Representation theory for polymatroids

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Abstract. We develop a theory of representations of (discrete) polymatroids over tracts in terms of Plücker coordinates and suitable Plücker relations. As special cases, we recover polymatroids themselves as polymatroid representations over the Krasner hyperfield $\mathbb{K}$ and M-convex functions as polymatroid representations over the tropical hyperfield $\mathbb{T}_0$.

We introduce and study several useful operations for polymatroid representations, such as translation and refined notions of minors and duality which have better properties than the existing definitions; for example, deletion and contraction become dual operations (up to translation) in our setting. We also prove an *idempotency principle* which asserts, roughly speaking, that polymatroids which are not translates of matroids are representable only over tracts in which $-1 = 1$.

The space of all representations of a polymatroid J, which we call the *thin Schubert cell* of J, is represented by an algebraic object called *universal tract* of J. When we restrict to just the 3-term Plücker relations, we obtain the *weak thin Schubert cell*, and passing to torus orbits yields the *realization space*. These are represented by the *universal pasture* and the *foundation* of J, respectively. We exhibit a canonical bijection between the universal tract and the universal pasture, which is new even in the case of matroids, and we show that the foundation of a polymatroid is generated by *cross ratios*. We also describe a (possibly incomplete) list of multiplicative relations between cross ratios.

Thin Schubert cells and realization spaces are canonically embedded in certain tori. Over idempotent tracts, we show that thin Schubert cells contain a canonical torus orbit and split naturally as a product of the realization space with this distinguished torus.

Contents

Introduction

Polymatroids were originally introduced in 1970 by Jack Edmonds [24] in the context of combinatorial optimization theory. Kazuo Murota demonstrated the importance of discrete polymatroids[1] to discrete convex analysis, via the equivalent notion of M-convex sets, and introduced M-convex functions as valuated generalizations. Much of this theory is developed in his book "Discrete Convex Analysis" [34].

Discrete polymatroids are treated more systematically from a combinatorial point of view analogous to matroid theory in Herzog and Hibi's article [30]. As a sampling of the many papers generalizing results from matroid theory to discrete polymatroids, we mention [13, 35, 36, 37].

Postnikov [41] investigated a class of polytopes called "generalized permutohedra", which arise as deformations of the standard permutohedron. These polytopes were subsequently recognized to be equivalent (up to translation) to base polytopes of polymatroids by an extension of the classical result by Gelfand, Goresky, MacPherson, and Serganova [28]. Since polymatroids are cryptomorphically equivalent to their base polytopes, generalized permutohedra can be viewed as another combinatorial representation of polymatroids.

In recent years, there has been an explosion of interest in polymatroids, motivated in part by Petter Brändén's discovery [14] that the support of a homogeneous multivariate stable polynomial is an M-convex set, that is, the set of bases of a discrete polymatroid. This generalized an earlier result of Choe, Oxley, Sokal, and Wagner [20] showing that the support

[1]In the body of this paper, we will be concerned solely with discrete polymatroids and so we omit the modifier "discrete", calling these objects simply "polymatroids". However, for the purposes of this historical introduction, we distinguish between polymatroids in the sense of Edmonds (which are certain real-valued submodular functions) with discrete polymatroids (a.k.a. M-convex sets or integer polymatroids) in the sense of Murota et. al.

of a *multi-affine* homogeneous stable polynomial is the set of bases of a matroid, and was generalized by Brändén and Huh, who showed that the support of a *Lorentzian* polynomial is always M-convex [17, Theorem 2.25]. The latter result is particularly interesting because, in fact, a subset of $\mathbb{N}^n$ is M-convex if and only if its normalized generating polynomial is a Lorentzian polynomial [17, Theorem 3.10].

Lorentzian polynomials have a close connection to combinatorial Hodge theory, and Hodge theory for matroids in the sense of Adiprasito–Huh–Katz [1] was recently extended to discrete polymatroids by Pagaria and Pezzoli [38]. Other papers exploring recently discovered connections between discrete polymatroids and combinatorial Hodge theory include the work of Crowley–Huh–Larson–Simpson–Wang [21], Crowley–Simpson–Wang [22], and Eur–Larson [25].

The relevance of, and interest in, discrete polymatroids is by no means confined to combinatorial Hodge theory. As a sampling of some other interesting applications of discrete polymatroids, we mention:

- Knutson and Tao's proofs of Horn's conjecture and the saturation conjecture [33] implicitly employ M-convex functions, which correspond to $\mathbb{T}_0$-representations of discrete polymatroids in our terminology (where $\mathbb{T}_0$ denotes the tropical hyperfield). The "hives" that provide the technical foundation for their work are precisely $\mathbb{T}_0$-representations of Δ_3^r. We discuss this example in greater detail in Section 4.6.
- Brändén [16] disproved a conjecture of Helton and Vinnikov, that any real zero polynomial admits a certain determinantal representation, by studying the discrete polymatroid which Gurvits had previously associated to a hyperbolic polynomial.
- Amini and Esteves showed that the tropicalization of linear series on an algebraic curve gives rise to certain families of tilings of vector spaces by discrete polymatroids [2].
- Farràs, Martí–Farré, and Padró provide applications of discrete polymatroids to cryptographic secret-sharing schemes [26].
- Discrete polymatroids arise naturally from the Klyaschko datum of a (framed) toric vector bundle, see [31].

Motivation for the present paper. The authors of the present text recently discovered [4, 5] some intriguing links between Lorentzian polynomials and representations of discrete polymatroids over tracts in the sense of this paper. For example, for every discrete polymatroid J, the dimension of the space L_J of Lorentzian polynomials with support J is equal to the rank of the finitely generated abelian group $\widehat{T}_J^\times$ defined in Section 5.3 below.[2] Moreover, L_J is always contained in the space $R_J(\mathbb{T}_2)$ of representations of J over the generalized triangular hyperfield $\mathbb{T}_2$ defined in Section 3.7 below.

These observations served as the primary motivation for working out the "foundational" results described in the present paper. Indeed, in order to precisely formulate and explore such a connection, one needs a rigorous development of the representation theory of discrete polymatroids over tracts, generalizing Baker and Bowler's theory [3] of matroids with coefficients and Baker and Lorscheid's subsequent work [9, 10, 11] on foundations of matroids. This is what we systematically set out to do below.

A word of warning. Linear representations of a discrete polymatroid J over a field F, as studied in [26, 37], differ from the F-representations of J defined in this paper. In the present work, we are primarily concerned with generalizing M-convex functions in the sense of Murota by viewing them as $\mathbb{T}_0$-representations; there does not appear to be a systematic way to generalize both this point of view and the traditional notion of linear representations over fields.

Content outline. Before we describe the contents of this text in detail, we provide the following overview of our results.

Part 1. We introduce and study several operations for polymatroids: duality and translates (Section 2.1), embedded minors (Section 2.2), which refine previous notions of polymatroid minors, permutation and extension of variables (Section 2.3), and direct sums (Section 2.5).

Part 2. We exhibit a characterization of polymatroids in terms of Plücker relations (Theorem 4.1), which leads to the notion of a *representation* of a polymatroid J over a tract F (Section 4.2). A feature of central importance is the *idempotency principle for proper polymatroids* (Proposition 4.7). M-convex functions are essentially the same as polymatroid representations over the tropical hyperfield (Section 4.5).

Part 3. We investigate thin Schubert cells of polymatroids and show that they are represented by the *universal tract* (Proposition 5.2). In an analogous way, the weak thin Schubert cell, defined by the 3-term Plücker relations, is represented by the *universal pasture* (Proposition 5.3). The *comparison theorem* (Theorem 5.4) establishes a bijection between the universal pasture and the universal tract . The realization space consists of polymatroid representations modulo rescaling and is represented by the *foundation* (Proposition 6.4). The foundation is generated by cross ratios (Theorem 8.4), for which

[2]As we will see, $\widehat{T}_J^\times$ is the multiplicative group of the "extended universal tract" of J, which is characterized by the property that $\mathrm{Hom}(\widehat{T}_J, F)$ is equal to the set of F-representations of J for every tract F.

we establish a (possibly incomplete) list of relations (Proposition 8.2). We extend the polymatroid operations from Part 1 to thin Schubert cells and realization spaces (Section 7).

Part 4. We establish and study canonical embeddings of representations spaces, thin Schubert cells, and realization spaces into tori (Section 9, Section 10, and Section 11, respectively). In the idempotent case, we study lineality spaces (Section 9.2) and a decomposition of thin Schubert cells into a product of the realization space with a torus orbit (Section 11.1).

Polymatroids. Discrete polymatroids appear in different cryptomorphic disguises in the literature: as a "rank function" $\mathrm{rk}\colon 2^{[n]} \to \mathbb{N}$ (where $2^{[n]}$ is the power set of $[n] = \{1, \dots, n\}$), as an integral polytope in $\mathbb{R}^n_{\geqslant 0}$, as a collection of "independent vectors" in $\mathbb{N}^n$, and as a collection of "bases" contained in the dilated discrete simplex

$$\Delta_n^r \;=\; \big\{\alpha \in \mathbb{N}^n \,\big|\, |\alpha| = r\big\}$$

for some $r \geqslant 0$ (the *rank of the polymatroid*) where $|\alpha| = \alpha_1 + \dots + \alpha_n$ ("M-convex set"). See Section 1 for an overview of these different descriptions of polymatroids, and of the relations between them.

Convention: We use the terms *polymatroid* and *M-convex set* interchangeably in this paper as synonyms for "discrete polymatroid". In particular, we omit the modifier "discrete" except when we wish to make comparisons to the literature.

Example. Matroids are polymatroids in a natural way: a matroid M defines the polymatroid $J = \{\sum_{i\in B} \varepsilon_i \mid B \text{ is a basis of } M\}$, where ε_i is the i-th standard basis vector of $\mathbb{N}^n$. We say that a polymatroid J is a *matroid* if it of this form.

For the purposes of the introduction, it suffices to understand polymatroids using a novel characterization in terms of Plücker relations in the Krasner hyperfield (Theorem A). In order to properly formulate this result, we first introduce the concept of a *tract*.

Tracts. Tracts were introduced by Baker and Bowler in [3] as a generalization of fields over which one can still develop a satisfying theory of matroid representations. Examples of matroid representations over a tract include matroids themselves, oriented matroids, valuated matroids, and linear subspaces of a vector space.

In this paper, we develop a theory of *polymatroid representations* over a tract.

Definition and examples of tracts. A *tract* is a multiplicatively written commutative monoid F with an absorbing element $0 \in F$ such that $F^\times = F - \{0\}$ forms an abelian group (the *unit group*), together with a subset N_F (the *null set*) of the group semiring $F^+ = \mathbb{N}[F^\times]$ that satisfies the following properties:

(1) N_F is an *ideal* of F^+, that is, $0 \in N_F$, $N_F + N_F = N_F$ and $F \cdot N_F = N_F$;

(2) there is an element $-1 \in F$ such that for all $a, b \in F$, one has $a + b \in N_F$ if and only if $b = (-1) \cdot a$.

We write $-a = (-1) \cdot a$ and call this element the *additive inverse of* a. We write $a - b$ for the element $a + (-b)$ of F^+.

Examples of tracts are fields F, whose addition gets replaced by the null set

$$N_F \;=\; \left\{ \sum a_i \,\middle|\, \sum a_i = 0 \text{ as elements of } F \right\},$$

the *Krasner hyperfield* $\mathbb{K} = \{0, 1\}$, with null set

$$N_{\mathbb{K}} \;=\; \mathbb{N} - \{1\} \;=\; \{0,\; 1+1,\; 1+1+1,\; 1+1+1+1, \dots\},$$

and the *tropical hyperfield* $\mathbb{T}_0 = \mathbb{R}_{\geqslant 0}$, with null set

$$N_{\mathbb{T}_0} \;=\; \left\{ \sum a_i \,\middle|\, \text{the maximum appears twice among } a_1, \dots, a_n \in \mathbb{R}_{\geqslant 0} \right\}.$$

Note that $-1 = 1$ in $\mathbb{K}$ and in $\mathbb{T}_0$. See Section 3 for a more comprehensive introduction to tracts.

Plücker relations for polymatroids. The fundamental insight that leads to our notion of polymatroid representations over tracts is the following result, which is Theorem 4.1.

We equip $\mathbb{N}^n$ with the component-wise partial order, where $\alpha \leqslant \beta$ if and only if $\alpha_i \leqslant \beta_i$ for all $i \in [n]$. For a subset $J \subseteq \Delta_n^r$, we define the infimum $\delta_J^- = \inf J$ and the supremum $\delta_J^+ = \sup J$. These are the elements of $\mathbb{N}^n$ whose components are given by

$$\delta_{J,i}^- \;=\; \min\{\alpha_i \mid \alpha \in J\} \quad\text{and}\quad \delta_{J,i}^+ \;=\; \max\{\alpha_i \mid \alpha \in J\}.$$

We write $\chi_J \colon \Delta_n^r \to \mathbb{K}$ for the characteristic function of J, defined by $\chi_J(\alpha) = 1$ if and only if $\alpha \in J$.

Theorem A. *A nonempty subset $J \subseteq \Delta_n^r$ is a polymatroid if and only if χ_J satisfies the Plücker relations*

$$\sum_{k=0}^{s} \chi_J(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \chi_J(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \in N_{\mathbb{K}}$$

for any $s = 2, \dots, r$, any $\alpha \in \Delta_n^{r-s}$, and any $i_0, \dots, i_s, j_2, \dots, j_s \in [n]$ such that

$$\delta_J^- \;\leqslant\; \alpha \qquad \textit{and} \qquad \alpha + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s} \;\leqslant\; \delta_J^+.$$

Polymatroid representations. The generalization of the Plücker relations in Theorem A from $\mathbb{K}$ to an arbitrary tract F involves a delicate choice of signs, unless $-1 = 1$ in F. It will turn out, *a posteriori*, that a polymatroid which is not "essentially" a matroid is only representable over tracts in which $-1 = 1$, see Proposition C. Thus the reader can ignore the power of -1 in the following definition if he/she wishes, without much loss of generality. The *support* of a function $\rho\colon \Delta_n^r \to F$ is the set of $\alpha \in \Delta_n^r$ such that $\rho(\alpha) \neq 0$.

Definition. Let $J \subseteq \Delta_n^r$ be a polymatroid and let F be a tract. A *strong Grassmann–Plücker representation of J over F* is a map $\rho\colon \Delta_n^r \to F$, whose support is J, such that ρ satisfies the *Plücker relations*

$$\sum_{k=0}^{s} (-1)^{k+\sigma(k)} \cdot \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \dots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \dots + \varepsilon_{j_s}) \in N_F$$

for any $2 \leqslant s \leqslant r$, any $\alpha \in \Delta_n^{r-s}$, any $1 \leqslant i_0 \leqslant \ldots \leqslant i_s \leqslant n$ and $1 \leqslant j_2 \leqslant \ldots \leqslant j_s \leqslant n$ such that

$$\delta_J^- \leqslant \alpha \qquad \text{and} \qquad \alpha + \varepsilon_{i_0} + \dots + \varepsilon_{i_s} + \varepsilon_{j_2} + \dots + \varepsilon_{j_s} \leqslant \delta_J^+.$$

where $\sigma(k)$ is the number of $k \in \{2, \ldots, s\}$ with $i_k < j_s$.

Remark. This definition turns out to be equivalent to Definition 4.3, which is stated using a slightly different formalism; see Section 4.4 for details.

> **Convention:** Unless otherwise noted, we will use the term "representation" as shorthand for "strong Grassmann–Plücker representation" throughout this paper. We will also frequently refer to (strong Grassmann–Plücker) representations of J over F as "F-representations of J".

Example. Let $J \subseteq \Delta_n^r$ be a polymatroid. The characteristic function $\chi_J\colon \Delta_n^r \to \mathbb{K}$ of J is the unique $\mathbb{K}$-representation of J. This establishes an equivalence between polymatroids and their (unique) $\mathbb{K}$-representations.

Another example of central interest are $\mathbb{T}_0$-representations, which are essentially the same as M-convex functions in the sense of Murota ([34]); cf. Section 4.5 for a definition. The following is Proposition 4.11.

Proposition B. *Let $\rho\colon \Delta_n^r \to \mathbb{T}_0$ be a map with support J. Then $-\log\rho\colon \Delta_n^r \to \mathbb{R} \cup \{\infty\}$ is an M-convex function if and only if J is a polymatroid and ρ is a $\mathbb{T}_0$-representation of J.*

Example (Hives)**.** Hives are combinatorial gadgets that were introduced by Knutson and Tao in [33] in their celebrated proof of the *saturation conjecture* for Littlewood–Richardson coefficients, which implies *Horn's conjecture* on the possible eigenvalues of a sum of Hermitian matrices. As pointed out by Brändén in [15], hives are naturally in bijection with M-concave functions (which are the negatives of M-convex functions) having support Δ_3^r, and thus with $\mathbb{T}_0$-representations of Δ_3^r.

Due to their explicit combinatorial nature, hives are easier to understand than $\mathbb{T}_0$-representations in general, and we encourage the interested reader to have a look at Section 4.6 before continuing with the introduction.

The idempotency principle for proper polymatroids. Let $\delta_J^- = \inf J$. The *reduction of* J is $\bar{J} = \{\alpha \in \mathbb{N}^r \mid \alpha + \delta_J^- \in J\}$. We say that J is a *proper polymatroid* if $\bar{J}$ is not a matroid. A tract F is *idempotent* if $-1 = 1$ and $1 + 1 + 1 \in N_F$ (in other words, if F is a $\mathbb{K}$-algebra). A tract F is *near-idempotent* if $-1 = 1$ and $1 + 1 + x \in N_F$ for some $x \in F^\times$. The following is Proposition 4.7.

Proposition C. *Let J be a proper polymatroid and suppose there exists a representation of J over F. Then F is near-idempotent. If $\delta_{J,i}^+ - \delta_{J,i}^- \geqslant 3$ for some $i \in [n]$, then F is idempotent.*

In other words, a polymatroid which is representable over some tract that is not near-idempotent must be a translate of a matroid. Note that a field (considered in the natural way as a tract) is never near-idempotent. Consequently, a polymatroid which is not the translate of a matroid is not representable over any field.

Since the representation theories of translates are essentially equivalent (cf. Theorem 7.1), and since the representation theory of matroids in the sense of this paper is well-understood:

We assume for the rest of the introduction that all tracts are near-idempotent.

In particular, we assume that $-1 = 1$, which allows us to suppress all signs, thus simplifying various expressions.

Discrepancy with other concepts of polymatroid representations. Just as matroids can be viewed as combinatorial abstractions of hyperplane arrangements, polymatroids can be viewed as combinatorial abstractions of subspace arrangements. More precisely, for any field K, let $V_1, \ldots, V_n$ be K-linear subspaces of a fixed K-vector space V. Then

$$\mathrm{rk}(S) := \mathrm{codim}_V(\cap_{i \in S} V_i)$$

is the rank function of a polymatroid P on $E = [n]$. The polymatroid P is a matroid if and only if every V_i is a hyperplane. A polymatroid arising in this way is said to be *linearly representable*, or *realizable*, over K, and the subspace arrangement is called a *linear representation*, or *realization*, of P over K, cf. [37, 26, 21].

As we see from the idempotency principle, the notion of (Grassmann–Plücker) polymatroid representations in this text differs from the concept of a linear representation, as a proper polymatroid has no Grassmann–Plücker representations over a field.

Thin Schubert cells and the universal tract. The last two parts of the paper are dedicated to the study of the spaces of all polymatroid representations. We consider several variations of such spaces (weak and strong representation spaces, weak and strong thin Schubert cells, and realization spaces); for the purpose of this introduction, we first discuss our main results in the context of thin Schubert cells, and turn to the other spaces afterwards.

The *representation space of J over F* is the set $\mathrm{R}_J(F)$ of all F-representations $\rho\colon \Delta_n^r \to F$ of J. The *thin Schubert cell of J over F* is the quotient $\mathrm{Gr}_J(F) = \mathrm{R}_J(F)/F^\times$ by the diagonal action of $F^\times$ on $\mathrm{R}_J(F)$. Both $\mathrm{R}_J(F)$ and $\mathrm{Gr}_J(F)$ are functorial in F (by composing an F-representation $\rho\colon \Delta_n^r \to F$ with a tract morphism $F \to F'$; cf. Section 5.2). The following is Proposition 5.2.

Proposition D. *Given a polymatroid J, the functor sending a tract F to the thin Schubert cell* $\mathrm{Gr}_J(F)$ *is represented by a tract T_J, i.e., there is a bijection* $\mathrm{Gr}_J(F) \to \mathrm{Hom}(T_J, F)$ *which is functorial in F. We call T_J the* universal tract of J.

The universal tract T_J is given by a simple construction: up to taking degree 0 elements, it is generated by symbols x_β (for $\beta \in J$), and its null set is generated by Plücker relations for the x_β. This explicit description makes T_J amenable to computations. On the other hand, the universal property of T_J implicit in Proposition D allows us to show that thin Schubert cells are functorial with respect to polymatroid embeddings (Theorem E), which we introduce in the following section.

Minors and polymatroid embeddings. Let $\delta_J^- = \inf J$ and $\delta_J^+ = \sup J$. The *translation* of J is $J + \tau = \{\alpha + \tau \mid \alpha \in J\}$, which is a polymatroid (and, in particular, contained in $\mathbb{N}^n$) provided that $\tau \geqslant -\delta_J^-$ (Lemma 2.1).

Let $\nu, \mu \in \mathbb{N}^n$ with $\mu + \delta_J^- \leqslant \alpha \leqslant \delta_J^+ - \nu$ for some $\alpha \in J$. The *deletion of ν in J* is

$$J\backslash\nu \;=\; \left\{\alpha \in J \,\middle|\, \alpha \leqslant \delta_J^+ - \nu\right\}$$

and the *contraction of μ in J* is

$$J/\mu \;=\; \left\{\alpha - \mu \in \mathbb{N}^n \,\middle|\, \alpha \in J,\ \delta_J^- + \mu \leqslant \alpha\right\}.$$

Both sets are polymatroids (Lemma 2.12).

An *embedded minor of J* is a sequence of deletions, contractions, and translations. While translations commute with both deletions and contractions, the exchange of deletions and contractions is more subtle since these operations have an irregular effect on δ_J^- and δ_J^+; cf. Proposition 2.15. Still, Proposition 2.17 attests that for given ν and μ, there are ν', μ' and τ' such that $(J/\mu)\backslash\nu = ((J\backslash\nu')/\mu') + \tau'$, which allows us to represent every embedded minor as $J\backslash\nu/\mu + \tau = ((J\backslash\nu)/\mu) + \tau$.

A *minor embedding* is an inclusion of polymatroids of the form

$$\iota_{J\backslash\nu/\mu+\tau}\colon\ J\backslash\nu/\mu+\tau\ \longrightarrow\ J,\qquad\text{with}\qquad \iota_{J\backslash\nu/\mu+\tau}(\alpha)\ =\ \alpha+\mu-\tau.$$

A permutation $\sigma\in S_n$ of $[n]$ induces a bijection $\iota_\sigma\colon\Delta_n^r\to\Delta_n^r$ by permuting the coordinates, which restricts to a bijection $\iota_\sigma\colon J\to\sigma(J)$ (called a *permutation of variables*), and $\sigma(J)$ is a polymatroid (Lemma 2.21).

The embedding $\iota_n\colon\mathbb{N}^n\to\mathbb{N}^{n+1}$ into the first n coordinates restricts to a bijection $\iota_n\colon J\to\iota_n(J)$ (called an *extension of variables*), and $\iota_n(J)$ is a polymatroid (Lemma 2.22). Its inverse bijection is called a *restriction of variables*.

Definition. A *polymatroid embedding* is a composition of minor embeddings with permutations, extensions, and restrictions of variables. Two polymatroids J and J' are *combinatorially equivalent* if there is a bijective polymatroid embedding $J\to J'$.

Note that the inverse of a bijective polymatroid embedding is again a polymatroid embedding. The following summarizes Theorem 7.1, Proposition 7.2, and Proposition 7.3.

Theorem E. *Let $\iota\colon J\to J'$ be a polymatroid embedding and let F be a tract. Precomposing an F-representation ρ of J' with ι yields a map $\iota^*\colon\ \mathrm{Gr}_{J'}(F)\to\mathrm{Gr}_J(F)$. If ι is bijective, then so is ι^*.*

As a consequence of Theorem E, it makes sense to talk about (embedded) minors of polymatroid representations, which are defined as $\rho\backslash\nu/\mu+\tau=\iota^*_{J\backslash\nu/\mu+\tau}(\rho)$.

Remark. Embedded minors in the sense of this text correspond to polymatroid truncations that appear in the work of Brändén–Huh (see Remark 2.20 for details).

Duality. Whittle has shown in [47] that there is no involution on polymatroids which interchanges deletion and contraction. Theorem E suggests an alternative perspective: perhaps polymatroid duality should only interchange deletion and contraction up to combinatorial equivalence. It turns out that this paradigm leads to a satisfactory notion of duality.

Definition. The *duality vector of* J is $\delta_J=\delta_J^-+\delta_J^+=\inf\ J+\sup\ J$. The *dual of* J is $J^*=\delta_J-J$.

The following summarizes Lemma 2.3, Proposition 2.14, and Theorem 7.6.

Theorem F. *Polymatroid duality satisfies the following properties:*

(1) $(J^*)^*=J$;
(2) $(J\backslash\nu)^*$ *and* J^*/ν *are combinatorially equivalent (by a translation);*
(3) there is a canonical bijection $\mathrm{Gr}_J(F)\to\mathrm{Gr}_{J^*}(F)$ *that is functorial in* J *and* F.

Direct sums. The *direct sum* $J_1 \oplus J_2$ of two polymatroids $J_1 \subseteq \Delta_{n_1}^{r_1}$ and $J_2 \subseteq \Delta_{n_2}^{r_2}$ is a polymatroid contained in $\Delta_{n_1+n_2}^{r_1+r_2}$ (see Section 2.5). The following is Theorem 7.7.

Theorem G. *There is a canonical bijection* $\mathrm{Gr}_{J_1 \oplus J_2}(F) \to \mathrm{Gr}_{J_1}(F) \times \mathrm{Gr}_{J_2}(F)$ *that is functorial in* F.

A polymatroid $J \subseteq \Delta_n^r$ is *nontrivial* if $n \geqslant 1$, and *indecomposable* if J is nontrivial and J is not the direct sum of two nontrivial polymatroids. Every polymatroid J has a unique decomposition into a direct sum $\bigoplus_{i=1}^{c(J)} J_i$ of indecomposable polymatroids J_i, which are unique up to combinatorial equivalence and a permutation of indices (Proposition 2.29).

The canonical torus embedding. The association $\rho \mapsto (\rho(\alpha))_{\alpha \in J}$ defines a canonical embedding $\mathrm{R}_J(F) \to (F^\times)^J$, which we show factors through a smaller subgroup $\mathrm{D}_J(F)$ of $(F^\times)^J$ defined as follows.

We say that a Plücker relation

$$\sum_{k=0}^{s} \quad \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \quad \in \quad N_F$$

is *degenerate* if it has exactly two nonzero terms, say

$$\rho(\beta - \varepsilon_{i_k})\rho(\gamma + \varepsilon_{i_k}) \qquad \text{and} \qquad \rho(\beta - \varepsilon_{i_\ell})\rho(\gamma + \varepsilon_{i_\ell})$$

for $k \neq \ell$, where $\beta = \alpha + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}$ and $\gamma = \alpha + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}$. By the uniqueness of additive inverses, and since we assume that $-1 = 1$, a degenerate Plücker relation corresponds to an equality of the form

$$\rho(\beta - \varepsilon_{i_k})\rho(\gamma + \varepsilon_{i_k}) \;=\; \rho(\beta - \varepsilon_{i_\ell})\rho(\gamma + \varepsilon_{i_\ell}).$$

The *degeneracy locus of* J *over* F is the subgroup

$$\mathrm{D}_J(F) \;=\; \big\{\rho \in (F^\times)^J \,\big|\, \rho \text{ satisfies all degenerate Plücker relations}\big\}$$

of $(F^\times)^J$, and it contains the image of the embedding $\mathrm{R}_J(F) \to (F^\times)^J$.

The Plücker embedding and the Polygrassmannian. The map $[\rho] \mapsto [\rho(\alpha)]_{\alpha \in J}$ defines a canonical embedding $\mathrm{Gr}_J(F) \to (F^\times)^J / F^\times$, which can be considered as a stratum of the projective space $\mathbb{P}(F^{\Delta_n^r}) = \big(F^{\Delta_n^r} - \{0\}\big)/F^\times$ (cf. Section 10 for details). Composing these two inclusions yields the Plücker embedding $\mathrm{Gr}_J(F) \to \mathbb{P}(F^{\Delta_n^r})$.

The union of the images of $\mathrm{Gr}_J(F)$ in $\mathbb{P}(F^{\Delta_n^r})$ for all polymatroids $J \subseteq \Delta_n^r$ defines the *Polygrassmannian* $\mathrm{PolyGr}(r,n)(F)$, which is in general larger than the Grassmannian; see Section 10.1 for details.

Weak thin Schubert cells and the universal pasture. A nontrivial result in matroid theory is that for many tracts of interest, including fields, $\mathbb{K}$, and $\mathbb{T}_0$, the thin Schubert cell $\mathrm{Gr}_J(F)$ of a matroid J is cut out by just the 3-*term Plücker relations*, which are of the form

$$\rho(\alpha+\varepsilon_j+\varepsilon_k)\cdot\rho(\alpha+\varepsilon_i+\varepsilon_l) \;-\; \rho(\alpha+\varepsilon_i+\varepsilon_k)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_l) \\ +\; \rho(\alpha+\varepsilon_i+\varepsilon_j)\cdot\rho(\alpha+\varepsilon_k+\varepsilon_l) \quad\in\quad N_F$$

for $\alpha \in \Delta_n^{r-2}$ and $1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n$ with $\delta_J^- \leqslant \alpha$ and $\alpha+\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l \leqslant \delta_J^+$ (where we can ignore the sign if F is near-idempotent).

Maps $\rho\colon \Delta_n^r \to F$ with support J that satisfy the 3-term Plücker relations are called *weak F-representations of J*, a notion that makes sense for all polymatroids. The *weak thin Schubert cell* $\mathrm{Gr}_J^w(F)$ is the space of $F^\times$-classes of weak F-representations of J.

Proposition H. *Given a polymatroid J, the functor sending a tract F to the weak thin Schubert cell $\mathrm{Gr}_J^w(F)$ is represented by a tract P_J, i.e., there is a bijection $\mathrm{Gr}_J^w(F) \to \mathrm{Hom}(P_J, F)$ which is functorial in F. We call P_J the* universal pasture of J.

Evidently, every (strong) F-representation is a weak F-representation, which yields an inclusion $\mathrm{Gr}_J(F) \subseteq \mathrm{Gr}_J^w(F)$, or, equivalently, a surjection $P_J \to T_J$. The following non-obvious result is Theorem 5.4:

Theorem I. *The canonical morphism $P_J \to T_J$ is a bijection.*

We deduce from this result that the functoriality of $\mathrm{Gr}_J(F)$ described in Theorem E also holds for $\mathrm{Gr}_J^w(F)$, and that $\mathrm{Gr}_J^w(F)$ is functorial with respect to duality (Theorem F) and direct sums (Theorem G). Theorem I also implies that $\mathrm{Gr}_J^w(F)$ is contained in the degeneracy locus $\mathrm{D}_J(F)$, and shows that the Tutte group of a matroid M (which coincides with $P_M^\times$) is canonically isomorphic to $T_M^\times$ (a result which was not previously known).

Excellent tracts. In general, the inclusion $\mathrm{Gr}_J(F) \subseteq \mathrm{Gr}_J^w(F)$ is not an equality. We call a tract F *excellent* if this inclusion is an equality for all polymatroids J.

Excellent tracts are closely related to perfect tracts F (see [3] for the definition), for which the equality $\mathrm{Gr}_J(F) = \mathrm{Gr}_J^w(F)$ holds whenever J is a matroid (see [3, Thm. 3.46]).

A tract F is *degenerate* if N_F contains every formal sum $\sum a_i$ with at least 3 nonzero terms $a_1, a_2, a_3 \in F^\times$. The following summarizes our present state of knowledge about excellent tracts (see Section 4.2.1, Corollary 4.8, Corollary 4.12, and Corollary 5.5).

Theorem J.

(1) *Every perfect tract that is not near-idempotent (for example, every field) is excellent.*

(2) *Every degenerate tract is excellent.*

(3) The Krasner hyperfield $\mathbb{K}$ and the tropical hyperfield $\mathbb{T}$ are excellent.

At the time of writing, we do not know the answers to the following questions:

Problem.

(1) Is every perfect tract excellent?

(2) Is every excellent tract perfect?

Realization spaces and the foundation. The *torus* $T(F) = (F^\times)^n$ acts on $\mathrm{Gr}_J^w(F)$ through the formula $(t.\rho)(\alpha) = (t_1^{\alpha_1} \cdots t_n^{\alpha_n}) \cdot \rho(\alpha)$. The *realization space of J over F* is the set $\underline{\mathrm{Gr}}_J^w(F)$ of $T(F)$-orbits of this action.

Proposition K. *For every polymatroid J, the functor sending a tract F to the realization space $\underline{\mathrm{Gr}}_J^w(F)$ is represented by a tract F_J, i.e., there is a bijection $\underline{\mathrm{Gr}}_J^w(F) \to \mathrm{Hom}(F_J, F)$ which is functorial in F. We call F_J the* foundation of J.

The foundation of a matroid has proven to be a valuable tool for the study of matroid representations (see, for example, [9, 10, 11]), in part because one knows an explicit presentation for it in terms of generators and relations, with the generators being certain canonical elements called *cross ratios*. This presentation is closely connected to Tutte's homotopy theory ([43, 44]; see also [7] for a "modern" account).

We succeed in partially generalizing this presentation to polymatroids (for the definition of cross ratios and a precise formulation, see Proposition 8.2 and Theorem 8.4).

Theorem L. *The foundation F_J of J is generated by the cross ratios for J. All types of relations between cross ratios that hold for matroids also hold for all polymatroids.*

At the time of writing we do not know an answer to the following:

Problem. Is the list of relations between cross ratios appearing in Proposition 8.2 complete? If not, then what is a complete list of relations?

The realization space $\underline{\mathrm{Gr}}_J(F)$ satisfies further properties similar to the thin Schubert cells: it is functorial with respect to polymatroid embeddings (Theorem 7.1, Proposition 7.2, Proposition 7.3), duality (Theorem 7.6), and direct sums (Theorem 7.7).

The lineality space. If F is idempotent, the characteristic map $\chi_J \colon \Delta_n^r \to F$ of J is an F-representation. The *lineality space of J over F* is the torus orbit $\mathrm{Lin}_J(F) = T(F).\chi_J$. The following is Theorem 11.2.

Theorem M. *Let* $c(J)$ *be the number of indecomposable components of* $J \subseteq \Delta_n^r$*. Then* $P_J \simeq F_J(x_1, \ldots, x_{n-c(J)})$ *is a free algebra over* F_J*, and this isomorphism defines a bijection*

$$\mathrm{Gr}_J^w(F) \ \simeq \ \underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^{n-c(J)}$$

which is functorial in F*. If* F *is idempotent, then* $\mathrm{Lin}_J(F) \simeq (F^\times)^{n-c(J)}$.

Acknowledgements. We thank Donggyu Kim, Ari Pomeranz, and Noah Solomon for helpful remarks on an early draft of this text. We thank the Korea Institute for Advanced Studies in Seoul and the Institute for Advanced Studies in Princeton for hosting us during our collaboration. M.B. was partially supported by NSF grant DMS-2154224 and a Simons Fellowship in Mathematics (1037306, Baker). J.H. is partially supported by the Oswald Veblen Fund and the Simons Investigator Grant. M.K. was partially supported by DFG grant 502861109. O.L. was partially supported by NSF grant DMS-1926686 and by the Institute for Advanced Studies.

Part 1. Polymatroids: duality and embedded minors

1. Polymatroids and M-convex sets

In this section, we review the concepts of (discrete and integral) polymatroids and their relation to M-convex sets.

We will consider the nonnegative orthant $\mathbb{R}^n_{\geqslant 0}$ in $\mathbb{R}^n$, together with the partial order $\leqslant$ defined by $\alpha \leqslant \beta$ iff $\alpha_i \leqslant \beta_i$ for all $i = 1, \ldots, n$. We define $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and we let ε_i denote the i-th standard basis vector of $\mathbb{R}^n$.

1.1. Polymatroids. Like matroids, polymatroids have several equivalent ("cryptomorphic") characterizations, for example in terms of rank functions and polytopes. For the purposes of this paper, the following approach seems the most economical.

A *polymatroid on* $[n]$ is a nonempty compact subset $\mathcal{P}$ of $\mathbb{R}^n_{\geqslant 0}$ that satisfies the following two axioms:

(P1) if $\beta \in \mathcal{P}$ and $\alpha \leqslant \beta$, then $\alpha \in \mathcal{P}$;
(P2) if $\alpha, \beta \in \mathcal{P}$ and $|\alpha| < |\beta|$, then there exist $i \in [n]$ and $r \in [0,\ \beta_i - \alpha_i]$ such that $\alpha + r\varepsilon_i \in \mathcal{P}$.

It follows from these axioms that $\mathcal{P}$ is a convex polytope in $\mathbb{R}^n_{\geqslant 0}$.

An *integral polymatroid on* $[n]$ is a polymatroid $\mathcal{P}$ on $[n]$ whose vertices have integer coordinates, i.e., $\mathcal{P}$ is the convex hull $\mathrm{conv}(S)$ of a finite subset S of $\mathbb{N}^n$.

