# SURFACES AND ISOMORPHISM

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## 1. Introduction

This is a note on the classification of surfaces which are topologically different, *e.g.* the sphere $\mathbb{S}^2$, and the torus $\mathbb{T}^2$. We first separately review notions of topology and abstract algebra in Sections 2 and 3. In Section 4, elements of algebraic topology are introduced, in particular Theorem 4.7 connecting topological equivalence to group isomorphism. We conclude in Section 5 with a demonstration that the surfaces $\mathbb{S}^2$ and $\mathbb{T}^2$ are distinct topologically via a proof that their respective fundamental groups are not isomorphic.

## 2. Topology and Homeomorphism

**Definition 2.1.** A *surface* is a metric space $X$ such that every point in $X$ has a neighborhood which is homeomorphic to the plane.

**Definition 2.2.** A *metric space* is a set $X$ with a map $d\colon X \times X \to \mathbb{R}$ satisfying the following properties:

- $d(x, x) = 0$
- If $x \neq y$, then $d(x, y) > 0$
- $d(x, y) = d(y, x)$ for all $x, y$
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z$.

**Definition 2.3.** A *topology* $\mathcal{T}$ on a set $X$ is a set of subsets of $X$ with the following properties:

- The empty set and $X$ belong to $\mathcal{T}$
- Any arbitrary union of members of $\mathcal{T}$ belongs to $\mathcal{T}$
- The intersection of any finite number of members of $\mathcal{T}$ belongs to $\mathcal{T}$.

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If a subset $U$ of $X$ is in the topology $\mathcal{T}$ of $X$, then it is called *open*. A set $U \subset X$ is called *closed* if its complement $X \setminus U$ is open. A topological space $(X, \mathcal{T})$ is a set $X$ together with a topology $\mathcal{T}$ on it.

**Definition 2.4.** A map $f : X \to Y$ from one topological space $(X, \mathcal{T}_1)$ to another $(Y, \mathcal{T}_2)$ is *continuous* if the preimage $f^{-1}(U)$ of any open set $U \in \mathcal{T}_2$ is open in $\mathcal{T}_1$.

**Definition 2.5.** A map $f : X \to Y$ from one metric space $(X, d_X)$ to another $(Y, d_Y)$ is *continuous* at $x \in X$ if for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$d_X(x, x') < \delta \quad \Rightarrow \quad d_Y(f(x), f(x')) < \epsilon.$$

The above two definitions of continuity agree whenever all topologies are induced by the respective metrics.

**Definition 2.6.** A map $d\colon X \to Y$ is a *homeomorphism* if $h$ is a bijection and both $h$ and $h^{-1}$ are continuous.

**Example 2.7.** The stereographic projection is a homeomorphism from the sphere minus a point to $\mathbb{C}$. Suppose the missing point is $(0, 0, 1)$, then the map is explicitly

$$(x_1, x_2, x_3) \mapsto \left(\frac{x_1}{1 - x_3}\right) + \left(\frac{x_2}{1 - x_3}\right) i.$$

## 3. Groups and Isomorphism

**Definition 3.1.** A *group* is a set $G$ together with an operation "$\cdot$" that satisfies the following:

- $g_1 \cdot g_2$ is defined and belongs to $G$ for all $g_1, g_2 \in G$
- $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ for all $g_1, g_2, g_3 \in G$
- There exists a unique $e \in G$ such that $e \cdot g = g \cdot e = g$ for all $g \in G$
- For each $g \in G$ there is a unique element $h$ such that $g \cdot h = h \cdot g = e$. This element is called "$g$ inverse" and is written as $h = g^{-1}$.

**Definition 3.2.** Let $G_1$ and $G_2$ be groups. A map $f : G_1 \to G_2$ is a *homomorphism* if $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in G_1$.

Notice that the "$\cdot$" on the left-hand side is the operation for $G_1$ and the "$\cdot$" on the right-hand side is the one for $G_2$.

**Definition 3.3.** A map $f$ is an *isomorphism* if $f$ is a bijective homomorphism.

**Example 3.4.** Logarithm is an example of isomorphism. Recall that for all positive real numbers $x, y$

$$\log(xy) = \log(x) + \log(y)$$

with $\log(1) = 0$ and $\log(1/x) = -x$. Therefore, we have an isomorphism of the multiplicative group of positive real numbers to the additive group on the whole real line.

