

KODAIRA VANISHING, MULTIPLIER IDEALS, SINGULARITIES

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We review multiplier ideals approach to classical vanishing theorems (Kodaira, Kawamata-Viehweg), and then review some analogues and related concepts like tight closure, test ideals, etc.

1. CLASSICAL VANISHING THEOREMS

In this section we only work over \mathbb{C} .

- For a line bundle L over an irreducible projective variety X , consider the semigroup $\mathbf{N}(L) = \{m \geq 0 \mid H^0(X, L^{\otimes m}) \neq 0\}$. If it is nonzero, there is largest natural number $e = e(L)$ such that all sufficiently large multiples of $e(L)$ appear in $\mathbf{N}(L)$.

Let Y_m be the closure of the complete linear series $\phi_m = \phi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}H^0(X, L^{\otimes m})$. The **litaka dimension** is $\kappa(L) = \max_{m \in \mathbf{N}(L)} \dim Y_m$, provided that X is normal and $\mathbf{N}(L) \neq 0$. If $\mathbf{N}(L) = 0$ then the dimension is set to be $-\infty$. If X is not normal then one passes to normalization. The **Kodaira dimension** is an example of litaka dimension where the dimension is for K_X , a canonical divisor on X . Over normal X , $\kappa(L)$ can be seen from asymptotics that $h^0(X, L^{\otimes m}) = \Theta(m^\kappa)$ for $m \in \mathbf{N}(L)$. We say L is **big** if $\kappa(L) = \dim X$. Note that definition is independent to taking integral multiples so one can define bigness for \mathbb{Q} -divisors.

- A smooth projective variety is of general type iff the canonical divisor is big.
- **Kodaira's lemma** says that for a big Cartier divisor D and an effective Cartier divisor F , $H^0(X, \mathcal{O}_X(mD - F)) \neq 0$ for big enough $m \in \mathbf{N}(D)$. This implies some criteria for bigness. Namely the following are equivalent.

* D is big.

- * For any ample integer divisor A , there is a positive integer $m > 0$ and an effective divisor N on X such that mD and $A + N$ are linearly equivalent.
- * There is an ample integer divisor A , a positive integer $m > 0$ and an effective divisor N on X such that mD and $A + N$ are linearly equivalent.
- * There is an ample integer divisor A , a positive integer $m > 0$ and an effective divisor N on X such that mD and $A + N$ are numerically equivalent.
- * For any coherent sheaf F , there is a positive integer $m = m(F)$ such that $F \otimes \mathcal{O}_X(mD)$ is generically globally generated.

Thus, bigness only depends on numerical equivalence class, and exponent of a big divisor is 1.

- Restriction of big divisor to a sufficiently general subvariety is big.
- (Siu’s numerical criterion for bigness) For D, E nef \mathbb{Q} -divisors on a projective variety X of dimension n , if $(D^n) > n(D^{n-1} \cdot E)$, then $D - E$ is big. This can be thought as a case of holomorphic Morse inequality.

Using this, a nef divisor on an irreducible projective variety is big iff its top self-intersection is strictly positive.

- From this we see that the set of big \mathbb{R} -divisors (also makes sense) forms an open convex cone, the big cone, inside $N^1(X)_{\mathbb{R}}$, numerical equivalence classes of \mathbb{R} -divisors. By the theorems above, numerically big divisors are not too far away from effective divisors; in particular the closure of the effective cone (convex cone span of effective divisors), which we call pseudoeffective cone, is the closure of the big cone, and the interior of it is the big cone.
- A line bundle on a complete scheme is **semiample** if some power of it is globally generated. Then we can consider the subsemigroup $M(L) \subset N(L)$ where M is now a collection of m where $L^{\otimes m}$ is free. Here again we can consider similarly $f(L)$ the exponent of $M(L)$.

Theorem 1.1 (Existence of semiample fibration). *Given a normal projective X with semiample L , there is an \mathcal{O} -connected (or fiber space, namely pushforward of \mathcal{O} is \mathcal{O}) $\phi : X \rightarrow Y$ such that for sufficiently large $m \in M(L)$, $Y_m = Y$ and $\phi_m = \phi$. Furthermore, there is an ample line bundle A on Y such that $\phi^* A = L^{\otimes f(L)}$.*

This is rather saying that $m \gg 0$ then ϕ_m stabilizes. Using this we can see that L is **finitely generated**, i.e. $R(L) = \bigoplus_{m \geq 0} H^0(X, L^{\otimes m})$ is finitely generated (it is generated in degrees $\leq f(L)$ by the corresponding theorem for ample lbs).

- The above theorem holds more generally up to birational equivalence.

Theorem 1.2 (Existence of Iitaka fibration). *Given a normal projective X with any line bundle L such that $\kappa(L) > 0$, for all sufficiently large $k \in N(L)$, the rational maps $\phi_k : X \dashrightarrow Y_k$ are birationally equivalent to a fixed \mathcal{O} -connected $\phi_{\infty} : X_{\infty} \rightarrow Y_{\infty}$ morphism of normal varieties, and the restriction of L to a very general fiber of ϕ_{∞} has Iitaka dimension 0. One has $\dim Y_{\infty} = \kappa(L)$.*

We say such fibration to be the **Iitaka fibration** which is unique up to birational equivalence. The Iitaka fibration of a variety means the Iitaka fibration for the canonical bundle of any nonsingular model of X .

- There is a notion of volume of a line bundle, namely $\text{vol}(L) = \limsup_{m \rightarrow \infty} h^0(X, L^{\otimes m}) / \frac{m^n}{n!}$. This is positive iff L is big, and for a nef L , $\text{vol}(L)$ can be computed by asymptotic Riemann-Roch, $\text{vol}(L) = \int_X c_1(L)^n$. Volume is numerical invariant, birational invariant and is continuous on $N^1(X)_{\mathbb{R}}$.
- A generalization can be done as follows. For an irreducible variety X , a **graded linear series** is a collection $\{V_m\}_{m \geq 0}$ of finite-dimensional vector subspaces $V_m \subset H^0(X, \mathcal{O}_X(mD))$, D being some integer divisor on X , such that $V_k V_l \subset V_{k+l}$, and V_0 contains all constant functions. This gives a graded \mathbb{C} -subalgebra $R(V) = \bigoplus V_m$ of $R(D)$. We call $R(V)$ the section ring associated to V . We can mimic every definition we mentioned above.
- Let \mathfrak{b}_m be the base ideal of V_m , namely the image of $V_m \otimes_{\mathbb{C}} L^* \rightarrow \mathcal{O}_X$ coming from the evaluation map $V_m \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow L$. Then $\mathfrak{b}_k \mathfrak{b}_l \subset \mathfrak{b}_{k+l}$, and this collection of ideals captures much of geometry of V . Mimicing this, we can define a **graded family of ideals** $\mathfrak{a} = \{\mathfrak{a}_m\}$ on X to be a collection of ideal sheaves \mathfrak{a}_m such that $\mathfrak{a}_k \mathfrak{a}_l \subset \mathfrak{a}_{k+l}$ and $\mathfrak{a}_0 = \mathcal{O}_X$. The **Rees algebra** if $\text{Rees}(\mathfrak{a}) = \bigoplus_{m \geq 0} \mathfrak{a}_m$. Given a graded family of ideals, one can get a graded linear series by

$$V_m = H^0(X, \mathcal{O}_X(mD) \otimes \mathfrak{a}_m).$$

- If \mathfrak{a} is finitely generated (i.e. Rees algebra is finitely generated) then its p -th Veronese embedding (i.e. forgetting everything except indices multiples of p) is trivially generated by \mathfrak{a}_p , for some p .
- Let $x \in X$, and suppose \mathfrak{a}_m vanish only at $x \in X$, independent of m . This means \mathfrak{a}_m is \mathfrak{m}_x -primary, and is of finite colength in \mathcal{O}_x . We can define the **multiplicity** of \mathfrak{a} to be $\text{mult}(\mathfrak{a}) = \limsup_{m \rightarrow \infty} \frac{\text{colength}_{\mathcal{O}_x}(\mathfrak{a}_m)}{m^n/n!}$. Mustata proved that $\text{mult}(\mathfrak{a}) = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m)}{m^n}$ where $e(\mathfrak{a}_m)$ is the Samuel multiplicity of \mathfrak{a}_m .

