulling back the divisors D_{F_j} under the inclusion $\iota : D_S \to X_{A_n}$. Since F_1 is not comparable to S, the pull-back of D_{F_1} to D_S is equivalent to zero. This leads to the contradiction that

$$\iota^* D_{F_1} \cdot \iota^* D_{F_2} \cdot \ldots \cdot \iota^* D_{F_r} \cdot [Y] = 0.$$

We now show that $Y = Y_{\mathscr{R}}$ for some realization $\mathscr{R}$ of M. Let i be an element of E, and let $H_i = \{z_i = 0\}$ be the corresponding hyperplane of $\mathbb{P}^n$. Proposition 4.7 shows that

$$\underbrace{\pi_1^{-1}(H_i) \cdot \ldots \cdot \pi_1^{-1}(H_i)}_{r} \cdot [Y] = \Big(\underbrace{\alpha \cup \cdots \cup \alpha}_{r} \cup \Delta_{\mathrm{M}}\Big) \cap [X_{A_n}] = 1.$$

The projection formula tells us that the image $\pi_1(Y)$ is an r-dimensional subvariety of $\mathbb{P}^n$ which has degree 1. In other words, the image is an r-dimensional linear subspace

$$\pi_1(Y) = \mathbb{P}^r \subseteq \mathbb{P}^n.$$

Write the equations defining the above linear embedding by

$$L : \mathbb{P}^r \longrightarrow \mathbb{P}^n, \qquad f = [f_0 : f_1 : \cdots : f_n].$$

Since Y is not contained in any torus-invariant hypersurface of X_{A_n}, the image $\pi_1(Y)$ is not contained in any torus-invariant hyperplane of $\mathbb{P}^n$. Therefore all the linear forms f_j are nonzero. Let $\mathscr{R}$ be the set of vectors $\{f_0, f_1, \ldots, f_n\}$ in the $(r+1)$-dimensional vector space $H^0\big(\mathbb{P}^r, \mathcal{O}(1)\big)$. This defines a loopless matroid N on E which is realizable over k.

By definition of the strict transform, $Y = Y_{\mathscr{R}}$. Applying the first part of the theorem to $Y_{\mathscr{R}}$, we have

$$[Y] = [Y_{\mathscr{R}}] = \Delta_N \cap [X_{A_n}],$$

Since the set of flats of a matroid determines the matroid, M = N. This proves the second assertion. ☐

We remark that, when M is realizable over k, the ring $A^*(\mathrm{M})$ is isomorphic to the Chow ring of the variety $Y_{\mathscr{R}}$ for any realization $\mathscr{R}$ of M over k.

EXAMPLE 6.4. We work with the permutohedral variety $X = X_{A_6}$ over the integers. Consider the embedding

$$L : \mathbb{P}^2 \longrightarrow \mathbb{P}^6,$$

$$[x_0 : x_1 : x_2] \longmapsto [x_0 : x_1 : x_2 : x_0 + x_1 : x_0 + x_2 : x_1 + x_2 : x_0 + x_1 + x_2].$$

The image of L is the intersection of the ten hyperplanes

$$\begin{aligned} &H_1 = \{z_5 = z_1 + z_2\},\ H_2 = \{z_4 = z_0 + z_2\},\ H_3 = \{z_3 = z_0 + z_1\}, \\ &H_4 = \{z_6 = z_2 + z_3\},\ H_5 = \{z_6 = z_1 + z_4\},\ H_6 = \{z_6 = z_0 + z_5\}, \\ &H_7 = \{2z_0 = z_3 + z_4 - z_5\},\ H_8 = \{2z_1 = z_3 - z_4 + z_5\}, \\ &H_9 = \{2z_2 = -z_3 + z_4 + z_5\},\ H_{10} = \{2z_6 = z_3 + z_4 + z_5\}. \end{aligned}$$

Let $\tilde{H}_j$ be the strict transform of H_j under the blowup $\pi_1 : X \longrightarrow \mathbb{P}^6$. For each prime number p, we have the commutative diagram over $\mathbb{Z}/p$

$$\begin{array}{ccc} Y_p & \longrightarrow & X_p \\ \Big\downarrow & & \Big\downarrow \pi_{1,p} \\ \mathbb{P}^r_p & \xrightarrow{\ L_p\ } & \mathbb{P}^n_p, \end{array}$$

where Y_p is the strict transform of the image of L_p under the blowup $\pi_{1,p}$. Write $\tilde{H}_{j,p}$ for the intersection of $\tilde{H}_j$ and X_p. For any prime number p, we have

$$\Big[\bigcap_{j=0}^{9} \tilde{H}_{j,p}\Big] = \Delta_M \cap [X_p],$$

where M is the rank 3 matroid on E whose rank 2 flats are

$$\{0,1,3\}, \quad \{0,2,4\}, \quad \{1,2,5\}, \quad \{0,5,6\}, \quad \{1,4,6\}, \quad \{2,3,6\}.$$