A *discrete polymatroid on* $[n]$ is a nonempty finite subset P of $\mathbb{N}^n$ that satisfies the following two axioms:

(DP1) if $\beta \in P$ and $\alpha \leqslant \beta$, then $\alpha \in P$;
(DP2) if $\alpha, \beta \in P$ and $|\alpha| < |\beta|$, then there exists $i \in [n]$ such that $\alpha_i < \beta_i$ and $\alpha + \varepsilon_i \in P$.

The elements of a discrete polymatroid P are called its *independent vectors*, and the maximal elements (w.r.t. $\leqslant$) are called its *bases*. Axiom (DP1) implies that a discrete polymatroid is determined by its set of bases, and axiom (DP2) implies that $|\beta| = |\beta'|$ for any two bases β, β' of P.

Taking the convex hull of a discrete polymatroid P, considered as a subset of $\mathbb{R}^n_{\geqslant 0}$, yields an integral polymatroid $\mathcal{P} = \operatorname{conv}(P)$. By [30, Thm. 3.4], we recover P as $\mathcal{P} \cap \mathbb{N}^n$. As a consequence of [34, Theorem 4.15], this establishes a bijective correspondence between discrete and integral polymatroids.

1.2. M-convex sets. Let $\Delta^r_n = \{\alpha \in \mathbb{N}^n \mid \alpha_1 + \cdots + \alpha_n = r\}$. An *M-convex set* of rank r on $[n]$ is a nonempty subset J of Δ^r_n such that for all $\alpha, \beta \in J$ and every $i \in [n]$ with $\alpha_i < \beta_i$, there exists an $j \in [n]$ with $\alpha_j > \beta_j$ such that J contains both $\alpha + \varepsilon_i - \varepsilon_j$ and $\beta - \varepsilon_i + \varepsilon_j$.

It follows directly from the definition that for an M-convex set $J \subseteq \Delta^r_n$, the subset $P_J = \{\alpha \in \mathbb{N}^n \mid \alpha \leqslant \beta \text{ for some } \beta \in J\}$ of $\mathbb{N}^n$ is a discrete polymatroid. *A priori,* the (symmetric) exchange axiom for M-convex sets seems stronger than the exchange axiom (DP2) for discrete polymatroids, since the symmetric exchange axiom requires both $\alpha + \varepsilon_i - \varepsilon_j$ and $\beta - \varepsilon_i + \varepsilon_j$ to be in J. However, it follows from [46, Thm. 2.7] that every discrete polymatroid is of the form P_J for some M-convex set J, which establishes a bijective correspondence between M-convex sets J and discrete polymatroids P_J.

Example 1.1. A matroid M of rank r on $[n]$ can be considered as the M-convex subset

$$J = \{\textstyle\sum_{i \in B} \varepsilon_i \mid B \text{ is a basis of } M\}$$

of Δ^r_n. In this sense, we consider matroids as particular kinds of M-convex sets. For simplicity, we say that an M-convex set J *is a matroid* if it stems from a matroid in the above sense. Note that an M-convex set J is a matroid if and only if $J \subseteq \{0,1\}^n$.

Another class of examples of M-convex sets are the sets $J = \Delta^r_n$, which are not matroids for $r \geqslant 2$. In particular, there are M-convex sets whose rank r is bigger than n, which does not occur for matroids.

1.3. Rank functions. Let $\mathcal{P} \subseteq \mathbb{R}^n_{\geqslant 0}$ be a polymatroid and let $2^{[n]}$ denote the power set of $[n]$. The *rank function of* $\mathcal{P}$ is the function $\mathbf{r}\colon 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ with values

$$\mathbf{r}(S) = \max\{\alpha_S \mid \alpha \in \mathcal{P}\},$$

for $S \subseteq [n]$, where $\alpha_S = \sum_{i \in S} \alpha_i$ (with $\alpha_\varnothing = 0$). The polymatroid $\mathcal{P}$ is characterized by $\mathbf{r}$ through the formula

$$\mathcal{P} \;=\; \left\{ \alpha \in \mathbb{R}_{\geqslant 0} \,\middle|\, \alpha_S \leqslant \mathbf{r}(S) \text{ for all } S \subseteq [n] \right\}.$$

By [30, Proposition 1.2], a function $\mathbf{r}\colon 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ is a rank function of a polymatroid if and only if it is *normalized*, i.e., $\mathbf{r}(\varnothing) = 0$, *non-decreasing*, i.e., $\mathbf{r}(S) \leqslant \mathbf{r}(T)$ whenever $S \subseteq T$, and *submodular*, i.e., $\mathbf{r}(S) + \mathbf{r}(T) \leqslant \mathbf{r}(S \cup T) + \mathbf{r}(S \cap T)$ for all $S, T \subseteq [n]$.

The polymatroid $\mathcal{P}$ is integral if and only if the image of its rank function $\mathbf{r}\colon 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ is contained in $\mathbb{N}$. If $\mathcal{P}$ is the integral polymatroid corresponding to an M-convex set J, then its rank function is given by the formula

$$\mathbf{r}(S) \;=\; \max \{ \alpha_S \mid \alpha \in J \}$$

for $S \subseteq [n]$.

1.4. Base polytopes and generalized permutohedra. An *(integral) generalized permutohedron* is a polytope $\mathcal{B} \subseteq \mathbb{R}^n$ such that all vertices of $\mathcal{B}$ belong to $\mathbb{Z}^n$ and all edges of $\mathcal{B}$ are parallel to $\varepsilon_i - \varepsilon_j$ for some $i \neq j$.

Generalized permutohedra are closely related to polymatroids. Given an integral polymatroid $\mathcal{P}$ on $[n]$ with associated M-convex set J, its *base polytope* $\mathcal{B}$ is defined as the convex hull of J, considered as elements of $\mathbb{N}^n \subseteq \mathbb{R}^n$. The base polytope $\mathcal{B}$ characterizes the matroid since $J = \mathcal{B} \cap \mathbb{N}^n$.

The following theorem is proved, for example, in [23] and [40]. It establishes a bijection (modulo translations) between discrete polymatroids and generalized permutohedra, and generalizes a well-known polytopal characterization of matroids due to Gelfand and Serganova [29].

Theorem. *A polytope $\mathcal{B} \subseteq \mathbb{R}^n$ is the base polytope of an integral polymatroid if and only if it is a generalized permutohedron and lies in the nonnegative orthant $\mathbb{R}^n_{\geqslant 0}$.*

1.5. Terminological convention for this paper. Similar to the usage of the term "matroid," which might refer to different characterizations—in terms of independent sets, bases, or a host of other quantities—we consider a polymatroid as an abstract mathematical object, which we typically describe in terms of its bases. Moreover, we assume from this point on, as a standing assumption whenever the term 'polymatroid' appears:

All polymatroids are discrete.

This means that we can describe a polymatroid in terms of its associated M-convex set J, which is our principal perspective on polymatroids.

2. Duality and embedded minors

In [47], Whittle introduces deletion and contraction operations for polymatroids, and discusses the existence and non-existence of a duality operation for polymatroids which interchanges these two operations, as is the case for matroids. The executive summary is that only polymatroids of a special shape allow for such a duality (cf. Remark 2.26).

In this section, we bypass the limitations which Whittle encountered by introducing a duality operation which only interchanges deletion and contraction "up to translation," leading to a more satisfactory theory. We also refine Whittle's notion of polymatroid minors.

2.1. Duality and translation. For the rest of this section, we fix an M-convex set $J \subseteq \Delta_n^r$. Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}^n$. We define $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and

$$J + \gamma = \{\alpha + \gamma \mid \alpha \in J\} \qquad \text{and} \qquad \gamma - J = \{\gamma - \alpha \mid \alpha \in J\}$$

as subsets of $\mathbb{Z}^n$.

Lemma 2.1. *If $J + \gamma \subseteq \mathbb{N}^n$ (resp. $\gamma - J \subseteq \mathbb{N}^n$), then $J + \gamma$ (resp. $\gamma - J$) is M-convex of rank $|\gamma| + r$ (resp. $|\gamma| - r$).*

Proof. Consider $\alpha + \gamma,\ \beta + \gamma \in J + \gamma$ with $\alpha, \beta \in J$ and $i \in [n]$ such that $(\alpha + \gamma)_i > (\beta + \gamma)_i$. Then $\alpha_i > \beta_i$, and by the M-convexity of J, there is a $j \in [n]$ such that $\alpha_j < \beta_j$ and $\alpha - \varepsilon_i + \varepsilon_j, \beta + \varepsilon_i - \varepsilon_j \in J$. Thus $(\alpha + \gamma)_j < (\beta + \gamma)_j$ and $\alpha + \gamma - \varepsilon_i + \varepsilon_j, \beta + \gamma + \varepsilon_i - \varepsilon_j \in J + \gamma$, which shows that $J + \gamma$ is M-convex. It is clear that the rank of $J + \gamma$ is $|\gamma| + r$.

The claim for $\gamma - J$ is proven by the same argument, but with the appropriate signs reversed. □

We consider the partial order on $\mathbb{Z}^n$ defined by $\alpha \leqslant \beta$ iff $\alpha_i \leqslant \beta_i$ for all $i \in [n]$. Note that every finite subset S of $\mathbb{Z}^n$ has a greatest lower bound $\delta_S^- = \inf S$ and a least upper bound $\delta_S^+ = \sup S$, whose respective coefficients are given by

$$\delta_{S,i}^- = \min\{\alpha_i \mid \alpha \in S\} \qquad \text{and} \qquad \delta_{S,i}^+ = \max\{\alpha_i \mid \alpha \in S\}.$$

Definition 2.2. The *duality vector of J* is $\delta_J = \delta_J^- + \delta_J^+$. The *dual of J* is $J^* = \delta_J - J$.

Before we discuss examples (see Remark 2.7 and Example 2.8 at the end of this section), we discuss several properties of polymatroid duality and compare it to matroid duality.

Lemma 2.3. *The dual J^* of J is M-convex of rank $|\delta_J| - r$ with duality vector $\delta_{J^*} = \delta_J$ and dual $J^{**} = J$.*

Proof. By Lemma 2.1, J^* is M-convex of rank $|\delta_J| - r$. The equality $\delta_{J^*} = \delta_J$ follows from

$$\delta^-_{J^*,i} \;=\; \min\{\delta_{J,i} - \alpha_i \mid \alpha \in J\} \;=\; \delta^-_{J,i} \;+\; \delta^+_{J,i} \;-\; \underbrace{\max\{\alpha_i \mid \alpha \in J\}}_{=\delta^+_{J,i}} \;=\; \delta^-_{J,i}$$

and

$$\delta^+_{J^*,i} \;=\; \max\{\delta_{J,i} - \alpha_i \mid \alpha \in J\} \;=\; \delta^-_{J,i} \;+\; \delta^+_{J,i} \;-\; \underbrace{\min\{\alpha_i \mid \alpha \in J\}}_{=\delta^-_{J,i}} \;=\; \delta^+_{J,i},$$

and $J^{**} = J$ follows from the equality $\delta_{J^*} - (\delta_J - \alpha) = \alpha$ for $\alpha \in J$. □

Lemma 2.4. *Let $\gamma \in \mathbb{Z}^n$ such that $J + \gamma \subseteq \mathbb{N}^n$. Then $\delta_{J+\gamma} = \delta_J + 2\gamma$ and $(J+\gamma)^* = J^* + \gamma$.*

Proof. The first claim follows from

$$\delta_{J+\gamma} \;=\; \delta^-_{J+\gamma} \;+\; \delta^+_{J+\gamma} \;=\; \delta^-_J \;+\; \gamma \;+\; \delta^+_J \;+\; \gamma \;=\; \delta_J \;+\; 2\gamma,$$

and the second claim follows from

$$\delta_{J+\gamma} \;-\; (\alpha+\gamma) \;=\; \delta_J \;+\; 2\gamma \;-\; \alpha \;-\; \gamma \;=\; (\delta_J - \alpha) \;+\; \gamma$$

for $\alpha \in J$, together with the observation that $\delta_{J+\gamma} - (\alpha+\gamma) \in (J+\gamma)^*$ and $\delta_J - \alpha \in J^*$. □

We say that an element $i \in [n]$ is *isolated in J* if $\delta^-_{J,i} = \delta^+_{J,i}$, and that J is *without isolated elements* if no element of $[n]$ is isolated in J.

Lemma 2.5. *Let J be a matroid. Then $i \in [n]$ is isolated in J if and only if i is a loop or a coloop.*

Proof. If i is a loop, then $\delta^-_{J,i} = \delta^+_{J,i} = 0$. If i is a coloop, then $\delta^-_{J,i} = \delta^+_{J,i} = 1$. In both cases, i is isolated in J. If i is not a loop nor a coloop, then there are $\alpha, \beta \in J$ such that $i \in \alpha$ and $i \notin \beta$. Thus $\delta^-_{J,i} = 0$ and $\delta^+_{J,i} = 1$, which shows that i is not isolated in J. □

Proposition 2.6. *If J is a matroid without isolated elements, then the matroid dual of J is equal to J^*.*

Proof. Since J is without isolated elements, we have $\delta^-_{J,i} = 0$ and $\delta^-_{J,i} = 1$ for all $i \in [n]$. Thus $\delta_{J,i} = 1$ for all i and $(\delta_J - \alpha)_i = 1 - \alpha_i$. Therefore the support of $\delta_J - \alpha$ is the complement of the support of α (as subsets of $[n]$), which agrees with the matroid dual of J. □

Remark 2.7. There is a discrepancy between matroid duality and polymatroid duality in the presence of isolated elements. The prototypical example is $J = \Delta^r_1 = \{r\varepsilon_1\}$, which has duality vector $\delta_J = \delta^-_J + \delta^+_J = (r) + (r) = (2r)$ and dual $J^* = \{2r\varepsilon_1 - r\varepsilon_1\} = \{r\varepsilon_1\} = J$. In particular, the loop $U_{0,1} = \Delta^0_1$ and the coloop $U_{1,1} = \Delta^1_1$ are self-dual as polymatroids, in contrast to matroid duality, which interchanges these two matroids. In any case, the matroid dual and polymatroid dual differ only by a translation.

Example 2.8. For $J = \Delta_2^r$, we have $\delta_J^- = 0$ and $\delta_J = \delta_J^+ = r\varepsilon_1 + r\varepsilon_2$. So $J^* = \Delta_2^r$ is self-dual, just as in the case of Δ_1^r. The situation is different for $n \geqslant 3$. If $J = \Delta_3^2$, then $\delta_J^- = 0$ and $\delta_J = \delta_J^+ = 2\varepsilon_1 + 2\varepsilon_2 + 2\varepsilon_3$. Thus

$$J^* \;=\; \big\{2\varepsilon_i + 2\varepsilon_j,\; 2\varepsilon_i + \varepsilon_j + \varepsilon_k \,\big|\, \{i,j,k\} = \{1,2,3\}\big\}.$$

2.2. Embedded minors. An *embedded minor* of a matroid M is a matroid of the form $M\backslash J/I$ with I independent and J coindependent, together with the data of I and J. In this section, we extend this concept to polymatroids.

Recall that $\delta_J = \delta_J^- + \delta_J^+$ with $\delta_J^- = \inf J$ and $\delta_J^+ = \sup J$.

Definition 2.9. The *reduction of* J is the M-convex set $\bar{J} := J - \delta_J^-$.

Definition 2.10. Let $\mu \in \mathbb{N}^n$. We say that μ is *effectively independent in* J if $\mu \leqslant \alpha - \delta_J^-$ for some $\alpha \in J$. We say that ν is *effectively coindependent in* J if $\nu \leqslant \delta_J^+ - \alpha$ for some $\alpha \in J$.

Let μ be effectively independent in J and let ν be effectively coindependent in J. The *contraction of* μ *in* J is

$$J/\mu \;=\; \{\alpha - \mu \mid \alpha \in J \text{ and } \mu \leqslant \alpha - \delta_J^-\},$$

and the *deletion of* ν *in* J is

$$J\backslash\nu \;=\; \{\alpha \mid \alpha \in J \text{ and } \nu \leqslant \delta_J^+ - \alpha\}.$$

Note that μ is effectively independent in J if and only if $\mu - \delta_J^-$ is independent in $\bar{J} := J - \delta_J^-$, and thus it differs from the usual notion of independence for polymatroids (cf. Section 1.1). We won't use the latter meaning of independence in this paper, however, so we will frequently omit the modifier "effectively" in what follows, for ease of terminology.

Contractions and deletions come with injections

$$\begin{array}{rccc} \iota_{J/\mu}\colon & J/\mu & \longrightarrow & J \\ & \alpha & \longmapsto & \alpha+\mu \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \iota_{J\backslash\nu}\colon & J\backslash\nu & \longrightarrow & J \\ & \alpha & \longmapsto & \alpha \end{array}$$

into J.

Example 2.11. Consider the M-convex set $J = \Delta_3^3\backslash\{(0,3,0)\}$, which has $\delta_J^- = 0$ and $\delta_J^+ = (3,2,3)$. The contractions and deletions of ε_3 and $2\varepsilon_3$ are illustrated in Figure 1, where we write ijk for the tuple $(i,j,k) \in J$. More examples can be found in Example 2.16.

Lemma 2.12. *Both* J/μ *and* $J\backslash\nu$ *are M-convex.*

Proof. Consider $\alpha - \mu, \beta - \mu \in J/\mu$ and $i \in [n]$ with $(\alpha-\mu)_i > (\beta-\mu)_i$. Then $\alpha, \beta \in J$ with $\mu + \delta_J^- \leqslant \alpha, \beta$ and $\alpha_i > \beta_i$. Since J is M-convex, there is a $j \in [n]$ such that $\alpha_j < \beta_j$ and

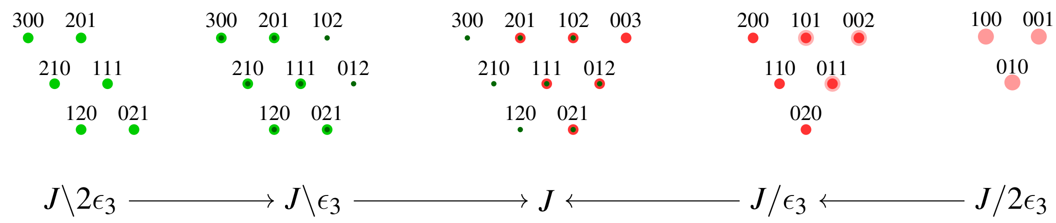

Figure 1. Some contractions and deletions of $J = \Delta_3^3 \backslash \{(0,3,0)\}$

$\alpha - \varepsilon_i + \varepsilon_j,\ \beta + \varepsilon_i - \varepsilon_j \in J$. Thus also $(\alpha - \mu)_j < (\beta - \mu)_j$. Since

$$\mu_i + \delta^-_{J,i} \ \leqslant\ \beta_i \ \leqslant\ \alpha_i - 1 \ =\ (\alpha - \varepsilon_i + \varepsilon_j)_i, \qquad \mu_i + \delta^-_{J,i} \ \leqslant\ \beta_i \ <\ (\beta + \varepsilon_i - \varepsilon_j)_i,$$
$$\mu_j + \delta^-_{J,j} \ \leqslant\ \alpha_j \ \leqslant\ \beta_j - 1 \ =\ (\beta + \varepsilon_i - \varepsilon_j)_j, \qquad \mu_j + \delta^-_{J,j} \ \leqslant\ \alpha_j \ <\ (\alpha - \varepsilon_i + \varepsilon_j)_j,$$

we have $\alpha - \mu - \varepsilon_i + \varepsilon_j,\ \alpha - \mu + \varepsilon_i - \varepsilon_j \in J/\mu$, which shows that J/μ is M-convex.

Consider $\alpha, \beta \in J\backslash\nu$ and $i \in [n]$ with $\alpha_i < \beta_i$. Since J is M-convex, there is a $j \in [n]$ such that $\alpha_j > \beta_j$ and $\alpha - \varepsilon_i + \varepsilon_j,\ \beta + \varepsilon_i - \varepsilon_j \in J$. Since

$$(\beta + \nu - \varepsilon_i + \varepsilon_j)_i \ =\ \beta_i + \nu_i + 1 \ \leqslant\ \alpha_i + \nu_i \ \leqslant\ \delta^+_{J,i},$$
$$(\alpha + \nu - \varepsilon_i + \varepsilon_j)_j \ =\ \alpha_j + \nu_j + 1 \ \leqslant\ \beta_j + \nu_j \ \leqslant\ \delta^+_{J,j},$$
$$(\beta + \nu + \varepsilon_i - \varepsilon_j)_j \ <\ \beta_j + \nu_j \ \leqslant\ \delta^+_{J,j}, \quad \text{and} \quad (\alpha + \nu + \varepsilon_i - \varepsilon_j)_i \ <\ \alpha_i + \nu_i \ \leqslant\ \delta^+_{J,i},$$

we have $\alpha + \nu + \varepsilon_i - \varepsilon_j,\ \beta + \nu + \varepsilon_i - \varepsilon_j \in J\backslash\nu$, which shows that $M\backslash\nu$ is M-convex. □

Lemma 2.13. *Let μ be independent and let ν coindepdendent in J, and let $\tau \in \mathbb{Z}^n$ be such that $\tau \geqslant -\delta^-_J$. Then*

$$(J + \tau)/\mu \ =\ (J/\mu) + \tau \qquad \text{and} \qquad (J + \tau)\backslash\nu \ =\ (J\backslash\nu) + \tau,$$

with all sets M-convex.

Proof. As subsets of $\mathbb{Z}^n$, we have

$$(J + \tau)/\mu \ =\ \{\alpha + \tau - \mu \mid \alpha + \tau \in J + \tau,\ \mu + \delta^-_{J+\tau} \leqslant \alpha + \tau\}$$
$$=\ \{\alpha - \mu + \tau \mid \alpha \in J,\ \mu + \delta^-_J \leqslant \alpha\} \ =\ (J/\mu) + \tau$$

and

$$(J + \tau)\backslash\nu \ =\ \{\alpha + \tau \mid \alpha + \tau \in J + \tau,\ \alpha + \tau \leqslant \delta^+_{J+\tau} - \nu\}$$
$$=\ \{\alpha + \tau \mid \alpha \in J,\ \alpha \leqslant \delta^+_J - \nu\} \ =\ (J\backslash\nu) + \tau.$$

Since $\tau \geqslant -\delta^-_J$, all of these set are contained in $\mathbb{N}^n$ and are thus M-convex. □

The following result shows that contractions and deletions behave well — up to translation — with respect to polymatroid duality. Note that μ is independent in J if and only if it is coindependent in J^*, and vice versa.

Proposition 2.14. *Let μ be independent and let ν coindependent in J. Then*

$$(J/\mu)^* \;=\; J^*\backslash\mu \;+\; (\delta_{J/\mu} + \mu - \delta_J) \quad \textit{and} \quad (J\backslash\nu)^* \;=\; J^*/\nu \;+\; (\delta_{J\backslash\nu} + \nu - \delta_J).$$

Proof. These equalities follow from the direct computations

$$\begin{aligned}(J/\mu)^* &= \left\{\delta_{J/\mu} - (\alpha - \mu) \,\middle|\, \alpha \in J,\ \mu \leqslant \alpha - \delta_J^-\right\} \\ &= \left\{\delta_J - \alpha \,\middle|\, \alpha \in J,\ \mu \leqslant \delta_J^+ - (\delta_J - \alpha)\right\} \;+\; (\delta_{J/\mu} + \mu - \delta_J) \\ &= J^*\backslash\mu \;+\; (\delta_{J/\mu} + \mu - \delta_J)\end{aligned}$$

and

$$\begin{aligned}(J\backslash\nu)^* &= \left\{\delta_{J\backslash\nu} - \alpha \,\middle|\, \alpha \in J,\ \nu \leqslant \delta_J^+ - \alpha\right\} \\ &= \left\{\delta_J - \alpha - \nu \,\middle|\, \alpha \in J,\ \nu \leqslant (\delta_J - \alpha) - \delta_J^-\right\} \;+\; (\delta_{J\backslash\nu} + \nu - \delta_J) \\ &= J^*/\nu \;+\; (\delta_{J\backslash\nu} + \nu - \delta_J).\end{aligned}$$

□

Note that if $\delta_{J/\mu} + \mu = \delta_J$ and $\delta_{J\backslash\nu} + \nu = \delta_J$, then the equalities from Proposition 2.14 simplify to $(J/\mu)^* = J^*\backslash\mu$ and $(J\backslash\nu)^* = J^*/\nu$, which resemble the analogous formulas from matroid theory. In general, we only have the following bounds on the duality vectors of minors. Example 2.16 shows that these bounds are attained.

Let $\mathbf{1} \in \mathbb{N}^n$ be the all-ones vector. Recall that $|\mu| = \mu_1 + \cdots + \mu_n$.

Proposition 2.15. *Let $\mu = \mu_1 + \mu_2$ be independent and let $\nu = \nu_1 + \nu_2$ be coindependent in J. Then μ_2 is independent in J/μ_1, ν_2 is coindependent in $J\backslash\nu_1$, and*

$$J/\mu \;=\; (J/\mu_1)/\mu_2 \qquad \textit{and} \qquad J\backslash\nu \;=\; (J\backslash\nu_1)\backslash\nu_2.$$

Moreover,

$$\begin{aligned}\delta_{J/\mu}^- &= \delta_J^-, & 0 &\leqslant \delta_J^+ - (\delta_{J/\mu}^+ + \mu) &&\leqslant |\mu|\cdot\mathbf{1} - \mu, \\ \delta_{J\backslash\nu}^+ &= \delta_J^+ - \nu, & 0 &\leqslant \delta_{J\backslash\nu}^- - \delta_J^- &&\leqslant |\nu|\cdot\mathbf{1} - \nu,\end{aligned}$$

and thus

$$\begin{aligned}0 &\leqslant \delta_J - \left(\delta_{J/\mu} + \mu\right) \leqslant |\mu|\cdot\mathbf{1} - \mu, \\ 0 &\leqslant \left(\delta_{J\backslash\nu} + \nu\right) - \delta_J \leqslant |\nu|\cdot\mathbf{1} - \nu.\end{aligned}$$

Proof. We first treat the case of contractions, and then derive the results for deletions by duality. We begin with the bounds for $\delta^{\pm}_{J/\varepsilon_i}$ for $i \in [n]$. Fix $\alpha \in J$ with $\alpha - \varepsilon_i \in J/\varepsilon_i$, i.e., $\alpha_i \geqslant \delta^-_{J,i} + 1$.

By Lemma 2.13, $(J - \delta^-_J)/\varepsilon_i = (J/\varepsilon_i) - \delta^-_J$, which allows us to assume that $\delta^-_J = 0$ for simplicity. Then $\delta^-_{J/\varepsilon_i} \geqslant 0 = \delta^-_J$ is evident. In order to establish the reverse inequality, consider $j \in [n]$ and $\beta \in J$ with $\beta_j = \delta^-_{J,j}$. If $\beta_i = \delta^-_{J,i} < \alpha_i$, then there is a $k \in [n]$ with $\beta_k > \alpha_k$ (and thus $k \neq i, j$) and $\beta' = \beta + \varepsilon_i - \varepsilon_k \in J$. Thus we can assume that $\beta_j = \delta^-_{J,j}$ and $\beta_i \geqslant \delta^-_{J,i} + 1$, i.e., $\beta - \varepsilon_i \in J/\varepsilon_i$. Therefore $\delta^-_{J/\varepsilon_i,j} \leqslant (\beta - \varepsilon_i)_j \leqslant \beta_j = \delta^-_{J,j}$, as desired.

Since $J/\varepsilon_i \subseteq J - \varepsilon_i$, we have $\delta^+_{J/\varepsilon_i} + \varepsilon_i \leqslant \delta^+_J$ and thus $0 \leqslant \delta^+_J - (\delta^+_{J/\varepsilon_i} + \varepsilon_i)$. In order to establish the upper bound, choose $j \in [n]$ and $\beta \in J$ such that $\beta_j = \delta^+_{J,j}$. If $\beta_i = \delta^-_{J,i} < \alpha_i$, the M-convexity of J implies that there is a $k \in [n]$ with $\beta_k > \alpha_k$ (and thus $k \neq i$, but possibly $k = j$) and $\beta' = \beta + \varepsilon_i - \varepsilon_k \in J$. Thus we can assume that $\beta_j \geqslant \delta^+_{J,j} - 1$ (and $\beta_j = \delta^+_{J,j}$ if $j = i$) and $\beta_i \geqslant \delta^-_{J,i} + 1$, i.e., $\beta - \varepsilon_i \in J/\varepsilon_i$. Therefore $\delta^+_{J,j} - 1 \leqslant \beta_j = (\beta - \varepsilon_i)_j \leqslant \delta^+_{J/\varepsilon_i,j}$ for $j \neq i$, and $\delta^+_{J,i} - 1 = \beta_i - 1 = (\beta - \varepsilon_i)_i \leqslant \delta^+_{J/\varepsilon_i,i}$, which establishes the desired upper bound $\delta^+_J - (\delta^+_{J/\varepsilon_i} + \varepsilon_i) \leqslant \mathbf{1} - \varepsilon_i$.

Next we establish the equality $J/\mu = (J/\mu_1)/\mu_2$ for $\mu_1 = \varepsilon_i$, by the following direct computation:

$$\begin{aligned}(J/\mu_1)/\mu_2 \; &= \; \big\{(\alpha - \mu_1) - \mu_2 \,\big|\, \alpha - \mu_1 \in J/\mu_1,\ (\alpha - \mu_1) - \mu_2 \geqslant \delta^-_{J/\mu_1}\big\} \\ &= \; \big\{\alpha - (\mu_1 + \mu_2) \,\big|\, \alpha \in J,\ \alpha - (\mu_1 + \mu_2) \geqslant \delta^-_J\big\} \; = \; J/\mu,\end{aligned}$$

where we use that $\delta^-_{J/\mu_1} = \delta^-_J$ for $\mu_1 = \varepsilon_i$. Note that since μ is independent in J, there exists $\alpha \in J$ with $(\alpha - \mu_1) - \mu_2 = \alpha - \mu \geqslant \delta^-_J = \delta^-_{J/\mu_1}$; thus, in particular, $\alpha - \mu_1 \in J/\mu_1$ and μ_2 is independent in J/μ_1, as claimed.

As a consequence, an induction over $s = |\mu|$ for $\mu = \varepsilon_{i_1} + \ldots + \varepsilon_{i_s}$ shows that $\delta^-_{J/\mu} = \delta^-_{J/(\mu - \varepsilon_{i_s})} = \ldots = \delta^-_J$ as well as $0 \leqslant \delta^+_J - (\delta^+_{J/\mu}) + \mu \leqslant |\mu| \cdot \mathbf{1} - \mu$. This in turn makes the above verification of $J/\mu = (J/\mu_1)/\mu_2$ valid for all μ_1. This establishes all claims of the proposition for contractions.

We proceed with deletions, and begin with the bounds for $(\delta_{J\backslash\nu} + \nu) - \delta_J$. By Proposition 2.14, we have

$$J\backslash\nu \; = \; (J^*/\nu)^* \; + \; (\delta_J - \delta_{J^*/\nu} - \nu),$$

and by Lemma 2.4 ($\delta_{J\backslash\nu} = \delta_{(J^*/\nu)^*} + 2\delta_J - 2\delta_{J^*/\nu} - 2\nu$) and Lemma 2.3 ($\delta_J = \delta_{J^*}$), we have

$$\big(\delta_{J\backslash\nu} + \nu\big) - \delta_J \; = \; \delta_{(J^*\backslash\nu)^*} + 2\delta_J - 2\delta_{J^*/\nu} - 2\nu + \nu - \delta_J \; = \; \delta_{J^*} - \big(\delta_{J^*/\nu} + \nu\big).$$

Applying the bounds for the contraction to $\delta_{J^*} - (\delta_{J^*/\nu} + \nu)$ yields the desired bounds for $(\delta_{J\backslash\nu} + \nu) - \delta_J$. Together with the trivial inequalities $0 \leqslant \delta^-_{J\backslash\nu} - \delta^-_J$ and $\delta^+_{J\backslash\nu} \leqslant \delta^+_J - \nu$, this implies the other two desired bounds for $\delta^-_{J\backslash\nu}$ and $\delta^+_{J\backslash\nu}$.

Finally, the equality $J\backslash\nu = (J\backslash\nu_1)\backslash\nu_2$, can also be deduced by duality from the previous results, using Proposition 2.14:

$$\begin{aligned}\big((J\backslash\nu_1)\backslash\nu_2\big)^* &= (J\backslash\nu_1)^*/\nu_2 \ + \ \big(\delta_{(J\backslash\nu_1)\backslash\nu_2} + \nu_2 - \delta_{J\backslash\nu_1}\big) \\ &= (J/\nu_1)/\nu_2 \ + \ \big(\delta_{J\backslash\nu_1} + \nu_1 - \delta_J\big) \ + \ \big(\delta_{(J\backslash\nu_1)\backslash\nu_2} + \nu_2 - \delta_{J\backslash\nu_1}\big) \\ &= J/\nu \ + \ \big(\delta_{J\backslash\nu} + \nu - \delta_J\big) \\ &= (J\backslash\nu)^*. \end{aligned}$$ □

Example 2.16. The only independent vector and the only coindependent vector for Δ_1^r is 0. Thus Δ_1^r does not have any proper minors.

For $J = \Delta_n^r$ with $n \geqslant 2$ and $r \geqslant 1$, we have $\delta_J^- = 0$ and $\delta_J = \delta_J^+ = r\varepsilon_1 + \cdots + r\varepsilon_n$. For $\mu = s\varepsilon_1$ with $0 \leqslant s \leqslant r$, we find

$$\Delta_n^r/s\varepsilon_1 \ = \ \Delta_n^{r-s} \qquad \text{and} \qquad \Delta_n^r\backslash s\varepsilon_1 \ = \ \Delta_n^r \setminus \{(r-s+1)\varepsilon_1 + \alpha \mid \alpha \in \Delta_n^{s-1}\}.$$

In the case of the contraction, the difference of the dimension vectors assumes the extremal value $\delta_J - (\delta_{J/\mu} + \mu) = |\mu^+| \cdot \mathbf{1} - \mu$ from Proposition 2.15. In the case of the deletion, we have $\delta^-_{\Delta_n^r\backslash s\varepsilon_1} = s\varepsilon_2$ if $n = 2$ and $\delta^-_{\Delta_n^r\backslash s\varepsilon_1} = 0$ for $n \geqslant 3$. Thus we have

$$\delta_{J\backslash\mu} \ = \ \begin{cases} (r-s)\varepsilon_1 + (r+s)\varepsilon_2 & \text{if } n = 2, \\ (r-s)\varepsilon_1 + r\varepsilon_2 + \ldots + r\varepsilon_n & \text{if } n \geqslant 3, \end{cases}$$

and the difference of the dimension vectors is

$$(\delta_{J\backslash\mu} \ + \ \mu) \ - \ \delta_J \ = \ \begin{cases} |\mu| \cdot \mathbf{1} - \mu & \text{if } n = 2, \\ 0 & \text{if } n \geqslant 3, \end{cases}$$

which assumes the different extremal values from Proposition 2.15, depending on the value of n.

An example in which the extremal value $\delta_J - \delta_{J/\mu} - \mu = 0$ appears is the M-convex set

$$J \ = U_{2,3}^+ := \ \big\{2\varepsilon_1,\ \varepsilon_1 + \varepsilon_2,\ \varepsilon_1 + \varepsilon_3,\ \varepsilon_2 + \varepsilon_3\big\}$$

in Δ_3^2. Here we have $\delta_J^- = 0$ and $\delta_J = 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$. For $\mu = \varepsilon_1$, we find $J/\varepsilon_1 = \Delta_3^1$ and $\delta_{J/\varepsilon_1} = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Thus $\delta_J - (\delta_{J/\mu} + \mu) = 0$.

The exchange of order of contractions and deletions involves a translation, as the following result shows.

Proposition 2.17. *Let $\mu, \nu \in \mathbb{N}^n$. Then the following are equivalent:*

(1) $\delta_J^- + \mu \leqslant \alpha \leqslant \delta_J^+ - \nu$ for some $\alpha \in J$;
(2) μ is independent in J and $\nu' = \sup\{\mathbf{0},\ \nu - \delta_J^+ + \delta_{J/\mu}^+ + \mu\}$ is coindependent in J/μ;

(3) ν is coindependent in J and $\mu' = \sup\{\mathbf{0},\ \delta_J^- - \delta_{J\backslash\nu}^- + \mu\}$ is independent in $J\backslash\nu$.

If these conditions are met, then

$$(J\backslash\nu)/\mu' \ = \ (J/\mu)\backslash\nu' \ + \ \sup\{-\mu,\ \ \delta_{J\backslash\nu}^- - \delta_J^-\}.$$

Proof. We begin with the equivalence of (1) and (2). First note that $\mu + \delta_J^- \leqslant \alpha$ for $\alpha \in J$ means that μ is independent in J and that $\beta = \alpha - \mu$ is in J/μ. Thus (1) is equivalent to μ being independent in J, together with the existence of a $\beta \in J/\mu$ such that

$$\beta \ \leqslant \ \delta_J^+ \ - \ \nu \ - \ \mu \ = \ \delta_{J/\mu}^+ \ - \ \big(\nu \ + \ \delta_{J/\mu}^+ \ - \ \delta_J^+ \ + \ \mu\big).$$

Since $\beta \leqslant \delta_{J/\mu}^+ - 0$, this latter inequality turns into $\beta \leqslant \delta_{J/\mu}^+ - \nu'$, which means that ν' is coindependent in J/μ. This establishes the equivalence between (1) and (2).

We continue with the equivalence of (1) and (3), which is proven similarly. The inequality $\alpha \leqslant \delta_J^+ - \nu$ for $\alpha \in J$ expresses that ν is coindependent in J and that $\alpha \in J\backslash\nu$. Thus (1) is equivalent to ν being coindependent in J, together with the existence of $\alpha \in J\backslash\nu$ such that

$$\delta_{J\backslash\nu}^- \ + \ \big(\delta_J^- \ - \ \delta_{J\backslash\nu}^- \ + \ \mu\big) \ = \ \delta_J^- \ \ + \ \mu \leqslant \ \alpha.$$

Since $\delta_{J\backslash\nu}^- + 0 \leqslant \alpha$, this latter inequality turns into $\delta_{J\backslash\nu}^- + \mu' \leqslant \alpha$, which means that μ' is independent in $J\backslash\nu$. This establishes the equivalence between (1) and (3).

