Bounds for a Taylor dynamo

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Agenda

1. Brief review of the Taylor equations
2. Well-posedness, Taylor’s constraint, energy balance
3. Brief review of the kinematic dynamo
4. “Nonlinear” dynamo bounds

Governing equations for the geodynamo

Magnetohydrodynamics in the magnetic diffusive time scale

$$\partial_t B - \nabla \times (u \times B) = \Delta B,$$

$$\frac{1}{\mathrm{Pr}_m}(\partial_t u + u \cdot \nabla u) + \frac{1}{\mathrm{E}_m}\hat{\mathbf{z}} \times u + \nabla p = \Delta u + \frac{1}{\mathrm{E}_m} J \times B + \mathrm{Ra}\, T\, \mathbf{g}$$

$$\frac{1}{q}(\partial_t T + u \cdot \nabla T) = \Delta T + \frac{1}{q} H$$

Dimensionless magnetic field B, velocity u, and temperature T

Governing equations for the geodynamo

In rapidly rotating settings *a la* Earth's core

$$\partial_t B - \nabla \times (u \times B) = \Delta B,$$

$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + \mathrm{Ra}\, T\mathbf{g}$$

$$\partial_t T + u \cdot \nabla T = q \Delta T + H$$

This is the basic reduction in JB Taylor, *Proc. R. Soc. Lond. A* '63.

Governing equations for the geodynamo

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$$\partial_t B - \nabla \times (u \times B) = \Delta B,$$

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Axis of rotation: $\hat{\mathbf{z}}$, gravity vector: $\mathbf{g}$, current $J = \nabla \times B$, heat source H

Governing equations for the geodynamo

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Constraints $\operatorname{div} B = 0$ and $\operatorname{div} u = 0$ is enforced. This determines p.

Governing equations for the geodynamo

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Is there **a unique evolution** for given initial and boundary conditions?

Well-posedness

Elements of well-posedness

For systems of partial differential equations, one may find:

classical solutions, which have derivatives defined pointwise. These are in the C^k spaces, or

weak solutions, which solve the equation in an integrated sense. These are functions in say, L^p or the Sobolev spaces $W^{k,p}$

In time evolution, classical solutions are **unique**, though may not exist for long. Weak solutions may be unique due some special structure.

Elements of well-posedness

C^k is the space of k-times differentiable functions f so that $\nabla^k f$ exists in the classical sense, so is continuous and bounded.

L^p is the space of functions f such that $|f|^p$ has **finite integral**, and L^∞ is the space of functions f which are **absolutely bounded**.

$W^{k,p}$ is the space of k-times **weakly differentiable** functions f in L^p so which have $\nabla^k f$ in L^p.

Let d be the dimension of space. The **Sobolev inequalities** imply the imbeddings $W^{1,p} \subset C^k$ if $p > d$ and $H^s \equiv W^{s,2} \subset C^k$ if $s > k + d/2$.

Context of well-posedness results

The incompressible Navier-Stokes equations

$$\partial_t u + u \cdot \nabla u + \nabla p = \Delta u, \quad \operatorname{div} \mathrm{u} = 0$$

2D: global in time unique C^∞ solutions, 3D: global in time L^2 solutions.

In the large Prandtl number limit, the Boussinesq-Navier-Stokes is

$$\partial_t T + u \cdot \nabla T = 0, \quad -\Delta u + \nabla p = T\mathbf{g}, \quad \operatorname{div} \mathrm{u} = 0$$

Global in time unique $L^1 \cap L^\infty$ solutions in 2D (G., *ARMA* '23) and 3D (Mecherbet-Sueur, *Ann. H. Lebesgue* '22).

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Global in time **unique** $L^1 \cap L^\infty$ solutions in 2D (G., *ARMA* '23) and 3D (Mecherbet-Sueur, *Ann. H. Lebesgue* '22).

The inviscid incompressible porous medium equation

$$\partial_t T + u \cdot \nabla T = 0, \quad u + \nabla p = T\mathbf{g}, \quad \operatorname{div} \mathrm{u} = 0$$

has e.g. **nonunique** L^∞ solutions (Székelyhidi, *Ann. Sci. E.N.S. '12)*

Regularity of the Moffatt equations

Moffatt and Loper (*Geophys. J. Int*. ’94) propose a relative to

$$\partial_t B + u \cdot \nabla B = B \cdot \nabla u + \Delta B,$$
$$\hat{\mathbf{z}} \times u + \nabla p = B \cdot \nabla B + T\mathbf{g}$$
$$\partial_t T + u \cdot \nabla T = q\Delta T + H$$

by splitting the magnetic field $B = \bar{B} + b$ into a mean part $\bar{B}$ and a fluctuating part b in MHD and then **linearize** dynamics around fixed $\bar{B}$, then take the Rossby number and magnetic Reynolds number small.

