INFORMATION IN FOURIER ANALYSIS

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Contents

1. Introduction

We briefly recount some results in two connected topics: Fourier analysis and analysis of the vibrating string. In particular, we focus on the localization of information in the Fourier transform and the propagation of data in the wave equation.

The Fourier transform of a function allows us to decompose a function of physical space into functions of frequency. The Plancherel formula tells us that this transform is in fact an isometry of ℓ_2. Yet, we have the uncertainty principle, which gives a limitation on simultaneous information about a function's variance in space and its transform's variance in frequency.

Vibrating strings which satisfy the wave equation propagate information in a particular manner; the influence of data localized to a specific region of space at a certain instance of time is limited to the backwards light cone. We prove the more precise statement for classical solutions to the wave equation, and finally the uniqueness of these solutions.

2. Fourier Transform

We consider the Fourier transform on the **Schwartz space** $\mathcal{S}(\mathbb{R})$ of functions. We say $f \in \mathcal{S}(\mathbb{R})$ if f is infinitely differentiable and all of its derivatives $f', f'', \ldots, f^{(l)}$ are rapidly decreasing. A function $f(x)$ is **rapidly decreasing** if $\forall k, l \geq 0$

$$\sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty. \tag{2.1}$$

For such functions, we define the **Fourier transform** to be

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(\xi) e^{2\pi i x \xi}\, d\xi. \tag{2.2}$$

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This is denoted as $f \to \hat{f}$. The rapidly decreasing property of f and its derivatives lends to the following nice properties on how Fourier transforms react to translation, rotation, and differentiation:

Proposition 2.3. *For $f \in \mathcal{S}(\mathbb{R})$, we have:*

(i) $f(x+h) \to \hat{f}(\xi)e^{2\pi i h\xi}$
(ii) $f(x)e^{-2\pi i x h} \to \hat{f}(\xi + h)$
(iii) $f(\delta x) \to \delta^{-1}\hat{f}(\delta^{-1}\xi)$ *for* $\delta > 0$
(iv) $f'(x) \to 2\pi i \xi \hat{f}(\xi)$
(v) $-2\pi i x f(x) \to \frac{d}{d\xi}\hat{f}(\xi)$.

Importantly, if $f(x) = e^{-\pi x^2}$, then $\hat{f}(\xi) = f(\xi)$. This together with (iii) from Proposition 2.3 tells us that $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$ has Fourier transform $e^{-\pi\delta\xi^2}$.

We call a family of kernels $\{K_\delta(x)\}$ **good** if:

— $\int_{-\infty}^{\infty} K_\delta(x)dx = 1$
— $\exists M > 0$ so that $\forall \delta : \int_{-\infty}^{\infty} |K_n(x)|\, dx < M$
— $\forall \eta > 0$, we have $\int_{|x|>\eta} |K_\delta(x)|\, dx \to 0$ as $\delta \to 0$.

In particular, the family defined by $K_\delta(x) = \delta^{-1/2}e^{-\pi x^2/\delta}$ as $\delta \to 0+$ is good.

Given $f, g \in \mathcal{S}(\mathbb{R})$, we define their **convolution** to be

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-t)g(t)\, dt. \tag{2.4}$$

Kernels are important because they approximate the identity via convolutions:

Theorem 2.5. *For $f \in \mathcal{S}(\mathbb{R})$, $(f * K_\delta)(x) \to f(x)$ uniformly as $\delta \to 0$.*

Another key property of the transform is the **multiplication formula**:

$$\int_{-\infty}^{\infty} f(x)\hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(y)g(y)\, dy \tag{2.6}$$

Together, these observations prove the **Fourier inversion formula**, which expresses a function via its transform:

Theorem 2.7. *If $f \in \mathcal{S}(\mathbb{R})$, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\, d\xi.$$

Proof. We first prove this for $x = 0$. Let $G_\delta(x) = e^{-\pi\delta x^2}$. By Proposition 2.3 (iii) and since $e^{-\pi x^2}$ is fixed under the Fourier transform, $\hat{G_\delta} = K_\delta$. By the multiplication formula,

$$\int_{-\infty}^{\infty} f(x)K_\delta(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(\xi)G_\delta(\xi)\, d\xi. \tag{2.8}$$

By Theorem 2.5, the left hand side converges uniformly to $f(0)$ as $\delta \to 0$. Meanwhile, the right hand side approaches $\int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi$, so we have the desired equality.