We turn to the last claim, assuming (1)–(3), which follows from the direct computation

$$\begin{aligned}(J\backslash\nu)/\mu \ &= \ \big\{\alpha - \mu' \ \big|\ \alpha \in J,\ \mu' + \delta_{J\backslash\nu}^- \leqslant \alpha \leqslant \delta_J^+ - \nu\big\} \\ &= \ \big\{\alpha - \mu \ \big|\ \alpha \in J,\ \mu + \delta_J^- \leqslant \alpha,\ \alpha - \mu \leqslant \delta_{J/\mu}^+ - \nu'\big\} \ + \ \big(\mu - \mu'\big) \\ &= \ (J/\mu)\backslash\nu' \ + \ \sup\{-\mu,\ \ \delta_{J\backslash\nu}^- - \delta_J^-\}.\end{aligned}$$

□

Example 2.18. Note that we need to use the supremum in the definitions of μ' and ν' in Proposition 2.17 since the tuples $\nu - \delta_J^+ + \delta_{J/\mu}^+ + \mu$ and $\delta_J^- - \delta_{J\backslash\nu}^- + \mu$ might have negative coefficients. An example where this happens is $J = \Delta_3^1$ with $\mu = \varepsilon_1$ and $\nu = \varepsilon_3$. In this case, we have $\nu - \delta_J^+ + \delta_{J/\mu}^+ + \mu = -\varepsilon_2$. Similarly, we have $\delta_{J^*}^- - \delta_{J^*\backslash\nu}^- + \mu = -\varepsilon_2$.

Definition 2.19. An *embedded minor of J* is an M-convex set of the form

$$J\backslash\nu/\mu \ + \ \tau \ = \ (J\backslash\nu)/\mu \ + \ \tau,$$

together with the *minor embedding*

$$\begin{aligned}\iota_{J\backslash\nu/\mu+\tau} :\ \ J\backslash\nu/\mu + \tau \ &\longrightarrow \ \ J \\ \alpha \ &\longmapsto \ \alpha + \mu - \tau,\end{aligned}$$

where $\tau \geqslant -\delta_{J\backslash\nu/\mu}^-$ and $\mu + \delta_{J\backslash\nu}^- \leqslant \alpha \leqslant \delta_J^+ - \nu$ for some $\alpha \in J$.

Note that the minor embedding $\iota_{J\backslash\nu/\mu+\tau}$ is an injective map. Note further that ν, μ and τ are uniquely determined by the minor embedding $\iota_{J\backslash\nu/\mu+\tau}$. In Section 7, we investigate the relationship between representations of J and representations of its embedded minors.

Remark 2.20 (Truncations with cubes and the multi-affine part)**.** A *cube in* $\mathbb{N}^n$ is a the non-empty intersection of $\mathbb{N}^n$ with a product of closed intervals, i.e. a subset of the form

$$I \;=\; I_{\beta,\gamma} \;=\; \{\alpha \in \mathbb{N}^n \mid \beta \leqslant \alpha \leqslant \gamma\}$$

with $\beta \leqslant \gamma$ in $\mathbb{N}^n$. As observed in the proof of [17, Lemma 2.8], the intersection of an M-convex set J with a cube I is M-convex, provided it is non-empty. In fact, such an intersection is an embedded minor of J:

$$J \cap I_{\beta,\gamma} \;=\; J\backslash\nu/\mu + \mu$$

for $\nu = \sup\{\mathbf{0},\ \delta_J^+ - \gamma\}$ and $\mu = \sup\{\mathbf{0},\ \beta - \delta_{J\backslash\nu}^-\}$. In particular, the *multi-affine part of* J is the intersection of J with the unit cube $I_{\mathbf{0},\mathbf{1}}$, which is M-convex whenever it is not empty (cf. [17, Cor. 3.5]).

2.3. Permutation and extension of variables. In this section, we discuss two additional operations on an M-convex set $J \subseteq \Delta_n^r$: permutation and extension of variables.

The proofs of the following two lemmas are immediate from the defining property of M-convex sets:

Lemma 2.21. *Let $\sigma \in S_n$ be a permutation of $[n]$ and $\iota_\sigma \colon \Delta_n^r \to \Delta_n^r$ the bijection defined by $\iota_\sigma(\alpha_i) = \alpha_{\sigma(i)}$ for $\alpha \in \Delta_n^r$ and $i \in [n]$. Then $\iota_\sigma(J)$ is M-convex.*

Lemma 2.22. *Let $\iota_{n+1} \colon [n] \to [n+1]$ be the inclusion given by $\iota_{n+1}(i) = i$ for $i \in [n]$. Then $\iota_{n+1}(J) \subseteq \Delta_{n+1}^r$ is M-convex.*

Definition 2.23. An M-convex set J' is said to be *elementary equivalent to* J if $J' = J + \tau$ for some $\tau \geqslant -\delta_J^-$, or $J' = \iota_\sigma(J)$ for some $\sigma \in S_n$, or $J' = \iota_n(J)$, or $J = \iota_n(J')$. Two M-convex sets J and J' are said to be *combinatorially equivalent* if there exists a chain of elementary equivalences $J = J_0 \sim \ldots \sim J_\ell = J'$. In this case, we say that J' is *of type* J.

An *elementary polymatroid embedding* is a map $\iota\colon J \to J'$ between polymatroids which is given by either a minor embedding $\iota_{J\backslash\nu/\mu+\tau}$, a permutation of variables ι_σ, an extension of variables ι_n, or the inverse ι_n^{-1} of an extension of variables. A *polymatroid embedding* is a map $\iota\colon J \to J'$ between polymatroids that is the composition of elementary polymatroid embeddings.

Lemma 2.24. *Two polymatroids J and J' are combinatorially equivalent if and only if there exists a bijective polymatroid embedding $J \to J'$.*

Proof. First note that each type of elementary equivalence induces a bijection $J \to J'$ between the respective M-convex sets, which is given by $\alpha \mapsto \alpha + \tau$ (translation), $\alpha \mapsto \iota_\sigma(\alpha)$ (permutation), and $\alpha \mapsto \iota_n(\alpha)$ (extension of variables), respectively. The inverse bijection is in each case also a polymatroid embedding, namely, $\alpha \mapsto \alpha - \tau$, $\alpha \mapsto \iota_{\sigma^{-1}}(\alpha)$, and ι_n^{-1}, respectively. Thus, if J and J' are combinatorially equivalent, the composition of the defining elementary equivalences yield a bijective polymatroid embedding $J \to J'$.

Conversely, we note that every elementary polymatroid embedding is injective and that the polymatroid embeddings $\iota_{J\backslash\varepsilon_\ell}\colon J\backslash\varepsilon_\ell \to J$ and $\iota_{J/\varepsilon_\ell}\colon J/\varepsilon_\ell \to J$ are not surjective. Thus, if $\iota\colon J \to J'$ is a bijective polymatroid embedding, it must be composed of elementary equivalences, which shows that J and J' are combinatorially equivalent. □

2.4. Comparison with Whittle's notion of minors. In [47], Whittle introduces single-element deletions and contractions for rank functions of (discrete) polymatroids; cf. also [37]. We recall Whittle's construction and compare it to our notions of deletion and contraction.

Let $\mathbf{r}\colon 2^{[n]} \to \mathbb{N}$ be the rank function of J, i.e., $\mathbf{r}(S) = \max\{\alpha_S \mid \alpha \in J\}$ for $S \subseteq [n]$. For ease of notation, and without loss of generality, we describe Whittle's operations only for the element n.

Let $\iota_{n-1}\colon \mathbb{N}^{n-1} \to \mathbb{N}^n$ be the inclusion of the first $n-1$ coordinates. The *deletion of n in* $\mathbf{r}$ is the function $\mathbf{r}\backslash n\colon 2^{[n-1]} \to \mathbb{N}$ given by $\mathbf{r}\backslash n(S) = \mathbf{r}(\iota_{n-1}(S))$ for $S \subseteq [n-1]$. The *contraction of n in* $\mathbf{r}$ is the function $\mathbf{r}/n\colon 2^{[n-1]} \to \mathbb{N}$ given by $\mathbf{r}/n(S) = \mathbf{r}(\iota_{n-1}(S) \cup n) - \mathbf{r}(n)$ for $S \subseteq [n-1]$.

Both $\mathbf{r}\backslash n$ and $\mathbf{r}/n$ are indeed rank functions of polymatroids, i.e., they satisfy

$$\mathbf{r}\backslash n(S) \;=\; \max\{\alpha_S \mid \alpha \in J_{\backslash n}\} \qquad \text{and} \qquad \mathbf{r}/n(S) \;=\; \max\{\alpha_S \mid \alpha \in J_{/n}\}$$

for all $S \subseteq [n-1]$ and (uniquely determined) M-convex sets $J_{\backslash n}$ and $J_{/n}$. The following result identifies these two M-convex sets (embedded into $\mathbb{N}^n$ via $\iota_{n-1}\colon \mathbb{N}^{n-1} \to \mathbb{N}^n$) with embedded minors of J in the sense of our paper.

Proposition 2.25. *Define $\mu = (\delta^+_{J,n} - \delta^-_{J,n}) \cdot \varepsilon_n$ and $\tau = \delta^-_n \cdot \varepsilon_n$. Then*

$$\iota_{n-1}(J_{\backslash n}) \;=\; J\backslash\mu \;-\; \tau \qquad \textit{and} \qquad \iota_{n-1}(J_{/n}) \;=\; J/\mu \;-\; \tau.$$

Proof. We only explain the proof the first identity; the proof of the second identity is analogous. (Alternatively, it can be deduced from the first identity by establishing suitable compatibilities between polymatroid duality in the sense of this paper, rank functions, and ι_{n-1}.)

We establish the first equality by identifying the rank function $\mathbf{r}_{\backslash n}$ of $\iota_{n-1}(J_{\backslash n})$ with the rank function $\mathbf{r}'$ of

$$J\backslash\mu \;-\; \tau \;=\; \left\{\alpha \in \Delta_n^{r-\delta^-_n} \,\middle|\, \alpha_n = 0,\; \alpha + \tau \in J\right\}.$$

We first show that $\mathbf{r}'(S) \leqslant \mathbf{r}_{\backslash n}(S)$ for all $S \subseteq [n]$. Note that evidently $\mathbf{r}_{\backslash n}(S-n) \leqslant \mathbf{r}_{\backslash n}(S)$, and that the above description of $J\backslash\mu - \tau$ yields $\mathbf{r}'(S-n) = \mathbf{r}'(S)$. Consider $\alpha \in J\backslash\mu - \tau$ with $\mathbf{r}'(S-n) = \alpha_{S-n}$ and let $\beta = \alpha + \tau \in J$. Then

$$\mathbf{r}'(S) \;=\; \mathbf{r}'(S-n) \;=\; \alpha_{S-n} \;=\; \beta_{S-n} \;\leqslant\; \mathbf{r}_{\backslash n}(S-n) \;\leqslant\; \mathbf{r}_{\backslash n}(S),$$

as desired.

Next we show that, for every $S \subseteq [n-1]$, there exists an $\alpha \in J$ with $\mathbf{r}_{\backslash n}(S) = \alpha_S$ and $\alpha_n = \delta_n^-$. For this, choose $\alpha \in J$ with $\mathbf{r}_{\backslash n}(S) = \alpha_S$ so that α_n is minimal, and choose $\beta \in J$ with $\beta_n = \delta_J^-$. If $\alpha_n > \delta_n^- = \beta_n$, there is a $k \in [n]$ such that $\alpha' = \alpha + \varepsilon_k - \varepsilon_n \in J$. This element satisfies $\mathbf{r}_{\backslash n}(\alpha) \leqslant \mathbf{r}_{\backslash n}(\alpha')$ and $\alpha'_n < \alpha_n$, which contradicts our assumptions on α. Thus $\alpha_n = \delta_J^-$, as desired.

In order to show that $\mathbf{r}'(S) \geqslant \mathbf{r}_{\backslash n}(S)$, we choose $\alpha \in J$ with $\mathbf{r}_{\backslash n}(S-n) = \alpha_{S-n}$ and $\alpha_n = \delta_J^-$. Then $\alpha - \tau \in J\backslash\mu - \tau$, and thus

$$\mathbf{r}'(S) \;\geqslant\; (\alpha - \tau)_S \;=\; \mathbf{r}_{\backslash n}(S),$$

which completes the proof. □

Remark 2.26. The main focus of Whittle's paper [47] is on duality operations which interchange deletion and contraction, in the sense that $(\mathbf{r}\backslash n)^* = \mathbf{r}^*/n$. This is a stricter requirement than what the duality operation, in the sense of this paper, satisfies. Under the additional assumption that duality is an involution, [47] shows that such a duality operation only exists for particular subclasses of polymatroids. The corresponding duality functions from [47] do not correspond to the duality in the sense of this paper (not even up to translation).

2.5. Direct sums. Let $J_1 \subseteq \Delta_{n_1}^{r_1}$ and $J_2 \subseteq \Delta_{n_2}^{r_2}$ be M-convex sets. Let $r = r_1 + r_2$ and $n = n_1 + n_2$. For $\alpha_1 \in J_1$ and $\alpha_2 \in J_2$, we define $\alpha_1 \oplus \alpha_2 \in \mathbb{N}^n$ by

$$(\alpha_1 \oplus \alpha_2)_i \;=\; \begin{cases} \alpha_{1,i} & \text{if } 1 \leqslant i \leqslant n_1, \\ \alpha_{2,i-n_1} & \text{if } n_1 < i \leqslant n. \end{cases}$$

Note that $|\alpha_1 \oplus \alpha_2| = r_1 + r_2 = r$ and thus $\alpha_1 \oplus \alpha_2 \in \Delta_n^r$. The *direct sum of J_1 and J_2* is the subset

$$J_1 \oplus J_2 \;=\; \{\alpha_1 \oplus \alpha_2 \in \Delta_n^r \mid \alpha_1 \in J_1,\ \alpha_2 \in J_2\}$$

of Δ_n^r, which inherits the exchange property from J_1 and J_2 and is therefore M-convex.

The direct sum of polymatroids behaves well with respect to the operations and constructions of the previous sections. The following properties are easy to verify, and will not be used elsewhere in this paper, so we omit the straightforward proofs.

Proposition 2.27. *The direct sum $J = J_1 \oplus J_2$ satisfies the following properties:*

(1) $\delta_J^- = \delta_{J_1}^- \oplus \delta_{J_2}^-$, $\delta_J^+ = \delta_{J_1}^+ \oplus \delta_{J_2}^+$ *and* $\delta_J = \delta_{J_1} \oplus \delta_{J_2}$*;*
(2) $J^* = J_1^* \oplus J_2^*$*;*
(3) $(J_1 \oplus J_2)\backslash(\nu_1 \oplus \nu_2)/(\mu_1 \oplus \mu_2) = (J_1\backslash\nu_1/\mu_1) \oplus (J_1\backslash\nu_2/\mu_2)$*;*
(4) $J_1 \oplus J_2$ *and* $J_2 \oplus J_1$ *are elementary equivalent (by a suitable permutation);*
(5) $\overline{J} = \overline{J}_1 \oplus \overline{J}_2$*, where* $\overline{J}$ *denotes the reduction of a polymatroid* J *(cf. Definition 2.9).*

2.5.1. *Decomposition into indecomposable summands.* An M-convex set $J \subseteq \Delta_n^r$ is *decomposable* if it is combinatorially equivalent to the direct sum $J_1 \oplus J_2$ of two polymatroids with nonempty ground sets $[n_1]$ and $[n_2]$. If J is not decomposable and its ground set is nonempty, then it is *indecomposable*. If J is a matroid, then it is indecomposable as an M-convex set if and only if it is connected as a matroid.

For $\alpha \in J$ and $S \subseteq [n]$, we write $\alpha_S = \sum_{i \in S} \alpha_i$; in particular, $|\alpha| = \alpha_{[n]}$. We say that a subset $S \subseteq [n]$ is a *direct summand of J* if there is an $r_S \in \mathbb{N}$ such that $\alpha_S = r_S$ for all $\alpha \in J$.

Lemma 2.28. *If* $S \subseteq [n]$ *is a direct summand of* J *and* $T = [n] - S$*, then* J *is elementary equivalent to* $J_1 \oplus J_2$ *(by a suitable permutation of variables) for two M-convex sets* J_1 *and* J_2 *with respective ground sets* $[\#S]$ *and* $[\#T]$*. In particular,* $J \subseteq \Delta_n^r$ *is indecomposable if and only if for every nonempty proper subset* $S \subsetneq [n]$*, there are elements* $\alpha, \beta \in J$ *with* $\alpha_S \neq \beta_S$*.*

Proof. By assumption there is an $r_1 \in \mathbb{N}$ such that $\alpha_S = r_1$ for all $\alpha \in J$. After permuting $[n]$, we can assume that $S = [n_1] \subseteq [n]$ for $n_1 = \#S$. Let $T = [n_2]$ for $n_2 = \#T = n - n_1$ and $r_2 = r - r_1$. Then every $\alpha \in J$ can be written as $\alpha = \alpha_{J_1} \oplus \alpha_{J_2}$ for $\alpha_{J_1} \in \Delta_{n_1}^{r_1}$ and $\alpha_{J_2} \in \Delta_{n_2}^{r_2}$. Define $J_1 = \{\alpha_{J_1} \mid \alpha \in J\}$ and $J_2 = \{\alpha_{J_2} \mid \alpha \in J\}$. We claim that $J = J_1 \oplus J_2$ is a decomposition of J.

First note that J_1 and J_2 are M-convex, since the fact that $\alpha_S = r = \beta_S$ for any two $\alpha, \beta \in J$ guarantees that when we apply the exchange axiom, we substitute an $i \in S$ by a $j \in S$. This allows us to deduce the exchange axiom for J_1 and J_2 from the exchange axiom for J.

It is evident that $J \subseteq J_1 \oplus J_2$. Consider $\alpha_1 \in J_1$ and $\alpha_2 \in J_2$. We need to show that $\alpha_1 \oplus \alpha_2 \in J$. By the definition of J_1 and J_2, there are $\beta_1 \in J_1$ and $\beta_2 \in J_2$ such that $\alpha_1 \oplus \beta_2, \beta_1 \oplus \alpha_2 \in J$. Using the fact that the exchange axiom substitutes an $i \in S$ by a $j \in S$, we can exchange the elements in the difference $\beta_1 - \alpha_1$ one by one, and by induction we obtain $\alpha_1 \oplus \alpha_2 \in J$ as desired. This establishes the first claim.

We turn to the second claim. If J is indecomposable and $S \subseteq [n]$ is a direct summand, then $S = \emptyset$ or $S = [n]$. Conversely, if J decomposes into the direct sum $J_1 \oplus J_2$ of two polymatroids with nonempty ground sets $[n_1]$ and $[n_2]$, then for every nonempty proper subset $S = [n_1]$ of $[n]$, $\alpha_S = |\alpha_1|$ is equal to the rank r_1 of J_1 for all $\alpha = \alpha_1 \oplus \alpha_2 \in J$. This establishes the reverse implication. □

Note that $(J_1 \oplus J_2) \oplus J_3 = J_1 \oplus (J_2 \oplus J_3)$, which allows us define the direct sum

$$\bigoplus_{i=1}^{m} J_i \;=\; J_1 \oplus \cdots \oplus J_m$$

of M-convex sets $J_1, \ldots, J_m$ unambiguously. As in Proposition 2.27, permuting the summands results in a direct sum that is elementary equivalent to $\bigoplus J_i$ (by a suitable permutation of variables).

Proposition 2.29. *Let J be an M-convex set. Then there are a unique positive integer m and indecomposable M-convex sets $J_1, \ldots, J_m$, which are unique up to combinatorial equivalence and a permutation of indices, such that J is combinatorially equivalent to $J_1 \oplus \cdots \oplus J_m$.*

Proof. By Lemma 2.28, every direct summand $S \subseteq [n]$ of J induces a decomposition into $J_1 \oplus J_2$ (up to permutation of variables). Using this fact repeatedly leads to a decomposition of J into a finite number of indecomposable M-convex sets.

The uniqueness follows from the fact that the intersection $S \cap T$ of two direct summands S and T of J is also a direct summand of $[n]$. To see this, we prove that $\alpha_{S\cap T} = \beta_{S\cap T}$ for all $\alpha, \beta \in J$ by induction on the *distance* $d(\alpha, \beta) = \frac{1}{2} \cdot \sum_{i\in[n]} |\alpha_i - \beta_i|$ between α and β.

If $d = d(\alpha, \beta) = 0$, then $\alpha_{S\cap T} = \beta_{S\cap T}$, as claimed. If $d > 0$, then $\alpha_i < \beta_i$ for some $i \in [n]$, and thus there is a $j \in [n]$ such that $\alpha' = \alpha + \varepsilon_i - \varepsilon_j \in J$. Since $d(\alpha', \beta) = d(\alpha, \beta) - 1$, the inductive hypothesis shows that $\alpha'_{S\cap T} = \beta_{S\cap T}$. Since $\alpha_S = \beta_S$ and $\alpha_T = \beta_T$, we have $i \in S$ if and only if $j \in S$, and $i \in T$ if and only if $j \in T$. Thus $\alpha_{S\cap T} = \alpha'_{S\cap T}$, and the result now follows by induction. □

Part 2. Representations of polymatroids

3. Tracts

Tracts were introduced in [3] as a streamlined way to systematically study matroids with coefficients. The axioms for a tract are similar to the axioms for a field, but we relax the requirement that addition is a binary operation. More precisely, we do not define the sum of two elements of a tract F, but rather declare certain formal sums $a_1 + \cdots + a_k$ of elements of F to be "null" and the rest to be non-null.

In this section, we review the definition and basic properties of tracts and provide a number of examples.

3.1. Definition of tracts.

A *pointed monoid* is a multiplicatively written commutative monoid F with identity element 1 and a distinguished *absorbing* element 0 that satisfies $0 \cdot a = 0$ for all $a \in F$. The *unit group of* F is the group

$$F^\times := \{a \in F \mid ab = 1 \text{ for some } b \in F\}$$

of all invertible elements in F.

A *pointed (abelian) group* is a pointed monoid F such that $F^\times = F - \{0\}$. The *ambient semiring* of a pointed group F is the group semiring

$$F^+ := \mathbb{N}[F^\times].$$

We denote its elements by $\sum n_a.a$, where $n_a \in \mathbb{N}$ and $n_a = 0$ for all but finitely many $a \in F^\times$. We sometimes write the sum $\sum n_a.a$ as $n_1.a_1 + \cdots + n_r.a_r$ or $\sum a_i$ (where a appears n_a times as a summand). The pointed group F embeds into F^+ by sending 0 to the empty sum (which is the additive identity element of F^+) and $a \in F^\times$ to $a = 1.a \in F^+$.

An *ideal of* F^+ is a subset I that contains 0 and is closed under addition and under multiplication by elements of F^+. For a subset $S \subseteq F^+$, we denote by $\langle S \rangle$ the ideal of F^+ generated by S.

A *tract*[3] is a pointed group F together with an ideal N_F of F^+, called the *null set of* F, such that for every $a \in F$ there is a unique $b \in F$ with $a + b \in N_F$. We write $-a$ for the unique element b with $a + b \in N_F$, and call it the *additive inverse of* a. We often write $a - b$ instead of $a + (-b)$. Typically, we denote a tract by F and suppress its null set N_F from the notation.

Note that the tract axioms imply that $a \in N_F$ if and only if $a = 0$; in particular $-0 = 0$. Furthermore, we have $(-1)^2 = 1$ and $a + b \in N_F$ if and only if $b = -a$.

A *tract morphism* is a multiplicative map $f\colon F_1 \to F_2$ with $f(0) = 0$ and $f(1) = 1$ such that $\sum f(a_i) \in N_{F_2}$ for all $\sum a_i \in N_{F_1}$. This defines the category Tracts.

3.2. First examples. Every field F is a tract by defining the null set of F as

$$N_F = \{\textstyle\sum a_i \mid \sum a_i = 0 \text{ in } F\}.$$

This construction extends to *partial fields* and *hyperfields*. Semple and Whittle's original definition of a partial field in [42] is in terms of a pointed group F together with a partially defined addition $+ : F \times F \dashrightarrow F$ that satisfies certain axioms. Equivalently, the partial addition can be captured in terms of a certain ring R that contains F as a multiplicative submonoid ([39, Thm. 5.1]). By [8], every partial field F can be viewed as a tract by defining the null set of F as

$$N_F = \{\textstyle\sum a_i \mid \sum a_i = 0 \text{ in } R\}.$$

[3] We deviate in this text from the definition of a tract in [3] by imposing the property that N_F is closed under addition. What we call a tract in this text should, strictly speaking, be called an *ideal tract* or *idyll* (cf. [8]).

Similarly a hyperfield F with hyperaddition $\boxplus : F \times F \to 2^F$ (where 2^F is the power set of F) is a tract with respect to the null set

$$N_F \;=\; \{\textstyle\sum a_i \mid 0 \in \boxplus\, a_i\}.$$

See [3, 8] for additional details on partial fields and hyperfields.

In each case, homomorphisms are naturally tract morphisms. More to the point, these constructions define fully faithful embeddings of the categories of fields, partial fields, and hyperfields into Tracts, which allows us to consider (partial / hyper-)fields as tracts by abuse of terminology.

Two concrete examples are the *regular partial field* $\mathbb{F}_1^\pm = \{0, 1, -1\}$, whose null set is

$$N_{\mathbb{F}_1^\pm} \;=\; \{n.1 + n.(-1) \mid n \in \mathbb{N}\} \;=\; \{0, \quad 1-1, \quad 1+1-1-1, \dots\},$$

and the *Krasner hyperfield* $\mathbb{K} = \{0, 1\}$, whose null set is

$$N_{\mathbb{K}} \;=\; \mathbb{N} - \{1\} \;=\; \{0, \quad 1+1, \quad 1+1+1, \quad 1+1+1+1, \dots\}.$$

Note that $-1 = 1$ in $\mathbb{K}$. The regular partial field $\mathbb{F}_1^\pm$ is an initial object of Tracts, and the Krasner hyperfield $\mathbb{K}$ is a terminal object of Tracts.

3.3. Subtracts. Let F be a tract. A *subtract of F* is a pointed submonoid A of F with $-1 \in A^\times = A - \{0\}$, which is a tract in its own right with respect to the null set

$$N_A \;=\; \{\textstyle\sum a_i \in N_F \mid a_i \in A\}.$$

3.4. Free algebras. Let k be a tract. A *k-algebra* is a tract F together with a tract morphism $\alpha_F \colon k \to F$. A k-linear morphism between k-algebras is a tract morphism $f \colon F_1 \to F_2$ between k-algebras F_1 and F_2 such that $\alpha_{F_2} = f \circ \alpha_{F_1}$.

Let $\{x_i\}_{i\in I}$ be a set. The *free k-algebra in $\{x_i\}$* is defined as follows: as a pointed monoid, it is

$$k(x_i) \;=\; k(x_i)_{i\in I} \;=\; \Big\{a \cdot \textstyle\prod x_i^{\varepsilon_i} \,\Big|\, a \in k,\ (\varepsilon_i) \in \bigoplus_I \mathbb{Z}\Big\} \;/\; \sim,$$

where $\sim$ is the equivalence relation generated by $0 := 0 \cdot \prod x_i^0 \sim 0 \cdot \prod x_i^{\varepsilon_i}$ for any $(\varepsilon_i) \in \bigoplus_I \mathbb{N}$. The association $a \mapsto a \cdot x_j^0$ defines an embedding of k as a submonoid of $k(x_i)$, which extends by linearity to an embedding $k^+ \to k(x_i)^+$. The nullset of $k(x_i)$ is the ideal $N_{k(x_i)}$ generated by the image of N_k in $k(x_i)^+$. We write $ax_{i_1}^{\varepsilon_{i_1}} \cdots x_{i_r}^{\varepsilon_{i_r}}$ for $a \cdot \prod x_i^{\varepsilon_i}$ with $\varepsilon_j = 0$ for $j \notin \{i_1, \dots, i_r\}$.

By construction, the inclusion $k \to k(x_i)$ is a tract morphism, which turns $k(x_i)$ into a k-algebra. It satisfies the expected universal property: every set-theoretic map $f_0 \colon \{x_i\} \to F$ into a k-algebra F extends uniquely to a k-linear morphism $f \colon k(x_i) \to F$ with $f(x_i) = f_0(x_i)$ (this is proven exactly as for pastures; cf. [9, Prop. 2.6]).

3.5. Quotients. Let F be a tract and $S \subseteq F^+$ be a subset that does not contain any element of $F^\times$. The *quotient of F by S* is the quotient monoid

$$F /\!\!/ \langle S \rangle \ := \ F/\sim,$$

where $\sim$ is the equivalence relation generated by the relations $ca \sim cb$ for all $a - b \in S$ and $c \in F$, together with the null set

$$N_{F/\!\!/\langle S \rangle} \ := \ \Big\langle \sum [ca_i] \ \Big| \ c \in F, \ \sum a_i \in N_F \cup S \Big\rangle,$$

where $[a]$ denotes the class of $a \in F$ in $F /\!\!/ \langle S \rangle$.

The quotient map $\pi_S \colon F \to F /\!\!/ \langle S \rangle$ is a tract morphism, which turns $F /\!\!/ \langle S \rangle$ into an F-algebra. It satisfies the expected universal property: every tract morphism $f\colon F \to F'$ with $\sum f(a_i) \in N_{F'}$ for all $\sum a_i \in S$ factors into $f = \bar{f} \circ \pi_S$ for a uniquely determined morphism $\bar{f}\colon F /\!\!/ \langle S \rangle \to F'$ (this is proven exactly as for pastures; cf. [9, Prop. 2.6]).

3.6. Tensor products. The category of tracts is complete and cocomplete. In a nutshell, $\mathbb{F}_1^\pm$ is an initial object, $\mathbb{K}$ is a terminal object, products are given by Cartesian products of the unit groups, equalizers are defined as the set-theoretic equalizers, and coequalizers can be constructed in terms of a quotient construction. The only subtle construction (similar to the constructions for rings) is the coproduct, or tensor product, of tracts, which is given by the following universal property.

The *tensor product* of a family $\{F_i\}_{i \in I}$ of tracts is a tract $\bigotimes F_i$, together with morphisms $\iota_i : F_i \to \bigotimes F_i$ (one for each $i \in I$), such that the induced map

$$\mathrm{Hom}\Big(\bigotimes_{i \in I} F_i, \ F' \Big) \ \longrightarrow \ \prod_{i \in I} \mathrm{Hom}(F_i, F')$$

is a bijection for all tracts F'. The construction of $\bigotimes F_i$ is analogous to the case of pastures ([9, Lemma 2.7]) and bands ([7, Prop. 1.42]).

3.7. More examples. Every tract F can be written in the form $F = \mathbb{F}_1^\pm(x_i) /\!\!/ \langle S \rangle$ by choosing a suitable set of generators x_i and a suitable set S of defining relations. Some examples are:

$\mathbb{F}_2 \ = \ \mathbb{F}_1^\pm /\!\!/ \langle 1 + 1 \rangle$	(the field with 2 elements)
$\mathbb{F}_3 \ = \ \mathbb{F}_1^\pm /\!\!/ \langle 1 + 1 + 1 \rangle$	(the field with 3 elements)
$\mathbb{S} \ = \ \mathbb{F}_1^\pm /\!\!/ \langle 1 + 1 - 1 \rangle$	(the *sign hyperfield*)
$\mathbb{U} \ = \ \mathbb{F}_1^\pm(x, y) /\!\!/ \langle x + y - 1 \rangle$	(the *near regular partial field*)
$\mathbb{D} \ = \ \mathbb{F}_1^\pm(z) /\!\!/ \langle z - 1 - 1 \rangle$	(the *dyadic partial field*)
$\mathbb{H} \ = \ \mathbb{F}_1^\pm(z) /\!\!/ \langle z^3 + 1, \ \ z^2 - z + 1 \rangle$	(the *hexagonal partial field*)
$\mathbb{G} \ = \ \mathbb{F}_1^\pm(z) /\!\!/ \langle z^2 - z - 1 \rangle$	(the *golden ratio partial field*)

If F is a pointed group, there are several general ways to define a tract structure on F.

We define the *trivial* tract structure on F by letting the null set be

$$N_F \;=\; \langle a + a \mid a \in F^\times \rangle,$$

so that in particular $1 = -1$. For example, this provides $\mathbb{R}_{\geqslant 0}$ with the structure of a tract. (And, from now on, when we write $\mathbb{R}_{\geqslant 0}$ as a tract, we consider it with the trivial tract structure.)

We define the *degenerate* tract structure on F by letting the null set be

$$N_F \;=\; \left\{ a_1 + \cdots + a_n \,\middle|\, a_2 = -a_1 \text{ or at least 3 terms are nonzero} \right\}.$$

The *tropical hyperfield* is the tract

$$\mathbb{T}_0 \;=\; \mathbb{R}_{\geqslant 0} /\!\!/ \langle \textstyle\sum a_i \mid \text{the maximum among } a_1, \ldots, a_n \text{ appears at least twice} \rangle,$$

the *triangular hyperfield* is the tract

$$\mathbb{T}_1 \;=\; \mathbb{R}_{\geqslant 0} /\!\!/ \langle \textstyle\sum a_i \mid a_1, \ldots, a_n \text{ are the side lengths of a (possibly degenerate) polygon} \rangle,$$

and the *degenerate triangular hyperfield* is the tract

$$\mathbb{T}_\infty \;=\; \mathbb{R}_{\geqslant 0} /\!\!/ \langle \textstyle\sum a_i \mid \text{the maximum appears twice or at least 3 terms are nonzero} \rangle.$$

Note that $\mathbb{T}_\infty$ is equal to the pointed group $F = \mathbb{R}_{\geqslant 0}$ endowed with the degenerate tract structure.

The tracts $\mathbb{T}_0, \mathbb{T}_1$, and $\mathbb{T}_\infty$ play a major role in the forthcoming papers [4] and [5], where we consider a continuous family of tracts $\mathbb{T}_q$ (for $q \in [0, \infty]$) that interpolates between them (a process which Viro calls *Litvinov–Maslov dequantization* in [45]). Concretely, for $q > 0$ the generalized triangular hyperfield $\mathbb{T}_q$ is defined as

$$\mathbb{T}_q \;=\; \mathbb{R}_{\geqslant 0} /\!\!/ \langle \textstyle\sum a_i \mid a_1^{1/q}, \ldots, a_n^{1/q} \text{ are the side lengths of a (possibly degenerate) polygon} \rangle.$$

We also define the *discrete tropical hyperfield* $\mathbb{T}_0^{\mathbb{Z}}$ as the subtract of $\mathbb{T}_0$ corresponding to the pointed submonoid $e^{\mathbb{Z}} \cup \{0\}$ of $\mathbb{T}_0 = \mathbb{R}_{\geqslant 0}$.

4. Representations of polymatroids

In this section, we extend the notions of strong and weak matroid representations over tracts (cf. [3, 9]) to polymatroids, using a novel characterization of polymatroids in terms of Plücker relations.

4.1. Plücker relations for polymatroids. We consider the characteristic function of a subset $J \subseteq \Delta_n^r$, defined by $\chi_J(\alpha) = 1$ if $\alpha \in J$ and $\chi_J(\alpha) = 0$ if not, as a function

$$\chi_J \colon \Delta_n^r \longrightarrow \mathbb{K}$$

into the Krasner hyperfield $\mathbb{K} = \mathbb{F}_1^{\pm} /\!\!/ \langle 1+1, 1+1+1 \rangle$, which has elements 0 and $1 = -1$.

Theorem 4.1. *A subset $J \subseteq \Delta_n^r$ is M-convex if and only if the characteristic function $\chi_J \colon \Delta_n^r \to \mathbb{K}$ of J satisfies the* Plücker relations

$$\sum_{k=0}^{s} \chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s}) \cdot \chi_J(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \in N_{\mathbb{K}}$$

for all $s \in \{2, \ldots, r\}$, all $\alpha \in \Delta_n^{r-s}$ with $\delta_J^- = \inf J \leqslant \alpha$, and all $i_0, \ldots, i_s, j_2, \ldots, j_s \in [n]$ such that $\alpha + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s} \leqslant \delta_J^+ = \sup J$.

Proof. Assume that J is M-convex. Since $N_{\mathbb{K}} = \mathbb{N} - \{1\}$, it suffices to show, for $\alpha \in \Delta_n^{r-s}$ with $s \in \{2, \ldots, r\}$ and $i_0, \ldots, i_s, j_2, \ldots, j_s \in [n]$, that either all terms in

$$\sum_{k=0}^{s} \chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s}) \cdot \chi_J(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$$

are zero or that at least two terms are nonzero. Assume that the sum contains a nonzero term $\chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s})\chi_J(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$. Then we define $i = i_k$, $\beta = \alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s}$, and $\gamma = \alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}$.

If $\beta_i \geqslant \gamma_i$, then there exists $k' \neq k$ with $i_{k'} = i_k$. Thus we find a second nonzero term $\chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_{k'}}} \cdots + \varepsilon_{i_s})\chi_J(\alpha + \varepsilon_{i_{k'}} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$ in the Plücker relation.

