

REALIZATIONS OF HOMOLOGY CLASSES AND PROJECTION AREAS

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ABSTRACT. The relationship between convex geometry and algebraic geometry has deep historical roots, tracing back to classical works in enumerative geometry. In this paper, we continue this theme by studying two interconnected problems regarding projections of geometric objects in four-dimensional spaces:

- (1) Let A be a convex body in $\mathbb{R}^4$, and let $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ be the areas of the six coordinate projections of A in $\mathbb{R}^2$. Which tuples of six nonnegative real numbers can arise in this way?
- (2) Let S be an irreducible surface in $(\mathbb{P}^1)^4$, and let $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ be the degrees of the six coordinate projections from S to $(\mathbb{P}^1)^2$. Which tuples of six nonnegative integers can arise in this way?

We show that these questions are governed by the Plücker relations for the Grassmannian $\text{Gr}(2, 4)$ over the triangular hyperfield $\mathbb{T}_2$. We extend our analysis by determining the homology classes in $(\mathbb{P}^m)^n$ proportional to the fundamental classes of irreducible algebraic surfaces, resolving the algebraic Steenrod problem in this setting. Our results lead to several conjectures on realizable homology classes in smooth projective varieties and on the projection volumes of convex bodies.

1. INTRODUCTION

The connection between convex geometry and intersection theory traces back to classical results of Newton [New03] and Minding [Min41] that relate the solutions of polynomial systems to their Newton polyhedra.¹ In this paper, we expand upon this classical theme by addressing two closely related problems: one concerning projection areas of convex bodies in $\mathbb{R}^4$, and the other involving homology classes of algebraic surfaces in $(\mathbb{P}^1)^4$, drawing inspiration from both convex and algebraic geometry.

Question 1.1. Let A be a convex body in $\mathbb{R}^4$, and let $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ be the areas of the six coordinate projections of A in $\mathbb{R}^2$. Which tuples of six nonnegative real numbers can arise in this way?

Question 1.2. Let S be an irreducible surface in $(\mathbb{P}^1)^4$, and let $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ be the degrees of the six coordinate projections from S to $(\mathbb{P}^1)^2$. Which tuples of six nonnegative integers can arise in this way?

¹For historical overviews and commentaries on Newton's and Minding's work from a modern perspective, see [Edw22, Essay 4.4] and [Min03, Section 2].

We answer Question 1.1 in Theorem 1.4, and Question 1.2 modulo positive multiples in Theorem 1.6. Based on our findings, we formulate a possible answer to the n -dimensional analogue of Question 1.1 in Conjecture 1.9, and provide an extension of Theorem 1.6 from $(\mathbb{P}^1)^4$ to $(\mathbb{P}^n)^m$ in Theorem 1.8.

Questions 1.1 and 1.2 find their solutions encoded in the Plücker relations for the Grassmannian $\text{Gr}(2, 4)$ over a triangular hyperfield $\mathbb{T}_2$. For general discussions of Plücker relations over triangular hyperfields and other algebraic objects, we refer to [BB19, BHKL]. For our purposes, the following explicit definition will suffice.

Definition 1.3. Let $\Delta(\mathbb{T}_2)$ be the set of $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{R}^4$ such that

$$p_{ij} \geq 0 \text{ and } \sqrt{p_{ij}p_{kl}} + \sqrt{p_{ik}p_{jl}} \geq \sqrt{p_{il}p_{jk}} \text{ for any } i, j, k, l.$$

The set $\Delta(\mathbb{T}_2)$ is a non-convex semialgebraic set whose image in the projective space of $\wedge^2 \mathbb{R}^4$ is homeomorphic to the 5-dimensional closed ball [BHKL]. This image is the set of $\mathbb{T}_2$ -valued points in the Grassmannian $\text{Gr}(2, 4)$ in the sense of [BL21], so one may view $\Delta(\mathbb{T}_2)$ as the set of $\mathbb{T}_2$ -valued points in the affine cone over the Grassmannian. The points in the interior $\Delta(\mathbb{T}_2)^\circ$ are called *nondegenerate*. These are the sextuples of positive numbers such that the square roots of $p_{12}p_{34}$, $p_{13}p_{24}$, $p_{14}p_{23}$ form the side lengths of a nondegenerate triangle.

We answer Question 1.1. Let π_{ij} be the coordinate projection of $\mathbb{R}^4$ onto the plane orthogonal to the standard basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$.

Theorem 1.4. The following holds for any $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{R}^4$.

(1) $\mathbf{p} \in \Delta(\mathbb{T}_2)^\circ$ if and only if there is a smooth convex body $A \subseteq \mathbb{R}^4$ that satisfies

$$p_{ij} = \left(\text{the area of the projection } \pi_{ij}(A) \right) \text{ for all } i < j.$$

(2) $\mathbf{p} \in \Delta(\mathbb{T}_2)$ if and only if there is a convex body $A \subseteq \mathbb{R}^4$ that satisfies

$$p_{ij} = \left(\text{the area of the projection } \pi_{ij}(A) \right) \text{ for all } i < j.$$

For any undefined terms in the theory of convex bodies, we refer to [Sch14]. One of our key tools will be the theory of mixed volumes [Sch14, Chapter 5]. To each pair of convex bodies A and B in $\mathbb{R}^4$, we associate a vector of real numbers

$$A \wedge B = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{R}^4,$$

where the components p_{ij} are defined as the mixed volumes of the projections of A and B :

$$p_{ij} = \left(\text{the mixed volume of } \pi_{ij}(A) \text{ and } \pi_{ij}(B) \text{ in } \mathbb{R}^2 \right).$$

With this notation, the basic properties of mixed volumes read

$$A \wedge B = B \wedge A \text{ and } (\lambda_1 A_1 + \lambda_2 A_2) \wedge B = \lambda_1 (A_1 \wedge B) + \lambda_2 (A_2 \wedge B) \text{ for } \lambda_1, \lambda_2 \geq 0.$$

Theorem 1.4 provides a characterization of vectors of the form $A \wedge A$. Our second main contribution is Theorem 3.7, which states that

$$\Delta(\mathbb{T}_2) \cap \wedge^2 \mathbb{Q}^4 = \{A \wedge B \mid A, B \text{ are rational polytopes in } \mathbb{R}^4\}.$$

We conjecture that, in fact, $\Delta(\mathbb{T}_2) \cap \wedge^2 \mathbb{Q}^4$ is the set of projection areas of rational polytopes in $\mathbb{R}^4$.

The parallel Question 1.2 in algebraic geometry can be rephrased as follows: For $E \subseteq [4]$, write φ_E for the coordinate projection $(\mathbb{P}^1)^4 \rightarrow (\mathbb{P}^1)^{4-|E|}$ that forgets the coordinates labelled by E . If S is an irreducible surface in $(\mathbb{P}^1)^4$ over the complex numbers, we can uniquely express its homology class as a nonnegative integral linear combination

$$[S] = p_{12}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^0] + \cdots + p_{34}[\mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1].$$

Which vectors of nonnegative integers $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$ can arise in this way? We call such homology classes *realizable*. Such questions are algebraic analogues of the *Steenrod problem* in topology [Eil49, Problem 25], which asks whether every homology class in any simplicial complex X is the image of the fundamental class of a closed oriented manifold by a map into the simplicial complex. In 1954, as part of his work on cobordism theory, Thom showed that in general the answer to this question depends on the chosen coefficients [Tho54]: the answer is positive for $H_*(X, \mathbb{Q})$ and negative for $H_*(X, \mathbb{Z})$. We introduce the necessary definitions adapted to the setting of algebraic geometry.

Definition 1.5. Let X be a complex smooth projective variety.²

- (1) A class $\eta \in H_{2k}(X, \mathbb{Z})$ is *realizable over $\mathbb{Z}$* if there is a subvariety $V \subseteq X$ with $\eta = [V]$.
- (2) A class $\eta \in H_{2k}(X, \mathbb{Q})$ is *realizable over $\mathbb{Q}$* if $\lambda\eta$ is realizable over $\mathbb{Z}$ for some $\lambda \in \mathbb{Q}_{\geq 0}$.
- (3) A class $\eta \in H_{2k}(X, \mathbb{R})$ is *realizable over $\mathbb{R}$* if it is a limit of classes realizable over $\mathbb{Q}$.

We provide a characterization of homology classes in $(\mathbb{P}^1)^4$ that are realizable over $\mathbb{Q}$.

Theorem 1.6. For $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{Q}^4$, consider the homology class

$$\eta(\mathbf{p}) := p_{12}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^0] + \cdots + p_{34}[\mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1] \in H_4((\mathbb{P}^1)^4, \mathbb{Q}).$$

The class $\eta(\mathbf{p})$ is realizable over $\mathbb{Q}$ if and only if $\mathbf{p}$ is in the set $\Delta(\mathbb{T}_2)$.

Compare Theorem 1.6 with Theorem 5.5, where we identify the set of complete intersection surface classes in $(\mathbb{P}^1)^4$, up to a rational multiple, with the set of rational points in

$$\Delta(\mathbb{T}_1) := \{\mathbf{p} \in \wedge^2 \mathbb{R}^4 \mid p_{ij} \geq 0 \text{ and } p_{ij}p_{kl} + p_{ik}p_{jl} \geq p_{il}p_{jk} \text{ for any } i, j, k, l\}.$$

As in the case of $\Delta(\mathbb{T}_2)$, the image of $\Delta(\mathbb{T}_1)$ in the projective space of $\wedge^2 \mathbb{R}^4$ is the set of $\mathbb{T}_1$ -valued points in the Grassmannian $\text{Gr}(2, 4)$, also homeomorphic to the 5-dimensional closed ball [BHL]. A boundary point of $\Delta(\mathbb{T}_2)$ is in $\Delta(\mathbb{T}_1)$ if and only if $p_{ij} = 0$ for some i and j .

²In this paper, a variety and its subvarieties are by definition reduced and irreducible.

Question 1.2, the realizability problem over $\mathbb{Z}$ for $(\mathbb{P}^1)^4$, is more subtle. For example, the homology class corresponding to $\mathbf{p} = (1, 1, 1, 1, 1, 3)$ is realizable over $\mathbb{Q}$ but not over $\mathbb{Z}$. To see this, note that the hypothetical surface S should satisfy

$$[\varphi_3(S)] = [\varphi_4(S)] = [\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0] + [\mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^1] + [\mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1].$$

Thus, the defining equations of $\varphi_3^{-1}\varphi_3(S)$ and $\varphi_4^{-1}\varphi_4(S)$ in an affine chart are of the form

$$*1 + *x_1 + *x_2 + *x_4 + *x_1x_2 + *x_1x_4 + *x_2x_4 + *x_1x_2x_4 = 0,$$

$$*1 + *x_1 + *x_2 + *x_3 + *x_1x_2 + *x_1x_3 + *x_2x_3 + *x_1x_2x_3 = 0,$$

where the $*$'s are placeholders for coefficients. For generic values of x_3 and x_4 , this system has at most 2 solutions, contradicting $p_{34} = 3$. In Proposition 5.2, we observe more generally that the realizability of $\mathbf{p}$ over $\mathbb{Z}$ implies

$$p_{ij} \leq p_{ik}p_{jl} + p_{il}p_{jk} \text{ for any } i, j, k, l.$$

For $\mathbf{p} \in \Delta(\mathbb{T}_2)^\circ$, we know no other obstructions to the realizability of $\mathbf{p}$ over $\mathbb{Z}$. In Section 6, using global surjectivity of the period map for marked complex K3 surfaces [Huy16, Chapter 7], we show that for any integral vector $\mathbf{p} \in \Delta(\mathbb{T}_2)$ and any integer $\lambda_1 > 1$, there is an integer $\lambda_2 > 0$ such that $\lambda_1\lambda_2^{-1}\eta(\mathbf{p})$ is realizable over $\mathbb{Z}$.

Definition 1.7. We say that a real symmetric matrix is *Lorentzian* if it has only nonnegative entries and has at most one positive eigenvalue.

Let $\mathbf{m}$ be a vector of positive integers $(m_1, \dots, m_n)$, and let $\mathbb{P}^{\mathbf{m}}$ be the product of projective spaces $\prod_{i=1}^n \mathbb{P}^{m_i}$. Let H_i be the pullback of the cohomology class of a hyperplane by the i -th projection $\mathbb{P}^{\mathbf{m}} \rightarrow \mathbb{P}^{m_i}$. As observed in [BHKL], a point $\mathbf{p} \in \wedge^2 \mathbb{R}^4$ is in $\Delta(\mathbb{T}_2)$ if and only if

$$L(\mathbf{p}) := \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ p_{12} & 0 & p_{23} & p_{24} \\ p_{13} & p_{23} & 0 & p_{34} \\ p_{14} & p_{24} & p_{34} & 0 \end{pmatrix}$$

is a Lorentzian matrix, see Proposition 3.1. Therefore, the following statement extends Theorem 1.6.

Theorem 1.8. For $\eta \in H_4(\mathbb{P}^{\mathbf{m}}, \mathbb{Q})$, consider the $n \times n$ symmetric matrix $L(\eta)$ with entries

$$L(\eta)_{ij} = \int_{\eta} H_i H_j \text{ for } 1 \leq i \leq j \leq n.$$

The class η is realizable over $\mathbb{Q}$ if and only if $L(\eta)$ is Lorentzian.

Extending Theorem 1.8, we propose in Conjecture 7.3 a numerical characterization of 2-dimensional universally pseudoeffective classes that are realizable over $\mathbb{R}$. For the case when X is the Grassmannian $\text{Gr}(d, n)$, see Theorem 7.7.

Based on our findings, we formulate a convex geometric analogue of Theorem 1.8 that extends Theorem 1.4 to any dimension $d \geq 3$. Extending the notation before, we write π_{ij} for the coordinate projection of $\mathbb{R}^d$ onto the coordinate subspace orthogonal to the standard basis vectors e_i and e_j .

Conjecture 1.9. For any $d \times d$ real symmetric matrix (p_{ij}) with nonnegative off-diagonal and zero diagonal entries, the following conditions are equivalent.

(1) There is a convex body $A \subseteq \mathbb{R}^d$ that satisfies

$$p_{ij} = \left(\text{the volume of the projection } \pi_{ij}(A) \right) \text{ for all } i \neq j.$$

(2) The matrix (p_{ij}) is Lorentzian.

By Corollary 2.5, condition (1) implies condition (2) in Conjecture 1.9. By Theorem 1.4, condition (2) implies condition (1) when $d = 4$. One may also formulate the weaker conjecture that (p_{ij}) is Lorentzian if and only if there exist convex bodies $A_1, \dots, A_{d-2} \subseteq \mathbb{R}^d$ such that

$$p_{ij} = \left(\text{the mixed volume of the projections } \pi_{ij}(A_1), \dots, \pi_{ij}(A_{d-2}) \right) \text{ for all } i \neq j.$$

See Remark 6.12 for the subtlety involved in formulating a similar conjecture for codimension 3 projections $\pi_{ijk} : \mathbb{R}^d \rightarrow \mathbb{R}^{d-3}$.

Notations. The field k will be the field of real or rational numbers. We will work with projective varieties over $\mathbb{C}$, and use singular homology and cohomology of their underlying topological spaces. All of our theorems remain valid over an arbitrary algebraically closed field if we replace the homology group by the group of algebraic cycles modulo homological or numerical equivalence, except in Theorem 6.3, where we need the ground field to be uncountable or of characteristic 0, and in Theorem 6.13, where we need the ground field to be of characteristic 0. A lattice polytope in $\mathbb{R}^d$ is a convex polytope all of whose vertices are in $\mathbb{Z}^d$. A rational polytope in $\mathbb{R}^d$ is a convex polytope all of whose vertices are in $\mathbb{Q}^d$.

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2. HOMOLOGY CLASSES AND VOLUMES

Let X be a complex smooth projective variety, S be an irreducible surface in X , and let $D_1, \dots, D_n$ be the cohomology classes of divisors on X .³ We consider the $n \times n$ intersection matrix $L(S) = L(S; D_1, \dots, D_n)$ defined by

$$L(S)_{ij} = \int_S D_i D_j.$$

The following proposition is a direct consequence of the Hodge Index Theorem [Har77, Section 5.1] applied to the resolution of singularities of S .

Proposition 2.1. The matrix $L(S)$ has at most one positive eigenvalue. If the linear span of D_i contains an ample class of X , then $L(S)$ has exactly one positive eigenvalue.