If $p \neq 2$, then

$$\bigcap_{j=1}^{10} \tilde{H}_{j,p} = Y_p.$$

If $p = 2$, then

$$\bigcap_{j=1}^{10} \tilde{H}_{j,p} = Y_p \cup S_p,$$

for some surface S_p in X_p. As a family of subschemes of X over $\mathrm{Spec}(\mathbb{Z})$, the latter is the limit of the former. When $p = 2$, we have

$$[Y_p] = \Delta_{\mathrm{N}} \cap [X_p],$$

where N is the rank 3 matroid on E whose rank 2 flats are

$$\{0,1,3\},\quad \{0,2,4\},\quad \{1,2,5\},\quad \{0,5,6\},\quad \{1,4,6\},\quad \{2,3,6\},\quad \{3,4,5\}.$$

The matroid N is realized by the Fano plane, the configuration of the seven nonzero vectors in the three-dimensional vector space over the field with two elements. This matroid is not realizable over fields with characteristic not equal to 2.

EXERCISE 6.5. Consider the quartic surface S in $\mathbb{P}^4$ defined by

$$z_0 + z_1 + z_2 + z_3 + z_4 = 0,$$
$$z_1z_2z_3z_4 + z_0z_2z_3z_4 + z_0z_1z_3z_4 + z_0z_1z_2z_4 + z_0z_1z_2z_3 = 0.$$

This quartic surface contains ten lines

$$z_{i_0} + z_{i_1} + z_{i_2} = z_{i_3} = z_{i_4} = 0$$

and ten double points

$$z_{i_0} = z_{i_1} = z_{i_2} = z_{i_3} + z_{i_4} = 0,$$

where $(i_0, i_1, i_2, i_3, i_4)$ is a permutation of $(0, 1, 2, 3, 4)$. Every line contains three of the ten points, and every point is contained in three of the ten lines. In fact, the incidence between the lines and the points is that of the Desargues configuration in a projective plane. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\iota_{\tilde{S}}} & X_{A_4} \\ {\scriptstyle \pi_S}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ S & \xrightarrow[\iota_S]{} & \mathbb{P}^4, \end{array}$$

where ι_S, $\iota_{\tilde{S}}$ are inclusions and π_S is the blowup of the ten singular points. The smooth surface $\tilde{S}$ is invariant under the Cremona symmetry of X_{A_4} and the action of the symmetric group on E. It contains twenty smooth rational curves with self-intersection (-2), namely the strict transforms of the ten lines and the exceptional curves over the ten singular points. Any two of the twenty curves are either disjoint or intersect transversely at one point. Show that the homology class of $\tilde{S}$ generates an extremal ray of the nef cone of X_{A_4}.

EXERCISE 6.6. Consider the quartic surface S in $\mathbb{P}^4$ defined by

$$z_1+z_2+z_3+z_4 = 0, \quad z_1z_2z_3z_4+z_0z_2z_3z_4+z_0z_1z_3z_4+z_0z_1z_2z_4+z_0z_1z_2z_3 = 0.$$

This quartic surface contains four lines of the form

$$z_{i_1} + z_{i_2} + z_{i_3} = z_{i_4} = z_0 = 0,$$

where (i_1, i_2, i_3, i_4) is a permutation of $(1, 2, 3, 4)$, six lines of the form

$$z_{i_1} + z_{i_2} = z_{i_3} = z_{i_4} = 0,$$

where (i_1, i_2, i_3, i_4) is a permutation of $(1, 2, 3, 4)$, six double points

$$z_0 = z_{i_1} = z_{i_2} = z_{i_3} + z_{i_4} = 0,$$

where (i_1, i_2, i_3, i_4) is a permutation of $(1, 2, 3, 4)$, and one triple point

$$z_1 = z_2 = z_3 = z_4 = 0.$$

The incidence between the ten lines and the seven points is that of the rank 2 flats and rank 3 flats of the matroid M on $\{0, 1, 2, 3, 4\}$ which has one minimal dependent set $\{1, 2, 3, 4\}$. We have a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\iota_{\tilde{S}}} & X_{A_4} \\ \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi_1} \\ S & \xrightarrow[\iota_S]{} & \mathbb{P}^4, \end{array}$$