If $\beta_i < \gamma_i$, then, by the exchange axiom for M-convex sets, there exists $j \in [n]$ such that $\gamma_j < \beta_j$ and such that both $\beta - \varepsilon_j + \varepsilon_i$ and $\gamma - \varepsilon_i + \varepsilon_j$ are in J. Since $\gamma_j < \beta_j$, we have $j = i_{k'}$ for some $k' \neq k$. Thus $\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_{k'}}} \cdots + \varepsilon_{i_s} = \beta - \varepsilon_j + \varepsilon_i$ and $\alpha + \varepsilon_{i_{k'}} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s} = \gamma - \varepsilon_i + \varepsilon_j$, which yields $\chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_{k'}}} \cdots + \varepsilon_{i_s})\chi_J(\alpha + \varepsilon_{i_{k'}} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$ as a second nonzero term in the Plücker relation. This shows that the characteristic function of an M-convex set satisfies the Plücker relations.

Conversely, assume that $J \subseteq \Delta_n^r$ is a subset whose characteristic function χ_J satisfies the Plücker relations. Consider $\beta, \gamma \in J$ with $\beta_i < \gamma_i$ for some $i \in [n]$. We need to show that there exists $j \in [n]$ such that $\gamma_j < \beta_j$ and such that both $\beta - \varepsilon_j + \varepsilon_i$ and $\gamma - \varepsilon_i + \varepsilon_j$ are in J. Let $\alpha = \inf\{\beta, \gamma\} \geqslant \delta_J^-$. We have $\alpha \in \Delta_n^{r-s}$ for some $1 \leqslant s \leqslant r$. If $s = 1$, then $\gamma = \beta + \varepsilon_i - \varepsilon_j$ for some $j \in [n]$ and the exchange property we're trying to show is trivially satisfied. Thus, we can assume that $s \in \{2, \ldots, r\}$. There are $i_1, \ldots, i_s, j_1, \ldots, j_s \in [n]$ (unique up to permutation) such that

$$\beta = \alpha + \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}, \quad \gamma = \alpha + \varepsilon_{j_1} + \cdots + \varepsilon_{j_s} \quad \text{and} \quad \{i_1, \ldots, i_s\} \cap \{j_1, \ldots, j_s\} = \varnothing,$$

and therefore $\alpha + \varepsilon_{i_1} + \cdots + \varepsilon_{i_s} + \varepsilon_{j_1} + \cdots + \varepsilon_{j_s} \leqslant \sup\{\beta, \gamma\} \leqslant \delta_J^+$. Assume without loss of generality that $i = j_1$ and define $i_0 = i$. Then the Plücker relation

$$\sum_{k=0}^{s} \chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s}) \cdot \chi_J(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \in N_{\mathbb{K}}$$

contains the nonzero term $\chi_J(\alpha + \varepsilon_{i_1} + \cdots + \varepsilon_{i_s})\chi_J(\alpha + \varepsilon_{i_0} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$, and therefore contains a second nonzero term $\chi_J(\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s})\chi_J(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$ for some $k \neq 0$. Thus J contains both

$$\alpha + \varepsilon_{i_0} + \cdots \widehat{\varepsilon_{i_k}} \cdots + \varepsilon_{i_s} = \beta - \varepsilon_j + \varepsilon_i \quad \text{and} \quad \alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s} = \gamma - \varepsilon_i + \varepsilon_j$$

for $j = i_k$. Since $i_k \notin \{j_1, \ldots, j_s\}$, we have $\gamma_j < \beta_j$. This shows that J is M-convex. □

Remark 4.2. Not all Plücker relations are needed in the characterization of M-convex sets in Theorem 4.1. It is visible from the proof that we only need those Plücker relations for which $\{i_1, \ldots, i_s\} \cap \{i_0, j_2, \ldots, j_s\} = \varnothing$ in order to imply M-convexity.

4.2. Polymatroid representations over tracts. The definition of polymatroid representations over arbitrary tracts, in which -1 might differ from 1, requires a suitable sign for the Plücker relations, which in turn depends on the ordering of the coordinates. This is best formulated in terms of r-tuples $\boldsymbol{\alpha} \in [n]^r$ instead of vectors $\alpha \in \Delta_n^r$. We can compare both viewpoints in terms of the surjection

$$(1) \qquad \begin{array}{cccc} \Sigma\colon & [n]^r & \longrightarrow & \Delta_n^r \\ & \boldsymbol{\alpha} & \longmapsto & \varepsilon_{\alpha_1} + \cdots + \varepsilon_{\alpha_r}. \end{array}$$

Furthermore, we use the shorthand notation $\boldsymbol{\alpha} i_1 \ldots i_s$ for $(\alpha_1, \ldots, \alpha_r, i_1, \ldots, i_s)$ where $\boldsymbol{\alpha} \in [n]^r$ and $i_1, \ldots, i_s \in [n]$.

Recall from Definition 2.9 that the *reduction* of an M-convex set J is the M-convex set $\bar{J} = J - \delta_J^-$, where $\delta_J^- = \inf J$.

Definition 4.3. Let $J \subseteq \Delta_n^r$ be an M-convex set. The *effective rank of* J is the rank $\bar{r} = r - |\delta_J^-|$ of $\bar{J}$. The *width of* J is $\omega_J = \delta_J^+ - \delta_J^- = \delta_{\bar{J}}^+$.

Let F be a tract. A *strong F-representation of* J is a function $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ that satisfies the following axioms:

(SR1) $\boldsymbol{\rho}(\boldsymbol{\alpha}) \in F^\times$ if and only if $\Sigma\boldsymbol{\alpha} \in \bar{J}$;
(SR2) $\boldsymbol{\rho}(i_{\sigma(1)}, \ldots, i_{\sigma(\bar{r})}) = \operatorname{sign}(\sigma) \cdot \boldsymbol{\rho}(i_1, \cdots, i_{\bar{r}})$ for every $\sigma \in S_{\bar{r}}$;
(SR3) $\boldsymbol{\rho}$ satisfies the *Plücker relation*

$$\sum_{k=0}^{s} (-1)^k \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} i_0 \ldots \widehat{i_k} \ldots i_s) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} i_k j_2 \ldots j_s) \quad \in \quad N_F$$

for all $2 \leqslant s \leqslant \bar{r}$, $\boldsymbol{\alpha} \in [n]^{\bar{r}-s}$ and $i_0, \ldots, i_s, j_2, \ldots, j_s \in [n]$ such that

$$\Sigma\boldsymbol{\alpha} i_0 \ldots i_s j_2 \ldots j_s \leqslant \omega_J.$$

A *weak F-representation of J* is a function $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ that satisfies the following axioms:

(WR1) $\boldsymbol{\rho}(\boldsymbol{\alpha}) \in F^\times$ if and only if $\Sigma\boldsymbol{\alpha} \in \bar{J}$;
(WR2) $\boldsymbol{\rho}(i_{\sigma(1)}, \ldots, i_{\sigma(\bar{r})}) = \operatorname{sign}(\sigma) \cdot \boldsymbol{\rho}(i_1, \cdots, i_{\bar{r}})$ for every $\sigma \in S_{\bar{r}}$;
(WR3) $\boldsymbol{\rho}$ satisfies the 3-*term Plücker relations*

$$\boldsymbol{\rho}(\boldsymbol{\alpha} jk) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} il) \;-\; \boldsymbol{\rho}(\boldsymbol{\alpha} ik) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} jl) \;+\; \boldsymbol{\rho}(\boldsymbol{\alpha} ij) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} kl) \quad \in \quad N_F$$

for all $\boldsymbol{\alpha} \in [n]^{\bar{r}-2}$, and $i, j, k, l \in [n]$ such that $\Sigma\boldsymbol{\alpha} ijkl \leqslant \omega_J$.

The tract F is called *excellent* if every weak F-representation of every M-convex set J is strong.

Remark 4.4. If J is a matroid, then this definition agrees with the notion of strong (resp. weak) F-representations of matroids in [10], which are also called a strong (resp. weak) Grassmann-Plücker functions in [3, 9].

Remark 4.5. Consider a function $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ that satisfies (SR2) and (SR3) with respect to the set

$$J := \{\Sigma\alpha \mid \boldsymbol{\rho}(\alpha) \neq 0\}.$$

Then J is M-convex and $\boldsymbol{\rho}$ is a strong F-representation of J. Indeed, the composition of $\boldsymbol{\rho}$ with the unique tract morphism $F \to \mathbb{K}$ satisfies the assumptions of Theorem 4.1 and has the same support as $\boldsymbol{\rho}$. This shows that J is M-convex, and then $\boldsymbol{\rho}$ is an F-representation of J by definition.

This extends a known fact for matroid representations to polymatroids. It fails for weak F-representations (in fact, already in the matroid case; see [8, Ex. 6.25]): for every tract F, there is a function $\boldsymbol{\rho}\colon [6]^3 \to F$ that satisfies (WR2) and (WR3), but that is not a weak F-representation of any M-convex set J.

4.2.1. *The unique $\mathbb{K}$-representation of a polymatroid.* Let J be an M-convex set and let $\chi_J\colon \Delta_n^r \to \mathbb{K}$ be its characteristic function. Then the map $\boldsymbol{\rho}_J\colon [n]^{\bar{r}} \to \mathbb{K}$, defined by $\boldsymbol{\rho}_J(i_1, \ldots, i_r) = \chi_J(\delta_J^- + \varepsilon_{i_1} + \ldots + \varepsilon_{i_r})$, is a strong $\mathbb{K}$-representation of J. More precisely, $\boldsymbol{\rho}$ is the unique strong (resp. weak) $\mathbb{K}$-representation of J, since it is entirely determined by axiom (SR1) (resp. by (WR1)). This shows, in particular, that $\mathbb{K}$ is excellent.

4.3. The idempotency principle for proper polymatroids. An M-convex set J is *a translate of a matroid* if $J = J' + \tau$ for a matroid J' and $\tau \in \mathbb{Z}^n$. Otherwise, we call J a *proper polymatroid* or *proper M-convex set*.

Let $\omega_J = \delta_J^+ - \delta_J^-$ be the width of J, $\bar{J} = J - \delta_J^-$ its reduction, $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^n$, and

$$U_{2,3}^+ \;=\; \{(2,0,0),\ (1,1,0),\ (1,0,1),\ (0,1,1)\} \;\subseteq\; \Delta_3^2.$$

Lemma 4.6. *The following are equivalent:*

(1) J is a translate of a matroid;
(2) $\bar{J}$ is a matroid;
(3) $\omega_J \leqslant \mathbf{1}$;
(4) J has no embedded minor of type Δ_2^2 or $U_{2,3}^+$.

Proof. We establish the circle of implications (3)⇒(2)⇒(1)⇒(4)⇒(3) in the following. Assume (3), i.e., $\omega_J \leqslant \mathbf{1}$. Since $\delta_{\bar{J}}^- = 0$ and ω_J is invariant under translates of J, we have $\delta_{\bar{J}}^+ = \omega_{\bar{J}} = \omega_J \leqslant \mathbf{1}$, which shows that $\bar{J}$ is a matroid and establishes (3)⇒(2).

Assume (2), i.e., $\bar{J}$ is a matroid. Then $J = \bar{J} + \delta_J^-$ is a translate of a matroid. This establishes (2)⇒(1).

Assume (1), i.e., $J = M + \tau$ is the translate of a matroid M. The matroid M does not have embedded minors of type Δ_2^2 or $U_{2,3}^+$, which are proper polymatroids. Since J and M have the same embedded minors, this establishes (1)⇒(4).

Assume (4) and let $i \in [n]$. Choose $\alpha, \beta \in J$ with $\alpha_i = \delta_{J,i}^+$ and $\beta_i = \delta_{J,i}^-$. If $\omega_{J,i} = \alpha_i - \beta_i \geqslant 2$, then applying exchange axiom repeatedly to α and β yields $k, l \in [n] \backslash \{i\}$ (not necessarily distinct) such that

$$\alpha - \varepsilon_i + \varepsilon_k, \qquad \alpha - \varepsilon_i + \varepsilon_l, \qquad \alpha - 2\varepsilon_i + \varepsilon_k + \varepsilon_l$$

are in J. Thus

$$J \setminus \delta_J^+ - \alpha + \varepsilon_k + \varepsilon_l \,/\, \alpha - 2\varepsilon_i \;=\; \{2\varepsilon_i,\ \ \varepsilon_i + \varepsilon_k,\ \ \varepsilon_i + \varepsilon_l,\ \ \varepsilon_k + \varepsilon_l\},$$

which is combinatorially equivalent to $\Delta_{2,2}$ or $U_{2,3}^+$, depending on whether $k = l$ or not. This contradicts our assumption (4), which shows that $\omega_{J,i} \leqslant 1$ and establishes (4)⇒(3). □

A tract F is *idempotent*[4] if $1 = -1$ (i.e., $1 + 1 \in N_F$) and $1 + 1 + 1 \in N_F$. Equivalently, a tract F is idempotent if and only if there exists a (necessarily unique) morphism $\mathbb{K} \to F$, i.e., if and only if F is an algebra over the Krasner hyperfield.

A tract F is *near-idempotent* if $1 = -1$ and if there is an $x \in F^\times$ with $1 + 1 + x \in N_F$. Every idempotent tract is near-idempotent. A typical example of a near-idempotent tract that is not idempotent is $\mathbb{F}_2 \otimes \mathbb{D} = \mathbb{F}_2(x)/\!\!/\langle 1 + 1 + x\rangle$.

[4]This terminology stems from the fact that an idempotent semifield (commutative, with 0 and 1) is naturally a tract whose nullset is generated by all relations of the form $b + \sum a_i$ for which $\sum a_i = b$ holds in F. This tract is idempotent in the sense of this text. More concisely, this construction defines a fully faithful functor from idempotent semifields to idempotent tracts.

Proposition 4.7 (Idempotency principle)**.** *Let F be a tract, $J \subseteq \Delta_n^r$ a proper M-convex set of effective rank $\bar{r}$, and $\boldsymbol{\rho}: [n]^{\bar{r}} \to F$ a weak F-representation of J. Then F is near-idempotent. If $\omega_{J,i} \geqslant 3$ for some $i \in [n]$, then F is idempotent.*

Proof. If J is not the translate of a matroid, then it contains an element of the form $\delta_J^- + \alpha + \varepsilon_i + \varepsilon_i$ for some $\alpha \in \Delta_n^{\bar{r}-2}$ and $i \in [n]$. Choose $\boldsymbol{\alpha} \in [n]^{\bar{r}-2}$ with $\Sigma\boldsymbol{\alpha} = \alpha$. Then by axiom (WR1), $\boldsymbol{\rho}(\boldsymbol{\alpha}ii) \in F^\times$ and by axiom (WR2), $\boldsymbol{\rho}(\boldsymbol{\alpha}ii) = -\boldsymbol{\rho}(\boldsymbol{\alpha}ii)$. Thus $1 = -1$ in F.

Let $\beta = \delta_J^- + \alpha + 2\varepsilon_i$ and $\gamma \in J$ with $\gamma_i = \delta_{J,i}^-$. Since $\gamma_i \leqslant \beta_i - 2$, we can apply the exchange axiom twice to find $k, l \in [n] - \{i\}$ such that all of

$$\beta - \varepsilon_i + \varepsilon_k, \qquad \beta - \varepsilon_i + \varepsilon_l, \qquad \beta - 2\varepsilon_i + \varepsilon_k + \varepsilon_l$$

are in J. Thus, in particular, $\alpha + 2\varepsilon_i + \varepsilon_k + \varepsilon_l \leqslant \omega_J$. By axiom (WR3), we have the Plücker relation

$$\boldsymbol{\rho}(\boldsymbol{\alpha}ii) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}kl) \ + \ \boldsymbol{\rho}(\boldsymbol{\alpha}ik) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}il) \ + \ \boldsymbol{\rho}(\boldsymbol{\alpha}il) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}ik) \quad \in \quad N_F.$$

Dividing all terms by $\boldsymbol{\rho}(\boldsymbol{\alpha}ik) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}il)$ yields $1 + 1 + x \in N_F$ for $x = \frac{\boldsymbol{\rho}(\boldsymbol{\alpha}ii) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}kl)}{\boldsymbol{\rho}(\boldsymbol{\alpha}ik) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}il)} \in F^\times$, which shows that F is near-idempotent.

If $\omega_{J,i} \geqslant 3$, then we can replace α as above by $\alpha - \varepsilon_i + \varepsilon_l$, which yields $\alpha + 3\varepsilon_i + \varepsilon_k \leqslant \omega_J$. Thus by axiom (WR3) we find the Plücker relation

$$\boldsymbol{\rho}(\boldsymbol{\alpha}ii) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}ik) \ + \ \boldsymbol{\rho}(\boldsymbol{\alpha}ii) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}ik) \ + \ \boldsymbol{\rho}(\boldsymbol{\alpha}ii) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}ik) \quad \in \quad N_F.$$

Dividing all terms by $\boldsymbol{\rho}(\boldsymbol{\alpha}ii) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}ik)$ yields $1+1+1 \in N_F$, which shows that F is idempotent. □

A tract F is called *perfect* if for every strong F-representation $\boldsymbol{\rho}$ of a matroid, the vectors of $\boldsymbol{\rho}$ are orthogonal to the covectors of $\boldsymbol{\rho}$ (for details, see [3, section 3.13]). The most important property of a perfect tract (from our perspective) is that every weak matroid representation over a perfect tract is strong (by [3, Thm. 3.46]). At the time of writing, we do not know whether this implies that every weak *poly*matroid representation over a perfect tract is strong. But the following result establishes this implication for a large class of perfect tracts (also see Section 4.2.1, Corollary 4.12, and Corollary 5.5).

Corollary 4.8. *Every perfect tract F that is not near-idempotent is excellent.*

Proof. If F is perfect but not near-idempotent, then every weak polymatroid representation is a weak matroid representation (Proposition 4.7), and therefore a strong matroid representation by [3, Thm. 3.46]. □

Examples of tracts to which Corollary 4.8 applies are fields, partial fields, and the sign hyperfield; all of these tracts are excellent.

4.4. Simplified description of near-idempotent polymatroid representations. Due to Proposition 4.7, only near-idempotent tracts F possess proper polymatroid representations, in which case they can be described in an equivalent but simplified way. Namely, if F is near-idempotent, we can identify a strong (or weak) F-representation $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ of an M-convex set J (with effective rank $\bar{r}$) with the function $\rho\colon \Delta_n^r \to F$ given by

$$\rho(\delta_J^- + \varepsilon_{i_1} + \ldots + \varepsilon_{i_{\bar{r}}}) \;=\; \boldsymbol{\rho}(i_1, \ldots, i_{\bar{r}}),$$

which does not depend on the ordering of $i_1, \ldots, i_{\bar{r}} \in [n]$ due to (SR2) (resp. (WR2)) and the fact that $1 = -1$ in F. Property (SR1) (resp. (WR1)) turns into the condition that the support of ρ is J. More concisely, this formula identifies functions $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ satisfying (SR1) and (SR2) (resp. (WR1) and (WR2)) with functions $\rho\colon \Delta_n^r \to F$ whose support is J.

The Plücker relations (SR3) for $\boldsymbol{\rho}$ turn into the relations

$$\sum_{k=0}^{s} \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \quad \in \quad N_F$$

for all $2 \leqslant s \leqslant r$, $\alpha \in \Delta_n^{r-s}$ with $\delta_J^- \leqslant \alpha$ and all $i_0, \ldots, i_s, j_2, \ldots, j_s \in [n]$ with $\alpha + \varepsilon_{i_0} + \ldots + \varepsilon_{i_s} + \varepsilon_{j_2} + \ldots + \varepsilon_{j_s} \leqslant \delta_J^+$. The 3-term Plücker relations (WR3) turn into the relations

$$\rho(\alpha + \varepsilon_j + \varepsilon_k) \cdot \rho(\alpha + \varepsilon_i + \varepsilon_l) \;+\; \rho(\alpha + \varepsilon_i + \varepsilon_k) \cdot \rho(\alpha + \varepsilon_j + \varepsilon_l) \\ +\; \rho(\alpha + \varepsilon_i + \varepsilon_j) \cdot \rho(\alpha + \varepsilon_k + \varepsilon_l) \quad \in \quad N_F$$

for all $\alpha \in \Delta_n^{r-2}$ with $\delta_J^- \leqslant \alpha$ and all $i, j, k, l \in [n]$ with $\alpha + \varepsilon_i + \varepsilon_j + \varepsilon_k + \varepsilon_l \leqslant \delta_J^+$.

We can extend this alternative perspective on polymatroid representations to arbitrary tracts in the following way.

Lemma 4.9. *Let F be a tract, $J \subseteq \Delta_n^r$ an M-convex set of effective rank $\bar{r} = r - |\delta_J^-|$, and $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ a function that satisfies (SR1) and (SR2). Let $\rho\colon \Delta_n^r \to F$ be the function with support J given by $\rho(\delta_J^- + \varepsilon_{i_1} + \ldots + \varepsilon_{i_{\bar{r}}}) = \boldsymbol{\rho}(i_1, \ldots, i_{\bar{r}})$ whenever $i_1 \leqslant \ldots \leqslant i_{\bar{r}}$. Then $\boldsymbol{\rho}$ is a strong F-representation of J if and only if ρ satisfies the Plücker relations*

$$\sum_{k=0}^{s} (-1)^{k+\sigma(k)} \cdot \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \in N_F$$

for all $2 \leqslant s \leqslant r$, $\alpha \in \Delta_n^{r-s}$, $1 \leqslant i_0 \leqslant \ldots \leqslant i_s \leqslant n$, and $1 \leqslant j_2 \leqslant \ldots \leqslant j_s \leqslant n$ such that $\delta_J^- \leqslant \alpha$ and $\alpha + \varepsilon_{i_0} + \ldots + \varepsilon_{i_s} + \varepsilon_{j_2} + \ldots + \varepsilon_{j_s} \leqslant \delta_J^+$, where $\sigma(k)$ is the number of $k \in \{2, \ldots, s\}$ with $i_k < j_s$.

The function $\boldsymbol{\rho}$ is a weak F-representation of J if and only if ρ satisfies the 3-term Plücker relations

$$\rho(\alpha + \varepsilon_j + \varepsilon_k) \cdot \rho(\alpha + \varepsilon_i + \varepsilon_l) \;-\; \rho(\alpha + \varepsilon_i + \varepsilon_k) \cdot \rho(\alpha + \varepsilon_j + \varepsilon_l)$$

$$+\ \rho(\alpha+\varepsilon_i+\varepsilon_j)\cdot\rho(\alpha+\varepsilon_k+\varepsilon_l) \quad \in \quad N_F$$

for all $\alpha \in \Delta_n^{r-2}$ *and* $1 \leqslant i \leqslant j \leqslant k \leqslant l \leqslant n$ *such that* $\delta_J^- \leqslant \alpha$ *and* $\alpha+\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l \leqslant \delta_J^+$.

Proof. If F is near-idempotent, then $-1 = 1$ and the requirements on the ordering of the i_k and j_ℓ become irrelevant. Thus the claim reduces to the discussion in the beginning of this section.

If F is not near-idempotent, then J is a matroid by the idempotency principle (Proposition 4.7) and thus $\delta_J^+ - \delta_J^- \leqslant \mathbf{1}$ by Lemma 4.6. Since the defining relations of $\boldsymbol{\rho}$ and ρ are translation invariant, we can assume that $\delta_J^- = 0$ and thus $\delta_J^+ \leqslant \mathbf{1}$. This means that $\boldsymbol{\rho}(i'_1,\dots,i'_r) = 0$ if the i'_k are not pairwise distinct. It also means that the Plücker relations for s, α, $1 \leqslant i_0 \leqslant \dots \leqslant i_s \leqslant n$, and $1 \leqslant j_2 \leqslant \dots \leqslant j_s \leqslant n$ are trivial (i.e., all terms are 0 or it is of the form $a - a \in N_F$) unless $1 \leqslant i_0 < \dots < i_s \leqslant n$ and $1 \leqslant j_2 < \dots < j_s \leqslant n$ (resp. $i < j < k < l$ in the case of 3-term Plücker relations). In the case of (SR3), we can further assume that $s = r$ and $\alpha = 0$, since if $i_1 = j_1, \dots, i_{r-s} = j_{r-s}$, the corresponding Plücker relation is equal to that for $\alpha = \varepsilon_{i_1} + \dots + \varepsilon_{i_{r-s}}$ (a fact which is particular to matroids and does not generalize to polymatroids).

Thanks to these simplifications, the Plücker relations assume their usual shape (for instance, cf. [8, Def. 3.1][5]), which reduces the lemma to the equivalence between (strong) matroid representations as alternating functions with domain $[n]^r$ and functions with domain $\binom{[n]}{r}$. The case of the 3-term Plücker relations (WR3) can be established in a similar vein. □

4.5. M-convex functions as representations over the tropical hyperfield. In this subsection, we will show that in the case of the tropical hyperfield $\mathbb{T}_0$, a $\mathbb{T}_0$-representation is essentially the same thing as an *M-convex function* in the sense of Murota. This is a function $f\colon \mathbb{Z}^n \to \mathbb{R}\cup\{\infty\}$ with nonempty support $J = \{\alpha \in \mathbb{Z}^n \mid f(\alpha) \neq \infty\}$ that satisfies the following *exchange axiom*: for $\alpha, \beta \in J$ and $k \in [n]$ with $\alpha_k > \beta_k$, there is an $l \in [n]$ with $\alpha_l < \beta_l$ and

$$f(\alpha) \ + \ f(\beta) \ \geqslant \ f(\alpha-\varepsilon_k+\varepsilon_l) \ + f(\beta+\varepsilon_k-\varepsilon_l). \tag{2}$$

It follows from this exchange axiom that J is an M-convex set ([34, Prop. 6.1]). Note that an M-convex function whose support J is a matroid is the same thing as a valuated matroid.

The following "local" characterization of M-convex functions (established in [34, Thm. 6.4]) facilitates our proofs. A function $f\colon \mathbb{Z}^n \to \mathbb{R}\cup\{\infty\}$ with M-convex support J of rank r is an M-convex function if and only if it satisfies the *local exchange axiom*: for $\alpha \in \Delta_n^{r-2}$ and all $i, j, k, l \in [n]$ such that $\{i,k\} \cap \{j,l\} = \emptyset$,

(3) $f(\alpha+\varepsilon_i+\varepsilon_k) + f(\alpha+\varepsilon_j+\varepsilon_l)$

[5]Note that the factor $(-1)^{\sigma(k)}$ stems from the permutation of $(i_k, j_2, \dots, j_s)$ that brings the coefficients into increasing order. This factor is missing in the Plücker relations in [8]—a mistake that requires correction.

$$\geqslant \min\{f(\alpha+\varepsilon_i+\varepsilon_j)+f(\alpha+\varepsilon_k+\varepsilon_l),\ f(\alpha+\varepsilon_i+\varepsilon_l)+f(\alpha+\varepsilon_j+\varepsilon_k)\}.$$

In the following, we identify $\mathbb{T}_0$-representations of J with functions $\rho\colon \Delta_n^r \to \mathbb{T}_0$ that have support J and satisfy the appropriate version of the Plücker relations; cf. Section 4.4 for details.

Proposition 4.10. *Let J be an M-convex set and $\rho\colon \Delta_n^r \to \mathbb{T}_0 = \mathbb{R}_{\geqslant 0}$ a function with support J. Then ρ is a weak $\mathbb{T}_0$-representation of J if and only if $f = -\log(\rho)$ is M-convex.*

Proof. We need to show that the 3-term Plücker relations from axiom (WR3) are equivalent to (3). The exchange relation (3) for f is equivalent to

$$\begin{aligned}(4)\quad &\rho(\alpha+\varepsilon_i+\varepsilon_k)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_l)\\ &\qquad\leqslant \max\{\rho(\alpha+\varepsilon_i+\varepsilon_j)\cdot\rho(\alpha+\varepsilon_k+\varepsilon_l),\ \rho(\alpha+\varepsilon_i+\varepsilon_l)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_k)\},\end{aligned}$$

which allows us to compare this condition directly with the 3-term Plücker relations

$$\begin{aligned}(5)\quad &\rho(\alpha+\varepsilon_i+\varepsilon_j)\cdot\rho(\alpha+\varepsilon_k+\varepsilon_l)\ +\ \rho(\alpha+\varepsilon_i+\varepsilon_k)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_l)\\ &\qquad\qquad +\ \rho(\alpha+\varepsilon_i+\varepsilon_l)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_k)\ \in\ N_{\mathbb{T}_0}.\end{aligned}$$

Note that the 3 terms in (4) and (5) agree. A permutation of i, j, k and l permutes these 3 terms. Such a permutation leaves (5) invariant, but changes (4).

If $\{i,k\}\cap\{j,l\}\neq\varnothing$, then $\rho(\alpha+\varepsilon_i+\varepsilon_k)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_l)$ is equal to one of the terms $\rho(\alpha+\varepsilon_i+\varepsilon_j)\cdot\rho(\alpha+\varepsilon_k+\varepsilon_l)$ and $\rho(\alpha+\varepsilon_i+\varepsilon_l)\cdot\rho(\alpha+\varepsilon_j+\varepsilon_k)$, and thus (4) is automatically satisfied. So we can remove the condition $\{i,k\}\cap\{j,l\}=\varnothing$ from the definition of an M-convex function.

Thus (4) holds for all permutations of i, j, k, and l (which is the same equation) if and only if the maximum among the 3 terms appears twice, which is equivalent to (5) for all permutations of i, j, k, and l. This verifies our claim. □

We use Proposition 4.10 to deduce the corresponding result for strong $\mathbb{T}_0$-representations.

Proposition 4.11. *Let J be an M-convex set and $\rho\colon \Delta_n^r \to \mathbb{T}_0 = \mathbb{R}_{\geqslant 0}$ a function with support J. Then ρ is a strong $\mathbb{T}_0$-representation of J if and only if $f = -\log(\rho)$ is M-convex.*

Proof. Assume that ρ is a strong $\mathbb{T}_0$-representation of J. Then it is, in particular, a weak $\mathbb{T}_0$-representation and, by Proposition 4.10, f is M-convex.

Conversely, assume that f is M-convex. We show that the exchange axiom (2) for M-convex functions implies the Plücker relations (SR3) for all $\alpha\in\Delta_n^{\bar{r}-s}$ and $i_0,\dots,i_s,j_2,\dots,j_s\in[n]$, i.e., the maximum appears at least twice in the formal sum

$$\sum_{k=0}^{s}\rho(\alpha+\varepsilon_{i_0}+\dots\widehat{\varepsilon_{i_k}}\dots+\varepsilon_{i_s})\cdot\rho(\alpha+\varepsilon_{i_k}+\varepsilon_{j_2}\dots+\varepsilon_{j_s}).$$

Fix a k such that $\rho(\beta) \cdot \rho(\gamma)$ assumes the maximum among these terms where

$$\beta = \alpha + \varepsilon_{i_0} + \ldots \widehat{\varepsilon_{i_k}} \ldots + \varepsilon_{i_s} \qquad \text{and} \qquad \gamma = \alpha + \varepsilon_{i_k} + \varepsilon_{j_2} \ldots + \varepsilon_{j_s}.$$

If $\beta_{i_k} \geqslant \gamma_{i_k} \geqslant 1$, then there exists an $l \neq k$ with $i_l = i_k$, and therefore also

$$\rho(\alpha + \varepsilon_{i_0} + \ldots \widehat{\varepsilon_{i_l}} \ldots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_l} + \varepsilon_{j_2} \ldots + \varepsilon_{j_s}) = \rho(\alpha + \varepsilon_{i_0} + \ldots \widehat{\varepsilon_{i_k}} \ldots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} \ldots + \varepsilon_{j_s})$$

assumes the maximum. If $\beta_{i_k} < \gamma_{i_k}$, then the exchange axiom (2) implies that there is an $l \neq k$ such that

$$f(\beta) + f(\gamma) \geqslant f(\beta - \varepsilon_k + \varepsilon_l) + f(\gamma + \varepsilon_k - \varepsilon_l)$$

or, equivalently,

$$\rho(\alpha + \varepsilon_{i_0} + \ldots \widehat{\varepsilon_{i_l}} \ldots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_l} + \varepsilon_{j_2} \ldots + \varepsilon_{j_s}) \geqslant \rho(\alpha + \varepsilon_{i_0} + \ldots \widehat{\varepsilon_{i_k}} \ldots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} \ldots + \varepsilon_{j_s}).$$

By the maximality of the latter term, this inequality must be an equality, which exhibits also in this case a second maximal term in the Plücker relation under consideration. This shows that ρ satisfies (SR3), which concludes the proof. □

Corollary 4.12. *The tropical hyperfield $\mathbb{T}_0$ is excellent.*

Proof. As an immediate consequence of Proposition 4.10 and Proposition 4.11, every weak $\mathbb{T}_0$-representation of J is strong. □

4.6. Hives. As mentioned in the introduction, hives are combinatorial gadgets that were introduced by Knutson and Tao in [33], and they are naturally in bijection with $\mathbb{T}_0$-representations of Δ_3^r. We explain the concept of a hive here and give an illustrative example, mainly following [19].

The r^{th} *hive triangle* is the triangular array depicted in Figure 2, consisting of $r+1$ "hive vertices" on each side and r^2 "small triangles".

A *rhombus* in the hive triangle is the union of two small triangles which share a common edge. There are three combinatorial types or orientations of rhombi, as depicted in Figure 3.

Each rhombus has two acute angles and two obtuse angles. Let H be the set of hive vertices and $\mathbb{R}^H$ the set of labelings of H by real numbers. Each rhombus gives rise to an inequality on $\mathbb{R}^H$ saying that the sum of the labels at the obtuse vertices must be greater than or equal to the sum of the labels at the acute vertices. A *hive* is a labeling in $\mathbb{R}^H$ that satisfies all rhombus inequalities. Of particular interest, in terms of the connection to the representation theory of GL_r, are the *integral hives*, which are hives for which all labels are integers.

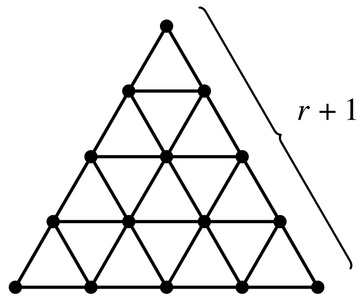

Figure 2. The r^{th} hive triangle.

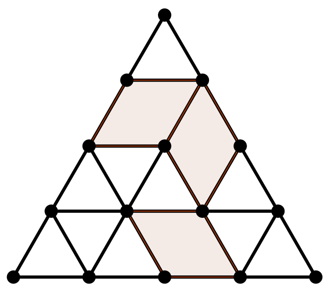

Figure 3. The three combinatorial types of rhombi.

Example 4.13. The rhombus inequalities say that the labeling given in Figure 4 is a hive if and only if $4 \leqslant x \leqslant 5$.

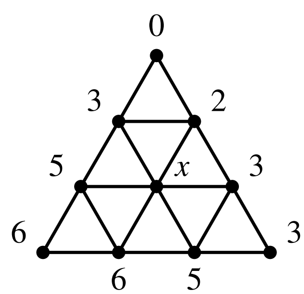

Figure 4. A labeling of the 3^{rd} hive triangle.

We coordinatize the hive triangle by letting $(r, 0, 0)$ denote the lower-left corner, $(0, r, 0)$ denote the lower-right corner, and $(0, 0, r)$ denote the top corner; see Figure 5. This identifies H with Δ_3^r.

With this coordinatization, the following observation of Brändén [15, Section 4] becomes a direct translation of Proposition 4.11 into the language of hives. Recall from Section 3.7 the

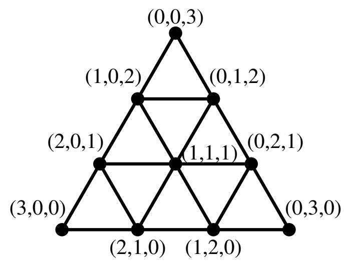

Figure 5. Coordinates on the 3rd hive triangle.

definition of the discrete tropical hyperfield as the tract $\mathbb{T}_0^{\mathbb{Z}} = \{0\} \cup \{e^i \mid i \in \mathbb{Z}\}$ with null set

$$N_{\mathbb{T}_0^{\mathbb{Z}}} = \{a_1 + \cdots + a_n \,|\, a_1, \ldots, a_n \text{ assumes its maximum at least twice}\}.$$

Proposition 4.14. *A function $\rho : H \to \mathbb{R}_{>0}$ is a $\mathbb{T}_0$-representation of Δ_3^r if and only if $\log \rho : H \to \mathbb{R}$ is a hive. A function $\rho : H \to e^{\mathbb{Z}}$ is a $\mathbb{T}_0^{\mathbb{Z}}$-representation of Δ_3^r if and only if $\log \rho : H \to \mathbb{R}$ is an integral hive.*

This allows us to translate the key results of [19] into the language of polymatroid representations.

Given an integer partition λ with at most r parts, we let $\lambda_1, \ldots, \lambda_k$ denote the parts in weakly decreasing order, i.e., $\lambda_1, \ldots, \lambda_k$ are integers with $k \leqslant r$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$, and we set $\lambda_{k+1} = \cdots = \lambda_r = 0$. We denote by $|\lambda| := \lambda_1 + \cdots + \lambda_k$ the integer being partitioned by λ, and we let V_λ denote the unique irreducible representation of GL_r with highest weight λ.

Given three such partitions λ, μ, ν with $|\nu| = |\lambda| + |\mu|$, we denote by $c_{\lambda\mu}^{\nu}$ the corresponding Littlewood–Richardson coefficient, i.e., the multiplicity of the representation V_ν in $V_\lambda \otimes V_\mu$.

The following result is, in a certain precise sense, equivalent to the celebrated Littlewood–Richardson rule, cf. [19, Appendix A].

Theorem 4.15 (Knutson–Tao)**.** *Let λ, μ, ν be integer partitions with at most r parts such that $|\nu| = |\lambda| + |\mu|$. Then the Littlewood–Richardson coefficient $c_{\lambda\mu}^{\nu}$ is equal to the number of representations $\rho : \Delta_3^r \to \mathbb{T}_0^{\mathbb{Z}}$ with logarithmic values (λ, μ, ν) on the "border" of Δ_3^r as in Figure 6.*

For example, if $\nu = (3, 2, 1)$, $\lambda = \mu = (2, 1)$, then by Example 4.13 there are two integral hives with the corresponding border labels (corresponding to $x = 4$ and $x = 5$). Thus $c_{\lambda\mu}^{\nu} = 2$.

