

HODGE-RIEMANN RELATIONS FOR POTTS MODEL PARTITION FUNCTIONS

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ABSTRACT. We prove that the Hessians of nonzero partial derivatives of the (homogenous) multivariate Tutte polynomial of any matroid have exactly one positive eigenvalue on the positive orthant when $0 < q \leq 1$. Consequences are proofs of the strongest conjecture of Mason and negative dependence properties for q -state Potts model partition functions.

1. INTRODUCTION

Several conjectures have been made regarding unimodality and log-concavity of sequences arising in matroid theory. Only recently have some of these been solved using combinatorial Hodge theory [AHK18, HSW18]. A conjecture that has resisted the approach of [AHK18] is the strongest conjecture of Mason regarding independent sets in a matroid [Mas72]. The purpose of this paper is to give a self-contained proof of the strongest conjecture avoiding, but inspired by, Hodge theory. We prove that the Hessian of the homogenous multivariate Tutte polynomial (or the q -state Potts model partition function) of a matroid has exactly one positive eigenvalue on the positive orthant when $0 < q \leq 1$. In a forthcoming paper we will take a more general approach and see that the results proved in this paper fit into a wider context¹.

Let n be an integer larger than 1, and let M be a matroid on $[n] = \{1, \dots, n\}$. Mason [Mas72] offered the following three conjectures of increasing strength. Several authors studied correlations in matroid theory partly in pursuit of these conjectures [SW75, Wag08, BBL09, KN10, KN11].

Conjecture. For any n -element matroid M and any positive integer k ,

- (1) $I_k(M)^2 \geq I_{k-1}(M)I_{k+1}(M)$,
- (2) $I_k(M)^2 \geq \frac{k+1}{k} I_{k-1}(M)I_{k+1}(M)$,
- (3) $I_k(M)^2 \geq \frac{k+1}{k} \frac{n-k+1}{n-k} I_{k-1}(M)I_{k+1}(M)$,

where $I_k(M)$ is the number of k -element independent sets of M .

¹In related forthcoming papers, Anari, Liu, Gharan and Vinzant have independently developed methods that overlap with our work. In particular, they also prove Mason's conjecture (3).

Conjecture (1) was proved in [AHK18], and Conjecture (2) was proved in [HSW18]. Note that Conjecture (3) may be written

$$\frac{I_k(\mathbb{M})^2}{\binom{n}{k}^2} \geq \frac{I_{k+1}(\mathbb{M})}{\binom{n}{k+1}} \frac{I_{k-1}(\mathbb{M})}{\binom{n}{k-1}},$$

and the equality holds when all $(k+1)$ -subsets of $[n]$ are independent in \mathbb{M} . Conjecture (3) is known to hold when n is at most 11 or k is at most 5 [KN11]. We refer to [Sey75, Dow80, Mah85, Zha85, HK12, HS89, Len13] for other partial results. We prove Conjecture (3) in Corollary 7 by uncovering concavity properties of the multivariate Tutte polynomial of \mathbb{M} .

Acknowledgements. Petter Brändén is a Wallenberg Academy Fellow supported by the Knut and Alice Wallenberg Foundation and Vetenskapsrådet. June Huh was supported by NSF Grant DMS-1638352 and the Ellentuck Fund.

2. THE HESSIAN OF THE MULTIVARIATE TUTTE POLYNOMIAL

Let $\text{rk}_{\mathbb{M}} : 2^{[n]} \rightarrow \mathbb{Z}_{\geq 0}$ be the rank function of \mathbb{M} . For a nonnegative integer k and a positive real parameter q , consider the degree k homogeneous polynomial in n variables

$$Z_{\mathbb{M}}^k = Z_{\mathbb{M}}^k(q, w_1, \dots, w_n) = \sum_A q^{-\text{rk}_{\mathbb{M}}(A)} \prod_{i \in A} w_i,$$

where the sum is over all k -element subsets A of $[n]$. We define the *homogeneous multivariate Tutte polynomial* of \mathbb{M} by

