Recitation notes 18.01A, Fall 2019

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Recitation 1: September 4 18.01A

Focus: Review of differentiation, logistics.

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Review of differentiation

1. The derivative of $f(x)$ at a point $x = a$ equals the instantaneous rate of change of f with respect to x. In formulas:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{1}$$

2. The **product rule** says that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x). \tag{2}$$

3. The **quotient rule** says that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}. \tag{3}$$

4. The **chain rule** says that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f\left(g(x)\right)\right) = f'\left(g(x)\right) \cdot g'(x). \tag{4}$$

5. $f'(a)$ equals the slope of the tangent line to the graph of $y = f(x)$ at the point $x = a$.

6. Useful derivatives to remember:

$$\boxed{\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^x\right) = e^x, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x)\right) = \frac{1}{x}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(x)\right) = \cos(x), \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos(x)\right) = -\sin(x), \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(x^a\right) = ax^{a-1}. \end{gathered}} \tag{5}$$

Problems

Problem 1. To see how the limit definition works, compute the derivative $f'(x)$ for $f(x) = x^2$, straight from the limit definition.

Problem 2. Compute $f'(x)$ if

a) $f(x) = (x^3 - 3x)(x^2 + 5)$.

b) $f(x) = e^{x^2+3x}$.

c) $f(x) = x\ln(x)$.

d) $f(x) = \tan(x)$.

e) $f(x) = \sin^3(x)$.

f) $f(x) = \frac{2x^3+1}{x+2}$.

Recitation 2: September 9 18.01A

Focus: Linear and quadratic approximations, l'Hopital's rule..

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Linear and quadratic approximations

Near a specific point $x = a$, we sometimes want to approximate a complicated function f with a simpler one, for instance a linear or quadratic function. There are two ways of doing this:

1. **Direct differentiation**

   (a) The best linear approximation to f at $x = a$ is

   $$f(a) + f'(a) \cdot (x - a). \tag{1}$$

   (b) The best quadratic approximation to f at $x = a$ is

   $$f(a) + f'(a) \cdot (x - a) + \frac{1}{2} f''(a) \cdot (x - a)^2. \tag{2}$$

2. **Use building blocks.** These are pre-computed using the formula above around the point 0

$$\boxed{\begin{aligned} e^u &\approx 1 + u + \frac{u^2}{2} \\ \sin(u) &\approx u \text{ (quadratic approximation)} \\ \cos(u) &\approx 1 - \frac{u^2}{2} \\ (1+u)^r &\approx 1 + ru + \frac{r(r-1)}{2} u^2 \\ \frac{1}{1-u} &\approx 1 + u + u^2 \\ \ln(1+u) &\approx u - \frac{u^2}{2} \end{aligned}} \tag{3}$$

L'Hopital's rule

To evaluate indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}$, (and $0^0, 0 \cdot \infty, \infty - \infty$):

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}, \tag{4}$$

if $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$, or $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$.

Problems

Problem 1.

a) Find the best quadratic approximation of $f(x) = xe^{-2x}$ at $x = 2$.

b) Find the best quadratic approximation of $f(x) = \frac{1}{1-2x}\frac{1}{1-3x}$ at $x = 0$.

c) Find the best linear approximation of $f(x) = \ln(2 + x)$ at $x = 0$.

d) Find the best quadratic approximation of $f(x) = \frac{\cos x}{\sqrt{1+x}}$ at $x = 0$. Also, approximate $f(0.1)$.

Problem 2. Evaluate the limits

a) $\lim_{x\to 1} \frac{4x^3-5x+1}{\ln x}$.

b) $\lim_{x\to\infty} \frac{x^5}{e^x}$.

c) $\lim_{x\to 0} (1 - \cos x)^{1-\cos x}$.

d) $\lim_{x\to 0} \frac{1}{x} - \frac{1}{e^x-1}$.

Recitation 3: September 11 18.01A

Focus: Integrals and Riemann sums.

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Integrals as Riemann sums

The integral

$$\int_a^b f(x)dx \tag{1}$$

equals the area under the curve $f(x)$ between a and b. We usually compute it using the fundamental theorem of calculus. A different way is to use Riemann sums: if $\Delta x = \frac{b-a}{n}$

$$\textbf{Left/lower Riemann sum} = f(a)\Delta x + f(a+\Delta x)\Delta x + \ldots + f(a+(n-1)\Delta x)\Delta x, \tag{2}$$

$$\textbf{Right/upper Riemann sum} = f(a+\Delta x)\Delta x + f(a+2\Delta x)\Delta x + \ldots + f(b)\Delta x. \tag{3}$$

The limit of both of these expression as $n \to \infty$ is $\int_a^b f(x)dx$.