The saturation theorem proved by Knutson and Tao is equivalent to the following statement:

Theorem 4.16 (Knutson–Tao)**.** *If there exists a representation $\rho : \Delta_3^r \to \mathbb{T}_0$ with given border labels in $\mathbb{T}_0^{\mathbb{Z}}$, then there exists a representation $\rho : \Delta_3^r \to \mathbb{T}_0^{\mathbb{Z}}$ with these border labels.*

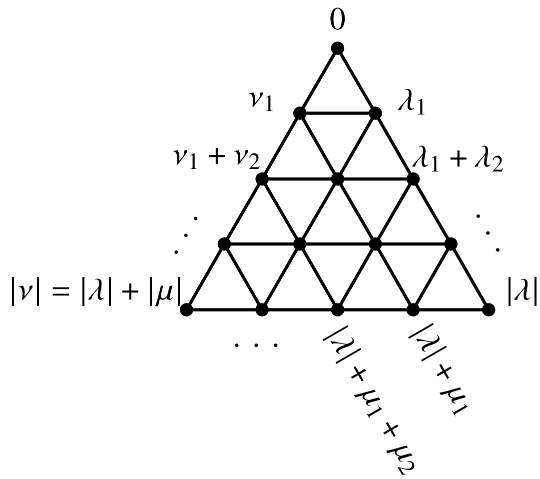

Figure 6. Border labels corresponding to a triple of integer partitions.

As discussed in Fulton's survey [27], the work of Klyachko [32], combined with the Knutson–Tao saturation theorem, implies the following result about eigenvalues of sums of Hermitian matrices which was previously known as "Horn's Conjecture":

Theorem 4.17. *There are $r \times r$ Hermitian matrices A, B, C with $A + B = C$ having respective eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$, $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r$, and $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_r$ if and only if there is a representation $\rho : \Delta_3^r \to \mathbb{T}_0^{\mathbb{Z}}$ with logarithmic border labels (λ, μ, ν).*

Part 3. Foundations of polymatroids

5. The universal tract and the universal pasture

In this section, we extend the notions of "universal tract" and "universal pasture", as introduced in [8], from matroids to polymatroids.

5.1. Representation spaces and thin Schubert cells. Let $J \subseteq \Delta_n^r$ be an M-convex set and F a tract. The *(strong) representation space of J over F* is the set $\mathrm{R}_J(F)$ of all strong F-representations of J, and the *weak representation space of J over F* is the set $\mathrm{R}_J^w(F)$ of all weak F-representations of J.

The multiplicative group $F^\times$ of F acts diagonally on both the weak and strong representation spaces of J over F (cf. Lemma 6.1 for a generalization). The *(strong) thin Schubert cell of J over F* is $\mathrm{Gr}_J(F) = \mathrm{R}_J(F)/F^\times$. The *weak thin Schubert cell of J over F* is $\mathrm{Gr}_J^w(F) = \mathrm{R}_J^w(F)/F^\times$.

Remark 5.1.

(1) If F is a field and M is a matroid, then $\mathrm{Gr}_M^w(F) = \mathrm{Gr}_M(F)$ corresponds to the usual notion of the thin Schubert cell of M, which consists of all points x of the Grassmannian $\mathrm{Gr}(r,n)(F)$ over F for which the Plücker coordinate x_α is nonzero precisely when α is a basis of M.
(2) If $F = \mathbb{T}_0$ is the tropical hyperfield and J is an M-convex set, we have $\mathrm{Gr}_J^w(\mathbb{T}_0) = \mathrm{Gr}_J(\mathbb{T}_0)$ since $\mathbb{T}_0$ is excellent (Corollary 4.12).
(3) If M is a matroid, then the association $\boldsymbol{\rho} \mapsto -\log \boldsymbol{\rho}$ (cf. Section 4.5) identifies $\mathrm{Gr}_M(\mathbb{T}_0)$ with the *local Dressian* Dr_M of all tropical linear spaces with underlying matroid M.

5.2. Functoriality. Let $f\colon F_1 \to F_2$ be a tract morphism and $\boldsymbol{\rho}\colon [n]^r \to F_1$ a strong (resp. weak) F_1-representation of an M-convex set $J \subseteq \Delta_n^r$. The *push-forward of $\boldsymbol{\rho}$ along f* is the function

$$f_*(\boldsymbol{\rho})\colon \quad \begin{array}{ccc} [n]^r & \longrightarrow & F_2 \\ \boldsymbol{\alpha} & \longmapsto & f(\boldsymbol{\rho}(\boldsymbol{\alpha})). \end{array}$$

Since tract morphisms preserve null sets as well as non-zero elements, $f_*(\boldsymbol{\rho})$ is a strong (resp. weak) F_2-representation of J. Thus $f\colon F_1 \to F_2$ defines maps

$$f_*\colon\ \mathrm{R}_J(F_1) \longrightarrow \mathrm{R}_J(F_2) \quad \text{and} \quad f_*\colon\ \mathrm{R}_J^w(F_1) \longrightarrow \mathrm{R}_J^w(F_2).$$

More precisely, taking the strong (resp. weak) representation space of J defines a functor $\mathrm{R}_J\colon$ Tracts $\to$ Sets (resp. $\mathrm{R}_J^w\colon$ Tracts $\to$ Sets).

Similarly, the strong (resp. weak) thin Schubert cell of J is functorial in F, i.e., a tract morphism $f\colon F_1 \to F_2$ induces maps

$$f_*\colon\ \mathrm{Gr}_J(F_1) \longrightarrow \mathrm{Gr}_J(F_2) \quad \text{and} \quad f_*\colon\ \mathrm{Gr}_J^w(F_1) \longrightarrow \mathrm{Gr}_J^w(F_2),$$

yielding functors Gr_J and Gr_J^w from Tracts to Sets.

5.3. The universal tract. Let $J \subseteq \Delta_n^r$ be an M-convex set of effective rank $\bar{r} = r - |\delta_J^-|$ and define $\mathbf{J} := \{\boldsymbol{\beta} \in [n]^{\bar{r}} \mid \Sigma\boldsymbol{\beta} \in \bar{J}\}$, where $\bar{J} = J - \delta_J^-$ is the reduction of J. (Here $\Sigma : [n]^{\bar{r}} \to \Delta_n^{\bar{r}}$ is the map defined in Equation (1).)

Let $\omega_J = \delta_J^+ - \delta_J^-$ be the width of J. The *extended universal tract of J* is the tract $\widehat{T}_J = \mathbb{F}_1^\pm(x_{\boldsymbol{\beta}} \mid \boldsymbol{\beta} \in \mathbf{J}) /\!\!/ \langle S \rangle$, where S consists of the Plücker relations

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\boldsymbol{\alpha} i_k j_2 \dots j_s}$$

for all $2 \leqslant s \leqslant \bar{r}$, $\boldsymbol{\alpha} \in [n]^{\bar{r}-s}$ and $i_0, \dots, i_s, j_2, \dots, j_s \in [n]$ with $\sum \boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s \leqslant \omega_J$, using the convention that $x_{\boldsymbol{\beta}} = 0$ if $\boldsymbol{\beta} \notin \mathbf{J}$.

The *universal representation of J* is the representation

$$\hat{\boldsymbol{\rho}}\colon\ [n]^{\bar{r}} \longrightarrow \widehat{T}_J$$

defined by $\hat{\boldsymbol{\rho}}(\boldsymbol{\alpha}) = x_\alpha$. It is a strong $\widehat{T}_J$-representation by the very definition of $\widehat{T}_J$.

The extended universal tract $\widehat{T}_J$ is graded by the multiplicative map

$$\deg\colon\ \widehat{T}_J \longrightarrow \mathbb{Z}$$

with $\deg(x_{\boldsymbol{\beta}}) = 1$ for $\boldsymbol{\beta} \in J$ and $\deg(0) = 0$.

The *universal tract of* J is the subtract

$$T_J \ = \ \left\{a \in \widehat{T}_J \,\middle|\, \deg(a) = 0\right\}$$

of $\widehat{T}_J$.

By the idempotency principle for proper polymatroids (Proposition 4.7), the existence of the universal representation implies that both $\widehat{T}_J$ and T_J are near-idempotent if J is not the translate of a matroid. If $\omega_{J,i} \geqslant 3$ for some $i \in [n]$, then both $\widehat{T}_J$ and T_J are idempotent.

Proposition 5.2. *Let $J \subseteq \Delta_n^r$ be an M-convex set with extended universal tract $\widehat{T}_J$ and universal tract T_J. Let F be a tract. Then composing the universal representation $\hat{\boldsymbol{\rho}}\colon [n]^{\bar{r}} \to \widehat{T}_J$ with a tract morphism $f\colon \widehat{T}_J \to F$ yields a bijection*

$$\Phi_{J,F}\colon\ \ \mathrm{Hom}(\widehat{T}_J,\ F) \ \longrightarrow\ \mathrm{R}_J(F),$$

which descends to a bijection

$$\overline{\Phi}_{J,F}\colon\ \ \mathrm{Hom}(T_J,\ F) \ \longrightarrow\ \mathrm{Gr}_J(F).$$

Both bijections are functorial in F.

Proof. This is proven exactly as for usual matroids, cf. [9, Thm. 6.15 and Prop. 6.23]. For completeness, we sketch the argument.

The inverse bijection $\Psi_{J,P}\colon\ \mathrm{R}_J(P) \to \mathrm{Hom}(\widehat{T}_J, P)$ to $\Phi_{J,P}$ is given as follows: an F-representation $\boldsymbol{\sigma}\colon [n]^{\bar{r}} \to F$ of J is mapped to the tract morphism $f : \widehat{T}_J \to F$ determined by $f(x_\alpha) = \boldsymbol{\sigma}(\boldsymbol{\alpha})$. Since $\boldsymbol{\sigma}$ satisfies the 3-term Plücker relations as a representation of the M-convex set J, it follows from the universal properties of free algebras and quotients (cf. Section 3.4 and Section 3.5) that the assignment $x_\alpha \mapsto \boldsymbol{\sigma}(\boldsymbol{\alpha})$ defines a tract morphism $f\colon \widehat{T}_J \to F$. By construction, $\Phi_{P,J}(f) = \boldsymbol{\sigma}$. Since $\widehat{T}_J$ is generated by the x_α over $\mathbb{F}_1^{\pm}$, f is uniquely determined by the images of the x_α, which completes the proof that $\Psi_{J,P}$ is the inverse bijection of $\Phi_{J,P}$.

The bijection $\Phi_{J,P}$ descends to a bijection $\overline{\Phi}_{J,P}\colon\ \mathrm{Hom}(T_J,\ P) \to \mathrm{Gr}_J(P)$ for the following reason: two morphisms $f_i\colon \widehat{T}_J \to F$ (for $i = 1, 2$) have the same restriction to $T_J = \{c \in \widehat{T}_J \mid \deg c = 0\}$ if and only if there exists $a \in P^\times$ such that $f_2(x_\alpha) = a f_1(x_\alpha)$ for all $\boldsymbol{\alpha} \in [n]^{\bar{r}}$. This is the case if and only if $\boldsymbol{\sigma}_2 = a\boldsymbol{\sigma}_1$ for the F-representations $\boldsymbol{\sigma}_i = f_i \circ \hat{\boldsymbol{\rho}}$ (for $i = 1, 2$), which means, by definition, that $[\boldsymbol{\sigma}_1] = [\boldsymbol{\sigma}_2]$ in $\mathrm{Gr}_J(P)$.

The functoriality of $\Phi_{J,P}$ in F follows from the fact that both $\mathrm{Hom}(\widehat{T}_J, -)$ and $\mathrm{R}_J(-)$ act on morphisms in terms of compositions of maps. Since a tract morphism $P \to Q$ restricts to a group homomorphism $P^\times \to Q^\times$, and since $\mathrm{Hom}(\widehat{T}_J, P)$ and $\mathrm{R}_J(P)$ are sets of $P^\times$-orbits, $\overline{\Phi}_{J,P}$ is also functorial. □

5.4. The universal pasture. Roughly speaking, the universal pasture of a polymatroid is the 3-term truncation of the universal tract, which only captures the 3-term Plücker relations. For simplicity, we refrain from introducing pastures in this text, and instead define the universal pasture as a tract. For more details on pastures, including their precise relationship to tracts, see [8, Section 6.4] and [9].

Let $J \subseteq \Delta_n^r$ be an M-convex set of effective rank $\bar{r}$, and let $\mathbf{J}$ be defined as above. The *extended universal pasture of* J is the tract $\widehat{P}_J = \mathbb{F}_1^\pm(x_{\boldsymbol{\beta}} \mid \boldsymbol{\beta} \in \mathbf{J}) /\!\!/ \langle S \rangle$, where S consists of the 3-term Plücker relations

$$x_{\alpha ij} \cdot x_{\alpha kl} \; - \; x_{\alpha ik} \cdot x_{\alpha jl} \; + \; x_{\alpha il} \cdot x_{\alpha jk}$$

for all $\boldsymbol{\alpha} \in [n]^{\bar{r}-2}$ and $i, j, k, l \in [n]$. The *universal representation of* J is the weak $\widehat{P}_J$-representation $\hat{\boldsymbol{\rho}}\colon [n]^{\bar{r}} \to \widehat{P}_J$ of J defined by $\hat{\boldsymbol{\rho}}(\boldsymbol{\beta}) = x_{\boldsymbol{\beta}}$.

Analogous to the extended universal tract, the extended universal pasture is graded by the multiplicative map $\deg\colon \widehat{P}_J \to \mathbb{Z}$ with $\deg(x_{\boldsymbol{\beta}}) = 1$ for $\boldsymbol{\beta} \in \mathbf{J}$ and $\deg(0) = 0$. The *universal pasture of* J is the subtract $P_J = \{a \in \widehat{P}_J \mid \deg(a) = 0\}$ of $\widehat{P}_J$.

By Proposition 4.7, $\widehat{P}_J$ and P_J are near-idempotent if J is not the translate of a matroid. If $\omega_{J,i} \geqslant 3$ for some $i \in [n]$, then both $\widehat{P}_J$ and P_J are idempotent.

Proposition 5.3. *Let $J \subseteq \Delta_n^r$ be an M-convex set with extended universal pasture $\widehat{P}_J$ and universal pasture P_J. Let F be a tract. Then composing the universal representation $\hat{\boldsymbol{\rho}}\colon [n]^{\bar{r}} \to \widehat{P}_J$ with a tract morphism $f\colon \widehat{P}_J \to F$ yields a bijection*

$$\Phi_{J,P}\colon \;\; \mathrm{Hom}(\widehat{P}_J, \; P) \;\longrightarrow\; \mathrm{R}_J^w(P),$$

which descends to a bijection

$$\overline{\Phi}_{J,P}\colon \;\; \mathrm{Hom}(P_J, \; P) \;\longrightarrow\; \mathrm{Gr}_J^w(P).$$

Both bijections are functorial in F.

Proof. The proof of Proposition 5.2 applies *mutatis mutandis*; we omit the details. □

5.5. The comparison map. Let $J \subseteq \Delta_n^r$ be an M-convex set, and let $\mathbf{J}$ be defined as defined above. Since the extended universal pasture $\widehat{P}_J = \mathbb{F}_1^\pm(x_{\boldsymbol{\beta}} \mid \boldsymbol{\beta} \in \mathbf{J}) /\!\!/ \langle S' \rangle$ of J is defined by the set S' of 3-term Plücker relations, which are a subset of the set S of all Plücker relations, which define the extended universal tract $\widehat{T}_J = \mathbb{F}_1^\pm(x_{\boldsymbol{\beta}} \mid \boldsymbol{\beta} \in \mathbf{J}) /\!\!/ \langle S \rangle$ of J, these two tracts come with a canonical morphism $\hat{\pi}_J\colon \widehat{P}_J \to \widehat{T}_J$, which is degree preserving and thus restricts to a morphism $\pi_J\colon P_J \to T_J$.

For M-convex sets $J \subseteq \Delta_n^r$ of rank $r \leqslant 2$, we have $S' = S$, and thus the canonical morphisms $\hat{\pi}_J$ and π_J are isomorphisms. This fails, in general, for M-convex sets of larger rank due to the presence of Plücker relations with 4 or more terms.

However, as we prove below (Theorem 5.4), the canonical maps $\hat{\pi}_J$ and π_J are bijective. As a preparation for the proof, we call a Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\alpha i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\alpha i_k j_2 \dots j_s}$$

(with $2 \leqslant s \leqslant \bar{r}$, $\boldsymbol{\alpha} \in [n]_n^{\bar{r}-s}$ and $i_0, \dots, i_s, j_2, \dots, j_s \in [n]$) *degenerate* if it has exactly two nonzero terms, i.e., there are exactly two indices k in $\{0, \dots, s\}$ for which both $\boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s$ and $\boldsymbol{\alpha} i_k j_2 \dots j_s$ are in $\mathbf{J}$.

Since $a - b \in N_{\widehat{T}_J}$ implies that $a = b$ in $\widehat{T}_J$ by the uniqueness of additive inverses, the degenerate Plücker relations enforce relations between the generators x_β of $\widehat{T}_J$ and, in the case of degenerate 3-term Plücker relations, of $\widehat{P}_J$. The following result implies that, in fact, the degenerate 3-term Plücker relations generate all relations between the generators of $\widehat{T}_J$. (It does not, however, imply that the non-degenerate Plücker relations are generated by the 3-term Plücker relations; in particular, the bijective morphism $P_J \to T_J$ is in general not an isomorphism.)

Theorem 5.4. *Let J be a polymatroid. Then the canonical morphisms $\hat{\pi}_J\colon \widehat{P}_J \to \widehat{T}_J$ and $\pi_J\colon P_J \to T_J$ are bijections.*

Proof. Since $\hat{\pi}_J(0) = 0$, it suffices to show that the restriction of $\hat{\pi}_J$ to $\widehat{P}_J^\times \to \widehat{T}_J^\times$ is bijective for the first claim. The second claim (about π_J) follows from the first claim by taking the respective degree 0 parts.

The groups $\widehat{T}_J^\times$ and $\widehat{P}_J$ are quotients of the free abelian group generated by -1 and the symbols $x_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \in \mathbf{J}$ by certain respective subgroups H_J and H_J^w (defined below). We show by an elementary induction over the number of terms of a Plücker relation that $H_J^w = H_J$, which shows that $\widehat{P}_J^\times \to \widehat{T}_J^\times$ is a bijection. Note that since $\hat{\pi}_J$ is degree preserving, this result implies at once that π_J is also a bijection.

Before we begin with the induction, we note that if J is a matroid, the claim follows from general results on perfect tracts. Namely, enriching the nullset of $\widehat{P}_J$ by the set S of all relations $\sum a_i$ with at least 3 nonzero terms yields a tract $F = \widehat{P}_J /\!\!/ \langle S \rangle$, which is perfect since it satisfies the modified strong fusion rule; see [12, Thm. 1.11]. By [3, Thm. 3.46], every weak F-representation of J is strong. Thus, by Proposition 5.2 and Proposition 5.3, we have canonical identifications

$$\mathrm{Hom}(\widehat{T}_J,\ F)\ =\ \mathrm{R}_J(F)\ =\ \mathrm{R}_J^w(F)\ =\ \mathrm{Hom}(\widehat{P}_J,\ F).$$

This means that the canonical projection $\pi_S\colon \widehat{P}_J \to \widehat{P}_J /\!\!/ \langle S\rangle = F$ factors through the surjection $\hat{\pi}_J\colon \widehat{P}_J \to \widehat{T}_J$. Since every relation in S has at least 3 nonzero terms, π_S is injective, and so is $\hat{\pi}_J$. This shows that $\hat{\pi}_J$ is a bijection. Since the translation $J \to \bar{J} = J - \delta_J^-$ induces isomorphisms between the respective universal tracts and universal pastures (cf. Theorem 7.1), this proof extends to all translates of matroids.

Even though the following proof does not rely on the previous discussion, we can use it to simplify matters: if J is not the translate of a matroid, then the idempotency principle for proper polymatroids (Proposition 4.7) implies that $1 = -1$ in $\widehat{T}_J$ and $\widehat{P}_J$.

We therefore assume that $-1 = 1$, which leads to a number of simplifications:

(1) The group $\widehat{T}_J$ is generated by the symbols $x_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \in \mathbf{J}$, i.e., we can remove the generator -1. Moreover, the generators $x_{\boldsymbol{\beta}}$ of $\widehat{T}_J$ are invariant under the permutation of the coefficients of $\boldsymbol{\beta}$, which allows us to define $x_\beta := x_{\boldsymbol{\beta}}$ for $\beta = \Sigma\boldsymbol{\beta} \in \bar{J}$, independently of the order of the coefficients of $\boldsymbol{\beta}$.

(2) The Plücker relations for the x_β turn into

$$\mathrm{Pl}(\alpha|i_0 \dots i_s|j_2 \dots j_s) := \sum_{k=0}^{s} \quad x_{\alpha+\varepsilon_{i_0}+\dots+\widehat{\varepsilon_{i_k}}+\dots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_k}+\varepsilon_{j_2}+\dots+\varepsilon_{j_s}} \quad \in \quad N_{\widehat{T}_J}$$

for $2 \leqslant s \leqslant \bar{r}$, $\alpha \in \Delta_n^{\bar{r}-s}$ and $i_0, \dots, i_s, j_2, \dots, j_s \in [n]$ with $\delta_J^- \leqslant \alpha$ and $\alpha + \varepsilon_{i_0} + \dots + \varepsilon_{i_s} + \varepsilon_{j_2} + \dots + \varepsilon_{j_s} \leqslant \omega_J$. In particular, we can drop the sign $(-1)^{k+\sigma(k)}$.

(3) The group $\widehat{T}_J^\times$ is the quotient of the free abelian group generated by the symbols x_β with $\beta \in J$ modulo the subgroup H_J generated by the *degenerate generalized cross ratios*

$$\frac{x_{\alpha+\varepsilon_{i_0}+\dots+\widehat{\varepsilon_{i_k}}+\dots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_k}+\varepsilon_{j_2}+\dots+\varepsilon_{j_s}}}{x_{\alpha+\varepsilon_{i_0}+\dots+\widehat{\varepsilon_{i_l}}+\dots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_l}+\varepsilon_{j_2}+\dots+\varepsilon_{j_s}}},$$

whose numerator and denominator are the two nonzero terms of a degenerate Plücker relation $\mathrm{Pl}(\alpha|i_0 \dots i_s|j_2 \dots j_s)$.

We show by induction over $s \geqslant 2$ that the degenerate generalized cross ratios lie in the subgroup H_J^w generated by the *degenerate cross ratios*

$$\frac{x_{\alpha+\varepsilon_i+\varepsilon_l} \cdot x_{\alpha+\varepsilon_j+\varepsilon_k}}{x_{\alpha+\varepsilon_i+\varepsilon_k} \cdot x_{\alpha+\varepsilon_j+\varepsilon_l}}$$

with $\alpha \in \Delta_r^{\bar{r}-2}$ (i.e., $s = 2$), which stem from degenerate 3-term Plücker relations $\mathrm{Pl}(\alpha|ikl|j)$. Since $\widehat{P}_J$ is the quotient of the free abelian group generated by the x_β modulo H_J^w, this proves the claim of the theorem.

The base case $s = 2$ is tautologically true. Thus we assume that $s \geqslant 3$, and we consider a degenerate Plücker relation $\mathrm{Pl}(\alpha|i_0 \dots i_s|j_2 \dots j_s)$. After permuting the indices, we can assume that the two nontrivial terms are indexed by $k = 0$ and $l = 1$, which means that we need to show that the generalized degenerate cross ratio

$$\frac{x_{\alpha+\varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_0}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s}}}{x_{\alpha+\varepsilon_{i_0}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_1}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s}}}$$

lies in H_J^w. If $\{i_2, \dots, i_s\} \cap \{j_2, \dots, j_s\}$ contains a common element, say $i_s = j_s$ (after rearranging indices), then the degenerate Plücker relation $\mathrm{Pl}(\alpha|i_0 \dots i_s|j_2 \dots j_s)$ is equal to the degenerate Plücker relation $\mathrm{Pl}(\alpha + \varepsilon_{i_s}|i_0 \dots i_{s-1}|j_2 \dots j_{s-1})$, up to a zero term. Thus the generalized degenerate cross ratio in question appears already for a shorter Plücker relation and lies in H_J^w by the inductive hypothesis. Therefore, we can assume that $\{i_2, \dots, i_s\} \cap \{j_2, \dots, j_s\} = \varnothing$ in the following.

Define $\beta = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}$ and $\gamma = \varepsilon_{i_1} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}$. Since $\mathrm{Pl}(\alpha|i_0 \dots i_s|j_2 \dots j_s)$ has only two nonzero terms for $k = 0$ and $l = 1$, the indices i_0 and i_1 do not appear in $\{i_2, \dots, i_s\}$ and thus $\gamma_{i_1} \geqslant \beta_{i_1} = 1$. Since $\{i_2, \dots, i_s\} \cap \{j_2, \dots, j_s\} = \varnothing$, we have $\gamma_{j_s} > \beta_{j_s}$, and the exchange axiom for M-convex sets yields an index in $\{i_2, \dots, i_s\}$, say i_s, such that both $\beta - \varepsilon_{i_s} + \varepsilon_{j_s}$ and $\gamma + \varepsilon_{i_s} - \varepsilon_{j_s}$ are in J.

For the ease of notation, we introduce the abbreviations $\zeta_R^{\widehat{S}} = x_\eta$ and $\xi_S^{\widehat{R}} = x_\vartheta$ for subsets $S \subseteq \{0, \dots, s\}$ and $R \subseteq \{2, \dots, s\}$, where

$$\eta = \alpha + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s} - \sum_{k\in S} \varepsilon_{i_k} + \sum_{k\in R} \varepsilon_{j_k} \text{ and } \vartheta = \alpha + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s} + \sum_{k\in S} \varepsilon_{i_k} - \sum_{k\in R} \varepsilon_{j_k}.$$

So the Plücker relation $\mathrm{Pl}(\alpha|i_0 \dots i_s|j_2 \dots j_s)$ reads

$$\underbrace{\zeta^{\widehat{0}} \cdot \xi_0}_{\neq 0} + \underbrace{\zeta^{\widehat{1}} \cdot \xi_1}_{\neq 0} + \underbrace{\zeta^{\widehat{2}} \cdot \xi_2}_{=0} + \cdots + \underbrace{\zeta^{\widehat{s}} \cdot \xi_s}_{=0} \quad \in \quad N_{\widehat{T}_J},$$

and the conclusion from the previous paragraph means that $\zeta_s^{\widehat{0s}} \cdot \xi_{1s}^{\widehat{s}} \neq 0$. Our goal is to show that the generalized degenerate cross ratio

$$\frac{\zeta^{\widehat{1}} \cdot \xi_1}{\zeta^{\widehat{0}} \cdot \xi_0} = \frac{x_{\alpha+\varepsilon_{i_1}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_0}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s}}}{x_{\alpha+\varepsilon_{i_0}+\varepsilon_{i_2}+\cdots+\varepsilon_{i_s}} \cdot x_{\alpha+\varepsilon_{i_1}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s}}}$$

is in H_J^w. We divide the proof into two cases: either $\zeta^{\widehat{s}} = 0$ or $\xi_s = 0$.

Case 1: $\zeta^{\widehat{s}} = 0$. Consider the 3-term Plücker relation

$$\mathrm{Pl}(\zeta^{\widehat{01s}}|i_0 i_1 i_s|j_s) := \underbrace{\zeta^{\widehat{0}}}_{\neq 0} \cdot \zeta_s^{\widehat{1s}} + \underbrace{\zeta^{\widehat{1}}}_{\neq 0} \cdot \underbrace{\zeta_s^{\widehat{0s}}}_{\neq 0} + \underbrace{\zeta^{\widehat{s}}}_{=0} \cdot \zeta_s^{\widehat{01}} \quad \in \quad N_{\widehat{T}_J}.$$

Since a relation in $N_{\widehat{T}_J}$ cannot have exactly one nonzero term, we conclude that $\zeta_s^{\widehat{1s}} \neq 0$, so that $\mathrm{Pl}(\zeta^{\widehat{01s}}|i_0 i_1 i_s|j_s)$ is degenerate and thus

$$\frac{\zeta^{\widehat{1}} \cdot \zeta_s^{\widehat{0s}}}{\zeta^{\widehat{0}} \cdot \zeta_s^{\widehat{1s}}} \quad \in \quad H_J^w.$$

Next we aim to show that the Plücker relation

$$\mathrm{Pl}(\alpha+\varepsilon_{j_s}|i_0 \dots i_{s-1}|j_2 \dots j_{s-1}) := \underbrace{\zeta_s^{\widehat{0s}} \cdot \xi_0}_{\neq 0} + \underbrace{\zeta_s^{\widehat{1s}} \cdot \xi_1}_{\neq 0} + \zeta_s^{\widehat{2s}} \cdot \xi_2 + \dots + \zeta_s^{\widehat{(s-1)s}} \cdot \xi_{s-1} \quad \in \quad N_{\widehat{T}_J}$$

is degenerate, i.e., $\zeta_s^{\widehat{ks}} \cdot \xi_k = 0$ for $k = 2, \dots, s-1$. We know that $\zeta^{\widehat{k}} \cdot \xi_k = 0$. If $\xi_k = 0$, then $\zeta_s^{\widehat{ks}} \cdot \xi_k = 0$, as desired. If $\zeta^{\widehat{k}} = 0$, then we consider the 3-term Plücker relation

$$\mathrm{Pl}(\zeta^{\widehat{0ks}}|i_0 i_1 i_s|j_s) := \underbrace{\zeta^{\widehat{0}}}_{\neq 0} \cdot \zeta_s^{\widehat{ks}} + \underbrace{\zeta^{\widehat{k}}}_{=0} \cdot \zeta_s^{\widehat{0k}} + \underbrace{\zeta^{\widehat{s}}}_{=0} \cdot \zeta_s^{\widehat{0s}} \quad \in \quad N_{\widehat{T}_J}.$$

Since a relation in $N_{\widehat{T}_J}$ cannot have exactly one nonzero term, we conclude that $\zeta_s^{\widehat{ks}} = 0$, which implies $\zeta_s^{\widehat{ks}} \cdot \xi_k = 0$ that in this case as well.

This shows that the Plücker relation $\mathrm{Pl}(\alpha + \varepsilon_{j_s}|i_0 \dots i_{s-1}|j_2 \dots j_{s-1})$ is degenerate. So the inductive hypothesis applies and shows that

$$\frac{\zeta_s^{\widehat{1s}} \cdot \xi_1}{\zeta_s^{\widehat{0s}} \cdot \xi_0} \quad \in \quad H_J^w.$$

Therefore

$$\frac{\zeta^{\widehat{1}} \cdot \xi_1}{\zeta^{\widehat{0}} \cdot \xi_0} = \frac{\zeta^{\widehat{1}} \cdot \zeta_s^{\widehat{0s}}}{\zeta^{\widehat{0}} \cdot \zeta_s^{\widehat{1s}}} \cdot \frac{\zeta_s^{\widehat{1s}} \cdot \xi_1}{\zeta_s^{\widehat{0s}} \cdot \xi_0} \quad \in \quad H_J^w,$$

which completes the inductive step in case 1.

Case 2: $\xi_s = 0$. The proof is analogous to case 1. Consider the 3-term Plücker relation

$$\mathrm{Pl}(\xi^{\widehat{s}}|i_0 i_1 i_s|j_s) := \underbrace{\xi_0}_{\neq 0} \cdot \underbrace{\xi_{1s}^{\widehat{s}}}_{\neq 0} + \underbrace{\xi_1}_{\neq 0} \cdot \xi_{0s}^{\widehat{s}} + \underbrace{\xi_s}_{=0} \cdot \xi_{01}^{\widehat{s}} \quad \in \quad N_{\widehat{T}_J}.$$

Then $\xi_s^{\widehat{1s}} \neq 0$ and $\mathrm{Pl}(\xi^{\widehat{s}}|i_0 i_1 i_s|j_s)$ is degenerate, which shows that

$$\frac{\xi_1 \cdot \xi_{0s}^{\widehat{s}}}{\xi_0 \cdot \xi_{1s}^{\widehat{s}}} \quad \in \quad H_J^w.$$

Next we aim to show that the Plücker relation

$$\mathrm{Pl}(\alpha+\varepsilon_{i_s}|i_0 \dots i_{s-1}|j_2 \dots j_{s-1}) := \underbrace{\zeta^{\widehat{0}} \cdot \xi_{0s}^{\widehat{s}}}_{\neq 0} + \underbrace{\zeta^{\widehat{1}} \cdot \xi_{1s}^{\widehat{s}}}_{\neq 0} + \zeta^{\widehat{2}} \cdot \xi_{2s}^{\widehat{s}} + \dots + \zeta^{\widehat{s-1}} \cdot \xi_{(s-1)s}^{\widehat{s}} \quad \in \quad N_{\widehat{T}_J}$$

is degenerate, i.e., $\zeta^{\widehat{k}} \cdot \xi^{\widehat{s}}_{ks} = 0$ for $k = 2, \dots, s-1$. We know that $\zeta^{\widehat{k}} \cdot \xi_k = 0$. If $\zeta^{\widehat{k}} = 0$, then $\zeta^{\widehat{k}} \cdot \xi^{\widehat{s}}_{ks} = 0$, as desired. If $\xi_k = 0$, then we consider the 3-term Plücker relation

$$\mathrm{Pl}(\xi^{\widehat{s}}|i_0 i_k i_s|j_s) := \underbrace{\xi_0}_{\neq 0} \cdot \xi^{\widehat{s}}_{ks} + \underbrace{\xi_k}_{=0} \cdot \xi^{\widehat{s}}_{0s} + \underbrace{\xi_s}_{=0} \cdot \xi^{\widehat{s}}_{0k} \quad \in \quad N_{\widehat{T}_J}.$$

Thus $\xi^{\widehat{s}}_{ks} = 0$, which implies $\zeta^{\widehat{ks}}_{s} \cdot \xi_k = 0$ also in this case.

This shows that the Plücker relation $\mathrm{Pl}(\alpha + \varepsilon_{i_s}|i_0 \dots i_{s-1}|j_2 \dots j_{s-1})$ is degenerate. So the inductive hypothesis applies and shows that

$$\frac{\zeta^{\widehat{1}} \cdot \xi^{\widehat{s}}_{1s}}{\zeta^{\widehat{0}} \cdot \xi^{\widehat{s}}_{0s}} \quad \in \quad H^w_J.$$

Therefore in case 2 we also have

$$\frac{\zeta^{\widehat{1}} \cdot \xi_1}{\zeta^{\widehat{0}} \cdot \xi_0} = \frac{\zeta^{\widehat{1}} \cdot \xi^{\widehat{s}}_{1s}}{\zeta^{\widehat{0}} \cdot \xi^{\widehat{s}}_{0s}} \cdot \frac{\xi_1 \cdot \xi^{\widehat{s}}_{0s}}{\xi_0 \cdot \xi^{\widehat{s}}_{1s}} \quad \in \quad H^w_J,$$

which completes the proof. □

Theorem 5.4 has several consequences, some of which will appear later in the paper. Right away, however, we gain a series of new examples of excellent tracts. We call a tract F *degenerate* if every formal sum $\sum a_i \in F^+$ with at least 3 nonzero terms is contained in N_F. An example of a degenerate tract is the degenerate triangular hyperfield $\mathbb{T}_\infty$.

Corollary 5.5. *Every degenerate tract is excellent.*

Proof. Let J be a polymatroid and $\hat{\pi}_J : \widehat{P}_J \to \widehat{T}_J$ the canonical morphism from the extended universal pasture to the extended universal tract of J. Let $\boldsymbol{\rho}: [n]^{\bar{r}} \to F$ be a weak F-representation of J with associated morphism $f_{\boldsymbol{\rho}}: \widehat{P}_J \to F$ (using Proposition 4.10). By Theorem 5.4, $\hat{\pi}_J$ is a bijection, which means that $f_{\boldsymbol{\rho}}$ defines a multiplicative map $\tilde{f}: \widehat{T}_J \to F$. Since N_F contains all relations with more than two nonzero terms, $\tilde{f}$ is automatically a tract morphism, which in turn corresponds to a strong F-representation of J by Proposition 4.11. This shows that $\boldsymbol{\rho}$ is a strong F-representation of J. Thus F is excellent. □

6. The foundation

The foundation F_M of a matroid M was introduced in [8]. It is characterized by the fact that $\mathrm{Hom}(F_M, -)$ represents the functor $\underline{\mathrm{Gr}}_M$: Tracts → Sets that sends a tract F to the set of F-rescaling classes of representations of M. In this section, we extend this concept from matroids to M-convex sets.

