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On the Distribution of Heat in Fibered Magnetic Fields

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Abstract: We study the equilibrium temperature distribution in a model for strongly magnetized plasmas in dimensions two and three. Provided the magnetic field is sufficiently structured (integrable in the sense that it is fibered by co-dimension one invariant tori, on most of which the field lines ergodically wander) and the effective thermal diffusivity transverse to the tori is small, it is proved that the temperature distribution is well approximated by a function that only varies across the invariant surfaces. The same result holds for "nearly integrable" magnetic fields up to a "critical" size. In this case, a volume of non-integrability is defined in terms of the temperature defect distribution and is related to the non-integrable structure of the magnetic field, confirming a physical conjecture of Paul et al (J Plasma Phys 88(1):905880107, 2022). Our proof crucially uses a certain quantitative ergodicity condition for the magnetic field lines on a full measure set of invariant tori, which is automatic in two dimensions for magnetic fields without null points and, in higher dimensions, is guaranteed by a Diophantine condition on the rotational transform of the magnetic field.

1. Introduction

The heat conduction in strongly magnetized plasmas is influenced locally by the direction of the magnetic field $B : \mathbb{R}^d \to \mathbb{R}^d$. In three dimensions, Braginskii [1] (see also [2,3]) derived an effective anisotropic diffusion equation for the temperature T in such an environment which, in steady state and free of heat sources, reads

$$\operatorname{div}(b\nabla_b T + \varepsilon \nabla_b^\perp T) = 0 \tag{1.1}$$

where, assuming the magnetic field has no null points $|B| \neq 0$, we introduced

$$b = B/|B| \qquad \nabla_b = b \cdot \nabla, \qquad \nabla_b^\perp = \nabla - b\nabla_b. \tag{1.2}$$

This equilibrium equation captures macroscopically the phenomenon that charged particle dynamics strongly influenced by B favors collisions aligned with b. In (1.1), the

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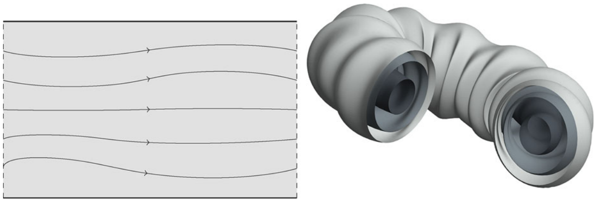

Fig. 1. Examples of fibered magnetic fields. Left: a 2d magnetic field without null points on a (topologically) annular domain—the periodic channel. Integral curves are levels of the streamfunction. Right: a 3d toroidal magnetic field; depicted in grayscale are distinct level surfaces of the first integral, ψ, the flux function

parameter $\varepsilon > 0$ represents the ratio $\kappa_\perp/\kappa_\parallel$ of the transverse diffusion coefficient to the longitudinal. In general it is a scalar function of local density and field magnitude $|B|$, however its magnitude is small in many applications of interest where $|B|$ is large. In our work, we consider this system on $\mathbb{R}^d$ with $d \geq 2$, and we treat ε as a constant and study the limit $\varepsilon \to 0$.

For arbitrary B, it is not immediate what emerges in the limit $\varepsilon \to 0$ of (1.1), given some fixed boundary conditions. We focus on toroidal "Arnold fibered" fields B. These are solenoidal vector fields B having the property that there is a smooth function $\psi : D \to \mathbb{R}$ defined in a bounded region $D \subset \mathbb{R}^d$ with $|\nabla\psi| \neq 0$ in D, whose level sets S_ψ are $(d-1)$-dimensional tori such that ψ is a first integral

$$B \cdot \nabla\psi = 0. \tag{1.3}$$

We shall term these fields *(toroidally) fibered*. In two dimensions, if B is divergence-free and sufficiently smooth, then $B = \nabla^\perp A$ for a "streamfunction" $A : D \to \mathbb{R}$ where $\nabla^\perp = (-\partial_y, \partial_x)$. If B has no nulls, then $|\nabla A| > 0$, so any non-vanishing divergence-free field in two dimensions is fibered by its streamfunction, e.g. $\psi = A$. See Fig. 1a. In three dimensions, its straightforward to write down explicit fibered fields, see (1.13) and Fig. 1b. Moreover, as we will later discuss, non-degenerate magnetohydrostatic (MHS) equilbria have this property.

The temperature equation (1.1) is to be solved in a toroidal shell D with boundaries $S_\pm$ that are level sets of the first integral ψ. Call $\psi_- := \inf_D \psi$ and $\psi_+ := \sup_D \psi$. Since ψ is non-degenerate by assumption, $S_\pm$ are the levels corresponding to the values $\psi_\pm$. To complete the problem, we impose Dirichlet boundary conditions for the temperature field $T : D \to \mathbb{R}$ on these surfaces. Overall, we consider the system

$$\begin{aligned} \operatorname{div}(b\nabla_b T + \varepsilon\nabla_b^\perp T) &= 0 \qquad && \text{in } D, \\ T &= T_\pm, && \text{on } S_\pm, \end{aligned} \tag{1.4}$$

for constants T_-, T_+. The system (1.4) is used in practice as an efficient method to visualize the flux surfaces of the magnetic field [2,4,5].

To state our main result concerning the convergence of T_ε, we use some notions from mixing to characterize the behavior of B via its trajectories on the flux surfaces S_ψ. Denoting $I = [\psi_-, \psi_+]$, we distinguish surfaces S_ψ whose label ψ is in the set

$$E(\gamma, M) := \left\{\psi \in I : \|u\|_{\dot{H}^{-\gamma}(S_\psi)} \leq M\|\nabla_B u\|_{L^2(S_\psi)}, \text{ for all } u \in H^1(S_\psi)\right\} \tag{1.5}$$

for some nonnegative γ and M, where $\dot{H}^{-\gamma}(S_\psi)$ denotes the homogeneous Sobolev space of index $-\gamma$ on S_ψ as defined in (2.10). The definition of these sets is motivated by a Diophantine condition, see (4.1) and the following discussion. The sets $E(\gamma, M)$ of labels may be empty or may have full measure, depending on B. We then define the collection

$$N(\gamma, M) = I \setminus E(\gamma, M), \tag{1.6}$$

of "non-ergodic" values of ψ. Note that if $M > M'$ then $N(\gamma, M) \subseteq N(\gamma, M')$.

Definition 1. We say that B satisfies the "coercive ergodicity condition" if, with $N(\gamma, M)$ defined as in (1.6), for some $c, \gamma > 0$, we have

$$\lim_{M\to\infty} M^c \mu(N(\gamma, M)) = 0, \tag{1.7}$$

where μ denotes the one-dimensional Lebesgue measure.

Our main result below roughly states that, provided B is ergodic on almost all of the surfaces S_ψ such that the ergodicity condition holds, the temperatures profiles T_ε indeed converge (in $H^1(D)$) to the effective temperature T_0. A consequence of our theorem is that the limiting temperature profile T_0 itself fibers B. This fact partially motivated the work of Paul–Hudson–Helander [5].

Theorem 1.1. *Let $d \geq 2$ and let B be toroidally fibered by ψ, and let D be the region bounded by two level sets $S_\pm$. For $\varepsilon > 0$, let $T_\varepsilon : D \to \mathbb{R}$ be the solution of system (1.4) for constants T_- and T_+. If the ergodicity condition from Definition 1 holds, then*

$$T_\varepsilon \to T_0 := \Theta(\psi) \qquad \text{in } H^1(D) \tag{1.8}$$

where $\Theta(\psi)$ is the solution of the one-dimensional boundary-value problem on $\psi \in [\psi_-, \psi_+]$:

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\left(\frac{\mathrm{d}\Theta}{\mathrm{d}\psi}\int_{S_\psi} |\nabla\psi|\, \mathrm{d}\mathscr{H}^{(d-1)}\right) = 0, \qquad \Theta(\psi_\pm) = T_\pm \tag{1.9}$$

where $\mathscr{H}^{(d-1)}$ denotes $(d-1)$ dimensional Hausdorff measure on S_ψ.

In particular, there exists a constant $C := C(D, B) > 0$ such that

$$\|T_\varepsilon - T_0\|_{H^1(D)} \leq C\varepsilon^{\frac{c}{2+c}}, \tag{1.10}$$

where c is the largest number so that there is a $\gamma > 0$ making condition (1.7) of Definition 1 hold.

Remark 1. If one weakens the ergodicity condition (1.7) to the condition that $\lim_{M\to\infty} \mu(N(\gamma, M)) = 0$, the same proof shows that (1.8) still holds but without the explicit rate (1.10). See Remark 4. We thank the anonymous reviewer for pointing this out.

Remark 2. By the previous remark, the coercive ergodicity condition (1.7) is not necessary if one only wants the convergence (1.8). It would be interesting to determine if this weaker condition is in fact necessary. We leave this to future work.

The proof is given in § 2. Briefly, if each integral curve of $B|_{S_\psi}$ covers S_ψ densely for some ψ, (that is, if S_ψ is an "irrational torus"), one encounters a small divisors problem; the operator ∇_B is bounded below on S_ψ but the lower bound may be arbitrarily small. However, for $\psi \in E(\gamma, M)$, this lower bound cannot be less than $1/M$. On the complement $N(\gamma, M)$, the operator ∇_B is not bounded below, but the ergodicity condition (1.7) ensures that the measures of the sets $N(\gamma, M)$ go to zero as M increases. The net result is one of homogenization to a one-dimensional limit profile adapted to the geometry of the invariant tori that satisfies an effective diffusion equation. See § 2 for further discussion.