Now we list some positivity theorems.

- (Lefschetz hyperplane theorem) For X nonsingular irreducible projective variety of dimension n and D an effective ample divisor, $H^i(X; \mathbb{Z}) \rightarrow H^i(D; \mathbb{Z})$ is an isomorphism for $i \leq n - 2$, and injective when $i = n - 1$. This follows from that $X - D$ is affine and Artin vanishing. It actually holds for homotopy groups too.
- (Hard Lefschetz theorem) For a smooth variety X of dimension n and a Kahler form ω , the k -fold iterate of cup-product with $\omega L^k : H^{n-k}(X; \mathbb{C}) \rightarrow H^{n+k}(X; \mathbb{C})$ is an isomorphism.
- (Barth's theorem) For $X \subset \mathbb{P}^r$ a nonsingular subvariety of dimension n and codimension $e = r - n$, the restriction gives an isomorphism $H^i(\mathbb{P}^r; \mathbb{C}) \xrightarrow{\sim} H^i(X; \mathbb{C})$ for $i \leq 2n - r = n - e$. This heuristically means that if codimension is small then one gets more topological restrictions.

Hartshorne's **conjecture** suspects that a smooth irreducible variety $X \subset \mathbb{P}^r$ is automatically complete intersection if $\dim X > 2r/3$. For example for $n \geq 4$ and codimension 2 Hartshorne's conjecture is equivalent to that any rank 2 vector bundle on \mathbb{P}^r splits as a direct sum of line bundles if $r \geq 7$.

- (Bertini theorem) For $f : X \rightarrow \mathbb{P}^r$ and X irreducible, if $L \subset \mathbb{P}^r$ is a general codimension d linear space, where $d < \dim \overline{f(X)}$, then $f^{-1}(L)$ is irreducible. More generally for any codimension d linear space $f^{-1}(L)$ is connected.
- (Connectedness theorem of Fulton-Hansen) For $f : X \rightarrow \mathbb{P}^r \times \mathbb{P}^r$ for irreducible X , if $\dim f(X) > r$, then $f^{-1}(\Delta)$ is connected. This has several applications.
 - An unramified morphism $f : X \rightarrow \mathbb{P}^r$ is always a closed embedding if X is complete irreducible of dimension n and $2n > r$.

- If $X \subset \mathbb{P}^r$ is irreducible of dimension larger than $r/2$ then X is algebraically simply connected.
- If $C \subset \mathbb{P}^2$ is a possibly reducible curve whose only singularities are simple nodal double points, then any finite cover $f : X \rightarrow \mathbb{P}^2$ branched only over C is abelian, i.e. Galois with abelian Galois group.
- (Grothendieck connectedness theorem) Let (A, \mathfrak{m}) be complete local Noetherian and $k \geq 1$. If $\text{Spec } A$ is k -connected, i.e. if $\dim A > k$ and $\text{Spec } A - \text{Spec } A/I$ is connected for every closed subset $\text{Spec } A/I$ of dimension $< k$, then $\text{Spec}(A/fA)$ is $(k - 1)$ -connected for any $f \in \mathfrak{m}$.

Now we discuss classical vanishing theorems.

- For vanishing theorems we need resolution of singularities. Given an irreducible X and an effective Cartier divisor D , there is a projective birational morphism $\mu : X' \rightarrow X$ where X' is nonsingular and μ has divisorial exceptional locus $\text{except}(\mu)$ such that $\mu^*D + \text{except}(\mu)$ is a divisor with SNC support (i.e. underlying reduced divisor is snc). We call this a **log resolution**.

A stronger statement is that one can construct X' via a sequence of blow-ups along smooth centers supported in the singular loci of D and X . The advantage of knowing how it's constructed is that we have more refined information about the map.

- We can assume that μ is isomorphism over $X - (\text{Sing}(X) \cup \text{Sing}(D))$.
- $\mu_*\mathcal{O}_{X'}(K_{X'}) = \mathcal{O}_X(K_X)$ and $R^j\mu_*\mathcal{O}_{X'}(K_{X'}) = 0$ for $j > 0$.
- If X is projective and H is ample, then for suitable integers $p \gg 0$ and $b_j \geq 0$, $\mu^*(pH) - \sum b_j E_j$ is ample, where the sum is taken over the exceptional divisors of μ .
- Also we need various coverings.
 - One can take “ m -th root” of divisors as a cyclic covering. To be more precise, locally over an affine variety X and $s \in \mathbb{C}[X]$ a nonzero regular function, we can consider $Y \subset X \times \mathbb{A}^1$ defined by $\{t^m = s\}$ where t is the variable of \mathbb{A}^1 . Then $\pi : Y \rightarrow X$ is a cyclic covering branched along $D = \text{div}(s)$, and for $s' = t|_Y \in \mathbb{C}[Y]$, one has $(s')^m = \pi^*s$. This construction globalizes, so that given any nonzero $s \in \Gamma(X, L^{\otimes m})$ and any X , there is a finite flat covering $\pi : Y \rightarrow X$ where $L' = \pi^*L$ carries $s \in \Gamma(Y, L')$ such that $(s')^m = \pi^*s$. Also $\text{div}(s')$ maps isomorphically to D .
 - More generally one can take m -th root of the whole line bundle;

Theorem 1.3 (Bloch-Gieseker covering). *For an irreducible quasiprojective variety X and a line bundle L over it, for any $m \geq 1$, there is a finite faithfully flat $f : Y \rightarrow X$ from a reduced irreducible variety Y together with a line bundle N over Y such that $f^*L = N^{\otimes m}$. If X is nonsingular we can take Y to be so too. If so and if $\dim X \geq 2$ then we can make a irreducible reduced smooth divisor pulls back to a reduced smooth divisor. One can prove this by first proving for very ample line bundles and use that any line bundle can be expressed as the difference of two very ample line bundles.*

A step further is the construction of **Kawamata covering**, which establishes, for a nonsingular quasiprojective variety X and a snc divisor $D = \sum D_i$ on X , given positive integers $m_1, \dots, m_t > 0$, a smooth variety Y and a finite flat covering $h : Y \rightarrow X$ such that $h^*D_i = m_i D'_i$ for some smooth divisor D'_i on Y , where $D' = \sum D'_i$ has snc.

- The reason why we can use these coverings and resolutions to get vanishing theorems is because of the following lemma.

Lemma 1.1 (Injectivity lemma). *Let $f : Y \rightarrow X$ be a finite surjective morphism of irreducible projective varieties. Let X be normal and E be a vector bundle on X . Then $H^j(X, E) \rightarrow H^j(Y, f^*E)$ is injective.*

This is because, at least when Y is normal, the trace map $\text{tr}_{Y/X} : f_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ gives a splitting of the natural inclusion $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$, so $H^j(X, E)$ is a direct summand of $H^j(X, E \otimes f_*\mathcal{O}_Y)$ which is by projection formula the same as $H^j(Y, f^*E)$.

- Now we state classical vanishing theorems.

Theorem 1.4 (Kodaira vanishing theorem). *Let X be smooth irreducible projective, and A be an ample divisor on X . Then $H^i(X, \mathcal{O}_X(K_X + A)) = 0$ for $i > 0$. Equivalently by Serre duality $H^j(X, \mathcal{O}_X(-A)) = 0$ for $j < \dim X$.*

Proof goes roughly as follows. Take a smooth divisor $D \in |mA|$ for $m \gg 0$. Let $p : Y \rightarrow X$ be the m -fold cyclic covering branched along D . Let $A' = p^*A$. Then there is a smooth ample divisor $D' \in |A'|$ and injectivity lemma says that we are reduced to the case where A is effective nonsingular. Now the theorem is then Lefschetz hyperplane theorem.

- But ampleness in Kodaira vanishing turns out to be too strong.