6.1. The realization space. The set $\mathrm{R}_J^w(F)$ of all weak F-representations of an M-convex set $J \subseteq \Delta_n^r$ is invariant under scalar multiplication by $F^\times$ and under rescaling by elements of the torus $T(F) := (F^\times)^n$. In more detail, given a weak F-representation $\boldsymbol{\rho}\colon [n]^r \to F$ of J and $a \in F^\times$, we define $a.\boldsymbol{\rho}\colon [n]^r \to F$ by the formula

$$(a.\boldsymbol{\rho})(\boldsymbol{\alpha}) \;=\; a \cdot \boldsymbol{\rho}(\boldsymbol{\alpha})$$

for $\boldsymbol{\alpha} \in \Delta_n^r$. Given $t = (t_1, \dots, t_n) \in T(F)$, we define

$$(t.\boldsymbol{\rho})(\boldsymbol{\alpha}) \;=\; \Big(\prod_{i=1}^n t_i^{\alpha_i}\Big) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}).$$

Lemma 6.1. *Both $a.\boldsymbol{\rho}$ and $t.\boldsymbol{\rho}$ are weak F-representations. If $\boldsymbol{\rho} \in \mathrm{R}_J(F)$, then both $a.\boldsymbol{\rho}$ and $t.\boldsymbol{\rho}$ are strong F-representations.*

Proof. This follows from the fact that for all $\boldsymbol{\alpha} \in [n]^{r-2}$ and $i, j, k, l \in [n]$, the corresponding 3-term Plücker relation for $a.\boldsymbol{\rho}$ (resp. for $t.\boldsymbol{\rho}$) is a multiple of the Plücker relation for $\boldsymbol{\rho}$ by a factor a^2 (resp. by a factor $(t_i t_j t_k t_l \cdot \prod_{m=1}^n t_m^{2\alpha_m})$). The same holds for Plücker relations with more terms, which establishes the latter claims. □

We thus have actions of $F^\times$ and of $T(F)$ on the set $\mathrm{R}_J^w(F)$ of all weak F-representations. If $F^\times$ is n-divisible, then the $F^\times$-orbit of an F-representation $\boldsymbol{\rho}$ is contained in the $T(F)$-orbit of $\boldsymbol{\rho}$. Since this is not always the case, it is useful to consider the action of $\widehat{T}(F) := F^\times \times T(F)$ on $\mathrm{R}_J^w(F)$ defined by

$$(a,t).\boldsymbol{\rho}(\boldsymbol{\alpha}) \;=\; a \cdot \Big(\prod_{i=1}^n t_i^{\alpha_i}\Big) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha}).$$

Definition 6.2. An *F-rescaling class of J* is the $\widehat{T}(F)$-orbit of a weak F-representation of J. The *realization space of J* is the set $\underline{\mathrm{Gr}}_J^w(F) = \mathrm{R}_J^w(F)/\widehat{T}(F)$ of all F-rescaling classes of J.

The torus action $\widehat{T}(F)$ on $\mathrm{R}_J^w(F)$ is functorial in F, and in particular the push-forward $f_*\colon \mathrm{R}_J^w(F_1) \to \mathrm{R}_J^w(F_2)$ for a tract morphism $f\colon F_1 \to F_2$ induces a map $f_*\colon \underline{\mathrm{Gr}}_J^w(F_1) \to \underline{\mathrm{Gr}}_J^w(F_2)$ between the respective realization spaces. Thus we may consider the realization space of J as a functor $\underline{\mathrm{Gr}}_J^w\colon$ Tracts $\to$ Sets.

6.2. The foundation of a polymatroid. The foundation represents the realization space $\underline{\mathrm{Gr}}_J^w$ of an M-convex set J just as the universal pasture represents the weak thin Schubert cell Gr_J^w (considered as a functor from Tracts to Sets). The advantage of the foundation over the universal tract and the universal pasture is that it is easier to compute, and it allows for several structural results that transfer the combinatorics of (poly)matroids into algebraic properties. At the same time, the foundation and the realization space still capture essential information about the thin Schubert cells (see Theorem 11.2).

We note that there is an analogous tract that represents the strong realization space $\underline{\mathrm{Gr}}_J$ of J, but for simplicity we omit a treatment of this theory.

The extended universal pasture $\widehat{P}_J$ of an M-convex set $J \subseteq \Delta_n^r$ is multi-graded by the group homomorphism

$$\deg_{[n]}: \ \widehat{P}_J \longrightarrow \mathbb{Z}^n$$

defined by $\deg_{[n]}(x_\alpha) = \Sigma\alpha$.

Definition 6.3. The *foundation of J* is the subtract

$$F_J = \left\{ a \in \widehat{P}_J \,\middle|\, \deg_{[n]}(a) = 0 \right\}$$

of $\widehat{P}_J$.

Note that $F_J \subseteq P_J$ since $\deg(a) = 0$ if $\deg_{[n]}(a) = 0$. Note further that the idempotency principle for proper polymatroids (Proposition 4.7) implies that F_J is near-idempotent if J is not the translate of a matroid. If $\omega_{J,i} \geqslant 3$ for some $i \in [n]$, then F_J is idempotent.

Proposition 6.4. *Let $J \subseteq \Delta_n^r$ be an M-convex set of effective rank $\bar{r}$ with extended universal pasture $\widehat{P}_J$, weak universal representation $\hat{\rho}\colon [n]^{\bar{r}} \to \widehat{P}_J$, and foundation F_J. Let F be a tract. Then there exists a unique bijection*

$$\underline{\Phi}_{J,F}: \ \mathrm{Hom}(F_J,\ F) \ \xrightarrow{\sim} \ \underline{\mathrm{Gr}}_J^w(F)$$

that satisfies $\underline{\Phi}_{J,F}(f|_{F_J}) = [f \circ \hat{\rho}]$ for every tract morphism $f\colon \widehat{P}_J \to F$. Moreover, this bijection is functorial in F.

Proof. The proof is similar to that of the bijection $\overline{\Phi}_{J,F}$ in Proposition 5.2. In the present case, the group $\widehat{T}(F) = (F^\times)^{n+1}$ acts on both $\mathrm{Hom}(\widehat{P}_J, F)$ and $\mathrm{R}_J^w(F)$, and $\Phi_{J,F}$ is $\widehat{T}(F)$-equivariant. So $\Phi_{J,F}$ descends to a functorial bijection $\underline{\Phi}_{J,F} : \mathrm{Hom}(F_J, F) \to \underline{\mathrm{Gr}}_J^w(F)$ between the respective sets of $\widehat{T}(F)$-orbits. □

Lemma 6.5. *Let $J \subseteq \Delta_n^r$ be an M-convex set with foundation F_J, universal pasture P_J, and extended universal pasture $\widehat{P}_J$. Then*

$$P_J \simeq F_J(x_1, \dots, x_s) \qquad \text{and} \qquad \widehat{P}_J \simeq F_J(x_0, x_1 \dots, x_s)$$

for some $0 \leqslant s \leqslant n$.

Proof. The proof is analogous to [9, Cor. 7.14]. We sketch the argument for completeness. If we multiply a 3-term Plücker relation

$$x_{\alpha ij} \cdot x_{\alpha kl} - x_{\alpha ik} \cdot x_{\alpha jl} + x_{\alpha il} \cdot x_{\alpha jk} \quad \in \quad N_{\widehat{P}_J}$$

(where $\alpha \in [n]^{\bar{r}-2}$ and $i, j, k, l \in [n]$ with $\sum \alpha ijkl \leqslant \omega_J$) by $(x_{\alpha ik} \cdot x_{\alpha jl})^{-1}$, we obtain

$$\frac{x_{\alpha ij} \cdot x_{\alpha kl}}{x_{\alpha ik} \cdot x_{\alpha jl}} \quad - \quad 1 \quad + \quad \frac{x_{\alpha il} \cdot x_{\alpha jk}}{x_{\alpha ik} \cdot x_{\alpha jl}},$$

which is contained in N_{F_J}. Therefore, the null sets of P_J and of $\widehat{P}_J$ are generated by the null set of N_{F_J}. The groups $P_J^\times / F_J^\times$ and $\widehat{P}_J^\times / F_J^\times$ are isomorphic to subgroups of the free abelian groups $\mathbb{Z}^n$ and $\mathbb{Z} \times \mathbb{Z}^n$, respectively, via the multi-degree map. Thus both $P_J^\times / F_J^\times$ and $\widehat{P}_J^\times / F_J^\times$ are themselves free abelian groups, and the lemma follows easily from this. □

We show in Theorem 11.2 that $s = n - c(J)$, where $c(J)$ is the number of indecomposable components of J.

6.3. First examples. We present here some examples of foundations of matroids. We postpone examples of foundations of proper polymatroids to Section 8.5, since they are based on further theory. For a more comprehensive list of foundations of matroids, see [11, Appendix A].

6.3.1. *Regular matroids.* The foundation of a regular matroid M is $F_M = \mathbb{F}_1^\pm$ ([8, Thm. 7.35]). In this case, $\underline{\mathrm{Gr}}_M^w(F) = \mathrm{Hom}(\mathbb{F}_1^\pm, F)$ is a point for every tract F. Thus $\mathrm{R}_M^w(F)$ consists of a single $\widehat{T}(F)$-orbit, which is in bijection with $(F^\times)^s$ for some $s \leqslant n+1$ by Lemma 6.5. In particular, we have

$$\mathrm{R}_{U_{2,2}}^w(F) \ \simeq \ F^\times \qquad \text{and} \qquad \mathrm{R}_{U_{2,3}}^w(F) \ \simeq \ (F^\times)^2$$

for the uniform rank 2 matroids $U_{2,2}$ and $U_{2,3}$ with 2 and 3 elements, respectively.

6.3.2. *The uniform rank 2 matroid on 4 elements.* The smallest matroid with a nontrivial realization space is $U_{2,4}$, whose foundation is the near-regular partial field $\mathbb{U} = \mathbb{F}_1^\pm(x, y) /\!\!/ \langle x + y - 1 \rangle$ ([9, Prop. 4.11]). Thus

$$\underline{\mathrm{Gr}}_{U_{2,4}}^w(F) \ = \ \mathrm{Hom}(\mathbb{F}_1^\pm(x,y)/\!\!/\langle x+y-1\rangle, \ F) \ = \ \big\{(a,b) \in (F^\times)^2 \,\big|\, a+b-1 \in N_F\big\}.$$

The torus orbit $\widehat{T}(F).\boldsymbol{\rho}$ of an F-representation $\boldsymbol{\rho}$ of $U_{2,4}$ is in bijection with $(F^\times)^3$.

6.3.3. *Binary matroids.* The foundation of a binary matroid M is $\mathbb{F}_1^\pm$ if M is regular and $\mathbb{F}_2$ otherwise ([8, Thm. 7.32]). In the latter case, $\underline{\mathrm{Gr}}_M^w(F)$ is a singleton if $-1 = 1$ in F and empty otherwise.

6.3.4. *Ternary matroids.* By [9, Thm. 6.28], the foundation F_M of a ternary matroid M is isomorphic to the coproduct, or *tensor product*, $F_1 \otimes \cdots \otimes F_r$ of tracts $F_1, \dots, F_r \in \{\mathbb{F}_3,\ \mathbb{H},\ \mathbb{D},\ \mathbb{U}\}$ (see Section 3.6 and Section 3.7 for definitions), which is the tract characterized by the functorial bijection

$$\underline{\mathrm{Gr}}^w_M(F) \ = \ \mathrm{Hom}(F_M,\ F) \ \simeq \ \mathrm{Hom}(F_1,\ F) \ \times \ \cdots \ \times \ \mathrm{Hom}(F_r,\ F).$$

The terms $\mathrm{Hom}(F_i, F)$ are of the following forms: $\mathrm{Hom}(\mathbb{F}_3, F)$ is a singleton if $1+1+1 \in N_F$ and empty otherwise,

$$\begin{aligned} \mathrm{Hom}(\mathbb{H}, F) \ &= \ \{a \in F^\times \mid a^2 - a + 1 \in N_F\}, \\ \mathrm{Hom}(\mathbb{D}, F) \ &= \ \{a \in F^\times \mid a + a - 1 \in N_F\}, \end{aligned}$$

and $\mathrm{Hom}(\mathbb{U}, F)$ is as described in Section 6.3.2.

7. Representations of embedded minors and duals

In this section, we compare representations of embedded minors $J\backslash\nu/\mu + \tau$ of J in terms of induced morphisms between the corresponding universal tracts, universal pastures, and foundations.

We fix an M-convex set $J \subseteq \Delta^r_n$ for the rest of this section and let $\mathbf{J} = \{\boldsymbol{\alpha} \in [n]^r \mid \Sigma\boldsymbol{\alpha} \in J\}$. As usual, we denote its duality vector by $\delta_J = \delta^-_J + \delta^+_J$ with $\delta^-_J = \inf J$ and $\delta^+_J = \sup J$. We denote its effective rank by $\bar{r} = r - |\delta^-_J|$. We write T_J for its universal tract, P_J for its universal pasture, and F_J for its foundation.

Since the canonical maps $F_J \longrightarrow T_J$ and $P_J \longrightarrow T_J$ are both injective (cf. Theorem 5.4), a morphism $T_J \longrightarrow T_{J'}$ between the universal tracts of two M-convex sets J and J' restricts to at most one morphism $P_J \longrightarrow P_{J'}$ between the respective universal pastures and to at most one morphism $F_J \longrightarrow F_{J'}$ between the respective foundations. For this reason, we begin with the description of the morphism $T_J \longrightarrow T_{J'}$ in the following results.

7.1. Minor embeddings. When we say that $J\backslash\nu/\mu + \tau$ is an embedded minor of J in the following, we assume that $\nu, \mu \in \mathbb{N}^n$ and $\tau \in \mathbb{Z}^n$ with $\tau \geqslant -\delta^-_J$, and that there is an $\alpha \in J$ such that $\mu + \delta^-_J \leqslant \alpha \leqslant \delta_{J/\mu} - \nu$.

Recall from Section 2.2 that the embedded minor $J\backslash\nu/\mu+\tau$ comes with the minor embedding

$$\begin{array}{cccc} \iota_{J\backslash\nu/\mu+\tau}\colon & J\backslash\nu/\mu + \tau & \longrightarrow & J \\ & \alpha & \longmapsto & \alpha + \mu - \tau. \end{array}$$

Theorem 7.1. *Let $J\backslash\nu/\mu + \tau$ be an embedded minor of J. Let $\bar{r}' = r - |\mu| - |\delta^-_{J\backslash\nu/\mu}|$ be the effective rank of $J\backslash\nu/\mu$. Fix $\boldsymbol{\gamma} \in [n]^{\bar{r}'-\bar{r}}$ with $\sum\boldsymbol{\gamma} = \delta^-_{J\backslash\nu/\mu} + \mu - \delta^-_J$. Then the association*

$x_\beta \mapsto x_{\gamma\beta}$ defines a morphism

$$\psi_{J\backslash\nu/\mu+\tau} \colon \ T_{J\backslash\nu/\mu+\tau} \ \longrightarrow \ T_J,$$

which does not depend on the choice of γ and which restricts (uniquely) to morphisms

$$\psi^w_{J\backslash\nu/\mu+\tau} \colon \ P_{J\backslash\nu/\mu+\tau} \ \longrightarrow \ P_J \qquad \text{and} \qquad \varphi_{J\backslash\nu/\mu+\tau} \colon \ F_{J\backslash\nu/\mu+\tau} \ \longrightarrow \ F_J.$$

In the case of a translation (i.e., if $\nu = \mu = 0$), all three morphisms are isomorphisms.

Proof. We can separate the formation of an embedded minor into deletion, contraction, and translation, which allows us to prove the claim for these cases separately. This is easiest for translations: we have $\delta^-_{J+\tau} = \delta^-_J + \tau$ and $\bar{r}' = \bar{r}$. Thus the association of the theorem is the identity map $x_\beta \mapsto x_\beta$. From the shape of the Plücker relations, it is evident that this map identifies $\widehat{T}_{J+\tau}$ with $\widehat{T}_J$ as tautologically isomorphic tracts. This isomorphism preserves (multi)degrees and the 3-term Plücker relations, and thus restricts to isomorphisms $T_{J+\tau} \to T_J$, $P_{J+\tau} \to P_J$, and $F_{J+\tau} \to F_J$.

As the next case we consider contractions J/μ. By Proposition 2.15, $\delta_{J/\mu} = \delta^-_J$, and thus $\Sigma\gamma = \mu$. We claim that the association $x_\beta \mapsto x_{\gamma\beta}$ defines a multiplicative map

$$\hat{\psi} \ = \ \hat{\psi}_{J/\mu} \colon \ \widehat{T}_{J/\mu} \ \longrightarrow \ \widehat{T}_J.$$

Firstly note that $x_{\gamma\beta} \neq 0$ for $\Sigma\boldsymbol{\beta} \in J/\mu$ since $\beta \mapsto \beta + \mu$ defines an injection $J/\mu \to J$. We need to show that $\hat{\psi}^+ \colon \widehat{T}^+_{J/\mu} \to \widehat{T}^+_J$ restricts to the respective nullsets, which can be tested on generators, i.e., elements of $T^+_{J/\mu}$ of the form

$$\mathrm{Pl}(\boldsymbol{\alpha}|i_0 \dots i_s|j_2 \dots j_s) := \sum_{k=0}^{s} (-1)^k \cdot x_{\boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\boldsymbol{\alpha} i_k j_2 \dots j_s} \tag{6}$$

for $2 \leqslant s \leqslant \bar{r}'$, $\boldsymbol{\alpha} \in \Delta_n^{\bar{r}'-s}$ and $i_0, \dots, i_s, j_2, \dots, j_s \in [n]$ such that $\Sigma\boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s \leqslant \omega_{J/\mu}$.

Note that every $\alpha \in J$ with $\alpha \geqslant \delta^-_J + \mu$ is in the image of $\iota_{J/\mu} \to J$. Thus a variable x_β that appears in (6) is nonzero if and only if $x_{\gamma\beta}$ is nonzero in $\widehat{T}_J$. Further, we have

$$\Sigma\boldsymbol{\gamma\alpha} i_0 \dots i_s j_2 \dots j_s \ \leqslant \ \omega_{J/\mu} + \mu \ \leqslant \ \omega_J$$

by Proposition 2.15. So the image of (6) under $\hat{\psi}$ is the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\boldsymbol{\gamma\alpha} i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\boldsymbol{\gamma\alpha} i_k j_2 \dots j_s},$$

which is in $N_{\widehat{T}_J}$. This shows that the map $\hat{\psi} \colon \widehat{T}_{J/\mu} \to \widehat{T}_J$ is a morphism of tracts.

Note that ψ depends on the choice of $\boldsymbol{\gamma}$ up to a sign, due to axiom (SR2). Since ψ is degree preserving, it restricts to the tract morphism $\psi_{J/\mu} \colon T_{J/\mu} \to T_J$, which does not depend on the choice of $\boldsymbol{\gamma}$. From the above argument, it is clear that $\hat{\psi}^+$ sends 3-term Plücker relations to

3-term Plücker relations, giving a tract morphism $\psi^w\colon P_{J/\mu} \to P_J$ between the respective universal pastures.

Since $P_{J/\mu}$ is generated by elements of the form $x_\beta/x_{\beta'}$ and the association $x_\beta/x_{\beta'} \mapsto x_{\gamma\beta}/x_{\gamma\beta'}$ is invariant under the rescaling action by $T(F) = (F^\times)^n$, the morphism $\psi^w\colon P_{J/\mu} \to P_J$ restricts to a morphism $\phi\colon F_{J/\mu} \to F_J$ between the respective foundations.

Finally, for deletions $J\backslash\nu$, the argument is analogous to that of contractions. In this case, $\Sigma\gamma = \delta^-_{J\backslash\nu} - \delta^-_J$. We claim that the association $x_\beta \mapsto x_{\gamma\beta}$ defines a tract morphism $\hat\psi\colon \widehat{T}_{J\backslash\nu} \to \widehat{T}_J$.

Since $J\backslash\nu \subseteq J$, $\hat\psi$ maps nonzero elements x_β of $\widehat{T}_{J\backslash\nu}$ to nonzero elements $x_{\gamma\beta}$ of $\widehat{T}_J$. We are left with verifying that $\hat\psi^+$ maps the generators

$$\mathrm{Pl}(\alpha|i_0\dots i_s|j_2\dots j_s) := \sum_{k=0}^{s} (-1)^k \cdot x_{\alpha i_0\dots\widehat{i_k}\dots i_s} \cdot x_{\alpha i_k j_2\dots j_s} \tag{7}$$

of the nullset of $\widehat{T}_{J\backslash\nu}$ to the nullset of $\widehat{T}_J$. Since

$$\Sigma\gamma\alpha i_0\dots i_s j_2\dots j_s \leqslant \delta^-_{J\backslash\nu} - \delta^-_J + \delta^+_{J\backslash\nu} - \delta^-_{J\backslash\nu} = \delta^+_J - \delta^-_J - \nu$$

by Proposition 2.15, we conclude that an element x_β in (7) is nonzero in $\widehat{T}_{J\backslash\nu}$ if and only if $x_{\gamma\beta}$ is nonzero in $\widehat{T}_J$, and that

$$\Sigma\gamma\alpha i_0\dots i_s j_2\dots j_s \leqslant \omega_J.$$

This shows that $\hat\psi^+$ maps (7) to the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\gamma\alpha i_0\dots\widehat{i_k}\dots i_s} \cdot x_{\gamma\alpha i_k j_2\dots j_s}$$

in $N_{\widehat{T}_J}$. This shows that the map $\hat\psi\colon \widehat{T}_{J\backslash\nu} \to \widehat{T}_J$ is a tract morphism.

The same arguments as in the case of contractions show that $\hat\psi$ induces tract morphisms $\psi\colon T_{J\backslash\nu} \to T_J$, $\psi^w\colon P_{J\backslash\nu} \to P_J$, and $\varphi\colon F_{J\backslash\nu} \to F_J$, which are independent of the choice of γ. □

Let $J\backslash\nu/\mu + \tau$ be an embedded minor of J and let F be a tract. Applying $\mathrm{Hom}(-, F)$ to the morphisms from Theorem 7.1 yields maps between the respective (strong and weak) thin Schubert cells and realization spaces by Proposition 5.2, Proposition 5.3, and Proposition 6.4, respectively:

$$\begin{array}{llll} \psi_{J\backslash\nu/\mu+\tau}\colon T_{J\backslash\nu/\mu+\tau} \longrightarrow T_J & \text{yields} & \psi^*_{J\backslash\nu/\mu+\tau}\colon \mathrm{Gr}_J(F) \longrightarrow \mathrm{Gr}_{J\backslash\nu/\mu+\tau}(F); \\ \psi^w_{J\backslash\nu/\mu+\tau}\colon P_{J\backslash\nu/\mu+\tau} \longrightarrow P_J & \text{yields} & \psi^{w,*}_{J\backslash\nu/\mu+\tau}\colon \mathrm{Gr}^w_J(F) \longrightarrow \mathrm{Gr}^w_{J\backslash\nu/\mu+\tau}(F); \\ \varphi_{J\backslash\nu/\mu+\tau}\colon F_{J\backslash\nu/\mu+\tau} \longrightarrow F_J & \text{yields} & \varphi^*_{J\backslash\nu/\mu+\tau}\colon \underline{\mathrm{Gr}}^w_J(F) \longrightarrow \underline{\mathrm{Gr}}^w_{J\backslash\nu/\mu+\tau}(F). \end{array}$$

We denote the image of a class $[\boldsymbol{\rho}]$ in a (strong or weak) thin Schubert cell or in the realization space under the corresponding map by $[\boldsymbol{\rho}]\backslash\nu/\mu+\tau$, and call it an *embedded minor of* $[\boldsymbol{\rho}]$.

7.2. Change of coordinates. In this section, we show that combinatorially equivalent M-convex sets have isomorphic universal tracts and foundations. We have established this already for translations in Theorem 7.1.

Proposition 7.2. *Let $\iota_n\colon \mathbb{N}^n \to \mathbb{N}^{n+1}$ be the embedding into the first n coordinates, $J' = \iota_n(J)$ and $\boldsymbol{\iota}_n\colon [n]^r \to [n+1]^r$ the tautological embedding. Then the association $x_\alpha \mapsto x_{\boldsymbol{\iota}_n(\alpha)}$ defines an isomorphism $T_J \to T_{J'}$, which restricts (uniquely) to isomorphisms $P_J \to P_{J'}$ and $F_J \to F_{J'}$.*

Proof. We claim that the association $x_\alpha \mapsto x_{\boldsymbol{\iota}_n(\alpha)}$ defines an isomorphism of tracts $\hat{\eta}\colon \widehat{T}_J \to \widehat{T}_{J'}$. Since $\iota_n\colon J \to J'$ is a bijection, x_α is nonzero in $\widehat{T}_J$ if and only if $x_{\boldsymbol{\iota}_n(\alpha)}$ is nonzero in $\widehat{T}_{J'}$. The image of the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\alpha i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\alpha i_k j_2 \dots j_s}$$

in $N_{\widehat{T}_J}$ under $\hat{\eta}^+$ is the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\boldsymbol{\iota}_n(\alpha i_0 \dots \widehat{i_k} \dots i_s)} \cdot x_{\boldsymbol{\iota}_n(\alpha i_k j_2 \dots j_s)}$$

in $N_{\widehat{T}_{J'}}$, where we note that $\omega_{J'} = \iota_n(\omega_J)$ and thus $\Sigma\boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s \leqslant \omega_J$ if and only if $\Sigma\boldsymbol{\iota}_n(\boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s) \leqslant \omega_{J'}$. This verifies that $\hat{\eta} : \widehat{T}_J \to \widehat{T}_{J'}$ is an isomorphism of tracts. It is clearly degree preserving and thus restricts to an isomorphism $T_J \to T_{J'}$, as well as isomorphisms $P_J \to P_{J'}$ (since it preserves 3-term Plücker relations) and $F_J \to F_{J'}$ (since it preserves the multi degree). □

Proposition 7.3. *Let $\sigma\colon \mathbb{N}^n \to \mathbb{N}^n$ be a coordinate permutation, $\boldsymbol{\sigma}\colon [n]^r \to [n]^r$ the induced bijection, and $J' = \sigma(J)$. Then the association $x_\alpha \mapsto x_{\boldsymbol{\sigma}(\alpha)}$ defines an isomorphism $T_J \to T_{J'}$, which restricts (uniquely) to isomorphisms $P_J \to P_{J'}$ and $F_J \to F_{J'}$.*

Proof. We claim that the association $x_\alpha \mapsto x_{\boldsymbol{\sigma}(\alpha)}$ defines an isomorphism of tracts $\hat{\tau}\colon \widehat{T}_J \to \widehat{T}_{J'}$. Since $\sigma\colon J \to J'$ is a bijection, x_α is nonzero in $\widehat{T}_J$ if and only if $x_{\boldsymbol{\sigma}(\alpha)}$ is nonzero in $\widehat{T}_{J'}$. The image of the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\alpha i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\alpha i_k j_2 \dots j_s}$$

in $N_{\widehat{T}_J}$ under $\hat{\tau}^+$ is the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\sigma(\alpha i_0 \ldots \widehat{i_k} \ldots i_s)} \cdot x_{\sigma(\alpha i_k j_2 \ldots j_s)}$$

in $N_{\widehat{T}_{J'}}$, where we note that $\omega_{J'} = \sigma(\omega_J)$ and thus $\Sigma\alpha i_0 \ldots i_s j_2 \ldots j_s \leqslant \omega_J$ if and only if $\Sigma\sigma(\alpha i_0 \ldots i_s j_2 \ldots j_s) \leqslant \omega_{J'}$. This verifies that $\hat{\tau}\colon \widehat{T}_J \to \widehat{T}_{J'}$ is an isomorphism of tracts. It is clearly degree preserving and thus restricts to an isomorphism $T_J \to T_{J'}$, as well as isomorphisms $P_J \to P_{J'}$ (since it preserves 3-term Plücker relations) and $F_J \to F_{J'}$ (since it preserves the multi degree). □

Applying $\mathrm{Hom}(-, F)$ to these isomorphisms yields due to Proposition 5.2, Proposition 5.3 and Proposition 6.4 canonical bijections

$$\begin{aligned} \mathrm{Gr}_{\iota_n(J)}(F) &\simeq \mathrm{Gr}_J(F), & \mathrm{Gr}_{\sigma(J)}(F) &\simeq \mathrm{Gr}_J(F), \\ \mathrm{Gr}^w_{\iota_n(J)}(F) &\simeq \mathrm{Gr}^w_J(F), & \mathrm{Gr}^w_{\sigma(J)}(F) &\simeq \mathrm{Gr}^w_J(F), \\ \underline{\mathrm{Gr}}^w_{\iota_n(J)}(F) &\simeq \underline{\mathrm{Gr}}^w_J(F), & \underline{\mathrm{Gr}}^w_{\sigma(J)}(F) &\simeq \underline{\mathrm{Gr}}^w_J(F) \end{aligned}$$

between the respective (strong and weak) thin Schubert cells and realization spaces.

Corollary 7.4. *The spaces and maps*

$$\mathrm{Gr}_J(F) \rightarrowtail \mathrm{Gr}^w_J(F) \twoheadrightarrow \underline{\mathrm{Gr}}^w_J(F)$$

are functorial in tract morphisms $F \to F'$ and polymatroid embeddings $J \to J'$.

Proof. The functoriality in F has been established in Section 5.2 and Section 6.1. The functoriality in polymatroid embeddings follows from Theorem 7.1, Proposition 7.2, and Proposition 7.3. □

Corollary 7.5. *Two combinatorially equivalent M-convex sets have isomorphic universal tract, universal pastures, and foundations.*

Proof. By Theorem 7.1, Proposition 7.2, and Proposition 7.3, two elementary equivalent M-convex sets have isomorphic universal tract, universal pastures, and foundations. The result follows by composing such isomorphisms. □

7.3. Duality. Let $\omega_J = \delta_J^- - \delta_J^+$ be the width of J and $d = |\omega_J|$. For $\boldsymbol{\beta} \in [n]^d$, we define the *signature of $\boldsymbol{\beta}$* as

$$\operatorname{sign} \boldsymbol{\beta} = \operatorname{sign} \sigma \in T_J,$$

where $\sigma \in S_d$ is a permutation such that $\boldsymbol{\beta}_{\sigma(1)} \leqslant \ldots \leqslant \boldsymbol{\beta}_{\sigma(d)}$. The signature of $\boldsymbol{\beta}$ is well-defined, since:

- σ is uniquely determined by strict inequalities between the $\boldsymbol{\beta}_i$ if J is the translate of a matroid;
- otherwise, $1 = -1$ by the idempotency principle (Proposition 4.7), and thus sign $\boldsymbol{\beta} = 1$, independently of the choice of σ.

Theorem 7.6. *Let J be M-convex, $\bar{r}$ its effective rank, and $d = |\omega_J|$. The association $x_{\boldsymbol{\beta}} \mapsto \operatorname{sign}(\boldsymbol{\beta}\boldsymbol{\beta}^*) \cdot x_{\boldsymbol{\beta}^*}$, where $\boldsymbol{\beta}^* \in [n]^{d-\bar{r}}$ satisfies $\Sigma\boldsymbol{\beta}^* = \delta_J - \Sigma\boldsymbol{\beta}$, defines an isomorphism $T_J \to T_{J^*}$, which is independent of the choices of the $\boldsymbol{\beta}$'s and which restricts (uniquely) to isomorphisms $P_J \to P_{J^*}$ and $F_J \to F_{J^*}$.*

Proof. Since $\alpha \mapsto \delta_J - \alpha$ defines a bijection $J \to J^*$, the association $x_{\boldsymbol{\beta}} \mapsto \operatorname{sign}(\boldsymbol{\beta}\boldsymbol{\beta}^*) \cdot x_{\boldsymbol{\beta}^*}$ with $\Sigma\boldsymbol{\beta}^* = \delta_J - \Sigma\boldsymbol{\alpha}$ defines a bijection between the variables of $\widehat{T}_J$ and $\widehat{T}_{J^*}$, where we note that axiom (SR2) identifies $\operatorname{sign}(\boldsymbol{\beta}\boldsymbol{\beta}^*) \cdot x_{\boldsymbol{\beta}^*}$ with $\operatorname{sign}(\boldsymbol{\beta}\tilde{\boldsymbol{\beta}}^*) \cdot x_{\tilde{\boldsymbol{\beta}}^*}$ if $\tilde{\boldsymbol{\beta}}^*$ is another choice of element in $[n]^{d-\bar{r}}$ with $\Sigma\tilde{\boldsymbol{\beta}}^* = \delta_J - \Sigma\boldsymbol{\alpha}$.

Consider the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s} \cdot x_{\boldsymbol{\alpha} i_k j_2 \dots j_s} \quad \in \quad \widehat{T}_J$$

with $2 \leqslant s \leqslant \bar{r}$, $\boldsymbol{\alpha} \in [n]^{\bar{r}-s}$, $i_0, \dots, i_s, j, \dots, j_s \in [n]$ with $\Sigma\boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s \leqslant \omega_J$. Let $\boldsymbol{\alpha}' \in [n]^{d-\bar{r}-s}$ with $\Sigma\boldsymbol{\alpha}' = \omega_J - \Sigma\boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s$. Then, evidently, $\Sigma\boldsymbol{\alpha}' i_0 \dots i_s j_2 \dots j_s \leqslant \omega_J$ and the association $x_{\boldsymbol{\beta}} \mapsto \operatorname{sign}(\boldsymbol{\beta}\boldsymbol{\beta}^*) \cdot x_{\boldsymbol{\beta}^*}$ sends $x_{\boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s}$ to $\eta \cdot x_{i_k j_2 \dots j_s \boldsymbol{\alpha}'}$ and $x_{\boldsymbol{\alpha} i_k j_2 \dots j_s}$ to $\eta \cdot x_{i_0 \dots \widehat{i_k} \dots i_s \boldsymbol{\alpha}'}$, where $\eta = \operatorname{sign}(\boldsymbol{\alpha} i_0 \dots i_s j_2 \dots j_s \boldsymbol{\alpha}')$. Thus the above Plücker relation corresponds to the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot x_{\boldsymbol{\alpha}' i_k j_2 \dots j_s} \cdot x_{\boldsymbol{\alpha}' i_0 \dots \widehat{i_k} \dots i_s} \quad \in \quad \widehat{T}_{J^*}.$$

This shows that the null sets of $\widehat{T}_J$ and $\widehat{T}_{J^*}$ agree, which establishes the desired isomorphism $\widehat{T}_J \simeq \widehat{T}_{J^*}$.

Since this isomorphism is degree preserving, it restricts to an isomorphism $T_J \simeq T_{J^*}$. Since the 3-term Plücker relations of T_J correspond to the 3-term Plücker relations of T_{J^*}, this isomorphism restricts further to an isomorphism $P_J \simeq P_{J^*}$. Since an element $\prod x_{\boldsymbol{\beta}_j}^{e_j}$ of P_J has multidegree zero if and only if its image $\prod x_{\boldsymbol{\beta}_j^*}^{e_j}$ in P_{J^*} has multidegree zero, $P_J \simeq P_{J^*}$ restricts to an isomorphism $F_J \simeq F_{J^*}$, which concludes the proof. □

Applying $\operatorname{Hom}(-, F)$ to these isomorphisms yields, by Proposition 5.2, Proposition 5.3 and Proposition 6.4, canonical bijections

$$\operatorname{Gr}_{J^*}(F) \;\simeq\; \operatorname{Gr}_J(F), \qquad \operatorname{Gr}^w_{J^*}(F) \;\simeq\; \operatorname{Gr}^w_J(F), \qquad \underline{\operatorname{Gr}}^w_{J^*}(F) \;\simeq\; \underline{\operatorname{Gr}}^w_J(F).$$

7.4. Direct sums. Thin Schubert cells and realization spaces of direct sums of M-convex sets decompose into products, as detailed in the following result.

Theorem 7.7. *Let* $J_1 \subseteq \Delta_{n_1}^{r_1}$ *and* $J_2 \subseteq \Delta_{n_2}^{r_2}$ *be M-convex and* $J = J_1 \oplus J_2 \subseteq \Delta_n^r$*. Then there are canonical bijections*

$$\begin{aligned} T_J &\simeq T_{J_1} \otimes T_{J_2}, & \mathrm{Gr}_J(F) &\simeq \mathrm{Gr}_{J_1}(F) \times \mathrm{Gr}_{J_2}(F), \\ P_J &\simeq P_{J_1} \otimes P_{J_2}, & \mathrm{Gr}_J^w(F) &\simeq \mathrm{Gr}_{J_1}^w(F) \times \mathrm{Gr}_{J_2}^w(F), \\ F_J &\simeq F_{J_1} \otimes F_{J_2}, & \underline{\mathrm{Gr}}_J^w(F) &\simeq \underline{\mathrm{Gr}}_{J_1}^w(F) \times \underline{\mathrm{Gr}}_{J_2}^w(F), \end{aligned}$$

which are tract isomorphisms (left column) and functorial in the tract F *(right column), respectively.*

Proof. The canonical isomorphism $T_J \simeq T_{J_1} \otimes T_{J_2}$ is equivalent to the functorial bijection $\mathrm{Gr}_J(F) \simeq \mathrm{Gr}_{J_1}(F) \times \mathrm{Gr}_{J_2}(F)$, since $\mathrm{Gr}_J(F) = \mathrm{Hom}(T_J, F)$ and

$$\mathrm{Gr}_{J_1}(F) \times \mathrm{Gr}_{J_2}(F) \;=\; \mathrm{Hom}(T_{J_1}, F) \times \mathrm{Hom}(T_{J_2}, F) \;=\; \mathrm{Hom}(T_{J_1} \otimes T_{J_2}, F)$$

by Proposition 5.2. The analogous equivalence holds for the other claims of the proposition.