where ι_S, $\iota_{\tilde{S}}$ are inclusions and π_S is the blowup of the seven singular points. The strict transforms of the ten lines are smooth rational curves in $\tilde{S}$ disjoint from each other. Four of the ten curves, those corresponding to the lines containing the triple point, have self-intersection (-2). The remaining six have self-intersection (-1). The six exceptional curves over the double points of S are smooth rational curves in $\tilde{S}$ with self-intersection (-2), and the exceptional curve over the triple point is an elliptic curve with self-intersection (-3). A curve corresponding to a line meets a curve corresponding to a singular point if and only if the line contains the point. In this case, the two curves intersect transversely at one point. Show that the homology class of $\tilde{S}$ generates an extremal ray of the nef cone of X_{A_4}.

7. The characteristic polynomial is the anticanonical image

The anticanonical linear system of the permutohedral variety X_{A_n} is basepoint free and big. The product map $\pi_1 \times \pi_2$ may be viewed as the anticanonical map of X_{A_n}, where π_1, π_2 are the blowups in the commutative diagram

$$\begin{array}{ccc} & X_{A_n} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\ \mathbb{P}^n & \overset{\text{Crem}}{\dashrightarrow} & \mathbb{P}^n. \end{array}$$

Write the reduced characteristic polynomial of a loopless matroid M as

$$\overline{\chi}_{\mathrm{M}}(q) = \chi_{\mathrm{M}}(q)/(q-1) = \sum_{l=0}^{r} (-1)^l \mu_{\mathrm{M}}^l q^{r-l}.$$

Proposition 2.20 shows that the reduced characteristic polynomial $\overline{\chi}_{\mathrm{M}}(q)$ represents the push-forward of the homology class of M under the anti-canonical mapping.

THEOREM 7.1. *Under the anticanonical map*

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n,$$

the homology class of M *push-forwards to its reduced characteristic polynomial* $\overline{\chi}_{\mathrm{M}}(q)$*:*

$$\Delta_{\mathrm{M}} \cap [X_{A_n}] \longmapsto \sum_{l=0}^{r} \mu_{\mathrm{M}}^{l} [\mathbb{P}^{r-l} \times \mathbb{P}^{l}].$$

Let X be an n-dimensional smooth complete variety over an algebraically closed field k. The group of numerical equivalence classes of d-dimensional cycles $N_d(X)$ is a finitely generated abelian group with several additional structures. In particular, it contains

(1) the set of prime classes, the classes of subvarieties,
(2) the set of effective classes, the nonnegative linear combinations of prime classes,
(3) the set of nef classes, the classes which intersect all codimension d primes nonnegatively.

The semigroups (2) and (3) define cones in the finite-dimensional vector space $N_d(X)_{\mathbb{R}}$, the pseudoeffective cone and the nef cone of X. When X is a toric variety, the group $N_d(X)$ and its subsets (2), (3) are determined by the fan of X. On the other hand, in general, the subset (1) depends on the field k, as we have seen in Theorem 6.3 for permutohedral varieties.

DEFINITION 7.2. A homology class $\xi \in N_d(X)_{\mathbb{R}}$ is said to be *prime* if some positive multiple of ξ is the class of a subvariety of X. Define

$$P_d(X) := \Big(\text{the closure of the set of prime classes in } N_d(X)_{\mathbb{R}}\Big).$$

The set $P_d(X)$ is a closed subset of the finite-dimensional vector space $N_d(X)_{\mathbb{R}}$ invariant under scaling by positive real numbers. It contains all extremal rays of the pseudoeffective cone of X in dimension d.

QUESTION 7.3. Suppose X is a smooth projective toric variety over an algebraically closed field k. Does $P_d(X)$ depend only on the fan of X and not on k?

The theorem of Kleiman says that a nef divisor class is a limit of ample divisor classes [**Kle66**]. This shows that

$$\mathrm{Nef}_{n-1}(X) \subseteq P_{n-1}(X) \subseteq \mathrm{Peff}_{n-1}(X).$$

The theorem of Boucksom-Demailly-Paun-Peternell says that a nef curve class is a limit of movable curve classes [**BDPP13**]. This shows that

$$\mathrm{Nef}_1(X) \subseteq P_1(X) \subseteq \mathrm{Peff}_1(X).$$

In general, the set $P_d(X)$ does not contain all the pseudoeffective nef classes of X. If X is a product of two projective spaces, then $P_d(X)$ is the set of log-concave sequences of nonnegative numbers with no internal zeros [**Huh12**].