**Definition 3.5.** For a set $A$ of $n$ elements, a *permutation* is a bijection $f : A \to A$.

A permutation group is a set of all the $n!$ permutations of set $A$ with the group operation being composition of maps. In fact, we can represent the elements of any finite abstract group by sets of permutations.

**Theorem 3.6** (Cayley's Theorem)**.** *Every finite group is isomorphic to a subgroup of a permutation group.*

## 4. Fundamental Group

**Definition 4.1.** Let $X$ and $Y$ be metric spaces and $I = [0, 1]$ be the unit interval. Two maps $f_0, f_1 : X \to Y$ are *homotopic* if there is a continuous map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in X$. We write $f_0 \sim f_1$.

**Definition 4.2.** Take $X = I$. A *loop* in $Y$ is a continuous map $f : I \to Y$ such that $f(0) = f(1) = y_0 \in Y$. Here $y_0$ is called the *basepoint.*

**Definition 4.3.** The fundamental group of space $Y$ is the group of equivalence classes under homotopic loops with basepoint $y_0$. The group is written as $\pi_1(Y, y_0)$.

The group operation of the fundamental group is composing loops. Define the new loop $h = f \cdot g$ by the following rule:

- If $x \in [0, 1/2]$, define $h(x) = f(2x)$,
- If $x \in [1/2, 1]$, let $x' = x - 1/2$ and define $h(x) = g(2x')$.

**Definition 4.4.** $Y$ is *path connected* if for any two points $y_0, y_1 \in Y$ there is a continuous map $f : I \to Y$ such that $f(0) = y_0$ and $f(1) = y_1$.

**Lemma 4.5.** *Suppose $y_0, y_1 \in Y$ are connected by a path. Then $\pi_1(Y, y_0)$ and $\pi_1(Y, y_1)$ are isomorphic groups.*

If a space Y is path connected, then the fundamental group $\pi_1(Y, y)$ is independent of the choice of basepoint $y$ so we may just write $\pi_1(Y)$.

**Proposition 4.6.** *Let $(Y, y_0)$ and $(Z, z_0)$ be two pointed spaces, and let $f : Y \to Z$ be a continuous map such that $f(y_0) = z_0$. Then there exists a homomorphism*

$$f_* : \pi_1(Y, y_0) \to \pi_1(Z, z_0).$$

This is a first connection between topology and algebra where the homomorphism $f_*$ is constructed from the continuous map $f$. If one desires $f_*$ to be an isomorphism, then we must demand that $f$ is actually a homeomorphism. The result of including bijectivity into the argument for Proposition 4.6 is the following:

**Theorem 4.7.** *Suppose $Y$ and $Z$ are path connected spaces. If $\pi_1(Y)$ and $\pi_1(Z)$ are not isomorphic, then $Y$ and $Z$ are not homeomorphic.*

The inverse of Theorem 4.7 is generally difficult. A famous example for three-dimensional manifolds was known as the Poincaré conjecture:

Every closed 3-manifold with trivial fundamental group is homeomorphic to $\mathbb{S}^3$.

This remains the only Millennium problem to be solved.

## 5. Spheres and Tori

The two-dimensional sphere $\mathbb{S}^2$ and torus $\mathbb{T}^2$ are topologically distinct. This is done by computing the fundamental groups of these surfaces. It turns out that

$$\pi_1(\mathbb{S}^2) \ncong \pi_1(\mathbb{T}^2). \tag{5.1}$$

We then apply Theorem 4.7.

**Proposition 5.2.** *Let $(Y, y_0)$ and $(Z, z_0)$ be two pointed spaces. Then,*

$$\pi_1(Y \times Z, (y_0, z_0)) = \pi_1(Y, y_0) \times \pi_1(Z, z_0).$$

**Proposition 5.3.** *For any basepoint $x \in \mathbb{S}^1$,*

$$\pi_1(\mathbb{S}^1, x) = \mathbb{Z}.$$

The proof that $\pi_1(\mathbb{S}^1, x)$ is merely nontrivial is subtle. This involves showing that the loop

$$f(t) = (\cos(2\pi t), \sin(2\pi t)) \tag{5.4}$$

is inequivalent to the identity loop. We refer to [1, 2] for details and discussion. The fundamental group of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is evident.

**Proposition 5.5.** *For any basepoint $x \in \mathbb{T}^2$,*

$$\pi_1(\mathbb{T}^2, x) = \mathbb{Z}^2.$$

**Definition 5.6.** A loop $g : I \to \mathbb{S}^2$ with basepoint $x$ is called *bad* if $g(I) = \mathbb{S}^2$. Otherwise, the loop $g$ is called *good*.