Let $\mathbf{m}$ be a vector of nonnegative integers $(m_1, \dots, m_n)$, and let $\mathbb{P}^{\mathbf{m}}$ be the product of projective spaces $\prod_{i=1}^n \mathbb{P}^{m_i}$. By the Künneth formula, we have

$$H^\bullet(\mathbb{P}^{\mathbf{m}}, \mathbb{Z}) \simeq \bigotimes_{i=1}^n \mathbb{Z}[H_i]/(H_i^{m_i+1}),$$

where H_i is the pullback of the hyperplane class by the i -th projection $\mathbb{P}^{\mathbf{m}} \rightarrow \mathbb{P}^{m_i}$.

Corollary 2.2. For any irreducible surface S in $\mathbb{P}^{\mathbf{m}}$, the matrix $L(S; H_1, \dots, H_n)$ is Lorentzian.

This corollary provides necessary restrictions on the classes in $H_4(\mathbb{P}^{\mathbf{m}}, \mathbb{Z})$ that are realizable over $\mathbb{Z}$. However, these conditions are not sufficient, see Proposition 5.2 and Example 5.1.

One of our key tools will be the theory of mixed volumes. For a sequence of convex bodies $A_1, \dots, A_d$ in $\mathbb{R}^d$, the *mixed volume* $MV(A_1, \dots, A_d)$ is given by

$$MV(A_1, \dots, A_d) := \frac{1}{d!} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} \text{Vol}(x_1 A_1 + \cdots + x_d A_d),$$

where Vol denotes the d -dimensional normalized volume in $\mathbb{R}^d$. The mixed volume enjoys several important properties: it is symmetric in its arguments, multilinear with respect to Minkowski sums with nonnegative coefficients, monotone with respect to inclusion of convex bodies, and satisfies the identity

$$MV(A, \dots, A) = \text{Vol}(A) \text{ for any convex body } A \subseteq \mathbb{R}^d.$$

For additional background and further properties of mixed volumes, we refer the reader to [Sch14, Chapter 5].

The connection to intersection theory is provided by toric geometry. We start by recalling the Bernstein–Khovanskii–Kushnirenko theorem [CLO05, Section 7.5]. Consider an algebraic torus $T \simeq (\mathbb{C}^*)^d$ with the lattice of characters $M_T \simeq \mathbb{Z}^d$. Let $P_1, \dots, P_d$ be a sequence of lattice

³The statements of this section hold over any algebraically closed field if we use the Chow groups in place of the singular homology groups.

polytopes in $M_T \otimes \mathbb{R}$. We write f_{P_i} for a general $\mathbb{C}$ -linear combination of characters in $P_i \cap M_T$, viewed as a function on T .

Theorem 2.3 (Bernstein–Khovanskii–Kushnirenko). The number of solutions $x \in T$ of the system of equations $f_{P_1}(x) = \dots = f_{P_d}(x) = 0$ is equal to the mixed volume $\text{MV}(P_1, \dots, P_d)$.

Let $\mathbf{m}$ be a vector of positive integers $(m_1, \dots, m_n)$ whose sum is $d \geq 2$. In the remainder of this section, let T be the product torus $\prod_{i=1}^n (\mathbb{C}^*)^{m_i}$ with the lattice of characters $\bigoplus_{i=1}^n \mathbb{Z}^{m_i}$. For each i , we set

$$\Delta_i := (\text{the convex hull of } \mathbf{0} \text{ and the standard basis vectors of } \mathbb{R}^{m_i}) \subseteq \mathbb{R}^d = \bigoplus_{i=1}^n \mathbb{R}^{m_i}.$$

We may view f_{Δ_i} as a function on the torus T of $\mathbb{P}^{\mathbf{m}}$, whose homogenization defines the inverse image of a hyperplane under the i -th projection $\mathbb{P}^{\mathbf{m}} \rightarrow \mathbb{P}^{m_i}$.

Theorem 2.4. Let $P_1, \dots, P_{d-2}$ be convex polytopes in $\mathbb{R}^d$.

- (1) If every P_i is a rational polytope, then there is $\lambda \in \mathbb{Q}_{>0}$ and an irreducible surface $S \subseteq \mathbb{P}^{\mathbf{m}}$ such that

$$\int_S H_i H_j = \lambda^{-1} \text{MV}(P_1, \dots, P_{d-2}, \Delta_i, \Delta_j) \text{ for all } i, j.$$

- (2) If every P_i is a lattice polytope, then there is $\lambda \in \mathbb{Z}_{>0}$ and an irreducible surface $S \subseteq \mathbb{P}^{\mathbf{m}}$ such that

$$\int_S H_i H_j = \lambda^{-1} \text{MV}(P_1, \dots, P_{d-2}, \Delta_i, \Delta_j) \text{ for all } i, j.$$

In the latter case, if $\dim(\sum_{i \in I} P_i) > |I|$ for all nonempty $I \subseteq [d]$, then we may take $\lambda = 1$.

By Poincaré duality, the constant and the mixed volumes in Theorem 2.4 determine the homology class of S in $\mathbb{P}^{\mathbf{m}}$. By varying the polytopes, we obtain many classes in $H_4(\mathbb{P}^{\mathbf{m}}, \mathbb{Q})$ that are realizable over $\mathbb{Q}$. In Section 5, we show that every class in $H_4((\mathbb{P}^1)^4, \mathbb{Q})$ that is realizable over $\mathbb{Q}$ can be realized in this way.

Proof. We use the language and basic constructions of toric geometry. For background, see [Ful93]. To keep our argument valid in arbitrary characteristic, we avoid applying Bertini's theorem on the smoothness of general members of a basepoint-free linear system.

It is enough to prove the statements on lattice polytopes. Let Y be a smooth projective toric variety of T whose normal fan refines the normal fans of P_i and Δ_i . Each P_i defines a basepoint-free divisor class D_i on Y and a map

$$f_i : Y \longrightarrow \mathbb{P}H^0(Y, \mathcal{O}_Y(D_i))^\vee,$$

whose image has dimension equal to that of P_i . To simplify notation, we write D_i for a general member in its linear system. First suppose that $\dim(\sum_{i \in I} P_i) > |I|$ for every nonempty index set I . Under this assumption, we prove that the intersection Z of all the D_i 's is an irreducible surface in Y .

We proceed inductively, assuming that the intersection $Z_i := D_1 \cap \cdots \cap D_i$ is irreducible of codimension i in Y for $i < k$. We claim that the dimension of $f_{i+1}(Z_i)$ is strictly larger than one.

For contradiction, suppose this is not the case. Then, the intersection $Z_i \cap D_{i+1} \cap E_{i+1}$ is empty, where E_{i+1} is another general member in the linear system of D_{i+1} . Let H be a very ample divisor class on Y corresponding to a polytope P . The Bernstein–Khovanskii–Kushnirenko theorem implies that

$$\text{MV}(P_1, \dots, P_i, P_{i+1}, P_{i+1}, P, \dots, P) = 0.$$

By translating P_i and P , we may assume they all contain the origin of $\mathbb{R}^d$. By our assumption and Rado’s theorem [Rad42, Theorem 1], picking general points in $P_1, \dots, P_i, P_{i+1}, P_{i+1}, P, \dots, P$, we obtain a basis of $\mathbb{R}^d$. This contradicts the fact that mixed volume is zero, as mixed volume of linearly independent line segments is nonzero and mixed volume is monotone under inclusions of convex bodies. This finishes the proof of the claim.

We conclude from Bertini’s irreducibility theorem [Jou83, Theorem 6.3 (4)] that Z_{i+1} is irreducible. This finishes the proof that $Z = Z_{d-2}$ is nonempty and irreducible. Writing S for the image of Z in $\mathbb{P}^m$, Theorem 2.3 guarantees that

$$\int_S H_i H_j = \int_Z H_i H_j = \text{MV}(P_1, \dots, P_{d-2}, \Delta_i, \Delta_j) \text{ for all } i, j.$$

In the general case, we show in Lemma 2.6 that $Z_i \cap D_{i+1}$ is a union of irreducible components that are algebraically equivalent to each other. We may therefore define Z_{i+1} to be any one of these irreducible components. Since algebraic equivalence implies homological equivalence [Ful98, Chapter 19], we have

$$\int_S H_i H_j = \int_Z H_i H_j = \lambda^{-1} \text{MV}(P_1, \dots, P_{d-2}, \Delta_i, \Delta_j) \text{ for all } i, j,$$

where λ is a positive rational number. □

Since convex bodies $A_1, \dots, A_{d-2} \subseteq \mathbb{R}^d$ can be approximated by rational polytopes, Proposition 2.2 implies the following classical result in convex geometry.⁴

Corollary 2.5. For any convex bodies $A_1, \dots, A_{d-2} \subseteq \mathbb{R}^d$, the $n \times n$ symmetric matrix with entries $\text{MV}(A_1, \dots, A_{d-2}, \Delta_i, \Delta_j)$ is Lorentzian.

By [Sch14, Theorem 5.3.1], when $m_i = 1$ for all i , we have

$$\text{MV}(A_1, \dots, A_{d-2}, \Delta_i, \Delta_j) = \text{MV}(\pi_{ij}(A_1), \dots, \pi_{ij}(A_{d-2})) \text{ for any } i \neq j.$$

Since the mixed volume is zero when $i = j$, condition (1) implies condition (2) in Conjecture 1.9.

The remainder of this section is devoted to a general lemma used in the proof of Theorem 2.4.

⁴The same argument works more generally when Δ_i ’s are arbitrary convex bodies. This is one of the main conclusions of Brunn–Minkowski theory, equivalent to the Alexandrov–Fenchel inequality. See Section 4 for a more complete discussion. We refer to [Sch14, Section 7.3] for Alexandrov’s original proof.

Lemma 2.6. If $f : Y \rightarrow X$ is a surjective map between irreducible projective varieties over an algebraically closed field, then the irreducible components of a general fiber of f are algebraically equivalent to each other in Y .

Proof. We first consider the case when $f_*\mathcal{O}_Y = \mathcal{O}_X$ and Y is normal. Under these assumptions, we show that a general fiber is irreducible. By [Laz04, Example 2.1.12], the function field of X is algebraically closed in the function field of Y . Therefore, the generic fiber of f is geometrically irreducible [Sta25, Lemma 10.47.8], and hence a general fiber of f is irreducible [Sta25, Lemma 37.27.5].

Next we consider the case when f is finite. If m is the purely inseparable degree of the function field of Y over the function field of X , for a general point $x \in X$,

$$[f^{-1}(x)] = m[y_1] + \cdots + m[y_k],$$

where y_i are the closed points in the fiber of f over x .

We now consider the general case. Let $g : Y' \rightarrow Y$ be the normalization. Consider the composition $Y' \xrightarrow{g} Y \xrightarrow{f} X$, and denote its Stein factorization by $Y' \xrightarrow{f'} X' \xrightarrow{g'} X$. We have the commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g \downarrow & & \downarrow g' \\ Y & \xrightarrow{f} & X \end{array}$$

where $g' : X' \rightarrow X$ is a finite morphism and $f' : Y' \rightarrow X'$ satisfies $f'_*(\mathcal{O}_{Y'}) = \mathcal{O}_{X'}$. The variety X' can be considered as the normalization of X in Y' . Consider the closed subset

$$V' = \{x' \in X' \mid f'^{-1}(x') \text{ is reducible or contained in the exceptional locus of } g\}.$$

The image of V' in X is a proper closed subset of X . Let x be a general point of X so that, in particular, x is not in the image of V' and the fiber of g' over x lies in the smooth locus of X' . Let $F_1, \dots, F_k$ be the irreducible components of the fiber of $g' \circ f'$ over x . Since a general fiber of g' consists of points of the same multiplicity and a general fiber of f' is irreducible, F_i are algebraically equivalent to each other in Y' . Since g is birational, this shows that the irreducible components of $f^{-1}(x)$ are algebraically equivalent to each other in Y . $\square$

3. PROJECTIONS AND MIXED VOLUMES OF FOUR-DIMENSIONAL BODIES

To each pair of convex bodies A and B in $\mathbb{R}^4$, we associate a vector of real numbers

$$A \wedge B = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{R}^4,$$

where the components p_{ij} are defined as the mixed volumes of the projections of A and B . With this notation, the basic properties of mixed volumes read

$$A \wedge B = B \wedge A \quad \text{and} \quad (\lambda_1 A_1 + \lambda_2 A_2) \wedge B = \lambda_1 (A_1 \wedge B) + \lambda_2 (A_2 \wedge B) \quad \text{for } \lambda_1, \lambda_2 \geq 0.$$

The main contributions of this section are Theorems 3.6 and 3.7, which characterize real and rational vectors of the form $A \wedge B$. We also provide a proof of Theorem 1.4 when $\mathbf{p}$ is in $\Delta(\mathbb{T}_2)^\circ$. The proof of Theorem 1.4 when $\mathbf{p}$ is in $\partial\Delta(\mathbb{T}_2)$ will be given in the next section.

These results have direct applications to realizability questions concerning $(\mathbb{P}^1)^4$, as we demonstrate in the following section.

The following reformulation of the Lorentzian condition for 4-by-4 matrices from [BHL] will play a critical role in the proofs of the above characterizations.

Proposition 3.1. A point $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{R}^4$ is in $\Delta(\mathbb{T}_2)$ if and only if

$$L(\mathbf{p}) := \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ p_{12} & 0 & p_{23} & p_{24} \\ p_{13} & p_{23} & 0 & p_{34} \\ p_{14} & p_{24} & p_{34} & 0 \end{pmatrix} \text{ is a Lorentzian matrix.}$$

Proof. A direct computation reveals that

$$\begin{aligned} \det L(\mathbf{p}) = & -(\sqrt{p_{12}p_{34}} + \sqrt{p_{13}p_{24}} + \sqrt{p_{14}p_{23}}) \cdot (-\sqrt{p_{12}p_{34}} + \sqrt{p_{13}p_{24}} + \sqrt{p_{14}p_{23}}) \\ & \cdot (\sqrt{p_{12}p_{34}} - \sqrt{p_{13}p_{24}} + \sqrt{p_{14}p_{23}}) \cdot (\sqrt{p_{12}p_{34}} + \sqrt{p_{13}p_{24}} - \sqrt{p_{14}p_{23}}). \end{aligned}$$

Note that any symmetric 3-by-3 matrix with nonnegative off-diagonal and zero diagonal entries is Lorentzian. Therefore, by Cauchy's interlacing theorem, $L(\mathbf{p})$ is Lorentzian if and only if the product

$$(-\sqrt{p_{12}p_{34}} + \sqrt{p_{13}p_{24}} + \sqrt{p_{14}p_{23}})(\sqrt{p_{12}p_{34}} - \sqrt{p_{13}p_{24}} + \sqrt{p_{14}p_{23}})(\sqrt{p_{12}p_{34}} + \sqrt{p_{13}p_{24}} - \sqrt{p_{14}p_{23}})$$

is nonnegative. Since the sum of every two factors is nonnegative, the product is nonnegative if and only if every factor is nonnegative. Such nonnegativity corresponds precisely to the triangle condition defining $\Delta(\mathbb{T}_2)$. $\square$

For the remainder of this section, we write $\mathbf{k}$ for $\mathbb{Q}$ or $\mathbb{R}$.

Definition 3.2. We define an action of the multiplicative group $\mathbf{k}_{>0} \times \mathbf{k}_{>0}^4$ on $\wedge^2 \mathbf{k}^4$ by

$$(\lambda, c_1, c_2, c_3, c_4) \cdot (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = \lambda(c_1 c_2 p_{12}, c_1 c_3 p_{13}, c_1 c_4 p_{14}, c_2 c_3 p_{23}, c_2 c_4 p_{24}, c_3 c_4 p_{34}).$$

We say that $\mathbf{p}$ and $\mathbf{q}$ are *equivalent over $\mathbf{k}$* if $\mathbf{p}$ and $\mathbf{q}$ are in the same orbit of $\mathbf{k}_{>0} \times \mathbf{k}_{>0}^4 \rtimes S_4$.

Note that the action of the diagonal $\mathbf{k}_{>0}$ is not subsummed by the action of $\mathbf{k}_{>0}^4$ if $\mathbf{k} = \mathbb{Q}$. Clearly, the action of $\mathbf{k}_{>0} \times \mathbf{k}_{>0}^4 \rtimes S_4$ preserves the sets $\Delta(\mathbb{T}_2)$ and $\Delta^\circ(\mathbb{T}_2)$. The next lemma shows that it also preserves the set of vectors of the form $A \wedge B$.