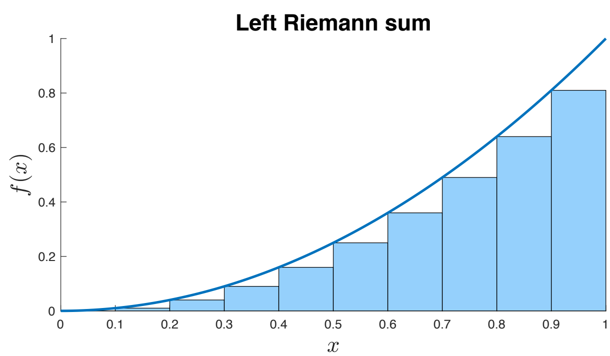


Figure 1: Left Riemann sum

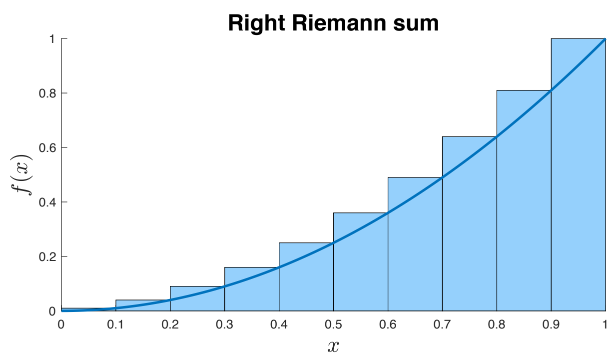


Figure 2: Right Riemann sum

Problems

Problem 1. For any real number r, write down the left Riemann sum for x^r on the interval $[1, 2]$.

Problem 2. Compute the sum

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{k+n}, \tag{4}$$

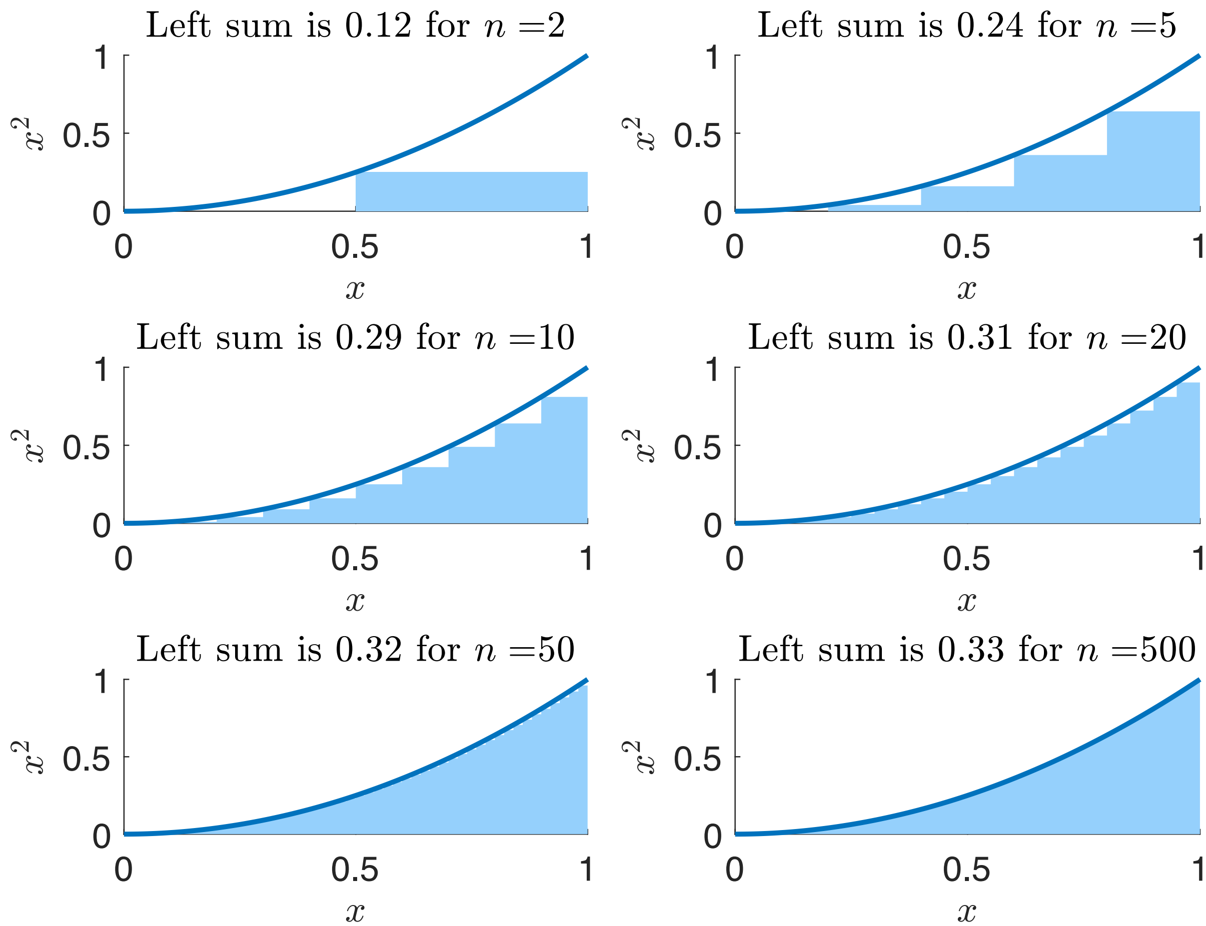
by identifying it as a Riemann some of some function on some interval.

Problem 3. Compute the integrals

a) $\int_0^2 \sqrt{4x+1}dx$. b) $\int_0^{2b} \frac{x}{\sqrt{x^2+b^2}}dx$. c) $\int_0^{\frac{\pi}{4}} \sin(4x)$. d) $\int_0^{\frac{\pi}{3}} \frac{\sin(\theta)}{\cos^2(\theta)}d\theta$.

$$\int_0^1 x^2 dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$$

Left sum is $\frac{1}{n}\left(f(0) + f(\frac{1}{n}) + \ldots + f(\frac{n-1}{n})\right) = \frac{1}{n}\left(0^2 + \left(\frac{1}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2\right)$



Recitation 4: September 16 18.01A

Focus: Second fundamental theorem of calculus and volumes of revolution.