$$Z_{\mathbb{M}} = Z_{\mathbb{M}}(q, w) = \sum_{k=0}^n Z_{\mathbb{M}}^{n-k} w_0^k,$$

which is a homogeneous polynomial of degree n in $w = (w_0, w_1, \dots, w_n)$. When $w_0 = 1$, the function $Z_{\mathbb{M}}$ agrees with the partition function of the q -state Potts model, or the random cluster model [Pem00, Sok05, Gri06]. The *Hessian* of $Z_{\mathbb{M}}$ is the matrix

$$\mathcal{H}_{Z_{\mathbb{M}}}(w) = \left(\frac{\partial^2 Z_{\mathbb{M}}}{\partial w_i \partial w_j} \right)_{i,j=0}^n.$$

When $w \in \mathbb{R}_{>0}^{n+1}$, the largest eigenvalue of $\mathcal{H}_{Z_{\mathbb{M}}}$ is simple and positive by the Perron-Frobenius theorem. We prove the following analogue of the Hodge-Riemann relations for $Z_{\mathbb{M}}$.

Theorem 1. The Hessian of $Z_{\mathbb{M}}$ has exactly one positive eigenvalue for all $w \in \mathbb{R}_{>0}^{n+1}$ and $0 < q \leq 1$.

It follows that the Hessian of $\log Z_{\mathbb{M}}$ is negative semidefinite on $\mathbb{R}_{\geq 0}^{n+1}$, and hence $\log Z_{\mathbb{M}}$ is concave on $\mathbb{R}_{\geq 0}^{n+1}$ when $0 < q \leq 1$ [AOV, Lemma 2.7]. We deduce Theorem 1 from the following more precise statement. Let $c = (c_0, c_1, \dots, c_n)$ be a sequence of $n+1$ positive real numbers. We say that c is *strictly log-concave* if

$$c_m^2 > c_{m-1} c_{m+1} \quad \text{for } 0 < m < n.$$

For any strictly log-concave sequence c as above, set

$$Z_{M,c} = \sum_{k=0}^n c_{n-k} Z_M^{n-k} w_0^k.$$

For $\alpha \in \mathbb{Z}_{\geq 0}^{n+1}$, we write $\partial_i = \frac{\partial}{\partial w_i}$ and $\partial^\alpha = \partial_0^{\alpha_0} \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$.

Theorem 2. If $\partial^\alpha Z_{M,c}$ is not identically zero, then

- (i) the Hessian of $\partial^\alpha Z_{M,c}$ is nonsingular for all $w \in \mathbb{R}_{>0}^{n+1}$ and $0 < q \leq 1$, and
- (ii) the Hessian of $\partial^\alpha Z_{M,c}$ has exactly one positive eigenvalue for all $w \in \mathbb{R}_{>0}^{n+1}$ and $0 < q \leq 1$.

Theorem 1 can be deduced from Theorem 2 for $\alpha = 0$ by approximating the constant sequence 1 by strictly log-concave sequences. Theorem 2 will be proved by induction on the degree of $\partial^\alpha Z_{M,c}$. For undefined matroid terminologies, see [Oxl11].

Lemma 3. Let $A = (a_{ij})_{i,j=1}^n$ be a symmetric matrix with at least one positive eigenvalue. The following statements are equivalent.

- (1) A has exactly one positive eigenvalue.
- (2) For any $u, v \in \mathbb{R}^n$ with $u^T A u > 0$, $(u^T A v)^2 \geq (u^T A u)(v^T A v)$.
- (3) There is a vector $u \in \mathbb{R}^n$ with $u^T A u > 0$, such that $(u^T A v)^2 \geq (u^T A u)(v^T A v)$ for all $v \in \mathbb{R}^n$.

Proof. Since A has a positive eigenvalue, (2) implies (3).

If (3) holds, then A is negative semidefinite on the hyperplane $\{v \in \mathbb{R}^n \mid u^T A v = 0\}$. Since A has a positive eigenvalue, Cauchy's interlacing theorem implies (1).