In the upcoming Corollaries 1.1, 1.2, we show that this condition holds for a large family of physically-relevant vector fields B. Whenever $d = 2$, the sets $N(\gamma, M)$ are empty for large enough M; that is, every surface S_ψ is ergodic in this setting (in three and higher-dimensions, the ergodicity condition need not be true in general). Thus c in bound (1.10) may be taken to ∞ for any $\gamma \geq 0$. It follows from our main theorem that, in this case, we have convergence of T_ε to the effective temperature T_0. More quantitatively:

Corollary 1.1. *Let $d = 2$ and let B be a non-vanishing divergence-free vector field. Then*

$$\|T_\varepsilon - T_0\|_{H^1(D)} \leq C\varepsilon, \tag{1.11}$$

where $T_0 = \Theta(\psi)$ where Θ is given by (1.9) and ψ is the streamfunction of B.

In three dimensions, an important example of fibered fields are the smooth solutions of the magnetohydrostatic equations

$$(\text{curl } B) \times B = \nabla p, \qquad \text{div } B = 0, \qquad \text{in } D \subset \mathbb{R}^3, \tag{1.12}$$

having the property that the pressure satisfies $\nabla p \neq 0$. As noted by Arnold [6,7] since $|\nabla p|$ is nonvanishing by assumption, each surface S_p is a smooth two-dimensional surface which admits two everywhere transverse non-vanishing tangent vector fields (curl B and B) and are thus two-dimensional tori or cylinders. In this setting B is fibered by its pressure, $\psi = p$. It is straightforward to construct fields B of this type which are axisymmetric, see e.g. [8,9]. It is an open problem (see [9,10]) to construct such smooth magnetohydrostatic equilibria with $|\nabla p| > 0$ outside of Euclidean symmetry.

More generally, in three dimensions, given a non-degenerate function $\psi : D \to \mathbb{R}$ whose level sets are tori along with functions $\theta, \phi : D \to \mathbb{R}$, any vector field of the form

$$B = \nabla\psi \times \nabla\theta + \nabla\phi \times \nabla\chi \tag{1.13}$$

is divergence-free. If χ is chosen so that $\chi = \chi(\psi, \phi)$, B is fibered by ψ, and known as "integrable" because the integral curves of B obey a Hamiltonian system with Hamiltonian χ, and this Hamiltonian is integrable in the usual sense[1] when $\partial_\theta \chi = 0$ (see (1.15) in the footnote). See Fig. 1, right panel. Fields of this form play an important role in the

[1] We suppose that with B as in (1.13), the functions θ, ϕ, ψ together form a coordinate system in D. Then for any smooth $u : D \to \mathbb{R}$, we have $\nabla u = \partial_\psi u \nabla\psi + \partial_\phi u \nabla\phi + \partial_\theta u \nabla\theta$ and so, writing $J = \nabla\psi \times \nabla\theta \cdot \nabla\phi$, which is nonvanishing by our assumption, we have the formula

$$(B \cdot \nabla)u = \left[\partial_\phi u + \iota(\psi, \theta, \phi)\partial_\theta u + \tau(\psi, \theta, \phi)\partial_\psi u\right] J, \tag{1.14}$$

where $\tau(\psi, \theta, \phi) := -\partial_\theta \chi(\psi, \theta, \phi)$ and where we have introduced the rotational transform $\iota(\psi, \theta, \phi) := \partial_\psi \chi(\psi, \theta, \phi)$. There is a simple interpretation of the function χ. Consider any integral curve of B, parametrized problem of confining a plasma with a magnetic field [11]. Such fields may sometimes be regarded as MHS solutions held steady by external forcing (e.g. by current carrying coils in some particular geometry) [12–14].

Suppose θ, ϕ form a coordinate system on S_ψ and so we write $u = u(\psi, \theta, \phi)$. Then if B is as in (1.13) with $\partial_\theta \chi = \partial_\phi \chi = 0$, it follows after writing $\iota(\psi) = \chi'(\psi)$,

$$(B \cdot \nabla)u = \left[\partial_\phi u + \iota(\psi)\partial_\theta u\right] J, \tag{1.16}$$

where $J = \nabla\psi \times \nabla\theta \cdot \nabla\phi$. Generally, by a theorem of Sternberg [15], if B is *any* nonvanishing divergence-free vector field fibered by a function ψ, (in particular, this includes the case $\chi = \chi(\psi, \phi)$ of (1.13)) then on each S_ψ there are coordinates θ, ϕ and a number $\iota = \iota(\psi)$ so that, expressed in these coordinates, B takes the form (1.16) for a function $J = J(\psi, \theta, \phi) > 0$. We call the function ι from (1.16) the rotational transform. Our main result in three dimensions, proven in § 4, is that provided ι is invertible with Lipschitz inverse, we have convergence $T_\varepsilon \to T_0$ in $H^1(D)$.

Corollary 1.2. *Let $d = 3$ and let B be toroidally fibered by ψ. Suppose that the rotational transform ι from (1.16) is invertible and for some $L > 0$*

$$|\psi_1 - \psi_2| \leq L|\iota(\psi_1) - \iota(\psi_2)| \tag{1.17}$$

holds for all $\psi_1, \psi_2 \in I$. Then the ergodicity condition (1.7) holds for any $\gamma > 1$ with $c = 1$. Consequently,

$$\|T_\varepsilon - T_0\|_{H^1(D)} \leq C\varepsilon^{\frac{1}{3}}, \tag{1.18}$$

where $T_0 = \Theta(\psi)$ where Θ is given by (2.6).

In other words, we show that the ergodicity condition holds for integrable Arnol'd fibered fields B with monotone rotational transform. Such fields are of specific interest in the plasma physics community, see the discussion in [12]. However such plasma equilibria, if they exist, may be unstable, or difficult to physically realize. Thus, it is important to also understand the behavior of non-integrable fields $\partial_\theta \chi \neq 0$. There is an obstruction: the behavior of particle transport (and thus of heat) in non-integrable fields can be quite complicated because non-integrable Hamiltonian systems may exhibit chaos.

In [5], the authors consider a model of non-integrable magnetic fields taking the form (1.13) where

$$\chi_\varepsilon(\psi, \theta, \phi) = \chi_0(\psi) + \varepsilon^a \chi_1(\psi, \theta, \phi), \tag{1.19}$$

and $a \geq 1/2$. A modification of the proof of Theorem 1.1 (see Sect. 5) gives the following generalization of Corollary 1.2 to fields of this type which are "weakly nonintegrable."

Footnote 1 continued
by ϕ. That is, we consider $\Psi(\phi), \vartheta(\phi)$ defined by

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\Psi = \frac{B \cdot \nabla\psi}{B \cdot \nabla\phi} = -\partial_\theta \chi, \qquad \frac{\mathrm{d}}{\mathrm{d}\phi}\vartheta = \frac{B \cdot \nabla\theta}{B \cdot \nabla\phi} = \partial_\psi \chi, \tag{1.15}$$

with the understanding that the quantities on the right-hand sides are evaluated at $(\psi, \theta, \phi) = (\Psi(\phi), \vartheta(\phi), \phi)$. Thus the integral curves of B satisfy a Hamiltonian system with Hamiltonian χ. Note that if $\partial_\theta \chi = 0$, the above system is integrable (has a conserved quantity) since ψ is constant along the flow. This also be seen from the formula (1.14).

We require the following anisotropic Sobolev spaces tailored to the invariant tori: $f \in L^2(D)$ which are finite in the norm

$$\|f\|^2_{H^{(0,\gamma)}} := \int_{\psi_-}^{\psi_+} \|f(\psi, \cdot)\|^2_{H^\gamma(S_\psi)}\, \mathrm{d}\psi. \tag{1.20}$$

Theorem 1.2. *Suppose that B has the form (1.13) where $\chi = \chi_\varepsilon$ is given by (1.19) for $a \geq 1/2$ and satisfies $\|\partial_\theta \chi_1\|_{L^\infty(D)} < 1$ and $\partial_\theta \chi_1|_{\partial D} = 0$. Moreover, denote B_0 the field when $\varepsilon = 0$, and assume that $\iota = \chi_0'$ satisfies the condition from Corollary 1.2. Then, there is a constant $C = C(L)$ such that*

$$\|\nabla^\perp_{b_0}(T_\varepsilon - T_0)\|_{L^2} + \|T_\varepsilon - T_0\|^2_{L^2} \leq C\varepsilon^{2a-1}\|\partial_\theta \chi_1\|^2_{L^2} + C\varepsilon^{\frac{1}{3}}\left(\|\Delta T_0\|^2_{H^{(0,\gamma)}} + \|\Delta T_0\|^2_{L^\infty}\right), \tag{1.21}$$

$$\|\nabla_{b_0}(T_\varepsilon - T_0)\|_{L^2} \leq C\varepsilon^{2a}\|\partial_\theta \chi_1\|^2_{L^2} + C\varepsilon^{\frac{4}{3}}\left(\|\Delta T_0\|^2_{H^{(0,\gamma)}} + \|\Delta T_0\|^2_{L^\infty}\right). \tag{1.22}$$

When $a > 1/2$, the estimate (1.21) implies that $T_\varepsilon \to T_0$ almost everywhere. When $a = 1/2$, we show the same result is true, and in this case, even though $\rho^* = \lim_{\varepsilon \to 0} T_\varepsilon - T_0 \in H^1(D)$ may not vanish almost everywhere, we find $\rho^*|_{S_\psi} = 0$ for all ψ except possibly for a family of ψ lying in a set of measure zero. Here, $\rho^*|_{S_\psi}$ denotes the trace of the function ρ^* on the surface S_ψ, and this quantity is well-defined whenever $\rho^* \in H^1(D)$, by the trace theorem. In fact, the support of $\rho^* = \lim_{\varepsilon \to 0} \rho_\varepsilon$ is contained in the collection of non-ergodic surfaces $N(\gamma) = I \setminus E(\gamma)$, which has measure zero, where $E(\gamma)$ denote the family of ergodic surfaces

$$E(\gamma) = \bigcup_{M>0} E(\gamma, M). \tag{1.23}$$

Corollary 1.3. *Under the hypotheses of Corollary 1.2 with $a = 1/2$, the sequence $\rho_\varepsilon = T_\varepsilon - T_0$ converges weakly in H^1 to a distribution ρ^* in $H^1(D)$ with the property that*

$$\rho^*|_{S\psi} = 0, \qquad \text{whenever } \psi \in E(\gamma). \tag{1.24}$$

That is, the support of ρ^* is contained in $N(\gamma) = I \backslash E(\gamma)$.

