Recitation notes 18.02A, IAP 2020

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Recitation 16: January 7 18.02A

Focus: Vector fields and line integrals in the plane.

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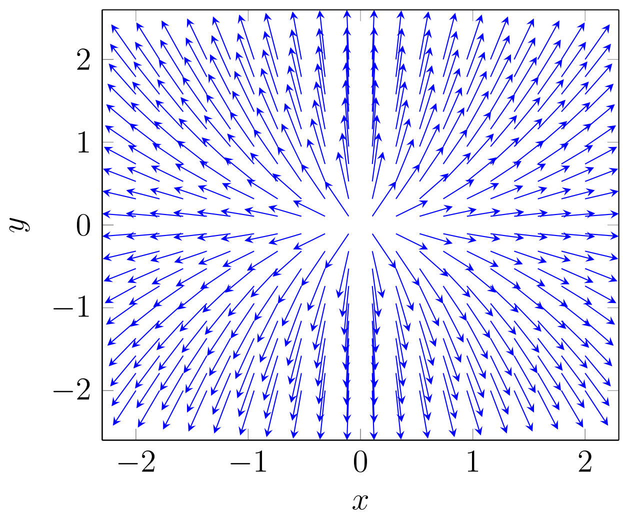
Vector fields in the plane

A vector field takes any point (x, y) in the plane and assigns to it a vector (blue in the figure). We can write this as

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}. \qquad (1)$$



Examples: F electromagnetic force between two charged particles, or gravitational force between two planets.

A special kind of vector field, called a **gradient field**, is one where there is some function f so that the vector field can be written on the form $\mathbf{F} = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Line integrals in the plane

A line integral of a vector field $\mathbf{F}$ along a curve C in the plane calculates the total work done by a force $\mathbf{F}$ on a particle that moves along the curve C. It is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{r}, \text{ or } \int_C M(x, y)dx + N(x, y)dy. \qquad (2)$$

To compute:

1. parametrize the curve C as $x = x(t)$, $y = y(t)$, where $t = t_1$ is the starting point of the curve and $t = t_2$ the end-point.
2. Write $dx = \frac{dx}{dt}dt$, $dy = \frac{dy}{dt}dt$.
3. Plug everything in in terms of t. Calculate the ordinary integral of the *one* variable t:

$$\int_C M(x, y)dx + N(x, y)dy = \int_{t_1}^{t_2} \left(M(x, y)\frac{dx}{dt} + N(x, y)\frac{dy}{dt} \right) dt. \qquad (3)$$

The line integral **does not** depend on the parametrization we choose. However, the value **does** depend on the path we take from start-point to end-point. Different paths give different values (in general).

Problems

Problem 1.
Evaluate the line integral $I = \int_C xy^2dx - (x+y)dy$, along the following paths from $(0,0)$ to $(1,2)$:

a) the straight line segment from $(0,0)$ to $(1,2)$,

b) the broken line from $(0,0)$ to $(1,0)$ to $(1,2)$.

c) the path $x = \frac{1}{8}y^3$ from $(0,0)$ to $(1,2)$.

Problem 2.
Calculate the work done by the force $\mathbf{F} = 2xy\mathbf{i} + (x^2+y^2)\mathbf{j}$ on a particle moving along the boundary of the semicircular region $x^2+y^2 \leq 1$, $y \geq 0$, described clockwise.

Problem 3.
Write down expressions for the following vector fields:

a) The force pointing towards a particle at (x,y), from the point $(1,1)$, with magnitude 1 over the distance between the points.

b) Each vector is parallel to $\mathbf{i}+\mathbf{j}$, but the magnitude is $e^{-x^2-y^2}$. Draw this vector field.

Problem 4.
Calculate the line integral $I = \int_C xydx + (x^2+y^2)dy$ along the following paths C from $(0,0)$ to $(1,1)$:

a) $y = x$

b) $y = x^2$

c) $x = y^2$

d) the broken line from $(0,0)$ to $(1,0)$ to $(1,1)$.

Problem 5.
Find the work done by the force $\mathbf{F}(x,y) = \frac{\mathbf{i}}{x^2+y^2} + \frac{\mathbf{j}}{x^2+y^2}$ on a particle moving around the upper half of the circle $x^2+y^2 = a^2$ from $(a,0)$ to $(-a,0)$.

Answers: **1a)** -2 **b)** -4 **c)** -1; **2)** 0; **4a)** 1 **b)** $\frac{13}{12}$ **c)** $\frac{14}{15}$ **d)** $\frac{4}{3}$; **5)** $-\frac{2}{a}$.