Theorem 1.5 (Kawamata-Viehweg vanishing theorem, easy version). *Let X be smooth projective, and D be a nef big divisor on X . Then $H^i(X, \mathcal{O}_X(K_X + D)) = 0$ for $i > 0$.*

Kawamata's proof is roughly as follows. We can pick $m \gg 0$ and H ample N effective so that mD and $H + N$ are linearly equivalent. Passing to resolution of singularities of (X, N) we can assume that N has SNC support but ampleness of H might be ruined; but we can modify so that for some $p \gg 0$ and $b_j \geq 0$ $\mu^*(pH) - \sum b_j F_j$ is ample. Adding $\sum b_j F_j$ gives no harm on snc support-ness so we can safely assume that N has snc support. Using Kawamata covering we can even assume that N is an SNC divisor. Now let $N = \sum N_i$, then the natural exact sequence

$$0 \rightarrow \mathcal{O}_X \left(-H - \sum_{i=1}^{k+1} N_i \right) \rightarrow \mathcal{O}_X \left(-H - \sum_{i=1}^k N_i \right) \rightarrow \mathcal{O}_{N_{k+1}} \left(-H - \sum_{i=1}^k N_i \right) \rightarrow 0$$

enables us to do induction from Kodaira vanishing.

- The Easy Kawamata-Viehweg has several applications. For example, one can use it to prove vanishing of higher direct images.

2. MULTIPLIER IDEALS

Now we introduce multiplier ideals and prove more general Kawamata-Viehweg vanishing theorem.

- Given a log-resolution $\mu : X' \rightarrow X$ (of some sort), we let $K_{X'/X} = K_{X'} - \mu^*K_X$, which is an effective divisor supported on the exceptional locus of μ . In particular, as $\mu_*\mathcal{O}_{X'}(K_{X'/X}) = \mathcal{O}_X$, if N is an effective integral divisor on X' , then $\mu_*\mathcal{O}_{X'}(K_{X'/X} - N) \subset \mathcal{O}_X$ is an ideal sheaf of \mathcal{O}_X .
- We have Kawamata-Viehweg vanishing theorem for \mathbb{Q} -divisors.

Theorem 2.1 (Kawamata-Viehweg vanishing for \mathbb{Q} -divisors). *For a nonsingular projective variety X and an integral divisor N over X , if N is numerically equivalent to $B + \Delta$ where B is a nef big \mathbb{Q} -divisor and $\Delta = \sum a_i \Delta_i$ is a \mathbb{Q} -divisor with snc support and $0 \leq a_i < 1$ for all i (or in short Δ is a **boundary divisor**), then $H^i(X, \mathcal{O}_X(K_X + N)) = 0$ for all $i > 0$.*

One uses a similar strategy of using coverings to reduce the number of fractional parts to reduce to the case of integral divisors.

- Now we define multiplier ideals. The **multiplier ideal sheaf** for an effective \mathbb{Q} -divisor D on a smooth variety X is $\mathcal{J}(D) := \mu_*(\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D]))$ where $\mu : X' \rightarrow X$ is a log resolution of D and $[-]$ is the round-down. Similarly one can define multiplier ideal for ideal sheaves (or many ideal sheaves) of \mathcal{O}_X . This definition arises as this is something you will see if you push down vanishing from log resolution. Although not clear from the definition, the multiplier ideal is independent of choice of log resolution (Esnault-Viehweg).

- If A is an integral divisor of X , then $\mathcal{J}(A) = \mathcal{O}_X(-A)$.
- If D is any \mathbb{Q} -divisor with snc support, then $\mathcal{J}(D) = \mathcal{O}_X(-[D])$.
- If $\dim X = n$, and $\mathfrak{m}_x \subset \mathcal{O}_X$ is the maximal ideal corresponding to $x \in X$. Then

$$\mathcal{J}(\mathfrak{m}_x^c) = \begin{cases} \mathfrak{m}_x^{[c]+1-n} & \text{if } c \geq n \\ \mathcal{O}_X & \text{otherwise} \end{cases}.$$

- The multiplier ideal detects the badness of singularity. The worse the singularity is, the deeper the multiplier ideal is.
- Multiplier ideal coming from linear series or ideal sheaves can be computed in terms of the \mathbb{Q} -divisors arising from sufficiently general elements of the series. Namely, for a linear series $|V| \subset |L|$ on a smooth variety X , for a rational number $c > 0$ and $k > c$ integer, if one picks general divisors $A_1, \dots, A_k \in |V|$ and $D = \frac{1}{k}(A_1 + \dots + A_k)$ then $\mathcal{J}(c|V|) = \mathcal{J}(cD)$. In terms of ideal sheaf a general element simply means in this case that you pick generators and then the linear combination is general.
- If you add $\varepsilon D'$ for effective D' and $\varepsilon \ll 1$, the multiplier ideal does not change.
- (Kollar-Bertini theorem I) For an effective \mathbb{Q} -divisor D on a smooth variety and if $|V|$ is a free linear series, then for a general divisor $A \in |V|$, $\mathcal{J}(D + cA) = \mathcal{J}(D)$ for every $0 < c < 1$.
- $\mathcal{J}(D + A) = \mathcal{J}(D) \otimes \mathcal{O}_X(-A)$ for A integral divisor.
- For $\mathfrak{a} \subset \mathcal{O}_X$, $\mathfrak{a} \subset \mathcal{J}(\mathfrak{a})$.
- (Birational transformation rule) If $f : Y \rightarrow X$ is a proper birational map of nonsingular varieties and if D is a \mathbb{Q} -divisor on X , then $\mathcal{J}(X, D) = f_*(\mathcal{J}(Y, f^*D) \otimes \mathcal{O}_Y(K_{Y/X}))$. Or, more symmetrically, $\mathcal{O}_X(K_X) \otimes \mathcal{J}(X, D) = f_*(\mathcal{O}_Y(K_Y) \otimes \mathcal{J}(Y, f^*D))$.
- For $D = \sum a_i D_i$ an effective \mathbb{Q} -divisor and $x \in X$, $\text{mult}_x D = \sum a_i \text{mult}_x D_i$. If D has high multiplicity for some point, then $\mathcal{J}(D)$ is forced to be bad. To be more precise, if $\text{mult}_x D \geq \dim X + p - 1$, then $\mathcal{J}(D) \subset \mathfrak{m}_x^p$. This is because if you build a log resolution involving blow-up at x , then $K_{X'/X}$ at x should have order $n - 1$. Similar discussion in terms of \mathbb{Q} -divisors involves the notion of **symbolic powers**. Namely, for $Z \subset X$ an irreducible subvariety, the p -th symbolic power of I_Z is

$$I_Z^{(p)} = \{f \in \mathcal{O}_X \mid \text{ord}_x(f) \geq p \text{ for a general point } x \in Z\}.$$

Then if an effective \mathbb{Q} -divisor D has $\text{mult}_Z D \geq e + p - 1$, where Z is a codimension e irreducible subvariety, then $\mathcal{J}(D) \subset I_Z^{(p)}$.

- As multiplier ideals detect singularity to some extent, we can define certain classes of singularities in terms of multiplier ideals. For an effective \mathbb{Q} -divisor D on a smooth variety X , (X, D) is **Kawamata log terminal** if $\mathcal{J}(X, D) = \mathcal{O}_X$, and is **log canonical** if $\mathcal{J}(X, (1 - \varepsilon)D) = \mathcal{O}_X$ for all $0 < \varepsilon < 1$. More generally, the **log-canonical threshold**

at $x \in X$ is $\text{lct}(D; x) = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(X, cD)_x \subset \mathfrak{m}_x\}$. Small lct means singularity is worse. Lct is a rational number, and lower semicontinuous function on Zariski topology.