Our final result relates directly to the work of [5]. In that paper, the authors consider the sets

$$\mathcal{N}(\varepsilon) = \{(\psi, \theta, \phi) : |\nabla_b T(\psi, \theta, \phi)|^2 \geq \varepsilon |\nabla_b^\perp T(\psi, \theta, \phi)|^2\}, \tag{1.25}$$

where $b = B/|B|$ with B as in (1.13), and study them as a proxy for the "non-integrability" of the field B. Using the estimates from the above section, we can get an upper bound on the measure of the set in the limit $\varepsilon \to 0$.

Proposition 1.1. *Define B as in (1.13) and $\mathcal{N}(\varepsilon)$ as in (1.25). Suppose that the boundary values $T_\pm$ from (1.9) satisfy $T_+ \neq T_-$. Under the hypotheses of Corollary 1.2, there is a constant C depending continuously on $\|T_0'\|_{L^\infty}$, $\|1/T_0'\|_{L^\infty}$, $\|T_0''\|_{L^\infty}$, $(1 - \|\nabla \chi_1\|_{L^\infty})^{-1}$ and $(1 - \gamma)^{-1}$ so that*

$$\mu(\mathcal{N}(\varepsilon)) \leq C\left(\varepsilon^{2a-1}\|\partial_\theta \chi_1\|^2_{L^\infty} + \varepsilon^{1/3}\|\Delta T_0\|^2_{H^{(0,\gamma)}}\right). \tag{1.26}$$

Note that the "integrable" case corresponds to taking $\chi_1 = 0$ and it follows that in this case

$$\lim_{\varepsilon \to 0} \mu(\mathcal{N}(\varepsilon)) = 0. \tag{1.27}$$

Since now $\chi = \chi_0$ is integrable, this agrees with the fact that the effective volume of non-integrability is zero. If χ_1 is nonzero, we get the same result with $a > 1/2$ but if $a = 1/2$ we instead have

$$\lim_{\varepsilon \to 0} \mu(\mathcal{N}(\varepsilon)) \leq C \|\partial_\theta \chi_1\|_{L^\infty}^2. \tag{1.28}$$

This exhibits a relationship between the volume of the set (1.25) and the above proxy for non-integrability of the Hamiltonian χ, captured by the θ-dependence of the perturbation χ_1.

Remark 3. A true measure of non-integrability would be the volume of the complement of the set of invariant tori that are perturbations of the unperturbed tori. The problem of determining this volume could, in principle, be treated by converse KAM theory (see e.g. [16]). We leave the problem of relating the above notion of non-integrability to this one to future work.

Our proof of the Proposition (see Sect. 5) partially confirms a conjecture announced in [5]. There, the authors conjecture that $\lim_{\varepsilon \to 0} \mu(\mathcal{N}(\varepsilon)) = 0$ precisely when B is an integrable field. The above result shows that, at least for the family of model fields (1.13) we consider here, integrability implies that the measure of these sets vanishes.

2. Proof of Theorem 1.1

The limiting profile $\Theta(\psi)$ is a function of the flux function ψ only, and is determined from the following heuristic. If B is fibered, expanding $T_\varepsilon = T_0 + \varepsilon T_1$ leads to

$$\operatorname{div}(b \nabla_b T_0) = 0, \tag{2.1}$$

$$\operatorname{div}(\nabla_b^\perp T_0) = -\operatorname{div}(b \nabla_b T_1). \tag{2.2}$$

Equation (2.1) is underdetermined for T_0; indeed noting that $b \cdot \nabla \psi = 0$ one sees that any function

$$T_0 = \Theta(\psi) \tag{2.3}$$

will satisfy (2.1). This arbitrariness of Θ may be eliminated by considering the second condition. Indeed, note that since b is tangent to each surface S_ψ, we have

$$\frac{\operatorname{div}(b \nabla_b T_1)}{|\nabla \psi|}\bigg|_{S_\psi} = \operatorname{div}_{S_\psi}\left(\left[\frac{b \nabla_b T_1}{|\nabla \psi|}\right]\bigg|_{S_\psi}\right) \tag{2.4}$$

where $\operatorname{div}_{S_\psi}$ denotes the divergence operator on S_ψ. See Lemma A.2. In light of this, equation (2.2) comes with the following compatibility condition: on each invariant torus S_ψ,

$$\int_{S_\psi} \frac{\Delta T_0}{|\nabla \psi|} \mathrm{d}\mathscr{H}^{(d-1)} = 0, \tag{2.5}$$

where $\mathrm{d}\mathscr{H}^{(d-1)}$ is the $(d-1)$ dimensional Hausdorff measure on S_ψ. Here we used that $\nabla_b^\perp T_0 = \nabla T_0$ since $b \cdot \nabla T_0 = 0$. Because T_0 is constant on S_ψ, it follows from Lemma (A.1) (see (A.6)) that the solvability requirement (2.5) and the boundary conditions of (1.4) are satisfied if Θ is the unique solution to (1.9). From (1.9), we deduce that the effective temperature distribution is given explicitly by

$$\Theta(\psi) = T_- + (T_+ - T_-)\frac{H(\psi;\ \psi_-)}{H(\psi_+;\psi_-)}, \quad \text{where} \quad H(\psi;\psi_-) := \int_{\psi_-}^{\psi} \frac{\mathrm{d}s}{\Gamma(s)}. \tag{2.6}$$

Here, Γ is given by

$$\Gamma(\psi) = \int_{S_\psi} |\nabla\psi|\, \mathrm{d}\mathscr{H}. \tag{2.7}$$

In two dimensions, $\Gamma(\psi)$ is simply the circulation of the vector field $B = \nabla^\perp\psi$ on the circle S_ψ:

$$\int_{S_\psi} |\nabla\psi|\, \mathrm{d}\mathscr{H} = \int_{S_\psi} B \cdot \mathrm{d}\ell. \tag{2.8}$$

Observing (2.5), we start with the following simple lemma. We first introduce the homogenous fractional Sobolev seminorms on S_ψ. For each S_ψ we pick coordinates $\theta_1, \dots, \theta_{d-1}$ on S_ψ such that θ_j maps S_ψ to $[0, 2\pi]$. For $k \in \mathbb{Z}^{d-1}$, we define

$$\widehat{u}(k) = \frac{1}{(2\pi)^{d-1}} \int_{[0,2\pi]^{d-1}} \prod_{j=1}^{d-1} e^{ik_j\theta_j} u(\theta_1, \dots, \theta_{d-1})\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{d-1}, \tag{2.9}$$

and then for $\gamma \in \mathbb{R}$, we define $\|\cdot\|_{\dot{H}^\gamma}$ by

$$\|u\|^2_{\dot{H}^\gamma(S_\psi)} = \sum_{k\in\mathbb{Z}^{d-1}\setminus 0} |k|^{2\gamma} |\widehat{u}(k)|^2. \tag{2.10}$$

Recall also that (1.20) defines the anisotropic spaces $H^{(0,\gamma)}$ tailored to the tori. The result is then

Lemma 2.1. *Suppose that $F \in H^{(0,\gamma)}(D) \cap L^\infty(D)$ for some $\gamma \geq 0$ and that F satisfies*

$$\int_{S_\psi} \frac{F}{|\nabla\psi|}\, \mathrm{d}\mathscr{H}^{(d-1)} = 0, \tag{2.11}$$

for all $\psi \in [\psi_-, \psi_+]$. There is a constant C depending only on D, B and ψ so that for any $M > 0$ we have

$$\left|\int_D F u\, \mathrm{d}\mu\right| \leq CM\|F\|_{H^{(0,\gamma)}} \|\nabla_b u\|_{L^2} + C\mu(N(\gamma, M))^{1/2} \|F\|_{L^\infty} \|u\|_{L^2}. \tag{2.12}$$

Proof. By the co-area formula (A.1), we have

$$\int_D Fu\,\mathrm{d}\mu = \int_{\psi_-}^{\psi_+}\int_{S_\psi}\frac{F}{|\nabla\psi|}u\,\mathrm{d}\mathscr{H}^{(d-1)}\mathrm{d}\psi. \tag{2.13}$$

