# Radiative Vlasov–Maxwell Equations

Peter Constantin and Hezekiah Grayer II

Abstract. In the radiative Vlasov–Maxwell equations, the Lorentz force is modified by the addition of radiation reaction forces. The radiation forces produce damping of particle energy but the forces are no longer divergence-free in momentum space, which has an effect of concentration to zero momentum. We prove unconditional global regularity of solutions for a class of radiative Vlasov–Maxwell equations with large initial data.

## 1. Introduction

The kinetic description of collisionless relativistic plasma is provided by the Vlasov–Maxwell equations. The problem of global regularity of solutions of the Vlasov–Maxwell equations for large data has been studied extensively, but remains unsolved. In this paper we prove global regularity for large data for solutions of radiative Vlasov–Maxwell equations.

Radiation reaction forces in the plasma dampen the energy of the particles. A rigorous self-consistent derivation of the particle dynamics and their radiation is fraught with fundamental challenges [30]. There are several models of radiation in the physical literature [18], [27] and a formal derivation from microscopic models [17]. Radiative forces are not accounted for in the classical Vlasov–Maxwell equations. These forces are significant for particles at large velocities.

The Vlasov–Maxwell equations are locally well posed [1]. Small data results have been obtained [12], [28], in which the plasma is initially dilute, the solutions remain small and smooth, disperse and their asymptotic behavior is free ([2–5]). This picture holds for nearly neutral data as well ([10],[8]). There are several recent results ([14, 21]) concerning the asymptotic behavior of small perturbations of steady states which do not depend

*Key words and phrases.* Vlasov–Maxwell, radiation, global regularity.
*MSC Classification: 35Q70, 35Q83.*

on the space variable. Existence of global weak solutions was obtained in [9].

For smooth large data, the possibility of spontaneous singularity formation has been the focus of many analytical works. In seminal papers, Glassey and Strauss [11, 13] proved that the only way singularities might arise in finite time is through concentration of particle density at very high velocity. Specifically, they proved that if the solution-averaged Lorentz factor $\langle \gamma \rangle$ is uniformly bounded, then no singularities can form in finite time from smooth and localized initial data. The solution-averaged Lorentz factor $\langle \gamma \rangle$ is a function of space and time representing the kinetic energy density of the particles. In [15] it was shown using Fourier analysis that the singularities are averted if the electromagnetic fields remain bounded. Several other results are based on Fourier methods [6], [7, 23].

A number of extensions of the results of Glassey and Strauss concern moments of the type $M_{\theta,q} = \|\langle \gamma^\theta \rangle\|_{L^q(dx)}$. In our notation,

$$\langle \gamma^\theta \rangle = \int_{\mathbb{R}^3} (\sqrt{1+|p|^2})^\theta f(x,p,t)\, dp$$

for an exponent $\theta$. The average of the kinetic energy density considered by Glassey and Strauss corresponds to $M_{1,\infty}$. In [22], control of $M_{\theta,q}$ where $\theta > 4/q$ and $6 \le q \le \infty$ is shown to be sufficient for regularity. Then, [29] uses [22] to show that control of $M_{0,\infty}$ is sufficient for regularity. This result was improved in [24] where it was shown that control of $M_{0,6}$ is sufficient for regularity. In [19], it is proven that the solutions remain smooth if a plane projection of the momenta is bounded through the evolution. The results of [16] imply that finiteness of $M_{3,2}$ is sufficient for regularity. In [20] it is shown that if $2 < q \le \infty$ and $\theta > 2/q$, then control of $M_{\theta,q}$ is sufficient for regularity, and if $1 \le q \le 2$ and $\theta > 8/q - 3$, then control of $M_{\theta,q}$ is sufficient for regularity and an improvement [25] shows that if $\theta > 3$, then control of $M_{\theta,1}$ is sufficient for regularity. Results of global regularity for cylindrical symmetry are announced in [31].

The Vlasov–Maxwell (VM) equations are formed by the Vlasov equation for the particle distribution function $f = f(x,p,t)$, coupled to the Maxwell equations for the electromagnetic (EM) fields $E = E(x,t)$ and $B = B(x,t)$. The particle dynamics is driven by the Lorentz force

$$F_L = E + v \times B.$$

The radiative Vlasov–Maxwell (RVM) equations are the same equations, except that the particles are forced by a total force

$$F = F_L + F_R$$

where $F_R$ is the radiation reaction force. The main result of this paper is

THEOREM 1. *Assume that the initial data $E_0(x)$ and $B_0(x)$ for the electromagnetic fields $E(x,t)$ and $B(x,t)$ and the initial data $f_0(x,p)$ for the particle distribution function $f(x,p,t)$ are smooth, compatible, and decay at spatial infinity. In addition assume*

$$f_0(x,0) = 0$$

*(the initial particle distribution vanishes at zero momentum) and*

$$\sup_{x,p} f_0(x,p)\exp(A_0|p|) < \infty$$

*holds for some $A_0 > 0$ large enough (the initial particle density decays uniformly exponentially at high momentum). Then, the solution of the RVM equations is globally smooth and there exist constants $C$ depending explicitly only on the initial data so that*

$$|E(x,t)| + |B(x,t)| + |\nabla_x E(x,t)| + |\nabla_x B(x,t)| \le C\exp(Ct)$$

*and*

$$f(x,p,t) + |\nabla_x f(x,p,t)| + \sqrt{1+|p|^2}|\nabla_p f(x,p,t)| \le C\exp(C\exp(Ct))$$

*hold for all $x, p$ and $t$.*

In this paper we address the main problem, which is to obtain global a priori bounds for large data. We do not strive for the most general function spaces, and do not provide a construction of solutions. The construction of solutions, asymptotic behavior for small data, and analysis of related models, will be discussed in forthcoming works. We consider for simplicity the single species model but the same proof applies to multiple species.

Some ideas of the proof and a comparison with the VM equations are given below. Unlike the VM equations, where the total force $F_L$ is divergence-free in $p$, $\mathrm{div}_p F_L = 0$, the radiative force's divergence

$$\mathrm{div}_p F_R \neq 0$$

is negative. Thus, unlike the VM case where $f$ is automatically bounded if initially so, in the RVM equations $f$ is not bounded uniformly and can (and will) grow in time. The danger is implosion, because the phase volume is contracting. On the other hand, the radiation reaction force causes the flux of the solution-averaged Lorentz factor to decay. Thus, the main danger of singularity formation in RVM, as opposed to VM, is coming not from high, but from low velocity. The radiation reaction force is used to obtain unconditional a priori bounds on the fluxes of momenta

$$\langle |v|[p]^n\rangle(x,t) = \int_{\mathbb{R}^3} |v|[p]^n f(x,p,t)\,dp \le M_n$$

(in our notation the Lorentz factor is $\gamma = [p] = \sqrt{1+|p|^2}$, with the normalized speed of light $c = 1$, the velocity is $v = p/[p]$ and $p$ is the momentum). These flux bounds are a direct consequence of unconditional a priori bounds on the particle distribution, which blow up like $|p|^{-3}$ near the origin, but decay exponentially at large $|p|$.

Once these bounds are obtained, we deduce "flux of energy"-type bounds on moments in terms of fluxes of moments and logarithms of gradients of $f$. Here we use the propagation of the condition $f(x, 0, t) = 0$ due to the annihilation of the contribution of the electric field at zero momentum. The moment bounds are then used in conjunction with the Glassey–Strauss method of representing the electromagnetic fields. We obtain bounds on the EM fields in terms of a choice of logarithms of gradients of $f$, in other words, in terms of a quantity

$$\min\left\{\log_+ \|\nabla_p f(t)\|_{L^\infty}, \sup_{s\le t} \log_+ \|\nabla_x f(s)\|_{L^\infty}\right\}.$$

The Glassey–Strauss representation for gradients is then used together with the EM bounds to obtain a priori estimates of the gradients of the EM fields. Finally, we apply the bounds on the EM fields and their gradients to bound the gradients of $f$, closing the argument. Ultimately, global regularity is a consequence of superlinear differential inequalities for the gradients of $f$, with doubly logarithmic nonlinearity.

The paper is organized as follows: After a section on notation and preliminaries (Section 2) where we describe the RVM equations, we make specific the form of the radiation reaction force $F_R$ and summarize its properties in Section 3. We recall the Glassey–Strauss representation in Section 4, and in Section 5 we derive moment bounds. In Section 6 we obtain bounds on the EM fields and in Section 7 we derive bounds for their gradients. In Section 8 we obtain the final gradient bounds on $f$ and conclude the proof of Theorem 1. In Appendix A we verify some properties of the Glassey–Strauss representation and in Appendix B we give the proofs of ODE lemmas.

