Recitation notes 18.02A, Fall 2019

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Recitation 1: October 21 18.02A

Focus: Vectors, scalar products.

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Vectors

Vectors are tools to help us understand what happens in $\mathbb{R}^2$ or $\mathbb{R}^3$ where it is hard to draw (or $\mathbb{R}^4$ where it is impossible to draw).

- (Vector addition) If $\mathbf{v} = (x, y, z)$, $\mathbf{w} = (a, b, c)$, then $\mathbf{v} + \mathbf{w} = (x + a, y + b, z + c)$.
- (Multiplication of a vector by a real number) If $\mathbf{v} = (x, y, z)$, and t is a real number, then $t\mathbf{v} = (tx, ty, tz)$.
- (Length of a vector) If $\mathbf{v} = (x, y, z)$, then $|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.
- (Dot product of two vectors, also called scalar *product*) The dot product takes two vectors and spits out a real number: if $\mathbf{v} = (x, y, z)$, $\mathbf{w} = (a, b, c)$, then $\mathbf{v} \cdot \mathbf{w} = xa + yb + zc$.
- (Vectors with special names) $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$.

Problem 1.
What are the $\mathbf{ij}$-components of a vector $\mathbf{v}$ of length 3 in the xy-plane, if it makes an angle of $30°$ with $\mathbf{i}$ and $60°$ with $\mathbf{j}$?

Problem 2.
Let $\mathbf{v} = (2, 1)$, $\mathbf{w} = (1, 2)$. Draw the vectors $\mathbf{v}, \mathbf{w}, -\mathbf{w}, \mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w}$.

Problem 3.
Show that the triangle with vertices $(4, 3, 6)$, $(-2, 0, 8)$ and $(1, 5, 0)$ is a right triangle. Find its area.

Recitation 2: October 23 18.02A

Focus: Projection of vectors, cross products, determinants.

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Dot product

1. (Dot product of two vectors, also called scalar *product*) The dot product takes two vectors and spits out a real number: if $\mathbf{v} = (x, y, z)$, $\mathbf{w} = (a, b, c)$, then $\mathbf{v} \cdot \mathbf{w} = xa + yb + zc$.

2. **Very important geometric meaning of dot product:** $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \cdot |\mathbf{w}| \cos\theta$, where θ is the angle between the two vectors $\mathbf{v}$ and $\mathbf{w}$. **In other words, we can easily calculate the angle between two vectors:** if $\mathbf{v} = (x, y, z)$, $\mathbf{w} = (a, b, c)$, then

$$\theta = \cos^{-1}\left(\frac{xa + yb + zc}{|\mathbf{v}||\mathbf{w}|}\right). \tag{1}$$

Projection of vectors

1. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is the component of $\mathbf{b}$ along $\mathbf{a}$, i.e. the length of the part of $\mathbf{b}$ in the same direction as $\mathbf{a}$. Computed by

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|a|}. \tag{2}$$

2. The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is the part of $\mathbf{b}$ in the same direction as $\mathbf{a}$. Computed by

$$\begin{aligned} \text{vector projection} &= (\text{scalar projection}) \cdot \ (\text{unit vector in direction of } \mathbf{a}) && (3) \\ &= (\text{scalar projection}) \cdot \frac{\mathbf{a}}{|\mathbf{a}|} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}\right) \frac{\mathbf{a}}{|\mathbf{a}|}. && (4) \end{aligned}$$

Determinants

The determinant is a real number calculated from two vectors $\mathbf{v_1} = (a, b)$, and $\mathbf{v_2} = (c, d)$ by

$$\det(\mathbf{v_1}, \mathbf{v_2}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{5}$$

The absolute value of this number equals the area of the parallelogram with sides $\mathbf{v_1}$, and $\mathbf{v_2}$. Alternatively, $\det(\mathbf{v_1}, \mathbf{v_2}) = |\mathbf{v_1}||\mathbf{v_2}| \sin\theta$.

Cross product

1. The cross product only exists in 3D. There, it takes two vectors $\mathbf{v_1} = (a_1, a_2, a_3)$, $\mathbf{v_2} = (b_1, b_2, b_3)$ and spits out another vector by the formula

$$\mathbf{v_1} \times \mathbf{v_2} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \tag{6}$$

   In the next lecture, you will learn a formula for this that is easier to remember.

2. **Fact:** $\mathbf{v_1} \times \mathbf{v_2}$ is a vector that is orthogonal to both $\mathbf{v_1}$ and $\mathbf{v_2}$.

3. **Fact:** $\mathbf{v_1} \times \mathbf{v_2}$ has length

$$|\mathbf{v_1} \times \mathbf{v_2}| = |\mathbf{v_1}| \cdot |\mathbf{v_2}| \sin(\theta). \tag{7}$$

1 Problems

Problem 1.
Find the angle between the two vectors $\mathbf{v} = (1, 2, 0)$ and $\mathbf{w} = (0, 3, 1)$.

Problem 2.
Calculate the vector projection of $(1, 2, 3)$ onto the vector $(1, 0, 1)$.

Problem 3.
Let $\mathbf{v_1} = (1, -1, 1)$, $\mathbf{v_2} = (1, 2, 2)$.

a) Calculate $\mathbf{v_3} = \mathbf{v_1} \times \mathbf{v_2}$.

b) Calculate the projection of $\mathbf{v_3}$ onto $\mathbf{v_1}$ and the projection of $\mathbf{v_3}$ onto $\mathbf{v_2}$.

c) What is the area of the parallelogram spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$?

Problem 4.
Find the area of the triangle with vertices at $(0, 0, 1)$, $(2, 3, 0)$ and $(-1, 1, 2)$.

Problem 5.
Let P be the point $(1, 2, 1)$. Calculate the point on the line through the origin and $(1, 1, 1)$ that is closest to P.