Now, for each ψ, we write

$$\begin{aligned}\int_{S_\psi}&\frac{F}{|\nabla\psi|}u\,\mathrm{d}\mathscr{H}^{(d-1)}\\&=\int_{[0,2\pi]^{d-1}}G(\psi,\theta_1,\ldots,\theta_{d-1})u(\psi,\theta_1,\ldots,\theta_{d-1})\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{d-1},\end{aligned} \tag{2.14}$$

where $G=\frac{F}{|\nabla\psi|}|h|^{1/2}$, where we are writing the metric on S_ψ as $h=h_{\alpha\beta}\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta$ and $|h|=\det h_{\alpha\beta}$. By Parseval's theorem,

$$\begin{aligned}\int_{[0,2\pi]^{d-1}}&G(\psi,\theta_1,\ldots,\theta_{d-1})u(\psi,\theta_1,\ldots,\theta_{d-1})\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{d-1}\\&=(2\pi)^{d-1}\sum_{k\in\mathbb{Z}^{d-1}}\widehat{G}(\psi,k)\widehat{u}(\psi,-k),\end{aligned} \tag{2.15}$$

where $\widehat{F},\widehat{u}$ are defined as in (2.9). Now we note that

$$\widehat{G}(\psi,0)=\int_{[0,2\pi]^{d-1}}\frac{F}{|\nabla\psi|}|h|^{1/2}\mathrm{d}\theta_1\cdots\mathrm{d}\theta_{d-1}=\int_{S_\psi}\frac{F}{|\nabla\psi|}\mathrm{d}\mathscr{H}^{(d-1)}=0, \tag{2.16}$$

by assumption. We therefore have

$$\int_{S_\psi}\frac{F}{|\nabla\psi|}u\,\mathrm{d}\mathscr{H}^{(d-1)}=(2\pi)^{d-1}\sum_{k\in\mathbb{Z}^{d-1}\setminus 0}\widehat{G}(\psi,k)\widehat{u}(\psi,-k), \tag{2.17}$$

and it follows that

$$\left|\int_D Fu\,\mathrm{d}\mu\right|\leq C\int_{\psi_-}^{\psi_+}\sum_{|k|\neq 0}|\widehat{G}(\psi,k)\widehat{u}(\psi,-k)|\,\mathrm{d}\psi. \tag{2.18}$$

Now we split $[\psi_-,\psi_+]=E(\gamma,M)\cup N(\gamma,M)$ and bound

$$\begin{aligned}\int_{\psi\in E(\gamma,M)}&\sum_{|k|\neq 0}|\widehat{G}(\psi,k)\widehat{u}(\psi,-k)|\;\mathrm{d}\psi\\&\leq\int_{\psi\in E(\gamma,M)}\|G\|_{H^\gamma(S_\psi)}\|u\|_{\dot{H}^{-\gamma}(S_\psi)}\,\mathrm{d}\psi\\&\leq M\int_{\psi\in E(\gamma,M)}\|G\|_{H^\gamma(S_\psi)}\|\nabla_B u\|_{L^2(S_\psi)}\,\mathrm{d}\psi\\&\leq M\|G\|_{H^{(0,\gamma)}(D)}\|\nabla_B u\|_{L^2(D)},\end{aligned} \tag{2.19}$$

by the definition of $E(\gamma, M)$ and

$$\begin{aligned}\int_{\psi\in N(\gamma,M)} \sum_{|k|\neq 0} |\widehat{G}(\psi,k)\widehat{u}(\psi,-k)|\,\mathrm{d}\psi &\le \|G\|_{L^2(N(\gamma,M))}\|u\|_{L^2(D)}\\ &\le \mu(N(\gamma,M))^{1/2}\|G\|_{L^\infty(D)}\|u\|_{L^2(D)}.\end{aligned} \tag{2.20}$$

This gives the result. □

Our main result, Theorem 1.1, will be a direct result of the following estimate.

Proposition 2.1. *Define* $T_0 = \Theta(\psi)$ *where* Θ *is given by (2.6) and let* $\rho = T_\varepsilon - T_0$. *Under the hypotheses of Theorem 1.1, there is a constant* C *depending only on the domain* D *so that for each* $\varepsilon > 0$ *and* $M > 0$, *we have*

$$\begin{aligned}&\|\nabla_b\rho\|_{L^2}^2 + \varepsilon\|\nabla_b^\perp\rho\|_{L^2}^2 + \varepsilon\|\rho\|_{L^2}^2\\ &\quad\le C\left(M^2\varepsilon^2 + \varepsilon\mu(N(\gamma,M))\right)\left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right)\end{aligned} \tag{2.21}$$

Proof. The remainder ρ satisfies

$$\operatorname{div}(b\nabla_b\rho) + \varepsilon\operatorname{div}(\nabla_b^\perp\rho) = -\varepsilon\Delta T_0, \qquad \text{in } D, \tag{2.22}$$

$$\rho|_{S_\pm} = 0, \tag{2.23}$$

where we wrote $\Delta T_0 = \operatorname{div}(\nabla_b^\perp T_0)$ since $b\cdot\nabla T_0 = 0$. If we multiply (2.22) by ρ and integrate over D, then use the co-area formula (A.1) we find

$$\int_D \left(|\nabla_b\rho|^2 + \varepsilon|\nabla_b^\perp\rho|^2\right)\mathrm{d}\mu = \varepsilon\int_D \Delta T_0\rho\,\mathrm{d}\mu = \varepsilon\int_{\psi_-}^{\psi_+}\int_{S_\psi}\frac{\Delta T_0}{|\nabla\psi|}\rho\,\mathrm{d}\mathscr{H}^{(d-1)}\mathrm{d}\psi. \tag{2.24}$$

Recall that we have defined T_0 so that $F = \Delta T_0$ satisfies the condition (2.11). We can therefore apply Lemma 2.1, and by (2.12) we have

$$\begin{aligned}&\varepsilon\left|\int_{\psi_0}^{\psi_1}\int_{S_\psi}\frac{\Delta T_0}{|\nabla\psi|}\rho\,\mathrm{d}\mathscr{H}^{(d-1)}\mathrm{d}\psi\right|\\ &\quad\le M\varepsilon\|\Delta T_0\|_{H^{(0,\gamma)}}\|\nabla_b\rho\|_{L^2} + \varepsilon\mu(N(\gamma,M))^{1/2}\|\Delta T_0\|_{L^\infty}\|\rho\|_{L^2}\\ &\quad\le \frac{1}{2}\left(M^2\varepsilon^2\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \frac{1}{\delta}\varepsilon\mu(N(\gamma,M))\|\Delta T_0\|_{L^\infty}^2\right) + \frac{1}{2}\|\nabla_b\rho\|_{L^2}^2 + \frac{\delta}{2}\varepsilon\|\rho\|_{L^2}^2\end{aligned} \tag{2.25}$$

for any $\delta > 0$. Since $\rho|_{\partial D} = 0$, by Poincaré's inequality we have

$$\|\rho\|_{L^2}^2 \le C_P\|\nabla\rho\|_{L^2}^2, \tag{2.26}$$

where C_P is the Poincaré constant for D. Taking δ so that δC_P is sufficiently small, we see from (2.24) and (2.25) that there is a constant $C > 0$ depending only on C_P so that

$$\begin{aligned}&\|\nabla_b\rho\|_{L^2}^2 + \varepsilon\|\nabla_b^\perp\rho\|_{L^2}^2 + \varepsilon\|\rho\|_{L^2}^2\\ &\quad\le C\left(M^2\varepsilon^2 + \varepsilon\mu(N(\gamma,M))\right)\left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right),\end{aligned} \tag{2.27}$$

after using (2.26) again to bound $\varepsilon\|\rho\|_{L^2}^2 \le C'\|\nabla_b\rho\|_{L^2}^2 + C'\varepsilon\|\nabla_b^\perp\rho\|_{L^2}^2$ for another constant C'. □

Proof of Theorem 1.1. If we take $M = \varepsilon^{-\frac{1}{2+c}}$, then writing $N_\varepsilon = N(\gamma, \varepsilon^{-\frac{1}{2+c}})$, (2.21) gives

$$\|\nabla_b^\perp \rho\|_{L^2} + \|\rho\|_{L^2}^2 \le C\varepsilon^{\frac{c}{2+c}} \left(1 + \varepsilon^{-\frac{c}{2+c}} \mu(N_\varepsilon)\right) \left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right), \quad (2.28)$$

$$\|\nabla_b \rho\|_{L^2} \le C\varepsilon^{1+\frac{c}{2+c}} \left(1 + \varepsilon^{-\frac{c}{2+c}} \mu(N_\varepsilon)\right) \left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right). \quad (2.29)$$

By assumption, $\lim_{\varepsilon \to 0} \varepsilon^{-\frac{c}{2+c}} \mu(N_\varepsilon) = 0$ and the result follows. □

Remark 4. If one replaces assumption (1.7) with the weaker assumption that $\lim_{M\to\infty} \mu(N(\gamma, M)) = 0$, a nearly identical argument gives that $\lim_{\varepsilon\to 0} \|\nabla\rho\|_{L^2} = 0$. Indeed, it is enough to take $M = M(\varepsilon)$ so that $\lim_{\varepsilon\to 0} M^2\varepsilon = 0$ and then note that by assumption $\lim_{\varepsilon\to 0} \mu(N(\gamma, M(\varepsilon))) = 0$.

3. Proof of Corollary 1.1: The 2d case

If $|B| > 0$ in D, B is fibered by its streamfunction ψ, $B = \nabla^\perp \psi$. We bound

$$\|u\|_{\dot{H}^0(S_\psi)}^2 = \sum_{k\in\mathbb{Z}\setminus 0} |\widehat{u}(k)|^2 \le \sum_{k\in\mathbb{Z}} |k|^2 |\widehat{u}(k)|^2 \le \frac{C}{\inf_{S_\psi} |B|^2} \|\nabla_B u\|_{L^2(S_\psi)}^2 \quad (3.1)$$

for a constant $C > 0$. Here we have used that B spans the tangent space to S_ψ at each point. Therefore, $E(0, M) = D$ whenever $M \ge \frac{C}{\inf_D |B|}$, and so $N(0, M)$ is empty in this case. Thus (1.7) holds for any $c \ge 0$ and the result follows.

4. Proof of Corollary 1.2: The 3d Integrable Case

We first show that if $\gamma > 1$, under the hypotheses of Corollary 1.2, the condition (1.7) holds with $c = 1$. We start by relating this condition to the "Diophantine" condition.