Recitation 17: January 9 18.02A

Focus: Gradient fields and conservative vector fields.

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Gradient fields

1. A vector field $\mathbf{F}$ is called a gradient field if there is some function f so that $\mathbf{F} = \nabla f$.

2. **Fundamental theorem of calculus for line integrals:** If $\mathbf{F} = \nabla f$ is a gradient field, and C a path from a point P to Q, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(Q) - f(P). \tag{1}$$

Conservative vector fields

Recall that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ between two points P and Q does not depend on the parametrization we choose, but in general depends on which path we take from P to Q.

1. A vector field $\mathbf{F}$ is called conservative if the line integral does **not** depend on the particular path we choose, but only on the end-points P and Q, i.e., if C_1 and C_2 are any two paths between P and Q, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

2. A vector field is conservative if and only $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$, for any closed curve C, i.e., a curve ending and starting at the same point.

3. **Important fact:** a vector field is conservative if and only if it is a gradient field.

Testing if F is conservative and finding the potential

If $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$, and M, N and all their derivatives M_x, M_y, N_x, N_y are continuous for all (x, y), then: $\mathbf{F}$ is conservative if and only if $M_y = N_x$.

If we have a field $\mathbf{F} = (M, N)$ that we know is conservative, how do we find the corresponding f? Two methods:

1. "Algebraic method": since $M = f_x$, find antiderivative of M with respect to x and call this $f = A(x, y) + g(y)$. $g(y)$ is an unknown constant of integration that we now determine. Differentiate this to get $f_y = A_y + g'(y) = N$. Solve for $g'(y)$ and integrate to get $g(y)$. The potential is then $f(x, y) = A(x, y) + g(y)$.

2. "Integration method": by the fundamental theorem, $f(x, y) = \int_{P_0}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$, where you can choose P_0 and any path from P_0 to (x, y) as your curve.

Problems

Problem 1.
Let $\mathbf{F} = (x, y)$.

a) Find a function f so that $\mathbf{F} = \nabla f$.

b) Using the fundamental theorem of calculus for line integrals, find the maximum possible work that the force field $\mathbf{F}$ can do in moving a particle between two points in the region $x^2 + y^2 \leq 4$

Problem 2.
Show that the vector field $2xy\mathbf{i} + (y^2 - x^2)\mathbf{j}$ is not conservative, by computing its line integral along two different paths from $(0, 0)$ to $(1, 1)$.

Problem 3.
For which values of a is $\mathbf{F} = (y^2 + 2x)\mathbf{i} + axy\mathbf{j}$ a gradient field. What is the potential function?

Problem 4.
Which of the following vector fields are conservative? For the ones that are, find a potential function.

a) $xy\mathbf{i} + xy^2\mathbf{j}$

b) $(2xy^2 + y^3, 2x^2y + 3xy^2)$

Problem 5.
Let $\mathbf{F} = (2x\cos(\pi y) + y^3, -\pi\sin(\pi y)x^2 + 3xy^2)$. Let C the upper semicircle from $(1, 0)$ to $(-1, 0)$.

a) Check that $M_y = N_x$. Is $\mathbf{F}$ conservative?

b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by using path independence and evaluating the integral along a simpler path.

c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ by using the fundamental theorem of calculus for line integrals.

Answers: **1a)** $\frac{x^2}{2} + \frac{y^2}{2}$ **b)** 2; **2)** E.g. along $y = x : 2/3$, along $y = x^2 : 1/3$; **3)** $a = 2$; potential $x^2 + xy^2$; **4a)** No; **b)** Yes, $x^2y^2 + xy^3$+constant; **5a)**Yes; **b)** 0; **c)** 0.

Recitation 18: January 14 18.02A

Focus: Green's theorem (tangential and normal forms), flux, divergence, curl.