Recitation 3: October 28 — 18.02A

Focus: Determinants, equations for lines and planes.

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Lines in $\mathbb{R}^3$

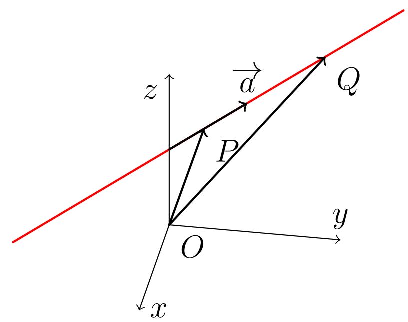
A line is determined by one point $P = (x_0, y_0, z_0)$ on it and its direction vector $\mathbf{a} = (a_1, a_2, a_3)$. The line consists of all the points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is parallel to $\mathbf{a}$, i.e. $\overrightarrow{PQ} = (x - x_0, y - y_0, z - z_0) = t\mathbf{a}$.

1. The line can be described in parametric form by

$$(x_0, y_0, z_0) + t(a_1, a_2, a_3). \tag{1}$$

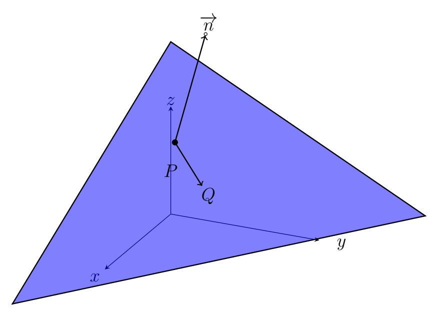
2. The line can alternatively be described by the symmetric equations

$$t = \frac{x - x_0}{a_1} = \frac{y - y_0}{a_2} = \frac{z - z_0}{a_3}. \tag{2}$$



Planes in $\mathbb{R}^3$

A plane is determined by one point $P = (x_0, y_0, z_0)$ on it and its normal vector $\mathbf{n} = (a_1, a_2, a_3)$. The plane consists of all those points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is perpendicular to $\mathbf{n}$, i.e. $0 = \mathbf{n} \cdot \overrightarrow{PQ} = a_1(x - x_0) + a_2(y - y_0) + a_3(z - z_0)$. Alternatively, a plane is determined by the equation $a_1 x + a_2 y + a_3 z = d$.



Determinants in 3D

In 3D, the determinant is a real number calculated from three vectors $\mathbf{v_1} = (a, b, c)$, $\mathbf{v_2} = (d, e, f)$ and $\mathbf{v_3} = (g, h, i)$ by

$$\det(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}. \tag{3}$$

The absolute value of this number equals the volume of the parallelepiped with sides $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$. Therefore, the determinant is zero if and only if $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ are all contained in the same plane.

Cross product

The cross product only exists in 3D. There, it takes two vectors $\mathbf{v_1} = (a, b, c)$, $\mathbf{v_2} = (d, e, f)$ and spits out another vector by

$$\mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ d & e & f \end{vmatrix}. \tag{4}$$

1 Problems

Problem 1.
Calculate

a) the determinant of the vectors $\mathbf{v_1} = (1, 2, 3)$, $\mathbf{v_2} = (0, -1, 2)$, $\mathbf{v_3} = (1, 1, 1)$.

b)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 3 & 4 \end{vmatrix} \tag{5}$$

Problem 2.
Let $\mathbf{v_1} = (0, -3, 2)$, $\mathbf{v_2} = (2, 2, 1)$. Calculate $\mathbf{v_1} \times \mathbf{v_2}$.

Problem 3.
Find the equation of

a) the plane that passes through the three points $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$.

b) the plane through $(0, 2, 3)$, $(1, 0, 1)$ and parallel to $(3, 1, 2)$.

Problem 4.

a) Find the equation of the line through $P_1 = (0, -1, -1)$ and $P_2 = (2, 3, 3)$

b) Find the point where the line intersects the three coordinate planes.

c) Find the point where this line intersects the plane $2x + y - 3z = 1$.

Recitation 4: October 30 18.02A

Focus: Intersections of lines and planes; distances between points, planes and lines.

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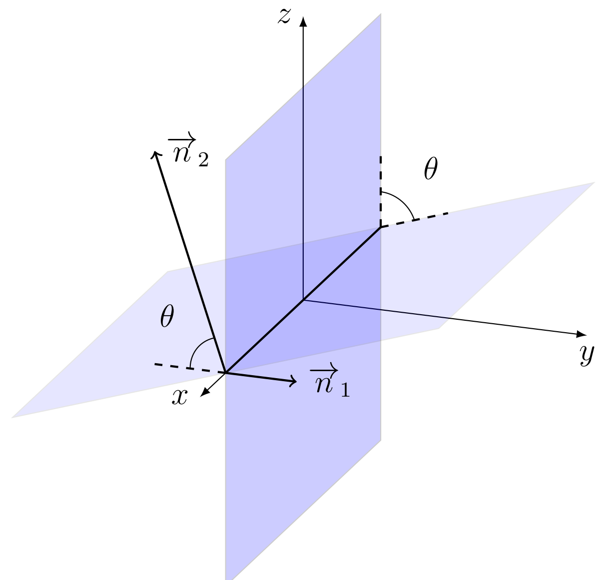
Remember: the normal vector to a plane $ax + by + cz = d$ can be read off by $\overrightarrow{n} = (a, b, c)$.

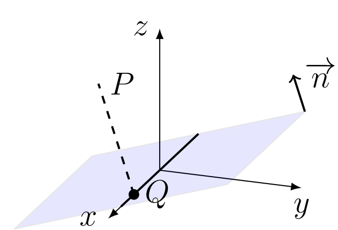
Intersections of lines and planes

1. A line always intersects a plane in a single point, unless the line is parallel to the plane or the line is contained in the plane. To find the intersection point, plug in the parametric equation for the line into the equation for the plane and solve for t.
2. Two planes intersect unless they are parallel which is equivalent to their normal vectors being parallel. To find the intersection line, find one point on the line and the direction vector of the line.
3. The angle between two intersecting planes is equal to the angle between their normal vectors.