## 2. Preliminaries: notation, the RVM equations

The radiative Vlasov–Maxwell equations are formed with the Vlasov equation

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_p(Ff) = 0 \tag{1}$$

with $f(x, p, t) \ge 0$, $(x, p, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ and

$$F = F_L + F_R \tag{2}$$

where $F_L$ is the Lorentz force

$$F_L = E + v \times B \tag{3}$$

and $F_R$ is the radiation reaction force, which will be discussed in the next section (see Definition 1). The velocity is denoted by $v$,

$$v = \frac{p}{\sqrt{1+|p|^2}} = \frac{p}{[p]}, \tag{4}$$

and the Lorentz factor $\gamma$ by $[p]$,

$$[p] = \sqrt{1+|p|^2}. \tag{5}$$

$E(x,t)$ and $B(x,t)$ are respectively the electric field and the magnetic field. They solve the Maxwell equations,

$$\begin{cases} \partial_t E - \nabla_x \times B = -j \\ \mathrm{div}_x E = \rho \\ \partial_t B + \nabla_x \times E = 0 \\ \mathrm{div}_x B = 0 \end{cases} \tag{6}$$

together with

$$\rho = \int f\,dp = \langle 1 \rangle \quad \text{and} \quad j = \int v f\,dp = \langle v \rangle. \tag{7}$$

Throughout the paper, for a function $\phi(x,p,t)$, we denote the solution average

$$\langle \phi \rangle(x,t) = \int \phi(x,p,t) f(x,p,t)\,dp. \tag{8}$$

The RVM equations are comprised of (1) with (2) and (6) with (7). Smooth solutions of RVM require the following compatibility conditions to be satisfied by the initial data: $f_0 \geq 0$,

$$\mathrm{div}_x E_0 = \int f_0\,dp \quad \text{and} \quad \mathrm{div}_x B_0 = 0. \tag{9}$$

## 3. The radiation reaction force

Let us write

$$\mathbf{K}(x,t) = (E(x,t), B(x,t)) \tag{10}$$

and

$$K^2 = |E|^2 + |B|^2 = |\mathbf{K}|^2. \tag{11}$$

Definition 1. In this paper, the *radiation reaction force* is

$$F_R(x,p,t) = -\chi(|p|)E(x,p,t) - MpK(x,t)$$

with $M > 2$ a constant. Here $0 \leq \chi \leq 1$ is a smooth cutoff,

$$\chi(r) = 1 \text{ for } r \leq R_0 \text{ and } \chi(r) = 0 \text{ for } r \geq R_1, |\chi'(r)| \leq 2.$$

Remark 1. Some of the examples of radiation reaction forces in the physical literature include ([18])

$$F_{LL} = -hv\gamma^2(|F_L|^2 - (v\cdot E)^2)$$

and the force due to inverse Compton scattering ([27])

$$F_{IC} = -hv\gamma^2 K^2.$$

The parameter $h > 0$ measures the relative intensity of the reaction, and is proportional to Planck's constant. These examples grow quadratically with the EM fields and vanish at $p = 0$. In the present work we use the term $-\chi E$ to mitigate the effect of the electric field at $p = 0$, and the linear growth of $F_R$ in the EM fields to close an a priori bound on the EM fields using a bootstrap argument. The form in Definition 1 was chosen for its simplicity, many other similar expressions, including modifications of $F_{LL}$ and $F_{IC}$ will provide the same effect.

The effect of our radiation reaction force as it pertains to regularity is as follows. Writing $\widehat{p} = p/|p|$, we find

$$\begin{aligned} F\cdot\widehat{p} &= (1-\chi(|p|))E\cdot\widehat{p} - MK|p| \\ &\le -K(x,t)(M|p| - (1-\chi(|p|))) \le 0 \end{aligned} \tag{12}$$

holds because

$$M \ge 2, \quad 1-\chi(r) \le 2r. \tag{13}$$

We note that

$$\mathrm{div}_p F_L = 0, \tag{14}$$

however, $\mathrm{div}_p F_R \neq 0$; in fact

$$-\mathrm{div}_p F = 3MK(x,t) + \chi'(|p|)E\cdot\widehat{p}. \tag{15}$$

We show in Section 5 that for large enough positive constants $A$,

$$\left(\frac{3}{|p|} + A\right) F\cdot\widehat{p} - \mathrm{div}_p F \le 0 \tag{16}$$

holds. This is a key property of $F$.

Observe that

$$|F(x,p,t)| \le (M+2)|p|K(x,t), \tag{17}$$

and differentiating, we find

$$|\nabla_p F(x,p,t)| + |\nabla_p\nabla_p F(x,p,t)| \le C(M+2)K(x,t). \tag{18}$$

Moreover,

$$|\nabla_x F(x,p,t)| \le C|p|(|\nabla_x E| + |\nabla_x B| + K(x,t)), \tag{19}$$

and

$$|\nabla_p\nabla_x F(x,p,t)| \le C(|\nabla_x E| + |\nabla_x B| + K(x,t)). \tag{20}$$

The properties (16)-(20) are sufficient to obtain global regularity.

## 4. On the Glassey–Strauss representation

Differentiating the Maxwell equations results in the wave equations

$$\Box E = -\partial_t j - \nabla_x \rho, \tag{21}$$

and

$$\Box B = \nabla_x \times j. \tag{22}$$

We write

$$\Box^{-1} g = \int_{|x-y|\le t} \frac{1}{|x-y|} g(y, t-|x-y|)dy. \tag{23}$$

We consider the the tangential derivatives $T_i$

$$T_i = \partial_i - \omega_i \partial_t \tag{24}$$

with $\omega = (y-x)/|y-x|$, which differentiate in directions parallel to the light cone,

$$T_i = \frac{\partial}{\partial y_i}(g(y, t-|x-y|)), \tag{25}$$

and the derivative

$$V = \partial_t - \omega \cdot \nabla_y \tag{26}$$

which differentiates in the running time $s$ along the light cone,

$$\frac{d}{ds} g(x+(t-s)\omega, s) = (Vg)(x+(t-s)\omega, s). \tag{27}$$

We note that

$$\omega \cdot T + V = 0. \tag{28}$$

Now we note that, if $g = Lh$ where $L$ is a vector field belonging to the linear span of $T_i$ and $V$ and of $h$ is bounded, then $\Box^{-1} g$ is bounded. This is done by integration by parts, using the representation (23) for $Vh$ and $Th$. The linear span can be with variable coefficients depending smoothly on $\omega$.

Glassey and Strauss [11] represent $E$ and $B$ using the linear wave equations and expressing $\partial_t$ and $\nabla_y$ as linear combinations of $S$ and $T_i$ where

$$S = \partial_t + v \cdot \nabla_y \tag{29}$$

is the streaming derivative, and where $T_i$ is the tangential derivative given in (24). The linear combinations are

$$\partial_i = T_i + \frac{\omega_i}{1+\omega\cdot v}(S - v\cdot T) \tag{30}$$

and

$$\partial_t = \frac{S - v\cdot T}{1+\omega \cdot v}. \tag{31}$$

This procedure results in two sets of expressions, one coming from the streaming derivative $S$ and one coming from the tangential derivatives $T_i$. The overall form is

$$\mathbf{K}(x,t) = (\mathbf{K}_T + \mathbf{K}_S)(x,t) + O(1) \tag{32}$$

where $O(1)$ represents a smooth function of $(x,t)$ which depends explicitly on the initial data. For the expressions coming from $S$, we have

$$\begin{aligned}\mathbf{K}_S(x,t) &= \int_{|x-y|\le t} a_S(\omega,v)(Sf)(y,p,t-|x-y|)\,dp\frac{dy}{|x-y|}\\ &= \int_0^t (t-s)\,ds\int_{|\omega|=1} a_S(\omega,v)(Sf)(x+(t-s)\omega,p,s)\,dp\;\,dS(\omega)\end{aligned} \tag{33}$$

where the kernel $a_S = a_S(\omega,v)$ is an explicit analytic tensor valued function satisfying

$$|\nabla_p a_S| \le C[p]. \tag{34}$$

The expressions coming from $T$ are

$$\begin{aligned}\mathbf{K}_T(x,t) &= \int_{|x-y|\le t} a_T(\omega,v)f(y,p,t-|x-y|)\,dp\frac{dy}{|x-y|^2}\\ &= \int_0^t ds\int_{|\omega|=1} a_T(\omega,v)f(x+(t-s)\omega,p,s)\,dp\,dS(\omega)\end{aligned} \tag{35}$$

where the kernel $a_T = a_T(\omega,v)$ is an explicit analytic tensor valued function satisfying

$$|a_T| \le C[p]. \tag{36}$$

For the gradient of the field, the representation ([11] Theorem 4), which is obtained via a similar procedure, has the form

$$\nabla_x\mathbf{K}(x,t) = ((\nabla_x\mathbf{K})_{TT} + (\nabla_x\mathbf{K})_{TS} + (\nabla_x\mathbf{K})_{SS})(x,t) + O(1) \tag{37}$$

where $O(1)$ represents a smooth function of $(x,t)$ which depends explicitly on the initial data. The terms are

$$(\nabla_x\mathbf{K})_{TT}(x,t) = \int_{|x-y|\le t} a_{TT}(\omega,v)f(y,p,t-|x-y|)\,dp\frac{dy}{|x-y|^3} \tag{38}$$

$$(\nabla_x\mathbf{K})_{TS}(x,t) = \int_{|x-y|\le t} a_{TS}(\omega,v)(Sf)(y,p,t-|x-y|)\,dp\frac{dy}{|x-y|^2} \tag{39}$$

$$(\nabla_x\mathbf{K})_{SS}(x,t) = \int_{|x-y|\le t} a_{SS}(\omega,v)(S^2f)(y,p,t-|x-y|)\,dp\frac{dy}{|x-y|}. \tag{40}$$

Above, the kernels $a_{TT}$, $a_{TS}$ and $a_{SS}$ are explicit tensor valued analytic functions which satisfy various properties (see [13] Lemma 4). In particular, their derivatives in $y$ and $p$ are bounded by powers of $[p]$.