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Divergence, curl and flux

1. The divergence of a vector field $\mathbf{F} = (M, N)$ is $\text{div}(\mathbf{F}) = M_x + N_y$.
2. The curl of $\mathbf{F} = (M, N)$ is defined as $\text{curl}(\mathbf{F}) = N_x - M_y$. $\text{curl}(\mathbf{F})$ measures how rotational $\mathbf{F}$ is.
3. $\mathbf{F}$ is a gradient/conservative if and only if $\text{curl}(\mathbf{F}) = 0$ (unless the domain of definition of $\mathbf{F}$ has holes, and if M, N and all their derivatives are continuous).
4. A curve is called closed if it starts and ends at the same point.
5. A curve is called simple if it does not intersect itself (except at the endpoints, if it is a closed curve).
6. A curve is called positively oriented if the region it encloses is always to the left.
7. The flux across a curve C is denoted by $\int_C \mathbf{F} \cdot \mathbf{n} ds$. This can be computed by a line integral
$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C M dy - N dx.$$

Green's theorem

Green's theorem relates line integrals over a given curve to double integrals on the region enclosed by the curve. There is one for work (tangential form) and one for flux (normal form).

If C is a simple, closed curve with positive orientation, enclosing a region R, then **Green's theorem in tangential form** says that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \text{curl}(\mathbf{F}) dx dy, \qquad \textbf{or in other words:}$$
$$\int_C M dx + N dy = \iint_R N_x - M_y dx dy.$$

If C is a simple, closed curve with positive orientation, enclosing a region R, then **Green's theorem in normal form** says that

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \text{div}(\mathbf{F}) dx dy, \qquad \textbf{or in other words:}$$
$$\int_C M dy - N dx = \iint_R M_x + N_y dx dy.$$

This measures the flux out of R.

Problems

Problem 1.
Use Green's theorem to calculate $\oint_C -y^3 dx + x^3 dy$, where C is the closed path formed by $y = x^3$, $y = x$, oriented counter-clockwise.

Problem 2.
Compute

a) The flux of $\mathbf{F} = (2x^2y^2, xy)$ across the curve $y = x^2$ for $0 \leq x \leq 1$.

b) The flux of $\mathbf{F} = (x^2, xy)$ out of the square with corners at $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Do this both directly and using Green's theorem.

Problem 3.

a) Let $\mathbf{F} = (y + e^{-x}, 2x - \cos(y))$. Compute the line integral of $\mathbf{F}$ along the broken curve from $P = (-1, 0)$ to $(0, 1)$ then to $Q = (1, 0)$.

b) Let C be the closed curve that starts at the origin, then follows the curve $x = \sin y$ to the point $(0, 2\pi)$ and then follows the curve $x = -\sin y$ back to the origin. Use Green's theorem to calculate $\oint_C x dx + xy dy$.

c) Let C_1 be the curve that goes counter-clockwise along the square of side length 2 centered at the origin, and C_2 the curve that goes clockwise along the square of side-length 1. Let R be the region between them. If someone tells you that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 2$, and $\iint_R \text{curl}(\mathbf{F}) dx dy = 3$, what is $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$?

Answers: **1** 2/5; **2a)** 1/4; **2b)** 3/2 **3a)** $e - 1 - e^{-1}$; **3b)** -4π; **3c)** 1.

Recitation 19: January 16 18.02A

Focus: Triple integrals, cylindrical coordinates, review.

Triple integrals

A triple integral of a function $f(x, y, z)$ over a region V in xyz-space is denoted by

$$\iiint_V f(x, y, z)dxdydz. \tag{1}$$

To compute

1. Find the limits of integration.
2. Compute the inner integral with respect to the first variable, then the second, followed by the third.

Just like for double integrals, we can change the order of integration to e.g., $dydzdx$. When we do this, we must also remember to change the limits!

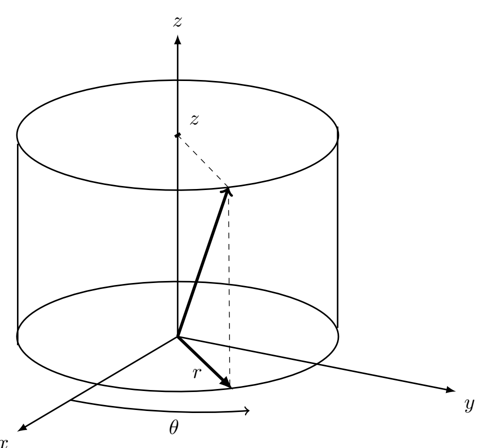
1. Limits in order $dzdxdy$: fix x and y and ask how z varies (in terms of x and y). Then find all possible values of x, y by projecting onto xy-plane.
2. Limits in order $dxdydz$ or $dydxdz$: fix z and ask how x and y vary (in terms of z). Then find all possible values of z.