Distance from a point to a plane

The distance between a point P and a plane is given by dropping the point perpendicularly (i.e. along the normal vector $\overrightarrow{n}$ of the plane) onto the plane. If we call the point on the plane that we reach Q, then the distance is $\left|\overrightarrow{PQ}\right|$.



Parametric equations for curves

A particle moving through $\mathbb{R}^2$ or $\mathbb{R}^3$ can be described parametrically, by saying that at time t, the particle is at

$$\mathbf{r}(t) = (x(t), y(t), z(t)). \tag{1}$$

One way of thinking about this is that at each time t, the parametric equation gives you a snapshot of where the particle is at that instant, namely at $\langle x(t), y(t), z(t)\rangle$. As t varies (for instance for $-\infty < t < \infty$), the particle traces out a curve.

The particle position $\mathbf{r}(t)$ is a vector-valued function, and we can take its derivative by taking the derivative of each coordinate. This is the velocity of the particle, denoted by $\mathbf{v}(t)$:

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)) = (x'(t), y'(t), z'(t)). \tag{2}$$

The speed of the particle is the length of the velocity vector, i.e. $|v(t)|$.

Problems

Problem 1.
Find the parametric equation for the intersection of the two planes $x + y - 3z = 2$ and $-x + 5y - z = 1$. What is the angle between these planes?

Problem 2.
Find the distance (i.e. shortest distance) between the point $P = (2, 1, 1)$ and the plane $-x + 2y + z = 2$.

Problem 3.
Find the distance between the two lines $(x, y, z) = (1, 0, 2)+t(1, 1, 1)$ and $(x, y, z) = (2, 2, 1)+ s(-1, 1, 0)$.

Problem 4.
The point P moves with constant speed 1 in the direction of the constant vector $4\mathbf{i} + 3\mathbf{j}$ towards increasing x-values. If at time $t = 0$ it is at (x_0, y_0), what is its position vector function $r(t)$?

Recitation 5: November 4 18.02A

Focus: Parametric equations for curves.

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Parametric equations for curves

A particle moving through $\mathbb{R}^2$ or $\mathbb{R}^3$ can be described parametrically, by saying that at time t, the particle is at

$$\mathbf{r}(t) = (x(t), y(t), z(t)). \tag{1}$$

One way of thinking about this is that at each time t, the parametric equation gives you a snapshot of where the particle is at that instant, namely at $(x(t), y(t), z(t))$. As t varies (for instance for $-\infty < t < \infty$), the particle traces out a curve.

The particle position $\mathbf{r}(t)$ is a vector-valued function, and we can take its derivative by taking the derivative of each coordinate. This is the velocity of the particle, denoted by $\mathbf{v}(t)$:

$$\mathbf{v}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}(t)) = (x'(t), y'(t), z'(t)). \tag{2}$$

The speed of the particle is the length of the velocity vector, i.e. $|v(t)|$.

We can differentiate the velocity, too. This is again a vector-valued function, and the derivative is the acceleration, denoted $\mathbf{a}(t)$:

$$\mathbf{a}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}(\mathbf{t})) = (x''(t), y''(t), z''(t)). \tag{3}$$

Problems

Problem 1.
Describe the motions given by each of the following position vector functions, as t goes from $-\infty$ to ∞. In each case, give the xy-equation of the curve along which the particle travels, and tell what part of the curve is actually traced out by the particle.
(a) $x = t^3$, $y = t^2$.
(b) $x = \sin^2 2t$, $y = \cos t$.

Problem 2.
A circle of radius b rolls on the *inside* of a circle of radius $a > b$. If the circle of radius b is attached at a point P on the x-axis at time $t = 0$, find an expression for the position of P at time t. (This is called a hypocycloid, and on the HW, you will study an epicycloid which is a circle rolling on the *outside* of another circle.)

Problem 3.
A machine consists of two rods. One big rod of length L is moving around the origin, and then a smaller rod of length ℓ is attached to its other end. The big rod makes an *angle* $\sin at$ relative to the x-axis, and the small rod makes an *angle* $\sin bt^2$ relative to the x-axis.
(a) What is the position vector from the origin to the endpoint of the smaller rod?
(b) What is the velocity vector for this endpoint?

Focus: Functions of several variables, partial derivatives.

Functions of several variables

For the remainder of the course, we will study functions that depend on two or more variables: $f(x, y)$, $f(x, y, z)$ et.c. We have two tools to visualize functions $f(x, y)$ of two variables:

1. **Drawing level curves** (2D picture consisting of many level curves). We draw those values of x and y for which $f(x, y) = 0$, which is a curve in the xy-plane, together with the value of x and y where $f(x, y) = 1$, $f(x, y) = -1$ et.c.
2. **Graphing the function** (3D picture). This is more informative than drawing level curves, but can be harder. It can sometimes help to draw slices separately, i.e., drawing $f(x, 0)$ in an xz-coordinate system and $f(0, y)$ in a yz-coordinate system.

Partial derivatives

The value of a function $f(x, y)$ changes when we change either x or y. This change is measured by partial derivatives.

1. The partial derivative of $f(x, y)$ with respect to x is the change of f when you change x, and keep y constant. Computed in the ordinary way, by treating y as a constant.
2. The partial derivative of $f(x, y)$ with respect to y is the change of f when you change y, and keep x constant. Computed in the ordinary way, by treating x as a constant.

Fact: For every function we will encouter in this class, the order in which we take partial derivatives does not matter:

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x, y)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x, y)\right). \tag{1}$$

Problems

Problem 1.
Draw level curves of the following functions, and graph them as well.
(a) $f(x, y) = 1 - x - y$.
(b) $f(x, y) = 1 - x^2$.
(c) $f(x, y) = \sqrt{x^2 + y^2}$.