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Second fundamental theorem of calculus

If $f(x)$ is a continuous function, then the function $A(x)$ defined by

$$A(x) = \int_a^x f(t)\mathrm{d}t \tag{1}$$

is differentiable with derivative $A'(x) = f(x)$.

Area between two curves

The area of the region between $f(x)$ and $g(x)$ in the interval $[a, b]$ is

$$\int_a^b f(x) - g(x)dx. \tag{2}$$

Volumes of revolution

Best remembered by drawing!

Volume obtained when revolving curve $f(x)$ between $x = a$ and $x = b$ around the x-axis

$$\int_a^b \pi f(x)^2 \mathrm{d}x \qquad \text{(Disk method)} \tag{3}$$

Volume obtained when revolving curve $f(x)$ between $x = a$ and $x = b$ around the y-axis

$$\int_a^b 2\pi x f(x) \mathrm{d}x \qquad \text{(Shell method)} \tag{4}$$

Problems

Problem 1. Let

$$F(x) = \int_0^x \frac{dt}{1+t^2}. \tag{5}$$

a) Is F even, odd or neither?

b) Let $f(x) = F'(x)$. Plot $f(t)$ on the interval $[-3, 3]$ (make sure to use the quadratic approximation close to $t = 0$). Is F increasing, decreasing or neither on $[-3, 3]$?

c) Sketch the graph of $F(x)$ on $[-3, 3]$.

d) For which x_m in the interval is $F(x)$ maximized? Show that

$$\frac{3}{10} \leq F(x_m) \leq 3. \tag{6}$$

Problem 2. Find the area bounded by the curves

a) $y = x^2$ and $x = y^2$.

b) $y = \sin x$, $y = \cos x$, $0 \leq x$ and $x \leq \frac{\pi}{2}$.

Problem 3. Calculate

a) Calculate the volume obtained when revolving the area enclosed by the curve $x^2 + y^2 = 1$ around the x-axis.

b) Find the volume of the region obtained by rotating the region $\sqrt{x} \leq y \leq 1$ for $x \geq 0$ around the y-axis.

c) A hole of radius $\sqrt{3}$ is bored through the center of a sphere of radius 2. Find the volume removed.

Recitation 5: September 18 18.01A

Focus: Arc lengths and surface areas.

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Arc length

The arc length of curve $f(x)$ between $x = a$ and $x = b$ is

$$\int_a^b \sqrt{1 + (f'(x))^2}\mathrm{d}x \tag{1}$$

Surface area

The surface area of the region obtained when rotating the function $f(x)$ between $x = a$ and $x = b$ around the x-axis is

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\mathrm{d}x \tag{2}$$

Average value

The average value of the function $f(x)$ on the interval between $x = a$ and $x = b$ is

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x. \tag{3}$$

Work

The work required when moving from $x = a$ to $x = b$ with a force $F(x)$ acting on you is

$$W = \int_a^b F(x)\mathrm{d}x. \tag{4}$$

Problem 1.
Compute

a) the arc length of the curve $y = \frac{1}{3}\sqrt{x}(3-x)$ for $1 \leq x \leq 2$.

b) the surface area of the region obtained by rotating $y = \frac{x^4}{4} + \frac{1}{8x^2}$ for $1 \leq x \leq 2$ around the x-axis.

c) the surface area of the region obtained by rotating $y = x^{\frac{1}{3}}$ for $0 \leq x \leq 1$ around the y-axis.

Problem 2.
Compute

a) $\int_0^1 xe^{-x^2}\mathrm{d}x$

b) $\int_0^\pi \sin(x)\cos^2(x)\mathrm{d}x$

Problem 3.
An amount of money A compounded continuously at interest rate r increases according to the law

$$A(t) = A_0 e^{rt}, \quad (t = \text{ time in years}) \tag{5}$$

What is the average amount of money in the bank over the course of T years?

Problem 4.
An extremely stiff spring is 12 inches long, and a force of 2,000 pounds extends it 1/2 inch. How many inch-pounds of work would be done in stretching it to 18 inches?

Problem - Bonus.
Compute the volume obtained when revolving the region $0 \leq y \leq x^2 + 1$ for $0 \leq x \leq 1$ around the y-axis using both the disk and the shell method.

Recitation 6: September 23 18.01A

Focus: Trigonometric integrals.

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Change of variables for integration

For tricky integrals, sometimes it helps to change the variables using the substitution $u = u(x)$:

$$\int_a^b f(u(x))u'(x)\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u, \tag{1}$$

or the substitution $x = x(t)$:

$$\int f(x)\mathrm{d}x = \int f(x(t))x'(t)\mathrm{d}t, \tag{2}$$

for some clever choices of $x(t)$ or $u(x)$. For definite integrals, remember to change the limits of integration to the limits in the new variable! For indefinite integrals, after solving the integral in terms of u or t, remember to re-express the answer in terms of x!

Trigonometric identities to remember

- $\sin^2(x) + \cos^2(x) = 1$
- $\tan^2(x) = 1 + \sec^2(x)$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 1 - 2\sin^2(x) = 2\cos^2(x) - 1$
- $\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = \sec^2(x) = 1 + \tan^2(x)$

Integrals of powers of $\sin(x)$ and $\cos(x)$

We can compute all integrals of the form

$$\int \sin^a(x)\cos^b(x)\mathrm{d}x. \tag{3}$$

If

1. a is odd, substitute $u = \cos x$.
2. b is odd, substitute $u = \sin x$.
3. both a and b are even, use the double-angle formula.