Assume (1), $u^T A u > 0$, and that u and v are linearly independent. Let $Q(w) = w^T A w$. The discriminant Δ of the polynomial $t \mapsto Q(tu + v)$ is $(u^T A v)^2 - (u^T A u)(v^T A v)$. If $\Delta < 0$, then Q is positive on the plane spanned by u and v . This contradicts the fact that A has exactly one positive eigenvalue, by Cauchy's interlacing theorem. Hence $\Delta \geq 0$, and (2) follows. \square

Lemma 4. Theorem 2 holds when the degree of $\partial^\alpha Z_{M,c}$ is two.

Proof. It is enough to consider the case $\partial^\alpha = \partial_0^{n-2-k} \prod_{i \in S} \partial_i$, where S is a k -element subset of $E = [n]$. Note that $\partial_i Z_M^\ell = q^{-r(\{i\})} Z_{M/i}^{\ell-1}$, where M/i is the contraction of M by i . We need to prove that the Hessian of the quadratic form

$$Q = \frac{q^{\sum_{i \in S} r(\{i\})}}{(n-k-2)!} \partial^\alpha Z_{M,c} = c_k \binom{n-k}{2} w_0^2 + (n-k-1)c_{k+1} Z_{M/S}^1(w) w_0 + c_{k+2} Z_{M/S}^2(w)$$

is nonsingular and has exactly one positive eigenvalue. By contraction, we may assume that $S = \emptyset$ and $k = 0$. Write $Q(w) = w^T A w$, where $2A = \mathcal{H}_Q$. We prove that the inequality in the third statement of Lemma 3 is satisfied with strict inequality whenever $u = (1, 0, \dots, 0)^T$, and

$v \in \mathbb{R}^{n+1}$ is not a multiple of u . From this follows that A is nonsingular and has exactly one positive eigenvalue. In other words, we will prove,

$$Z_M^1(w)^2 > 2t \frac{n}{n-1} Z_M^2(w) \quad \text{for all } w \in \mathbb{R}^n \setminus \{0\}, \text{ where } t = \frac{c_0 c_2}{c_1^2}. \quad (\text{a})$$

Let E_0 be the set of loops in E , and let E_1, E_2, \dots, E_ℓ be the parallel classes of M . By the change of variables $w_j \rightarrow qw_j$ for all non-loops j , we get $Z_M^1 = e_1(E)$ and

$$Z_M^2 = e_2(E) - (1-q)(e_2(E_1) + \dots + e_2(E_\ell)), \quad (\text{b})$$

where $e_k(U)$ denotes the degree k elementary symmetric polynomial in the variables indexed by $U \subseteq E$.

We prove (a) for $t = 1$ with $>$ replaced by \geq . Moreover, we prove that if $Z_M^1(w) = 0$ for $w \neq 0$, then $Z_M^2(w) < 0$. The inequality (a) for $t = \frac{c_0 c_2}{c_1^2}$ then follows since $0 < \frac{c_0 c_2}{c_1^2} < 1$. Note that for $q = 1$ the desired inequality is an instance of the Cauchy-Schwarz inequality:

$$(w_1 + \dots + w_n)^2 \leq n(w_1^2 + \dots + w_n^2), \quad w \in \mathbb{R}^n. \quad (\text{c})$$

By (b), the inequality therefore reduces to the case when $e_2(E_1) + \dots + e_2(E_\ell) < 0$. By monotonicity in q it suffices to consider the case $q = 0$. Then the inequality reduces to

$$e_1(E)^2 \leq n \sum_{i=1}^{\ell} e_1(E_i)^2 + n \sum_{j \in E_0} w_j^2,$$

which follows from (c). Suppose $Z_M^1(w) = 0$ for $w \neq 0$. It remains to prove $Z_M^2(w) < 0$. Since $e_1(E) = 0$ and $w \neq 0$, it follows from the identity $e_1(E)^2 = 2e_2(E) + \sum_{i=1}^n w_i^2$ that $e_2(E) < 0$. Again the proof reduces to the case when $e_2(E_1) + \dots + e_2(E_\ell) < 0$, by (b). We have already proved that $Z_M^2(w) \leq 0$ when $q = 0$. But then $Z_M^2(w) < 0$ when $0 < q \leq 1$, by (b). This completes the proof of the lemma. \square

We prepare the proof of Theorem 2 with a lemma.