Now, let $F(y) = f(y+x)$ so that

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(\xi)\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}\, d\xi, \tag{2.9}$$

by Proposition 2.3 (i). □

Corollary 2.10. *The Fourier transform is a bijection on the Schwartz space.*

3. Plancherel Formula

In fact, the Fourier transform is an isomorphism on ℓ_2 with norm

$$\|f\| := \left(\int_{\mathbb{R}} |f(x)|^2\, dx\right)^{1/2}. \tag{3.1}$$

This result is known as **Plancherel formula**:

Theorem 3.2. *If $f \in \mathcal{S}(\mathbb{R})$, then $\|f\| = \|\hat{f}\|$.*

Its proof requires some basic properties of convolutions:

Proposition 3.3. *For $f, g \in \mathcal{S}(\mathbb{R})$:*

(i) $f * g \in \mathcal{S}(\mathbb{R})$
(ii) $f * g = g * f$
(iii) $\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Proof of Plancherel formula. Let $h = f * \overline{f(-x)}$. Then by Proposition 3.3 (iii), $\hat{h}(\xi) = |\hat{f}(\xi)|^2$. From the convolution formula (2.4),

$$h(0) = \int_{-\infty}^{\infty} f(t)\bar{f}(t)\, dt = \|f\|. \tag{3.4}$$

Now we apply the Fourier inversion formula (Theorem 2.7) with $x = 0$. □

4. Uncertainty Principle

The more information we have on the variance of a function in the spatial variable $x \in \mathbb{R}$, the less we have on such for its transform in the frequency variable $\xi \in \mathbb{R}$. This is illustrated by the uncertainty principle:

Theorem 4.1. *Consider a function $\psi \in \mathcal{S}(\mathbb{R})$ with $\|\psi\| = 1$. Then,*

$$\left(\int_{-\infty}^{\infty} x^2|\psi(x)|^2\, dx\right)\left(\int_{-\infty}^{\infty} \xi^2|\hat{\psi}(\xi)|^2\, d\xi\right) \geq \frac{1}{16\pi^2}.$$

Here we show this result in $\mathbb{R}^d$ where, for $f \in \mathcal{S}(\mathbb{R}^d)$

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(\xi)e^{2\pi i x\cdot\xi}\, d\xi \quad \text{and} \quad \|f\| := \left(\int_{\mathbb{R}^d} |f(x)|^2\, dx\right)^{1/2}, \tag{4.2}$$

the Plancherel formula (Theorem 3.2) holds.

Theorem 4.3. *Suppose $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfies $\|\psi\| = 1$. Then,*

$$\left(\int_{\mathbb{R}^d} x^2|\psi(x)|^2\, dx\right)\left(\int_{\mathbb{R}^d} \xi^2|\hat{\psi}(\xi)|^2\, d\xi\right) \geq \frac{1}{16\pi^2}.$$

Proof. Integration by parts gives:

$$\begin{aligned}\int_{\mathbb{R}^d} |\psi(x)|^2\, dx &= -\int_{\mathbb{R}^d} x\cdot\nabla|\psi(x)|^2\, dx \\ &= -\int_{\mathbb{R}^d} (x\cdot\nabla\psi(x)\overline{\psi(x)} + x\cdot\overline{\nabla\psi(x)}\psi(x))\, dx.\end{aligned} \tag{4.4}$$

Because $\|\psi\| = 1$, we have

$$\begin{aligned}\frac{1}{2} &\le \left| \int_{\mathbb{R}^d} x \cdot \nabla\psi(x)\overline{\psi(x)}\, dx \right| \\ &\le \|\nabla\psi\| \|x\overline{\psi}\|,\end{aligned} \tag{4.5}$$

where the second inequality follows from the Cauchy-Schwarz inequality. By Plancherel formula and Proposition 2.3 (iv) in $\mathbb{R}^d$,

$$\|\nabla\psi\| = 2\pi\|\xi\hat{\psi}\|. \tag{4.6}$$

We conclude that

$$\|\xi\hat{\psi}\|^2\|x\psi\|^2 \ge \frac{1}{16\pi^2}. \tag{4.7}$$

□

5. Data in the Wave Equation

The **wave equation** on $\mathbb{R}^{d+1}$ for coordinates $(x_1, \dots, x_d, t)$ is given by:

$$\Box u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0. \tag{5.1}$$

The **Cauchy problem** for the wave equation asks for solutions $u(x,t)$ to $\Box u = 0$ with initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x). \tag{5.2}$$