Regularity of the Moffatt equations

Moffatt and Loper (*Geophys. J. Int*. ’94) propose the simplification

$$
\begin{aligned}
0 &= \bar{B}\cdot\nabla u + \Delta b,\\
\hat{\mathbf{z}}\times u + \nabla p &= \bar{B}\cdot\nabla b + T\mathbf{g}\\
\partial_t T + u\cdot\nabla T &= q\Delta T + H
\end{aligned}
$$

$q > 0$: global in time L^∞ solutions

$q = 0$: solution map is not Lipschitz continuous in any $W^{k,p} \subset\subset L^2$

Due to a series of papers by Friedlander-Rusin-Vicol, ’11, ’11, ’14, ’15

Regularity of a linearized Taylor equations

Gallagher and Gérard-Varet '17 simplify the Taylor equations

$$\partial_t B - \nabla \times (u \times B) = \Delta B,$$
$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + \mathrm{Ra}\, T\mathbf{g}$$
$$\partial_t T + u \cdot \nabla T = q\Delta T + H$$

by considering $\mathrm{Ra} = 0$ and $H = 0$, and then linearizing $B = \bar{B} + b$.

Regularity of a linearized Taylor equations

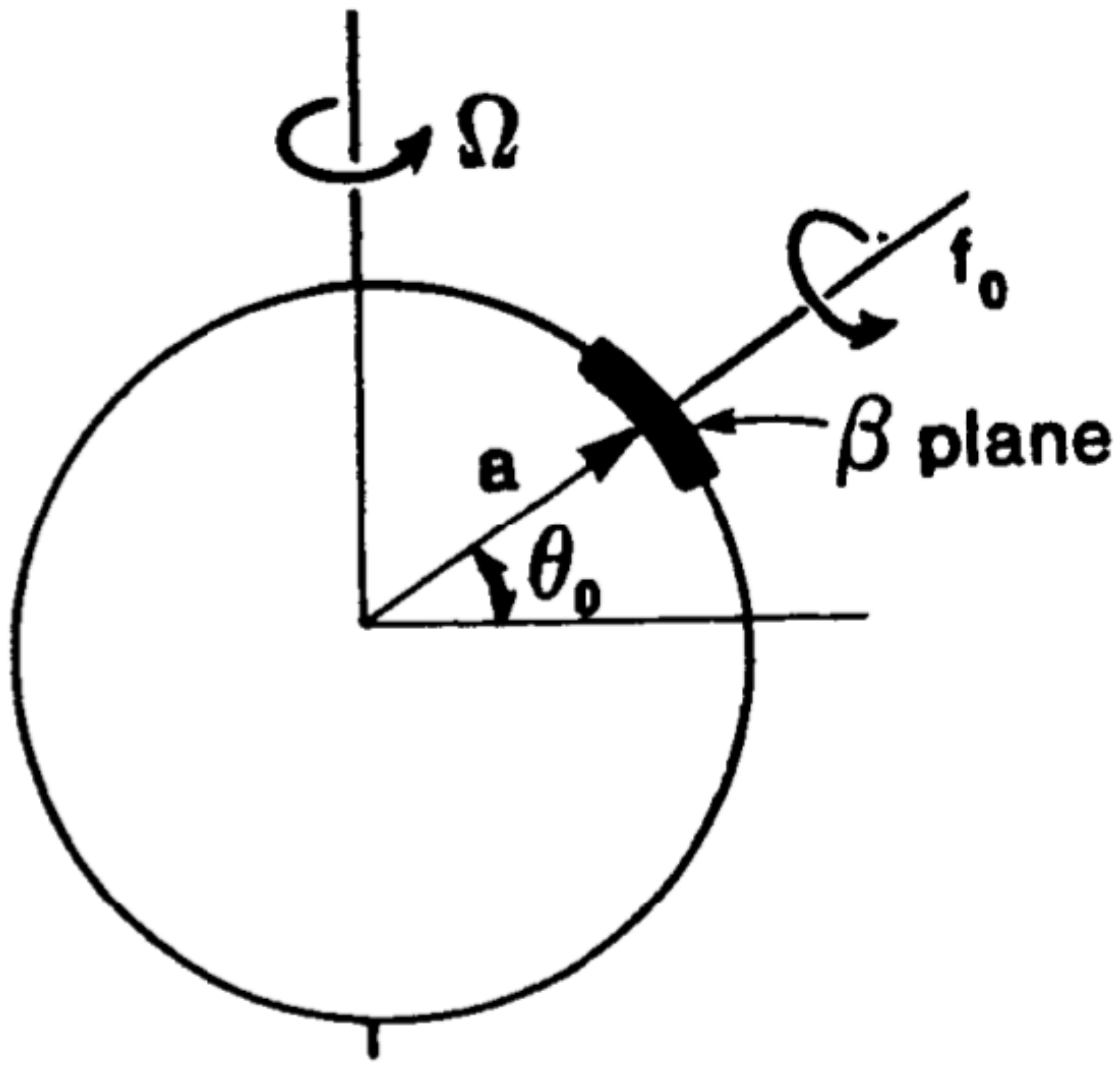
Gallagher and Gérard-Varet ’17 simplify the Taylor equations to