Applications: The mass of a shape V with density δ is given by

$$\iiint_V \delta(x, y, z)dxdydz. \tag{2}$$

Cylindrical coordinates

Instead of specifying a point (x, y, z) in xyz-space with its cartesian coordinates, we can also specify it by giving z together with (x, y) in polar coordinates, i.e., (r, θ, z). Certain integrals are more easily computed in cylindrical coordinates, if either the integrand or the domain of integration is easier to express in cylindrical coordinates.



To compute the integral $\iiint_V f(x, y, z)dxdydz$:

1. Write $f(x, y, z)$ in terms of r, θ and z.
2. Write limits of integration for region V in terms of r, θ and z.
3. Set $dxdydz = rdrd\theta dz$ or $dxdydz = rdzdrd\theta$. **Warning: Do not forget the factor r!**

Problems

Problem 1.

a) Evaluate

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} y dz dy dx. \tag{3}$$

Also write the limits of this integral also in the order $dxdydz$.

b) Set up the double integral computing the mass of the shape determined by the surfaces $x = y$, $x = 1$, $x = 2$, $y = 0$, $z = 0$, $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates. The density is $\delta(x, y, z) = \frac{y}{\sqrt{x^2+y^2}}$.

c) Evaluate the triple integral

$$\iiint_V x^2 y dx dy dz \tag{4}$$

where V is bounded the the surfaces $y = x^2$, $x = y$, $z = 0$ and $z + x + y = 5$. Also set up the integral in cylindrical coordinates.

Problem 2.

a) Let $\mathbf{F} = (\cos(x)\sin(y) + x, \sin(x)\cos(y) + 1)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $x = t^3$, $y = t^2$, for $0 \le t \le 2$.

b) Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = x^2(\mathbf{i} + \mathbf{j})$ and C is the rectangle joining $(0, 0), (2, 0), (0, 1), (2, 1)$ traversed clockwise.

c) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $F = x\mathbf{i} + xy\mathbf{j}$ and C is the part of the curve $y = e^x$ for $0 \le x \le 1$.

Answers: **1a)** $1/24$ and $\int_0^1 \int_0^{1-z} \int_0^{1-z-y} y dx dy dz$; **1b)** $7/6$; **1c)** ≈ 0.1 and $\int_0^{\pi/4} \int_0^{\sin\theta/\cos^2\theta} \int_0^{5-r(\cos\theta+\sin\theta)} r^4 \cos^2\theta \sin\theta$; **2a)** $\sin(8)\sin(4) + 36$; **2b)** -4 **2c)** $\frac{3+e^2}{4}$.

Recitation 20: January 21 18.02A

Focus: Triple integrals, cylindrical coordinates, spherical coordinates.

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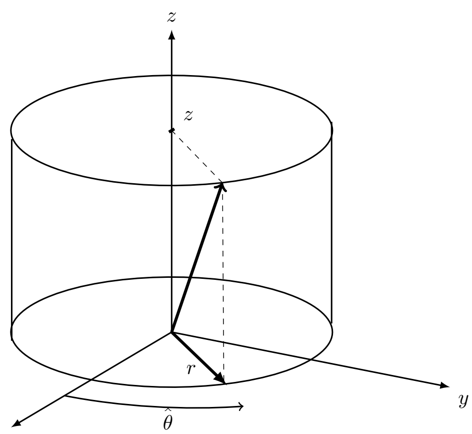
Cylindrical coordinates

Instead of specifying a point (x, y, z) in xyz-space with its cartesian coordinates, we can also specify it by giving z together with (x, y) in polar coordinates, i.e., (r, θ, z). Certain integrals are more easily computed in cylindrical coordinates, if either the integrand or the domain of integration is easier to express in cylindrical coordinates.



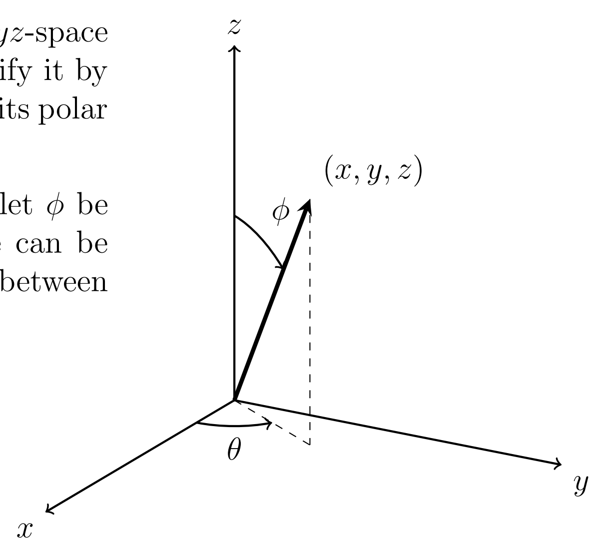
To compute the integral $\iiint_V f(x, y, z)dxdydz$:

1. Write $f(x, y, z)$ in terms of r, θ and z.
2. Write limits of integration for region V in terms of r, θ and z.
3. Set $dxdydz = rdrd\theta dz$ or $dxdydz = rdzdrd\theta$. **Warning: Do not forget the factor r!**

Spherical coordinates

Instead of specifying a point (x, y, z) in xyz-space with its cartesian coordinates, we can also specify it by giving its distance ρ to the origin, together with its polar angle θ and angle to the z-axis ϕ, i.e., (ρ, θ, ϕ).

Here, $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$. We do not let ϕ be bigger than π, since all points with this angle can be described by a ϕ in $[0, \pi]$ and $\theta + \pi$. To convert between the coordinates, use



$$x = \rho \sin\phi \cos\theta,$$
$$y = \rho \sin\phi \sin\theta,$$
$$z = \rho \cos\phi.$$

To compute the integral $\iiint_V f(x, y, z)dxdydz$:

1. Write $f(x, y, z)$ in terms of r, θ and ϕ.
2. Write limits of integration for region V in terms of ρ, θ and ϕ.
3. Set $dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$.

Problems

Problem 1.
Find the z-coordinate for the centroid of the region below the unit sphere and above the cone $z^2 = x^2 + y^2$.

Problem 2.
Change the following integral to spherical coordinates:

$$\int_0^{2\pi} \int_2^3 \int_0^{\sqrt{9-r^2}} r dz dr d\theta + \int_0^{2\pi} \int_0^2 \int_{\sqrt{4-r^2}}^{\sqrt{9-r^2}} r dz dr d\theta.$$

Problem 3.
The paraboloid $z = x^2 + y^2$ is shaped like a wine-glass, and the plane $z = 2x$ slices off a finite piece D of the region above the paraboloid (i.e., inside the wine-glass). Calculate its volume.

Problem 4.
Describe the region given in spherical coordinates by $0 \le \rho \le 2\sin\phi$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. Also calculate its volume.

Answers: 1) $\frac{3}{16(1-\frac{1}{\sqrt{2}})}$; **2)** $\int_0^{2\pi} \int_0^{\pi/2} \int_2^3 \rho^2 \sin\phi d\rho d\phi d\theta$; **3)** $\pi/2$ **4)** $2\pi^2$.

Recitation 21: January 23 18.02A

Focus: Flux in 3D, surface integrals.

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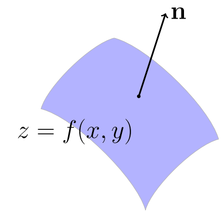
Flux integrals **Please read supplementary notes V9.**

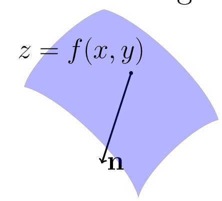
Let **F** be a vector field in 3D. Given a surface S in xyz-space, we can compute the flux of **F** through S. This is denoted by

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS. \qquad (1)$$

Here, **n** is a **unit** normal to the surface S, and the flux is the flux in the same direction as **n**. If we e.g. want the flux out of a sphere, we choose the normal pointing outwards, and if we want the flux into the sphere, we choose the normal pointing inwards.



Upwards flux along **n**



Downwards flux along **n**

1. If S is the graph of $f(x, y)$ and **n** is the normal pointing upwards, then

$$\mathbf{n} dS = (-f_x, -f_y, 1) dxdy. \qquad (2)$$

   If **n** is the normal pointing downwards, then

$$\mathbf{n} dS = -(-f_x, -f_y, 1) dxdy = (f_x, f_y, -1) dxdy. \qquad (3)$$

   NOTE: no need to separate n and dS.

2. Sometimes it is convenient to find an expression for the unit normal **n** directly. We then need to plug in an expression for dS:
   (a) If $z = f(x, y)$, $dS = \sqrt{1 + (f_x)^2 + (f_y)^2} dxdy$,
   (b) If the surface is a cylinder of radius a, $dS = adzd\theta$,
   (c) If the surface is a sphere of radius a, $dS = a^2 \sin\phi d\phi d\theta$.
   (d) If the surface is a part of the xy-plane, $dS = dxdy$. Similarly for xz-plane, and yz-plane.