## 5. Moment bounds

In this section we use the radiation reaction force to obtain bounds for moments

$$m_n(x,t) = \langle [p]^n \rangle = \int [p]^n f(x,p,t)\, dp. \tag{41}$$

The charge density $\rho$ corresponds to $m_0(x,t)$ and, as a consequence of the Vlasov equation (1), it obeys the conservation equation

$$\partial_t \rho + \mathrm{div}_x j = 0. \tag{42}$$

For higher moments, from the Vlasov equation (1), we have

$$\frac{\partial}{\partial t} m_n + \mathrm{div}_x \langle v[p]^n \rangle = n \langle (v \cdot F)[p]^{n-1} \rangle, \tag{43}$$

where we used

$$v = \nabla_p [p] \tag{44}$$

and integrated by parts $\int [p]^n \mathrm{div}_p (Ff) dp$. A key element of the proof is provided by the unconditional a priori control of the fluxes $vm_n$ of the moments $m_n$,

$$vm_n(x,t) = \int f(x,p,t)|v|[p]^n dp = \langle |v|[p]^n \rangle \tag{45}$$

in terms of the initial data.

Theorem 2. *Let $(f,E,B)$ be a smooth solution of the RVM equations on $[0,T]$. Assume that there exists constant $C_0$ such that*

$$0 \le |p|^3 f_0(x,p) \exp(A|p|) \le C_0$$

*holds for some*

$$A \ge \frac{3+2R_0}{(M-2)(R_0)^2}.$$

*Then, for any $n \ge 0$*

$$\sup_{0 \le t \le T} \|vm_n(\cdot,t)\|_{L^\infty} \le M_n$$

*holds with constants $M_n$ depending explicitly only on $n$, $A$ and $C_0$.*

Theorem 2 is a corollary of the a priori estimate:

Theorem 3. *Let $(f,E,B)$ be a smooth solution of the RVM equations on $[0,T]$. Assume that there exists a constant $C_0$ such that*

$$0 \le |p|^3 f_0(x,p) \exp A|p| \le C_0 \tag{46}$$

*holds for some*

$$A \ge \frac{3+2R_0}{(M-2)(R_0)^2}. \tag{47}$$

*Then,*

$$0 \le f(x,p,t) \le C_0 |p|^{-3} \exp(-A|p|) \tag{48}$$

*holds for* $t \le T$.

Proof. The path map is defined by the ordinary differential equations

$$\begin{cases} \dfrac{dX}{dt}(a,\pi,t) = v(P(a,\pi,t)), & X(a,\pi,0) = a, \\ \\ \dfrac{dP}{dt}(a,\pi,t) = F(X(a,\pi,t), P(a,\pi,t), t), & P(a,\pi,0) = \pi. \end{cases} \tag{49}$$

These represent the characteristic curves of the operator

$$D_t = \partial_t + v \cdot \nabla_x + F \cdot \nabla_p. \tag{50}$$

Note that

$$|X(a,\pi,t) - a| < t, \tag{51}$$

because $|v| < 1$. This property implies that the decay of $f$ at spatial infinity is controlled for finite time, as long as $F$ is Lipschitz continuous.

We fix a single characteristic $X(a,\pi,t)$ and $P(a,\pi,t)$. The equation (1) implies

$$\begin{aligned} &\frac{d}{dt} f(X(a,\pi,t), P(a,\pi,t), t) = \\ &\qquad -(\mathrm{div}_p F(X(a,\pi,t), P(a,\pi,t), t)) f(X(a,\pi,t), P(a,\pi,t), t). \end{aligned} \tag{52}$$

For the purpose of economy of notation, let us write

$$r(t) = |P(a,\pi,t)|, \tag{53}$$

for the momentum magnitude,

$$k(t) = K(X(a,\pi,t), t), \tag{54}$$

for the field strength and

$$f(t) = f(X(a,\pi,t), P(a,\pi,t), t) \tag{55}$$

for the probability density on characteristics. These quantities depend on initial data $a$ and $\pi$.

In view of (15), (52) results in

$$\frac{d}{dt} \log f(t) \le (3M + |\chi'(r(t))|) k(t). \tag{56}$$

For further economy, we suppress that $r, k, f$ are evaluated at $t$. Using (12) we have

$$\begin{aligned}\frac{dr}{dt} &\leq -Mkr + (1-\chi(r))k \\ &\leq -k(Mr - (1-\chi(r))) \\ &\leq 0\end{aligned} \tag{57}$$

where we use the facts that $M \geq 2$ and $(1-\chi(r)) \leq |\chi'(r)|r \leq 2r$. Let us consider the function

$$\Phi(r) = Ar + \log r^3. \tag{58}$$

We have that

$$\begin{aligned}\frac{d}{dt}(\Phi(r) + \log f) &= \Phi'(r)\frac{dr}{dt} + \frac{d}{dt}\log f \\ &\leq -\left(A + \frac{3}{r}\right)(Mkr - (1-\chi)k) + 3Mk + |\chi'|k \\ &\leq -A(Mkr - (1-\chi)k) + \frac{3}{r}(1-\chi)k + |\chi'|k \\ &\leq 0.\end{aligned} \tag{59}$$

The last inequality follows because $A$ is large enough (47). Indeed, the supports of $\chi'$ and of $(1-\chi)$ are included in $r \geq R_0$, and

$$Mkr - (1-\chi)k \geq k(M-2)r \geq k(M-2)R_0 \tag{60}$$

there, while $k((1-\chi)3/r + |\chi'|) \leq k(3/R_0 + 2)$. We deduce that

$$\frac{d}{dt}(r^3 f \exp Ar) \leq 0. \tag{61}$$

We obtained on each characteristic

$$\begin{aligned}&|P(a,\pi,t)|^3 f(X(a,\pi,t), P(a,\pi,t), t) \\ &\qquad \leq (f_0(a,\pi)|\pi|^3 \exp A|\pi|) \exp(-A|P(a,\pi,t)|).\end{aligned} \tag{62}$$

Straightfoward from (52) and $f_0 \geq 0$ is

$$f(X(a,\pi,t), P(a,\pi,t), t) \geq 0. \tag{63}$$

Reading (62) and (63) at $x = X(a,\pi,t)$, $p = P(a,\pi,t)$ where $(x,p,t)$ is arbitrary in view of the fact that the flow map is invertible (due to the inverse map theorem of Hadamard, see e.g. [26]) we deduce (48). □

We show that bounds on moment fluxes imply bounds on moments which depend logarithmically on gradients of $f$ in either $x$ or $p$. We define

$$G_1(t) = \sup_{0\leq s\leq t}\sup_{x,p} |\nabla_x f(x,p,s)| + 2, \tag{64}$$

and

$$G_2(t) = \sup_{x,p} |\nabla_p f(x,p,t)| + 2. \tag{65}$$

THEOREM 4. *Let $(f, E, B)$ be a smooth solution of the RVM equations on $[0, T]$. Assume that (48) holds and that the initial data satisfies*

$$f_0(x, 0) = 0.$$

*Then,*

$$m_n(x,t) \le CM_n + C_n \log G_2(t)$$

*holds for $t \le T$ with a constant $C_n$ depending continuously and explicitly only on $n$ and initial data.*

PROOF. We note first that $f(x, 0, t) = 0$ holds as long as the solution is smooth (because both $v$ and $F$ vanish at $p = 0$). Then, we write

$$\int f(x,p,t)\,dp = \int_{|p|\le R} (f(x,p,t) - f(x,0,t))\,dp + \int_{|p|\ge R} f(x,p,t)\,dp. \tag{66}$$

Using (48) which implies that $\int_{|p|\ge R} f(x,p,t)dp \le C_0 \log \frac{1}{R} + \frac{C_0}{A}$, and optimizing in $R$ we obtain

$$\rho(x,t) \le C \log G_2(t). \tag{67}$$

We have proved the claim for $m_0 = \rho$. For higher moments, we observe

$$\begin{aligned} m_n(x,t) &\le \sqrt{2}(vm_n(x,t)) + (\sqrt{2})^n \int_{|p|\le 1} f(x,p,t)dp \\ &\le \sqrt{2}(vm_n(x,t)) + (\sqrt{2})^n \rho(x,t). \end{aligned} \tag{68}$$

So, the bound on $m_0$ implies bounds on all higher moments, in view of Theorem 2. □