We may assume that $c = 1$ by coordinate scaling. By Proposition 2.3 (iv) in $\mathbb{R}^d$ and the wave equation, the solution u has Fourier coefficients $\hat{u}(\xi, t)$ which satisfy:

$$-4\pi^2|\xi|^2\hat{u}(\xi,t) = \frac{\partial^2 \hat{u}}{\partial t^2}(\xi,t). \tag{5.3}$$

For a fixed $\xi \in \mathbb{R}^d$, this is an ordinary differential equation which has the general solution

$$\hat{u}(\xi,t) = A(\xi)\cos(2\pi|\xi|t) + B(\xi)\sin(2\pi|\xi|t). \tag{5.4}$$

Meanwhile, the initial conditions become $\hat{u}(\xi,0) = \hat{f}(\xi)$ and $\partial_t\hat{u}(\xi,0) = \hat{g}(\xi)$. So we must have $A(\xi) = \hat{f}(\xi)$ and $B(\xi) = (2\pi|\xi|)^{-1}\hat{g}(\xi)$. Altogether, this yields solution

$$\hat{u}(\xi,t) = \hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi|\xi|t). \tag{5.5}$$

Now, we can use the Fourier inversion formula in $\mathbb{R}^d$ to get a solution u for the initial-value problem, (5.1) with (5.2), in terms of f and g:

Theorem 5.6. *A solution to the Cauchy problem for the wave equation is*

$$u(x,t) = \int_{\mathbb{R}^d} \left[\hat{f}(\xi)\cos(2\pi|\xi|t) + \frac{\hat{g}(\xi)}{2\pi|\xi|}\sin(2\pi|\xi|t) \right] e^{2\pi i x\cdot\xi}\, d\xi.$$

In fact, the solution $u(x,t)$ of the Cauchy problem in a region known as the **backward light cone**:

$$\mathcal{L}_{B(x_0,r_0)} \equiv \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |x - x_0| \le r_0 - t, 0 \le t \le r_0\} \tag{5.7}$$

is uniquely determined by the initial data on the base region $B(x_0, r_0)$, the closed ball centered at $x_0 \in \mathbb{R}^d$ with radius r_0 in the hyperplane of $\{t = 0\}$. More precisely:

Theorem 5.8. *Suppose that $u(x,t)$ is a C^2 function on the closed upper half-plane $\{(x,t) : x \in \mathbb{R}^d, t \geq 0\}$ that solves the wave equation $\Box u = 0$. If*

$$u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$$

for all $x \in B(x_0, r_0)$, then $u(x,t) = 0$ for all $(x,t) \in \mathcal{L}_{B(x_0,r_0)}$.

Proof. The proof has two main steps; (a) we discover an expression for the growth of an energy E for the solution u and then (b) through estimates discover that E and thus u vanish on the interior of the backwards light cone.

a) Assume that u is real. For $0 \leq t \leq r_0$ let

$$B_t(x_0, r_0) \equiv \{x : |x - x_0| \leq r_0 - t\} \tag{5.9}$$

and

$$Du(x,t) := \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial u}{\partial t}\right). \tag{5.10}$$

Consider the nonnegative **energy** of the string

$$E(t) := \frac{1}{2}\int_{B_t(x_0,r_0)} |Du|^2 = \frac{1}{2}\int_{B_t(x_0,r_0)} \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2. \tag{5.11}$$

Now with integration by parts,

$$\frac{d}{dt}\frac{1}{2}\int_{B_t(x_0,r_0)} \left(\frac{\partial u}{\partial t}\right)^2 = \int_{B_t(x_0,r_0)} \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} dt - \frac{1}{2}\int_{\partial B_t} |Du|^2 d\sigma(y). \tag{5.12}$$

By the chain rule,

$$\frac{d}{dt}\sum_{j=1}^{d}\left(\frac{\partial u}{\partial x_j}\right)^2 = 2\sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}. \tag{5.13}$$