Problem 2.
(a) For $f(x, y) = \sin(x^2 y + 3y)$, compute f_x, f_y, f_{xx}, f_{xy}, f_{yx}.
(b) Show that $w = e^{ax}\sin(ay)$ satisfies $w_{xx} + w_{yy} = 0$.
(c) Find *one example* of a function $f(x, y)$ that can be written on the form $f(x, y) = g(x)h(y)$ that satisfies both $f_x + 9f = 0$ and $f_y + f = 0$.

Problem 3.
Where does the graph of the function $f(x, y) = x^2 + y^2$ intersect
(a) the line $(1, 1, 1) + t(2, 0, -1)$?
(b) the plane $z - 2x - 2y = 2$?

Recitation 7: November 13 18.02A

Focus: Tangent planes, linear approximations, gradients and directional derivatives.

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Gradients

Given a function $f(x, y)$, the gradient of f at a point (x_0, y_0) is defined by

$$\nabla f(x_0, y_0) := \left(\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)\right). \tag{1}$$

In 3D, the gradient of $f(x, y, z)$ at a point (x_0, y_0, z_0) is defined by

$$\nabla f(x_0, y_0, z_0) := \left(\frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0)\right). \tag{2}$$

The reason why we care enough about the gradient to name it, is because it has the following important properties:

1. If (x_0, y_0) is a point on the level curve $f(x, y) = C$ (for some constant C), then $\nabla f(x_0, y_0)$ is perpendicular to the level curve at (x_0, y_0). In 3D, same but for level surfaces.
2. At a point (x_0, y_0), the direction in which f increases fastest is along $\nabla f(x_0, y_0)$ and the direction in which f decreases fastest is along $-\nabla f(x_0, y_0)$.
3. It can be used to compute directional derivatives.

Directional derivatives

Let $\mathbf{u}$ be a unit vector, i.e. $|\mathbf{u}| = 1$. The directional derivative of f at a point (x_0, y_0) in the direction $\mathbf{u}$ is the rate of change of f in the direction $\mathbf{u}$, starting at (x_0, y_0). It is denoted by $D_{\mathbf{u}} f(x_0, y_0)$ and can be calculated by

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}. \tag{3}$$

Note: u must have length 1! If your u has a different length, replace it by $\frac{\mathbf{u}}{|\mathbf{u}|}$.

Tangent planes

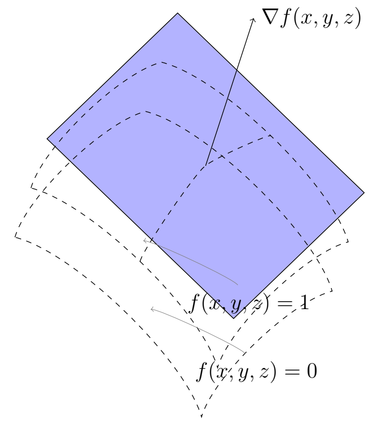
The tangent plane to the function $z = f(x, y)$ at a point (x_0, y_0, z_0) is given by

$$z - z_0 = f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0). \tag{4}$$

In other words, this is a plane containing the point (x_0, y_0, z_0) and with normal vector

$$(-f_x(x_0, y_0), -f_y(x_0, y_0), 1). \tag{5}$$

Surfaces can also be defined implicitly, e.g., by $f(x,y,z) = \text{constant}$. This means that the surface is a level surface of the function f. The tangent plane can then be found by using the fact that ∇f is a normal to the level surface.



Linear approximations

In 1D, the best linear approximation of a function $f(x)$ at the point x_0 is given by the tangent line to $f(x)$ at x_0.

In 2D, the best linear approximation to a function $f(x,y)$ is given by the tangent plane to $f(x,y)$ at (x_0,y_0), i.e. by

$$f(x_0,y_0) + f_x(x_0,y_0)\cdot(x-x_0) + f_y(x_0,y_0)\cdot(y-y_0). \tag{6}$$

Out of all the polynomials that are linear in both x and y, this is the polynomial that is the closest to the graph of $f(x,y)$, for x close to x_0 and y close to y_0.

Problems

Problem 1.
Find the equations of the tangent planes of the surfaces
a) $w = y^2/x$ at the point $(x_0,y_0,w_0) = (1,2,4)$.
b) $2x^2 + y^2 + z^2 = 1$ at $(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
c) $xyz + x^2 - 2y^2 + z^3 = 14$ at the point $(5,-2,3)$

Problem 2.
Let $f(x,y) = x^2 + 4y^2 - 4x$.
a) Calculate the directional derivative at the point $(1,2)$ in the direction parallel to the line $y = x$.
b) Find all points (x_0,y_0) where the direction of fastest increase of f is perpendicular to $(1,1)$.
c) Find all points (x_0,y_0) where the direction of fastest increase of f is parallel to $(1,1)$.

Recitation 8: November 18 18.02A

Focus: Maximizing/minimizing functions of several variables, second derivative test, method of least squares.

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Finding maxima/minima

The maximum/minimum of f in a region R with boundary curve C is attained either inside the region, or on the boundary. To find:

1. Find all critical points inside the region, i.e. points (a, b) such that $\nabla f(a, b) = 0$. If the maximum/minimum is attained (strictly) inside the region, it is attained at a critical point.

2. Check values of f on the boundary C (using 18.01A knowledge). If the maximum/minimum is attained on the boundary, you find it in this step.

3. Compare the function values at all the critical points to the maximum/minimum on the boundary, and pick the biggest/smallest one.