We proceed to show the fundamental group of the sphere is the trivial group "0" consisting of only the identity (loop). In other words, we will show every loop on the sphere is homotopic to a point.

**Proposition 5.7.** *Any good loop is homotopic to a point.*

*Proof.* Let $g : I \to \mathbb{S}^2$ be a good loop. Then there exists a point $p \in \mathbb{S}^2$ such that $p \notin g(I)$. It follows that $g(I)$ is a subset of $\mathbb{S}^2 \setminus \{p\}$. With the stereographic projection (see Example 2.7), we have $\mathbb{S}^2 \setminus \{p\}$ is homeomorphic to the plane. The plane is in fact simply connected, and this follows immediately by the construction (5.11) in the next proof. ☐

**Proposition 5.8.** *Let $[a, b]$ be an interval, and let $H$ be a hemisphere in $\mathbb{S}^2$. Let $f : [a, b] \to H$ be a continuous map. Then, there exists a homotopy $F : [a, b] \times [0, 1] \to H$ such that*

- *$F(a, t)$ and $F(b, t)$ are independent of $t$,*
- *$F(x, 0) = f(x)$ for all $x$,*
- *$f_1 : [a, b] \to H$ is contained in a circular arc joining $f(a)$ to $f(b)$.*

*Proof.* First, we will show that the hemisphere is homeomorphic to the disk. View the sphere $\mathbb{S}^2$ as embedded in $\mathbb{R}^3$ where its center is at the origin. We fix the orientation such that the pole of the hemisphere $H \subset \mathbb{S}^2$ lies directly on the $z$ axis. In that case consider the projection $P : \mathbb{R}^3 \to \mathbb{R}^2$

$$(x, y, z) \mapsto (x, y) \tag{5.9}$$

by taking the first two coordinates in $\mathbb{R}^3$ as its coordinates in $\mathbb{R}^2$. By restricting $P$ to $H \subset \mathbb{R}^3$, we have a bijection from the hemisphere into the open disk

$$D = \{(x, y) \colon x^2 + y^2 < 1\}. \tag{5.10}$$

In the inverse, the $z$ coordinate is $z = \sqrt{1 - (x^2 + y^2)}$. One may see that $P$ is continuous as well as its inverse because any open set in $H$ is projected onto an open set in $D$, and similar happens in the inverse.

Next, we will show the disk is simply connected, meaning all paths connecting two points in $D$ are homotopic to each other. Let $g_0$ and $g_1$ be two paths with the same orientation connecting points $x_1$ and $x_2$, then a homotopy is

$$g_t(s) = t g_1(s) + (1 - t) g_0(s). \tag{5.11}$$

It follows that $D$ is simply connected, and by the homeomorphism above, that the hemisphere $H$ is simply connected. In particular, any path $f$ in the hemisphere is homotopic to a circular arc joining $f(a)$ to $f(b)$. □

**Proposition 5.12.** *Let $g$ be an arbitrary loop on $\mathbb{S}^2$. There is a finite partition*

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

*such that $g$ maps each interval $[t_i, t_{i+1}]$ into a hemisphere.*

*Proof.* Since $g$ is continuous on the compact interval $I$, then $g$ is in fact uniformly continuous. Therefore, there exists $\delta > 0$ such that whenever $|t - t'| < \delta$ it follows that

$$d_{\mathbb{S}^2}(g(t), g(t')) < \pi \tag{5.13}$$

for all $t, t' \in I$. We choose the intervals such that $t_i = i/n$ where $n = \lceil 1/\delta \rceil$ and $\lceil s \rceil$ is the least integer greater than $s \in \mathbb{R}$. □

**Corollary 5.14.** *An arbitrary loop on $\mathbb{S}^2$ is loop homotopic to a good loop.*

*Proof.* First, we use Proposition 5.12 to partition an arbitrary loop into a finite number of paths each on a hemisphere. Then, we use Proposition 5.8 to show each path is homotopic to a circular arc. A finite number of circular arcs cannot cover the whole sphere, and so we have found a homotopy into a good loop. □

**Theorem 5.15.** *For any basepoint $x \in \mathbb{S}^2$,*

$$\pi_1(\mathbb{S}^2, x) = 0.$$

*Proof.* Observing both Proposition 5.7 and Corollary 5.14, it follows an arbitrary loop on $\mathbb{S}^2$ is homotopic to a point. □

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## References

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