Lemma 3.3. Let A_1, A_2 , and A be convex polytopes with vertices in $\mathbf{k}^4$, and let $\mathbf{p} \in \wedge^2 \mathbf{k}^4$.

- (1) If $\mathbf{p}$ is equivalent to $A_1 \wedge A_2$ over $\mathbf{k}$, then $\mathbf{p} = B_1 \wedge B_2$ for some convex polytopes B_1 and B_2 with vertices in $\mathbf{k}^4$.

(2) If $\mathbf{p}$ is equivalent to $A \wedge A$ over $\mathbb{R}$, then $\mathbf{p} = B \wedge B$ for some convex polytope B in $\mathbb{R}^4$.

Analogous statements hold for convex bodies A_1, A_2 , and A in $\mathbb{R}^4$.

Proof. It is enough to consider the action of $\mathbf{k}_{>0} \times \mathbf{k}_{>0}^4$. For the first statement, suppose that $\mathbf{p} = (\lambda, c_1, c_2, c_3, c_4) \cdot A_1 \wedge A_2$. We take

$$B_1 = (c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}) \cdot A_1 \text{ and } B_2 = \lambda c_1 c_2 c_3 c_4 (c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}) \cdot A_2.$$

For the second statement, suppose that $\mathbf{p} = (\lambda, c_1, c_2, c_3, c_4) \cdot A \wedge A$. We take

$$B = \lambda^{1/2} c_1^{1/2} c_2^{1/2} c_3^{1/2} c_4^{1/2} (c_1^{-1}, c_2^{-1}, c_3^{-1}, c_4^{-1}) \cdot A. \quad \square$$

The following lemma characterizes the points in $\Delta(\mathbb{T}_2) \cap \wedge^2 \mathbf{k}^4$ that have at least one zero entry, up to equivalence over $\mathbf{k}$.

Lemma 3.4. If $\mathbf{p}$ is a nonzero vector in $\Delta(\mathbb{T}_2) \cap \wedge^2 \mathbf{k}^4$ with a zero entry, then $\mathbf{p}$ is equivalent over $\mathbf{k}$ to exactly one of the following six points:

$$(1, 1, 1, 1, 1, 0), (1, 1, 0, 1, 0, 0), (0, 1, 1, 1, 1, 0), (1, 1, 1, 0, 0, 0), (1, 1, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0).$$

Each one of these points is of the form $A \wedge A$ for some lattice polytope A .

Proof. We prove the first statement. If $\mathbf{p} \in \Delta(\mathbb{T}_2)$ has a zero entry, then, either all three products $p_{12}p_{34}, p_{13}p_{24}, p_{14}p_{23}$ are zero, or exactly one of them is zero. In the former case, it is straightforward to verify that $\mathbf{p}$ is equivalent over $\mathbf{k}$ to one of the following four points:

$$(1, 1, 0, 1, 0, 0), (1, 1, 1, 0, 0, 0), (1, 1, 0, 0, 0, 0), (1, 0, 0, 0, 0, 0).$$

In the latter case, we may assume without loss of generality that $p_{34} = 0$ and thus $p_{13}p_{24} = p_{14}p_{23}$ is nonzero. Then $\mathbf{p}$ is equivalent over $\mathbf{k}$ to exactly one of the points

$$(0, 1, 1, 1, 1, 0) \text{ or } (1, 1, 1, 1, 1, 0),$$

depending on whether p_{12} is zero or nonzero, respectively. This proves the first statement.

The second statement is also a straightforward computation:

$$A_2 \wedge A_2 = (1, 1, 1, 1, 1, 0) \text{ when } A_2 = \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4),$$

$$A_3 \wedge A_3 = (1, 1, 0, 1, 0, 0) \text{ when } A_3 = \text{conv}(\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4),$$

$$A_4 \wedge A_4 = (0, 1, 1, 1, 1, 0) \text{ when } A_4 = \text{conv}(\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4),$$

$$A_5 \wedge A_5 = (1, 1, 1, 0, 0, 0) \text{ when } A_5 = \text{conv}(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4),$$

$$A_6 \wedge A_6 = (1, 1, 0, 0, 0, 0) \text{ when } A_6 = \text{conv}(\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_4),$$

$$A_7 \wedge A_7 = (1, 0, 0, 0, 0, 0) \text{ when } A_7 = \text{conv}(\mathbf{0}, \mathbf{e}_3, \mathbf{e}_4). \quad \square$$

We now proceed to prove Theorem 1.4. The first statement, the necessity of the condition $\mathbf{p} \in \Delta(\mathbb{T}_2)$, directly follows from Propositions 3.1 and Corollary 2.5. For the second statement,

the sufficiency of the condition $\mathbf{p} \in \Delta(\mathbb{T}_2)^\circ$, we construct a convex polytope A in $\mathbb{R}^4$ satisfying $\mathbf{p} = A \wedge A$. The construction relies on the following symmetrization lemma.

Lemma 3.5. For any vector of positive real numbers $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$, there is a point $\mathbf{q} = (q_{12}, q_{13}, q_{14}, q_{14}, q_{13}, q_{12})$ equivalent to $\mathbf{p}$ over $\mathbb{R}$.

Proof. Let λ be the reciprocal of $\sqrt{p_{12}p_{13}p_{14}p_{23}p_{24}p_{34}}$, and take

$$c_1 = \sqrt{p_{23}p_{24}p_{34}}, \quad c_2 = \sqrt{p_{13}p_{14}p_{34}}, \quad c_3 = \sqrt{p_{12}p_{14}p_{24}}, \quad c_4 = \sqrt{p_{12}p_{13}p_{23}}.$$

It is straightforward to check that $(\lambda, c_1, c_2, c_3, c_4) \cdot \mathbf{p}$ is equal to

$$(\sqrt{p_{12}p_{34}}, \sqrt{p_{13}p_{24}}, \sqrt{p_{14}p_{23}}, \sqrt{p_{14}p_{23}}, \sqrt{p_{13}p_{24}}, \sqrt{p_{12}p_{34}}).$$

□

There are exactly three ways to partition a set of four elements into two subsets of two elements each. The corresponding line segments in $\mathbb{R}^4$ will be the building blocks of A :

$$D_{12|34} := \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4), \quad D_{13|24} := \text{conv}(\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4), \quad D_{14|23} := \text{conv}(\mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3).$$

A straightforward computation shows:

$$\begin{aligned} D_{12|34} \wedge D_{13|24} &= (2, 2, 0, 0, 2, 2), & D_{12|34} \wedge D_{12|34} &= (0, 0, 0, 0, 0, 0), \\ D_{12|34} \wedge D_{14|23} &= (2, 0, 2, 2, 0, 2), & D_{13|24} \wedge D_{13|24} &= (0, 0, 0, 0, 0, 0), \\ D_{13|24} \wedge D_{14|23} &= (0, 2, 2, 2, 2, 0), & D_{14|23} \wedge D_{14|23} &= (0, 0, 0, 0, 0, 0). \end{aligned}$$

Proof of Theorem 1.4 when $\mathbf{p}$ is in $\Delta(\mathbb{T}_2)^\circ$. We first construct a convex polytope with projection areas given by $\mathbf{p}$. By Lemmas 3.3 and 3.5, we may assume that the given point $\mathbf{p}$ in $\Delta(\mathbb{T}_2)^\circ$ is of the form $(p_{12}, p_{13}, p_{14}, p_{14}, p_{13}, p_{12})$. By permuting the coordinates, we may suppose that $p_{12} \geq p_{13} \geq p_{14}$, which gives

$$a := p_{12} - p_{13} \geq 0, \quad b := p_{13} - p_{14} \geq 0, \quad c := p_{13} + p_{14} - p_{12} > 0.$$

Note that c is positive because $\mathbf{p}$ is in the interior of $\Delta(\mathbb{T}_2)$. From the computation above,

$$(xD_{12|34} + yD_{13|24} + zD_{14|23}) \wedge (xD_{12|34} + yD_{13|24} + zD_{14|23}) = (p_{12}, p_{13}, p_{14}, p_{14}, p_{13}, p_{12})$$

for indeterminates x, y, z , translates into the system of equations

$$xy = u := (a + b + d)/4, \quad xz = v := (a + d)/4, \quad yz = w := d/4, \quad \text{where } d = c/2 > 0.$$

This system has a unique positive solution

$$x = \sqrt{uv/w}, \quad y = \sqrt{uw/v}, \quad z = \sqrt{vw/u}.$$

This proves that there is a convex polytope A such that $A \wedge A = \mathbf{p}$.

We now show that there is a smooth convex body A with $A \wedge A = \mathbf{p}$. Choose smooth families of smooth convex bodies $D_{12|34}(\epsilon), D_{13|24}(\epsilon), D_{14|23}(\epsilon)$ that converge to the line segments $D_{12|34}, D_{13|24}, D_{14|23}$ as ϵ goes to zero, and consider the Minkowski sum

$$A = xD_{12|34}(\epsilon) + yD_{13|24}(\epsilon) + zD_{14|23}(\epsilon).$$

With u, v, w defined as above, the system of equations $A \wedge A = \mathbf{p}$ translates to

$$xy + f(x, y, z, \epsilon) = u, \quad xz + g(x, y, z, \epsilon) = v, \quad yz + h(x, y, z, \epsilon) = w,$$

where f, g, h are smooth functions satisfying

$$f(x, y, z, 0) = g(x, y, z, 0) = h(x, y, z, 0) = 0 \text{ for all } x, y, z > 0.$$

We consider the function $F : \mathbb{R}_{\geq 0}^4 \rightarrow \mathbb{R}^3$ given by

$$F(x, y, z, \epsilon) := (xy + f(x, y, z, \epsilon), xz + g(x, y, z, \epsilon), yz + h(x, y, z, \epsilon)).$$

By modifying the families $D_{12|34}(\epsilon), D_{13|24}(\epsilon), D_{14|23}(\epsilon)$ if necessary, we may suppose that F extends to a smooth function on a neighborhood of the point

$$(\sqrt{uv/w}, \sqrt{uw/v}, \sqrt{vw/u}, 0) \in F^{-1}(u, v, w).$$

For example, we may take $D_{12|34}(\epsilon)$ to be the ellipsoid of maximal volume inscribed in the ϵ -neighborhood of $D_{12|34}$. The leading maximal minor of the Jacobian of F at the point is $-2\sqrt{uvw}$, which is nonzero. Thus, by the implicit function theorem, there is a germ of a smooth curve $(x(\epsilon), y(\epsilon), z(\epsilon))$ passing through the point that satisfies, for all sufficiently small ϵ ,

$$F(x(\epsilon), y(\epsilon), z(\epsilon), \epsilon) = (u, v, w).$$

This shows that there are positive numbers x, y, z and ϵ that satisfy $A \wedge A = \mathbf{p}$. $\square$

We now characterize vectors of mixed areas of coordinate projections of pairs of convex bodies in $\mathbb{R}^4$:

Theorem 3.6. We have the equalities

$$\Delta(\mathbb{T}_2) = \{A \wedge B \mid A, B \text{ are convex bodies in } \mathbb{R}^4\} = \{A \wedge B \mid A, B \text{ are convex polytopes in } \mathbb{R}^4\}.$$

Proof. As previously noted, the necessity of the condition $\mathbf{p} \in \Delta(\mathbb{T}_2)$ follows from Proposition 3.1 and Corollary 2.5. We prove the sufficiency. By Theorem 1.4, we may suppose that $\mathbf{p} \in \partial\Delta(\mathbb{T}_2)$. By Lemma 3.4, we may suppose that every entry of $\mathbf{p}$ is positive. By Lemma 3.5, we may suppose that $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{14}, p_{13}, p_{12})$. By permuting coordinates, we may suppose that $p_{12} = p_{13} + p_{14}$. Under these assumptions, we find that

$$(p_{13}D_{13|24} + p_{14}D_{14|23}) \wedge (2^{-1}D_{12|34}) = (p_{12}, p_{13}, p_{14}, p_{14}, p_{13}, p_{12}). \quad \square$$

We now characterize vectors of mixed areas of coordinate projections of pairs of rational polytopes in $\mathbb{R}^4$:

Theorem 3.7. We have the equality

$$\Delta(\mathbb{T}_2) \cap \wedge^2 \mathbb{Q}^4 = \{A \wedge B \mid A, B \text{ are rational polytopes in } \mathbb{R}^4\}.$$

The main technical challenge is the absence of Lemma 3.5 for $\mathbf{k} = \mathbb{Q}$. We begin by analyzing the symmetric case.

Lemma 3.8. If $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{14}, p_{13}, p_{12}) \in \Delta(\mathbb{T}_2)$ is a vector of nonnegative rational numbers satisfying $p_{12} \geq p_{13} \geq p_{14}$, then there is a rational polytope A such that $\mathbf{p} = A \wedge D_{12|34}$.

Proof. We consider the lattice polytopes

$$P := D_{13|24} + D_{14|23}, \quad Q := D_{13|24}, \quad R := \text{conv}(\mathbf{e}_1, \mathbf{e}_2) + \text{conv}(\mathbf{e}_3, \mathbf{e}_4).$$

These polytopes give the following projection mixed volumes

$$P \wedge D_{12|34} = (4, 2, 2, 2, 2, 4), \quad Q \wedge D_{12|34} = (2, 2, 0, 0, 2, 2), \quad R \wedge D_{12|34} = (2, 2, 2, 2, 2, 2).$$

Therefore, $x := (p_{12} - p_{13})/2 \geq 0$, $y := (p_{13} - p_{14})/2 \geq 0$, $z := (p_{13} + p_{14} - p_{12})/2 \geq 0$ satisfy

$$(xP + yQ + zR) \wedge D_{12|34} = (p_{12}, p_{13}, p_{14}, p_{14}, p_{13}, p_{12}). \quad \square$$

We continue with an analysis of the boundary case.

Lemma 3.9. If $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \partial\Delta(\mathbb{T}_2)$ is a vector of positive rational numbers, then there is a vector $\mathbf{q} = (q_{12}, q_{13}, q_{14}, q_{14}, q_{13}, q_{12}) \in \partial\Delta(\mathbb{T}_2)$ of positive rational numbers equivalent to $\mathbf{p}$ over $\mathbb{Q}$.

Proof. Without loss of generality, we may assume $p_{14} = p_{23} = 1$ and

$$\sqrt{p_{12}p_{34}} + 1 = \sqrt{p_{13}p_{24}}.$$

Taking the square, we see that $\sqrt{p_{12}p_{34}}$ is rational, and hence, so is $\sqrt{p_{13}p_{24}}$. Take

$$c_1 = \sqrt{p_{12}p_{34}}\sqrt{p_{13}p_{24}}, \quad c_2 = p_{13}\sqrt{p_{12}p_{34}}, \quad c_3 = p_{12}\sqrt{p_{13}p_{24}}, \quad c_4 = p_{12}p_{13}.$$

It is straightforward to check that $(c_1^{-1}c_4^{-1}, c_1, c_2, c_3, c_4) \cdot \mathbf{p}$ is equal to

$$(\sqrt{p_{12}p_{34}}, \sqrt{p_{13}p_{24}}, 1, 1, \sqrt{p_{13}p_{24}}, \sqrt{p_{12}p_{34}}). \quad \square$$

We conclude this section with the proof of Theorem 3.7.

Proof of Theorem 3.7. If $\mathbf{p}$ has at least one zero entry, the existence follows from Lemma 3.4. If $\mathbf{p}$ has only positive entries and satisfies a degenerate triangle inequality, the existence follows from Lemmas 3.8 and 3.9.

Suppose $\mathbf{p}$ has only positive entries and satisfies the nondegenerate triangle inequalities. By Lemma 3.5, we have $\mathbf{p}$ is equivalent to $\mathbf{q} = (q_{12}, q_{13}, q_{14}, q_{14}, q_{13}, q_{12})$ over $\mathbb{R}$, where

$$q_{12} = \sqrt{p_{12}p_{34}}, \quad q_{13} = \sqrt{p_{13}p_{24}}, \quad q_{14} = \sqrt{p_{14}p_{23}}.$$

Without loss of generality, we may suppose that $q_{12} \geq q_{13} \geq q_{14}$.