- More generally one can define **jumping numbers**, namely increasing sequence of rational numbers $\xi_i(D; x)$ such that $\mathcal{J}(X, cD)_x$ changes at that point. Thus $\xi_1(D; x)$ is lct. All jumping numbers are rational numbers.
- For $\mathfrak{a} \subset \mathcal{O}_X$ an ideal sheaf vanishing at a single point $x \in X$ (so that \mathfrak{a} is \mathfrak{m}_x -primary), the Samuel multiplicity $e(\mathfrak{a})$ satisfies the inequality

$$e(\mathfrak{a}) \geq \left(\frac{n}{\text{lct}(\mathfrak{a}; x)} \right)^n.$$

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Theorem 2.2 (Mustata). *Let $\mathfrak{a} \subset \mathcal{O}_X$ and let $\text{Arc}_m(\mathfrak{a})$ be the m -th order jet scheme along \mathfrak{a} , namely the moduli space of maps $\text{Spec } \mathbb{C}[t]/(t^{m+1}) \rightarrow V(\mathfrak{a})$. Then*

$$\text{lct}(\mathfrak{a}; X) = \dim X - \sup_{m \geq 0} \frac{\dim \text{Arc}_m(\mathfrak{a})}{m+1}.$$

- One example of explicit computation of multiplier ideals is as follows. For $X = \mathbb{C}^n$ and $\mathfrak{a} \subset \mathbb{C}[X]$ a **monomial ideal**, i.e. ideal generated by monomials, let $P(\mathfrak{a}) \subset \mathbb{R}^n$ be the convex hull in \mathbb{R}^n of the set of all exponent vectors of monomials in \mathfrak{a} . For any rational $c > 0$, $\mathcal{J}(c\mathfrak{a})$ is the monomial ideal generated by all monomials x^v where $v + (1, 1, \dots, 1)$ is in the interior of $cP(\mathfrak{a})$.
- There is analytic construction of multiplier ideal. There is a generalization of the notion to singular (but normal) ambient space. Namely the definition cannot literally carry on because K_X is not a Cartier divisor so it cannot be pulled back, but one can add additional “boundary divisor” Δ to make it (\mathbb{Q} -)Cartier. In particular it is usual to consider pair (X, Δ) instead of just X .
- $\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D)\mathcal{O}_H$ for $H \subset X$ nonsingular irreducible hypersurface not contained in the support of D . Namely, restriction can only make singularities worse.
- Multiplier ideals are subadditive wrt D . Analogously for ideal sheaves we have Mustata’s summation theorem, which says that $\mathcal{J}(X, (\mathfrak{a} + \mathfrak{b})^c) \subset \sum_{\lambda+\mu=c} \mathcal{J}(X, \mathfrak{a}^\lambda)\mathcal{J}(X, \mathfrak{b}^\mu)$. This says that lct is subadditive.
- One has something analogous to birational transformation rule for multiplier ideals.

Now we discuss vanishing results.

Theorem 2.3 (Local vanishing). *For a \mathbb{Q} -divisor D on a smooth variety X , and for a log-resolution $\mu : X' \rightarrow X$ of D , $R^{>0}\mu_*\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D]) = 0$.*

This is because, at least when X, X' are projective, Kawamata-Viehweg says $H^{>0}(X', \mathcal{O}_{X'}(K_{X'} + \mu^*A - [\mu^*D])) = 0$ for sufficiently positive ample divisor A (so that $\mu^*(A - D)$ is big nef), which says that $R^{>0}\mu_*\mathcal{O}_{X'}(K_{X'} + \mu^*A - [\mu^*D]) = R^{>0}\mu_*\mathcal{O}_{X'}(K_{X'/X} - [\mu^*D]) \otimes \mathcal{O}_X(K_X + A)$ is zero.

Theorem 2.4 (Global vanishing, frequently referred as Nadel vanishing). *For a \mathbb{Q} -divisor D on a smooth projective variety X , if L is an integral divisor such that $L - D$ is big nef, then $H^{>0}(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{J}(D)) = 0$.*

This is just spectral sequence + local vanishing. From this, one has Skoda’s theorem:

Theorem 2.5 (Skoda). *For $\mathfrak{a} \subset \mathcal{O}_X$ and X nonsingular of dimension n , if $m \geq n$ is an integer, then $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a}\mathcal{J}(\mathfrak{a}^{m-1})$. More generally the same holds if you multiply this with \mathfrak{b}^c for a nonzero ideal $\mathfrak{b} \subset \mathcal{O}_X$ and a positive rational number c .*

In particular, this implies the celebrated Briancon-Skoda theorem, which says that in the situation of Skoda's theorem, the integral closure of \mathfrak{a}^m is contained in \mathfrak{a}^{m+1-n} . Here the integral closure of an ideal means, if $\nu : X^+ \rightarrow X$ is the normalization of the blowup of X along \mathfrak{a} , $\nu_*(\mathfrak{a}\mathcal{O}_{X^+})$.

Kollar proved that if Θ is a theta divisor of a ppav (namely Θ is ample with $h^0(A, \mathcal{O}_A(\Theta)) = 1$), then (A, Θ) is log canonical.

3. MISCELLANY

We record some miscellaneous vaguely related results before we move on further.

3.1. Deligne-Illusie. Deligne-Illusie proved Kodaira vanishing using algebraic proof of E_1 -degeneration of Hodge-de Rham spectral sequence. To be precise it seems like such implication is observed first by Esnault-Viehweg and Deligne-Illusie provided a simple algebraic proof of degeneration.

Theorem 3.1 (Deligne-Illusie). *Let S be an \mathbb{F}_p -scheme. Assume that there is a flat lift T of S over $\mathbb{Z}/p^2\mathbb{Z}$. Let X/S be smooth, and let $F : X \rightarrow X^{(p)}$ be the relative Frobenius of X/S . If $X^{(p)}$ has a smooth lift over T , then the complex of $\mathcal{O}_{X^{(p)}}$ -modules $\tau_{<p}F_*\Omega_{X/S}$ is decomposable in the derived category of $\mathcal{O}_{X^{(p)}}$ -modules.*

Before we prove the theorem, we first prove why this gives E_1 -degeneration of Hodge-to-de Rham spectral sequence in characteristic 0, and how it can prove Kodaira vanishing theorem.

Theorem 3.2. *Let K be a field of characteristic zero, and let X be a smooth proper K -scheme. Then the Hodge-to-de Rham spectral sequence of X over K ,*

$$E_1^{p,q} = H^q(X, \Omega_{X/K}^p) \Rightarrow H_{\text{dR}}^*(X/K),$$

degenerates at E_1 .

Proof. Because all cohomology groups involved are finite-dimensional vector spaces, we see that the degeneration will follow if we prove dimensions match for Hodge decomposition, i.e. $h^n = \sum_{i+j=n} h^{i,j}$. Indeed this is how transcendently one can prove degeneration using Hodge decomposition of Hodge theory.

By "spreading out", X comes from a smooth proper $S = \text{Spec } A$ -scheme \mathfrak{X} where $A \subset K$ is a sub- \mathbb{Z} -algebra of finite type. Generic smoothness implies that A can be assumed to be smooth over \mathbb{Z} (although it can miss finitely many primes of \mathbb{Z}). Now the strategy is that we can do reduction mod any sufficiently large prime p , so we choose an appropriate prime p and we transfer algebraic data of $h^{i,j}$'s and h^n to mod p fiber, and prove E_1 -degeneration for mod p fiber instead by using Theorem 3.1; the \mathbb{Z} -smoothness even ensures that the mod p fiber is liftable to W_2 , so we can apply the Theorem.

However, E_1 -degeneration mod p happens with certain restriction. Namely, there are actual examples where mod p E_1 -degeneration fails. If you take in the statement of Theorem 3.1 $S = \text{Spec } k$ for a perfect field of characteristic p and take cohomology of the isomorphism in derived category, then we get an isomorphism

$$\bigoplus_{i+j=n, i < p} H^j(X^{(p)}, \Omega_{X^{(p)}/k}^i) \xrightarrow{\sim} H^n(X^{(p)}, F_*\Omega_{X/k}^n).$$

Absolute Frobenius is an isomorphism for $\text{Spec } k$, so $\dim_k H^j(X, \Omega_{X/k}^i) = \dim_k H^j(X^{(p)}, \Omega_{X^{(p)}/k}^i)$ follows from the compatibility of base change and formation of Ω^i 's. On the other hand, note that the relative Frobenius $F : X \rightarrow X^{(p)}$ is a homeomorphism. Thus, $H^n(X^{(p)}, F_*\Omega_{X/k}^i)$ is canonically isomorphic to $H^n(X, \Omega_{X/k}^i) = H_{\text{dR}}^n(X/k)$. So what this proves is that as long as $n < p$ the desired equality of dimensions holds. This will not annoy us if $\dim X < p$ (this is basically Fontaine-Laffaille condition).