Problems

Problem 1.
Compute

a) $\displaystyle\int_{\frac{3\pi}{2}}^{2\pi} \sin^3(x)\mathrm{d}x$ b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4(x)\mathrm{d}x$ c) $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2(x)\cos^2(x)\mathrm{d}x$

Problem 2.
Compute

a) $\displaystyle\int \frac{x^3}{\sqrt{9-x^2}}\mathrm{d}x$ b) $\displaystyle\int \frac{1}{(4+x^2)^2}\mathrm{d}x$ c) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\mathrm{d}x$

Problem 3.
Compute

a) $\displaystyle\int_0^1 e^x\left(e^x+2\right)^9\mathrm{d}x$ b) $\displaystyle\int_1^e \frac{\mathrm{d}x}{x\sqrt{\ln x}}$ c) $\displaystyle\int_1^2 \frac{2x+1}{\sqrt{x^2+x+2}}\mathrm{d}x$

Recitation 7: September 25 18.01A

Focus: Partial fractions and review.

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Partial fraction expansion

How to compute the indefinite integral of any rational function, i.e.

$$\int \frac{f(x)}{g(x)} \mathrm{d}x \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials.

Today: $g(x)$ product of two distinct linear factors, i.e. $g(x) = (x-a)\cdot(x-b)$ and compute $\displaystyle\int \frac{f(x)}{(x-a)\cdot(x-b)}\mathrm{d}x$ when $f(x)$ has degree ≤ 1.

1. Find a and b. For today: do this by inspection.
2. Write $\dfrac{f(x)}{(x-a)\cdot(x-b)} = \dfrac{A}{x-a} + \dfrac{B}{x-b}$.
3. Solve for the coefficients A, B by clearing denominators: $f(x) = A\cdot(x-b) + B\cdot(x-a)$ and plugging in $x = a$ and $x = b$, respectively.
4. Integrate each term in the partial fraction expansion:

$$\int \frac{f(x)}{(x-a)\cdot(x-b)} = \int \frac{A}{x-a} + \frac{B}{x-b}\mathrm{d}x = A\ln|x-a| + B\ln|x-b| + C. \tag{2}$$

Don't forget the absolute value signs!

Problems

Problem 1.
Compute

a) $\displaystyle\int \frac{\mathrm{d}x}{x^2-9}$

b) $\displaystyle\int \frac{x+2}{x^2+3x}\mathrm{d}x$

Problem 2.
Compute the quadratic approximation of

a) $\displaystyle\int_0^x \frac{e^t-1}{t}\mathrm{d}t$ at $x=0$.

b) $\displaystyle\frac{\sqrt{1-2x}}{\cos x}$ at $x=0$.

Problem 3.
Compute

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{k}{n^{4/3}}+\frac{1}{n^{1/3}}\right)^3. \tag{3}$$

Problem 4.
Compute

a) the volume when revolving the part of the curve $y=2-x$ for $1\le x\le 2$ around the y-axis

b) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\mathrm{d}x$

Recitation 8: September 30 18.01A

Focus: Partial fractions and integration by parts.

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Partial fraction expansion

How to compute the indefinite integral of any rational function, i.e.

$$\int \frac{f(x)}{g(x)}\mathrm{d}x \tag{1}$$

where $f(x)$ and $g(x)$ are polynomials.

> **Important fact:** *any polynomial can be factored as a product of linear terms, and quadratic terms with no real roots.*

If degree of $f(x) <$ degree of $g(x)$:

1. Factor $g(x)$ into a product of linear terms, and quadratic terms with no real roots.
2. Write $\frac{f(x)}{g(x)}$ equal to a sum of terms, one term for each factor in step 1 in the following way:
   - A non-repeated linear factor, i.e. a factor $(x-a)$ gives the term $\frac{A}{x-a}$.
   - A repeated linear factor e.g. $(x-a)^2$ gives the *two* terms $\frac{A}{x-a} + \frac{B}{(x-a)^2}$; the term $(x-a)^3$ gives the *three* terms $\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{(x-a)^3}$, and so on.
   - A quadratic factor without real roots x^2+ax+b gives the term $\frac{Ax+B}{x^2+ax+b}$. **NOTE:** to use this case, make sure that $x^2+ax+b=0$ has no real solutions.
3. Solve for the coefficients $A, B, C, \ldots$ by clearing denominators.
4. Integrate each term in the partial fraction expansion.

If degree of $f(x) \geq$ degree of $g(x)$: do polynomial long division to write $\frac{f(x)}{g(x)} = Q(x) + \frac{R(x)}{g(x)}$ where degree of $R(x) <$ degree of $g(x)$. Then perform the steps above.

Integrals to use

$$\text{Non-repeated linear factor:} \quad \int \frac{1}{x-a}\mathrm{d}x = \ln|x-a| + C \tag{2}$$

$$\text{Repeated linear factor:} \quad \int \frac{1}{(x-a)^n}\mathrm{d}x = \frac{-1}{n-1}\frac{1}{(x-a)^{n-1}} + C, \qquad \text{for } n \neq 1, \tag{3}$$

$$\text{Quadratic factor:} \quad \int \frac{1}{x^2+a^2}\mathrm{d}x = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \tag{4}$$

$$\text{or:} \quad \int \frac{x}{x^2+a^2}\mathrm{d}x = \frac{1}{2}\ln(x^2+a^2) + C. \tag{5}$$

For $\frac{1}{x^2+ax+b}$ or $\frac{x}{x^2+ax+b}$ complete the square: $x^2+ax+b = (x+\frac{a}{2})^2 + b - \frac{a^2}{4}$ and use the last two integrals.