Choose rational polytopes $R_{12}, R_{13}, R_{14}, R_{23}, R_{24}, R_{34}$ in $\mathbb{R}^4$ such that

$$\begin{aligned} R_{12} \wedge D_{12|34} &= (0, 1, 1, 1, 1, 1), & R_{23} \wedge D_{12|34} &= (1, 1, 1, 0, 1, 1), \\ R_{13} \wedge D_{12|34} &= (1, 0, 1, 1, 1, 1), & R_{24} \wedge D_{12|34} &= (1, 1, 1, 1, 0, 1), \\ R_{14} \wedge D_{12|34} &= (1, 1, 0, 1, 1, 1), & R_{34} \wedge D_{12|34} &= (1, 1, 1, 1, 1, 0). \end{aligned}$$

For example, we may choose the convex polytopes

$$\begin{aligned} R_{12} &= \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_1, \mathbf{0}), & R_{23} &= \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_3, \mathbf{e}_4), \\ R_{13} &= \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_4), & R_{24} &= \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_1, \mathbf{e}_3), \\ R_{14} &= \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2, \mathbf{e}_3), & R_{34} &= \text{conv}(\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_3, \mathbf{0}). \end{aligned}$$

For any point $\mathbf{v} \in \wedge^2 \mathbb{R}^4$, we define a cone centered at $\mathbf{v}$ by

$$\mathcal{C}(\mathbf{v}) = \left\{ \mathbf{v} + \sum_{ij} c_{ij} R_{ij} \wedge D_{12|34} \mid c_{ij} \geq 0 \right\},$$

and denote its interior by $\mathcal{C}^\circ(\mathbf{v})$. We write S for the subspace of $\wedge^2 \mathbb{R}^4$ consisting of points

$$(a_1, a_2, a_3, a_3, a_2, a_1) \text{ for } a_1, a_2, a_3 \in \mathbb{R}.$$

Take $\mathbf{q}_- = \mathbf{q} - \varepsilon(1, 1, 1, 1, 1, 1)$, where $\varepsilon > 0$ is chosen small enough so that $\mathbf{q}_-$ has positive entries and satisfies the nondegenerate triangle inequalities. Clearly, $\mathbf{q}$ is in $\mathcal{C}^\circ(\mathbf{q}_-)$.

Since $\mathcal{C}(\mathbf{q}_-)$ is a full-dimensional cone whose interior intersects S , there is a rational point arbitrarily close to $\mathbf{q}_-$ in the intersection $\mathcal{C}^\circ(\mathbf{q}_-) \cap S$. Thus, there is a rational point

$$\mathbf{r} = (r_{12}, r_{13}, r_{14}, r_{14}, r_{13}, r_{12}) \text{ satisfying } r_{12} \geq r_{13} \geq r_{14} \text{ and } \mathbf{q} \in \mathcal{C}_\mathbf{r}^\circ.$$

By Lemma 3.8, we have $\mathbf{r} = A \wedge D_{12|34}$ for some rational polytope A . Thus, any rational point in $\mathcal{C}_\mathbf{r}$ is of the form

$$(A + c_{12}R_{12} + c_{13}R_{13} + c_{14}R_{14} + c_{23}R_{23} + c_{24}R_{24} + c_{34}R_{34}) \wedge D_{12|34}$$

for some nonnegative rational numbers c_{ij} . Since $\mathbb{Q}_{\geq 0}^4$ is dense in $\mathbb{R}_{\geq 0}^4$, there is a point arbitrarily close to $\mathbf{q}$ that is equivalent to $\mathbf{p}$ over $\mathbb{Q}$. Clearly, we may take this rational point to be in $\mathcal{C}_\mathbf{r}^\circ$. We conclude by Lemma 3.3. $\square$

4. REALIZING THE BOUNDARY OF $\Delta(\mathbb{T}_2)$

In this section, we finish the proof of Theorem 1.4. We construct a family of convex bodies $A \subseteq \mathbb{R}^4$ such that $A \wedge A$ sweeps out $\partial\Delta(\mathbb{T}_2) \simeq S^4$ and show that any such A necessarily has a singular point in its boundary. By Lemma 3.4, we may suppose that the given vector in $\partial\Delta(\mathbb{T}_2)$ has all entries positive. By Lemmas 3.3 and 3.5, we may suppose that the given vector is of the form $(s+1, s, 1, 1, s, s+1)$ for some $s \geq 1$. We will give an explicit description of the vertices of a convex polytope A as a function of s .

Finding a convex polytope A satisfying $A \wedge A = (s+1, s, 1, 1, s, s+1)$ is nontrivial. Nevertheless, once the vertices of A are correctly guessed as a function of s , verifying that A has the required property is straightforward, as demonstrated below. In the remainder of this section, we show how recent breakthroughs on the equality conditions of the Alexandrov–Fenchel inequality in [SvH22, SvH23] lead to a specific set of vertices of A .

Proposition 4.1. For any $s \geq 1$, there is a convex polytope $A \subseteq \mathbb{R}^4$ such that

$$A \wedge A = (s+1, s, 1, 1, s, s+1).$$

Proof. Take $c = \sqrt{1 - 1/s}$, and consider the 4×16 matrices

$$L := \begin{bmatrix} 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \end{bmatrix},$$

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ -2 & 0 & 0 & -2 & -1 & 0 & -1 & -1 & -1 & -2 & 0 & -1 & -1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & 1 & 1 & 1 & 0 \\ -2 & 0 & 0 & 0 & -1 & -2 & -1 & -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Let A be the convex hull of the 16 columns of $L + cM$ in $\mathbb{R}^4$. With patience, one can check that

$$|\pi_{12}(A)| = |\pi_{34}(A)| = 8 - 4c^2, \quad |\pi_{13}(A)| = |\pi_{24}(A)| = 4, \quad |\pi_{14}(A)| = |\pi_{23}(A)| = 4 - 4c^2,$$

where $|\pi_{ij}(A)|$ is the normalized volume in $\mathbb{R}^2$. We have

$$A \wedge A \sim \left(\frac{2-c^2}{1-c^2}, \frac{1}{1-c^2}, 1, 1, \frac{1}{1-c^2}, \frac{2-c^2}{1-c^2} \right) = (s+1, s, 1, 1, s, s+1). \quad \square$$

We now explain how we obtained the vertices of A . Let Δ_i be the line segment joining $\mathbf{0}$ and the standard basis vector $\mathbf{e}_i$. The main observation here is that $A \wedge A = (s+1, s, 1, 1, s, s+1)$ corresponds to the equality case

$$\text{MV}(A, A, \Delta_1 + \Delta_2, \Delta_3 + \Delta_4)^2 = \text{MV}(A, A, \Delta_1 + \Delta_2, \Delta_1 + \Delta_2) \text{MV}(A, A, \Delta_3 + \Delta_4, \Delta_3 + \Delta_4),$$

because both sides are equal to $4(s+1)^2$. Unlike many classical geometric inequalities, which typically admit simple equality characterizations, the Alexandrov–Fenchel inequality has a rich family of equality cases that depends on the relative position of singular boundary points on the convex bodies [Sch14, Section 7.6]. Two recent breakthroughs of Shenfeld–van Handel in [SvH22] and [SvH23] settled respectively the special case of Minkowski inequality for arbitrary convex bodies and the general case of Alexandrov–Fenchel inequality for convex polytopes.

The statement we need is [SvH22, Theorem 1.3], which confirms Schneider’s conjecture from [Sch85] in the setting of Minkowski’s quadratic inequality. Let P, Q , and A be convex bodies in $\mathbb{R}^n$. A nonzero vector $\mathbf{u} \in \mathbb{R}^n$ is called an r -extreme normal vector of A if there do not exist linearly independent normal vectors $\mathbf{u}_1, \dots, \mathbf{u}_{r+2}$ at a boundary point of A such that $\mathbf{u} = \mathbf{u}_1 + \dots + \mathbf{u}_{r+2}$.

Theorem 4.2 (Shenfeld–van Handel). Suppose that A is n -dimensional and $\text{MV}(A, \dots, A, Q, Q)$ is positive. In this case,

$$\text{MV}(A, \dots, A, P, Q)^2 = \text{MV}(A, \dots, A, P, P) \text{MV}(A, \dots, A, Q, Q)$$

holds if and only if there are $\lambda \geq 0$ and $\mathbf{t} \in \mathbb{R}^n$ such that P and $\lambda Q + \mathbf{t}$ have the same supporting hyperplanes in all 1-extreme normal directions of A .

When A is a polytope containing the origin in its interior, $\mathbf{u}$ is a 1-extreme normal direction if and only if there exists $c > 0$ such that $c\mathbf{u}$ lies on the 1-skeleton of the polar dual of A . The *support function* of a convex body $P \subseteq \mathbb{R}^n$ is the convex function

$$h_P(\mathbf{u}) := \max_{\mathbf{x} \in P} \langle \mathbf{x}, \mathbf{u} \rangle.$$

In terms of the support functions, the equality condition in Theorem 4.2 is that, for some $\lambda \geq 0$ and $\mathbf{t} \in \mathbb{R}^n$, we have

$$h_P(\mathbf{u}) = \lambda h_Q(\mathbf{u}) + \langle \mathbf{t}, \mathbf{u} \rangle \text{ for every 1-extreme normal direction } \mathbf{u} \text{ of } A.$$

Note that λ is necessarily equal to the ratio $MV(A, \dots, A, P, Q)/MV(A, \dots, A, Q, Q)$ because

$$MV(A, \dots, A, P, Q) = MV(A, \dots, A, \lambda Q + \mathbf{t}, Q) = \lambda MV(A, \dots, A, Q, Q).$$

When $P = \Delta_1 + \Delta_2$ and $Q = \Delta_3 + \Delta_4$, Theorem 4.2 says the following.

Corollary 4.3. If A is a 4-dimensional convex body such that $A \wedge A = (s+1, s, 1, 1, s, s+1)$ for $s \geq 1$, then there exists $\mathbf{t} \in \mathbb{R}^4$ such that

$$\max(u_1, 0) + \max(u_2, 0) - \max(u_3, 0) - \max(u_4, 0) = \langle \mathbf{t}, \mathbf{u} \rangle$$

holds for all 1-extreme directions $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of A .

Corollary 4.3 poses a strong constraint on the convex bodies corresponding to a point in $\partial\Delta(\mathbb{T}_2)$. When A is smooth, every normal direction is 1-extreme, so no smooth convex body A satisfies $A \wedge A = (s+1, s, 1, 1, s, s+1)$ for $s \geq 1$. When A is a convex polytope, we can use Corollary 4.3 to determine the facet directions of A . We illustrate this for $\mathbf{t} = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$.

Lemma 4.4. Let B be a convex polytope in $\mathbb{R}^4$ containing the origin in its interior. Suppose that all of its vertices and edges are contained in the hypersurface

$$\mathcal{T} := \left\{ \mathbf{u} \in \mathbb{R}^4 \setminus 0 \mid \max(u_1, 0) + \max(u_2, 0) - \max(u_3, 0) - \max(u_4, 0) = \frac{1}{2}(u_1 + u_2 - u_3 - u_4) \right\}.$$

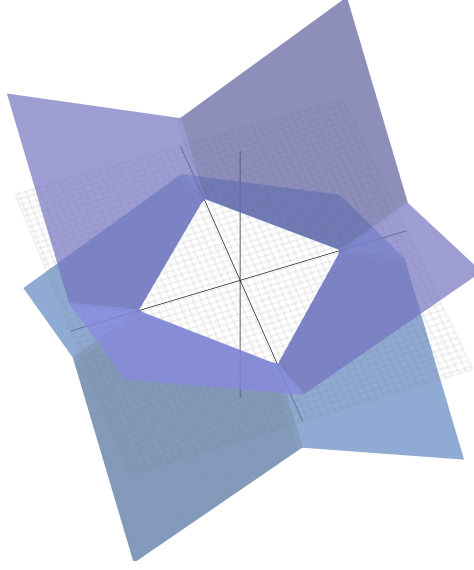
Then B has at most 16 vertices, each lying on a ray generated by a column of the matrix

$$N := \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \end{bmatrix}.$$

The proof is straightforward, given the piecewise linear structure of $\mathcal{T}$, shown below in Figure 1.

Applying Lemma 4.4 to the polar dual B of A , we see that A must be of the form

$$A = \bigcap_{i=1}^{16} \left\{ \mathbf{x} \in \mathbb{R}^4 \mid \langle \mathbf{x}, \mathbf{v}_i \rangle \geq a_i \right\},$$

FIGURE 1. The hypersurface $\mathcal{T}$ in the affine chart $(u_1/u_4, u_2/u_4, u_3/u_4)$

where $\mathbf{v}_i$ are the column vectors of the matrix N and $a_1, \dots, a_{16}$ are some undetermined numbers. The area of the projection $\pi_{ij}(A)$ is a function of the 16 variables a_i . The combinatorial type of A subdivides $\mathbb{R}^{16}$ into a finite number of polyhedral regions, and in each region, the area of $\pi_{ij}(A)$ is a quadratic polynomial in the variables a_i .

In principle, one can compute the subdivision of $\mathbb{R}^{16}$ and the corresponding piecewise quadratic polynomial. For our purposes, it is enough to consider the following 1-parameter family $A = A(c)$ given by the vector

$$(a_1, \dots, a_{16}) = -(1, 1, 1, 1, 1, 1 + 2c, 1, 1 - 2c, 1, 1 + 2c, 1 + 2c, 1, 1, 1, 1 - 2c, 1 - 2c), \quad c \in [0, 1].$$

One can check that its vertices are given by the matrices L and M in Proposition 4.1, and

$$|\pi_{12}(A)| = |\pi_{34}(A)| = 8 - 4c^2, \quad |\pi_{13}(A)| = |\pi_{24}(A)| = 4, \quad |\pi_{14}(A)| = |\pi_{23}(A)| = 4 - 4c^2.$$

The method of finding A does not depend on the specific choice of $\mathbf{t}$ or on the assumption that A is a convex polytope. This provides a way to classify all convex bodies A such that $A \wedge A = \mathbf{p}$ for a fixed $\mathbf{p} \in \partial\Delta(\mathbb{T}_2)$.

5. REALIZABLE CLASSES IN $(\mathbb{P}^1)^4$

For $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{Q}^4$, we write $\eta(\mathbf{p})$ for the homology class

$$p_{12}[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^0 \times \mathbb{P}^0] + \dots + p_{34}[\mathbb{P}^0 \times \mathbb{P}^0 \times \mathbb{P}^1 \times \mathbb{P}^1] \in H_4((\mathbb{P}^1)^4, \mathbb{Q}).$$

Theorem 1.6 states that $\eta(\mathbf{p})$ is realizable over $\mathbb{Q}$ if and only if $\mathbf{p} \in \Delta(\mathbb{T}_2)$.

Proof of Theorem 1.6. We only need to collect the pieces we already have. The necessity of the condition $\mathbf{p} \in \Delta(\mathbb{T}_2)$ follows from Corollary 2.2 and Proposition 3.1. The sufficiency of the condition $\mathbf{p} \in \Delta(\mathbb{T}_2)$ follows from Theorem 2.4 Theorem 3.7. $\square$

As noted before, $\mathbf{p} \in \Delta(\mathbb{T}_2) \cap \wedge^2 \mathbb{Z}^4$ does not imply that $\eta(\mathbf{p})$ is realizable over $\mathbb{Z}$.

Example 5.1. Consider the point $\mathbf{p} = (1, 1, 1, 1, 1, 4) \in \partial\Delta(\mathbb{T}_2)$. Proposition 5.2 below shows that $\eta = \eta(\mathbf{p})$ is not realizable over $\mathbb{Z}$. On the other hand, consider the lattice polytope

$$P := \text{conv}(2\mathbf{e}_1, 2\mathbf{e}_2, 2\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4) \subseteq \mathbb{R}^4,$$

which has projection areas $(2, 2, 2, 2, 2, 8)$. Therefore, by Theorem 2.4, 2η is realizable over $\mathbb{Z}$. In fact, by Theorem 6.13, the class $p\eta$ is realizable over $\mathbb{Z}$ for any prime number p .