We have discovered that the energy grows according to

$$E'(t) = \int_{B_r(x_0,r_0)} \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t}\, dx - \frac{1}{2}\int_{\partial B_t} |Du|^2\, d\sigma(y). \tag{5.14}$$

By the product rule,

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\right) = \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} + \frac{\partial^2 u}{\partial x_j^2}\frac{\partial u}{\partial t}, \tag{5.15}$$

and so it follows that

$$\sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} = \sum_{j=1}^{d}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\right) - \frac{\partial^2 u}{\partial x_j^2}\frac{\partial u}{\partial t}\right). \tag{5.16}$$

So,

$$\begin{aligned} &\int_{B_r(x_0,r_0)} \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{d}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} = \\ &\qquad \int_{B_t(x_0,r_0)} \frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{d}\frac{\partial^2 u}{\partial x_j^2}\right) + \sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\right). \end{aligned} \tag{5.17}$$

Since u satisfies the wave equation $\Box u = 0$, the first summand on the right hand side vanishes. Thus from equation (5.14), we get

$$E'(t) = \int_{B_t(x_0,r_0)} \sum_{j=1}^{d} \frac{\partial}{\partial x_j}\left(\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t}\right) - \frac{1}{2}\int_{\partial B_t} |Du|^2 \, d\sigma(y). \tag{5.18}$$

b) By the divergence theorem, this equals

$$\int_{\partial B_t(x_0,r_0)} \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial t} v_j d\sigma(y) - \frac{1}{2}\int_{\partial B_t} |Du|^2 \, d\sigma(y), \tag{5.19}$$

where v_j is the jth coordinate of the normal to $B_t(x_0, r_0)$. By Cauchy-Schwarz,

$$\left(\nabla u, v\frac{\partial u}{\partial t}\right)^2 \leq \left(|Du|^2 - \left|\frac{\partial u}{\partial t}\right|^2\right)\left|v\frac{\partial u}{\partial t}\right|^2, \tag{5.20}$$

where $\nabla u = (\partial_{x_1} u, \ldots \partial_{x_d} u)$, and on the left hand side is the inner product on $\mathbb{R}^d$. Next, $|v\partial_t u|^2 = |v|^2|\partial_t u|^2$, and since v is a unit normal vector, $|v| = 1$. Altogether,

$$\left(\nabla u, v\frac{\partial u}{\partial t}\right)^2 \leq \left(|Du|^2 - \left|\frac{\partial u}{\partial t}\right|^2\right)\left|\frac{\partial u}{\partial t}\right|^2. \tag{5.21}$$

Since $\partial u/\partial t$ is merely a coordinate of Du, clearly $|Du|^4/4$ bounds the right hand side above. Therefore we have $(\nabla u, v\partial_t u) \leq |Du|^2/2$ which implies

$$E'(t) \leq 0. \tag{5.22}$$

However, $E(0) = 0$ and therefore $E(t) = 0$ in general, such that u vanishes everywhere in the backwards light cone $\mathcal{L}_{B(x_0,r_0)}$. □

Corollary 5.23. *Suppose there are two C^2 functions w, v which satisfy the wave equation on the closed upper half plane, and that*

$$w(x,0) = v(x,0) \quad \textit{and} \quad \frac{\partial w}{\partial t}(x,0) = \frac{\partial v}{\partial t}(x,0),$$

for all $x \in B(x_0, r_0)$. Then, $w = v$ in $\mathcal{L}_{B(x_0,r_0)}$.

Proof. Since the wave equation is linear, their difference $u = w - v$ solves the wave equation. As w and v agree on $B(x_0, r_0)$, u vanishes on the base. By Theorem 5.8, u must then also vanish on the whole backward light cone, so we are done. □

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For the reading program, I read through the first 7 chapters of Stein and Shakarchi's Fourier analysis [1], and this paper summarizes my favorite results and problems. The statements of propositions and theorems, as well as their proofs, are based off this book. The problem solutions in sections 4 and 5 are my own work.

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References

[1] Stein, Elias M., and Rami Shakarchi. *Fourier analysis: an introduction.* Princeton Lectures in Analysis Vol. 1. Princeton University Press, 2011.