Classifying critical points

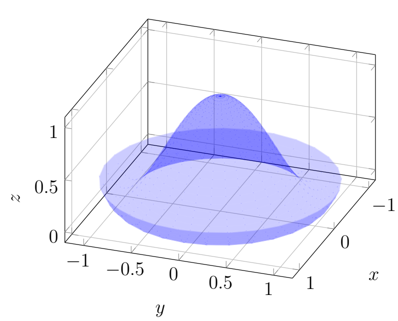
The above procedure finds the maximum/minimum of f in all of the region R (also called global maximum/minimum). If we want to understand each critical point better, we can classify each of these as local maximum/minimum or saddle points. The second derivative test: for each critical point (x_0, y_0), compute $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$. If

1. $D > 0$, $f_{xx} < 0$, it is a local maximum.

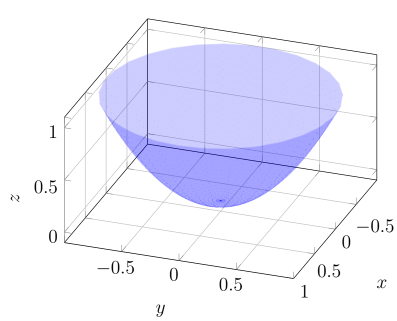
2. $D > 0$, $f_{xx} > 0$, it is a local minimum.

3. $D < 0$, it is a saddle point

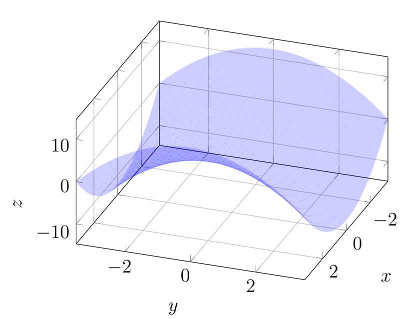
4. $D = 0$, the test is inconclusive.



Local maximum



Local minimum



Saddle point

Problems

Problem 1.
Find the maximum and minimum of $f(x, y) = x^2 + y^3 - 6xy$ on the region R given by $0 \leq x \leq 4, 0 \leq y \leq x$.

Problem 2.
Classify the critical points of $f(x, y) = x^2y + y^2x + xy$ as either local maximum, minimum or saddle points.

Problem 3. Find the global max of $2x - x^2 + 2y^2 - y^4$ in all of the xy-plane.

Problem 4.
A scientist is measuring the pollution (x) in three different lakes together with the amount of healthy fish in the respective lakes (y). Find the line $y = ax + b$ that best fits the given data $(x, y) = (1, 3)$, $(2, 2)$ and $(3, 2)$.

Recitation 9: November 20 18.02A

Focus: Lagrange multipliers.

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Lagrange multipliers

2D

Lagrange multipliers are tools that allow us to maximize/minimize functions $f(x, y)$ subject to a constraint of the form $g(x, y) = C$, for some constant C. At a maximum/minimum (x_0, y_0), there is a real number λ (the Lagrange multiplier) such that

- $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$
- $g(x_0, y_0) = C$.

To find max/min:

1. Solve these three equations for x_0, y_0 and λ. Although we usually do not care about the actual value of λ, it is usually needed to solve for x_0 and y_0.
2. **Check which of all solutions gives the largest/smallest value of f.** These are then the maxima/minima.

3D

The same equations work for functions $f(x, y, z)$, $g(x, y, z)$ in 3D:

- $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$
- $g(x_0, y_0, z_0) = C$.

To maximize/minimize functions $f(x, y, z)$ subject to two constraints of the form $g(x, y, z) = C$, $h(x, y, z) = D$, for some constants C, D, there are real numbers λ, μ (the Lagrange multipliers) such that

- $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$
- $g(x_0, y_0, z_0) = C$, $h(x_0, y_0, z_0) = D$.

Problems

Problem 1.
Maximize and minimize xy subject to the constraint $x^2 + 4y^2 = 4$.

Problem 2.
What is the minimum and the maximum of $81x^2 + y^2$ in the region $4x^2 + y^2 \leq 9$?

Problem 3.
Find the points on the curve $x^2 + xy + y^2 = 3$ that are closest to the origin and farthest away from the origin.

Problem 4.
Consider the curve given as the intersection of the ellipsoid $\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{25} = 1$ and the plane $x + y = -z$. Find the the maximum and minimum distance of points on the curve to the origin.

Recitation 10: November 25 18.02A

Focus: Chain rules for multivariable functions.

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Chain rule

Single variable chain rule: If $y = f(x)$ and $x = g(t)$, then $f(x) = f(g(t))$ and

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}. \tag{1}$$

Multivariable chain rules:

1. If $w = w(x, y)$, $x = x(t)$ and $y = y(t)$, then $w = w(x(t), y(t))$ and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}. \tag{2}$$

2. If $w = w(x, y)$, $x = x(u, v)$ and $y = y(u, v)$, then $w = w(x(u, v), y(u, v))$ and

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial u} \tag{3}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial v}. \tag{4}$$

Implicit functions

A function $z(x, y)$ can be defined by an implicit equation such as $x^3 + y^3 + z^3 + 6xyz = 9$. We can find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ **without having to explicitly solve for** z **as a function of** x **and** y, by taking $\frac{\partial}{\partial x}$ everywhere and solving for $\frac{\partial z}{\partial x}$.

Double integrals

If R is a region in the xy-plane, then the volume under the graph of the function $f(x, y)$ over the region R is given by the double integral

$$\int\int_R f(x, y) dA. \tag{5}$$

To compute the integral:

1. Draw the region R.
2. Slice R either horizontally or vertically.
3. For vertical slices: for each x, integrate $f(x, y)$ over y. Then integrate over x:

$$\int\int f(x, y) dy dx. \tag{6}$$

For horizontal slices: for each y, integrate $f(x, y)$ over x. Then integrate over y:

$$\int\int f(x, y) dx dy. \tag{7}$$

Problems

Problem 1 (2E-1).
In the following, find $\frac{dw}{dt}$ for the composite function $w = f(x(t), y(t), z(t))$ in two ways:

i) use the chain rule, then express your answer in terms of t by using $x = x(t)$, etc.;

ii) express the composite function f in terms of t, and differentiate.

a) $w = xyz, x = t, y = t^2, z = t^3$
b) $w = x^2 - y^2, x = \cos t, y = \sin t$.