Surface integrals

If a function $f(x, y, z)$ is defined on a surface S, we can integrate f over S by plugging in the expression for dS:

$$\iint_S f(x, y, z)dS. \tag{4}$$

Problem 1.

a) Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n}dS$ when $\mathbf{F} = (y^2, -xy, z)$, and S is the part of $z = x^2 + y^2$ above the disk of radius 3 centered at $(0, 0)$ in the xy-plane. Choose $\mathbf{n}$ as the upwards normal.

b) Set up the flux integral if $\mathbf{F} = (x, 1, z)$, and S is the graph of $z = e^{x-y}$ above the region in the xy-plane bounded by $y = x$ and $y = x^2$. Choose $\mathbf{n}$ as the downwards normal.

Problem 2.
Compute $\displaystyle\iint_S 3dS$, when S is the graph of $z = xy$ over the part of the unit circle in the first quadrant.

Problem 3.
Compute the flux $\iint_S \mathbf{F} \cdot \mathbf{n}dS$ when

a) $\mathbf{F} = (2x, 2y, 2z)$ and S is the part $y \geq 0$, $2 \leq z \leq 3$ of the cylinder with radius 2 around the z-axis. Choose the normal pointing away from the origin.

b) $\mathbf{F} = (-y, x, z)$ and S is the part of the sphere of radius 3 at the origin with $z \geq 0$ together with the bottom. Choose the normal pointing **into** the sphere.

Problem 4.
Set up the integral for the flux into S when $\mathbf{F} = (1, 0, 1)$ and S is the part of the cylinder of radius 3 and height 10 below the plane $y + z = 5$ (the sides only).

Answers: **1a)** $\frac{81\pi}{2}$; **1b)** $\int_0^1 \int_{x^2}^{x} (x-2)e^{x-y}dydx$; **2)** $\frac{\pi}{2}\left(2^{3/2} - 1\right)$; **3a)** 8π; **3b)** -18π ; **4)** $\int_0^{2\pi} \int_0^{5-3\sin\theta} -3\cos\theta dz d\theta$.

Recitation 22: January 28 18.02A

Focus: Divergence theorem, line integrals in 3D.

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Divergence theorem

The divergence of a field $\mathbf{F} = (M, N, P)$ is

$$\mathrm{div}(F) = \nabla \cdot \mathbf{F} = M_x + N_y + P_z. \tag{1}$$

The divergence theorem (also sometimes known as Gauss's theorem) is a 3D analogue of Green's theorem for 2D flux.

1. (Green's theorem for 2D flux) If C is a closed curve and $\mathbf{n}$ is the normal with outward orientation, then the flux out of the region R enclosed by C is

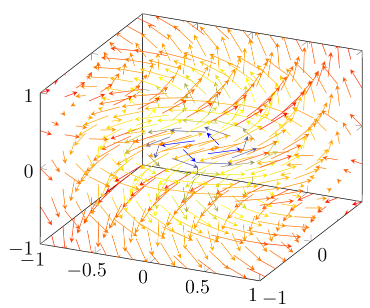
$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \iint_R \mathrm{div}(\mathbf{F}) dx dy \tag{2}$$

2. (Divergence theorem) If S is a **closed surface** and $\mathbf{n}$ is the normal with **outward** orientation, then the flux out of the domain R enclosed by S is

$$\oiint_C \mathbf{F} \cdot \mathbf{n} dS = \iiint_R \mathrm{div}(\mathbf{F}) dx dy dz \tag{3}$$

If the surface S is not closed, we can close it by adding another surface (just like we did for Green's theorem when the curve was not closed).

Line integrals in 3D



Let $\mathbf{F} = (M, N, P)$ be a vector field, and C a path. A line integral of a vector field $\mathbf{F}$ along a curve C in xyz-space calculates the total work done by a force $\mathbf{F}$ on a particle that moves along the curve C. It is denoted by

$$\int_C \mathbf{F} \cdot d\mathbf{r}, \text{ or } \int_C M(x,y,z)dx + N(x,y,z)dy + P(x,y,z)dz.$$

To compute:

1. parametrize the curve C as $x = x(t)$, $y = y(t)$, $z = z(t)$, where $t = t_1$ is the starting point of the curve and $t = t_2$ the end-point.
2. Write $dx = \frac{dx}{dt}dt$, $dy = \frac{dy}{dt}dt$, $dz = \frac{dz}{dt}dt$.
3. Plug everything in in terms of t. Calculate the ordinary integral of the *one* variable t:

$$\int_C M dx + N dy + P dz = \int_{t_1}^{t_2} \left(M(x,y,z)\frac{dx}{dt} + N(x,y,z)\frac{dy}{dt} + P(x,y,z)\frac{dz}{dt} \right) dt. \tag{4}$$

Problem 1.

a) A region is enclosed by the surface $z = 1 - x^2 - y^2$ and the xy-plane. Calculate the flux into this region (including through the bottom part) by the field $\mathbf{F} = (2x^3 + y^3)\mathbf{i} + (y^3 + z^3)\mathbf{j} + 3y^2z\mathbf{k}$.

b) Find the flux out of the part of the unit sphere at the origin with $y \geq 0$, by the field $\mathbf{F} = (2xz^3, \frac{2}{3}y^3z, x^2z^2)$ (not including the part of the sphere with $y = 0$).

c) Find the flux in through the sides of the cylinder (not including the top or bottom) around the z-axis with radius a and height h. The field $\mathbf{F}$ is $\mathbf{F} = (\ln(e^x + e^y), \ln(e^x + e^y), x^2 + y^2 + z^2)$.

Problem 2.
Compute the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ when

a) C is the path $\big((t+1)^2, 3t+1, \cos(\frac{\pi}{2}t)\big)$ from $(4, 4, 0)$ to $(9, 7, -1)$, and $\mathbf{F} = (y, x, z^2)$.

b) C is given by $z = \theta, r = 2\theta$ from $\theta = 0$ to $\theta = \pi/2$ and $\mathbf{F} = (-z, 0, x)$.

c) C is a curve on the unit sphere centered at the origin, described by $\theta = 2t$, $\phi = 2t$ from $t = 0$ to $t = \pi/4$ and $\mathbf{F} = (2z, z, 3z)$.

Answers: **1a)** $-\pi$; **1b)** 0; **1c)** $-\pi a^2 h$; **2a)** 140/3; **2b)** $2\pi - 4$; **2c)** $-1/6$.

Recitation 23: January 30 18.02A

Focus: Conservative fields in 3D, Stokes' theorem, review.

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Conservative vector fields in 3D

1. The curl of a field $\mathbf{F} = (M, N, P)$ in 3D is defined by

$$\text{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = (P_y - N_z, M_z - P_x, N_x - M_y).$$

   Here, the z-component is what we previously defined as the curl of a 2D vector field (M, N).

2. A vector field $\mathbf{F}$ is called conservative if the line integral does **not** depend on the particular path we choose, but only on the end-points P and Q, i.e., if C_1 and C_2 are any two paths between P and Q, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$.

3. If $\mathbf{F} = (M, N, P)$, and M, N, P and all their derivatives are continuous for all (x, y, z), then: $\mathbf{F}$ is conservative if and only if $\text{curl}(\mathbf{F}) = 0$.

4. If we have a field $\mathbf{F} = (M, N, P)$ that we know is conservative, how do we find the corresponding potential function f? "Algebraic method":

   (a) since $M = f_x$, find antiderivative of M with respect to x and call this $f = A(x, y, z) + g(y, z)$. $g(y, z)$ is an unknown constant of integration that we now determine.

   (b) Differentiate this to get $f_y = A_y + g_y(y, z) = N$. Solve for $g_y(y, z)$ and integrate to get $g(y, z) + h(z)$. $h(z)$ is again an unknown constant of integration that we determine.

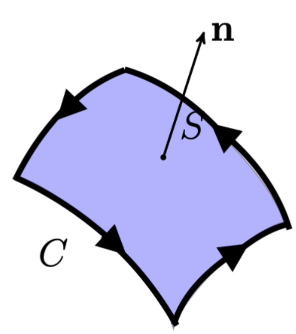
   (c) Differentiate with respect to z to get $f_z = A_z + g_z + h'(z)$. Solve for $h'(z)$ and integrate to get $h(z)$. The potential is then $f(x, y, z) = A(x, y, z) + g(y, z) + h(z)$.

Stokes' theorem

Stokes' theorem relates the line integral of a field $\mathbf{F}$ over a closed curve C to the flux of the field $\text{curl}(\mathbf{F})$ through the surface S enclosed by C.