Let $I = [\psi_-, \psi_+]$. Fix $\iota = \iota(\psi)$ with $\iota \in L^\infty(I)$. Let $|(m, n)| = \sqrt{m^2 + n^2}$. We define

$$D(\gamma, M) = \left\{\psi \in I : |m + \iota(\psi)n| \ge \frac{1}{M|(m, n)|^\gamma} \quad \text{for all} \quad (m, n) \in \mathbb{Z}^2 \setminus \{0\}\right\}. \quad (4.1)$$

If $\psi \in D(\gamma, M)$ for some γ, M, we will say that ψ is "Diophantine" and that the surface S_ψ is a "Diophantine surface". With $m + \iota(\psi)n$ replaced by $\omega \cdot (m, n)$ for $\omega \in \mathbb{R}^2$, these sets play a fundamental role in the proof of the celebrated KAM theorem [17]. Note that if $\psi \in D(\gamma, M)$, the flow of B is ergodic in the usual sense by e.g. Theorem 3.1 from [18], since in particular $\psi \in D(\gamma, M)$ requires that $\iota(\psi)$ is irrational. The condition in (4.1) can therefore be thought of as a quantitative measure of the ergodicity of the flow of B. Similarly, one can think of the "coercive ergodicity" assumption from Definition 1 as the requirement that the restriction of the flow of B to "most" surfaces S_ψ is (quantitatively) ergodic, in the above sense.

These sets are empty if $\gamma < 1$ by the classical Dirichlet approximation theorem (see [19, Theorem 9.1]) but it turns out that if ι is bi-Lipschitz and $\gamma > 1$ they have positive measure; in fact the complement of $\cup_{M>0} D(\gamma, M)$ has zero measure, as the next result shows.

Lemma 4.1. *Let $\iota : I \to \mathbb{R}$ be an invertible function and suppose there is $L > 0$ so that*

$$\frac{1}{L}|\psi_1 - \psi_2| \leq |\iota(\psi_1) - \iota(\psi_2)| \leq L|\psi_1 - \psi_1|. \tag{4.2}$$

Define $D(\gamma, M)$ as in (4.1) and let μ denote the one-dimensional Lebesgue measure. If $\gamma > 1$ and $M > 0$, there is a constant K depending only on $\mu(I)$, L, and $1/(\gamma - 1)$ so that

$$\mu(I \setminus D(\gamma, M)) \leq K\left(\frac{1}{M^{\frac{1}{1+\gamma}}} + \frac{1}{M}\right). \tag{4.3}$$

Taking $M \to \infty$ in (4.3) shows that the set of ψ which fails the condition (4.1) for *all* $M > 0$ has zero measure. Equivalently, the set of ψ satisfying the condition in (4.1) for *some* M has full measure, though in general the complement may be nonempty.

Proof. This result follows from a straightforward modification of the argument from e.g. [19, Theorem 9.3]. We include the details here for the convenience of the reader. For $(m, n) \in \mathbb{Z}^2 \setminus (0, 0)$, we define

$$\Pi_{(m,n)}(\gamma, M) = \left\{\psi \in I : |m + \iota(\psi)n| < \frac{1}{M|(m, n)|^\gamma}\right\}, \tag{4.4}$$

so that

$$I \setminus D(\gamma, M) \subseteq \bigcup_{(m,n)\neq(0,0)} \Pi_{(m,n)}(\gamma, M). \tag{4.5}$$

If $n \neq 0$, $\Pi_{(m,n)}(\gamma, M)$ is contained in the interval $[\psi_1, \psi_2] \subset I$ where ψ_1 are such that $\iota(\psi_1) = \frac{1}{n}\frac{1}{M|(m,n)|^\gamma} + \frac{m}{n}$ and $\iota(\psi_2) = -\iota(\psi_1)$ (such ψ_1, ψ_2 exist and are unique since by (4.2) ι is invertible). These are the maximal and minimal values of ψ such that (4.4) holds for a given (m, n). Therefore

$$\mu(\Pi_{(m,n)}(\gamma, M)) \leq |\psi_1 - \psi_2| \leq \frac{L}{2M|n||(m, n)|^\gamma}. \tag{4.6}$$

If $n = 0$, $\Pi_{(m,0)}(\gamma, M) = I$ when $|m| < M^{-1/(1+\gamma)}$ and it is empty otherwise. Therefore

$$\begin{aligned}\sum_{(m,n)\neq(0,0)} \mu(\Pi_{(m,n)}(\gamma, M)) &\leq \sum_{m\neq 0} \mu(\Pi_{(m,0)}(\gamma, M)) + \sum_{(m,n)\neq(0,0),n\neq 0} \mu(\Pi_{(m,n)}(\gamma, M)) \\ &\leq |I|\frac{1}{M^{1/(1+\gamma)}} + \frac{L}{M} + \frac{L}{2M}\sum_{|m|,|n|\geq 1} \frac{1}{|n||(m, n)|^\gamma},\end{aligned} \tag{4.7}$$

since there are two terms in the second sum on the first line with $m = 0$. The last sum here is bounded by

$$\begin{aligned}2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\frac{1}{(m^2+n^2)^{\gamma/2}} &\leq 2\sum_{m=1}^{\infty}\int_1^\infty \frac{\mathrm{d}z}{z(m^2+z^2)^{\gamma/2}} \\ &= 2\sum_{m=1}^{\infty}\frac{1}{m^\gamma}\int_{1/m}^\infty \frac{\mathrm{d}w}{w(1+w^2)^{\gamma/2}}.\end{aligned} \tag{4.8}$$

Now we bound

$$\int_{1/m}^{1} \frac{\mathrm{d}w}{w(1+w^2)^{\gamma/2}} \lesssim \int_{1/m}^{1} \frac{\mathrm{d}w}{w} = \log m, \tag{4.9}$$

and

$$\int_{1}^{\infty} \frac{\mathrm{d}w}{w(1+w^2)^{\gamma/2}} \leq C((\gamma-1)^{-1}) \tag{4.10}$$

for a continuous function C. It follows that the above sum is bounded by

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(m^2+n^2)^{\gamma/2}} \leq \sum_{m=1}^{\infty} \frac{1}{m^{\gamma}}(\log m + C(\gamma)), \tag{4.11}$$

which is finite for $\gamma > 1$.

In this case we therefore have

$$\begin{aligned}\mu\left(I \setminus D(\gamma, M)\right) &\leq \mu\left(\bigcup\nolimits_{(m,n)\neq(0,0)} \Pi_{m,n}(\gamma, M)\right) \\ &\leq C_1\left(\frac{1}{M^{1/(1+\gamma)}} + \frac{1}{M}\right),\end{aligned} \tag{4.12}$$

for a constant C_1 depending only on $|I|$, L and $1/(\gamma-1)$, which proves (1.26). □

Remark 5. If one replaces the assumption that ι is bi-Lipschitz with the (essentially weaker) assumption that ι is C^1 but has only finitely many critical points, a slight modification of the above proof gives (the slightly weaker result) $\mu(\cup_M I \setminus D(\gamma, M)) = 0$. Indeed, for any $\epsilon > 0$, let $B_\epsilon(x_i)$ denote the ball of radius ϵ around each critical point x_i. On $I \setminus \cup_{x_i} B_\epsilon(x_i)$, ι is locally bi-Lipschitz. Applying the above argument to $I \setminus \cup_{x_i} B_\epsilon(x_i)$ and taking $M \to \infty$, one gets

$$\mu(\cup_M I \setminus D(\gamma, M)) \leq C\epsilon$$

for a constant C depending on the number of critical points. Since ϵ was arbitarary this gives the stated result. Combining this argument with Remark 4 and the upcoming argument gives an alternative version of Corollary 1.2 where the assumption 1.17 is replaced with the above-mentioned assumption, but where the conclusion is the slightly weaker result that $\|T_\varepsilon - T_0\|_{H^1(D)} \to 0$ in place of (1.18). We thank the anonymous reviewer for pointing this out.

The values of $\psi \in D(\gamma, M)$ are sometimes called "strongly non-resonant", and the surfaces S_ψ with $\psi \in D(\gamma, M)$ are called "non-resonant flux surfaces".

Proof of Corollary 1.2. We first consider a field of the form (1.13) when $\chi = \chi(\psi)$ and where θ, ϕ form a coordinate system on each S_ψ. With $\iota(\psi) = \chi'(\psi)$, in this setting we have

$$(B \cdot \nabla)u = J\left[\partial_\phi + \iota(\psi)\partial_\theta\right]u, \qquad J = \nabla\psi \times \nabla\theta \cdot \nabla\phi \tag{4.13}$$

where $J \neq 0$ by assumption. We claim that the Diophantine surfaces are ergodic, in the sense that there is a constant C with

$$D(\gamma, CM) \subseteq E(\gamma, M) \tag{4.14}$$

for any $M > 0$. It follows from this claim that $N(\gamma, M) \subseteq I \backslash D(\gamma, CM)$ and so the condition (1.7) holds for any $c < 1$, since (4.3) then implies that

$$\mu(N(\gamma, M)) \le \frac{C}{M}, \qquad M \ge 1. \tag{4.15}$$

We now prove (4.14). Whenever $\psi \in D(\gamma, M)$, for any smooth function $u : S_\psi \to \mathbb{R}$, we have

$$\|u\|^2_{\dot{H}^{-\gamma}(S_\psi)} = \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \frac{|\widehat{u}(m,n)|^2}{\left(m^2+n^2\right)^\gamma} \le M^2 \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} |m+\iota(\psi)n|^2 |\widehat{u}(m,n)|^2. \tag{4.16}$$

Now we note that by (4.13),

$$\left(\widehat{\nabla_{B/J} u}\right)(\psi, m, n) = -2\pi i (m + \iota(\psi)n)\widehat{u}(\psi, m, n). \tag{4.17}$$

It follows that if $\psi \in D(\gamma, M)$, there are constants C_1, C_2 so that

$$\|u\|^2_{\dot{H}^{-\gamma}(S_\psi)} \le C_1 M^2 \|\nabla_{B/J} u\|^2_{L^2(S_\psi)} \le C_2 M^2 \|\nabla_b u\|^2_{L^2(S_\psi)}, \tag{4.18}$$

and this gives (4.14).