Problem 2.
Let $w = f(x, y)$, and make the change of variables $x = u^2 - v^2$, $y = 2uv$. Show

$$(w_x)^2 + (w_y)^2 = \frac{(w_u)^2 + (w_v)^2}{4(u^2 + v^2)}. \tag{8}$$

Problem 3.
Let $w = f(x, y)$, and make the change of variables $x = u - v$, $y = v^3$. Express w_{vv} in terms of w_{xx}, w_{yy} et.c.

Problem 4.
Calculate $\frac{\partial z}{\partial x}$ at the point $(1, 0, 1)$, where z is implicitly defined by $x^2 z + 3\sin(yz) + y/x = 1$.

Problem 5 (2E-2).
Information about the gradient of an unknown function $f(x, y)$ is given; x and y are in turn functions of t. Use the chain rule to find out additional information about the composite function $w = f(x(t), y(t), z(t))$, without trying to determine f explicitly.
c) $\nabla f = (1, -1, 2)$ at $(1, 1, 1)$. Let $x = t, y = t^2, z = t^3$; find $\frac{df}{dt}$ at $t = 1$.

Recitation 11: November 27 18.02A

Focus: Double integrals.

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Double integrals

If R is a region in the xy-plane, then the volume under the graph of the function $f(x,y)$ over the region R is given by the double integral

$$\int\int_R f(x,y)dA. \tag{1}$$

To compute the integral:

1. $\boxed{\text{Draw the region } R.}$
2. Slice R either horizontally or vertically.
3. For vertical slices: for each x, integrate $f(x,y)$ over y. Then integrate over x:

$$\int\int f(x,y)dydx. \tag{2}$$

For horizontal slices: for each y, integrate $f(x,y)$ over x. Then integrate over y:

$$\int\int f(x,y)dxdy. \tag{3}$$

One of these choices can be easier than the other! **Warning: when changing the order of integration from e.g. $dxdy$ to $dydx$, you must also change the limits of integration!**

Applications

1. Area of $R = \displaystyle\int\int_R 1dA$.
2. Volume between $z = f(x,y)$ and $z = g(x.y)$ over region $R = \displaystyle\int\int_R (f(x,y) - g(x.y))\,dA$.
3. Average value of $f(x.y)$ on region $R = \dfrac{1}{\text{Area}(R)}\displaystyle\int\int_R f(x,y)dA$.
4. Total mass m of metal plate R with variable density $\rho(x,y)$ is $m = \displaystyle\int\int_R \rho(x,y)dA$.
5. Center of mass (x_0, y_0) of the plate is given by

$$x_0 = \frac{1}{m}\int\int_R x\rho(x,y)dA, \tag{4}$$
$$y_0 = \frac{1}{m}\int\int_R y\rho(x,y)dA. \tag{5}$$

Problems

Problem 1 (3A-3c).

Compute the integral $\int\int_R y dA$ where R is the triangle with vertices at $(\pm 1, 0), (0, 1)$.

Problem 2 (3A-5).

Evaluate each of the following iterated integrals, by changing the order of integration (begin by figuring out what the region R is, and sketching it).

a) $\int_0^{\frac{1}{4}} \int_{\sqrt{t}}^{\frac{1}{2}} \frac{e^u}{u} du dt$

b) $\int_0^1 \int_{x^{\frac{1}{3}}}^1 \frac{1}{1+u^4} du dx.$

Problem 3.

a) Find the volume above the xy-plane bounded by the paraboloid $z = x^2 + y^2$ and the planes $x = \pm 1$, $y = \pm 1$.

b) Find the volume above the xy-plane bounded by the cylinder $y = 4 - x^2$ and the planes $y = 3x, z = x + 4$.

Recitation 12: December 2 18.02A

Focus: Double integrals.

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Double integrals

If R is a region in the xy-plane, then the volume under the graph of the function $f(x, y)$ over the region R is given by the double integral

$$\int\int_R f(x,y)dA. \tag{1}$$

To compute the integral:

1. $\boxed{\text{Draw the region } R.}$
2. Slice R either horizontally or vertically.
3. For vertical slices: for each x, integrate $f(x, y)$ over y. Then integrate over x:

$$\int\int f(x,y)dydx. \tag{2}$$

For horizontal slices: for each y, integrate $f(x, y)$ over x. Then integrate over y:

$$\int\int f(x,y)dxdy. \tag{3}$$

One of these choices can be easier than the other! **Warning: when changing the order of integration from e.g. $dxdy$ to $dydx$, you must also change the limits of integration!**

Applications

1. Area of $R = \int\int_R 1dA$.
2. Volume between $z = f(x, y)$ and $z = g(x.y)$ over region $R = \int\int_R (f(x,y) - g(x.y))\, dA$.
3. Average value of $f(x.y)$ on region $R = \frac{1}{\text{Area}(R)} \int\int_R f(x,y)dA$.
4. Total mass m of metal plate R with variable density $\rho(x, y)$ is $m = \int\int_R \rho(x,y)dA$.
5. Center of mass (x_0, y_0) of the plate is given by

$$x_0 = \frac{1}{m}\int\int_R x\rho(x,y)dA, \tag{4}$$

$$y_0 = \frac{1}{m}\int\int_R y\rho(x,y)dA. \tag{5}$$

Problems

Problem 1.
a) Compute the volume above the xy-plane bounded by the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 2$.
b) Find the volume of the solid in the first octant bounded by the cylinder $4y = x^2$ and the planes $x = 0$, $z = 0$, $y = 4$ and $x - y + 2z = 2$.