Lemma 5. Let F be a degree d homogeneous polynomial in $\mathbb{R}[w_0, w_1, \dots, w_n]$. If $w \in \mathbb{R}_{>0}^{n+1}$ and $\mathcal{H}_{\partial_i F}(w)$ has exactly one positive eigenvalue for each $i = 0, 1, \dots, n$, then

$$\ker \mathcal{H}_F(w) = \bigcap_{i=0}^n \ker \mathcal{H}_{\partial_i F}(w).$$

Proof. We fix $w \in \mathbb{R}_{>0}^{n+1}$ and write \mathcal{H}_F for $\mathcal{H}_F(w)$. We may suppose $d \geq 3$. By Euler's formula for homogeneous functions,

$$(d-2)\mathcal{H}_F = \sum_{i=0}^n w_i \mathcal{H}_{\partial_i F},$$

and hence the kernel of \mathcal{H}_F contains the intersection of the kernels of $\mathcal{H}_{\partial_i F}$.

For the other inclusion, let z be a vector in the kernel of \mathcal{H}_F . By Euler's formula again,

$$(d-2)e_i^T \mathcal{H}_F = w^T \mathcal{H}_{\partial_i F},$$

where e_i is the i -th standard basis vector in \mathbb{R}^{n+1} , and hence $w^T \mathcal{H}_{\partial_i F} z = 0$. We have $w^T \mathcal{H}_{\partial_i F} w > 0$ because $w \in \mathbb{R}_{>0}^{n+1}$ and $\partial_i F$ has nonnegative coefficients. It follows that $\mathcal{H}_{\partial_i F}$ is negative semidefinite on the kernel of $w^T \mathcal{H}_{\partial_i F}$, by e.g. Lemma 3. In particular,

$$z^T \mathcal{H}_{\partial_i F} z \leq 0, \text{ with equality if and only if } \mathcal{H}_{\partial_i F} z = 0.$$

To conclude, we write zero as the positive linear combination

$$0 = (d-2) \left(z^T \mathcal{H}_F z \right) = \sum_{i=0}^n y_i \left(z^T \mathcal{H}_{\partial_i F} z \right).$$

Since every summand in the right-hand side is non-positive by the previous analysis, we must have $z^T \mathcal{H}_{\partial_i F} z = 0$ for every i , and hence $\mathcal{H}_{\partial_i F} z = 0$ for each i . \square

Proof of Theorem 2. The proof is by induction on the degree m of $F = \partial^\alpha Z_{M,c}$. The case when $m = 2$ is Lemma 4. By relabeling the variables we may assume that w_0, w_1, \dots, w_n are the active variables in F . Suppose the theorem is true when the degree of F is at most m , where $m \geq 2$.

Suppose F has degree $m + 1$. We first prove (i). By induction, the Hessian of any derivative of F is non-singular and has exactly one positive eigenvalue. Hence (i) for F follows from Lemma 5.

When $q = 1$, F has the form

$$F = (\ell - 1)! c_{\ell-1} e_{m+1}([n]) + \ell! c_\ell e_m([n]) w_0 + \frac{1}{2} (\ell + 1)! c_{\ell+1} e_{m-1}([n]) w_0^2 + \dots.$$

If we choose c so that $c_i = 0$ unless $i \in \{\ell - 1, \ell\}$, $c_{\ell-1} = 1/(\ell - 1)!$ and $c_\ell = 1/\ell!$, then F is equal to the degree $m + 1$ elementary symmetric polynomial in w_0, w_1, \dots, w_n . The Hessian of F evaluated at the all ones vector is equal to a constant multiple of the matrix J_{n+1} , which has all diagonal entries equal to zero and all off-diagonal entries equal to 1. Clearly J_{n+1} is nonsingular and has exactly one positive eigenvalue. We may approximate c with a strictly log-concave positive sequence. This implies that there is a strictly log-concave sequence c for which the Hessian of F is nonsingular and has exactly one positive eigenvalue when $w = (1, \dots, 1)^T$ and $q = 1$. Since (i) holds for all $0 < q \leq 1$ and $w \in \mathbb{R}_{>0}^{n+1}$, and (ii) holds for at least one choice of the parameters, by continuity of the eigenvalues, (ii) holds for all $0 < q \leq 1$ and $w \in \mathbb{R}_{>0}^{n+1}$. \square