Integration by parts

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x. \tag{6}$$

Use when the right hand side can be computed easily, but the left hand side cannot.

Problems

Problem 1.

Compute (a) $\displaystyle\int \frac{x^3+2}{x^2+2x}\mathrm{d}x$ (b) $\displaystyle\int \frac{3}{x^3+4x^2+5x}\mathrm{d}x$ (c) $\displaystyle\int \frac{x}{(x^2+1)(x+1)^2}\mathrm{d}x$

Problem 2.

(a) Evaluate $\displaystyle\int x^a \ln x \mathrm{d}x$ for $a \neq -1$. (b) Evaluate $\displaystyle\int \frac{x}{\cos^2(x)}\mathrm{d}x$.

Recitation 9: October 2 18.01A

Focus: Integration by parts and improper integrals.

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Integration by parts

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x. \tag{1}$$

Use when the right hand side can be computed easily, but the left hand side cannot.

Improper integrals

An improper integral is either an integral with one or two endpoints equal to $\pm\infty$:

$$\int_a^\infty f(x)dx, \qquad \int_{-\infty}^b f(x)dx, \qquad \int_{-\infty}^\infty f(x)dx, \tag{2}$$

or an integral where the integrand approaches $\pm\infty$ at one of the endpoints of the interval, e.g.

$$\int_0^1 \frac{1}{x^p}dx. \tag{3}$$

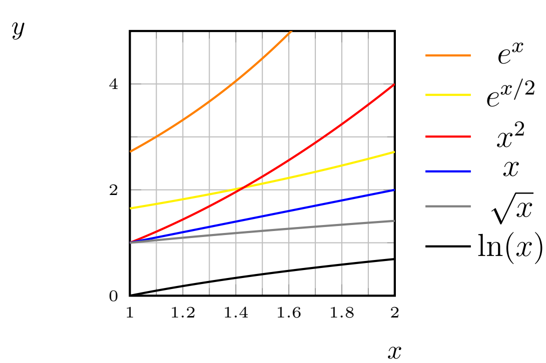
Comparison theorem

If $0 \le f(x) \le g(x)$ for all $x \ge a$ then $\int_a^\infty f(x)dx \le \int_a^\infty g(x)dx$ so if

- $\int_a^\infty g(x)dx$ converges, then so does $\int_a^\infty f(x)dx$.
- $\int_a^\infty f(x)dx$ diverges, then so does $\int_a^\infty g(x)dx$.

Good functions to compare to:

$$\boxed{\begin{aligned}&\int_1^\infty \frac{1}{x^p}dx \text{ converges if } p > 1 \text{ and diverges if } p \le 1.\\ &\int_0^\infty e^{-ax}dx \text{ converges if } a > 0.\end{aligned}} \tag{4}$$



Problems

Problem 1.
Compute $\int \frac{x}{(x^2+1)(x+1)^2}\mathrm{d}x$

Problem 2. a) Compute $\int \ln x \mathrm{d}x$.

b) Let $I_n = \int x^n \cos(x) dx$. Express I_n in terms of I_{n-2}.

Problem 3.
Which of the following integrals are convergent?

(a) $\int_3^\infty \frac{\ln x}{x^2}\mathrm{d}x$ (b) $\int_3^\infty \frac{\ln x}{x}\mathrm{d}x$ (c) $\int_1^\infty e^{-x}\ln(x)\mathrm{d}x$

Problem 4.
Show that $\int_1^\infty e^{-2x}\left(\cos^2(x)+1\right)\mathrm{d}x < \frac{1}{e^2}$

Recitation 9: October 2 — 18.01A

Focus: Integration by parts and improper integrals.

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- The integral
$$\int_0^1 \frac{1}{x^p} \tag{1}$$
is improper because the integrand tends to infinity when x tends to 0. It is convergent when $p < 1$ and divergent when $p \geq 1$. Here the limits are 0 and 1

- The integral
$$\int_1^\infty \frac{1}{x^p} dx \tag{2}$$
is improper because the upper limit of integration is $+\infty$. It converges if $p > 1$ and diverges if $p \leq 1$.

Recitation 10: October 7 18.01A

Focus: Improper integrals and infinite series.

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Asymptotic comparison theorem for improper integrals

If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = C$ for some non-zero finite number C, then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ either both converge, or both diverge.

Infinite series

An infinite series is a series with infinitely many terms, e.g.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots. \tag{1}$$

The series is *divergent* if summing all the terms gives ∞, and *convergent* if summing all the terms gives a finite number. For a series to converge, it is not enough that the terms go to zero. They have to go to zero fast enough, too.

Convergence tests

1. If a_n does not tend to zero as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges (no need for any other test).
2. (Integral test) If $f(x) \geq 0$ for $x \geq a$ and $f(x)$ is a decreasing function, then $\sum_{n=a}^{\infty} f(n)$ converges/diverges $\Leftrightarrow \int_a^\infty f(x)dx$ converges/diverges.
3. (Direct comparison) If $0 \leq a_n \leq b_n$, then
   - If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
   - If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges as well.

Good facts to remember

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$ by the integral test.
- (Geometric series) $1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$ if $r \neq 1$.