We estimate in terms of $G_1$ the space-time average of $m_n$,

$$\overline{m}_n(x,t) = \frac{1}{4\pi t} \int_0^t \int_{|\omega|=1} m_n(x + (t-s)\omega, s)\,dS(\omega)ds. \tag{69}$$

Let us denote the region

$$\Gamma(x,t) = \{(y,s) \,:\, 0 \le s \le t, |x-y| \le t-s\}. \tag{70}$$

Fixing $n$ and the vertex $(x,t)$, we consider the quantity

$$Q(s) = \int_{|x-y|\le t-s} m_n(y,s) \frac{dy}{|x-y|^2} \tag{71}$$

and take the time derivative. Differentiating, we find

$$\frac{dQ}{ds} = -\frac{1}{(t-s)^2}\int_{|x-y|=t-s} m_n(y,s)\,dS(y) + \int_{|x-y|\le t-s} \frac{\partial m_n}{\partial s}(y,s)\frac{dy}{|x-y|^2}. \tag{72}$$

Then by the moment evolution law (43) and the property (17) of $F$

$$\begin{aligned}
&\int_{|x-y|\le t-s} \frac{\partial m_n}{\partial s}(y,s)\frac{dy}{|x-y|^2}\\
&\qquad = \int_{|x-y|\le t-s} n\langle (v\cdot F)[p]^{n-1}\rangle(y,s) - \mathrm{div}_y\langle v[p]^n\rangle(y,s)\frac{dy}{|x-y|^2}\\
&\qquad \le \int_{|x-y|\le t-s} C_nK(y,s) - \mathrm{div}_y\langle v[p]^n\rangle(y,s)\frac{dy}{|x-y|^2}
\end{aligned} \tag{73}$$

where $C_n$ depends only on $n$ and the a priori moment flux bound $M_n$ in Theorem 2. Then, integrating by parts

$$\begin{aligned}
&-\int_{|x-y|\le t-s} \mathrm{div}_y\langle v[p]^n\rangle(y,s)\frac{dy}{|x-y|^2}\\
&= -\frac{1}{(t-s)^2}\int_{|x-y|=t-s} \omega\cdot\langle v[p]^n\rangle(y,s)\,dS(y)\\
&\quad + P.V.\int_{|x-y|\le t-s} \frac{2}{|y-x|^3}\omega\cdot\langle v[p]^n\rangle(y,s)\,dy\\
&\quad + \lim_{\varepsilon\to 0}\langle v[p]^n\rangle(x,s)\cdot\int_{|\omega|=1} \omega\,dS(\omega).
\end{aligned} \tag{74}$$

We observe that on the right hand side of the equality above, the first term is bounded by $M_n$, and the last term vanishes. Then integrating the equation (72) with respect to $ds$ with (74) and (73) in hand yields upon dividing by $4\pi t$

$$\begin{aligned}
&\overline{m}_n(x,t)\\
&\le C_n + C_n\frac{1}{t}\int_0^t \|\mathbf{K}(s)\|_{L^\infty}\,ds + \frac{1}{2\pi t}P.V.\int_{\Gamma(x,t)} \frac{1}{|x-y|^3}\omega\cdot\langle v[p]^n\rangle\,dy\,ds.
\end{aligned} \tag{75}$$

Here, $t>0$ and $C_n$ depends only on $n$ and the initial data. Indeed, $(4\pi t)^{-1}Q(0)$ is bounded uniformly in $(x,t)$ for smooth data.

Theorem 5. *Let $(f,E,B)$ be a smooth solution of the RVM equations on $[0,T]$. Assume that the initial data $f_0(x,p)$ obeys $f_0(x,0)=0$ and the decay* *condition* (46). *Then*

$$\overline{m}_n(x,t) \le C_n \frac{1}{t}\int_0^t K_\infty(s)ds + C_T(1+\log G_1(t))$$

*and*

$$\overline{m}_n(x,t) \le C_n(1+\log G_2(t))$$

*holds for* $t \le T$ *with constant* $C_n$ *depending continuously and explicitly only on* $n$ *and initial data and* $C_T$ *depending on* $n$*, initial data and* $T$*.*

Proof. The bound for $\overline{m}_n$ in terms of $G_2$ is an immediate consequence of Theorem 4.

To show the bound for $\overline{m}_n$ in terms of $G_1$, we estimate the principal value integral in (75) as follows. For fixed $s$, we split the spatial integral into the regions $|x-y| \le \delta$ and $\delta \le |x-y| \le t-s$. The value $\delta = \delta(s)$ is chosen below.

The integral on $\delta \le |x-y| \le t-s$ is bounded by

$$\left|\int_{\delta\le|x-y|\le t-s} \frac{1}{|x-y|^3}\omega\cdot\langle v[p]\rangle(s)\,dy\right| \le CM_n \log\left(\frac{t-s}{\delta}\right). \tag{76}$$

For $|x-y|\le\delta$ and $|p| \ge |x-y|^{-\kappa}$, we evaluate

$$\int_{|p|\ge|x-y|^{-\kappa}} |v|[p]^n f(y,p,s)dp \le |x-y|^{k\kappa} v m_{n+k}(y,s) \tag{77}$$

and thus the contribution of this term is bounded,

$$\begin{aligned}\left|\int_{|x-y|\le\delta}\frac{1}{|x-y|^3}\omega\cdot\int_{|p|\ge|x-y|^{-\kappa}} v[p]^n f(y,p,s)\,dp\,dy\right| \\ \le CM_{n+k}\int_{|x-y|\le\delta}|x-y|^{k\kappa-3}\,dy \\ \le CM_{n+k}\delta^{k\gamma}.\end{aligned} \tag{78}$$

We are left with the integral for $|x-y| \le \delta$ and $|p| \le |x-y|^{-\kappa}$. Because the unit sphere average $\int_{|\omega|=1}(\omega\cdot v)f(x,p,s)dS(\omega)$ vanishes, we have

$$\begin{aligned}\left|\int_{|x-y|\le\delta}\frac{1}{|x-y|^3}\omega\cdot\int_{|p|\le|x-y|^{-\kappa}} v[p]^n f(y,p,s)\,dp\,dy\right| \\ \le C\sup_{y,p}|\nabla f(s)|\int_{|x-y|\le\delta}\frac{dy}{|x-y|^2}\int_{|p|\le|x-y|^{-\kappa}}[p]^n\,dp \\ \le C\sup_{y,p}|\nabla f(s)|\delta^{1-(n+3)\kappa}.\end{aligned} \tag{79}$$

By choosing $0 < \kappa < \frac{1}{n+3}$ and $\delta = (t-s)/(2+\sup_{y,p}|\nabla_x f(s)|)$, we find that the time average of the principal value integral is bounded as

$$\left|\frac{1}{2\pi t}\int_0^t P.V.\int_{|x-y|\le t-s}\frac{1}{|x-y|^3}\omega\cdot\langle v[p]\rangle(s)\,dy\,ds\right| \le C_T(1+\log G_1(t)). \tag{80}$$

With this estimate and inequality (75), we have shown the bound in terms of $G_1$. □

## 6. Electromagnetic field bounds

Theorem 6. *Let $(f, E, B)$ be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys $f_0(x, 0) = 0$ and the decay condition (46). Let*

$$K_\infty(t) = \sup_{0\le s\le t}\|\mathbf{K}(\cdot, s)\|_{L^\infty}.$$

*Then*

$$K_\infty(t) \le C_1\left(1+\min\{\log G_1(t), \log G_2(t)\}\right)$$

*holds for $t \le T$ with a constant $C_1$ depending continuously and explicitly only on initial data and $T$.*

Proof. We use the Glassey–Strauss representation (32) for $\mathbf{K}$ and bound the integrals $\mathbf{K}_S$ and $\mathbf{K}_T$.

To bound the integral $\mathbf{K}_S$ with kernel $a_S$, we first use the Vlasov equation (1), $Sf = -\mathrm{div}_p(Ff)$, to integrate by parts in $p$, so

$$\int a_S Sf\,dp = \int(\nabla_p a_S)Ff\,dp \tag{81}$$

pointwise in $(y, s)$. Then, properties (17) and (34) imply

$$\begin{aligned}\left|\int a_S Sf\,dp\right| &\le C\int [p]|p||\mathbf{K}|f\,dp\\ &\le CM_2\|\mathbf{K}(s)\|_{L^\infty}\end{aligned} \tag{82}$$

because $|p|[p] = |v|[p]^2$. Therefore, $\mathbf{K}_S$ has the bound

$$\begin{aligned}\left|\int_{|x-y|\le t} a_S Sf\,dp\frac{dy}{|x-y|}\right| &\le CM_2\int_0^t (t-s)\|\mathbf{K}(s)\|_{L^\infty}\,ds\\ &\le CM_2T\int_0^t\|\mathbf{K}(s)\|_{L^\infty}\,ds.\end{aligned} \tag{83}$$

To bound the integral $\mathbf{K}_T$ with kernel $a_T$, we use Theorem 5 because $\langle a_T\rangle$ does not generally have a pointwise bound by a moment flux. In particular, property (36) implies

$$\left|\int_{|x-y|\le t} a_T f\,dp\frac{dy}{|x-y|^2}\right| \le CT\,\overline{m}_1(x, t) \tag{84}$$

pointwise in $(x, t)$ and then we apply Theorem 5 for $n = 1$ using the bound in terms of either $G_1$ or $G_2$. Using the bound in terms of $G_2$, we obtain

$$\left| \int_{|x-y|\le t} a_T f \, dp \frac{dy}{|x-y|^2} \right| \le C_1 M_1 T \log G_2(t). \tag{85}$$

On the other hand, from the bound in terms of $G_1$ we have

$$\left| \int_{|x-y|\le t} a_T f \, dp \frac{dy}{|x-y|^2} \right| \le C_1 \int_0^t \|\mathbf{K}(s)\|_{L^\infty} \, ds + C_1 \log G_1(t). \tag{86}$$

To conclude, we apply the estimate (83) for $\mathbf{K}_S$ with either estimate (85) or (86) for $\mathbf{K}_T$ in the Glassey–Strauss representation, and use the Grönwall inequality. □

## 7. Gradient bounds for electromagnetic fields

Now that we know the bounds for the moments in Theorem 2 and the uniform $L^\infty$ bound on $\mathbf{K}$ in Theorem 6, we can use the Glassey–Strauss representations (37) for the spatial gradients of $E$ and $B$ which we denote by $\nabla_x \mathbf{K}$.