In Theorem 6.13, we prove a more precise version of Theorem 1.6: If $\mathbf{p}$ is a primitive integral vector in $\Delta(\mathbb{T}_2)$ and p is a prime number, then $\eta(\mathbf{p})$ is realizable over $\mathbb{Z}$ or $p\eta(\mathbf{p})$ is realizable over $\mathbb{Z}$.⁵

Proposition 5.2. If $\eta(\mathbf{p})$ is realizable over $\mathbb{Z}$ and $p_{ij} > 0$, then $p_{kl} \leq p_{ik}p_{jl} + p_{il}p_{jk}$.

Note that $\sqrt{p_{ij}p_{kl}} \leq \sqrt{p_{ik}p_{jl}} + \sqrt{p_{il}p_{jk}}$ implies $p_{kl} \leq p_{ik}p_{jl} + p_{il}p_{jk}$ when $p_{ij} > 1$. For $i = 1$ and $j = 2$, the points

$$(0, 0, 1, 0, 1, 1), (0, 0, 0, 1, 1, 1), (0, 0, 0, 0, 1, 1), (0, 0, 0, 0, 0, 1) \in \partial\Delta(\mathbb{T}_2)$$

fail to satisfy both the assumption and the conclusion of Proposition 5.2. The corresponding homology classes are realizable over $\mathbb{Z}$ by Lemma 3.4 and Theorem 2.4. When $\mathbf{p} \in \Delta(\mathbb{T}_2)^\circ$, we know no other obstructions to the realizability of $\eta(\mathbf{p})$ over $\mathbb{Z}$ than the ones from Theorem 1.6 and Proposition 5.2.

As before, for $E \subseteq [4]$, we write φ_E for the coordinate projection $(\mathbb{P}^1)^4 \rightarrow (\mathbb{P}^1)^{4-|E|}$ that forgets the coordinates labelled by E . Following the usual abuse of notation, we use the same symbol $[V]$ for the Poincaré dual of the homology fundamental class $[V]$.

Proof. Let S be an irreducible surface in $(\mathbb{P}^1)^4$ with fundamental class $\eta(\mathbf{p})$. We show that $p_{34} \leq p_{13}p_{24} + p_{14}p_{23}$ when $p_{12} > 0$. Since $p_{12} > 0$, the coordinate projections $\varphi_3(S)$ and $\varphi_4(S)$ are irreducible hypersurfaces in $(\mathbb{P}^1)^3$, and their cohomology classes satisfy

$$d_3[\varphi_3(S)] = p_{24}H_1 + p_{14}H_2 + p_{12}H_4, \quad d_4[\varphi_4(S)] = p_{23}H_1 + p_{13}H_2 + p_{12}H_3,$$

where d_i is the degree of the map $S \rightarrow \varphi_i(S)$. The inverse images $\varphi_3^{-1}\varphi_3(S)$ and $\varphi_4^{-1}\varphi_4(S)$ are irreducible hypersurfaces in $(\mathbb{P}^1)^4$, and they are necessarily distinct because $p_{12} > 0$. Thus

⁵An integral vector is said to be *primitive* if its entries are coprime, that is, when the greatest common divisor of the entries is 1. In contrast to the other arguments in this paper, the proof of this statement requires the base field to have characteristic zero.

$\varphi_3^{-1}\varphi_3(S)$ and $\varphi_4^{-1}\varphi_4(S)$ intersect generically transversally, and

$$\begin{aligned} d_3 d_4 [\varphi_3^{-1}\varphi_3(S) \cap \varphi_4^{-1}\varphi_4(S)] &= (p_{24}H_1 + p_{14}H_2 + p_{12}H_4)(p_{23}H_1 + p_{13}H_2 + p_{12}H_3) \\ &= (p_{13}p_{24} + p_{14}p_{23})H_1H_2 + p_{12}(p_{24}H_1H_3 + p_{23}H_1H_4 + p_{14}H_2H_3 + p_{13}H_2H_4 + p_{12}H_3H_4). \end{aligned}$$

Since S is an irreducible component of the intersection of $\varphi_3^{-1}\varphi_3(S)$ and $\varphi_4^{-1}\varphi_4(S)$,

$$\deg(\varphi_{12} : S \rightarrow (\mathbb{P}^1)^2) \leq \deg(\varphi_{12} : \varphi_3^{-1}\varphi_3(S) \cap \varphi_4^{-1}\varphi_4(S) \rightarrow (\mathbb{P}^1)^2).$$

Thus, by Poincaré duality, the above computation gives

$$p_{34} \leq d_3 d_4 p_{34} \leq p_{13}p_{24} + p_{14}p_{23}. \quad \square$$

Remark 5.3. A complete characterization of the classes in $H_4(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z})$ realizable over $\mathbb{Z}$ is given in [Huh13].

Consider now the homology classes of surfaces in $(\mathbb{P}^1)^4$ obtained by intersecting classes of hypersurfaces. We refer to these as *complete intersection classes*.

Definition 5.4. Let $\Delta(\mathbb{T}_1)$ be the set of $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{R}^4$ such that

$$p_{ij} \geq 0 \text{ and } p_{ij}p_{kl} + p_{ik}p_{jl} \geq p_{il}p_{jk} \text{ for any } i, j, k, l.$$

We use $\Delta(\mathbb{T}_1)$ to characterize the complete intersection classes in $(\mathbb{P}^1)^4$, up to a positive rational multiple:

Theorem 5.5. For any nonzero $\mathbf{p} \in \wedge^2 \mathbb{Q}^4$, some positive rational multiple of $\eta(\mathbf{p})$ is a complete intersection class if and only if $\mathbf{p} \in \Delta(\mathbb{T}_1)$.

It is not true that $\eta(\mathbf{p})$ is always a complete intersection class for any $\mathbf{p} \in \Delta(\mathbb{T}_1) \cap \wedge^2 \mathbb{Z}^4$. For example, $\eta(1, 1, 1, 1, 1, 1)$ is not a complete intersection class, while $\eta(2, 2, 2, 2, 2, 2)$ is a complete intersection class.

Proof. It will be useful to work with $H^\bullet((\mathbb{P}^1)^4, \mathbb{Q})$ instead of $H_\bullet((\mathbb{P}^1)^4, \mathbb{Q})$. We set

$$(q_{34}, q_{24}, q_{23}, q_{14}, q_{13}, q_{12}) := (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}),$$

and, for any distinct indices i, j, k, l , define

$$q_{ij|kl} := q_{ik}q_{jl} + q_{il}q_{jk} - q_{ij}q_{kl} = p_{ik}p_{jl} + p_{il}p_{jk} - p_{ij}p_{kl}.$$

Suppose S is a complete intersection surface in $(\mathbb{P}^1)^4$ obtained by intersecting hypersurfaces A and B . There are nonnegative integers a_i and b_i such that

$$[A] = a_1H_1 + a_2H_2 + a_3H_3 + a_4H_4, \quad [B] = b_1H_1 + b_2H_2 + b_3H_3 + b_4H_4,$$

If $[S]$ is equal to $\eta(\mathbf{p})$, then

$$[S] = \sum_{i < j} q_{ij}H_iH_j = \sum_{i < j} (a_ib_j + a_jb_i)H_iH_j.$$

Therefore, we obtain, for example,

$$p_{14}p_{23} + p_{13}p_{24} - p_{12}p_{34} = q_{14}q_{23} + q_{13}q_{24} - q_{12}q_{34} = 2(a_1a_2b_3b_4 + b_1b_2a_3a_4) \geq 0.$$

This proves the necessity of the condition $\mathbf{p} \in \Delta(\mathbb{T}_1)$.

For the converse, suppose that $\mathbf{p}$ is an integral point in $\Delta(\mathbb{T}_1)$. We use the following factorization in $H^4((\mathbb{P}^1)^4, \mathbb{Q})$:

$$(2q_{14}q_{24}q_{34}) \cdot (q_{12}H_1H_2 + q_{13}H_1H_3 + q_{14}H_1H_4 + q_{23}H_2H_3 + q_{24}H_2H_4 + q_{34}H_3H_4) = \\ (q_{14}H_1 + q_{24}H_2 + q_{34}H_3) \cdot (q_{14|23}q_{14}H_1 + q_{13|24}q_{24}H_2 + q_{12|34}q_{34}H_3 + 2q_{14}q_{24}q_{34}H_4).$$

The coefficients q_{ij} and $q_{ij|kl}$ are nonnegative because $\mathbf{p} \in \Delta(\mathbb{T}_1)$. Therefore, if $q_{14}q_{24}q_{34}$ is nonzero, then $2q_{14}q_{24}q_{34} \eta(\mathbf{p})$ is a complete intersection class.

It remains to consider the cases when $q_{il}q_{jl}q_{kl} = 0$ for all i, j, k, l . Up to symmetry, the analysis reduces to two cases: The case when $q_{14} = q_{24} = q_{34} = 0$ and the case when $q_{12} = q_{34} = 0$.

In the first case, we may assume without loss of generality that $q_{12} > 0$. We have

$$q_{12} \cdot (q_{12}H_1H_2 + q_{13}H_1H_3 + q_{23}H_2H_3) = (q_{12}H_1 + q_{23}H_3) \cdot (q_{12}H_2 + q_{13}H_3),$$

so $q_{12} \eta(\mathbf{p})$ is a complete intersection class.

In the second case, we have $q_{13}q_{24} = q_{14}p_{23}$. Thus, we may assume that $q_{13}, q_{24}, q_{23}, q_{14}$ are nonzero, since otherwise we reduce to the previous case. We have

$$q_{13} \cdot (q_{13}H_1H_3 + q_{14}H_1H_4 + q_{23}H_2H_3 + q_{24}H_2H_4) = (q_{13}H_1 + q_{23}H_2) \cdot (q_{13}H_3 + q_{14}H_4),$$

so $q_{13} \eta(\mathbf{p})$ is a complete intersection class. $\square$

Remark 5.6. Characterizing complete intersection classes in $H_{2k}(\mathbb{P}^m, \mathbb{Q})$ or $H_{2k}(\mathbb{P}^m, \mathbb{R})$ is an elementary but challenging problem. For example, we do not know how to characterize complete intersection classes up to a positive multiple in $H_4((\mathbb{P}^2)^3, \mathbb{R})$. If η is a complete intersection surface class in $(\mathbb{P}^2)^3$, then

$$p_{12}p_{13}p_{23} \geq 4p_{11}p_{22}p_{33} \quad \text{and} \quad p_{jk}p_{ik} \geq p_{ij}p_{kk} \quad \text{and} \quad p_{ij}^2 \geq 2p_{ii}p_{jj},$$

where $p_{ij} = \int_S H_i H_j$ for all i, j . Which other inequalities hold for complete intersection classes?

Remark 5.7. Let A_1, A_2, A_3, A_4 be lattice polygons in $\mathbb{R}^2$. If f_i and g_i are general Laurent polynomials with Newton polygon A_i , then the closure of the image of the map

$$(f_1, g_1) \times (f_2, g_2) \times (f_3, g_3) \times (f_4, g_4) : (\mathbb{C}^*)^2 \longrightarrow (\mathbb{P}^1)^4$$

has the homology class given by $\lambda^{-1}(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) \in \wedge^2 \mathbb{Z}^4$ for some $\lambda > 0$, where

$$p_{ij} = \text{MV}(A_i, A_j) \quad \text{for all } i < j.$$

In [AS23, Theorem 3.2], Averkov and Soprunov show that

$$\Delta(\mathbb{T}_1) = \{\mathbf{p} \in \wedge^2 \mathbb{R}^4 \mid p_{ij} = \text{MV}(A_i, A_j) \text{ for convex bodies } A_1, A_2, A_3, A_4 \subseteq \mathbb{R}^2\}.$$

This gives an additional geometric interpretation of the locus $\Delta(\mathbb{T}_1)$.

6. REALIZABLE CLASSES IN PRODUCTS OF PROJECTIVE SPACES

As before, let $\mathbf{m}$ be a vector of positive integers $(m_1, \dots, m_n)$, and let $\mathbb{P}^{\mathbf{m}}$ be the product of projective spaces $\prod_{i=1}^n \mathbb{P}^{m_i}$. In this section, we prove Theorem 1.8: A class $\eta \in H_4(\mathbb{P}^{\mathbf{m}}, \mathbb{Q})$ is realizable over $\mathbb{Q}$ if and only if $L(\eta)$ is Lorentzian, where $L(\eta)$ is the $n \times n$ symmetric matrix with entries

$$L(\eta)_{ij} = \int_{\eta} H_i H_j \text{ for } 1 \leq i \leq j \leq n.$$

In [CFS25], the set of traceless Lorentzian matrices was identified with a massless Mandelstam region and parametrized by the future timelike cone in Minkowski space. To adapt this observation to the setting of Theorem 1.8, we construct in Theorem 6.3 a smooth projective surface of given Picard rank that contains no negative curves, and on which every nonzero nef divisor is numerically equivalent to a semiample divisor.⁶ In Theorem 6.13, we prove a more precise version of Theorem 1.8 for $(\mathbb{P}^1)^n$ with $2 \leq n \leq 11$: If $L(\eta)$ is an integral Lorentzian matrix with coprime entries and p is a prime number, then η is realizable over $\mathbb{Z}$ or $p\eta$ is realizable over $\mathbb{Z}$.

6.1. Prolific surfaces. Consider the vector space $\mathbb{R}^{1+d}$ with coordinate functions $x_0, x_1, \dots, x_d$ equipped with the nondegenerate symmetric bilinear pairing

$$\mathbf{x} \cdot \mathbf{y} = (x_0, x_1, \dots, x_d) \cdot (y_0, y_1, \dots, y_d) = x_0 y_0 - x_1 y_1 - \dots - x_d y_d.$$

The *future timelike cone* $\mathcal{C}$ is the subset of vectors $\mathbf{x}$ satisfying $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $x_0 \geq 0$. The future timelike cone is a self-dual convex cone:

$$\mathcal{C} = \{ \mathbf{x} \in \mathbb{R}^{1+d} \mid \mathbf{x} \cdot \mathbf{y} \geq 0 \text{ for all } \mathbf{y} \in \mathcal{C} \}.$$

Thus, $\mathcal{C}$ consists of $\mathbf{x}$ satisfying $\mathbf{x} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{y} \geq 0$ for any given $\mathbf{y}$ in the interior of $\mathcal{C}$.

We consider the analogous cone for a smooth projective surface Y . For any unexplained terms, we refer to [Laz04]. The intersection product defines a nondegenerate symmetric bilinear pairing on the real Néron–Severi space $\text{NS}(Y)_{\mathbb{R}}$ with exactly one positive eigenvalue. We consider the *future timelike cone*

$$\mathcal{C}(Y) := \left\{ D \in \text{NS}(Y)_{\mathbb{R}} \mid \int_Y D^2 \geq 0 \text{ and } \int_Y DH \geq 0 \right\},$$

where H is the class of a fixed ample divisor on Y . The cone $\mathcal{C}(Y)$ contains the ample cone, and it is independent of the choice of H .

In general, determining the cone of nef divisors on a smooth projective surface with large Picard number can be challenging. An important exception occurs when the surface does not contain any *negative curve*, that is, a reduced and irreducible curve with negative self-intersection number.

⁶A Cartier divisor on X is said to be *nef* if it intersects every irreducible curve in X nonnegatively, and *semiample* if some positive multiple of it is basepoint-free.

Lemma 6.1. If Y does not contain any negative curve, then the cone of nef divisors on Y is equal to $\mathcal{C}(Y)$.

Proof. In any case, the nef cone is contained in $\mathcal{C}(Y)$. If Y does not contain any negative curve, then the nef cone is self-dual with respect to the intersection pairing [Laz04, Example 1.4.33], so it must be equal to the self-dual cone $\mathcal{C}(Y)$. $\square$

Definition 6.2. We say that a smooth projective surface Y is *prolific* if Y does not contain any negative curve and every nef divisor on Y is numerically equivalent to a semiample divisor.

Let $\mathbb{P}^m$ be a product of n projective spaces, and let Y be a prolific surface of Picard rank at least $n + 3$. We will show in the next subsection that every realizable class in $H_4(\mathbb{P}^m, \mathbb{Q})$ is a rational multiple of $\varphi_*[Y]$ for some $\varphi : Y \rightarrow \mathbb{P}^m$.

Theorem 6.3. For any positive integer k , there is a prolific surface Y with Picard rank k .