Problems

Problem 1.
Using the asymptotic comparison theorem, determine if the following integrals converge or diverge (a) $\displaystyle\int_1^\infty \frac{\sqrt{x^3+3x+2}}{(x^8+1)^{1/3}}dx$ (b) $\displaystyle\int_1^\infty 1-\cos\left(\frac{1}{x}\right)dx$.

Problem 2.
Determine if the following series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{5^{n/2}}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+5)^{3/2}}$ (c) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{(n+1)^{5/4}}$

(d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{(n+1)^2}$ (e) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\dots$.

Problem 3.
Using the integral test, determine if the following series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$ (b) $\displaystyle\sum_{n=1}^{\infty}ne^{-n}$ (b) $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan(n)}{1+n^2}$

Recitation 11: October 9 18.01A

Focus: Infinite series and power series.

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Convergence tests

1. If a_n does not tend to zero as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges (no need for any other test).
2. (Ratio test) Assume $a_n > 0$ and calculate $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.
   - If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
   - If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
   - If $L = 1$, then there are examples of series which converge, and of series which diverge. **You must then use a different test to establish convergence/divergence!**
3. (Integral test) If $f(x) \geq 0$ for $x \geq a$ and $f(x)$ is a decreasing function, then $\sum_{n=a}^{\infty} f(n)$ converges/diverges $\Leftrightarrow \int_a^{\infty} f(x)dx$ converges/diverges.
4. (Direct comparison) If $0 \leq a_n \leq b_n$, then
   - If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.
   - If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges as well.
5. (Asymptotic comparison) If $a_n \geq 0$, $b_n \geq 0$ and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, for some non-zero and finite number L, then $\sum_{n=1}^{\infty} a_n$ converges/diverges if and only if $\sum_{n=1}^{\infty} b_n$ converges/diverges.
6. (Alternating series test) If $a_n > 0$ for all n and if $a_1 \geq a_2 \geq a_3 \geq \ldots$ with $\lim_{n\to\infty} a_n = 0$, then the series $a_1 - a_2 + a_3 - a_4 + a_5 - \ldots$ converges.

Power series

The power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \tag{1}$$

defines a function for those x-values where the infinite series obtained by plugging in this value of x converges. Every power series has a radius of convergence R such that

- If $|x| < R$, the series converges
- If $|x| > R$, the series diverges
- If $|x| = R$, it is unclear if the series diverges or converges for $|x| = R$ (compare to the ratio test when the limit of the ratio is 1).

The radius of convergence can be calculated by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|. \tag{2}$$

Note: the quotient is the upside-down version of the one in the ratio test! Also, $R = \infty$ is allowed!

Good facts to remember

- (Geometric series) $1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$ if $r \neq 1$.
- $r + r^2 + \ldots r^{n-1} = r(1 + r + \ldots + r^{n-2}) = r\frac{1-r^{n-1}}{1-r}$
- In general, $a + ar + ar^2 + \ldots ar^{n-1} = a\frac{1-r^n}{1-r}$

Problems

Problem 1.
Determine if the following series converge or diverge:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{2n^3-3}$ (c) $\displaystyle\sum_{n=1}^{\infty} (e^{\frac{1}{n}} - 1)$

Problem 2.
Using the ratio test, determine for which positive a and p the sum $\displaystyle\sum_{n=1}^{\infty} \frac{a^n}{n^p}$ converges.

Problem 3.
Calculate the radius of convergence R of $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}} x^n$. Is the series convergent or divergent at the endpoints of the interval of convergence, i.e. at $x = \pm R$?

Problem 4.
For which x does the series

$$\sum_{n=1}^{\infty} \frac{100^n}{n!} x^{\frac{n}{2}+1} \tag{3}$$

converge?

Problem 5.
Is the series

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln n}{n} \tag{4}$$

absolutely convergent, conditionally convergent or divergent?

Recitation 12: October 16 18.01A

Focus: Taylor series.

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Operations on power series

- Can add power series term by term:

$$\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots\right) + \left(b_0 + b_1x + b_2x^2 + b_3x^3 + \dots\right)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + \dots.$$

- Can differentiate power series term by term:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots\right) = a_1 + 2a_2x + 3a_3x^2 + \dots.$$

- Can integrate power series term by term:

$$\int_0^x \left(a_0 + a_1t + a_2t^2 + a_3t^3 + \dots\right) \mathrm{d}t = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \dots.$$

Taylor series

1. Every function $f(x)$ that we have seen in this course can be written as an infinite series on some interval $(-R, R)$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n, \tag{1}$$

   whenever $|x| \le R$.

2. Good Taylor series to remember:

$$\boxed{\begin{aligned} \frac{1}{1-x} &= 1 + x + x^2 + x^3 \dots, \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} \dots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots. \end{aligned}} \tag{2}$$

Problems

Problem 1.
Find the first four terms of the Taylor series of

a) e^{3x} b) $e^x \sin x$ c) $\sin(e^x - 1)$ d) $\int_x^{x^2} \frac{\cos(t)-1}{t}\mathrm{d}t$.