Now the point is that we can take any sufficiently large prime, and the dimension does not change even if you take a fiber, so you can ignore this issue. This is basically it, if we show that $h^{i,j}$ and h^n do not change after taking mod p fibers. But, if we let $f : \mathcal{X} \rightarrow S$ be the structure morphism, then $R^j f_* \Omega_{\mathcal{X}/S}^i$ and $R^n f_* \Omega_{\mathcal{X}/S}^i$ are generically locally free, so one might as well restrict to a nonempty open set (finite intersection of Zariski dense open sets) to assume that all these sheaves are free. Then $h^{i,j}$ and h^n evidently stay the same for every reduction as the ranks of those sheaves stay the same. We are thus done. \square

Now we can prove Kodaira vanishing. Actually there are two ways, and we exhibit both, because they focus on different things. The other approach will be exhibited in the next section.

Proof of Kodaira vanishing, following Deligne-Illusie. Let X/K be a smooth projective scheme of pure dimension d , and let L be an ample line bundle over X . We want to prove that $H^j(X, L \otimes \Omega_{X/K}^i) = 0$ for $i + j > d$ and $H^j(X, L^{\otimes p^n} \otimes \Omega_{X/K}^i) = 0$ for $i + j < d$ (Nakano vanishing).

We use the same strategy as above, namely we do spread-out and transfer the data to mod p situation. Here we need that any ample invertible sheaf even comes from an ample invertible sheaf over spread out stuff, which is certainly true. Then literally the same proof can be applied, because ampleness is preserved by basechange, if we prove the analogous vanishing statement mod p . And indeed such thing is true if $d < p$, the same restriction as mod p E_1 -degeneration.

The proof of mod p Kodaira-Nakano vanishing (k perfect field of characteristic p) under $d < p$ is as follows. By ampleness, there is $n \geq 0$ such that $H^j(X, L^{\otimes p^n} \otimes \Omega_{X/k}^i) = 0$ for all $j > 0$, or by Serre duality, $H^j(X, L^{\otimes -p^n} \otimes \Omega_{X/k}^i) = 0$ for all $j < d$. We will show that we can decrease n one by one to reach $L^{\otimes -1}$. By letting $L^{\otimes p^{n-1}}$ to be L , we can therefore assume that $n = 1$.

Let L' be the inverse image of L over $X^{(p)}$ via the natural map $X^{(p)} \rightarrow X$. Then the same argument as the proof of degeneration implies that

$$H^r(X^{(p)}, L' \otimes F_*\Omega_{X/k}^i) \cong \bigoplus_{i+j=r} H^j(X^{(p)}, L' \otimes \Omega_{X^{(p)}/k}^i).$$

The RHS is the pullback of $H^j(X, L \otimes \Omega_{X/k}^i)$ via absolute Frobenius of k , so ETS $H^r(X^{(p)}, L' \otimes F_*\Omega_{X/k}^i) = 0$. On the other hand, as $L' \otimes F_*\Omega_{X/k}^i = F_*(F^*L' \otimes \Omega_{X/k}^i) \cong F_*(L^{\otimes p} \otimes \Omega_{X/k}^i)$, so $H^j(X^{(p)}, L' \otimes F_*\Omega_{X/k}^i) = 0$ for all $i + j < d$. Thus, by the spectral sequence abutting to hypercohomology, we see that $H^r(X^{(p)}, L' \otimes F_*\Omega_{X/k}^i) = 0$ for all $r < d$. We are done. \square

Proof of Theorem 3.1. Recall that for any $f : X \rightarrow Y$ over \mathbb{F}_p , there is a **Cartier homomorphism**

$$\gamma : \bigoplus \Omega_{X^{(p)}/Y}^i \rightarrow \bigoplus \mathcal{H}^i F_* \Omega_{X/Y}^*$$

where $F : X \rightarrow X^{(p)}$ is the relative Frobenius of X/Y , which is a unique homomorphism of graded $\mathcal{O}_{X^{(p)}}$ -algebras such that $\gamma^0 = F^* : \mathcal{O}_{X^{(p)}} \rightarrow F_*\mathcal{O}_X$ and $\gamma^1(1 \otimes ds) = s^{p-1}ds$ for ds coming from $\Omega_{X/Y}^1$. If f is smooth, γ is an isomorphism and often also written as “ C^{-1} ”.

Using Cartier isomorphism, decomposability is equivalent to providing an arrow of $D(\text{Mod}_{\mathcal{O}_{X^{(p)}}})$

$$\varphi : \bigoplus_{i < p} \Omega_{X^{(p)}/S}^i[-i] \rightarrow F_* \Omega_{X/S}^*$$

inducing C^{-1} on \mathcal{H}^i for all $i < p$. Note that as the Cartier isomorphism is essentially trying to do “ $\frac{1}{p}F^*$ ”, one can really construct the arrow if F lifts globally from S to smooth lifts of $X^{(p)}$ and X over T , because then dividing the pullback of lift of F by p makes sense (you can divide multiples of p in $\mathbb{Z}/p^2\mathbb{Z}$ by p). In this case as the arrow is defined degree by degree one even has the decomposability for the whole $F_* \Omega_{X/S}^*$.

However in general one cannot expect this. Instead, as there is no local obstruction in lifting, we would like to prove that, as we are given a smooth lift Z' of $X^{(p)}$, one can glue φ^1 **in the derived category** to get a homomorphism

$$\varphi_{Z'}^1 : \Omega_{X^{(p)}/S}^1[-1] \rightarrow F_* \Omega_{X/S}^*$$

inducing C^{-1} over \mathcal{H}^1 . We pick a covering \mathcal{U} of X where there is a local lifting Z_i of U_i and $G_i : Z_i \rightarrow Z$ of $F|_{U_i}$. As this is in the level of derived categories, we can instead try to define a homomorphism

$$\varphi_{Z', (\mathcal{U}, (G_i))}^1 : \Omega_{X^{(p)}/S}^1[-1] \rightarrow C(\mathcal{U}, F_* \Omega_{X/S}^*),$$

where the target is the total complex of the Cech bicomplex. In particular its egree 1 part is $C^1(\mathcal{U}, F_* \mathcal{O}_X) \oplus C^0(\mathcal{U}, F_* \Omega_{X/S}^1)$. We want the second component of $\varphi_{Z', (\mathcal{U}, (G_i))}^1$ to be $\varphi_{G_i}^1$'s coming from the global lifting. Then what we need in the first component is, for each i, j , a homomorphism

$$h_{ij} : \Omega_{X'/S}^1|_{U_{ij}^{(p)}} \rightarrow F_* \Omega_{X/S}^1|_{U_{ij}},$$

such that $f_j - f_i = dh_{ij}$ and $h_{ij} + h_{jk} = h_{ik}$. This comes from the fact that liftings form a torsor over Hom and there is no “automorphism” so cocycle condition is automatically satisfied.

Now from derived category level we can still define $\varphi_{Z'}^i$, for $i < p$, as the composition

$$\Omega_{X^{(p)}/S}^i[-i] \rightarrow (\Omega_{X^{(p)}/S})^{\otimes i}[-i] \xrightarrow{(\varphi_{Z'}^1)^{\otimes i}} (F_* \Omega_{X/S}^*)^{\otimes i} \xrightarrow{\amalg} F_* \Omega_{X/S}^*$$

where the first arrow is the antisymmetrization arrow

$$\omega_1 \wedge \cdots \wedge \omega_i \mapsto \frac{1}{i!} \sum_{\sigma \in S_i} \text{sgn}(\sigma) \omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(i)}.$$

Here the derived tensor product becomes ordinary tensor products because both sheaves are locally free. This is the desired arrow. \square

Miscellany to miscellany:

- **Ordinary** If X and F are lifted to $W_2(k)$, then X is **ordinary**, whose definition is that $H^i(X, B^i \Omega_{X/S}^*) = 0$ where B is coboundary (i.e. image of de Rham differentials). This is always the case when X is affine, but it is a very special condition when X is proper. This coincides with the ordinarity of abelian varieties.