It is enough to consider the case $k \geq 3$. We write $k = n + 2$ for a positive integer n . For the remainder of this subsection, fix an elliptic curve (E, e) with $\text{End}(E) \simeq \mathbb{Z}$ and an integer $d \geq 3$. Let $H_0, H_1, \dots, H_n$ be the inverse images of e under the coordinate projections $E^{n+1} \rightarrow E$. The 0-th coordinate will play a distinguished role, and we will occasionally emphasize this by writing $E^{n+1} = E \times E^n$. We set

$$Y := \left(\text{the intersection of } (n-1) \text{ general members in the linear system of } \sum_{i=1}^n dH_i \right).$$

The surface Y is of the form $E \times C$, where C is a complete intersection curve in E^n . By Bertini's theorem, C is smooth and irreducible. We show that Y has the required properties.

Lemma 6.4. The surface Y does not contain any negative curve.

Proof. Since the elliptic curve E acts on Y by addition on the first factor, the only possible rigid curves are fibers of the projection $Y \rightarrow C$. However, any such curve moves in an algebraic family over C , so Y does not contain any rigid curve. $\square$

Lemma 6.5. For distinct indices i and j , consider the divisor

$$\Delta_{ij} := \{(x_0, x_1, \dots, x_n) \in E^{n+1} \mid x_i = x_j\}.$$

Then $[H_i]$ for all i and $[\Delta_{ij}]$ for all distinct i and j form a basis of $\text{NS}(E^{n+1})_{\mathbb{Q}}$.

Proof. Let p_1 and p_2 be the projections from $E \times E^n$ to E and E^n , and let q_1 and q_2 be the inclusions from $E \simeq E \times e^n$ and $E^n \simeq e \times E^n$ to $E \times E^n$. By [Sch94, Theorem 3.9], we have

$$\text{NS}(E \times E^n)_{\mathbb{Q}} / (p_1^* \text{NS}(E)_{\mathbb{Q}} + p_2^* \text{NS}(E^n)_{\mathbb{Q}}) \simeq \text{Hom}(E, E^n) \otimes \mathbb{Q}.$$

The pullbacks $p_1^*, p_2^*, q_1^*, q_2^*$ define and split the short exact sequence

$$0 \longrightarrow \text{NS}(E)_{\mathbb{Q}} \oplus \text{NS}(E^n)_{\mathbb{Q}} \longrightarrow \text{NS}(E \times E^n)_{\mathbb{Q}} \longrightarrow \text{Hom}(E, E^n) \otimes \mathbb{Q} \longrightarrow 0.$$

Since $\dim \operatorname{Hom}(E, E^n) \otimes \mathbb{Q} = n$ by the assumption on E , by induction, we see that

$$\dim \operatorname{NS}(E \times E^n)_{\mathbb{Q}} = 1 + n + \frac{1}{2}n(n+1) = \frac{1}{2}(n+1)(n+2).$$

To see the linear independence of $[H_i]$ and $[\Delta_{ij}]$, we intersect the divisor classes against the coordinate elliptic curves $E_k \simeq E$ and the diagonal elliptic curves $E_{kl} \simeq E$ in E^{n+1} for all $k \neq l$:

$$\begin{aligned} \int_{E_k} H_i &= \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases} \\ \int_{E_{kl}} H_i &= \begin{cases} 1 & \text{if there are exactly two distinct elements in } i, k, l, \\ 0 & \text{otherwise,} \end{cases} \\ \int_{E_k} \Delta_{ij} &= \begin{cases} 1 & \text{if there are exactly two distinct elements in } i, j, k, \\ 0 & \text{otherwise,} \end{cases} \\ \int_{E_{kl}} \Delta_{ij} &= \begin{cases} 1 & \text{if there are exactly three distinct elements in } i, j, k, l \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For any rational numbers a_i and b_{ij} , we see that the conditions

$$\begin{aligned} \int_{E_k} \left(\sum_i a_i H_i + \sum_{i < j} b_{ij} \Delta_{ij} \right) &= a_k + \sum_{j \neq k} b_{jk} = 0, \\ \int_{E_l} \left(\sum_i a_i H_i + \sum_{i < j} b_{ij} \Delta_{ij} \right) &= a_l + \sum_{j \neq l} b_{jl} = 0, \\ \int_{E_{kl}} \left(\sum_i a_i H_i + \sum_{i < j} b_{ij} \Delta_{ij} \right) &= a_k + a_l + \left(-b_{kl} + \sum_{j \neq k} b_{jk} \right) + \left(-b_{kl} + \sum_{j \neq l} b_{jl} \right) = 0 \end{aligned}$$

together imply $b_{kl} = 0$ for all distinct k and l , which in turn implies $a_k = 0$ for all k . $\square$

For the remainder of this subsection, we set $\gamma := d^{n-1}(n-1)!$ and $\Delta_i := \Delta_{0i}$ for $i \neq 0$.

Lemma 6.6. For distinct indices $i \neq 0$ and $j \neq 0$, we have

$$\int_Y \Delta_i \Delta_j = 2\gamma, \quad \int_Y \Delta_i H_0 = \int_Y \Delta_i H_1 = \gamma, \quad \int_Y H_0 H_1 = \gamma.$$

Proof. Since H_0, H_i , and Δ_i intersect transversely, we have

$$[H_i][\Delta_i] = [H_0][\Delta_i] = [H_i][H_0] \text{ and } [\Delta_i][\Delta_i] = [H_i][H_i] = [H_0][H_0] = 0.$$

Thus, for example, we have

$$\begin{aligned}
 \int_Y \Delta_1 \Delta_2 &= d^{n-1} \int_{E^{n+1}} \Delta_1 \Delta_2 (H_1 + \cdots + H_n)^{n-1} \\
 &= \gamma \left[\int_{E^{n+1}} \Delta_1 \Delta_2 H_1 (H_3 \cdots H_n) + \int_{E^{n+1}} \Delta_1 \Delta_2 H_2 (H_3 \cdots H_n) \right] \\
 &= \gamma \left[\int_{E^{n+1}} \Delta_2 H_0 H_1 (H_3 \cdots H_n) + \int_{E^{n+1}} \Delta_1 H_0 H_2 (H_3 \cdots H_n) \right] \\
 &= \gamma \left[\int_{E^{n+1}} H_0 H_1 H_2 (H_3 \cdots H_n) + \int_{E^{n+1}} H_0 H_1 H_2 (H_3 \cdots H_n) \right] = 2\gamma.
 \end{aligned}$$

The other computations are similar and simpler. For example, we have

$$\begin{aligned}
 \int_Y \Delta_i H_0 &= d^{n-1} \int_{E^{n+1}} H_0 H_i (H_1 + \cdots + H_n)^{n-1} = \gamma \int_{E^{n+1}} H_0 H_1 H_2 \cdots H_n = \gamma, \\
 \int_Y \Delta_i H_1 &= d^{n-1} \int_{E^{n+1}} \Delta_i H_1 (H_1 + \cdots + H_n)^{n-1} = \gamma \int_{E^{n+1}} \Delta_i H_1 H_2 \cdots H_n = \gamma. \quad \square
 \end{aligned}$$

We use the computation in Lemma 6.6 to obtain an explicit basis of $\text{NS}(Y)_{\mathbb{Q}}$. We write ι^* for the pullback map $\text{NS}(E \times E^n)_{\mathbb{Q}} \rightarrow \text{NS}(Y)_{\mathbb{Q}}$.

Lemma 6.7. The Néron–Severi space $\text{NS}(Y)_{\mathbb{Q}}$ has basis

$$[D_1] := \iota^*[\Delta_1], \dots, [D_n] := \iota^*[\Delta_n], \quad [D_{n+1}] := \iota^*[H_0], \quad [D_{n+2}] := \iota^*[H_1].$$

Proof. Recall that $Y = E \times C$ for $C \subseteq E^n$. Arguing as before for distinct $i \neq 0$ and $j \neq 0$,

$$\int_C \Delta_{ij} = \int_C 2H_i = 2\gamma, \text{ and hence } \iota^*[\Delta_{ij}] = 2\iota^*[H_i] = 2[D_{n+2}].$$

Lemma 6.5 thus implies that $[D_1], \dots, [D_{n+2}]$ span $\text{im}(\iota^*)$.

Each $[D_i]$ is either a pullback of the diagonal in $E \times E$ or a pullback of a divisor on E , so its self-intersection is zero. Therefore, by Lemma 6.6, the intersections between

$$[D_i] - [D_{n+1}] - [D_{n+2}], \quad [D_{n+1}] - [D_{n+2}], \quad [D_{n+1}] + [D_{n+2}], \quad i = 1, \dots, n,$$

are given by the diagonal matrix with diagonal entries $-2\gamma, \dots, -2\gamma, 2\gamma$. For example,

$$\begin{aligned}
 ([D_1] - [D_{n+1}] - [D_{n+2}]) &([D_2] - [D_{n+1}] - [D_{n+2}]) = 0, \\
 ([D_1] - [D_{n+1}] - [D_{n+2}]) &([D_{n+1}] - [D_{n+2}]) = 0, \\
 ([D_1] - [D_{n+1}] - [D_{n+2}]) &([D_{n+1}] + [D_{n+2}]) = 0.
 \end{aligned}$$

Since this matrix is nondegenerate, $[D_1], \dots, [D_{n+2}]$ form a basis of $\text{im}(\iota^*)$.

It remains to show that $\text{im}(\iota^*) = \text{NS}(Y)_{\mathbb{Q}}$. The key step in the proof is the following classical result, going back to Severi⁷:

⁷Severi's original arguments in [Sev13] and [Sev27] were incomplete. A complete proof was later provided by Ciliberto and van der Geer in [CvdG92]. The same statement was generalized to arbitrary uncountable algebraically closed ground fields in [Ban17]. When the ground field is $\mathbb{Q}$, the existence of the curve C was established in [Koc18].

Let S be a smooth projective surface. If C is a very general curve in the linear system of a very ample divisor, then the kernel of the natural map $\text{Jac}(C) \rightarrow \text{Alb}(S)$ is a simple abelian variety.

Let S be the intersection of $(n-2)$ general members in the linear system of $\sum_{i=1}^n dH_i$ in E^n . By the Lefschetz hyperplane theorem, the Albanese variety of S is isomorphic to E^n ,⁸ and hence the Jacobian of C is isogenous to $A \times E^n$ for some simple abelian variety A of large dimension. Therefore, by [Sch94, Theorem 3.9], the surface $Y = E \times C$ satisfies

$$\text{NS}(Y)_{\mathbb{Q}}/(\text{NS}(E)_{\mathbb{Q}} + \text{NS}(C)_{\mathbb{Q}}) \simeq \text{Hom}(E, \text{Jac}(C)) \otimes \mathbb{Q} \simeq \text{Hom}(E, E^n) \otimes \mathbb{Q}.$$

It follows that the dimension of $\text{NS}(Y)_{\mathbb{Q}}$ is equal to that of $\text{im}(\iota^*)$. $\square$

Proof of Theorem 6.3. It remains to show that any nef divisor D is numerically equivalent to a semiample divisor. By replacing D with a positive multiple if necessary, we may write

$$[D] = d_1[D_1] + \cdots + d_{n+1}[D_{n+1}] + d_{n+2}[D_{n+2}] \text{ for } d_i \in \mathbb{Z}.$$

By Lemma 6.6, the integers d_i are determined by the intersection numbers

$$\frac{1}{\gamma} \int_Y DD_i = \begin{cases} 2s_0 - 2d_i + d_{n+1} + d_{n+2} & \text{when } 1 \leq i \leq n, \\ s_0 + d_{n+2} & \text{when } i = n+1, \\ s_0 + d_{n+1} & \text{when } i = n+2, \end{cases}$$

where $s_0 := d_1 + \cdots + d_n$. If the self-intersection of D is positive, then D is ample by Lemma 6.1. Thus, we may and will assume that the self-intersection of D is zero:

$$\frac{1}{2\gamma} \int_Y D^2 = (s_0 + d_{n+1})(s_0 + d_{n+2}) - \left(\sum_{j=1}^n d_j^2 \right) = 0.$$

Since D is nef, the integer $s_0 + d_{n+1}$ is nonnegative. If it is zero, then $d_1 = \cdots = d_n = 0$ by the displayed formula, so D is numerically equivalent to a nonnegative multiple of the semiample divisor D_{n+2} . Thus, without loss of generality, we may suppose that $s_0 + d_{n+1}$ is positive.

We use the group structure of E to define a map

$$\Psi : E \times E^n \longrightarrow E, \quad (x_0, x_1, \dots, x_n) \longmapsto (s_0 + d_{n+1})x_0 - d_1x_1 + \cdots - d_nx_n.$$

We gather some intersection-theoretic information on the fiber F of Ψ over e :

- (1) When i is an element of $\{1, \dots, n\}$, the intersection product of F and $(H_1 \cdots H_{i-1})\Delta_i(H_{i+1} \cdots H_n)$ counts the number of points x_0 in E such that $(s_0 + d_{n+1} - d_i)x_0$ is a given general point. Since there are $(s_0 + d_{n+1} - d_i)^2$ such points, we have

$$\int_{E^{n+1}} F(H_1 \cdots H_{i-1})\Delta_i(H_{i+1} \cdots H_n) = (s_0 + d_{n+1} - d_i)^2.$$

⁸For the case over an arbitrary algebraically closed ground field, see [Gro05, Corollaire XII.3.6].

- (2) When j is an element of $\{1, \dots, n\}$, the intersection product of F and $(H_1 \cdots H_{j-1})H_0(H_{j+1} \cdots H_n)$ counts the number of points x_j in E such that $-d_j x_j$ is a given general point. Since there are d_j^2 such points, we have

$$\int_{E^{n+1}} F(H_1 \cdots H_{j-1})H_0(H_{j+1} \cdots H_n) = d_j^2.$$

- (3) When i and j are distinct elements of $\{1, \dots, n\}$, the intersection product of F and $(H_1 \cdots H_{j-1})\Delta_i(H_{j+1} \cdots H_n)$ counts the number of points x_j in E such that $-d_j x_j$ is a given general point. Since there are d_j^2 such points, we have

$$\int_{E^{n+1}} F(H_1 \cdots H_{j-1})\Delta_i(H_{j+1} \cdots H_n) = d_j^2.$$

Thus, for $1 \leq i \leq n$, we have

$$\frac{1}{\gamma} \int_Y \iota^*(F) D_i = \frac{1}{(n-1)!} \int_{E^{n+1}} F \Delta_i (H_1 + \cdots + H_n)^{n-1} = (s_0 + d_{n+1} + d_i)^2 - d_i^2 + \sum_{j=1}^n d_j^2,$$

$$\frac{1}{\gamma} \int_Y \iota^*(F) D_{n+1} = \frac{1}{(n-1)!} \int_{E^{n+1}} F H_0 (H_1 + \cdots + H_n)^{n-1} = \sum_{j=1}^n d_j^2,$$

$$\frac{1}{\gamma} \int_Y \iota^*(F) D_{n+2} = \frac{1}{(n-1)!} \int_{E^{n+1}} F H_1 (H_1 + \cdots + H_n)^{n-1} = (s_0 + d_{n+1})^2.$$

Comparing the intersection numbers for $\iota^*(F)$ with the intersection numbers for D , we see that

$$\frac{1}{\gamma} \int_Y \iota^*(F) D_i = \frac{1}{\gamma} \int_Y (s_0 + d_{n+1}) D D_i \text{ for } 1 \leq i \leq n+2.$$

Since the intersection pairing is nondegenerate, $(s_0 + d_{n+1})D$ is numerically equivalent to $\iota^*(F)$, which is semiample. $\square$

Remark 6.8. The proof of Lemma 6.7, and hence the proof of Theorem 6.3, is valid if the ground field is uncountable or of characteristic 0. Is there a prolific surface Y over $\overline{\mathbb{F}}_p$ of any given Picard rank?

6.2. Classes realizable over $\mathbb{Q}$. We use a prolific surface Y of large Picard rank to prove Theorem 1.8. The key remaining tool is Meyer's theorem on indefinite rational quadratic forms [Cas78, Chapter 6]:

If a homogeneous quadratic equation $Q(x_1, \dots, x_n) = 0$ with rational coefficients in five or more variables has a nonzero real solution, then it has a nonzero rational solution.

For a rational quadratic form Q , we write $(s_+(Q), s_-(Q), s_0(Q))$ for the signature of Q .