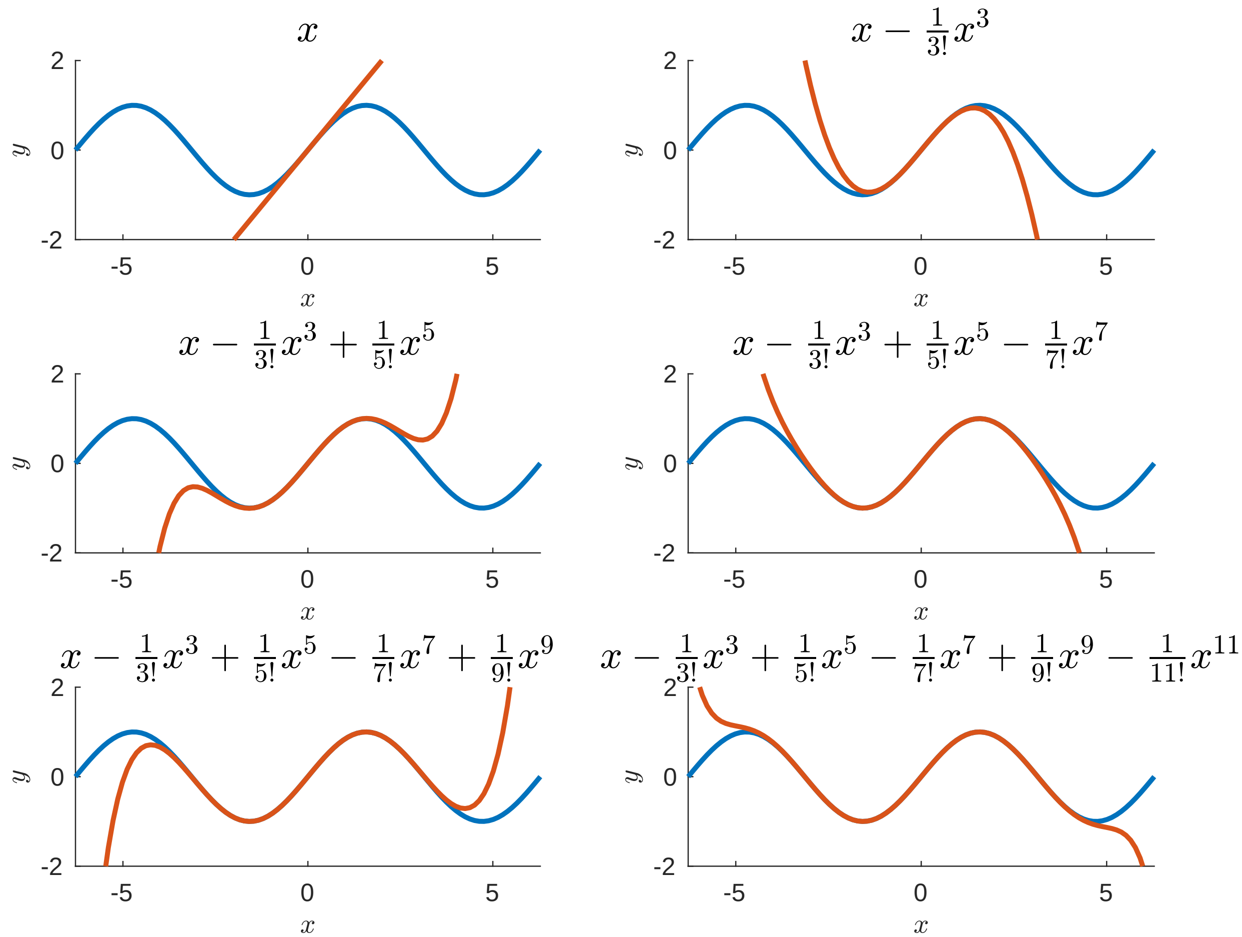

Figure 1: Taylor series of $\sin(x)$.

Problem 2.
Compute

a) $\displaystyle\sum_{n=0}^{\infty} \frac{3^{\frac{n-1}{2}}}{n!}$.

b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)3^{n-\frac{1}{2}}}$. Hint: identify the power series with the anti-derivative of a simpler function.

Extra midterm practice problems 18.01A

Problem 1. Evaluate the following limits

a) $\lim\limits_{x\to 0} \dfrac{xe^{3x}}{\sin(2x)}$ b) $\lim\limits_{x\to\infty} \left(\cos\left(\dfrac{1}{x}\right)\right)^{x^2}$ c) $\lim\limits_{x\to 1} \dfrac{\ln x}{(x-1)^3}$

Problem 2. Compute the following limits by writing them as Riemann sums

a) $\lim\limits_{n\to\infty} \left(\dfrac{n}{(n+1)^2} + \dfrac{n}{(n+2)^2} + \ldots + \dfrac{n}{(n+(n-1))^2} + \dfrac{n}{(n+n)^2}\right)$

b) $\lim\limits_{n\to\infty} \left(\dfrac{\sqrt{0}}{n\sqrt{n}} + \dfrac{\sqrt{1}}{n\sqrt{n}} + \ldots + \dfrac{\sqrt{2}}{n\sqrt{n}} + \ldots + \dfrac{\sqrt{n-1}}{n\sqrt{n}}\right)$

Problem 3. Compute the quadratic approximation of

a) $\dfrac{(1+\sin x)^{\frac{3}{2}}}{1+2x}$ at $x=0$

b) $\displaystyle\int_1^x \frac{e^t-1}{t}\mathrm{d}t$ at $x=1$

c) $\displaystyle\int_0^x \frac{e^t-1}{t}\mathrm{d}t$ at $x=0$

d) $x\sin(\frac{\pi}{2}x)$ at $x=1$

Problem 4. Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{e^x} \sin(t)\mathrm{d}t. \tag{1}$$

Problem 5. Compute the arc length of the curve $x = \frac{1}{3}\left(y^2+2\right)^{3/2}$ for $1 \le y \le 2$.