To be a little more precise, there are several interrelated notions. We say X is **de Rham-decomposable** if the decomposition of de Rham complex into direct sum of Hodge cohomologies holds for the full complex (not the truncated complex). We say X is **parallelizable** if $\Omega_{X/k}^1$ is free. We say X is **Frobenius-decomposable** if $\mathcal{O}_{X^{(p)}} \rightarrow F_* \mathcal{O}_X$ admits a retraction. Note that obstruction to liftability to W_2 is the vanishing of the extension

$$0 \rightarrow \mathcal{O}_{X^{(k)}} \rightarrow F_* \mathcal{O}_X \rightarrow F_* Z \Omega_{X/k}^1 \xrightarrow{C} \Omega_{X'/k}^1 \rightarrow 0,$$

so Frobenius-decomposability implies that X is liftable over W_2 . There are many results regarding relations between these concepts.

- Smooth proper parallelizable k -scheme is Frobenius-decomposable iff ordinary iff F^* on the top cohomology is an isomorphism.
- Smooth proper ordinary parallelizable k -scheme lifts over W_2 . If furthermore X is connected and $k = \bar{k}$, then X is a Galois étale quotient of order a power of p of an abelian variety.
- Smooth projective ordinary parallelizable k -scheme has unique lift of the pair (X, F) over W_n for any $n \geq 2$ and even $W(k)$. This is a generalization of Serre-Tate canonical lifting.
- As indicated, if X is smooth proper and liftable over W_2 , then X is ordinary.
- **Gerbes** This is not relevant but I read appendix of Tannakian Categories by Deligne and Milne.
 - A functor $\alpha : F \rightarrow A$ is a **fibered category** if for every morphism $a : V \rightarrow U$ in A and $F' \in \text{Obj}(F_V)$ an inverse image a^*F' of F' exists, where F_V is the category whose objects are $F'' \in \text{Obj}(F)$ with $\alpha(F'') = U$ and whose morphisms are morphisms sent to id_U via α , and the inverse image is some universal object that satisfies the analogous universal property for pullbacks, and the composite of two cartesian morphisms is cartesian, where a cartesian morphism is a morphism through which objects can be pulled back.
 - For a faithfully flat morphism $a : V \rightarrow U$ of affine S -schemes, $\alpha : F \rightarrow \text{Aff}_S$ fibered category, and $F \in \text{Obj}(F_U)$, a **descent datum** on F relative to a is an isomorphism $\phi : p_1^*(F) \rightarrow p_2^*(F)$ satisfying cocycle condition. These form a category of descent data $\text{Desc}(V/U)$. There is a natural functor $F_U \rightarrow \text{Desc}(V/U)$ under which F maps to (a^*F, ϕ) . α is a **stack** when this functor is an equivalence of categories for all a . This means morphisms and objects given fpqc-locally patch to global objects.
 - A **gerbe** over S is a stack $G \rightarrow \text{Aff}_S$ such that, all morphisms are isomorphisms in every category G_U , there is a faithfully flat $U \rightarrow S$ such that G_U is nonempty, and any two objects of G_U are locally isomorphic. A **neutral** gerbe is when G_S is nonempty.
 - Given an fpqc sheaf of groups F/S , the fibered category $\text{Tors}(F) \rightarrow \text{Aff}_S$, for which $\text{Tors}(F)_U$ is the category of right F -torsors on U , is a neutral gerbe. Conversely given a neutral gerbe G , if there is $Q \in \text{Obj}(G_S)$ and if $F = \text{Aut}(Q)$ is a sheaf of commutative groups on S , then G is isomorphic to $\text{Tors}(F)$.
 - For a gerbe G/S , and for $U \rightarrow S$ in Aff_S and $x \in G_U$, the presheaf $V \mapsto \text{Aut}(x|_V)$ is a sheaf of groups over U for fpqc topology. If x, y are isomorphic objects in G_U , then we have an isomorphism $\text{Aut}(x) \rightarrow \text{Aut}(y)$ unique up to inner automorphism. Thus we are led to consider the category LI_U whose objects are sheaves of groups on U and the morphisms $F \rightarrow G$ are the sections of the quotient sheaf $G \backslash \text{Hom}(F, G)/F$ where F and G act by inner automorphism. In this way, we get a fibered category $\text{LI} \rightarrow \text{Aff}_S$ in which the morphisms patch. Then by standard procedure we can add objects to get a fibered category $\text{LIEN} \rightarrow \text{Aff}_S$ in which the objects also patch, i.e. which is a stack. An object of LIEN_U is called a **band** or a **lien** over U . A gerbe defines up to a unique isomorphism a band B over S . In this case we say G is banded by B , or that it is a B -gerbe.

- Generalization by Illusie to relative and semistable situation with more general coefficients in F -crystals gives an algebraic proof of the following theorem on VHS.

Theorem 3.3 (Zucker). *For a nonsingular curve S/\mathbb{C} and a vector bundle V underlying a polarizable VHS of weight m over S , if $j : S \hookrightarrow \bar{S}$ is Nagata compactification, then there is a natural polarizable Hodge structure of weight $m + i$ on $H^i(\bar{S}, j_* V)$.*

3.2. Esnault-Viehweg. Now we prove the Kodaira vanishing following Esnault-Viehweg. It does use the same degeneration at E_1 of (log) Hodge-de Rham spectral sequence, but the rest of argument does not rely on characteristic p methods (in particular it does not appeal to mod p Kodaira vanishing). Unlike Deligne-Illusie proof of Kodaira vanishing, this proof seems more convincing to me that vanishing theorems should follow from E_1 degeneration.

WLOG $K = \bar{K}$. Let $n \gg 0$ be such that $L^{\otimes n}$ is very ample, $L^{\otimes n} = \mathcal{O}_X(H)$. Let $s \in H^0(X, L^{\otimes n})$ be the corresponding section. Take the cyclic cover $\pi : Y \rightarrow X$ taking the “ n -th root of s ” as discussed before (the construction works equally well over K). Then the Galois group $\text{Gal}(Y/X)$ is cyclic of order n ; let σ be a generator of G , which acts on Y and $D = (\pi^* H)_{\text{red}}$, so on $\pi_* \Omega_{Y/K}^i$ and $\pi_* \Omega_{Y/K}^i(\log D)$. Here the **sheaf of log differentials** can be defined by using $\Omega_{Y/K}^i(* D)$, the pushforward of $\Omega_{Y-D/K}^i$, which can have arbitrarily bad poles along D , and take meromorphic forms α such that α and $d\alpha$ both have at worst simple poles along D . Both sheaves decompose in a direct sum of sheaves of eigenvectors of σ , and we will use it to get vanishing theorems.

We will from now on often omit “/ K ”.

- As in the ordinary case, the log differential sheaves are wedge powers of Ω^1 . Namely $\Omega_X^i(\log D) = \wedge^i \Omega_X^1(\log D)$.

Locally if D is cut out by f_1, \dots, f_s and if X has local parameters f_1, \dots, f_n then $\Omega_X^1(\log D)$ is spanned by $\frac{df_1}{f_1}, \dots, \frac{df_s}{f_s}, f_{s+1}, \dots, f_n$.

- Suppose we are given an integrable log connection $\nabla : \mathcal{E} \rightarrow \Omega_X^1(\log D) \otimes \mathcal{E}$ for a vector bundle \mathcal{E} . Suppose further that f_1, \dots, f_s are actually globally defined, so that we define $D_i = V(f_i)$. Then we can define the **residue map** along D_i to be the composed map

$$\text{Res}_{D_i}(\nabla) : \mathcal{E} \xrightarrow{\nabla} \Omega_X^1(\log D) \otimes \mathcal{E} \xrightarrow{\beta_i \otimes \text{id}} \mathcal{O}_{D_i} \otimes \mathcal{E},$$

where β_i picks out the coefficient of $\frac{df_i}{f_i}$. This map factors through the restriction map $\mathcal{E} \rightarrow \mathcal{O}_{D_i} \otimes \mathcal{E}$, because the map is \mathcal{O}_X -linear and the resulting sheaf is \mathcal{O}_{D_i} -module.