We now establish the claims in the right-hand column. We begin with the bijection $\mathrm{Gr}_J(F) \simeq \mathrm{Gr}_{J_1}(F) \times \mathrm{Gr}_{J_2}(F)$. Consider representations $\boldsymbol{\rho}_1 \colon [n_1]^{r_1} \to F$ and $\boldsymbol{\rho}_2 \colon [n_2]^{r_2} \to F$ of J_1 and J_2, respectively, and define $\boldsymbol{\rho} \colon [n]^r \to F$ as the function that satisfies axiom (SR2),

$$\boldsymbol{\rho}(\boldsymbol{\alpha}\boldsymbol{\beta}') \;=\; \boldsymbol{\rho}_1(\boldsymbol{\alpha}) \cdot \boldsymbol{\rho}_2(\boldsymbol{\beta})$$

whenever $\boldsymbol{\alpha} \in [n_1]^{r_1}$, $\boldsymbol{\beta} \in [n_2]^{r_2}$ and $\boldsymbol{\beta}'_\ell = \boldsymbol{\beta}_\ell + n_1$ for all $\ell \in [r_2]$, and $\boldsymbol{\rho}(\boldsymbol{\gamma}) = 0$ whenever $\#\{\ell \in [r] \mid \boldsymbol{\gamma}_\ell \in [n_1]\} \neq r_1$. It is immediate from the definitions that the function $\boldsymbol{\rho}$ satisfies axioms (SR1) and (SR2) of a strong F-representation of J. Consider the Plücker relation

$$\sum_{k=0}^{s} (-1)^k \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s) \cdot \boldsymbol{\rho}(\boldsymbol{\alpha} i_k j_2 \dots j_s) \in N_F,$$

which contains a nontrivial term only if there is a k such that

$$\#\big\{\ell \in [r] \,\big|\, \boldsymbol{\alpha} i_0 \dots \widehat{i_k} \dots i_s)_\ell \in [n_1]\big\} \;=\; \#\big\{\ell \in [r] \,\big|\, (\boldsymbol{\alpha} i_k j_2 \dots j_s)_\ell \in [n_1]\big\} \;=\; r_1.$$

Depending on whether $i_k \leqslant n_1$ or $i_k > n_1$, this Plücker relation is equivalent to a corresponding Plücker relation for $\boldsymbol{\rho}_1$ or $\boldsymbol{\rho}_2$, respectively. Carrying out this comparison carefully leads to the conclusion that the Plücker relations for $\boldsymbol{\rho}$ are equivalent to the Plücker relations for $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$.

Therefore, the $F^\times$-class $[\boldsymbol{\rho}]$ is in $\mathrm{Gr}_J(F)$, and every class in $\mathrm{Gr}_J(F)$ stems from a unique pair of classes $[\boldsymbol{\rho}_1] \in \mathrm{Gr}_{J_1}(F)$ and $[\boldsymbol{\rho}_2] \in \mathrm{Gr}_{J_2}(F)$, which establishes the bijection $\mathrm{Gr}_J(F) \simeq \mathrm{Gr}_{J_1}(F) \times \mathrm{Gr}_{J_2}(F)$. It is evident that this bijection is functorial in F.

The canonical bijection $\mathrm{Gr}_J(F)^w \simeq \mathrm{Gr}_{J_1}^w(F) \times \mathrm{Gr}_{J_2}^w(F)$ can be established analogously (one only considers the 3-term Plücker relations for $s = 2$). The canonical bijection $\underline{\mathrm{Gr}}_J^w(F) \simeq$

$\underline{\mathrm{Gr}}^w_{J_1}(F) \times \underline{\mathrm{Gr}}^w_{J_2}(F)$ follows from this, since it is invariant under the action of the torus $(F^\times)^n \simeq (F^\times)^{n_1} \times (F^\times)^{n_2}$. □

8. Generators and relations for the foundation

A fundamental result about the foundation F_M of a matroid M is that it is generated as a tract over $\mathbb{F}_1^\pm$ by the cross ratios of M. In this section we generalize this result to polymatroids. Moreover, in the matroid case, we know a complete system of relations between the cross ratios, which determines the foundation. We show that these relations extend to the polymatroid case. At the time of writing, it is unknown to us whether this set of relations is complete.

8.1. Cross ratios. Let $J \subseteq \Delta_n^r$ be an M-convex set and $\mathbf{J} = \{\boldsymbol{\beta} \in [n]^r \mid \sum \boldsymbol{\beta} \in J\}$. Consider $\boldsymbol{\alpha} \in [n]^{r-2}$ and $i, j, k, l \in [n]$ such that $\boldsymbol{\alpha} ik,\ \boldsymbol{\alpha} jk,\ \boldsymbol{\alpha} il,\ \boldsymbol{\alpha} jl \in \mathbf{J}$. Then the element

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\boldsymbol{\alpha}} = \frac{x_{\boldsymbol{\alpha} ik} \cdot x_{\boldsymbol{\alpha} jl}}{x_{\boldsymbol{\alpha} il} \cdot x_{\boldsymbol{\alpha} jk}}$$

of $\widehat{P}_J$ is invertible and has multidegree $\deg_{[n]}\left(\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_{\boldsymbol{\alpha}}\right) = 0$; thus it is contained in $F_J^\times$. Note that a permutation of the coefficients of $\boldsymbol{\alpha}$ leads to a simultaneous sign change of all 4 terms in the definition of $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_{\boldsymbol{\alpha}}$, which shows that this element only depends on $\alpha = \sum \boldsymbol{\alpha} \in \Delta_n^{r-2}$.

Definition 8.1. Let Ω_J be the collection of all tuples (α, i, j, k, l) with $\alpha \in \Delta_n^{r-2}$ and $i, j, k, l \in [n]$ such that all of

$$\alpha + \varepsilon_i + \varepsilon_k, \qquad \alpha + \varepsilon_i + \varepsilon_l, \qquad \alpha + \varepsilon_j + \varepsilon_k, \qquad \alpha + \varepsilon_j + \varepsilon_l$$

are in J. We call $(\alpha, i, j, k, l) \in \Omega_J$ *non-degenerate* if also $\alpha + \varepsilon_i + \varepsilon_j$ and $\alpha + \varepsilon_k + \varepsilon_l$ are in J; otherwise we call (α, i, j, k, l) *degenerate*. We define $\Omega_J^\circ \subseteq \Omega_J$ as the subset of all non-degenerate elements.

Let $(\alpha, i, j, k, l) \in \Omega_J$ and $\boldsymbol{\alpha} \in [n]^{r-2}$ with $\sum \boldsymbol{\alpha} = \alpha$. The *cross ratio for* (α, i, j, k, l) is the element

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha} = \frac{x_{\boldsymbol{\alpha} ik} \cdot x_{\boldsymbol{\alpha} jl}}{x_{\boldsymbol{\alpha} il} \cdot x_{\boldsymbol{\alpha} jk}} \quad \in \quad F_J^\times.$$

8.2. Relations. The relations of [9, Thm. 4.21] between the cross ratios of a matroid extend to all polymatroids, as detailed in the following result.

Proposition 8.2. *Let $J \subseteq \Delta_n^r$ be an M-convex set with foundation F_J. Then the following relations between the cross ratios hold:*

(CRσ) *If* $(\alpha, i, j, k, l) \in \Omega_J^\diamond$*, then* $\big(\alpha, \sigma(i), \sigma(j), \sigma(k), \sigma(l)\big) \in \Omega_J^\diamond$ *for every permutation* $\sigma \in S_4$ *and*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\alpha = \begin{bmatrix} k & l \\ i & j \end{bmatrix}_\alpha = \begin{bmatrix} j & i \\ l & k \end{bmatrix}_\alpha = \begin{bmatrix} l & k \\ j & i \end{bmatrix}_\alpha.$$

(CR-) *If J has an embedded minor that is isomorphic or dual to the Fano matroid, then*

$$1 = -1.$$

If J is a proper polymatroid, then

$$1 = -1 \qquad \text{and} \qquad 1 + 1 + 1 \in N_{F_J}.$$

(CR+) *If* $(\alpha, i, j, k, l) \in \Omega_J^\diamond$*, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\alpha + \begin{bmatrix} i & k \\ j & l \end{bmatrix}_\alpha - 1 \in N_{F_J}.$$

(CR0) *If* $(\alpha, i, j, k, l) \in \Omega_J$ *is degenerate, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\alpha = 1.$$

(CR1) *If* $(\alpha, i, j, k, l) \in \Omega_J^\diamond$*, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\alpha \cdot \begin{bmatrix} i & j \\ l & k \end{bmatrix}_\alpha = 1.$$

(CR2) *If* $(\alpha, i, j, k, l) \in \Omega_J^\diamond$*, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\alpha \cdot \begin{bmatrix} i & l \\ k & j \end{bmatrix}_\alpha \cdot \begin{bmatrix} i & k \\ j & l \end{bmatrix}_\alpha = -1.$$

(CR3) *If* $(\alpha, i, j, k, l), (\alpha, i, j, l, m), (\alpha, i, j, m, k) \in \Omega_J$*, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_\alpha \cdot \begin{bmatrix} i & j \\ l & m \end{bmatrix}_\alpha \cdot \begin{bmatrix} i & j \\ m & k \end{bmatrix}_\alpha = 1.$$

(CR4) *If* $(\alpha + \varepsilon_m, i, j, k, l), (\alpha + \varepsilon_k, i, j, l, m), (\alpha + \varepsilon_l, i, j, m, k) \in \Omega_J$ *for* $\alpha \in \Delta_n^{r-3}$*, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha+\varepsilon_m} \cdot \begin{bmatrix} i & j \\ l & m \end{bmatrix}_{\alpha+\varepsilon_k} \cdot \begin{bmatrix} i & j \\ m & k \end{bmatrix}_{\alpha+\varepsilon_l} = 1.$$

(CR5) *If* $(\alpha + \varepsilon_p, i, j, k, l), (\alpha + \varepsilon_q, i, j, k, l) \in \Omega_J^\diamond$ *for* $\alpha \in \Delta_n^{r-3}$ *and if both* $(\alpha + \varepsilon_i, k, l, p, q)$, $(\alpha + \varepsilon_j, k, l, p, q) \in \Omega_J$ *are degenerate, then*

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha+\varepsilon_p} = \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha+\varepsilon_q}.$$

Proof. These relations between cross ratios are known if J is a matroid; cf. [9, Thm. 4.21]. The proof for polymatroids is similar. For completeness we sketch the argument.

The relations (CRσ) and (CR1)–(CR4) follow from a direct verification. Relation (CR-) is proven for matroids in [9, Thm. 4.21]; for proper polymatroids it follows from the idempotency principle Proposition 4.7. In order to show (CR+) and (CR0), we divide the 3-term Plücker relation

$$x_{\alpha ik} \cdot x_{\alpha jl} \; - \; x_{\alpha il} \cdot x_{\alpha jk} \; + \; x_{\alpha ij} \cdot x_{\alpha kl} \quad \in \quad N_{\widehat{P}_J}$$

by $x_{\alpha il} \cdot x_{\alpha jk}$ (where we assume that $i \leqslant j \leqslant l \leqslant k$; the other cases are similar). This yields the relation (CR+) if (α, i, j, k, l) is non-degenerate (where $\alpha = \sum \boldsymbol{\alpha}$) and which yields

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha} - 1 \quad \in \quad N_{F_J}$$

if (α, i, j, k, l) is degenerate, and thus (CR0). Relation (CR5) follows from the direct computation

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha+\varepsilon_p} \cdot \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha+\varepsilon_q}^{-1} = \frac{x_{\alpha ikp} \cdot x_{\alpha jlp} \cdot x_{\alpha ilq} \cdot x_{\alpha jkq}}{x_{\alpha ilp} \cdot x_{\alpha jkp} \cdot x_{\alpha ikq} \cdot x_{\alpha jlq}} = \begin{bmatrix} k & l \\ p & q \end{bmatrix}_{\alpha+\varepsilon_i} \cdot \begin{bmatrix} k & l \\ q & p \end{bmatrix}_{\alpha+\varepsilon_j} = 1,$$

where we apply (CR1) to express the inverse of $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_{\alpha+\varepsilon_q}$ and (CR0) to identify the latter two degenerate cross ratios with 1. □

In the matroid case, the foundation is completely described by Proposition 4.7 (cf. [9, Thm. 4.21]):

(1) The cross ratios generate the foundation as a tract over $\mathbb{F}_1^{\pm}$.
(2) Relations (CRσ), (CR-) and (CR0)–(CR5) form a complete system of multiplicative relations between the cross ratios.
(3) The 3-term relations (CR+) generate the null set of the foundation.

The last result (3) follows at once from the definition of the extended universal pasture in terms of 3-term Plücker relations and generalizes to polymatroids; cf. the proof of Lemma 6.5. The first result (1) is a consequence of Tutte's path theorem ([43, (5.1)]), and we generalize (1) to polymatroids in Theorem 8.4. The second result (2) is a consequence of Tutte's homotopy theorem ([43, (6.1)]), and we do not know at the time of writing if it generalizes to all polymatroids.

Problem 8.3. Do the relations (CRσ), (CR-) and (CR0)–(CR5) form a complete system of multiplicative relations between the cross ratios for every polymatroid? If not, then what are the missing relations?

8.3. Generators.

Theorem 8.4. *Let J be an M-convex set. Then the foundation F_J of J is generated by the cross ratios $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ with $(\alpha, i, j, k, l) \in \Omega_J^\diamond$ over $\mathbb{F}_1^\pm$.*

Proof. For ease of notation, we begin with the following simplification. The association $x_\alpha \to x_\alpha$ defines an isomorphism of foundations $F_{J+\tau} \to F_J$, where $J + \tau$ is a translate of J (cf. Theorem 7.1 for a detailed proof). This map identifies the cross ratio $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ in $F_{J+\tau}$ with the corresponding cross ratio $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ in F_J. This allows us to subtract δ_J^- from J and assume without loss of generality that $\delta_J^- = 0$.

We proceed by induction on δ_J^+ with respect to the partial order of $\mathbb{N}^n$. If $\delta_J^+ \leqslant \mathbf{1}$, then J is a matroid, for which the result is proven in [9, Thm. 4.21].

If $d = \delta_{J,i}^+ \geqslant 2$ for some $i \in [n]$, then $J \backslash \varepsilon_i$ has $\delta_{J\backslash\varepsilon_i}^+ \leqslant \delta_J^+ - \varepsilon_i < \delta_J^+$, and thus we can assume by the inductive hypothesis that $F_{J\backslash\varepsilon_i}$ is generated by its cross ratios over $\mathbb{F}_1^\pm$. By the idempotency principle (Proposition 4.7), $1 = -1$ in F_J. This allows us to label the generators x_β of the extended universal pasture $\widehat{P}_J$ of J by elements $\beta \in J$, which simplifies the notation in the arguments to follow. Recall from the proof of Theorem 7.1 that the minor embedding $J\backslash\varepsilon_i \to J$ induces a morphism $\widehat{T}_{J\backslash\varepsilon_i} \to \widehat{T}_J$ (which does not depend on any choice of signs in this case since $-1 = 1$), and thus a morphism $\widehat{P}_{J\backslash\varepsilon_i} \to \widehat{P}_J$ (using the bijection $\widehat{T}_J \to \widehat{P}_J$ from Theorem 5.4).

Consider an element $x = \prod_{\beta\in J} x_\beta^{m_\beta}$ in F_J. We claim that in $\widehat{P}_J$, the elements x_β with $\beta_i = d$ can be expressed as a product of elements in the image of $\widehat{P}_{J\backslash\varepsilon_i} \to \widehat{P}_J$ with a cross ratio. Since $F_{J\backslash\varepsilon_i}$ is generated by cross ratios over $\mathbb{F}_1^\pm$, this substitution of the variable x_β with $\beta_i = d$ shows that x can be written as a product of cross ratios.

In order to find such an expression for x_β, we consider some $\gamma \in J$ with $\gamma_i = 0$, which exists by our assumption that $\delta_J^- = 0$. Then $\gamma_i = 0 < 2 \leqslant \beta_i$, and thus the M-convexity of J implies that there exists $k \in [n]$ with $\gamma_k > \beta_k$ and $\beta' := \beta - \varepsilon_i + \varepsilon_k \in J$. Since $\gamma_i = 0 < 1 \leqslant \beta_i'$, we find $l \in [n]$ with $\gamma_l > \beta_l'$ and

$$\beta'' := \beta' - \varepsilon_i + \varepsilon_l = \beta - 2\varepsilon_i + \varepsilon_k + \varepsilon_l \in J.$$

Since $\beta_l'' > \beta_l$ and i is the only index in $[n]$ that satisfies $\beta_i'' < \beta_i$, also $\beta''' := \beta - \varepsilon_i + \varepsilon_l \in J$. This shows that $(\alpha, i, k, i, l) \in \Omega_J$ for $\alpha = \beta - 2\varepsilon_i$, and thus

$$\begin{bmatrix} i & k \\ i & l \end{bmatrix}_\alpha = \frac{x_{\alpha ii} \cdot x_{\alpha kl}}{x_{\alpha il} \cdot x_{\alpha ki}} \in J.$$

Therefore, dividing x by the cross ratio $\left[\begin{smallmatrix} i & k \\ i & l \end{smallmatrix}\right]_\alpha$ replaces x_β by $x_{\alpha il}x_{\alpha ki}/x_{\alpha kl}$, which is in the image of $\widehat{P}_{J\backslash\varepsilon_i} \to \widehat{P}_J$. This completes the inductive step and concludes the proof of the theorem. □

8.4. Non-degenerate cross ratios. Since degenerate cross ratios are equal to 1 (Proposition 8.2), the foundation is generated by the non-degenerate cross ratios (Theorem 8.4). Non-degenerate cross ratios stem from certain types of embedded minors of J, which we classify in this section. This has been done in the matroid case (cf. [9]): every non-degenerate cross is in the image of the canonical map $F_{J\backslash\nu/\mu} \longrightarrow F_J$ for a minor $J\backslash\nu/\mu$ of type $U_{2,4}$. In the polymatroid case, we find two additional types. Recall that

$$U_{2,3}^{+} \;=\; \big\{(2,0,0),\ (1,1,0),\ (1,0,1),\ (0,1,1)\big\}.$$

In Section 8.5.1, we determine the foundation of both $U_{2,3}^{+}$ and Δ_2^2 as $\mathbb{K}(x)/\!\!/\langle x+1+1\rangle$.

Proposition 8.5. *A cross ratio $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ in F_J that is not equal to* 1 *is in the image of the canonical map $F_{J'} \longrightarrow F_J$ for an embedded minor $J' = J\backslash\nu/\mu+\tau$ of J that is of type $U_{2,4}$, $U_{2,3}^{+}$, or Δ_2^2.*

Proof. Let $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ be a cross ratio of F_J that is not equal to 1. Whenever we find an embedded minor J' of J such that $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ is in the image of the canonical map $F_{J'} \longrightarrow F_J$, we can replace J by J' until we have arrived at one of the three M-convex sets in the claim of the proposition.

To begin with, the cross ratio $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_\alpha$ in F_J is the image of the cross ratio $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right] \in F_{J/\alpha}$ under the canonical map $\varphi_{J/\alpha}\colon F_{J/\alpha} \longrightarrow F_J$, which allows us to assume that $\alpha = 0$ and that J is of rank 2.

By (CR0), $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]$ has to be non-degenerate, i.e., J contains

$$S \;=\; \big\{\,\varepsilon_i+\varepsilon_j,\ \ \varepsilon_i+\varepsilon_k,\ \ \varepsilon_i+\varepsilon_l,\ \ \varepsilon_j+\varepsilon_k,\ \ \varepsilon_j+\varepsilon_l,\ \ \varepsilon_k+\varepsilon_l\,\big\}.$$

Any other element of J does not matter, which means that $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]$ lies in the image of $F_{J\backslash\nu} \longrightarrow F_J$ for $\nu = \delta_J^+ - \sup S$. This allows us to assume that $\delta_J^+ = \sup S$ and, after permuting $[n]$ and restricting the support suitably, that $[n] = \{i,j,k,l\}$.

Since

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix} \;=\; \frac{x_{ik}\cdot x_{jl}}{x_{il}\cdot x_{jk}}$$

is not equal to 1, we can assume that $i \neq j$ and that $k \neq l$. If i, j, k and l are pairwise distinct, then $J = U_{2,4}$.

If two of i, j, k and l are equal, then we can apply (CRσ) and (CR1) (i.e., passing to the multiplicative inverse) and assume that $i \neq j = k \neq l$. In this case, J is one of the last two M-convex sets of the proposition, depending on whether $i \neq l$ or $i = l$. □

8.5. More examples of foundations. In this section we compute the foundations of some proper polymatroids.

8.5.1. *The foundation of $U^+_{2,3}$ and of Δ^2_2.* Consider $J = U^+_{2,3}$ or $J = \Delta^2_2$, which is a proper polymatroid in either case. Thus its foundation F_J is near-idempotent by Proposition 4.7. Consider a cross ratio

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix} = \frac{x_{ik} \cdot x_{jl}}{x_{il} \cdot x_{jk}} \quad \in \quad F_J$$

of J that is not equal to 1. Then we have $i \neq j$ and $k \neq l$ and it is non-degenerate by (CR0), and so are all cross-ratios obtained by permuting i, j, k, l by (CRσ). After reordering the rows (using (CRσ)) and possibly exchanging k and l (which exchanges $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]$ by its inverse by (CR1)), we can assume that $i \neq j = k \neq l$. This implies that $i \neq l$ if $J = U_{2,3}$ and $i = l$ if $J = \Delta^2_2$. Finally, since

$$\begin{bmatrix} i & j \\ j & l \end{bmatrix} = \begin{bmatrix} l & j \\ j & i \end{bmatrix}$$

by (CRσ), we find that the foundation is generated by $\left[\begin{smallmatrix} i & j \\ j & l \end{smallmatrix}\right]$, or, equivalently, by its multiplicative inverse $x = \left[\begin{smallmatrix} i & j \\ l & j \end{smallmatrix}\right]$. Note that the degree of x_{jj} in $\left[\begin{smallmatrix} i & j \\ j & l \end{smallmatrix}\right] = \frac{x_{ij}x_{jl}}{x_{il}x_{jj}}$ maps the powers of x bijectively to $\mathbb{Z}$, which shows that there are no further multiplicative relations.

The unique Plücker relation of J is

$$x_{il} \cdot x_{jj} \ + \ x_{ij} \cdot x_{jl} \ + \ x_{ij} \cdot x_{jl} \quad \in \quad N_{\widehat{P}_J},$$

which is equivalent to $1 + 1 + x \in N_{F_J}$, after dividing by $x_{ij} \cdot x_{jl}$. We therefore find that

$$F_J \ = \ \mathbb{F}_2(x) /\!\!/ \langle 1 + 1 + x \rangle \ \simeq \ \mathbb{F}_2 \otimes \mathbb{D}$$

for both $J = U^+_{2,3}$ and $J = \Delta^2_2$.

8.5.2. *The foundation of $\Delta^2_3 \backslash \varepsilon_2$.* Consider the proper polymatroid

$$J \ = \ \Delta^2_3 \backslash \varepsilon_2 \ = \ \big\{(2,0,0),\ (1,1,0),\ (1,0,1),\ (0,1,1),\ (0,0,2)\big\},$$

whose foundation is near-idempotent. By Proposition 8.5, the nontrivial cross ratios stem from the embedded minors of J of types $U_{2,4}$, $U^+_{2,3}$, and Δ^2_2, which are

$$\begin{aligned} J\backslash\varepsilon_1 &= \big\{(1,1,0),\ (1,0,1),\ (0,1,1),\ (0,0,2)\big\} && \text{(type } U^+_{2,3}\text{);} \\ J\backslash\varepsilon_2 &= \big\{(2,0,0),\ (1,0,1),\ (0,0,2)\big\} && \text{(type } \Delta^2_2\text{);} \\ J\backslash\varepsilon_3 &= \big\{(2,0,0),\ (1,1,0),\ (1,0,1),\ (0,1,1)\big\} && \text{(type } U^+_{2,3}\text{).} \end{aligned}$$

The corresponding cross ratios are $x = \left[\begin{smallmatrix} 2 & 3 \\ 1 & 3 \end{smallmatrix}\right]$, $y = \left[\begin{smallmatrix} 1 & 3 \\ 1 & 3 \end{smallmatrix}\right]$, $z = \left[\begin{smallmatrix} 2 & 1 \\ 3 & 1 \end{smallmatrix}\right]$, which satisfy $x + 1 + 1$, $y + 1 + 1$, $z + 1 + 1 \in N_{F_J}$ by Section 8.5.1. By (CR3), we have

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \ = \ 1,$$

and thus (using (CRσ) and (CR1)), $y = xz$. There is no further relation between x (which involves x_{11}) and z (which involves x_{33}). Thus we find that the foundation of $J = \Delta_3^2 \backslash \varepsilon_2$ is

$$F_J \;=\; \mathbb{F}_2(x,z)/\!\!/\langle 1+1+x,\ 1+1+xz,\ 1+1+z\rangle.$$

8.5.3. *The foundation of* Δ_3^2. The polymatroid $J = \Delta_3^2$ has 3 embedded minors $J\backslash(\varepsilon_i+\varepsilon_j)$ of type $U_{2,3}^+$ and 3 embedded minors $J\backslash 2\varepsilon_k$ of type $\Delta_{2,2}$, where $\{i,j,k\} = \{1,2,3\}$. We denote the corresponding cross ratios by $x_k = \left[\begin{smallmatrix} i & k \\ j & k\end{smallmatrix}\right]$ and $y_k = \left[\begin{smallmatrix} i & j \\ i & j\end{smallmatrix}\right]$. For the same reasons as in the previous example, they satisfy the relations $y_k = x_i x_j$, $x_k+1+1 \in N_{F_J}$, and $y_k+1+1 \in N_{F_J}$, which determines the foundation F_J of Δ_3^2 as

$$\mathbb{F}_2(x_1,x_2,x_3)/\!\!/\langle x_1+1+1,\ x_2+1+1,\ x_3+1+1,\ x_1x_2+1+1,\ x_1x_3+1+1,\ x_2x_3+1+1\rangle.$$

8.5.4. *The foundation of* Δ_2^3. The polymatroid $J = \Delta_2^3$ has 2 embedded minors of type Δ_2^2, which are J/ε_1 and J/ε_2, and none of types $U_{2,3}^+$ or $U_{2,4}$. The corresponding cross ratios are $x = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2\end{smallmatrix}\right]_{\varepsilon_1}$ and $y = \left[\begin{smallmatrix} 1 & 1 \\ 2 & 2\end{smallmatrix}\right]_{\varepsilon_2}$, and their respective inverses.

By the idempotency principle, the foundation F_J of J is idempotent and, in particular, $-1 = 1$. Since the variables x_{111} and x_{222} only occur in one of x and y, taking the multidegree in x_{111} and x_{222} defines an group isomorphism $F_J^\times \to \mathbb{Z}^2$.

The Plücker relations are parametrized by $\alpha \in \{\varepsilon_1,\varepsilon_2\}$ and $i,j,k,l \in \{1,2\}$ such that $\alpha+\varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon_l \leqslant \omega_J = (3,3)$. If $\alpha = \varepsilon_1$, then up to permutation of i,j,k,l, we have either $i = 1$ and $j = k = l = 2$ or $i = j = 1$ and $k = l = 2$. In the former case, the Plücker relation is equivalent to $1+1+1 \in N_{F_J}$ (which we know already from the idempotency principle), and the latter relation is equivalent to $1+1+x \in N_{F_J}$. Similarly the Plücker relations for $\alpha = \varepsilon_2$ yield $1+1+1 \in N_{F_J}$ and $1+1+y \in N_{F_J}$. Thus we find

$$F_{\Delta_2^3} \;=\; \mathbb{K}(x,y)/\!\!/\langle 1+1+x,\ 1+1+y\rangle.$$

Part 4. Canonical embeddings

In this last part of the paper, we describe canonical embeddings of the representation space $\mathrm{R}_J^w(F)$ and of the realization space $\underline{\mathrm{Gr}}_J^w(F)$ into tori, by which we mean sets of the form $(F^\times)^N$ for some N. This is of particular interest if F (and thus $F^\times$) carries a topology, since the torus embeddings then endow the representation / realization spaces with a topology (cf. [4] and [5], where such a study is carried out in the case of triangular hyperfields).

Further, we discuss the Plücker embedding $\mathrm{pl}_J\colon\ \mathrm{Gr}_J^w(F) \to \mathbb{P}^N(F)$ (for $N = \#\Delta_n^r - 1$) and a decomposition of $\mathrm{R}_J^w(F)$ into the product of $\underline{\mathrm{Gr}}_J^w(F)$ with a torus.

9. The torus embedding of the representation space

We call a group isomorphic to $(F^\times)^s$ (for some $s \geqslant 0$) a *torus over* F. Let $J \subseteq \Delta_n^r$ be an M-convex set, $\bar{J}$ its reduction, and $\bar{r}$ its effective rank. By the definition of a strong (resp. weak) F-representation of J as a function $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$, the representation space $\mathrm{R}_J(F)$ (resp. $\mathrm{R}_J^w(F)$) can be considered as a subspace of $F^{n^{\bar{r}}}$. Axiom (SR1) (or, equivalently, (WR1)) determines the non-vanishing coordinates of $\boldsymbol{\rho}$ as those whose indices α lie in $\mathbf{J} = \{\alpha \in [n]^{\bar{r}} \mid \alpha_1 + \cdots + \alpha_r \in \bar{J}\}$. This results in the canonical embedding

$$\mathrm{R}_J(F) \ \subseteq\ \mathrm{R}_J^w(F) \ \longrightarrow\ (F^\times)^{\mathbf{J}}$$

of the (strong resp. weak) representation space into the torus $(F^\times)^{\mathbf{J}}$. Axiom (SR2) (or, equivalently, (WR2)) implies that $\mathrm{R}_J(F)$ (resp. $\mathrm{R}_J^w(F)$) is contained in the subgroup $C_{\mathbf{J}}(F)$ of $(F^\times)^{\mathbf{J}}$ that consists of all $\boldsymbol{\rho} \in (F^\times)^{\mathbf{J}}$ that satisfy

$$\boldsymbol{\rho}(i_{\sigma(1)}, \dots, i_{\sigma(r)}) = \mathrm{sign}(\sigma) \cdot \boldsymbol{\rho}(i_1, \dots, i_r),$$

which is a subgroup of $(F^\times)^{\mathbf{J}}$. Choosing an $\boldsymbol{\alpha} \in \mathbf{J}$ with $\sum \boldsymbol{\alpha} = \alpha$ for each $\alpha \in \bar{J}$, e.g. the unique such $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}_1 \leqslant \ldots \leqslant \boldsymbol{\alpha}_r$, yields the torus embedding $\mathrm{R}_J^w(F) \hookrightarrow (F^\times)^{\bar{J}}$. Note that if $1 \neq -1$ in F, then the signs of the coordinates depend on the choices of the $\boldsymbol{\alpha}$.

9.1. The degeneracy locus. There is yet a smaller subgroup of $\mathrm{D}_J(F) \subseteq (F^\times)^{\mathbf{J}}$ that contains the representation space. It is cut out by the degenerate 3-term Plücker relations, which force the two nonzero terms to be additive inverses of each other. These relations are of the form

$$\boldsymbol{\rho}(\alpha ik) \cdot \boldsymbol{\rho}(\alpha jl) \ =\ \boldsymbol{\rho}(\alpha il) \cdot \boldsymbol{\rho}(\alpha jk)$$

for $\alpha \in \Delta_n^{\bar{r}-2}$, assuming that $\boldsymbol{\rho}(\alpha ij) \cdot \boldsymbol{\rho}(\alpha kl) = 0$. The *degeneracy locus of J over F* is defined as the subgroup

$$\mathrm{D}_J(F) \ =\ \left\{ \boldsymbol{\rho} \in (F^\times)^{\mathbf{J}} \,\middle|\, \begin{array}{c} \boldsymbol{\rho} \text{ satisfies (WR2) and all} \\ \text{degenerate 3-term Plücker relations} \end{array} \right\}$$

of $(F^\times)^{\mathbf{J}}$. Summing up, this yields a chain of inclusions

$$\mathrm{R}_J(F) \ \subseteq\ \mathrm{R}_J^w(F) \ \subseteq\ \mathrm{D}_J(F) \ \subseteq\ (F^\times)^{\mathbf{J}} \ \subseteq\ F^{n^r}.$$

The following fact verifies that the degeneracy locus does not get smaller if we require *all* degenerate Plücker relations to hold.

Corollary 9.1. *Let J be an M-convex set of F a tract. Then*

$$\mathrm{D}_J(F) \ =\ \left\{ \boldsymbol{\rho} \in (F^\times)^{\mathbf{J}} \,\middle|\, \boldsymbol{\rho} \textit{ satisfies (WR2) and all degenerate Plücker relations} \right\}.$$

Proof. This follows at once from Theorem 5.4: every degenerate Plücker relation is contained in the ideal generated by the degenerate 3-term Plücker relations. □

Proposition 9.2. *Let J be an M-convex set with extended universal pasture $\widehat{P}_J$ and let F be a tract. Then there is a canonical bijection*

$$\mathrm{D}_J(F) \longrightarrow \{\textit{group homomorphisms } f\colon \widehat{P}_J^\times \to F^\times \textit{ with } f(-1) = -1\}.$$

If F is a degenerate tract, then $\mathrm{D}_J(F) = \mathrm{R}_J^w(F) = \mathrm{R}_J(F)$.

Proof. Let G be the (multiplicatively written) free abelian group generated by symbols -1 and x_β for $\beta \in \mathbf{J}$. An element $\rho \in \mathrm{D}_J(F)$ defines a group homomorphism $f_\rho\colon G \to F^\times$ with $f_\rho(-1) = -1$ and $f_\rho(x_\beta) = \rho(\beta)$. Since $(f_\rho(-1))^2 = (-1)^2 = 1$ and

$$f_\rho\left(\frac{x_{\alpha ik}\cdot x_{\alpha jl}}{x_{\alpha il}\cdot x_{\alpha jk}}\right) = \frac{\rho(\alpha ik)\cdot\rho(\alpha jl)}{\rho(\alpha il)\cdot\rho(\alpha jk)} = 1$$

whenever $\rho(\alpha ik)\cdot\rho(\alpha jl)$ and $\rho(\alpha il)\cdot\rho(\alpha jk)$ are the two nonzero terms of a degenerate 3-term Plücker relation, the group homomorphism f_ρ factors through a uniquely determined group homomorphism

$$\bar{f}_\rho\colon \widehat{P}_J^\times = G/H_J^w \longrightarrow F$$

with $\bar{f}_\rho(-1) = -1$, where H_J^w is the subgroup of G generated by the elements $(-1)^2$ and the degenerate cross ratios $\frac{x_{\alpha ik}\cdot x_{\alpha jl}}{x_{\alpha il}\cdot x_{\alpha jk}}$.

This defines the canonical map given in the statement of the proposition. It is injective since ρ can be recovered from $\bar{f}_\rho$ via $\rho_\beta = \bar{f}_\rho(x_\beta)$. It is surjective since all defining relations between the generators x_β of $\widehat{P}_J^\times$ hold for the coefficients ρ_β of $\rho \in \mathrm{D}_J(F)$.

The claim $\mathrm{D}_J(F) = \mathrm{R}_J^w(F)$ follows from the fact that the non-degenerate 3-term Plücker relations are vacuous if F is degenerate. By Corollary 5.5, F is excellent, i.e., $\mathrm{R}_J^w(F) = \mathrm{R}_J(F)$. □

9.2. The lineality space. If F is idempotent, then $\mathrm{R}_J(F)$ contains a certain subgroup of the ambient torus $(F^\times)^{\mathbf{J}}$, which we call the lineality space.

Recall from Section 6.1 that the torus $\widehat{T}(F) = F^\times \times (F^\times)^n$ acts on $\mathrm{R}_J(F)$ by the formula $(a,t).\rho(\alpha) = a\cdot\left(\prod_{i=1}^{\bar{r}} t_{\alpha_i}\right)\cdot\rho(\alpha)$.

If F is idempotent, then there is a (necessarily unique) morphism $i_F\colon \mathbb{K} \to F$. The composition of the unique $\mathbb{K}$-representation $\chi_J\colon [n]^{\bar{r}} \to \mathbb{K}$ of J with i_F yields the *trivial F-representation* $\chi_{J,F}\colon [n]^{\bar{r}} \to F$ given by $\chi_{J,F}(\alpha) = 1$ if $\sum\alpha \in \bar{J}$ and $\chi_{J,F}(\alpha) = 0$ otherwise.

Definition 9.3. The *lineality space of* $\mathrm{R}_J(F)$ is the orbit $\mathrm{Lin}_J(F) = \widehat{T}(F).\chi_{J,F}$, which is a subgroup of $(F^\times)^{\mathbf{J}}$ that is contained in $\mathrm{R}_J(F)$. The thin Schubert cell $\mathrm{Gr}_J(F) = \mathrm{R}_J(F)/F^\times$ contains the quotient torus $\widehat{T}(F).\chi_{J,F}/F^\times$, which we call the *lineality space of* $\mathrm{Gr}_J(F)$.

Example 9.4. If J is a matroid, then the bijection $-\log\colon \operatorname{Gr}_J(F) \to \operatorname{Dr}_J$ (cf. Section 4.5) identifies the lineality space of $\operatorname{Gr}_J(\mathbb{T}_0)$ with the lineality space of the local Dressian Dr_J, which consists of all valuated matroids with underlying matroid J (cf. [18] for details on local Dressians and their lineality spaces).