Lemma 6.9. Let P be a rational quadratic form on $\mathbb{Q}^m$, and let Q be a rational quadratic form on $\mathbb{Q}^n$ such that

$$s_+(P) \leq s_+(Q), \quad s_-(P) \leq s_-(Q), \quad \text{and} \quad s_+(P) + s_-(P) + 3 \leq s_+(Q) + s_-(Q).$$

Then there is a $\mathbb{Q}$ -linear map $f : \mathbb{Q}^m \rightarrow \mathbb{Q}^n$ such that $P(x) = Q(f(x))$ for all $x \in \mathbb{Q}^m$.

Proof. By taking a complement of the kernel of Q in $\mathbb{Q}^n$, we can reduce to the case when Q is nondegenerate. We prove the statement by induction on m under this assumption, the case $m = 0$ being trivial.

Suppose P is not identically zero, since otherwise the assertion is trivial. In this case, $n \geq 4$, and there is $x_0 \in \mathbb{Q}^m$ such that $P(x_0)$ is nonzero. We introduce a new variable z , and apply Meyer's theorem to the quadratic equation

$$R(y, z) := Q(y) - P(x_0)z^2 = 0.$$

Since Q is nondegenerate, R is nondegenerate, so the existence of a rational solution for $R = 0$ implies the existence of a dense set of rational solutions for $R = 0$. We choose a nonzero $q \in \mathbb{Q}$ and $y_0 \in \mathbb{Q}^n$ such that

$$Q(y_0) = P(x_0)q^2 = P(qx_0).$$

Let $x_0^\perp$ be the orthogonal complement to x_0 in $\mathbb{Q}^m$, and let $y_0^\perp$ be the orthogonal complement to y_0 in $\mathbb{Q}^n$. Since $Q(y_0)$ and $P(x_0)$ have the same sign, the induction hypothesis applies to the restrictions of P and Q to these orthogonal complements. The outcome is that there is a $\mathbb{Q}$ -linear map $f : x_0^\perp \rightarrow y_0^\perp$ such that $P(x) = Q(f(x))$ for all $x \in x_0^\perp$. The unique $\mathbb{Q}$ -linear extension $f : \mathbb{Q}^m \rightarrow \mathbb{Q}^n$ satisfying $f(qx_0) = y_0$ has the desired property. $\square$

Proof of Theorem 1.8. Let $\mathbf{m} = (m_1, \dots, m_n)$ be a vector of positive integers, and let $\eta \in H_4(\mathbb{P}^{\mathbf{m}}, \mathbb{Q})$ be a class with $L(\eta)$ Lorentzian. We show that some rational multiple of η is realizable over $\mathbb{Z}$.

By Theorem 6.3, there is a prolific surface Y of Picard rank $n + 3$. By Lemma 6.9, we can find $[A_1], \dots, [A_n] \in \text{NS}(Y)_{\mathbb{Q}}$ such that $L(\eta)_{ij} = \int_Y A_i A_j$ for all i and j . Choose an ample divisor H on Y such that $\int_Y A_i H$ is nonzero for all i , and set

$$[B_i] := \begin{cases} +[A_i] & \text{if } \int_Y A_i H \text{ is positive,} \\ -[A_i] & \text{if } \int_Y A_i H \text{ is negative.} \end{cases}$$

Since Y does not contain any negative curve, Lemma 6.1 determines the nef cone of Y . We deduce that $[B_i]$ is nef for every i because $\int_Y B_i H$ and $\int_Y B_i^2$ are nonnegative. Since the intersection between any nef divisors is nonnegative, we have $L(\eta)_{ij} = \int_Y B_i B_j$ for all i and j .

Since Y is prolific, we may suppose that B_i are semiample $\mathbb{Q}$ -divisors satisfying $L(\eta)_{ij} = \int_Y B_i B_j$ for all i and j . Choose a positive integer ℓ such that each ℓB_i is the pullback of a hyperplane under the map

$$\varphi_i : Y \longrightarrow \mathbb{P}H^0(Y, \mathcal{O}_Y(\ell B_i))^\vee.$$

The image of φ_i is a curve if and only if $L(\eta)_{ii} = 0$.

If the dimension of the target projective space is at most m_i , we replace φ_i by the composition of φ_i with a linear inclusion into $\mathbb{P}^{m_i}$. If the dimension of the target projective space is larger than m_i , we replace φ_i by the composition of φ_i with a general linear projection onto $\mathbb{P}^{m_i}$. Since

the image of φ_i is a curve whenever $m_i = 1$, such compositions do not change the class of the pullback of a hyperplane. Combining all the maps, we get

$$\varphi := \prod_{i=1}^n \varphi_i : Y \longrightarrow \mathbb{P}^m.$$

By construction, $[\varphi(Y)] = \frac{\ell^2}{\lambda} \eta$, where λ is the degree of the map $Y \rightarrow \varphi(Y)$. $\square$

Remark 6.10. As noted before, Theorem 6.3 remains valid over any uncountable algebraically closed field. From this, one can deduce that Theorem 1.8 remains valid over any algebraically closed field if we replace $H_4(\mathbb{P}^m, \mathbb{Q})$ by the Chow group $\text{CH}_2(\mathbb{P}^m, \mathbb{Q})$. To see this, let k_1 be an algebraically closed field, and let k_2 be an uncountable algebraically closed field extension of k_1 . Suppose that Z_2 is an irreducible subvariety of $\mathbb{P}^m$ over k_2 . Fixing an ample divisor of $\mathbb{P}^m$, the subvariety Z_2 defines a k_2 -point of the Hilbert scheme $\text{Hilb}^P(\mathbb{P}^m)$, where P is the Hilbert polynomial of Z_2 with respect to the fixed ample divisor [Gro95]. The field extension $k_1 \subseteq k_2$ induces an injective map $\text{Hilb}^P(\mathbb{P}^m)(k_1) \rightarrow \text{Hilb}^P(\mathbb{P}^m)(k_2)$ whose image is Zariski dense. Since being geometrically integral is an open condition [Gro65, Théorème 12.2.1], there is a k_1 -point of $\text{Hilb}^P(\mathbb{P}^m)$ such that the corresponding k_1 -subscheme Z_1 is integral and $[Z_1] = [Z_2]$.

Let X be a d -dimensional irreducible projective variety over an algebraically closed field k . For any collection of nef $\mathbb{Q}$ -divisors $D = (D_1, \dots, D_n)$ on X , consider the polynomial

$$f_H(x_1, \dots, x_n) = \frac{1}{d!} \int_X (x_1 D_1 + \dots + x_n D_n)^d.$$

A *volume polynomial over k* is any limit of such polynomials. A volume polynomial over k is a Lorentzian polynomial in the sense of [BH20]. Theorem 1.8 implies that the converse holds when $d = 2$.

Corollary 6.11. For a quadratic polynomial f , the following conditions are equivalent:

- (1) f is a volume polynomial over some k .
- (2) f is a volume polynomial over any k .
- (3) f is a Lorentzian polynomial.

Remark 6.12. When $d \geq 3$, it is not known if the set of volume polynomials over $\mathbb{K}$ depends on k . Not every cubic Lorentzian polynomial is a volume polynomial over k for some k . The following example was given in [Huh23, Example 14]:

$$f = 14x_1^3 + 6x_1^2x_2 + 24x_1^2x_3 + 12x_1x_2x_3 + 6x_1x_3^2 + 3x_2x_3^2.$$

This example highlights the subtlety involved in formulating a reasonable conjecture that characterizes, for example, the realizable classes in $H_6((\mathbb{P}^3)^3, \mathbb{R})$.

6.3. Classes realizable over $\mathbb{Z}$. Let n be an integer satisfying $2 \leq n \leq 11$. This will be the standing assumption in this subsection. We use K3 surfaces to construct many classes in $H_4((\mathbb{P}^1)^n, \mathbb{Z})$ that are realizable over $\mathbb{Z}$. An integral matrix is said to be *primitive* if its entries are coprime, that is, when the greatest common divisor of the entries is 1.

Theorem 6.13. Let η be a class in $H_4((\mathbb{P}^1)^n, \mathbb{Z})$. If $L(\eta)$ is a primitive Lorentzian matrix and p is a prime number, then η is realizable over $\mathbb{Z}$ or $p\eta$ is realizable over $\mathbb{Z}$.

Unlike the other arguments in this paper, our proof of Theorem 6.13 essentially relies on the fact that we are working over a field of characteristic 0. We do not know if Theorem 6.13 remains valid if we work over any algebraically closed field or if we drop the assumption that $n \leq 11$.

We deduce Theorem 6.13 from the following existence result.

Proposition 6.14. Let L be a nondegenerate rank n integral Lorentzian matrix with zero diagonal entries. If the quadratic form $\mathbf{x}L\mathbf{x}^T$ does not take the value -2 on $\mathbb{Z}^n$, then there is a complex K3 surface Y and divisors $D_1, \dots, D_n$ on Y satisfying the following conditions:

- (1) The intersection product $\int_Y D_i D_j$ is equal to L_{ij} for all i and j .
- (2) The classes $[D_1], \dots, [D_n]$ form a $\mathbb{Z}$ -basis of the Néron–Severi group $\text{NS}(Y)_{\mathbb{Z}}$.
- (3) The surface Y does not contain any negative curve.
- (4) The divisors $D_1, \dots, D_n$ are basepoint-free.

Proof. The main ingredient is [Mor84, Corollary 2.9, Remark 2.11]:

If $n \leq 11$, then every even lattice of signature $(1, n - 1)$ occurs as the Néron–Severi group of some complex algebraic K3 surface equipped with the intersection pairing.

Its proof relies on the global surjectivity of the period mapping for K3 surfaces [Huy16, Section 7.4] and Nikulin’s lattice embedding theorem [Nik79, Theorem 1.10.1].

Since the diagonal entries of L are zero, the quadratic form $\mathbf{x}L\mathbf{x}^T$ is even. We thus get a complex K3 surface Y and divisors $D_1, \dots, D_n$ satisfying conditions (1) and (2). Any negative curve on a K3 surface must have self-intersection -2 [Huy16, Section 2.1.3], so the surface Y must also satisfy condition (3).

Choose an ample divisor H on Y such that $\int_Y D_i H$ is nonzero for all i . Since Y does not contain any negative curve, Lemma 6.1 determines the nef cone of Y . We deduce that either D_i is nef or $-D_i$ is nef for each i , because the self-intersections are zero. By replacing every D_i by $-D_i$ if necessary, we may suppose without loss of generality that D_1 is nef. We show in this case that D_i is nef for every i . This gives condition (4), because every nef divisor D of self-intersection zero on a K3 surface is basepoint-free [Huy16, Chapter 2, Proposition 3.10].

Suppose $D_1, \dots, D_k$ and $-D_{k+1}, \dots, -D_n$ are nef divisors. If D_i is chosen from the first group and D_j is chosen from the second group, then $L_{ij} = \int_Y D_i D_j$ is nonpositive, so L_{ij} must

be zero. This means that L is a symmetric block-diagonal matrix with nonnegative entries, so L has at least two positive eigenvalues if the decomposition is nontrivial. Since L is Lorentzian, the block decomposition is trivial, so every D_i is nef. $\square$

Remark 6.15. By [Huy16, Remark 17.2.16], we may find such a K3 surface over $\overline{\mathbb{Q}}$. This shows that Theorem 6.13 remains valid over any algebraically closed field of characteristic 0.

Proof of Theorem 6.13. We prove the following more precise statement:

If $L(\eta)$ is a nonzero integral Lorentzian matrix and the quadratic form $\mathbf{x}L(\eta)\mathbf{x}^T$ does not take the value -2 on $\mathbb{Z}^n$, then $\lambda^{-1}\eta$ is realizable over $\mathbb{Z}$ for some positive integer λ .

Applying this statement to $p\eta$ when $L(\eta)$ is primitive, we see that $\lambda^{-1}\eta$ is realizable over $\mathbb{Z}$ for some positive integer λ , where λ is necessarily 1 or p .

We first consider the case when $L(\eta)$ is nondegenerate. By Proposition 6.14, there is a complex K3 surface Y and divisors $D_1, \dots, D_n$ with the stated properties. We choose a pencil in the linear system of D_i for each i to define a map

$$\varphi : Y \longrightarrow (\mathbb{P}^1)^n.$$

By construction, $[\varphi(Y)] = \lambda^{-1}\eta$, where λ is the degree of the map $Y \rightarrow \varphi(Y)$.

When $L(\eta)$ is degenerate, consider the nondegenerate quadratic form $M(\eta)$ on $\mathbb{Z}^n / \ker L(\eta)$ induced by $L(\eta)$. By [Mor84, Corollary 2.9, Remark 2.11], there is a K3 surface Y and an isometry

$$\mathbb{Z}^n / \ker L(\eta) \simeq \text{NS}(Y)_{\mathbb{Z}}.$$

Let $[D_1], \dots, [D_n]$ be the divisor classes on Y corresponding to the image of the standard basis vectors of $\mathbb{Z}^n$, so that the intersection matrix of $D_1, \dots, D_n$ is equal to $L(\eta)$. As in the proof of Proposition 6.14, replacing every D_i by $-D_i$ if necessary, we may assume that each D_i is a nef divisor. By [Huy16, Chapter 2, Proposition 3.10], each D_i is basepoint-free, and we can apply the same construction as in the nondegenerate case. $\square$

7. REALIZABILITY IN GRASSMANNIAN AND OTHER VARIETIES

Let $B_k(X) \subseteq H_{2k}(X, \mathbb{R})$ be the subspace spanned by the k -dimensional algebraic cycles on X with real coefficients. We consider the closed subset

$$\mathcal{R}_k(X) := \{k\text{-dimensional classes on } X \text{ that are realizable over } \mathbb{R}\} \subseteq B_k(X).$$

We write $\mathcal{R}^k(X)$ for $\mathcal{R}_{d-k}(X)$, where d is the dimension of X . In general, $\mathcal{R}_k(X)$ may exhibit sporadic structures in that it may be different from the closure of its interior in $B_k(X)$.

Proposition 7.1. For any smooth projective variety X , we have

$$\mathcal{R}^1(X)^\circ = \text{Mov}^1(X)^\circ,$$

where $\text{Mov}^1(X)^\circ$ denotes the interior of the movable cone of divisors in $\text{NS}(X)_\mathbb{R}$.⁹

In particular, when X is a surface, the interior of $\mathcal{R}^1(X)$ is precisely the ample cone of X , and every ray in $\mathcal{R}^1(X) \setminus \overline{\mathcal{R}^1(X)}^\circ$ is isolated.

Proof. Notice that when $[D]$ is in the interior of either $\text{Mov}^1(X)$ or $\mathcal{R}^1(X)$, then $[D]$ is in the interior of the pseudoeffective cone, and hence D is big [Laz04, Theorem 2.2.26].

Suppose that D is an irreducible divisor whose class is in $\mathcal{R}^1(X)^\circ$. Since D is big, there is a smallest integer m such that mD is linearly equivalent to E for some effective divisor E different from mD . By the minimality of m , no component of E is equal to D . Thus, the base locus of the linear system $|D|$ must have codimension at least 2, and hence D is movable.

Suppose that D is an effective divisor whose class is in $\text{Mov}^1(X)^\circ$. Since D is big, there is a positive integer r such that the base locus of the linear system $|rD|$ has codimension at least 2 and the induced rational map $\varphi : X \dashrightarrow \mathbb{P}^n$ is generically injective. Let U be the open subset of X where φ is well-defined, and let H be a general hyperplane of $\mathbb{P}^n$. Then, by Bertini's irreducibility theorem, the closure of $U \cap \varphi^{-1}H$ is an irreducible divisor of X linearly equivalent to rD , so $[D]$ is realizable over $\mathbb{Q}$. $\square$

It follows that the interior of $\mathcal{R}^1(X)$ is convex for any smooth projective variety X . On the other hand, the interior of $\mathcal{R}^2(X)$ is typically not convex, as we have seen in the case when $X = (\mathbb{P}^1)^4$.

In the following example, we observe that the interior of $\mathcal{R}_1(X)$ need not be connected.