Problem 2.
Calculate the centroid for the region R between $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$.

Problem 3.
Calculate the total mass and center of mass for the region R bounded by the parabola $y = x^2$ and the line $y = x$, where the density at a point (x, y) is $\sqrt{xy}$.

Recitation 13: December 4 18.02A

Focus: Polar coordinates for double integrals.

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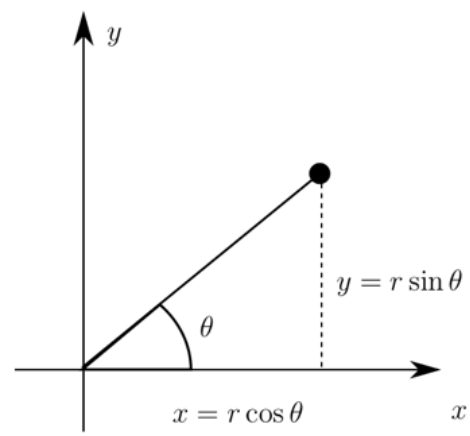
Polar coordinates

Instead of specifying the xy-coordinates of a point in the xy-plane, we can also specify the distance r of the point to the origin and the angle θ it makes to the positive x-axis.

In formulas:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned} \tag{1}$$



If we want to transform the other way, we have the formulas

$$\begin{aligned} r &= \sqrt{x^2+y^2}, \\ \theta &= \operatorname{atan}\left(\frac{y}{x}\right). \end{aligned} \tag{2}$$

Double integrals in polar coordinates

Certain integrals are more easily computed in polar coordinates, if either the integrand or the domain of integration is easier to express in polar coordinates.

To compute the integral $\int\int_R f(x,y)dA$:

1. Write $f(x,y)$ in terms of r and θ.
2. Write limits of integration for region R in terms of r and θ.
3. Set $dA = rdrd\theta$. **Warning: Do not forget the factor r!**

1 Problems

Problem 1.
a) Find the volume of the domain under the graph of xy and over the region R, which is the first-quadrant portion of the interior of $x^2 + y^2 = a^2$.

b) Compute

$$\int\int_R \cos(x^2 + y^2)dA,$$

over the region R, which is the first-quadrant portion of the interior of $x^2 + y^2 = 1$, bounded by the line $x = y$.

c) Compute

$$\int\int_R \frac{1}{\sqrt{1 - x^2 - y^2}}dA,$$

where R is the right half-disk of radius $\frac{1}{2}$ centered at $(0, \frac{1}{2})$.

Problem 2.
Consider two circles of radius 1, one centered at $(0, 0)$ and the other centered at $(1, 0)$; calculate the double integral for the area of intersection between the two circles.

Recitation 14: December 9 18.02A

Focus: Surface area, change of variables in double integrals.

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Change of variables in double integrals

For one-variable integrals, we can change the variable of integration by

$$\int f(x)dx = \int f(x(t))\frac{dx}{dt}dt, \tag{1}$$

where we also change the limits of integration.

For double integrals, we can change the variables from x, y to new variables u, v where $u(x, y)$ and $v(x, y)$ are functions of both x and y.

The steps for computing the integral $\int\int_R f(x, y)dxdy$ in the new variables are:

1. Express $f(x, y)$ in terms of u and v.

2. Express the region R in terms of u and v.

3. Put $dxdy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| dudv$ (the lines denote the absolute value). Here, the quantity $\frac{\partial(x,y)}{\partial(u,v)}$ is called the Jacobian for the change of variables, and is defined by

$$\frac{\partial(x, y)}{\partial(u, v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}. \tag{2}$$

   To remember the formula $dxdy = \frac{\partial(x, y)}{\partial(u, v)}dudv$, notice the "cancellation" of u and v in the right hand side.

Sometimes it is easier to compute

$$\frac{\partial(u, v)}{\partial(x, y)} = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \tag{3}$$

than $\frac{\partial(x,y)}{\partial(u,v)}$. We can then use the following fact in step 3:

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}. \tag{4}$$

Surface area using double integrals

The surface area of the portion of the graph $z = f(x, y)$ above the region R in the xy-plane is given by

$$\int\int_R \sqrt{1 + (f_x)^2 + (f_y)^2}dxdy. \tag{5}$$

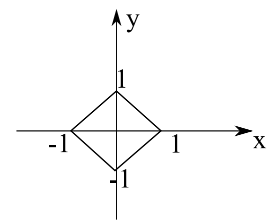
Problems

Problem 1.

Using an appropriate change of variables $u = x - y$, $v = x + y$, compute the integral

$$\int\int_R \frac{(x-y)^2}{(x+y+3)^3} dxdy, \tag{6}$$

where R is shown in the figure.



Problem 2 (3D-4).

Evaluate $\int\int_R (2x-3y)^2(x+y)^2 dxdy$, where R is the triangle bounded by the positive x-axis, negative y-axis, and line $2x - 3y = 4$, by making a change of variable $u = x + y$, $v = 2x - 3y$.

Problem 3.

Find the surface area of the surface $z = 1 - x^2 - y^2$ that lies above the unit disk R in the xy-plane.

Problem 4.

Find the surface area of the part of the cylinder $x^2 + z^2 = a^2$ that lies in the first octant and between the planes $y = 3x$ and $y = 5x$.

For more practice: please do 3D-2, 3D-3 of the supplementary notes and let me know if you want suggestions of more problems!

Recitation 15: December 11 18.02A

Focus: Review.

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Problems

Problem 1.
Find the maximum/minimum of $f(x, y, z) = xy - z^2$ on the region $x^2 + y^2 + z^2 \leq 1$.

Problem 2.
Calculate the volume between the surfaces $z = 3 - x^2 - y^2$ and $z = 2\sqrt{x^2 + y^2}$ (shown in the figure), for x, y in the first quadrant.