We now show how to get the same result for any non-vanishing divergence-free fibered field provided the rotational transform satisfies the bound (1.17). We first show that the rotational transform is well-defined in this setting; that is, that we can find coordinates so that B takes the form (1.16).

In light of (A.7), since $\operatorname{div} B = 0$, it follows that $|\nabla\psi|^{-1}$ is an integral invariant of $B|_{S_\psi}$ (namely $U = |\nabla\psi|^{-1}$ is a conserved density along the flow of $B|_{S_\psi}$ on S_ψ) and so by Sternberg's theorem (Theorem 1 of [15]), B is orbitally conjugate to a constant vector field on S_ψ. That is, there are coordinates (θ, ϕ) on S_ψ so that in these coordinates, there is a nonvanishing function $J = J(\theta, \phi)$ so that on S_ψ, B takes the form

$$(B \cdot \nabla^T)u = J\left(\partial_\phi + \iota\partial_\theta\right)u, \qquad u \in C^\infty(S_\psi) \tag{4.19}$$

for a real number ι (compare with (1.14)), where ∇^T denotes the tangential gradient on S_ψ, given by $\nabla^T u = (\nabla - \nabla\psi/|\nabla\psi|^2\nabla\psi \cdot \nabla)u$ when u is a function defined in a neighborhood of S_ψ. Applying this theorem on each S_ψ then gives $\iota = \iota(\psi)$.

The above proof goes through with a minor change, which is that we want to replace the fractional Sobolev norm $\|u\|_{\dot{H}^\gamma}$, which was defined relative to a fixed coordinate system (because the Fourier coefficients $\widehat{u}(k)$ depend on the choice of coordinates in (2.9)), with a fractional Sobolev norm $\|u\|_{\tilde{\dot{H}}^\gamma}$ defined relative to the coordinates guaranteed by Sternberg's theorem. This means we want to modify the definition of $E(\gamma, M)$ and define

$$\widetilde{E}(\gamma, M) = \left\{\psi \in I : \|u\|_{\tilde{\dot{H}}^{-\gamma}(S_\psi)} \le M\|\nabla_B u\|_{L^2(S_\psi)}, \text{ for all } u \in H^1(S_\psi)\right\}. \tag{4.20}$$

It is clear that the proof of Theorem 1.1 goes through without change if we replace the sets $E(\gamma, M)$ with $\widetilde{E}(\gamma, M)$. It is also clear from the formula (4.19) and the above argument that there is a constant $C > 0$ so that (4.14) holds with E replaced with $\widetilde{E}$, if the rotational transform $\iota = \iota(\psi)$ from (4.19) satisfies (1.17). ◻

5. Proof of Theorem 1.2, Corollary 1.3 and Proposition 1.1: the non-integrable case

We now consider the non-integrable case $\chi_1 \neq 0$ of (1.19). With B as in (1.13)–(1.19), we set

$$B_0 = \nabla\psi \times \nabla\theta + \iota(\psi)\nabla\phi \times \nabla\psi, \qquad B_1 = \nabla\phi \times \nabla\chi_1 \tag{5.1}$$

where $\iota(\psi) = \chi_0'(\psi)$. We then write $b = B/|B|$ in the form

$$b = b_0 + \varepsilon^a b_1, \qquad b_0 = \frac{B_0}{|B|}, \quad b_1 = \frac{B_1}{|B|}. \tag{5.2}$$

5.1. Proof of Theorem 1.2. We start by recording a simple estimate.

Lemma 5.1. *If $\|\nabla\chi_1\|_{L^\infty} < 1$ and $a \geq 1/2$, there is a constant $C > 0$ so that for any function $u \in H^1(D)$, we have*

$$\frac{1}{C}\left(\|\nabla_b u\|_{L^2}^2 + \varepsilon\|\nabla_b^\perp u\|_{L^2}^2\right) \leq \|\nabla_{b_0} u\|_{L^2}^2 + \varepsilon\|\nabla_{b_0}^\perp u\|_{L^2}^2 \leq C\left(\|\nabla_b u\|_{L^2}^2 + \varepsilon\|\nabla_b^\perp u\|_{L^2}^2\right). \tag{5.3}$$

Proof. This follows after writing $\nabla_{b_0} = \nabla_b - \varepsilon^a\nabla_{b_1}$, noting that $|b_1| \leq C|\nabla\chi_1|$, and

$$|\nabla_{b_0} u| \leq |\nabla_b u| + \varepsilon^a|\nabla\chi_1||\nabla u|, \qquad |\nabla_{b_0}^\perp u| = |(\nabla - b_0\nabla_{b_0})u| \leq |\nabla_b^\perp u| + \varepsilon^a|\nabla\chi_1||\nabla u| \tag{5.4}$$

for smooth u. It follows that for $u \in H^1(D)$,

$$\|\nabla_{b_0} u\|_{L^2}^2 + \varepsilon\|\nabla_{b_0}^\perp u\|_{L^2}^2 \leq \|\nabla_b u\|_{L^2}^2 + \varepsilon\|\nabla_b^\perp u\|_{L^2}^2 + \varepsilon^{2a}\|\nabla\chi_1\|_{L^\infty}^2\|\nabla u\|_{L^2}^2. \tag{5.5}$$

Provided $2a \geq 1$, this gives

$$(1 - \varepsilon\|\nabla\chi_1\|_{L^\infty})\|\nabla_{b_0} u\|_{L^2}^2 + \varepsilon(1 - \|\nabla\chi_1\|_{L^\infty}^2)\|\nabla_{b_0}^\perp u\|_{L^2}^2 \leq \|\nabla_b u\|_{L^2}^2 + \varepsilon\|\nabla_b^\perp u\|_{L^2}^2, \tag{5.6}$$

which is the second bound in (5.3). The first bound is nearly identical. □

Proof of Corollary 1.2. Since $\nabla_{b_0} T_0 = 0$ we have

$$\begin{aligned}\operatorname{div}(b\nabla_b T_0) + \varepsilon\operatorname{div}(\nabla_b^\perp T_0) &= \varepsilon^a\operatorname{div}(b\nabla_{b_1} T_0) + \varepsilon\Delta T_0 - \varepsilon\operatorname{div}(b\nabla_b T_0)\\ &= (\varepsilon^a - \varepsilon^{1+a})\operatorname{div}(b\nabla_{b_1} T_0) + \varepsilon\operatorname{div}(\nabla T_0).\end{aligned} \tag{5.7}$$

Then $\rho = T_\varepsilon - T_0$ satisfies

$$\operatorname{div}(b\nabla_b\rho) + \varepsilon\operatorname{div}(\nabla_b^\perp\rho) = (\varepsilon^{1+a} - \varepsilon^a)\operatorname{div}(b\nabla_{b_1} T_0) - \varepsilon\operatorname{div}(\nabla T_0) \qquad \text{in } D, \tag{5.8}$$

$$\rho|_{S_\pm} = 0. \tag{5.9}$$

Since $\partial_\theta\chi_1|_{\partial D} = 0$ by assumption, b is tangent to ∂D and so if we multiply this by ρ and integrate over D, we find

$$\int_D |\nabla_b\rho|^2 + \varepsilon|\nabla_b^\perp\rho|^2\,\mathrm{d}\mu = \varepsilon\int_D \Delta T_0\rho\,\mathrm{d}\mu + (\varepsilon^a - \varepsilon^{1+a})\int_D \nabla_{b_1} T_0\nabla_b\rho\,\mathrm{d}\mu, \tag{5.10}$$

after integrating by parts in the second term on the right-hand side. Since $|\nabla_{b_1} T_0| \le |\partial_\theta \chi_1||T_0'|$, for any $\delta > 0$ we have

$$\|\nabla_b \rho\|_{L^2}^2 + \varepsilon \|\nabla_b^\perp \rho\|_{L^2}^2 \le \frac{\varepsilon^{2a}}{2\delta} \|\partial_\theta \chi_1\|_{L^2}^2 \|T_0'\|_{L^\infty}^2 + \frac{\delta}{2} \|\nabla_b \rho\|_{L^2}^2 + \varepsilon \left| \int_{\psi_-}^{\psi_+} \int_{S_\psi} \frac{\operatorname{div} \nabla T_0}{|\nabla \psi|} \rho \, \mathrm{d}\mathscr{H}^{(d-1)} \mathrm{d}\psi \right|. \tag{5.11}$$

Arguing as in (2.25)-(2.27) and using (5.3), this implies that there is a constant $C > 0$ so that for any $M > 0$,

$$\|\nabla_{b_0} \rho\|_{L^2}^2 + \varepsilon \|\nabla_{b_0}^\perp \rho\|_{L^2}^2 + \varepsilon \|\rho\|_{L^2}^2 \le C\varepsilon^{2a} \|\partial_\theta \chi_1\|_{L^2}^2 \|T_0'\|_{L^2}^2 + C\left(M^2\varepsilon^2 + \varepsilon\mu(N(\gamma, M))\right)\left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right). \tag{5.12}$$

If we take $M = \varepsilon^{-1/3}$ and use the estimate (4.15) for $\mu(N(\gamma, M))$, we find

$$\|\nabla_{b_0}^\perp \rho\|_{L^2} + \|\rho\|_{L^2}^2 \le C\varepsilon^{2a-1} \|\partial_\theta \chi_1\|_{L^2}^2 + C\varepsilon^{\frac{1}{3}} \left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right), \tag{5.13}$$

$$\|\nabla_{b_0} \rho\|_{L^2} \le C\varepsilon^{2a} \|\partial_\theta \chi_1\|_{L^2}^2 + C\varepsilon^{\frac{4}{3}} \left(\|\Delta T_0\|_{H^{(0,\gamma)}}^2 + \|\Delta T_0\|_{L^\infty}^2\right). \tag{5.14}$$

□

5.2. Proof of Corollary 1.3. Before proving Corollary 1.3, we collect some preliminary results. First, from the uniform bounds (1.21) and (1.22), it follows that that the sequence $T_\varepsilon - T_0$ has weak limit ρ^* in H^1 and that $\nabla_b(T_\varepsilon - T_0)$ converges strongly to 0 in L^2.