Whether or not a weak F-representation $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ belongs to the lineality space $\operatorname{Lin}_J(F)$ can be evaluated in terms of the triviality of the *cross ratios of* $\boldsymbol{\rho}$, which are the elements

$$\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha,\boldsymbol{\rho}} = \frac{\boldsymbol{\rho}(\boldsymbol{\alpha} ik)\cdot\boldsymbol{\rho}(\boldsymbol{\alpha} jl)}{\boldsymbol{\rho}(\boldsymbol{\alpha} il)\cdot\boldsymbol{\rho}(\boldsymbol{\alpha} jk)} \quad \in \quad F^\times$$

for $(\alpha,i,j,k,l) \in \Omega_J$ and $\boldsymbol{\alpha} \in \Delta_n^{\bar{r}-2}$ with $\sum\boldsymbol{\alpha} = \alpha$. If $f_{\boldsymbol{\rho}}\colon \widehat{P}_J \to F$ is the tract morphism associated with $\boldsymbol{\rho}$ (see Proposition 5.3), which is given by $f_{\boldsymbol{\rho}}(x_{\boldsymbol{\beta}}) = \boldsymbol{\rho}(\beta)$, then

$$f_{\boldsymbol{\rho}}\Big(\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha}\Big) = f_{\boldsymbol{\rho}}\Big(\frac{x_{\boldsymbol{\alpha} ik}\cdot x_{\boldsymbol{\alpha} jl}}{x_{\boldsymbol{\alpha} il}\cdot x_{\boldsymbol{\alpha} jk}}\Big) = \frac{\boldsymbol{\rho}(\boldsymbol{\alpha} ik)\cdot\boldsymbol{\rho}(\boldsymbol{\alpha} jl)}{\boldsymbol{\rho}(\boldsymbol{\alpha} il)\cdot\boldsymbol{\rho}(\boldsymbol{\alpha} jk)} = \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha,\boldsymbol{\rho}}.$$

Proposition 9.5. *Let F be idempotent, let $\boldsymbol{\rho} \in \mathrm{R}_J^w(F)$, and let $f_{\boldsymbol{\rho}}\colon \widehat{P}_J \to F$ the associated tract morphism. Then $\boldsymbol{\rho} \in \operatorname{Lin}_J(F)$ if and only if $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_{\alpha,\boldsymbol{\rho}} = 1$ for all $(\alpha,i,j,k,l) \in \Omega_J$.*

Proof. It is clear from the definition that the cross ratios of $\boldsymbol{\rho}$ are invariant under the action of $\widehat{T}(F)$. The "only if" direction now follows from the obvious fact that the cross ratios of the trivial F-representation are all equal to one.

Conversely, we have $\boldsymbol{\rho} \in \operatorname{Lin}_J(F) = \widehat{T}(F).\chi_{J,F}$ if and only if $[\boldsymbol{\rho}] = [\chi_{J,F}]$ as classes of the realization space $\underline{\operatorname{Gr}}_J^w(F)$, i.e., if and only if the restriction of $f_{\boldsymbol{\rho}}$ to $F_J \to F$ agrees with the tract morphism $f_{\chi_{J,F}}\colon F_J \to F$, using Proposition 6.4. Now if $\left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_{\alpha,\boldsymbol{\rho}} = 1$ for all $(\alpha,i,j,k,l) \in \Omega_J$, then $f_{\boldsymbol{\rho}}|_{F_J^\times}\colon F_J^\times \to F$ is indeed trivial because $F_J^\times$ is generated by -1, whose image in F is $-1 = 1$, and all cross ratios by Theorem 8.4. □

10. The Plücker embedding for thin Schubert cells

We define the *projective N-space over F* as the quotient

$$\mathbb{P}^N(F) = \big\{(a_0,\dots,a_N) \in F^{N+1} \,\big|\, a_i \in F^\times \text{ for some } 0 \leqslant i \leqslant N\big\} \,/\, F^\times.$$

We denote elements of $\mathbb{P}^N(F)$ by

$$[a_1 : \ldots : a_N] := F^\times \cdot (a_0,\dots,a_N).$$

Quotienting out the domain and the codomain of the canonical embedding $\mathrm{R}_J^w(F) \to F^{n^r}$ (cf. Section 9) by the diagonal action of $F^\times$ yields the *effective Plücker embedding*

$$\operatorname{Gr}_J(F) \subseteq \operatorname{Gr}_J^w(F) \longrightarrow \mathbb{P}^{n^{\bar{r}}-1}(F) = \big\{[x_{\boldsymbol{\alpha}}] \,\big|\, \boldsymbol{\alpha} \in [n]^{\bar{r}}\big\},$$

which sends $\big[\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F\big] \in \operatorname{Gr}_J^w(F)$ to $[\boldsymbol{\rho}(\boldsymbol{\alpha})]_{\boldsymbol{\alpha}\in[n]^{\bar{r}}}$.

10.1. The Polygrassmannian for idempotent fusion tracts. The thin Schubert cells $\mathrm{Gr}_J(F)$ for various M-convex subsets $J \subseteq \Delta_n^r$ (with fixed r and n) glue to a subvariety of $\mathbb{P}^{n^r-1}$, which we call the Polygrassmannian. If F is not near-idempotent, then the idempotency principle (Proposition 4.7) implies that $\mathrm{Gr}_J(F)$ is nonempty only if J is the translate of a matroid. So in this case, the Polygrassmannian is a union of "translates" of usual Grassmannians. If F is near-idempotent, then the Polygrassmannian is larger than (a union of translates of) usual Grassmannians.

In order to realize the embedding of $\mathrm{Gr}_J(F)$ into $\mathbb{P}^{n^r-1}$ (as opposed to $\mathbb{P}^{n^{\bar{r}}-1}$, as in the effective Plücker embedding), we need to "translate" $[\boldsymbol{\rho}] \in \mathrm{Gr}_J(F)$ to a function $\boldsymbol{\rho}\colon [n]^r \to F$ by adding coefficients that correspond to δ_J^- in a certain sense, as explained below. Unless J is a matroid, the signs of the coordinates of the "translate" of $\boldsymbol{\rho}$ depends on the ordering of $[n]$ if $1 \neq -1$ in F. To avoid these complications, we concentrate on the near-idempotent case in the following, which is the only genuinely interesting case as explained above.

Thus, let us assume for the remainder of this discussion that F is near-idempotent. By Lemma 4.9, a function $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ that satisfies axioms (SR1) and (SR2) is a strong F-representation if and only if the function $\rho\colon \Delta_n^r \to F$ with support J, given by $\rho(\delta_J^- + \sum \boldsymbol{\alpha}) = \boldsymbol{\rho}(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \mathbf{J}$, satisfies the Plücker relations

$$\sum_{k=0}^{s} \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \in N_F$$

for all $2 \leqslant s \leqslant r$, $\alpha \in \Delta_n^{r-s}$ and $i_0, \cdots, i_s, j_2, \ldots, j_s \in [n]$ such that $\delta_J^- \leqslant \alpha$ and $\alpha + \varepsilon_{i_0} + \ldots + \varepsilon_{i_s} + \varepsilon_{j_2} + \ldots + \varepsilon_{j_s} \leqslant \delta_J^+$.

This identifies F-representations of different M-convex subsets $J \subseteq \Delta_n^r$ (for fixed r and n) with functions $\rho\colon \Delta_n^r \to F$ on the same domain. However, the defining relations depend on J because of the bounds that involve δ_J^- and δ_J^+. The following result resolves this issue by removing these bounds. It comes with the price that we need to assume the *fusion rule* for F, which is

(FR) If $\sum a_i - c,\ c + \sum b_j \in N_F$, then $\sum a_i + \sum b_j \in N_F$.

We call a tract that satisfies the fusion rule (FR) a *fusion tract*. Note that the most interesting examples of tracts are fusion tracts, which include all hyperfields and partial fields and, in particular, all examples mentioned in this text. Note that we also need to assume that F is idempotent.

Lemma 10.1. *Let F be an idempotent fusion tract. Let $J \subseteq \Delta_n^r$ be an M-convex set, $\bar{J}$ its reduction, and $\bar{r} = r - |\delta_J^-|$ its effective rank. Let $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ a function that satisfies axioms (SR1) and (SR2) and $\rho\colon \Delta_n^r \to F$ the function with support J given by $\rho(\delta_J^- + \sum \boldsymbol{\alpha}) = \boldsymbol{\rho}(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in [n]^{\bar{r}}$. Then $\boldsymbol{\rho}$ is a strong F-representation of J if and only if ρ satisfies the Plücker* *relations*

$$\sum_{k=0}^{s} \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) \in N_F$$

for all $2 \leqslant s \leqslant r$, $\alpha \in \Delta_n^{r-s}$ *and* $i_0, \cdots, i_s, j_2, \ldots, j_s \in [n]$.

Proof. By Lemma 4.9, we only need to verify that ρ satisfies the Plücker relations of the claim if $\delta_J^- \nleqslant \alpha$ or if $\alpha + \varepsilon_{i_0} + \ldots + \varepsilon_{i_s} + \varepsilon_{j_2} + \ldots + \varepsilon_{j_s} \nleqslant \delta_J^+$. These two cases can be proven analogously (and are, more concisely, related by polymatroid duality). We only explain the proof for $\delta_J^- \nleqslant \alpha$, proceeding by induction on $d^+ = d^+(\alpha, \delta_J^-) = \sum_{i \in [n]} \max\{0, \delta_{J,i}^- - \alpha_i\}$.

If $d^+ = 0$, then $\delta_J^- \leqslant \alpha$, so the corresponding Plücker relation holds by assumption. If $d^+ > 0$, then $\alpha_\ell < \delta_{J,\ell}^-$ for some $\ell \in [n]$ and the expression

$$\text{(8)} \quad \mathrm{Pl}(\alpha | i_0 \ldots i_s | j_2 \ldots j_s) := \sum_{k=0}^{s} \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$$

is either identically 0, and thus in N_F, or

$$\beta = \alpha + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s} \qquad \text{and} \qquad \gamma = \alpha + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}$$

satisfy $\beta_\ell \geqslant \delta_{J,\ell}^-$, $\gamma_\ell \geqslant \delta_{J,\ell}^- - 1$, and $\beta_\ell + \gamma_\ell \geqslant 2\delta_{J,\ell}^-$.

We begin with the case where $\gamma_\ell = \delta_{J,\ell}^- - 1$, and thus $\beta_\ell \geqslant \delta_{J,\ell}^- + 1 \geqslant \alpha_\ell + 2$. Then (8) is of the form

$$\sum_{k \text{ s.t. } i_k = \ell} \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$$
$$= \big(\underbrace{1 + \cdots + 1}_{m\text{-times}}\big) \cdot \rho(\alpha + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s} - \varepsilon_\ell) \cdot \rho(\alpha + \varepsilon_\ell + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s})$$

for $m = \beta_\ell - \alpha_\ell \geqslant 2$, which is in N_F since F is idempotent and thus $m.1 = 1 + \ldots + 1 \in N_F$.

In the other case, $\gamma_\ell \geqslant \delta_{J,\ell}^- \geqslant \alpha_\ell + 1$, we have $\beta_\ell \geqslant \delta_{J,\ell}^- \geqslant \alpha_\ell + 1$ and thus ℓ appears in both $\{i_0, \ldots, i_s\}$ and $\{j_2, \ldots, j_s\}$, say $i_0 = j_2 = \ell$. If $\beta_\ell = \alpha_\ell + 1$, then

$$\rho(\alpha + \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_0} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}) = 0$$

and thus (8) equals $\mathrm{Pl}(\alpha + \varepsilon_\ell | i_1 \ldots i_s | j_3 \ldots j_s)$, which is in N_F by the inductive hypothesis since $d^+(\alpha + \varepsilon_\ell, \delta_J^-) = d^+(\alpha, \delta_J) - 1 < d^+$. If $\beta_\ell \geqslant \alpha_\ell + 2$, then ℓ appears at least twice in $\{i_0, \ldots, i_s\}$, say $i_0 = i_1 = \ell$, and (8) is of the form

$$\rho(\beta - \varepsilon_\ell) \cdot \rho(\gamma + \varepsilon_\ell) + \rho(\beta - \varepsilon_\ell) \cdot \rho(\gamma + \varepsilon_\ell)$$
$$+ \sum_{k=2}^{s} \rho(\alpha - \varepsilon_{i_k} + \varepsilon_{i_0} + \cdots + \varepsilon_{i_s}) \cdot \rho(\alpha + \varepsilon_{i_k} + \varepsilon_{j_2} + \cdots + \varepsilon_{j_s}),$$

which is in N_F by the fusion rule (FR) applied to the two terms

$$\rho(\beta-\varepsilon_\ell)\cdot\rho(\gamma+\varepsilon_\ell)\ +\ \rho(\beta-\varepsilon_\ell)\cdot\rho(\gamma+\varepsilon_\ell)\ +\ \rho(\beta-\varepsilon_\ell)\cdot\rho(\gamma+\varepsilon_\ell),$$

which is in N_F since F is idempotent, and

$$\rho(\beta-\varepsilon_\ell)\cdot\rho(\gamma+\varepsilon_\ell)\ +\ \sum_{k=2}^{s}\ \rho(\alpha-\varepsilon_{i_k}+\varepsilon_{i_0}+\cdots+\varepsilon_{i_s})\cdot\rho(\alpha+\varepsilon_{i_k}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s}),$$

which is $\mathrm{Pl}(\alpha+\varepsilon_\ell|i_1\dots i_s|j_3\dots j_s)$ and thus in N_F by the inductive hypothesis. □

Definition 10.2. Let F be an idempotent fusion tract. The *Polygrassmannian of rank r on $[n]$ over F* is the subset $\mathrm{PolyGr}(r,n)(F)$ of $\mathbb{P}^N(F)$ (for $N=\#\Delta_n^r-1=\binom{n+r}{r}-1$) that consists of all classes $[\rho\colon\Delta_n^r\to F]$ of functions that satisfy the Plücker relations

$$\sum_{k=0}^{s}\ \rho(\alpha-\varepsilon_{i_k}+\varepsilon_{i_0}+\cdots+\varepsilon_{i_s})\cdot\rho(\alpha+\varepsilon_{i_k}+\varepsilon_{j_2}+\cdots+\varepsilon_{j_s})\in N_F$$

for all $2\leqslant s\leqslant r$, $\alpha\in\Delta_n^{r-s}$ and $i_0,\cdots,i_s,j_2,\dots,j_s\in[n]$.

As a consequence of Lemma 10.1 and Remark 4.5, $\mathrm{PolyGr}(r,n)(F)$ is the union of the thin Schubert cells $\mathrm{Gr}_J(F)$ for the various M-convex subsets $J\subseteq\Delta_n^r$. More precisely, the association $\boldsymbol{\rho}\mapsto\rho$ established by Lemma 10.1 defines a bijection

$$\coprod_{\substack{J\subseteq\Delta_n^r\\ \text{M-convex}}}\mathrm{Gr}_J(F)\ \longrightarrow\ \mathrm{PolyGr}(r,n)(F).$$

In particular, $\mathrm{PolyGr}(r,n)(\mathbb{K})$ is canonically in bijection with the collection of all M-convex subsets $J\subseteq\Delta_n^r$.

Remark 10.3. In fact, the Plücker relations endow the Polygrassmannian with the structure of a band scheme $\mathrm{PolyGr}(r,n)$ whose F-rational points are naturally identified with $\mathrm{PolyGr}(r,n)(F)$. This perspective can be extended to an interpretation of $\mathrm{PolyGr}(r,n)$ as the fine moduli space of polymatroids, in the vein of [8], in terms of a suitable theory of polymatroid bundles for band schemes.

Note that we refrain from making an analogous definition for the *weak* Polygrassmannian, because the weak Plücker relations (WR3) do not by themselves imply that the support of ρ is a polymatroid. This leads to a less satisfactory picture from an algebro-geometric point of view: the space of weak F-representations of M-convex subsets of Δ_n^r is not represented by a band scheme. (A similar phenomenon occurs already for matroids, so there is nothing unique in this regard about the case of polymatroids.)

10.1.1. *The Polydressian.* Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the min-plus-algebra. Taking coefficientwise inverse logarithms defines the bijection $-\log\colon \mathbb{P}^N(\mathbb{T}_0) \to \overline{\mathbb{R}}^N/\overline{\mathbb{R}}$, where $N = \#\Delta_n^r - 1$. We define the *Polydressian* as the image $\mathrm{PolyDr}(r,n) = -\log(\mathrm{PolyGr}(r,n)(\mathbb{T}_0))$ of the Polygrassmannian over $\mathbb{T}_0$ under this map. The Polydressian $\mathrm{PolyDr}(r,n)$ contains the Dressian $\mathrm{Dr}(r,n)$, which is the union of the local Dressians $\mathrm{Dr}_J = -\log\big(\mathrm{Gr}_J(\mathbb{T}_0)\big)$ for which J is a matroid. It follows from the results in Section 4.5 that the Polydressian $\mathrm{PolyDr}(r,n)$ is the set of M-convex functions $\Delta_n^r \to \mathbb{R} \cup \{\infty\}$ modulo constant functions.

In rank $r = 1$, every polymatroid is a matroid and thus $\mathrm{Dr}(1,n) = \mathrm{PolyDr}(1,n)$. For rank $r \geqslant 2$ and $n \geqslant 1$, the Polydressian is strictly larger. For example, $\mathrm{Dr}(r,1)$ is empty for $r \geqslant 2$ since there is no matroid of rank $r \geqslant 2$ on $[1]$, but $\mathrm{PolyDr}(r,1)$ is a singleton that contains the class of the trivial valuation $v\colon \Delta_1^r \to \overline{\mathbb{R}}$ with $v(r,0) = 0$.

A richer example is $\mathrm{PolyDr}(2,2)$. The Dressian $\mathrm{Dr}(2,2)$ is a singleton that consists of the class of the trivial valuation $v : U_{2,2} \to \overline{\mathbb{R}}$ with $v(1,1) = 0$, where $U_{2,2} = \{(1,1)\}$ is the uniform matroid of rank 2 on $[2]$. The Polydressian PolyDr has six strata $\mathrm{Dr}_J = -\log(\mathrm{Gr}_J(\mathbb{T}_0))$, labeled by the six M-convex subsets

$$\{(2,0)\}, \quad \{(1,1)\}, \quad \{(0,2)\}, \quad \{(2,0),(1,1)\}, \quad \{(2,0),(1,1)\}, \quad \Delta_2^2$$

of Δ_2^2. The (logarithmic) Plücker relations are trivially satisfied for all valuations of these M-convex sets, with the unique exception of Δ_2^2, whose valuations must satisfy the condition that the minimum among

$$v(2,0) + v(0,2), \qquad v(1,1) + v(1,1), \qquad \text{and} \qquad v(1,1) + v(1,1)$$

occurs at least twice, which means that $2v(1,1) \geqslant v(2,0) + v(0,2) \neq \infty$. See Figure 7 for an illustration of $\mathrm{PolyDr}(2,2)$.

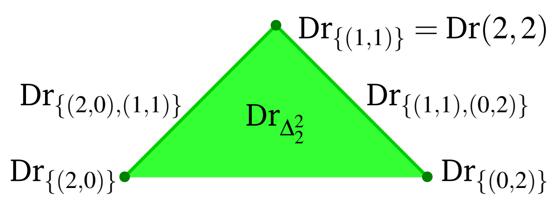

Figure 7. The Polydressian $\mathrm{PolyDr}(2,2)$

10.2. The Tutte rank. If M is a matroid, then $P_M^\times$ is canonically isomorphic to the Tutte group of M; cf. [8, Thm. 6.27]. This justifies the following definition of the Tutte group of a polymatroid.

Definition 10.4. Let J be an M-convex set. The *Tutte group of J* is the abelian group $P_J^\times$, and the *Tutte rank of J* is its free rank $\tau(J) = \mathrm{rk}\, P_J^\times$.

As an immediate consequence of Theorem 5.4, the Tutte group is also canonically isomorphic to $T_J^\times$. The Tutte rank $\tau(J)$ is finite since $P_J^\times$ is finitely generated.

Corollary 10.5. *Let $J \subseteq \Delta_n^r$ be an M-convex set. Then $-\log\big(\mathrm{Gr}_J^w(\mathbb{T}_\infty)\big)$ is a real vector space whose dimension equals the Tutte rank $\tau(J)$ of J.*

Proof. Since $\mathbb{T}_\infty$ is degenerate and every group homomorphism $f\colon \widehat{P}_J \to \mathbb{T}_\infty^\times$ maps the torsion element -1 to $1 = -1$ in $\mathbb{T}_\infty^\times = \mathbb{R}_{>0}$, Proposition 9.2 implies that

$$\mathrm{R}_J^w(\mathbb{T}_\infty) \;=\; \mathrm{D}_J(\mathbb{T}_\infty) \;=\; \mathrm{Hom}_{\mathbb{Z}}(\widehat{P}_J^\times,\, \mathbb{R}_{>0}).$$

In conclusion $-\log\big(\mathrm{Gr}_J^w(\mathbb{T}_\infty)\big) = \mathrm{Hom}_{\mathbb{Z}}(P_J^\times,\, \mathbb{R})$ is a real vector space of dimension $\tau(J) = \mathrm{rk}(P_J^\times)$. □

Remark 10.6. We prove in [4] that for every M-convex set J, the dimension of the space of Lorentzian polynomials with support J is equal to $\tau(J) + 1$.

11. The canonical torus embedding for realization spaces

Let $J \subseteq \Delta_n^r$ be M-convex, $\bar{J}$ its reduction, $\bar{r}$ its effective rank, and $\mathbf{J} = \{\boldsymbol{\alpha} \in [n]^{\bar{r}} \mid \sum \boldsymbol{\alpha} \in \bar{J}\}$. The non-degenerate cross ratios of an F-representation $\boldsymbol{\rho}\colon [n]^{\bar{r}} \to F$ of J determine the map

$$\begin{array}{cccc} \varpi_J: & \underline{\mathrm{Gr}}_J^w(F) & \longrightarrow & (F^\times)^{\Omega_J^\diamond} \\ & [\boldsymbol{\rho}] & \longmapsto & \Big[(\alpha, i, j, k, l) \mapsto \left[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right]_{\alpha, \boldsymbol{\rho}}\Big]. \end{array}$$

Let $\eta_J\colon (F^\times)^{\mathbf{J}} \to (F^\times)^{\Omega_J^\diamond}$ be the group homomorphism that maps a function $\boldsymbol{\rho}\colon \mathbf{J} \to F^\times$ to the function $\eta_J(\boldsymbol{\rho})\colon \Omega_J \to F^\times$ given by

$$(\alpha, i, j, k, l) \;\longmapsto\; \begin{bmatrix} i & j \\ k & l \end{bmatrix}_{\alpha,\rho} \;=\; \frac{\boldsymbol{\rho}(\boldsymbol{\alpha} ik)\cdot \boldsymbol{\rho}(\boldsymbol{\alpha} jl)}{\boldsymbol{\rho}(\boldsymbol{\alpha} il)\cdot \boldsymbol{\rho}(\boldsymbol{\alpha} jk)},$$

where $\boldsymbol{\alpha} \in [n]^{\bar{r}-2}$ such that $\sum \boldsymbol{\alpha} = \alpha$. Let $\mathrm{pl}_J\colon\ \mathrm{R}_J^w \to (F^\times)^{\mathbf{J}}$ be the canonical torus embedding (cf. Section 9) and $\pi_J\colon\ \mathrm{R}_J^w(F) \to \underline{\mathrm{Gr}}_J^w(F)$ the quotient map.

Proposition 11.1. *The map ϖ_J is injective and*

$$\begin{array}{ccc} \mathrm{R}_J^w(F) & \xrightarrow{\ \mathrm{pl}_J\ } & (F^\times)^{\mathbf{J}} \\ {\scriptstyle \pi_J}\big\downarrow & & \big\downarrow{\scriptstyle \eta_J} \\ \underline{\mathrm{Gr}}_J^w(F) & \xrightarrow{\ \varpi_J\ } & (F^\times)^{\Omega_J^\diamond} \end{array}$$

commutes.

Proof. By Proposition 6.4, a class $[\boldsymbol{\rho}] \in \mathrm{R}^w_J(F)$ is uniquely determined by the associated morphism $f_{\boldsymbol{\rho}}\colon F_J \to F$ (cf. Theorem 8.4), which in turn is uniquely determined by the image of the non-degenerate cross ratios $f_\rho([\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}]_\alpha) = [\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}]_{\alpha,\boldsymbol{\rho}}$ since F_J is generated by the non-degenerate cross ratios (Theorem 8.4). Thus ϖ_J is injective.

The diagram commutes since

$$\begin{aligned}(\varpi_J \circ \pi_J)(\boldsymbol{\rho}) \;=\; \varpi_J([\boldsymbol{\rho}]) \;&=\; \left\{[\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}]_{\alpha,\boldsymbol{\rho}} \,\middle|\, (\alpha,i,j,k,l) \in \Omega^\diamond_J\right\} \\ &=\; \eta_J\big(\{\boldsymbol{\rho}(\boldsymbol{\beta}) \mid \boldsymbol{\beta} \in \mathbf{J}\}\big) \;=\; (\eta_J \circ \mathrm{pl}_J)(\boldsymbol{\rho})\end{aligned}$$

for all $\boldsymbol{\rho} \in \mathrm{R}^w_J(F)$. □

11.1. A decomposition of the representation space. By Lemma 6.5, the extended universal pasture is a free algebra over the foundation. The choice of an isomorphism $\widehat{P}_J \simeq F_J(x_1,\dots,x_s)$ yields a bijection

$$\begin{aligned}\mathrm{Gr}^w_J(F) \;=\; \mathrm{Hom}(P_J,F) \;&\simeq\; \mathrm{Hom}(F_J(x_1,\dots,x_s),F) \\ &=\; \mathrm{Hom}(F_J,F) \;\times\; \mathrm{Maps}\,\big(\{x_1,\dots,x_s\},F^\times\big) \;=\; \underline{\mathrm{Gr}}^w_J(F)\times(F^\times)^s\end{aligned}$$

which is functorial in F and which commutes with the projections onto $\underline{\mathrm{Gr}}^w_J(F)$. More precisely, the following holds.

Theorem 11.2. *Let $J \subseteq \Delta^r_n$ be M-convex and $J = \bigoplus_{i=1}^{c(J)} J_i$ a decomposition into indecomposable M-convex sets $J_i \subseteq \Delta^{r_i}_{n_i}$. Let $\boldsymbol{\rho}\colon [n]^r \to F$ be an F-representation of J. Then the stabilizer of $[\boldsymbol{\rho}]$ under the action of $T(F) = (F^\times)^n$ on $\mathrm{Gr}^w_J(F)$ is*

$$\mathrm{Stab}_{T(F)}([\boldsymbol{\rho}]) \;=\; \big\{t \in T(F)\,\big|\, t_{n_{i-1}+1} = \dots = t_{n_i} \text{ for all } i = 1,\dots,c(J)\big\},$$

where $n_0 = 0$, and the orbit $T(F).[\boldsymbol{\rho}]$ is in bijective correspondence with $(F^\times)^{n-c(J)}$.

In particular, $P_J \simeq F_J(x_1,\dots,x_{n-c(J)})$. If F is idempotent, then $\mathrm{Lin}_J(F) \simeq (F^\times)^{n-c(J)}$.

Proof. We begin with the first claim. An element $t \in T(F)$ is in the stabilizer of $[\boldsymbol{\rho}]$ iff $[t.\boldsymbol{\rho}] = [\boldsymbol{\rho}]$, i.e., iff there is an element $c \in F^\times$ such that $(t.\boldsymbol{\rho})(\boldsymbol{\beta}) = c \cdot \boldsymbol{\rho}(\boldsymbol{\beta})$ for all $\boldsymbol{\beta} \in J$, which means that $\prod_{j=1}^n t_j^{\boldsymbol{\beta}_j} = c$.

By Theorem 7.7, $\boldsymbol{\rho}(\boldsymbol{\beta}) = \prod_{i=1}^{c(J)} \boldsymbol{\rho}_i(\boldsymbol{\beta}|_{J_i})$ for certain F-representations $\boldsymbol{\rho}_i\colon [n_i]^{r_i} \to F$ of J_i and $\boldsymbol{\beta}|_{J_i} = (\boldsymbol{\beta}_{n_{i-1}+1},\dots,\boldsymbol{\beta}_{n_i})$.

Assume that $t_{n_{i-1}+1} = \dots = t_{n_i}$ for all $i = 1,\dots,c(J)$ and define $c = \prod_{i=1}^{c(J)} t_{n_i}^{r_i}$. Then $\prod_{j=1}^n t_j^{\boldsymbol{\beta}_j} = c$ for all $\boldsymbol{\beta} \in J$ since $|\boldsymbol{\beta}|_{J_i}| = r_i$, and thus $t \in \mathrm{Stab}_{T(F)}([\boldsymbol{\rho}])$.

Conversely, assume that $t \in \mathrm{Stab}_{T(F)}([\boldsymbol{\rho}])$. We want to show that $t_{n_{i-1}+1} = \dots = t_{n_i}$ for each $i = 1,\dots,c(J)$, which can be established separately for each component J_i. After replacing J by J_i, this allows us to assume that J is indecomposable for simplicity. We establish the claim by showing through an induction on $s = 1, \dots, n$ that there is a subset $S \subseteq [n]$ of cardinality s that contains t_n and such that $t_i = t_n$ for all $i \in S$.

If $s = 1$, then $S = \{t_n\}$ satisfies the claim. In order to establish the inductive step, assume that S is a proper subset of $[n]$ that contains t_n and such that $t_i = t_n$ for all $i \in S$. By Lemma 2.28, there are $\beta, \gamma \in J$ with $\beta_S < \gamma_S$. Applying the exchange axiom repeatedly yields a sequence $\beta = \beta^{(0)}, \dots, \beta^{(m)} = \gamma$ of elements in J with $\sum_{i\in[n]} |\beta_i^{(j)} - \beta_i^{(j-1)}| = 2$ for all $j = 1, \dots, m$. In particular, there is a j for which $\beta_S^{(j)} = \beta_S^{(j-1)} + 1$.

After replacing β by $\beta^{(j-1)}$ and γ by $\beta^{(j)}$, this means that there are $k \in S$ and $\ell \in [n] - S$ such that $\gamma_k = \beta_k + 1$, $\gamma_\ell = \beta_\ell - 1$, and $\gamma_i = \beta_i$ for all $i \neq k, \ell$. Since $t \in \mathrm{Stab}_{T(F)}(\boldsymbol{\rho})$, we have $\prod_{i\in[n]} t_i^{\beta_i} = \prod_{i\in[n]} t_i^{\gamma_i}$, and thus $t_\ell = t_k$. This shows that $S \cup \{\ell\}$ satisfies the claim of the induction, which establishes the inductive step. This completes the proof of the first claim of the proposition.

The orbit $T(F).[\boldsymbol{\rho}]$ is in bijection with $T(F)/\mathrm{Stab}_{T(F)}(\boldsymbol{\rho})$, which is isomorphic (as a group) to $(F^\times)^{n-c(J)}$, which establishes the second claim. For a finite proper extension of $\mathbb{K}$ (e.g. $F = \mathbb{H} \otimes \mathbb{K}$), counting the elements of $\underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^{n-c(J)} = \underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^s$ implies that $s = n - c(J)$, where $s \in \mathbb{N}$ is chosen so that $\widehat{P}_J \simeq F_J(x_1, \dots, x_s)$. Thus $P_J \simeq F_J(x_1, \dots, x_{n-c(J)})$. If F is idempotent, we can choose $\boldsymbol{\rho}$ to be the trivial F-representation $\chi_{J,F} \colon [n]^r \to F$, which yields $\mathrm{Lin}_J(F) = T(F).[\chi_{J,F}] \simeq (F^\times)^{n-c(J)}$. This establishes the last claim. □

12. Topologies for representation spaces and realization spaces

If a tract F comes with a topology, then this induces a topology on the (weak) representation and realization spaces. If the topology of F is sufficiently nice, then this topology has several equivalent characterizations. The following definition is analogous to the notion of a topological idyll (cf. [6]).

Definition 12.1. A *topological tract* is a tract F together with a topology such that the following holds:

- (TP1) $\{0\} \subseteq F$ is a closed subset.
- (TP2) The multiplication $m \colon F \times F \to F$ is continuous (where $F \times F$ carries the product topology).
- (TP3) The inversion $i \colon F^\times \to F^\times$ is continuous (where $F^\times \subseteq F$ carries the subspace topology).
- (TP4) For all $n \in \mathbb{N}$, the set $\{(a_1, \dots, a_n) \in F^n \mid a_1 + \dots + a_n \in N_F\}$ is a closed subset of F^n (where F^n carries the product topology).

Note that (TP1) is equivalent to $F^\times$ being an open subset of F. Axioms (TP2) and (TP3) imply that $F^\times$ is a topological group. Examples of topological tracts are topological fields and the triangular hyperfields $\mathbb{T}_q$ (for all $0 \leqslant q < \infty$) with the euclidean topology for $\mathbb{T}_q = \mathbb{R}_{\geqslant 0}$. Every tract is a topological tract with respect to the discrete topology.

If F is a topological tract and F' an arbitrary tract, we endow $\mathrm{Hom}(F', F)$ with the compact-open topology where we equip F' with the discrete topology. In other words, $\mathrm{Hom}(F', F)$ is endowed with the coarsest topology such that for every $a \in F'$, the map

$$\begin{array}{rccc} \mathrm{ev}_a\colon & \mathrm{Hom}(F',\ F) & \longrightarrow & F \\ & [f\colon F' \to F] & \longmapsto & f(a) \end{array}$$

is continuous. In this way, we obtain a topology on $\underline{\mathrm{Gr}}^w_J(F) = \mathrm{Hom}(\widehat{F}_J, F)$, and similarly on (weak) representation spaces and (weak) thin Schubert cells. (For the sake of brevity, we restrict our discussion to the weak setup.)

Proposition 12.2. *Let F be a topological tract. Then $\mathrm{R}^w_J(F)$ is a closed subspace of $(F^\times)^J$ (considered with the product topology).*

Proof. For every $\alpha \in J$, let x_α be the generator of $\widehat{P}_J$ indexed by the element $\boldsymbol{\alpha} \in [n]^r$ with $\Sigma\boldsymbol{\alpha} = \alpha$ whose entries are ordered increasingly. Then the bijection $\mathrm{Hom}(\widehat{P}_J, F) \to \mathrm{R}^w_J(F)$ is given by $\varphi \mapsto (\mathrm{ev}_{x_\alpha}(\varphi))_{\alpha\in J}$, and thus is continuous. Because -1 and the x_α generate the unit group of $\widehat{P}_J$, every value of $\varphi \in \mathrm{Hom}(\widehat{P}_J, F)$ can be expressed in terms of products and inverses of constants and the $\mathrm{ev}_{x_\alpha}(\varphi)$. Since $F^\times$ is a topological group, this shows that the map inverse to $\mathrm{Hom}(\widehat{P}_J, F) \to \mathrm{R}^w_J(F)$ is also continuous. The subspace $\mathrm{R}^w_J(F)$ being closed in $(F^\times)^J$ follows directly from the definition of a topological tract. □

Remark 12.3. Note that also $\mathrm{R}^w_J(\mathbb{T}_\infty) = \mathrm{D}_J(\mathbb{T}_\infty)$ is a closed subset of $(\mathbb{T}^\times_\infty)^J = \mathbb{R}^J_{>0}$, even though the order topology for $\mathbb{T}_\infty = \mathbb{R}_{\geqslant 0}$ fails to satisfy (TP4).

Proposition 12.4. *Let T be a topological tract and $F = \operatorname{colim} \mathcal{F}$ for a finite diagram $\mathcal{F}$ of tracts, then the canonical bijection $\mathrm{Hom}(F, T) \to \lim \mathrm{Hom}(\mathcal{F}, T)$ is a homeomorphism.*

Proof. This follows from [7, Thm. 3.5]. □

Corollary 12.5. *Let F be a topological tract, $J \subseteq \Delta^r_n$ an M-convex set with $c(J)$ indecomposable components and $s = n - c(J)$. Then there are (non-canonical) homeomorphisms $\mathrm{R}^w_J(F) \to \underline{\mathrm{Gr}}^w_J(F) \times (F^\times)^{s+1}$ and $\mathrm{Gr}^w_J(F) \to \underline{\mathrm{Gr}}^w_J(F) \times (F^\times)^s$ such that the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{R}^w_J(F) & \longrightarrow & \mathrm{Gr}^w_J(F) & \longrightarrow & \underline{\mathrm{Gr}}^w_J(F) \\ \downarrow & & \downarrow & & \downarrow \\ \underline{\mathrm{Gr}}^w_J(F) \times (F^\times)^{s+1} & \longrightarrow & \underline{\mathrm{Gr}}^w_J(F) \times (F^\times)^s & \longrightarrow & \underline{\mathrm{Gr}}^w_J(F) \end{array} \quad . \tag{9}$$

Here the maps in the first row take elements to their equivalence classes and the maps in the second row are coordinate projections.

Proof. By Lemma 6.5 we have

$$P_J \simeq F_J(x_1, \dots, x_s) \simeq F_J \otimes \mathbb{F}_1^{\pm}(x_1, \dots, x_s).$$

Thus by the preceding proposition we have a homeomorphism

$$\mathrm{Gr}_J^w(F) = \mathrm{Hom}(P_J, F) \simeq \mathrm{Hom}(F_J, F) \times \mathrm{Hom}(\mathbb{F}_1^{\pm}(x_1, \dots, x_s), F) = \underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^s,$$

and similarly for $\widehat{P}_J$. □

Corollary 12.6. *Let F be a topological tract and $J \subseteq \Delta_n^r$ an M-convex set. The spaces $\mathrm{Gr}_J^w(F)$ and $\underline{\mathrm{Gr}}_J^w(F)$ carry the quotient topology of $\mathrm{R}_J^w(F)$.*

Proof. It follows from Corollary 12.5 that the natural maps $\mathrm{R}_J^w(F) \to \mathrm{Gr}_J^w(F)$ and $\mathrm{R}_J^w(F) \to \underline{\mathrm{Gr}}_J^w(F)$ are open and continuous, which implies the claim. □

Corollary 12.7. *Let F be a topological tract, $J \subseteq \Delta_n^r$ an M-convex set with $c(J)$ indecomposable components, and $s = n - c(J)$. Let $\widehat{T}(F) = F^\times \times T(F)$ be the extended torus acting on $\mathrm{R}_J^w(F)$. Then the orbits of $F^\times$ and $\widehat{T}(F)$ are closed in $\mathrm{R}_J^w(F)$, and the orbits are homeomorphic to $F^\times$ and $(F^\times)^{s+1}$, respectively.*

Proof. This follows from Corollary 12.5 because the fibers of the maps $\underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^{s+1} \to \underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^s$ and $\underline{\mathrm{Gr}}_J^w(F) \times (F^\times)^{s+1} \to \underline{\mathrm{Gr}}_J^w(F)$ in the second row of Equation (9) are closed and homeomorphic to $F^\times$ and $(F^\times)^{s+1}$, respectively. □

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