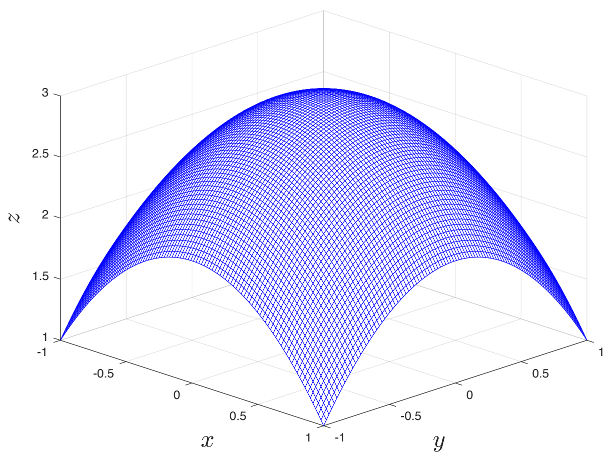

Figure 1: $z = 3 - x^2 - y^2$

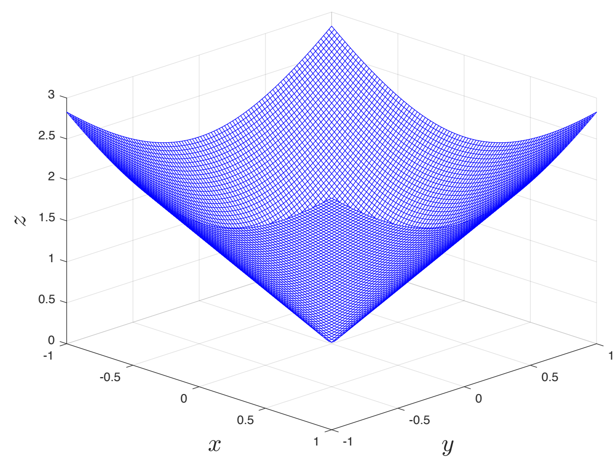

Figure 2: $z = 2\sqrt{x^2 + y^2}$

Problem 3.
Let $w = f(x, y)$ be a function of x and y. Suppose $\nabla w = (2, -1)$ at $x = 0, y = 2$. If we perform the change of variables $x = u^2 - v^2$, $y = 2uv$, what are $\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}$ at $u = 1, v = 1$?

Problem 4.
Evaluate $\displaystyle\int\int_R \frac{xy}{(4x - y)^2} dxdy$, where R is the region enclosed by the four curves $4x - y = 1, 4x - y = 5$ and $2x - y = 2, 2x - y = 3$.

Problem 5.
Evaluate $\displaystyle\int\int_R \frac{1}{x^2} dxdy$, where R is the region with boundaries $y^2 - x^2 = 1, y^2 - x^2 = 4$ and $y = 4x, y = 6x$ using the change of variables $u = y^2 - x^2$, $v = y/x$.

Midterm review problems

Problem 1. Denote by P the plane passing through the points $(1,2,3)$, $(3,1,2)$ and $(1,1,0)$. Find the

a) point where P intersects the line through $(-1,0,1)$ and $(2,1,1)$.

b) line where P intersects the plane given by $-2x+4y-2z=2$. Also calculate the angle between the two planes.

c) distance between P and the point $Q=(2,3,1)$.

Problem 2. Find the tangent plane to

a) $f(x,y)=\arctan\left(\frac{x}{y}\right)$ at the point $(4,4)$. Approximate $f(4.1,3.9)$.

b) the surface defined by $x^4yz^3=2$ at $(1,2,1)$.

Problem 3. Let $f(x,y)=x^3+2x^2-y^2+2y$. For which x,y is

a) the direction of fastest increase of f at (x,y) perpendicular to $(1,1)$?

b) the direction of fastest increase of f at (x,y) parallel to $(1,1)$?

Problem 4. a) Calculate the directional derivative of $\ln(x^2+y^2)$ at $(3,2)$ in the direction of $(5,1)$.

b) For which directions $\overrightarrow{u}$ is the directional derivative of $f(x,y)=x^2-y^2$ at the point $(1/2,1/2)$ in the direction $\overrightarrow{u}$ equal to 1?

Problem 5. a) Suppose a ball starts at the origin $(0,0)$ and is thrown at an angle of $\theta=\frac{\pi}{4}$ with the positive x-axis and with speed s, after which it only feels acceleration g in the negative y-direction. Calculate its position vector $\mathbf{r}(t)$ as a function of time t.

b) Suppose a fly starts out at $(1,0)$ and orbits the ball with fixed radius and constant angular speed ω counterclockwise around the ball. Calculate the position vector and the velocity vector of the fly as a function of time t.

Final review problems

Problem 1. A function $z(x,y)$ is defined implicitly by

$$zx^2y^2 + 3\sin(\frac{\pi}{2}xyz) = 2. \tag{1}$$

Compute $\frac{\partial z}{\partial y}$ at the point $(1,1,2)$.

Problem 2. Find and classify the critical points of $f(x,y) = 2x^4 + y^2 + xy + 1$.

Problem 3. Find the maximum and minimum of the function $f(x,y) = \frac{1}{3}x^3 + y^2 - 3x$ on the region given by triangle with vertices at $(-2,0), (0,2), (0,-2)$.

Problem 4. Calculate

a) $\displaystyle\int_0^{1/4}\int_{\sqrt{t}}^{1/2} \frac{\sin(u)}{u} du dt$.

b) the volume removed by drilling a hole of radius b into a sphere of radius a.

c) $\displaystyle\int\int xdxdy$, where R is the region bounded by the curves $x - 2y = 1$, $x - 2y = 3$, and $x + y = 4, x + y = 5$.

Problem 5. Find the maximum and minimum of $2x^2 + y + y^2$ on the circle $x^2 + y^2 = 1$.