Theorem 7. *Let $(f, E, B)$ be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x, p)$ obeys $f_0(x, 0) = 0$ and the decay condition (46). Then*

$$\|\nabla_x \mathbf{K}(\cdot, t)\|_{L^\infty} \le C_1 \log G_1(t) \log G_2(t)$$

*holds for $t \le T$ with a constant $C_1$ depending continuously and explicitly only on initial data and $T$.*

Proof. We use the representation (37) for the gradient $\nabla_x \mathbf{K}$ and bound the integrals $(\nabla_x \mathbf{K})_{TT}$, $(\nabla_x \mathbf{K})_{TS}$ and $(\nabla_x \mathbf{K})_{SS}$.

The simplest term to bound is the integral $(\nabla_x \mathbf{K})_{TS}$ whose kernel $a_{TS}$ satisfies ([13], Lemma 4)

$$|\nabla_p a_{TS}| \le C[p]^4. \tag{87}$$

After using $Sf = -\mathrm{div}_p(Ff)$ to integrate by parts, we find

$$\int a_{TS} Sf \, dp = \int (\nabla_p a_{TS}) Ff \, dp. \tag{88}$$

The properties (17) and (87) then imply

$$\begin{aligned} \left| \int a_{TS} Sf \, dp \right| &\le C \int [p]^4 |p| |\mathbf{K}| f \, dp \\ &\le C M_5 \|\mathbf{K}(s)\|_{L^\infty} \end{aligned} \tag{89}$$

with the fact $|p|[p]^4 = |v|[p]^5$. Therefore, $(\nabla_x \mathbf{K})_{TS}$ has the bound

$$\left|\int_{|x-y|\le t} a_{TS} Sf\, dp \frac{dy}{|x-y|^2}\right| \le CM_5 \int_0^t \|\mathbf{K}(s)\|_{L^\infty}\, ds \le CM_5 T K_\infty(t). \tag{90}$$

In order to bound $(\nabla_x \mathbf{K})_{SS}$, we first rewrite $S^2 f$ appealing twice to the Vlasov equation $Sf = -\mathrm{div}_p(Ff)$. Pointwise,

$$\begin{aligned} S(Sf) &= -S(\mathrm{div}_p(Ff)) \\ &= \nabla_x(Ff) : \nabla_p v - \mathrm{div}_p(S(Ff)) \\ &= \nabla_x(Ff) : \nabla_p v - \mathrm{div}_p(fSF) - \mathrm{div}_p(FSf) \\ &= \nabla_x(Ff) : \nabla_p v - \mathrm{div}_p(fSF) + \mathrm{div}_p(F\mathrm{div}_p(Ff)). \end{aligned} \tag{91}$$

We thus have three terms entering the expression of $(\nabla_x \mathbf{K})_{SS}$. For $n = 0, 1, 2$, the kernel $a_{SS}$ satisfies ([13], Lemma 4)

$$|\nabla_p^n a_{SS}| \le C[p]^4. \tag{92}$$

For the last term, integrating by parts in $p$ twice gives

$$\int a_{SS}\, \mathrm{div}_p(F\mathrm{div}_p(Ff))\, dp = \int F\cdot\nabla_p(F\cdot\nabla_p a_{SS}) f\, dp. \tag{93}$$

Then, properties (17), (18) and (92) imply

$$\begin{aligned} \left|\int a_{SS}\mathrm{div}_p(F\mathrm{div}_p(Ff))\, dp\right| &\le C\int |p|[p]^4(|\nabla_p F| + |F|)|\mathbf{K}| f\, dp \\ &\le C\int |v|[p]^5|\mathbf{K}|^2 f\, dp \\ &\le CM_5\|\mathbf{K}(s)\|_{L^\infty}^2. \end{aligned} \tag{94}$$

The bound for the last term is therefore

$$\begin{aligned} &\left|\int_{|x-y|\le t} a_{SS}\, \mathrm{div}_p(F\mathrm{div}_p(Ff)) dp \frac{dy}{|x-y|}\right| \\ &\qquad \le CM_5 \int_0^t (t-s)\|\mathbf{K}(s)\|_{L^\infty}^2 ds \\ &\qquad\qquad \le CM_5 (TK_\infty(t))^2 \end{aligned} \tag{95}$$

For the second term, integration by parts in $p$ yields

$$-\int a_{SS}\, \mathrm{div}_p(fSF)\, dp = \int (\nabla_p a_{SS})(SF) f\, dp. \tag{96}$$

From the Maxwell equations,

$$\begin{aligned} SE &= v\cdot\nabla_x E + \nabla_x \times B - j, \\ SB &= v\cdot\nabla_x B - \nabla_x \times E, \end{aligned} \tag{97}$$

and so from property (92), noting that $S\chi = 0$,

$$\begin{aligned}\left|\int a_{SS}\,\mathrm{div}_p(fSF)\,dp\right| &\le C\int |p|[p]^4(|SE|+|SB|)f\,dp\\ &\le CM_5(M_0+\|\nabla_x \mathbf{K}(s)\|_{L^\infty}).\end{aligned} \tag{98}$$

The bound for the second term is then

$$\begin{aligned}&\left|\int_{|x-y|\le t} a_{SS}\,\mathrm{div}_p(fSF)\,dp\frac{dy}{|x-y|}\right|\\ &\qquad\le CM_5\int_0^t (t-s)(M_0+\|\nabla_x \mathbf{K}(s)\|_{L^\infty})\,ds\\ &\qquad\qquad\le CM_5T\int_0^t (M_0+\|\nabla_x \mathbf{K}(s)\|_{L^\infty})\,ds\end{aligned} \tag{99}$$

For the first term, to integrate by parts the quantity

$$\nabla_x(Ff) : \nabla_p v = \partial_i(F_j f)\frac{\partial v_i}{\partial p_j} \tag{100}$$

where $\partial_i = \partial/\partial x_i$, we recall the decomposition of derivatives

$$\partial_i = T_i + \left(\frac{\omega_i}{1+v\cdot\omega}\right)(v\cdot T - S). \tag{101}$$

Repeated indices indicate summation. We then write

$$a_{SS}(\nabla_x(Ff) : \nabla_p v) = A^{ij}T_i(F_j f) + b^j S(F_j f) \tag{102}$$

as the sum of two expressions.

The latter expression is

$$b^j S(F_j f) = \frac{\partial v_i}{\partial p_j}\left(\frac{\omega_i}{1+v\cdot\omega}\right) a_{SS} S(F_j f) \tag{103}$$

which becomes

$$b^j S(F_j f) = b^j F_j S f + b^j f S F_j. \tag{104}$$

Observe that each term on the right hand side above may be treated in a similar fashion to terms previously discussed; we use the Vlasov equation to integrate by parts in $p$ and use property (17) to deduce

$$\left|\int_{|x-y|\le t} b^j F_j S f\,dp\frac{dy}{|x-y|}\right| \le CM_6T^2K_\infty(t), \tag{105}$$

and we use properties (18) and (97) to arrive at

$$\left|\int_{|x-y|\le t} b^j f S F_j\,dp\frac{dy}{|x-y|}\right| \le CM_6T\int_0^t (M_0+\|\nabla_x \mathbf{K}(s)\|_{L^\infty})\,ds \tag{106}$$

The former expression is

$$A^{ij}T_i(F_j f) = \frac{\partial v_i}{\partial p_j}\left(T_i + \frac{\omega_i}{1+v\cdot\omega}v\cdot T\right)(F_j f). \tag{107}$$

Each $T_i$ is a total $y$ derivative, and so integrating by parts in $y$ gives

$$\int_{|x-y|\le t} A^{ij} T_i(F_j f)\, dp \frac{dy}{|x-y|} = -\int_{|x-y|\le t} \tilde{A}^j(F_j f)\, dp \frac{dy}{|x-y|^2} + O(1) \quad (108)$$

where $O(1)$ represents a function of $(x,t)$ which depends explicitly on the initial data. On the right hand side is the kernel $\tilde{A}^j = r^2\partial/\partial y_i(A^{ij}/r)$ where $r = |x-y|$, which in particular satisfies $|\tilde{A}^j| \le C[p]^4$ (see [13] Lemma 4). The estimate for this expression is then by property (17)

$$\begin{aligned} \left| \int_{|x-y|\le t} A^{ij} T_i(F_j f)\, dp \frac{dy}{|x-y|} \right| &\le C \int_{|x-y|\le t} |v|[p]^5 |\mathbf{K}| dp \frac{dy}{|x-y|^2} + O(1) \\ &\le C M_5 \int_0^t \|\mathbf{K}(s)\|_{L^\infty}\, ds + O(1) \\ &\le C_0(1 + M_5 T K_\infty(t)) \end{aligned} \quad (109)$$

where $C_0$ depends only on the initial data. Taking together (109), (105) and (106) gives us a bound on the first term entering the expression of $(\nabla_x \mathbf{K})_{SS}$, while the second and last term have bounds (99) and (95).