If we knew that $\nabla_B \rho^*$ was smooth, it would follow that $\nabla_B \rho^* = 0$ everywhere. We however only know that $\nabla_B \rho^*$ is in L^2 and in particular the restriction $\nabla_B \rho^*|_{S_\psi}$ need not be defined. The following result shows that $\nabla_B \rho^*|_{S_\psi} = 0$ in a weak sense.

Lemma 5.2. *Let $\rho^* = \lim_{\varepsilon\to 0} T_\varepsilon - T_0$. For any ψ and any $v \in C^2(S_\psi)$, with $B' = B|\nabla\psi|^{-1}$,*

$$\int_{S_\psi} \rho^* \operatorname{div}_{S_\psi}(B'v) \, \mathrm{d}\mathscr{H}^{(d-1)} = 0. \tag{5.15}$$

Remark 6. The statement of (5.15) holds with B' replaced by B (or indeed any multiple of B) but it is more convenient for our purposes to use B' since $\operatorname{div}_{S_\psi}(B'v) = |\nabla\psi| \operatorname{div}(Bv) = |\nabla\psi|\nabla_B v$, by (A.7) and the fact that $\operatorname{div} B = 0$. See Lemma 5.3.

Proof. The idea is to use Stokes' theorem to write (5.15) in terms of an integral in the interior of D and to integrate by parts in the interior to exploit the fact that $\nabla_B \rho^* = 0$ almost everywhere.

Fix a surface $S_{\psi'}$. Define a cutoff function $Q \in C^\infty(\mathbb{R})$ such that $Q(z) = 1$ when $|z| \le 1$ and $Q(z) = 0$ when $|z| > 2$. Define

$$Q_\delta(\psi) = Q((\psi - \psi')/\delta), \tag{5.16}$$

so that $Q_\delta(\psi)$ vanishes when $|\psi - \psi'| < 2\delta$ and $Q_\delta(\psi) \equiv 1$ when $|\psi - \psi'| \leq \delta$. Let $D_-(\psi') = D \cap \{\psi \leq \psi'\}$. Then the boundary of $D_-(\psi')$ is $S_{\psi'} \cup S_{\psi_-}$, and if δ is sufficiently small, Q_δ vanishes identically on S_{ψ_-}.

Let V denote the constant extension of v to $D_-(\psi')$, $V(\psi, \theta, \phi) = v(\psi', \theta, \phi)$ for all $(\psi, \theta, \phi) \in D_-(\psi')$. By Stokes' theorem, since the outer unit normal to $S_{\psi'}$ is $\nabla\psi/|\nabla\psi|$, we have

$$\int_{S_{\psi'}} \rho^* \operatorname{div}_{S_{\psi'}}(B'u)\, \mathrm{d}\mathscr{H}^{(d-1)} = \int_{S_{\psi'}} (n \cdot n)\, \rho^* \operatorname{div}_{S_{\psi'}}(B'V) Q_\delta\, \mathrm{d}\mathscr{H}^{(d-1)}$$
$$= \int_{D_-} Q_\delta \operatorname{div}\left(\rho^* \operatorname{div}_{S_{\psi'}}(B'V)\frac{\nabla\psi}{|\nabla\psi|}\right) \mathrm{d}\mu$$
$$+ \int_{D_-} \rho^* \operatorname{div}_{S_{\psi'}}(B'V)\, \frac{\nabla\psi}{|\nabla\psi|} \cdot (\nabla Q_\delta)\, \mathrm{d}\mu. \tag{5.17}$$

The first term here is bounded by

$$\| \operatorname{div}_{S_{\psi'}}(B'V)\|_{H^1(|\psi-\psi'|\leq\delta)} \|\rho^*\|_{H^1(|\psi-\psi'|\leq\delta)} \leq C\delta^{1/2} \|v\|_{C^2(S_{\psi'})} \|\rho^*\|_{H^1(D)}. \tag{5.18}$$

As for the second term, we use the identity (A.7) to write

$$\operatorname{div}_{S_\psi}(B'V) = \frac{1}{|\nabla\psi|} \operatorname{div}(BV), \tag{5.19}$$

and since $\frac{\nabla\psi}{|\nabla\psi|} \cdot (\nabla Q_\delta) = \frac{1}{\delta}|\nabla\psi| Q'_\delta$ and b is tangent to $S_{\psi'}$, the second term in (5.17) is

$$\frac{1}{\delta}\int_{D_-} \rho^* \operatorname{div}_{S_{\psi'}}(B'V)\, |\nabla\psi| Q'_\delta\, \mathrm{d}\mu = \frac{1}{\delta}\int_{D_-} \rho^* \operatorname{div}(BV) Q'_\delta\, \mathrm{d}\mu$$
$$= -\frac{1}{\delta}\int_{D_-} \nabla_B(\rho^* Q'_\delta) V\, \mathrm{d}\mu, \tag{5.20}$$

after integrating by parts. We have therefore shown that for any $\delta > 0$,

$$\left|\int_{S_{\psi'}} \rho^* \operatorname{div}_{S_\psi}(B'V)\, \mathrm{d}\mathscr{H}^{(d-1)}\right| \leq C\delta^{1/2} \|v\|_{C^2(S_{\psi'})} \|\rho^*\|_{H^1(D)}$$
$$+ \frac{1}{\delta} C \left(\|\nabla_B \rho^*\|_{L^2} + \|\nabla_B Q'_\delta\|_{L^2}\right) \|V\|_{L^2}$$
$$= C\delta^{1/2} \|v\|_{C^2(S_{\psi'})} \|\rho^*\|_{H^1(D)}, \tag{5.21}$$

where we used that $\nabla_B u = 0$ whenever $u = u(\psi)$ and that $\nabla_B \rho^* = 0$ in L^2. Taking $\delta \to 0$ gives the claim. □

The condition (5.15) nearly says that $\rho^* = 0$ on each S_ψ in the sense of distributions, but in order to conclude this we would need to know that any test function v can be written in the form $v = \operatorname{div}_{S_\psi}(Bw)$ for some test function w. This need not be possible on an arbitrary surface S_ψ, but by the next Lemma it is possible provided S_ψ is a Diophantine surface. We set

$$D(\gamma) = \bigcup_{M>0} D(\gamma, M). \tag{5.22}$$

Note that by (4.14), $D(\gamma) \subseteq E(\gamma)$ where $E(\gamma) = \cup_{M>0} E(\gamma, M)$ denotes the collection of all ergodic values of ψ. Note also that by Lemma 4.1, the complement $I \setminus D(\gamma)$ has zero measure when $\gamma > 1$.

Lemma 5.3. *Fix $\gamma > 1$ and define $D(\gamma)$ as in (5.22). If $\psi \in D(\gamma)$ and $v \in H^{s+\gamma}(S_\psi)$ for some $s \geq 0$, there is $w \in H^s(S_\psi)$ satisfying*

$$\mathrm{div}_{S_\psi}(B'w) = v, \qquad B' = B|\nabla\psi|^{-1}. \tag{5.23}$$

Remark 7. The equation (5.23) is sometimes known as the "magnetic differential equation". Lemma 5.3 simply says that you can solve this equation on good (sufficiently ergodic) flux surfaces.

Proof. We start by using (A.7) to write

$$\mathrm{div}_{S_\psi}(B'w) = \frac{1}{|\nabla\psi|}\mathrm{div}(Bw) = \frac{1}{|\nabla\psi|}\nabla_B w, \tag{5.24}$$

and so (5.23) takes the form

$$\nabla_{B/J} w = V, \qquad V = J|\nabla\psi|v, \tag{5.25}$$

where recall $J = |g|^{-1/2} = |(\nabla\psi \times \nabla\theta) \cdot \nabla\phi|$. We now define

$$\widehat{w}(m, n) = \frac{i}{2\pi}\frac{1}{m + \iota(\psi)n}\widehat{V}(m, n). \tag{5.26}$$

Because $\psi \in D(\gamma)$, for some $M > 0$, we have the bound

$$|\widehat{w}(m, n)|^2 \leq M^2\left(m^2 + n^2\right)^\gamma |\widehat{V}(m, n)|^2, \tag{5.27}$$

so that in particular $w \in H^s(S_\psi)$ whenever $v = V/(J|\nabla\psi|) \in H^{s+\gamma}(S_\psi)$. By the identity (4.17), it follows that w satisfies (5.23). □