- Given a divisor $B = \sum_{j=1}^s \mu_j D_j$, ∇ induces a connection ∇^B with log poles on $\mathcal{E}(B)$ by

$$\nabla^B \left(\prod_{j=1}^s f_j^{-\mu_j} e \right) = \prod_{j=1}^s f_j^{-\mu_j} \nabla(e) + d \left(\prod_{j=1}^s f_j^{-\mu_j} \right) \otimes e.$$

Then the residue satisfies

$$\text{Res}_{D_j}(\nabla^B) = \text{Res}_{D_j}(\nabla) - \mu_j \text{id}_{D_j}.$$

We say that (∇, \mathcal{E}) satisfies the condition (*) if for all $B \geq D$, $\text{Res}_{D_j}^B$ is an automorphism of $\mathcal{E}|_{D_j}$. We say that (∇, \mathcal{E}) satisfies the condition (!) if for all $B \leq 0$, $\text{Res}_{D_j}(\nabla^B)$ is an automorphism of $\mathcal{E}|_{D_j}$. In other words, (*) holds iff no positive integer is an eigenvalue of $\text{Res}_{D_j}(\nabla)$, and (!) holds iff no nonpositive integer is an eigenvalue of $\text{Res}_{D_j}(\nabla)$.

- Note that if $\text{Res}_{D_j}(\nabla) : \mathcal{E}|_{D_j} \rightarrow \mathcal{E}|_{D_j}$ is an isomorphism then the inclusion $(\Omega_X^*(\log D) \otimes \mathcal{E}(-D_j), \nabla^{-D_j}) \rightarrow (\Omega_X^*(\log D) \otimes \mathcal{E}, \nabla)$ is a quasi-isomorphism, because one can factor this

map into successive composition of natural inclusions $\mathcal{E}^{v+1} \rightarrow \mathcal{E}^v$ where \mathcal{E}^v is the complex

$$\begin{aligned} \mathcal{E}(-D_j) &\rightarrow \Omega_X^1(\log D) \otimes \mathcal{E}(-D_j) \rightarrow \cdots \rightarrow \Omega_X^{v-1}(\log D) \otimes \mathcal{E}(-D_j) \\ &\rightarrow \Omega_X^v(\log(D - D_j)) \otimes \mathcal{E} \rightarrow \Omega_X^{v+1}(\log D) \otimes \mathcal{E} \rightarrow \cdots \rightarrow \Omega_X^n(\log D) \otimes \mathcal{E}, \end{aligned}$$

and the cone (quotient in this case) is by elementary calculation the complex

$$0 \rightarrow \Omega_{D_j}^v(\log(D - D_j)|_{D_j}) \otimes \mathcal{E} \xrightarrow{(-1)^v \otimes \text{Res}_{D_j}(\nabla)} \Omega_{D_j}^v(\log(D - D_j)|_{D_j}) \otimes \mathcal{E} \rightarrow 0,$$

which has no cohomology by assumption. Therefore, in particular (*) implies that

$$(\Omega_X^*(\log D) \otimes \mathcal{E}, \nabla) \rightarrow (\Omega_X^*(\log D) \otimes \mathcal{E}(B), \nabla^B),$$

is a quasi-iso for $B \geq 0$, and (!) implies that

$$(\Omega_X^*(\log D) \otimes \mathcal{E}(B), \nabla^B) \rightarrow (\Omega_X^*(\log D) \otimes \mathcal{E}, \nabla),$$

is a quasi-iso for $B \leq 0$.

Now consider again the situation of cyclic cover Y/X of L, H, D, s . Eventually we can actually prove something more general, in the setting as follows.

- L is a line bundle,
- $H = \sum_{j=1}^r \alpha_j H_j$ is an effective divisor with normal crossings support such that $L^n \mathcal{O}_X(H)$ for some $n > 0$.
- There is $s \in H^0(X, L^n)$ whose zero divisor is H .

One way of neatly describing the cyclic cover is as follows. Note that $s \in H^0(X, L^n)$ gives the dual $s^\vee : L^{-n} \rightarrow \mathcal{O}_X$, and this enables us to give an \mathcal{O}_X -algebra structure on $A' = \bigoplus_{i=0}^{n-1} L^{-i}$ (namely, it is the quotient of $\bigoplus_{i=0}^{\infty} L^{-i}$ by the ideal generated by $s^\vee(l) - l$ for local sections l of L^{-n}). Then $Y \rightarrow X$ is the normalization of $\text{Spec}_X(A') \rightarrow X$, the relative Spec of A' over X . This is characterised as the unique finite normal map $\pi : Y \rightarrow X$ such that it factors through $\text{Spec}_X(A')$ where the morphism $A' \rightarrow \pi_* \mathcal{O}_Y$ of \mathcal{O}_X -algebras is isomorphic over a dense open subscheme of X . But it turns out that Y has a more direct description.

Proposition 3.1. *The morphism $A' \rightarrow \pi_* \mathcal{O}_Y$ of \mathcal{O}_X -algebras is the same as the inclusion*

$$\bigoplus_{i=0}^{n-1} L^{-i} \rightarrow \bigoplus_{i=0}^{n-1} (L^{(i)})^{-1},$$

where we define $L^{(i)} = L^{(i,H)} = L^i(-[\frac{i}{n}H])$.

Thus in the case of classical Kodaira vanishing, this is saying that $\text{Spec}_X(A')$ and Y are actually the same ($[\frac{i}{n}H] = 0$ for all $0 \leq i < n$).

Proof. Note that the RHS, which we will denote as A , has a multiplication so that the inclusion is homomorphism, because the multiplication can be given by

$$(L^{(i)})^{-1} \otimes (L^{(j)})^{-1} = L^{-i-j}([\frac{i}{n}H] + [\frac{j}{n}H]) \rightarrow L^{-i-j}([\frac{i+j}{n}H]) = (L^{(i+j)})^{-1},$$

just because $[\frac{i}{n}H] + [\frac{j}{n}H] \leq [\frac{i+j}{n}H]$, and also if $k \geq n$ we have

$$L^{(k)} = L^k(-[\frac{k}{n}H]) = L^{k-n}(-[\frac{k-n}{n}H]) \otimes L^n(-H) \xrightarrow{\sim} L^{(k-n)}$$

where the map is realized by s^\vee .

As we can check this after removing a codimension 2 or larger subscheme, so in particular we can assume H_{red} is nonsingular. Then it reduces to prove that $\text{Spec}_X(A) \rightarrow X$ is finite and

$\text{Spec}_X(A)$ is normal. We can check this locally so we can assume that $X = \text{Spec } B$ and H_{red} has only one component, say $H = \alpha_1 H_1$. We can also assume that L is free and H_1 is a principal divisor. Choose $L^i \cong \mathcal{O}_X$ for all i and $H_1 = \text{div}(f_1)$. Then $s = u f_1^{\alpha_1}$ for some $u \in B^*$, under the isomorphism $L^n \cong \mathcal{O}_X$. Note that in this case we should also have in mind the case of empty divisor.

In any case, A' is then identified with $B[t]/(t^n - s)$, and $A = \bigoplus_{i=0}^{n-1} B t^i f_1^{-\lfloor \frac{i}{n} \alpha_1 \rfloor}$. If $\alpha_1 = 0$ (or H is trivial) or $\alpha_1 = 1$, then trivially $A = A'$, and evidently $A = A'$ is fiberwise nonsingular. Now in general we see that A' is a subring of $R = B[t_0, t_1]/(t_0^n - u, t_1^n - f_1)$ via identifying t with $t_0 t_1^{\alpha_1}$. Note R is finite and fiberwise nonsingular over B .