Theorems 1 and 2 suggest that there is an algebraic structure satisfying the Poincaré duality and the hard Lefschetz theorem whose degree 1 Hodge-Riemann form is given by the Hessian of Z_M . We refer to [Huh18] for a discussion of the one positive eigenvalue condition and the Hodge-Riemann relations.

3. CONSEQUENCES

We collect some corollaries of Theorem 2. It has been conjectured that the q -state Potts model should exhibit negative dependence properties when $0 < q \leq 1$, see [Pem00, Sok05, Gri06,

[Wag08]. However, no substantial results on negative dependence have been proved so far. By the next theorem we see that q -state Potts models are *ultra log-concave* for $0 < q \leq 1$.

Corollary 6. For any $0 < m < n$ and any $0 < q \leq 1$, we have

$$\frac{Z_M^m(q, w)^2}{\binom{n}{m}^2} \geq \frac{Z_M^{m+1}(q, w)}{\binom{n}{m+1}} \frac{Z_M^{m-1}(q, w)}{\binom{n}{m-1}}, \quad \text{for all } w \in \mathbb{R}_{\geq 0}^n.$$

Proof. Let \mathcal{H} denote the Hessian of $\partial_0^{n-m-1} Z_M$ at $w \in \mathbb{R}_{>0}^{n+1}$. Then $(w^T \mathcal{H} e_0)^2 \geq (w^T \mathcal{H} w)(e_0^T \mathcal{H} e_0)$, where $e_0 = (1, 0, 0, \dots)^T$, by Theorem 2 and the second statement of Lemma 3. By Euler's formula for homogeneous functions,

$$w^T \mathcal{H} e_0 = m \partial_0^{n-m} Z_M(w), \quad w^T \mathcal{H} w = (m+1) m \partial_0^{n-m-1} Z_M(w), \quad \text{and } e_0^T \mathcal{H} e_0 = \partial_0^{n-m+1} Z_M(w).$$

The proof follows by continuity, letting $w_0 = 0$. \square

Let \mathcal{J}_M^m be the collection of independent sets of M of size m . The m -th generating function of M is the homogeneous polynomial in n variables

$$f_M^m(w) = \sum_{I \in \mathcal{J}_M^m} \prod_{i \in I} w_i, \quad w = (w_1, \dots, w_n).$$

Note that $f_M^m(1, \dots, 1)$ is the number of independent sets of M of size m .

Corollary 7. For every $0 < m < n$ and every $w \in \mathbb{R}_{\geq 0}^n$, we have

$$\frac{f_M^m(w)^2}{\binom{n}{m}^2} \geq \frac{f_M^{m+1}(w)}{\binom{n}{m+1}} \frac{f_M^{m-1}(w)}{\binom{n}{m-1}}.$$

Proof. The proof is immediate from Corollary 6 and the identity $f_M^m(w) = \lim_{q \rightarrow 0} Z_M^m(q, qw)$. \square

Let ℓ be the number of rank one flats of M . The simplification \underline{M} of M is a matroid on $[\ell]$ whose lattice of flats is isomorphic to that of M [Oxl11, Section 1.7]. Applying Corollary 7 to the simplification \underline{M} , we get the stronger inequality

$$\frac{f_M^m(w)^2}{f_M^{m+1}(w) f_M^{m-1}(w)} \geq \frac{\binom{\ell}{m}^2}{\binom{\ell}{m+1} \binom{\ell}{m-1}} \geq \frac{\binom{n}{m}^2}{\binom{n}{m+1} \binom{n}{m-1}} \quad \text{for all } w \in \mathbb{R}_{\geq 0}^n,$$

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