THEOREM 7.4. *If ξ is an element in the homology group*

$$\xi = \sum_j x_j \left[\mathbb{P}^{d-j} \times \mathbb{P}^j\right] \in A_d(X), \quad X = \mathbb{P}^{n-m} \times \mathbb{P}^m,$$

then some positive integer multiple of ξ is the class of a subvariety if and only if the x_j form a nonzero log-concave sequence of nonnegative integers with no internal zeros.

Therefore, in the vector space $N_d(X)_\mathbb{R}$ of numerical cycle classes in the product of two projective spaces, the elements of the subset $P_d(X)$ correspond to log-concave sequences of nonnegative real numbers with no internal zeros, while the elements of the cones $\mathrm{Nef}_d(X)$ and $\mathrm{Peff}_d(X)$ correspond to sequences of nonnegative real numbers. Note that Theorems 7.1 and 7.4 together imply the log-concavity conjectures when M is realizable over some field.

EXAMPLE 7.5. There is no five-dimensional subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class

$$\xi = 1[\mathbb{P}^5 \times \mathbb{P}^0] + 2[\mathbb{P}^4 \times \mathbb{P}^1] + 3[\mathbb{P}^3 \times \mathbb{P}^2] + 4[\mathbb{P}^2 \times \mathbb{P}^3] + 2[\mathbb{P}^1 \times \mathbb{P}^4] + 1[\mathbb{P}^0 \times \mathbb{P}^5],$$

although $(1, 2, 3, 4, 2, 1)$ is a log-concave sequence with no internal zeros. This follows from the classification of the quadro-quadric Cremona transformations of Pirio and Russo [**PR12**]. On the other hand, the proof of Theorem 7.4 shows that there is a five-dimensional subvariety of $\mathbb{P}^5 \times \mathbb{P}^5$ which has the homology class $48\,\xi$.

Recall that the anticanonical push-forward of the homology class of a matroid M in X_{A_n} is the reduced characteristic polynomial $\overline{\chi}_\mathrm{M}(q)$:

$$\pi_1 \times \pi_2 : X_{A_n} \longrightarrow \mathbb{P}^n \times \mathbb{P}^n, \qquad \Delta_\mathrm{M} \cap [X_{A_n}] \longmapsto \overline{\chi}_\mathrm{M}(q).$$

Therefore, the coefficients of the reduced characteristic polynomial $\overline{\chi}_\mathrm{M}(q)$ form a log-concave sequence if and only if

$$(\pi_1 \times \pi_2)_*\, \Delta_\mathrm{M} \cap [X_{A_n}] \in P_r(\mathbb{P}^n \times \mathbb{P}^n).$$

We ask whether the same inclusion holds in the permutohedral variety X_{A_n}.

QUESTION 7.6. For any matroid M and any algebraically closed field k, do we have

$$\Delta_\mathrm{M} \cap [X_{A_n}] \in P_r(X_{A_n})?$$

In view of Theorem 6.3, the question asks whether every matroid is realizable over every field, perhaps not as an integral homology class, but as a limit of homology class with real coefficients. Since $P_r(X_{A_n})$ maps to $P_r(\mathbb{P}^n \times \mathbb{P}^n)$ under the anticanonical push-forward, Question 7.6 can be viewed as a strengthening of the log-concavity conjectures.

EXAMPLE 7.7. Let E be a finite subset of a field containing k . Call a subset of E independent if it is algebraically independent over k . This defines a matroid M *algebraic* over k, see [**Oxl11**, Chapter 6].

Non-algebraic matroids exist. Ingleton and Main showed that the *Vámos matroid*, the matroid on $\{0, 1, \ldots, 7\}$ whose maximal independent sets are

$$\binom{E}{4} \setminus \Big\{\{0,1,2,3\}, \{2,3,4,5\}, \{4,5,6,7\}, \{0,1,6,7\}, \{0,1,4,5\}\Big\},$$

is not algebraic over any field. Many more examples can be found in [**Oxl11**, Appendix]. An argument of Josephine Yu [**Yu17**] can be used to show that:

If M *is not algebraic over k, then no multiple of Δ_{M} is the class of a subvariety of X_{A_n} over k.*

It is a challenging problem to decide whether the Bergman fan of the Vámos matroid is contained in $P_3(X_{A_7})$.

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