Proof of Corollary 1.3. Fix $\gamma > 1$ and define $D(\gamma)$ as in (5.22). It follows from Lemma 5.3 that given $v \in C^\infty(S_\psi)$, there is $w \in C^\infty(S_\psi)$ so that with $\mathrm{div}_{S_\psi}(B|\nabla\psi|w) = v$. It then follows from Lemma 5.15 that

$$\int_{S_\psi} \rho^* v \, \mathrm{d}\mathscr{H}^{(d-1)} = \int_{S_\psi} \rho^* \,\mathrm{div}_{S_\psi}(B|\nabla\psi|w) \, \mathrm{d}\mathscr{H}^{(d-1)} = 0, \tag{5.28}$$

as required. □

5.3. Proof of Proposition 1.1: effective volume of non-integrability. It follows immediately from the equation (1.9) (or, equivalently, (2.6)) for $T_0 = \Theta(\psi)$ that either T_0 is constant or else T_0' is nonvanishing. Since $T_0|_{S_+} = T_+ \neq T_- = T_0|_{S_-}$ we have that T_0' is nonvanishing and in particular T_0' is bounded below. With $\lambda = \min_D |T_0'|^{-1}$, for any set $N \subseteq D$ we therefore have

$$\mu(N) \leq \lambda^2 \int_N |T_0'|^2 \,\mathrm{d}\mu \leq C\left(\lambda^2 \int_N |\nabla_b^\perp T|^2 \,\mathrm{d}\mu + \lambda^2 \int_N |\nabla_b^\perp \rho|^2 \,\mathrm{d}\mu\right), \tag{5.29}$$

where we used that $T_0' = |\nabla\psi|^{-2}\nabla T_0\cdot\nabla\psi$ and that $|\nabla T_0| \le |\nabla_b^\perp T| \le |\nabla_b^\perp T| + |\nabla_b^\perp \rho|^2$. In particular, with $N = N(\varepsilon)$, since $\|\nabla_b^\perp T\|^2_{L^2(N(\varepsilon))} \le \varepsilon^{-1}\|\nabla_b T\|^2_{L^2(N(\varepsilon))}$,

$$\begin{aligned}\mu(N(\varepsilon)) &\le C\lambda^2\left(\varepsilon^{-1}\int_{N(\varepsilon)}|\nabla_b T|^2\,\mathrm{d}\mu + \int_{N(\varepsilon)}|\nabla_b^\perp\rho|^2\,\mathrm{d}\mu\right)\\ &\le C\lambda^2\left(\varepsilon^{-1}\int_{N(\varepsilon)}|\nabla_b T_0|^2\,\mathrm{d}\mu + \varepsilon^{-1}\int_D|\nabla_b\rho|^2\,\mathrm{d}\mu + \int_D|\nabla_b^\perp\rho|^2\,\mathrm{d}\mu\right)\\ &\le C\lambda^2\varepsilon^{-1}\left(\int_{N(\varepsilon)}|\nabla_b T_0|^2\,\mathrm{d}\mu + \int_D|\nabla_b\rho|^2\,\mathrm{d}\mu + \varepsilon\int_D|\nabla_b^\perp\rho|^2\,\mathrm{d}\mu\right).\end{aligned} \tag{5.30}$$

Since $b = b_0 + \varepsilon^a b_1$ and $\nabla_{b_0}T_0 = 0$, we have

$$\|\nabla_b T_0\|^2_{L^2(D)} \le \varepsilon^{2a}\|\nabla_{b_1}T_0\|^2_{L^2(D)} \le \varepsilon^{2a}\|\partial_\theta\chi_1\|^2_{L^2(D)}\|T_0'\|^2_{L^\infty(D)}. \tag{5.31}$$

To handle the second and third terms in (5.30), we use (1.21)-(1.22) combined with (5.3). By (5.31) we therefore have

$$\mu(N(\varepsilon)) \le C\lambda^2\left(\varepsilon^{2a-1}\|\partial_\theta\chi_1\|^2_{L^2(D)}\|T_0'\|^2_{L^\infty(D)} + \varepsilon^{1/3}\|\Delta T_0\|^2_{H^{(0,\gamma)}(D)}\right). \tag{5.32}$$

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Appendix A. Geometric Identities from the Co-area Formula

In this section we collect some geometric formulas that we will use repeatedly. In what follows, we fix $\psi : D \to \mathbb{R}$ such that $|\nabla\psi| \neq 0$ on D and so that the level surfaces S_ψ are codimension one manifolds which foliate D. Let $\psi_- = \inf_D \psi$ and $\psi_+ = \sup_D \psi$. We will use the co-area formula

$$\int_D u\,\mathrm{d}\mu = \int_{\psi_-}^{\psi_+}\int_{S_\psi}\frac{u}{|\nabla\psi|}\,\mathrm{d}\mathscr{H}^{(d-1)}\mathrm{d}\psi, \tag{A.1}$$

see e.g. [20]. We start with a simple result that generalizes Lemma E.2 from [21].

Lemma A.1. *If $F \in H^2(D)$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\int_{S_\psi}F\,\mathrm{d}\mathscr{H}^{(d-1)} = \int_{S_\psi}\mathrm{div}\left(\frac{\nabla\psi}{|\nabla\psi|}F\right)\frac{\mathrm{d}\mathscr{H}^{(d-1)}}{|\nabla\psi|}. \tag{A.2}$$

Proof. We start with the observation that

$$\int_{S_{\psi_2}} F \, \mathrm{d}\mathscr{H}^{(d-1)} - \int_{S_{\psi_1}} F \, \mathrm{d}\mathscr{H}^{(d-1)} = \int_{D_{\psi_1,\psi_2}} \operatorname{div}\left(\frac{\nabla\psi}{|\nabla\psi|}F\right) \mathrm{d}\mu, \tag{A.3}$$

where $D_{\psi_1,\psi_2} = \cup_{\psi_1 \le \psi' \le \psi_2} S_{\psi'}$ denotes the region bounded by the surfaces S_{ψ_1}, S_{ψ_2}. Indeed, by the divergence theorem,

$$\int_{D_{\psi_1,\psi_2}} \operatorname{div}\left(\frac{\nabla\psi}{|\nabla\psi|}F\right) \mathrm{d}\mu = \int_{S_{\psi_2}} n^{S_{\psi_2}} \cdot \frac{\nabla\psi}{|\nabla\psi|} F \, \mathrm{d}\mathscr{H}^{(d-1)} + \int_{S_{\psi_1}} n^{S_{\psi_1}} \cdot \frac{\nabla\psi}{|\nabla\psi|} F \, \mathrm{d}\mathscr{H}^{(d-1)}, \tag{A.4}$$

where n^{S_ψ} denotes the outward-pointing unit normal to S_ψ. Then

$$n^{S_{\psi_2}} = \frac{\nabla\psi}{|\nabla\psi|}\Big|_{S_{\psi_2}} \quad \text{and} \quad n^{S_{\psi_1}} = -\frac{\nabla\psi}{|\nabla\psi|}\Big|_{S_{\psi_1}}, \tag{A.5}$$

so (A.4) gives (A.3). Dividing (A.3) by $\psi_2 - \psi_1$ and taking the limit gives (A.2). □

In particular, if $F = F(\psi)$ is constant on S_ψ, writing $\Delta F = \operatorname{div}(\nabla F) = \operatorname{div}(\nabla\psi F')$ we have

$$\int_{S_\psi} \Delta F \frac{\mathrm{d}\mathscr{H}^{(d-1)}}{|\nabla\psi|} = \frac{\mathrm{d}}{\mathrm{d}\psi}\left(\int_{S_\psi} |\nabla\psi| F' \mathrm{d}\mathscr{H}^{(d-1)}\right) = \frac{\mathrm{d}}{\mathrm{d}\psi}\left(\left[\int_{S_\psi} |\nabla\psi| \mathrm{d}\mathscr{H}^{(d-1)}\right] F'\right). \tag{A.6}$$

Another consequence of the formula (A.1) is the following

Lemma A.2. *Let X be a vector field defined in D with the property that $X|_{S_\psi}$ is tangent to S_ψ. Then the divergence $\operatorname{div} X$ in D is related to the divergence operator $\operatorname{div}_{S_\psi}$ on S_ψ by*

$$\frac{\operatorname{div} X}{|\nabla\psi|}\Big|_{S_\psi} = \operatorname{div}_{S_\psi}\left(\left[\frac{X}{|\nabla\psi|}\right]\Big|_{S_\psi}\right). \tag{A.7}$$

In particular, if X is divergence-free in D, then $\varrho := |\nabla\psi|^{-1}|_{S_\psi}$ is a density conserved by X on S_ψ.

Proof. This can be seen by working in local coordinates but it is simpler to use (A.1) and note that if $u \in C^\infty(D)$ is any test function then

$$\begin{aligned} -\int_D \operatorname{div} X u \, \mathrm{d}\mu &= \int_D (X\cdot\nabla)u \, \mathrm{d}\mu = \int_{\psi_-}^{\psi_+} \int_{S_\psi} (X\cdot\nabla)u \frac{\mathrm{d}\mathscr{H}^{(d-1)}}{|\nabla\psi|} \\ &= \int_{\psi_-}^{\psi_+} \int_{S_\psi} (X\cdot\nabla^T)u \frac{\mathrm{d}\mathscr{H}^{(d-1)}}{|\nabla\psi|} \\ &= -\int_{\psi_-}^{\psi_+} \int_{S_\psi} \operatorname{div}_{S_\psi}\left(\frac{X}{|\nabla\psi|}\right) u \, \mathrm{d}\mathscr{H}^{(d-1)}, \end{aligned}$$

where we used that $X \cdot \nabla = X \cdot \nabla^T$ on S_ψ, where ∇^T denotes the tangential gradient on S_ψ,

$$\nabla^T U = \left(\nabla - \frac{\nabla \psi}{|\nabla \psi|^2} \nabla \psi \cdot \nabla \right) u \tag{A.8}$$

whenever u is an extension of U from S_ψ to a neighborhood of S_ψ. By (A.1), the left-hand side is

$$-\int_{\psi_-}^{\psi_+} \int_{S_\psi} \frac{\operatorname{div} X}{|\nabla \psi|} u \, \mathrm{d}\mathscr{H}^{(d-1)}. \tag{A.9}$$

Then (A.7) follows since u is arbitrary. □

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