Therefore, $(\nabla_x \mathbf{K})_{SS}$ has the bound

$$\left| \int_{|x-y|\le t} a_{SS}(S^2 f)\, dp \frac{dy}{|x-y|} \right| \le C_T \left( 1 + K_\infty(t)^2 + \int_0^t \|\nabla_x \mathbf{K}(s)\|_{L^\infty} ds \right) \quad (110)$$

where $C_T$ depends only on the initial data and $T$.

To bound $(\nabla_x \mathbf{K})_{TT}$, we write the integral as

$$(\nabla_x \mathbf{K})_{TT}(x,t) = \int_0^t \frac{ds}{t-s} \int_{|\omega|=1} a_{TT}(\omega, v) f(x + (t-s)\omega, p, s) dp\, dS(\omega) \quad (111)$$

We split the integral on the backwards light cone into two pieces: the base piece on $0 \le s \le t-\delta$, and tip piece on $t-\delta \le s \le t$, where $\delta$ is chosen below. The properties of the kernel $a_{TT}$ ([13], Lemma 4),

$$|a_{TT}| \le C[p]^3 \quad (112)$$

and

$$\int_{|\omega|=1} a(v,\omega) dS(\omega) = 0, \quad (113)$$

imply for the base piece

$$\left| \int_0^{t-\delta} \frac{ds}{t-s} \int_{|\omega|=1} a_{TT} f\, dp\, dS(\omega) \right| \le C(M_4 + \log G_2(t)) \log\left(\frac{t}{\delta}\right). \quad (114)$$

For the tip piece, we first note

$$
\begin{aligned}
\left|\int_{|p|\geq(t-s)^{-\kappa}} a_{TT} f\, dp\right| &\leq C \int_{|p|\geq(t-s)^{-\kappa}} [p]^3 f\, dp \\
&\leq C(t-s)^{\alpha} \int_{|p|\geq(t-s)^{-\kappa}} |p|^{\frac{\alpha}{\kappa}} [p]^3 f\, dp \\
&\leq CM_n (t-s)^{\alpha}
\end{aligned}
\tag{115}
$$

where $n = \lceil 4 + \alpha/\kappa \rceil$, and $\alpha, \kappa$ are numbers chosen freely. We let $\alpha > 0$ so that

$$
\left|\int_{t-\delta}^{t} \frac{ds}{t-s} \int_{|\omega|=1} \int_{|p|\geq(t-s)^{-\kappa}} a_{TT} f\, dp\, dS(\omega)\right| \leq CM_n \delta^{1-\alpha}. \tag{116}
$$

Then, we choose $\kappa < \frac{1}{6}$ such that

$$
\begin{aligned}
&\left|\int_{t-\delta}^{t} \frac{ds}{t-s} \int_{|\omega|=1} \int_{|p|\leq(t-s)^{-\kappa}} a_{TT} f\, dp\, dS(\omega)\right| \\
&\qquad\leq \sup_{s\leq t} \sup_{x,p} |\nabla_x f(x,p,s)| \int_{t-\delta}^{t} \int_{|p|\leq(t-s)^{-\kappa}} [p]^3\, dp \\
&\qquad\leq\ \delta^{1-6\kappa} \sup_{s\leq t} \sup_{x,p} |\nabla_x f(x,p,s)|.
\end{aligned}
\tag{117}
$$

With $\alpha = 1$, we choose here $\delta = t(2 + \sup_{s\leq t,x,p} |\nabla_x f(x,p,s)|)^{-1/(1-6\kappa)}$ in view of the above.

Therefore, we have the following bound for $(\nabla_x \mathbf{K})_{TT}$

$$
\left|\int_{0}^{t} \frac{ds}{t-s} \int_{|\omega|=1} a_{TT} f\, dp\, dS(\omega)\right| \leq C_T \log G_1(t) \log G_2(t). \tag{118}
$$

Putting together estimates (90), (110) and (118), we obtain

$$
|\nabla_x \mathbf{K}(x,t)| \leq C_T \left( \log G_1(t) \log G_2(t) + \int_0^t \|\nabla_x \mathbf{K}(s)\|_{L^\infty}\, ds \right) \tag{119}
$$

where we chose to bound $K_\infty^2$ by the product $C \log G_1 \log G_2$ in view of Theorem 5. Using the Grönwall inequality, we conclude the proof. ☐

## 8. Proof of Theorem 1

THEOREM 8. *Let $(f, E, B)$ be a smooth solution of the RVM equations on $[0, T]$. Assume that the initial data $f_0(x,p)$ obeys $f_0(x,0) = 0$ and the decay condition (46). Then*

$$
\|\nabla_x f(\cdot,t)\|_{L^\infty} + \|[p]|\nabla_p f(\cdot,t)|\|_{L^\infty} \leq C \exp(C \exp(Ct))
$$

*holds for $t \leq T$ with a constant $C$ depending continuously and explicitly only on initial data.*

Proof. We consider the quantities

$$W(t) = \sup_{s\le t} \|\nabla_x f(s)\|_{L^\infty} + 3 \tag{120}$$

and

$$Z(t) = \sup_{s\le t} \||p||\nabla_p f(s)| + (1+|p|)f(s)\|_{L^\infty} + 3. \tag{121}$$

Below we show $W$ and $Z$ obey the certain differential inequalities. We write (1) as

$$D_t f = -(\mathrm{div}_p F) f \tag{122}$$

and take derivatives in $x$ and in $p$:

$$D_t(\partial_{x_i} f) = -(\partial_{x_i} F)\cdot \nabla_p f - (\mathrm{div}_p F)(\partial_{x_i} f) - (\partial_{x_i}(\mathrm{div}_p F)) f \tag{123}$$

and

$$D_t(\partial_{p_i} f) = -(\partial_{p_i} v)\cdot \nabla_x f - (\mathrm{div}_p F)(\partial_{p_i} f) - (\partial_{p_i} F)\cdot \nabla_p f - (\partial_{p_i}(\mathrm{div}_p F)) f \tag{124}$$

We deduce inequalities for quantities

$$w = |\nabla_x f| + 3 \tag{125}$$

and

$$z = (1+|p|)f + |p||\nabla_p f| + 3. \tag{126}$$

Using the estimates (17), (18), (19) and (20), we find that

$$D_t w \le C(Kw + (K + |\nabla_x \mathbf{K}|)z) \tag{127}$$

and

$$D_t z \le C(w + Kz) \tag{128}$$

from equations (123) and (124).

To see this, first multiply the equation (123) by $\partial_{x_i} f/|\nabla_x f|$ and add in $i$ to obtain,

$$\begin{aligned} D_t|\nabla_x f| &\le |\mathrm{div}_p F||\nabla_x f| + |\nabla_x F||\nabla_p f| + |\nabla_x \mathrm{div}_p F| f \\ &\le C(K|\nabla_x f| + (K + |\nabla_x \mathbf{K}|)(|p||\nabla_p f| + |f|)) \\ &\le C(Kw + (K + |\nabla_x \mathbf{K}|)z). \end{aligned} \tag{129}$$

This implies (127). Then, we multiply (124) by $|p|\partial_{p_i} f/|\nabla_p f|$ and add in $i$ to obtain

$$\begin{aligned} |p| D_t|\nabla_p f| &\le 2|\nabla_x f| + |p||\mathrm{div}_p F||\nabla_p f| + |p||\nabla_p \mathrm{div}_p F| f \\ &\le 2|\nabla_x f| + C(K|p||\nabla_p f| + K|p| f) \\ &\le C(w + Kz). \end{aligned} \tag{130}$$