Example 7.2. A general quintic threefold Y in $\mathbb{P}^4$ contains a line L with the normal bundle

$$N_{L/Y} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

See, for example, [EH16, Proposition 6.30]. Let X be the blow up of Y along L , and denote the pullback of the hyperplane class by H and the exceptional divisor by E , so that

$$E \simeq \mathbb{P}^1 \times \mathbb{P}^1 \text{ and } N_{E/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1).$$

The Chow ring of X is determined by the relations

$$\int_Y H^3 = 5, \quad \int_Y H^2 E = 0, \quad \int_Y H E^2 = -1, \quad \int_Y E^3 = 2, \quad \text{and} \quad H^2 + 10HE + 5E^2 = 0.$$

The effective cone of divisors on X is generated by $H - E$ and E , and its dual cone of movable curves on X is given by

$$\text{Mov}_1(X) = \text{cone} \left(\frac{H^2}{5}, \frac{H^2}{5} - HE \right) \subseteq \mathcal{R}_1(X).$$

⁹Recall that the *movable cone of divisors* is the closure of the cone generated by classes of effective Cartier divisors L such that the base locus of the complete linear system $|L|$ has codimension at least 2. We thank Chenyang Xu for suggesting this statement.

It is easy to check that the class of any irreducible curve in X not contained in E is in this cone. In general, the cone of movable curves is the closure of the pushforwards of complete intersection of ample divisors under projective birational mappings [BDPP13]. In this case, one can check that any class in this cone can be approximated by the classes of complete intersections of ample divisors on X . This gives one component of $\mathcal{R}_1(X)^\circ$. The other component of $\mathcal{R}_1(X)^\circ$ is given by the curves contained in $E \simeq \mathbb{P}^1 \times \mathbb{P}^1$:

$$\iota_* \text{Mov}_1(E) = \text{cone} \left(HE, \frac{H^2}{5} + HE \right) \subseteq \mathcal{R}_1(X).$$

Together they generate the effective cone of curves in X . The shaded region in Figure 2 shows the classes of irreducible curves C in X :

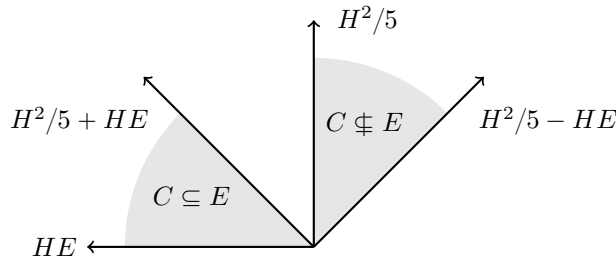


FIGURE 2. The classes of irreducible curves in X

The *pseudoeffective cone* in $B_k(X)$ is the closure of the cone generated by the k -dimensional effective cycles on X , and the elements of this cone are called *pseudoeffective*. An element $\eta \in B_k(X)$ is called *universally pseudoeffective* if $\pi^*\eta$ is pseudoeffective for any morphism $\pi : Y \rightarrow X$ from a smooth projective variety Y . The cone of k -dimensional universally pseudoeffective classes on X is a higher-codimension analogue of the cone of nef divisors.¹⁰ In [FL17], Fulger and Lehmann show several desirable properties of the cone of universally pseudoeffective classes. A divisor class is universally pseudoeffective if and only if it is nef, and a curve class is universally pseudoeffective if and only if it is in the closure of the cone of movable curves.

We propose a general conjecture concerning 2-dimensional realizable classes on a smooth complex projective variety X . The idea is that $\mathcal{R}_2(X)$ has a predictable shape within the cone of universally pseudoeffective cycles. Set $H^{1,1}(X)_{\mathbb{R}} := H^2(X, \mathbb{R}) \cap H^{1,1}(X)$.

Conjecture 7.3. The following are equivalent for any universally pseudoeffective $\eta \in B_2(X)$.

- (1) The class η is realizable over $\mathbb{R}$.

¹⁰A k -dimensional universally pseudoeffective class is *nef*: Its intersection with any codimension k subvariety of X is nonnegative. Grothendieck asked whether nef classes are always pseudoeffective (and thus universally pseudoeffective). Debarre, Ein, Lazarsfeld, and Voisin provided a negative answer to this question in [DELV11] when $k = 2$.

(2) The matrix representation of the bilinear form

$$H^{1,1}(X)_{\mathbb{R}} \times H^{1,1}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad (\alpha, \beta) \longmapsto \int_{\eta} \alpha \wedge \beta$$

has at most one positive eigenvalue.

That (1) implies (2) follows from the Hodge–Riemann relations for $H^{1,1}(S)$, where $S \rightarrow X$ is any map from a smooth projective surface. Since every Lorentzian matrix is a limit of Lorentzian matrices with rational entries, Theorem 1.8 confirms Conjecture 7.3 when X is a product of projective spaces.

The following example demonstrates that, in Conjecture 7.3, it is essential to require that η is universally pseudoeffective rather than merely pseudoeffective.

Example 7.4. Let $\pi : X \rightarrow \mathbb{P}^3$ be the blowup of $\mathbb{P}^3$ along a line L , and consider the class

$$\eta(a, b) = a[H] + b[E] \in H_4(X, \mathbb{Z}),$$

where H is the pullback of a hyperplane and E is the exceptional divisor. The Chow ring of X is determined by the relations

$$\int_X H^3 = 1, \quad \int_X H^2 E = 0, \quad \int_X H E^2 = -1, \quad \int_X E^3 = -2, \quad \text{and} \quad (H - E)^2 = 0.$$

Using these, it is straightforward to check that the bilinear form of $\eta(a, b)$ has at most one positive eigenvalue on $H^2(X, \mathbb{R})$ for any a and b . On the other hand, $\eta(a, b)$ is realizable over $\mathbb{Z}$ if and only if

$$(a > 0 \text{ and } b \leq 0 \text{ and } a + b > 0) \text{ or } (a, b) = (1, -1) \text{ or } (a, b) = (0, 1).$$

Indeed, if $b > 0$, any fiber over a point in L intersects $\eta(a, b)$ negatively, so an effective cycle with class $\eta(a, b)$ should contain E as a component.

The following example demonstrates that a universally pseudoeffective $\eta \in H_4(X, \mathbb{Z})$ satisfying the condition (2) in Conjecture 7.3 need not be realizable over $\mathbb{Q}$.

Example 7.5 (Mumford’s example). Take a smooth projective surface Y and a divisor D on Y with self-intersection zero such that the intersection of D and any irreducible curve in Y is positive [Laz04, Example 1.5.2]. Let Z be a smooth projective curve, and take the nef class η on $X := Y \times Z$ given by $\pi^*(D)$, where π is the projection $X \rightarrow Y$. As any nef class is a limit of ample classes [Laz04, Section 1.4], the class η is realizable over $\mathbb{R}$.

We check that η is not realizable over $\mathbb{Q}$. Suppose $\lambda\pi^*(D)$ is the class of an irreducible surface S for some positive rational number λ . By the projection formula, the intersection of S with a fiber of π has degree zero, so S must be of the form $\pi^{-1}(C)$ for some irreducible curve C in Y . Since the intersection of D and C in Y is positive, the intersection of $\pi^*(D)$ and S is homologous to a positive sum of fibers of π , contradicting that the self-intersection of D is zero.

We give an explicit universally pseudoeffective $\eta \in H_4(X, \mathbb{Z})$ for a 6-dimensional smooth projective toric variety X for which the validity of Conjecture 7.3 is unknown. See [Huh18] for background and unexplained terms. We recall that, if X is a toric variety, or more generally a spherical variety, then a class is universally pseudoeffective if and only if it is nef [FL17, Example 4.5].

Example 7.6 (Matroid classes). Let X_n be the complex smooth projective toric variety of the n -dimensional permutohedron. The Bergman fan of any simple rank 3 matroid M on $n + 1$ elements defines a class $\Delta_M \in H_4(X_n, \mathbb{Z})$ that is nef and effective [Huh18, Section 5]. The Hodge–Riemann relations for M in [AHK17, Theorem 1.4] shows that Δ_M satisfies the condition (2) of Conjecture 7.3. If the matroid M is realizable over $\mathbb{C}$, then Δ_M is realizable over $\mathbb{Z}$ [Huh18, Section 6]. Yu observes in [Yu17] that, if the matroid M is not realizable over $\mathbb{C}$, then Δ_M is not realizable over $\mathbb{Q}$. Take any M that is not realizable over $\mathbb{C}$. For example, take the matroid of the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ on seven elements. Is Δ_M realizable over $\mathbb{R}$?

When X is a homogeneous space G/P of Picard rank 1, then every effective class is universally pseudoeffective, and the condition (2) of Conjecture 7.3 is automatically satisfied for all effective classes. We verify Conjecture 7.3 for Grassmannians by showing that any effective class in $H_4(\text{Gr}(k, n), \mathbb{Z})$ is realizable over $\mathbb{Z}$, unless it is a multiple of some Schubert class.

We suppose $2 \leq k \leq n - 2$, since otherwise the assertion is trivial. Fix subspaces

$$F_i := \text{span}(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_i) \subseteq \mathbb{C}^n \text{ and } G_i := \text{span}(\mathbf{e}_n, \mathbf{e}_{n-1}, \dots, \mathbf{e}_{i+1}) \subseteq \mathbb{C}^n,$$

and consider the corresponding Schubert varieties in $\text{Gr}(k, n)$:

$$S_1 := \{k\text{-dimensional subspaces } V \subseteq \mathbb{C}^n \text{ such that } F_{k-2} \subseteq V \subseteq F_{k+1}\},$$

$$S_2 := \{k\text{-dimensional subspaces } V \subseteq \mathbb{C}^n \text{ such that } F_{k-1} \subseteq V \subseteq F_{k+2}\},$$

$$T_1 := \{k\text{-dimensional subspaces } V \subseteq \mathbb{C}^n \text{ such that } \dim V \cap G_{k-1} \geq 2\},$$

$$T_2 := \{k\text{-dimensional subspaces } V \subseteq \mathbb{C}^n \text{ such that } \dim V \cap G_{k+1} \geq 1\}.$$

The classes of S_1 and S_2 form a $\mathbb{Z}$ -basis of $H_4(\text{Gr}(k, n), \mathbb{Z})$. For integers a_1 and a_2 , we set

$$\eta(a_1, a_2) := a_1 [S_1] + a_2 [S_2].$$

The intersections between cycles of dimension 2 and codimension 2 are given by

$$[S_1] \cdot [T_1] = [S_2] \cdot [T_2] = [\text{point}] \text{ and } [S_1] \cdot [T_2] = [S_2] \cdot [T_1] = 0.$$

Theorem 7.7. The class $\eta(a_1, a_2)$ is realizable over $\mathbb{Z}$ if and only if one of the following holds:

- (1) $a_1 > 0$ and $a_2 > 0$, or $(a_1, a_2) = (1, 0)$ or $(a_1, a_2) = (0, 1)$, when $k = 2$ and $n = 4$.
- (2) $a_1 \geq 0$ and $a_2 > 0$, or $(a_1, a_2) = (1, 0)$, when $k = 2$ and $n > 4$.
- (3) $a_2 \geq 0$ and $a_1 > 0$, or $(a_1, a_2) = (0, 1)$, when $k = n - 2$ and $n > 4$.
- (4) $a_1, a_2 \geq 0$ and $(a_1, a_2) \neq (0, 0)$, when $n - 2 > k > 2$.

Compare Theorem 7.7 with the conjectural bound $a_1 \leq 3a_2$ for homology classes of smooth surfaces in $\text{Gr}(2, 4)$ with $a_1, a_2 > 0$ in [DR91]. In [Gro92], Gross proves that the following holds for the class $\eta(a_1, a_2)$ of any smooth surface $S \subseteq \text{Gr}(2, 4)$ with $a_1, a_2 > 0$: If S is not of general type or $a_2 \leq 19$, then $a_1 \leq 3a_2$.

Proof. Suppose that $\eta(a_1, a_2)$ is realizable over $\mathbb{Z}$. We must have $a_1, a_2 \geq 0$ and $(a_1, a_2) \neq (0, 0)$ because the effective cone of $\text{Gr}(k, n)$ is generated by $[S_1]$ and $[S_2]$.

We first show that $\eta(a_1, 0)$ is realizable over $\mathbb{Z}$ in $\text{Gr}(2, n)$ only if $a_1 = 1$. By Grassmannian duality, this implies that $\eta(0, a_2)$ is realizable over $\mathbb{Z}$ in $\text{Gr}(n-2, n)$ only if $a_2 = 1$. These results follow from, for example, the characterisation of Schur rigid classes in [RT12]. For the sake of completeness, we provide a direct argument. Suppose Y is an irreducible surface in $\text{Gr}(2, n)$ with $[Y] = \eta(a_1, 0)$. Consider the union of projective lines

$$Z = \bigcup_{V \in Y} \mathbb{P}(V) \subseteq \mathbb{P}(\mathbb{C}^n).$$

Since Z is the image of a projective bundle over Y , it is irreducible. We use Kleiman's generic transversality [Kle74] to extract information on Z from $[Y]$:

- (1) The intersection of Y and a general translate of T_2 is empty. Thus, there is a codimension 3 linear subspace of $\mathbb{P}(\mathbb{C}^n)$ that does not intersect Z . Therefore, Z is a surface.
- (2) The intersection of Y and a general translate of T_1 consists of a_1 distinct points. Thus, the intersection of Z with a general hyperplane of $\mathbb{P}(\mathbb{C}^n)$ contains a_1 distinct lines.

By Bertini's theorem [Jou83, Theorem 7.1], this general hyperplane section of Z is an irreducible curve, so a_1 is necessarily 1.

We next show that $\eta(a_1, 0)$ is realizable over $\mathbb{Z}$ in $\text{Gr}(k, n)$ for all $a_1 > 0$ when $k > 2$. By Grassmannian duality, this implies that $\eta(0, a_2)$ is realizable over $\mathbb{Z}$ in $\text{Gr}(k, n)$ for all $a_2 > 0$ when $n-2 > k$. Suppose $k > 2$, and consider the Schubert variety

$$\mathbb{P}^k \simeq \{k\text{-dimensional subspaces } V \subseteq \mathbb{C}^n \text{ with } V \subseteq F_{k+1}\} \subseteq \text{Gr}(k, n).$$

Since $k > 2$, there is an irreducible surface Y of degree a_1 in this $\mathbb{P}^k$. Clearly, $[Y] = \eta(a_1, 0)$.

When $a_1, a_2 > 0$, the realizability can be reduced to the case when $n = 4$ and $k = 2$: We have

$$\text{Gr}(2, 4) \simeq \{k\text{-dimensional subspaces } V \subseteq \mathbb{C}^n \text{ with } F_{k-2} \subseteq V \subseteq F_{k+2}\} \subseteq \text{Gr}(k, n),$$

and the pushforward map of the inclusion sends $\eta(a_1, a_2)$ to $\eta(a_1, a_2)$.

We construct an explicit irreducible surface Y in $\text{Gr}(2, 4)$ whose homology class is $\eta(a_1, a_2)$ for given positive integers a_1, a_2 . By Grassmannian duality, we may assume that $a_1 \geq a_2$. Consider the subspace

$$V(x, y) := \text{span}(\mathbf{e}_1 + x^{a_1}\mathbf{e}_3 + x^{a_2}\mathbf{e}_4, \mathbf{e}_2 + y\mathbf{e}_3 + x\mathbf{e}_4) \subseteq \mathbb{C}^4,$$

and let Y be the closure in $\text{Gr}(2, 4)$ of the subset

$$\{V(x, y), x, y \in \mathbb{C}\} \subseteq \text{Gr}(2, 4).$$

By Poincaré duality, it is enough to show that $[T_1] \cdot [Y] = a_1$ and $[T_2] \cdot [Y] = a_2$.

We compute the intersection of Y with T_1 . This amounts to counting the number of subspaces $V(x, y)$ that are perpendicular to a given general vector (c_1, c_2, c_3, c_4) , that is, counting the number of solutions of the system of equations

$$c_1 + c_3x^{a_1} + c_4x^{a_2} = 0 \quad \text{and} \quad c_2 + c_3y + c_4x = 0.$$

Since $a_1 \geq a_2 > 0$, the first equation has a_1 solutions, and the second equation uniquely determines y given x . This gives $[T_1] \cdot [Y] = a_1$.

We compute the intersection product of Y with T_2 . This amounts to counting the number of subspaces $V(x, y)$ that contain a given general vector (c_1, c_2, c_3, c_4) , that is, counting the number of solutions of the system of equations

$$c_1x^{a_1} + c_2y = c_3 \quad \text{and} \quad c_1x^{a_2} + c_2x = c_4.$$

Since $a_2 > 0$, the second equation has a_2 solutions, and the first equation uniquely determines y given x . This gives $[T_2] \cdot [Y] = a_2$. $\square$

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