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl}(\mathbf{F}) \cdot \mathbf{n} dS. \tag{1}$$

Here, $\mathbf{n}$ is the normal to the surface S with orientation chosen according to the right-hand rule: when we travel along C with S to our left, the normal $\mathbf{n}$ has to point upwards.



Problem 1.
Are the following fields conservative? If so, compute the corresponding potential functions and the work done by $\mathbf{F}$ on a particle moving from $(0,0,0)$ to $(1,1,2)$.

a) $\mathbf{F} = (2xyz + \cos(x), x^2z + z, x^2y + y)$

b) $\mathbf{F} = (x^2y^2z, x^3y^2z, z)$

c) $\mathbf{F} = (y^2z + x, 2xyz + \sin(z), xy^2 + y\cos(z) + 1)$

Problem 2.
Use Stokes' theorem to calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ if $\mathbf{F} = (y, z, x)$ and C is

a) the portion of the plane $x + 2y + 3z = 4$ cut out by the cylinder $x^2 + y^2 = 1$ traversed counter-clockwise when seen from positive z-axis,

b) the portion of the surface $z = 2 - x^2 - y^2$ cut out by the cylinder $x^2 + y^2 = 1$ traversed counter-clockwise when seen from positive z-axis.

Problem 3. Compute

a) the flux of the field $\mathbf{F} = (x^3, y^2)$ out of the unit circle in the xy-plane,

b) the work done by $\mathbf{F} = (xy^2, x)$ on a particle moving counter-clockwise around the triangle with vertices $(0,0)$, $(1,1)$, $(2,0)$,

c) the flux of $\mathbf{F} = (x^2y, y^2)$ across the curve $y = x^2$, $0 \le x \le 1$,

d) the work of $\mathbf{F} = (x^2y, y^2)$ on a particle moving along the curve $y = x^2$, $0 \le x \le 1$.

Answers: **1a)** $x^2yz + \sin(x) + yz + c$, work $= 4 + \sin(1)$; **1b)** No; **1c)** $xy^2z + x^2/2 + y\sin(z) + z + c$, work $= 9/2 + \sin(2)$; **2a)** -2π; **2b)** $-\pi$; **3a)** $3\pi/4$; **3b)** $1/3$; **3c)** $2/15$; **3d)** $8/15$.

Final review problems

Problem 1. a) Evaluate $\displaystyle\int_C \mathbf{F}\cdot\mathbf{n}ds$, when $\mathbf{F} = (x^2y, xy^2)$ and C is the curve $(t+1)\mathbf{i} + t^2\mathbf{j}$, for $0 \le t \le 1$.

b) Let C be the boundary of the triangle with vertices at $(0,0), (1,0), (0,1)$, traversed counter-clockwise. If $\mathbf{F} = (x^2y, y^2x)$, compute the flux out through C as well as the work done by $\mathbf{F}$ on a particle moving one lap around C.

c) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = (2xy, x^2+z, yz)$ and C is the line from $(1,1,1)$ to $(2,1,-2)$.

Problem 2. Let D be the region above the cone $z = \sqrt{x^2+y^2}$ and under the unit sphere at the origin. Find the average distance from to the origin to the points in D.

Problem 3. Use the divergence theorem to calculate the flux of $\mathbf{F} = (z^2x, \frac{1}{3}y^3 + \tan z, x^2z + y^2)$ over the top half of the sphere $x^2+y^2+z^2 = 1$.

Problem 4.
Compute the flux $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ when

a) $\mathbf{F} = (x,y,z)$ and S is the part $0 \le z \le 3$ of the cylinder with radius 2 around the z-axis. Choose the outwards normal.

b) $\mathbf{F} = (-y,x,z)$ and S is the top half of the sphere of radius 3 at the origin. Choose the outwards normal.

Problem 5. Is the field $\mathbf{F} = (2xy - 2xz + 2x, x^2 + 3y^2 + z, -x^2 + 4z^3 + y)$ conservative? If so, find the corresponding potential function.

Problem 6. Compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$ using Stokes' theorem. Here, C is the intersection of the cone $z = 2\sqrt{x^2+y^2}$ with the surface $z = 3 - x^2 - y^2$ and $\mathbf{F} = (y,z,x)$. C is traversed counter-clockwise as seen from above.

Answers

1. a) 19/15
   b) Flux: 1/6, work: 0.
   c) 9/2
2. 3/4
3. $13\pi/20$
4. (a) 24π
   (b) 18π
5. $x^2y - x^2z + x^2 + y^3 + zy + z^4$
6. $-\pi$