$$
\begin{gathered}
\partial_t b - \nabla \times (u \times b) = \Delta b, \\
\hat{\mathbf{z}} \times u + \nabla p = \operatorname{curl} b \times \bar{B} + \operatorname{curl} \bar{B} \times b \\
\operatorname{div} b = 0 \qquad \operatorname{div} u = 0
\end{gathered}
$$

by considering $\mathrm{Ra} = 0$ and $H = 0$, and then linearizing $B = \bar{B} + b$.

On the periodic domain $\mathbb{T}^3$, there is global existence of L^2_σ solutions b.

Realistic domains



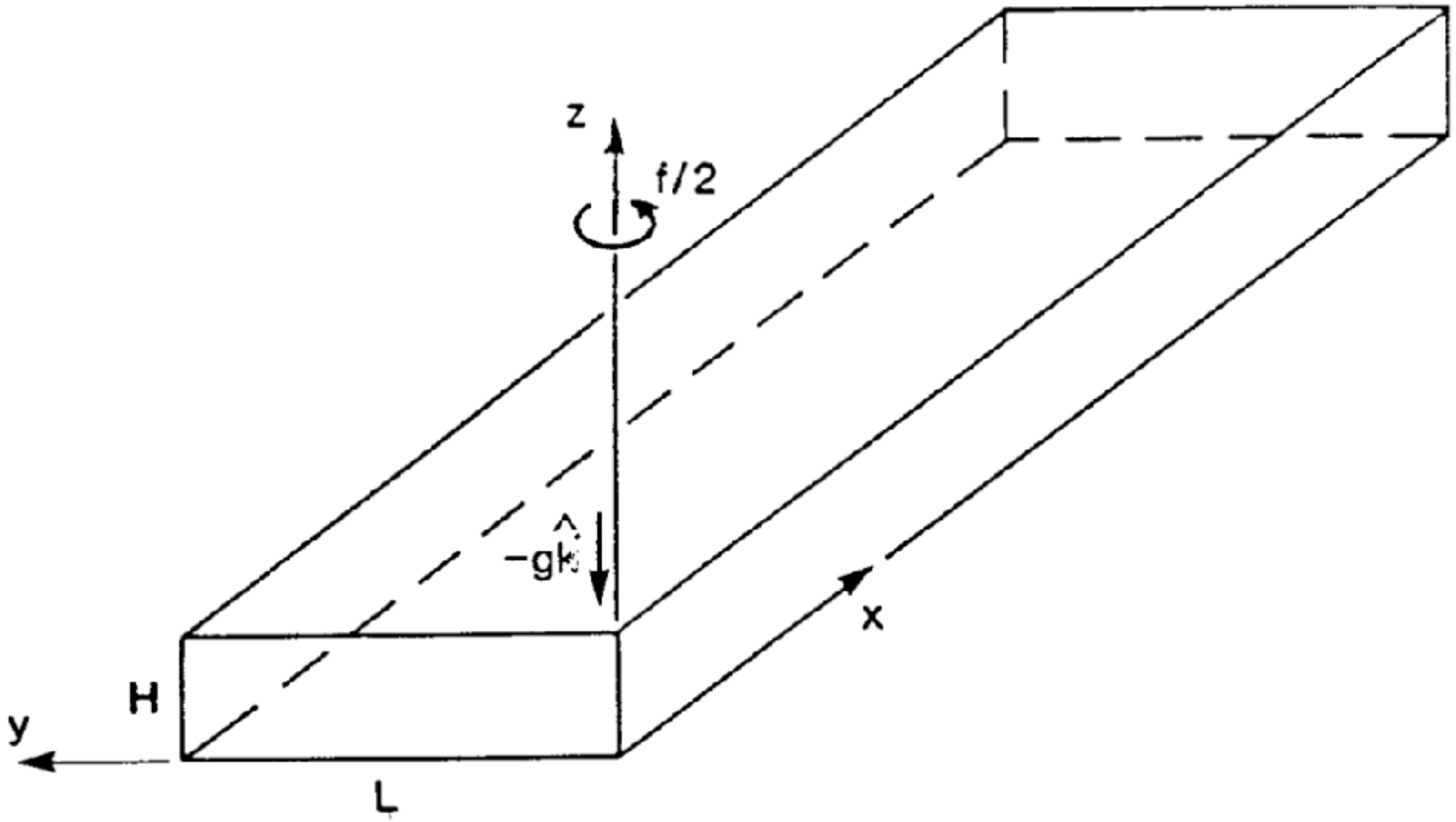


Illustration of the β-plane and the plane layer, Ghil-Childress ‘87

Taylor's constraint in the sphere

Taylor '63 showed that a necessary condition of a solution to

$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + T\mathbf{g}, \quad \operatorname{div} u = 0$$

on the sphere D, with no-slip boundary conditions $u \cdot n = 0$ on ∂D.

In cylindrical coordinates (r, ϕ, z) so that $D \equiv \{r^2 + z^2 \leq 1\}$,

$$\int_{r=s} (J \times B)_\phi \, \mathrm{d}r \, \mathrm{d}\phi = 0, \quad 0 < s \leq 1$$

must be satisfied by any classical solution to the Taylor equations.

Abstract MAC Balance

In terms of Coriolis operator $\mathscr{C}u = \hat{\mathbf{z}} \times u$ and projector $\mathbb{P} : L^2 \to L^2_\sigma$

$$\mathscr{C}u = \mathbb{P}(J \times B + \mathrm{Ra}\, T\mathbf{g})$$

is the momentum equation. Then Taylor's constraint abstractly is

$$\mathbb{P}(J \times B + \mathrm{Ra}\, T\mathbf{g}) \in \mathrm{Range}\, \mathscr{C} \qquad \text{in } L^2_\sigma$$

A solution $u \in L^2_\sigma$ then admits the decomposition