Problem 6. Compute the surface area when rotating the curve

a) $y = x^3$ for $0 \le x \le 1$ around the x-axis

b) $y = \frac{1}{4}x^2$ for $0 \le x \le 2\sqrt{3}$ **around the y-axis**.

Problem 7. Compute the volume of the region obtained when revolving the region bounded by the two curves $y = a^3 - x^3$ and $y = 0$ around

a) the x-axis b) the y-axis

Problem 8. Set up an integral for (but **do not evaluate**) the average distance from a point on the curve $y = 1-x$ for $0 \le x \le 1$, to the origin.

Answers

1. (a) $1/2$ (b) $e^{-1/2}$ (c) $+\infty$

2. (a) $1/2$ (b) $2/3$

3. (a) $1 - \frac{x}{2} + \frac{11}{8}x^2$ (c) $0 + x + \frac{1}{4}x^2$
   (b) $0 + (e-1)\cdot(x-1) + \frac{1}{2}(x-1)^2$ (d) $1 + (x-1) - \frac{\pi^2}{8}(x-1)^2$

4. $e^x \sin(e^x) - \sin(x)$

5. $10/3$

6. (a) $\frac{\pi}{27}\left(10^{3/2} - 1\right)$ (b) $\frac{56\pi}{3}$

7. (a) $\frac{9\pi}{14}a^7$ (b) $\frac{3\pi}{5}a^5$

8. $\displaystyle\int_0^1 \sqrt{2x^2 - 2x + 1}\mathrm{d}x.$

Midterm review problems

Problem 1. a) Find the linear approximation for e^x/x at $x = 1$.

b) Find the quadratic approximation for $\dfrac{e^x}{\cos(x)}$ at $x = 0$.

Problem 2. Evaluate

a) $\lim\limits_{x\to 0} \dfrac{e^x - 1}{\sin(x)}$

b) $\lim\limits_{x\to 0} x^{\sqrt{x}}$

Problem 3. Find the limit

$$\lim_{n\to\infty} \left(\frac{e^{0/n}}{n} + \frac{e^{1/n}}{n} + \ldots + \frac{e^{(n-1)/n}}{n} \right) \tag{1}$$

Problem 4. Take the region between the curves $y = x^2$ and $y = x^3$ in the first quadrant and compute its volume if we rotate it around the y-axis.

Problem 5. a) A solid is formed by rotating about the x-axis the region under the graph of $y = e^x$ and over the interval $0 \le x \le 1$. Compute the volume of the solid.

b) Compute the arc length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ for $1 \le y \le e$

c) Compute the surface area when revolving the curve $y = x^3$ around the x-axis, for $0 \le x \le 1$.

d) Write an integral for the surface area when revolving the curve $y = x^3$ around the y-axis, for $0 \le x \le 1$. (Do not compute it!)

Problem 6. Compute the integrals

a) $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\mathrm{d}x$ b) $\displaystyle\int \sin^3(x)\cos^2(x)\mathrm{d}x$ c) $\displaystyle\int \frac{\cos(e^{-x})}{e^x}\mathrm{d}x$

Problem 7. Calculate

a) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \tan(t^2)\mathrm{d}t$ b) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2} \tan(t^2)\mathrm{d}t$ c) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{x^2} \tan(t^2)\mathrm{d}t$

Answers

1. (a) $e + 0 \cdot (x-1) = e$ (b) $1 + x + x^2$

2. (a) 1 (b) 1

3. $e - 1$

4. $\frac{\pi}{10}$

5. (a) $\frac{\pi}{2}(e^2 - 1)$ (c) $\frac{\pi}{27}\left(10^{3/2} - 1\right)$

   (b) $\frac{e^2+1}{4}$ (d) $\displaystyle\int_0^1 2\pi y^{1/3}\sqrt{1 + \frac{1}{9}y^{-4/3}}$

6. (a) 1 (b) $\frac{1}{5}\cos^5(x) - \frac{1}{3}\cos^3(x) + C$ (c) $-\sin(e^{-x}) + C$

7. (a) $\tan(x^2)$ (b) $2x\tan(x^4)$ (c) $2x\tan(x^4) - \tan(x^2)$

Final review session 18.01A

Problem 1. For which positive b and which p does the series

$$\sum_{n=1}^{\infty} \frac{b^n}{n^p}$$

converge?

Problem 2. Is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

absolutely convergent, conditionally convergent or divergent?

Problem 3. Use the integral test to determine if the series

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

is convergent or divergent.

Problem 4. Evaluate

$$\int \frac{e^{3x}}{e^x - 2} dx.$$

Problem 5. Compute

$$\int \frac{x}{x^2 + 2x + 1} dx.$$

Problem 6. Compute the first four terms of the Taylor series of the function $f(x) = \cos(e^x - 1)$ at $x = 0$ (i.e. the coefficients a_0, a_1, a_2, a_3).

Problem 7. Find the value of

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(2n-1)\sqrt{3}^{2n-1}}.$$

Hint: identify the power series with the anti-derivative of a simpler function.

Answers

1. Converges if $b < 1$, for all p, **or** if $b = 1$ and $p > 1$.
2. Conditionally convergent.
3. Convergent
4. $\frac{e^{2x}}{2} + 2e^x + 4\ln|e^x - 2| + C$
5. $\ln|x+1| + \frac{1}{1+x} + C$
6. The first four terms are $1 - \frac{1}{2}x^2 - \frac{1}{2}x^3$.
7. $\frac{\pi}{6}$.