We used $|p||\nabla_p v| < 2$, which is immediate from (144) and $|v| < 1$. This implies, with (122), the estimate (128). Now we have that

$$W(t) = \sup_{s\le t} \|w(s)\|_{L^\infty}, \tag{131}$$

and

$$Z(t) = \sup_{s\le t} \|z(s)\|_{L^\infty}. \tag{132}$$

Taking the supremum in time of (127) and (128), we find

$$\begin{aligned} &\sup_{s\le t} \|D_t w(s)\|_{L^\infty} \\ &\le C\left(K_\infty(t)W(t) + \left(K_\infty(t) + \sup_{s\le t}\|\nabla_x \mathbf{K}(s)\|_{L^\infty}\right) Z(t)\right) \end{aligned} \tag{133}$$

and

$$\sup_{s\le t} \|D_t z(s)\|_{L^\infty} \le C(K_\infty(t)Z(t) + W(t)) \tag{134}$$

We use now

LEMMA 1. *Let $g = g(t)$ be a positive Lipschitz function of $t \in [0, T]$ and let $G(t) = \sup_{s\le t} g(s)$. Then, $G = G(t)$ is Lipschitz and*

$$\limsup_{h\to 0} \frac{G(t+h) - G(t)}{h} \le \liminf_{\varepsilon\to 0} \sup_{s\le t+\varepsilon} |g'(s)|.$$

By Lemma 1, differentiation under $\sup_{s\le t}$ for Lipschitz functions of time is permissible. We have thus

$$\frac{dW}{dt} \le \liminf_{\varepsilon\to 0} \sup_{s\le t+\varepsilon} \|D_t w(s)\|_{L^\infty} \tag{135}$$

and

$$\frac{dZ}{dt} \le \liminf_{\varepsilon\to 0} \sup_{s\le t+\varepsilon} \|D_t z(s)\|_{L^\infty} \tag{136}$$

holds for almost all $t$. Then, using Theorem 6 and Theorem 7 and the continuity of the upper bounds allowing to set $\varepsilon = 0$, we arrive at the ODE system

$$\frac{dW}{dt} \le C((\log W)W + (\log W)(\log Z)Z) \tag{137}$$

and

$$\frac{dZ}{dt} \le C((\log Z)Z + W). \tag{138}$$

We apply Lemma 2:

LEMMA 2. *Let $W = W(t)$ and $Z = Z(t)$ be nondecreasing, Lipschitz functions of $t \ge 0$. Let $W(0) = W_0$ and $Z(0) = Z_0$ and suppose*

$$\min\{\log W_0, \log Z_0\} \ge 1.$$

*Assume that $W(t)$ and $Z(t)$ obey differential inequalities*

$$\frac{dW}{dt} \le C((\log W)W + (\log W)(\log Z)Z)$$

*and*

$$\frac{dZ}{dt} \le C((\log Z)Z + W).$$

*Then the functions $W$ and $Z$ satisfy*

$$W + Z \le C\exp(C\exp(Ct))$$

*where $C$ depends only on $W_0$ and $Z_0$.*

Remark 2. In contrast, the ODE

$$\frac{dY}{dt} = Y(\log Y)^2$$

blows up in finite time.

□

The proof of Theorem 1 is completed now by applying the bounds of Theorem 8 to the bounds on the EM fields in Theorems 6 and 7.

## Appendix A: Checking the nonlinear Glassey–Strauss representation

Here we derive (33) and (35) and check the properties (34) and (36). The expressions for $E$ coming from $S$, $E_S$ are [11] p.63,

$$(E_S)_i = -\int dp \int_0^t \int_{|\omega|=1} \left(\frac{\omega_i + v_i}{1 + (\omega\cdot v)}\right)(Sf)(x - r\omega, p, t-r)\,r\,dr\,dS(\omega) \tag{139}$$

where $\omega = \widehat{y-x}$. Using the equation (1), denoting

$$N(y,p,s) = F(y,p,s)f(y,p,s), \tag{140}$$

and integrating by parts in (139) we obtain

$$(E_S)_i = -\int dp \int_0^t \int_{|\omega|=1} \partial_{p_j}\left(\frac{\omega_i + v_i}{1 + (\omega\cdot v)}\right)N_j(x - r\omega, p, t-r)\,r\,dr\,dS(\omega) \tag{141}$$

which we write as

$$(E_S)_i = -\int dp \int_0^t \frac{1}{t-s}\int_{|x-y|=t-s} N(y,s)\cdot\nabla_p\left(\frac{\omega_i + v_i}{1 + (\omega\cdot v)}\right)dS(y)\,ds \tag{142}$$

The expressions (142) for $E_S$ are nonlinear because they employ (1). The expression for $E_T$ [11] p. 63 is

$$(E_T)_i = -\int dp \int_0^t \frac{1}{(t-s)^2} \int_{|x-y|=t-s} f(y,s) \frac{1}{[p]^2} \left( \frac{\omega_i + v_i}{(1+(\omega\cdot v))^2} \right) dS(y) ds \tag{143}$$

Note that $E_T$ is linear in $f$, because it comes without use of the equation of evolution of $f$. There are analogous representations for $B$. The main point here is to verify (34) and (36). We observe that

$$\partial_{p_i} v_k = \frac{1}{\sqrt{1+|p|^2}} (\delta_{ik} - v_i v_k) = [p]^{-1} (\mathbb{I} - v\otimes v)_{ik} \tag{144}$$

and

$$|v|^2 = 1 - \frac{1}{[p]^2}. \tag{145}$$

We note the following facts. First,

$$\partial_{p_j} \left( \frac{1}{1+\omega\cdot v} \right) = \frac{v_j}{[p](1+\omega\cdot v)} - \frac{\omega_j + v_j}{[p](1+\omega\cdot v)^2} \tag{146}$$

and

$$\partial_{p_j} \left( \frac{\omega_i + v_i}{1+(\omega\cdot v)} \right) = \frac{1}{[p]} \frac{\left(\delta_{ij} + v_j\omega_i\right)}{1+(\omega\cdot v)} - \frac{1}{[p]} \frac{(\omega_i+v_i)(\omega_j+v_j)}{(1+(\omega\cdot v))^2}. \tag{147}$$

These are done by direct calculation, inserting $\omega + v$ terms. The second observation is that

$$\frac{|\omega+v|^2}{(1+(\omega\cdot v))^2} = \frac{(1-|v|)^2 + 2|v|\delta}{(1-|v|)^2 + |v|^2\delta^2 + 2(1-|v|)|v|\delta} \tag{148}$$

where

$$\delta = 1 + \omega\cdot\widehat{p} = 1 + \cos\theta. \tag{149}$$

Multiplying the numerator by $1-|v|$ and using $(1-|v|)^3 \le (1-|v|)^2$ in the numerator we see that the resulting fraction is less than 1, and therefore, after taking square roots we have,

$$\frac{|\omega+v|}{1+(\omega\cdot v)} \le \sqrt{2}[p] \tag{150}$$

where we used

$$(1-|v|)^{-1} = [p]^2(1+|v|) \le 2[p]^2. \tag{151}$$

Also, from $1+(\omega\cdot v) = 1+|v|\cos\theta \ge 1-|v|$ and (151) we have that

$$0 \le \frac{1}{1+\omega\cdot v} \le 2[p]^2. \tag{152}$$

Thus, the second term in (147) obeys

$$\left|\frac{1}{[p]}\frac{(\omega_i+v_i)(\omega_j+v_j)}{(1+(\omega\cdot v))^2}\right| \le 2[p] \tag{153}$$

and the first term in (147) is bounded in view of (152) by $4[p]$. This implies

$$\left|\partial_{p_j}\left(\frac{(\omega_i+v_i)}{1+(\omega\cdot v)}\right)\right| \le 6[p]. \tag{154}$$

Note also that

$$\left|\frac{1}{[p]^2}\left(\frac{\omega_i+v_i}{(1+(\omega\cdot v))^2}\right)\right| \le 2\sqrt{2}[p]. \tag{155}$$

We verified thus the bounds (34) and (36) in the representation of the electric field. After use of the equation (1) and integration by parts, the magnetic field representation [11] p.63, yields

$$B_S = \int dp\int_0^t \frac{1}{t-s}\int_{|x-y|=t-s} N(y,s)\cdot\nabla_p\left(\frac{\omega\times v}{1+(\omega\cdot v)}\right) dS(y)ds \tag{156}$$

From (146) and the inequalities (150) and (152) and because $|\omega\times v|\le|\omega+v|$ we have

$$\left|\nabla_p\left(\frac{\omega\times v}{1+(\omega\cdot v)}\right)\right| \le 10[p] \tag{157}$$

Finally, the representation of $B_T$ from [11] is

$$B_T = \int dp\int_0^t \frac{1}{(t-s)^2}\int_{|x-y|=t-s}\left(\frac{\omega\times v}{[p]^2(1+(\omega\cdot v))^2}\right) f(y,s)dS(y)ds \tag{158}$$

and we have

$$\left|\frac{\omega\times v}{[p]^2(1+(\omega\cdot v))^2}\right| \le 2\sqrt{2}[p], \tag{159}$$

concluding the verification of the inequalities (34) and (36).

## Appendix B: ODE Lemmas

We prove here Lemma 1 and Lemma 2.