$$u = u_M + u_A + u_g$$

Into **magnetostrophic** u_M, **Archimedean** u_A, and **geostrophic** u_g parts.

Abstract MAC Balance

A solution $u \in L^2_\sigma$ then admits the decomposition

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Into **magnetostrophic** u_M, **Archimedean** u_A, and **geostrophic** u_g parts.

- $u_M \in L^2_\sigma$ is the unique solution to $\mathscr{C} u_M = \mathbb{P}(J \times B)$
- $u_A \in L^2_\sigma$ is the unique solution to $\mathscr{C} u_A = \mathbb{P}(\mathrm{Ra}\, T\mathbf{g})$
- $u_g \in L^2_\sigma$ is a solution to $\mathscr{C} u_g = 0$

In Gallagher-Gérard-Varet, where $\mathrm{Ra} = 0$, this structure gives a key cancellation in their L^2 energy estimate of b in the linearized equations

An energy balance for Taylor's equations

For a solution of Taylor's equations, consider the **magnetic energy**

$$E_M = \frac{1}{2}\int_D |B|^2 \, \mathrm{d}V.$$

For insulating or perfectly conducting boundary conditions, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}E_M = -\int_D |\nabla \times B|^2 \, \mathrm{d}V + \mathrm{Ra}\int_D Tu \cdot \mathbf{g} \, \mathrm{d}V$$

This holds for very general D. If specifically D is the sphere, then

$$\int_D Tu \cdot \mathbf{g} \, \mathrm{d}V = \int_D Tu_M \cdot \mathbf{g} \, \mathrm{d}V$$

Dynamo criteria

The kinematic dynamo problem

Consider a magnetic field governed by the induction equation

$$\partial_t B - \nabla \times (u \times B) = \Delta B$$

where u is a given smooth flow in the domain D.

What criteria on u are necessary for dynamo action?