Proof of Lemma 1. If $G(t)=g(s)$ with $s<t$, then $g(s')\le g(s)$ for all $s\le s'\le t$ (otherwise, $G(t)$ would have been attained at $s'$ not at $s$) and therefore $G(s')=g(s)$ for $s'\in[s,t]$ and the left derivative of $G'(t-0)$ of $G$ at $t$ vanishes. If $g(t)<G(t)$ then $G(s)=G(t)$ for a small interval of $s>t$ and so $G'(t)=0$.

If $g(t)=G(t)$ then for any $\varepsilon>0$ we have

$$g(s)-g(t)\le(s-t)L_\varepsilon \tag{160}$$

for all $t < s \le t+\varepsilon$, where $L_\varepsilon = \sup_{t\le s\le t+\varepsilon} |g'(s)|$. We take $0 < h < \varepsilon$, write $g(s) \le g(t) + hL_\varepsilon$ for $s \le t+h$ and take the supremum in $s$ to deduce

$$G(t+h) \le G(t) + hL_\varepsilon. \tag{161}$$

Thus $G'(t+0) \le L_\varepsilon$. Because $\varepsilon > 0$ is arbitrary, we have

$$G'(t+0) \le \liminf_{\varepsilon\to 0} L_\varepsilon.$$

Finally, if $G(t) = g(t)$ and $g(s) < G(t)$ for all $s < t$ then

$$G(s_0) \le G(t) + \sup_{s'\le t} |g'(s')|(t - s_0) \tag{162}$$

holds by taking supremum of

$$g(s) \le g(t) + \sup_{s'\le t} |g'(s')|(t - s) \tag{163}$$

for $s \le s_0 < t$. This concludes the argument. □

Proof of Lemma 2. Consider $\widetilde{Z} = Z\log Z$. The differential inequality for $W$ then reads

$$\frac{dW}{dt} \le C((\log W)W + (\log W)\widetilde{Z}) \tag{164}$$

and from the differential inequality for $Z$ we have

$$\frac{d\widetilde{Z}}{dt} \le C(\log Z + 1)(W + \widetilde{Z}). \tag{165}$$

Now take $\overline{W} = W + \widetilde{Z}$. Because $\widetilde{Z} \ge Z$ we have $\log Z \le \log\widetilde{Z} \le \log\overline{W}$. We also have $\log W \le \log\overline{W}$, so we obtain

$$\frac{d\overline{W}}{dt} \le C(\log\overline{W} + 1)\overline{W} \tag{166}$$

and thus $\overline{W}$ is bounded by a double exponential of time. □

**Acknowledgments.** The authors thank J. Burby, G. W. Hammett, M. W. Kunz and A. Spitkovsky for helpful discussions. The work of PC was partially supported by NSF grant DMS-2106528 and by a Simons Collaboration Grant 601960. The work of HG was supported by NSF grant DGE-2039656 and by the Ford Foundation.

## References

[1] Kiyoshi Asano. On local solutions of the initial value problem for the Vlasov–Maxwell equation. *Communications in Mathematical Physics*, 106:551–568, 1986.

[**2**] Léo Bigorgne. Sharp asymptotic behavior of solutions of the 3D Vlasov–Maxwell system with small data. *Communications in Mathematical Physics*, 376(2):893–992, 2020.

[**3**] Léo Bigorgne. Asymptotic properties of the solutions to the Vlasov–Maxwell system in the exterior of a light cone. *International Mathematics Research Notices*, 2021(5):3729–3793, 2021.

[**4**] Léo Bigorgne. Scattering map for the Vlasov–Maxwell system around source-free electromagnetic fields. *arXiv preprint arXiv:2312.12214*, 2023.

[**5**] Léo Bigorgne. Global existence and modified scattering for the solutions to the Vlasov–Maxwell system with a small distribution function. *Analysis & PDE*, 18(3):629–714, 2025.

[**6**] François Bouchut, François Golse, and Christophe Pallard. Classical solutions and the Glassey–Strauss theorem for the 3D Vlasov–Maxwell system. *Archive for Rational Mechanics and Analysis*, 170: 1–15, 2003.

[**7**] François Bouchut, François Golse, and Christophe Pallard. Nonresonant smoothing for coupled wave+transport equations and the Vlasov–Maxwell system. In *Dispersive Transport Equations and Multiscale Models*, pages 37–50. Springer, 2004.

[**8**] Emile Breton. Modified scattering for small data solutions to the Vlasov–Maxwell system: a short proof. *arXiv preprint arXiv:2503.01677*, 2025.

[**9**] Ronald J. DiPerna and Pierre-Louis Lions. Global weak solutions of Vlasov–Maxwell systems. *Communications on Pure and Applied Mathematics*, 42(6):729–757, 1989.

[**10**] Robert T. Glassey and Jack W. Schaeffer. Global existence for the relativistic Vlasov–Maxwell system with nearly neutral initial data. *Communications in Mathematical Physics*, 119(3):353–384, 1988.

[**11**] Robert T. Glassey and Walter A. Strauss. Singularity formation in a collisionless plasma could occur only at high velocities. *Archive for Rational Mechanics and Analysis*, 92:59–90, 1986.

[**12**] Robert T. Glassey and Walter A. Strauss. Absence of shocks in an initially dilute collisionless plasma. *Communications in Mathematical Physics*, 113(2):191–208, 1987.

[**13**] Robert T. Glassey and Walter A. Strauss. Large velocities in the relativistic Vlasov–Maxwell equations. *J. Fac. Sci. Univ. Tokyo Sect. IA Math*, 36(3):615–627, 1989.

[**14**] Daniel Han-Kwan, Toan T. Nguyen, and Frédéric Rousset. Linear landau damping for the Vlasov–Maxwell system in $\mathbb{R}^3$. *arXiv preprint arXiv:2402.11402*, 2024.

[**15**] Sergiu Klainerman and Gigliola Staffilani. A new approach to study the Vlasov–Maxwell system. *Commun. Pure Appl. Anal*, 1(1):103–125, 2002.

[**16**] Markus Kunze. Yet another criterion for global existence in the 3D relativistic Vlasov–Maxwell system. *Journal of Differential Equations*, 259(9):4413–4442, 2015.

[**17**] L S Kuz'menkov. The Bogolyubov hierarchy of equations for relativistic systems. Radiation damping of waves in a plasma. In *Soviet Physics Doklady*, volume 23, pages 469–471, 1978.

[**18**] Lev Davidovich Landau and Evgeny Mikhailovich Lifshitz. *The Classical Theory of Fields*, volume 2 of *Course of Theoretical Physics*. Pergamon, Amsterdam, fourth edition, 1975. ISBN 978-0-08-025072-4.

[**19**] Jonathan Luk and Robert M. Strain. A new continuation criterion for the relativistic Vlasov–Maxwell system. *Communications in Mathematical Physics*, 331(3):1005–1027, 2014.

[**20**] Jonathan Luk and Robert M. Strain. Strichartz estimates and moment bounds for the relativistic Vlasov–Maxwell system. *Archive for Rational Mechanics and Analysis*, 219:445–552, 2016.

[**21**] Toan T. Nguyen. A new framework for particle-wave interaction. *arXiv preprint arXiv:2410.13703*, 2024.

[**22**] Christophe Pallard. On the boundedness of the momentum support of solutions to the relativistic Vlasov–Maxwell system. *Indiana University Mathematics Journal*, pages 1395–1409, 2005.

[**23**] Christophe Pallard. Non-resonant coupling of wave-transport systems in distorted geometry. *Journal of Hyperbolic Differential Equations*, 2(01):109–128, 2005.

[**24**] Christophe Pallard. A refined existence criterion for the relativistic Vlasov–Maxwell system. *Communications in Mathematical Sciences*, 13(2):347–354, 2015.

[**25**] Neel Patel. Three new results on continuation criteria for the 3D relativistic Vlasov–Maxwell system. *Journal of Differential Equations*, 264(3):1841–1885, 2018.

[**26**] Michael Ruzhansky and Mitsuru Sugimoto. On global inversion of homogeneous maps. *Bulletin of Mathematical Sciences*, 5:13–18, 2015.

[**27**] George B. Rybicki and Alan P. Lightman. *Radiative Processes in Astrophysics*. John Wiley & Sons, 1991.

[**28**] Jack Schaeffer. A small data theorem for collisionless plasma that includes high velocity particles. *Indiana University Mathematics Journal*, pages 1–34, 2004.

[**29**] Reinel Sospedra-Alfonso and Reinhard Illner. Classical solvability of the relativistic Vlasov–Maxwell system with bounded spatial density. *Mathematical Methods in the Applied Sciences*, 33(6):751–757, 2010.

[**30**] Herbert Spohn. *Dynamics of charged particles and their radiation field.* Cambridge University Press, 2004.

[**31**] Xuecheng Wang. Global solution of the 3D relativistic Vlasov–Maxwell system for large data with cylindrical symmetry. *arXiv preprint arXiv:2203.01199*, 2022.

Department of Mathematics, Princeton University, Princeton, NJ 08544
*Email address*: const@math.princeton.edu

Program in Applied and Computational Mathematics, Princeton University, Princeton, NJ 08544
*Email address*: hgrayer@math.princeton.edu