What **qualitative** or **quantitative** conditions u allow

$$\frac{\mathrm{d}}{\mathrm{d}t} E_M \geq 0 \ ?$$

Qualitative criteria

Consider a magnetic field governed by the induction equation

$$\partial_t B - \nabla \times (u \times B) = \Delta B$$

where u is a given velocity field in the domain D. Then **no dynamo** if

- B is axisymmetric in the whole space $\mathbb{R}^3$
- B is independent of a Cartesian coordinate in $\mathbb{T}^3$
- u is purely toroidal in a sphere
- u is planar in a Cartesian domain

Reviews by Elsasser ’46 and Moffatt ‘78

Quantitative criteria

Consider a magnetic field governed by the induction equation

$$\partial_t B - \mathrm{R}_m \nabla \times (u \times B) = \Delta B$$

where u is a given velocity field in the sphere D. Then **no dynamo** if

- R_m is smaller than a factor depending on $\|\nabla u\|_{L^\infty}$ (Backus ’58)
- R_m is smaller than $\pi\|u\|_{L^\infty}$ (Childress ’69)
- R_m is smaller than $\sqrt{3/8}/(\|\nabla u\|_{L^2} R^2) \approx 0.612$ in enstrophy normalized settting (Proctor, *Geophys. Astrophys. Fluid Dyn.* ’79)

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- R_m is smaller than $C^{-3/2}\sqrt{3/4} \approx 3.1009$ in enstrophy normalized settting (Luo-Chen-Li-Jackson, *Proc. R. Soc. A* '20). C is a constant.

The Taylor dynamo problem

Consider a magnetic field governed by the induction equation

$$\partial_t B - \nabla \times (u \times B) = \Delta B$$

where u satisfies **Taylor's equation**

$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + \mathrm{Ra}\, T\mathbf{g}, \quad \operatorname{div} u = 0$$

and T is a given temperature field in the domain D.

What **qualitative** or **quantitative** conditions on T allow

$$\frac{\mathrm{d}}{\mathrm{d}t} E_M \geq 0\,?$$

Qualitative criteria

Consider a magnetic field governed by the induction equation

$$\partial_t B - \nabla \times (u \times B) = \Delta B$$

where u satisfies **Taylor's equation**

$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + \mathrm{Ra}\, T\mathbf{g}, \quad \mathrm{div}\, u = 0$$

and T is a given temperature field in the **whole space** $\mathbb{R}^3$ with $\mathbf{g} = \mathbf{r}$.

Then from pointwise relation $\mathbf{g} = r\hat{\phi} \times \hat{\mathbf{z}}$, here (r, ϕ, z) are cylindrical

$$\frac{\mathrm{d}}{\mathrm{d}t} E_M = -\int_D |\nabla \times B|^2 \mathrm{d}V + \int_D T u \cdot \mathbf{g}\, \mathrm{d}V$$

Qualitative criteria

Consider a magnetic field governed by the induction equation

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Then from pointwise relation $\mathbf{g} = r\hat{\phi} \times \hat{\mathbf{z}}$, here (r, ϕ, z) are cylindrical

$$\int_D Tu \cdot \mathbf{g}\, \mathrm{d}V = -\int_D T\hat{\phi} \cdot u \times \hat{\mathbf{z}}\, r \mathrm{d}V = \int_0^1 r^2 \int_0^h \int_0^{2\pi} (J \times B)_\phi T\, \mathrm{d}\phi\, \mathrm{d}z\, \mathrm{d}r$$

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and T is a given temperature field in the **whole space** $\mathbb{R}^3$ with $\mathbf{g} = \mathbf{r}$.

$$\boxed{T = T(r) \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} E_M \leq 0}$$

Quantitative criteria

Consider a magnetic field governed by the induction equation

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$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + \mathrm{Ra}\, T\mathbf{g}, \quad \operatorname{div} u = 0$$

and T is a given temperature field in **plane layer** $D \equiv \mathbb{T}^2 \times [0,1]$

$$\frac{\mathrm{d}}{\mathrm{d}t} E_M \leq \left[C_p \mathrm{Ra} \left\| \int_0^z \nabla_{x,y} T \,\mathrm{d}\zeta \right\|_{L^p} - 1 \right] \int |\nabla \times B|^2 \,\mathrm{d}V$$

holds for known constant C_p depending on $p \geq 3$.

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where u satisfies **Taylor's equation**

$$\hat{\mathbf{z}} \times u + \nabla p = J \times B + \mathrm{Ra}\, T\mathbf{g}, \quad \operatorname{div} u = 0$$

and T is a given temperature field in **plane layer** $D \equiv \mathbb{T}^2 \times [0,1]$

$$\boxed{\mathrm{Ra} \leq \frac{1}{C_p \|\nabla_{x,y} T\|_{L^p}} \quad \Longrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} E_M \leq 0}$$

where C_p is a known constant depending on $p \geq 3$