Note that R has an action of $(\mathbb{Z}/n\mathbb{Z})^2$ by twisting t_0 and t_1 by powers of n -th roots of unity (we fix a primitive n -th root of unity ζ_n for convenience). Then A' is actually equal to $R^{H'}$ where $H' \subset (\mathbb{Z}/n\mathbb{Z})^2$ is the kernel of $(\mathbb{Z}/n\mathbb{Z})^2 \xrightarrow{(i,j) \mapsto i + \alpha_1 j} \mathbb{Z}/n\mathbb{Z}$. So this is the quotient which is also normal and finite. \square

Note that this proof says that $\pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{n-1} (L^{(i)})^{-1}$ is the decomposition of $\pi_* \mathcal{O}_Y$ into $G = \text{Gal}(Y/X)$ -eigenspaces.

Proposition 3.2. *Suppose H_{red} is nonsingular. Then the differential $d : \mathcal{O}_Y \rightarrow \Omega_Y^1$ on Y induces a log integrable connection*

$$\pi_*(d) : \pi_* \mathcal{O}_Y = \bigoplus_{i=0}^{n-1} (L^{(i)})^{-1} \rightarrow \pi_* \Omega_Y^1 = \bigoplus_{i=0}^{n-1} \Omega_X^1(\log H^{(i)}) \otimes (L^{(i)})^{-1},$$

compatible with the direct sum decomposition, where $H^{(i)} = \sum_{1 \leq j \leq r, \frac{i \alpha_j}{n} \in \mathbb{Z}} H_j$. If we denote the i -th component of $\pi_*(d)$ as $\nabla^{(i)} : (L^{(i)})^{-1} \rightarrow \Omega_X^1(\log H^{(i)}) \otimes (L^{(i)})^{-1}$, then $\nabla^{(i)}$ is a log integrable connection with residue

$$\text{Res}_{H_j}(\nabla^{(i)}) = \left(\frac{i \alpha_j}{n} - \left[\frac{i \alpha_j}{n} \right] \right) \text{id}_{\mathcal{O}_{H_j}}.$$

Proof. We need to first see why $\pi_* \Omega_Y^1 = \bigoplus_{i=0}^{n-1} \Omega_X^1(\log H^{(i)}) \otimes (L^{(i)})^{-1}$. We will use that $\pi_* \Omega_Y^1(\log \pi^* H) = \bigoplus_{i=0}^{n-1} \Omega_X^1(\log H) \otimes (L^{(i)})^{-1}$, which comes from Hurwitz's formula, $\pi^* \Omega_Y^1(\log H) = \Omega_Y^1(\log \pi^* H)$. This fact can be proven as follows. We work locally so we can assume that $X = \text{Spec } B$ and $H = \alpha_1 H_1$. Then if n divides α_1 , f_1 is the defining equation of $(\pi^* H_1)_{\text{red}}$ (f_1 as in the above proof), so Hurwitz's formula follows. Splitting the cyclic cover into two parts, we can thus assume that $(n, \alpha_1) = 1$. Then from our setting of local computation one can see that $(\pi^* H_1)_{\text{red}}$ has a defining equation g with $g^n = u^a f_1$ for some $a \in \mathbb{N}$. So

$$n \frac{dg}{g} = \frac{df_1}{f_1} + a \frac{du}{u}.$$

As $a \frac{du}{u} \in \Omega_X^1$, $\frac{df_1}{f_1}$ and $\pi^* \Omega_X^1$ generate $\Omega_Y^1(\log \pi^* H)$.

Now we want to detect $\pi_* \Omega_Y^1$ inside $\pi_* \Omega_Y^1(\log \pi^* H)$. Still we work locally as in the above paragraph. Local section of $\Omega_X^1(\log H) \otimes (L^{(i)})^{-1}$ is written as ϕg_i for $\phi \in \Omega_X^1(\log H)$ and $g_i = t^i f_1^{-\lfloor \frac{i \alpha_1}{n} \rfloor}$. Since $g_i^n = u^i f_1^{i \alpha_1 - n \lfloor \frac{i \alpha_1}{n} \rfloor}$, it has a zero along $(\pi^* H_1)_{\text{red}}$ if and only if $\frac{i \alpha_1}{n} \notin \mathbb{Z}$. This explains why $\pi_* \Omega_Y^1 = \bigoplus_{i=0}^{n-1} \Omega_X^1(\log H^{(i)}) \otimes (L^{(i)})^{-1}$.

Now from this we see that

$$\frac{dg_i}{g_i} = \frac{i}{n} \frac{du}{u} + \left(\frac{i \alpha_1}{n} - \left[\frac{i \alpha_1}{n} \right] \right) \frac{df_1}{f_1}.$$

As local section of $(L^{(i)})^{-1}$ is of form $g_i\phi$ for g_i as above and $\phi \in \mathcal{O}_X$, we see that

$$\text{Res}_{H_1}(\nabla^{(i)})(g_i\phi) = \left(\frac{i}{n}\alpha_1 - \left[\frac{i\alpha_1}{n}\right]\right)g_i\phi|_{D_1}.$$

□

Now we can extend the log integrable connection to general case where we don't assume nonsingularity of H_{red} as we can extend over codimension 2 or more subscheme.

Now a little more observation will give the Kodaira vanishing.

Proposition 3.3. *We retain the same notation, with additional assumption that $H = \sum H_j$ is a reduced divisor. Let $H' = \sum_{j=1}^r \mu_j H_j \geq 0$. Then $H^b(X, L^{-k}(-H')) \rightarrow H^b(X, L^{-k})$ is surjective, for $1 \leq k < n$.*

This will obviously imply the classical Kodaira vanishing by setting $H' = H$ as the LHS will be zero by ampleness.

Proof. As indicated before, we need degeneration of log de Rham complex, which we will just assume. Note that from our reducedness assumption, $L^k = L^{(k)}$. Also, from our calculation of residues, $\nabla^{(k)}$ satisfies the condition (!).

The surjectivity will follow for any (E, ∇) that has E_1 -degeneration and satisfies (!) ($E = L^{-k}$, $\nabla = \nabla^{(k)}$). From (!), we know that

$$\Omega_X^*(\log H) \otimes E(-H') \rightarrow \Omega_X^*(\log H) \otimes E$$

is a quasi-isomorphism. If you pick 0-th part of those complexes we get a natural commutative diagram of hypercohomologies

$$\begin{array}{ccc} H^b(X, \Omega_X^*(\log H) \otimes E) & \longrightarrow & H^b(X, E) \\ \parallel & & \uparrow \\ H^b(X, \Omega_X^*(\log H) \otimes E(-H')) & \longrightarrow & H^b(X, E(-H')) \end{array}$$

where the right vertical arrow is what we need to show to be surjective. But because Hodge-to-de Rham spectral sequence degenerates at E_1 , the top horizontal arrow is actually a quotient map (i.e. de Rham cohomology has a filtration with subquotients exactly Hodge cohomologies). Therefore the right vertical arrow is also surjective. □

One sees in the above proof that one can use induction on the depth of filtration to similarly deduce Kodaira-Nakano vanishing (i.e. the deepest part is zero then the next deepest subquotient is actually a quotient, etc.).

3.3. Kodaira vanishing for Deligne-Mumford stacks. We note that Kawamata-Viehweg vanishing theorem follows from Kodaira vanishing theorem for Deligne-Mumford stacks.

Theorem 3.4. *Suppose k is a field, and X/k is a smooth projective variety of pure dimension d . Let $E = \sum a_i D_i \in \text{Div}(X)_{\mathbb{Q}}$ be an ample \mathbb{Q} -divisor, where D_i 's are the distinct irreducible components. Suppose the support of $\{E\}$, the fractional part of E , is a normal crossings divisor D . Then, $H^i(X, \Omega_X^j(\log D) \otimes \mathcal{O}_X(-[E])) = 0$, for $i + j < d$ (and $i + j < p$ if k is perfect of characteristic p , where in this case the smooth log scheme $(X, \mathcal{M}_{X,D})$ has a log smooth lifting to $W_2(k)$, where $\mathcal{M}_{X,D}$ is the log structure on X associated to D).*

Proof. Write $\{E\} = \sum_i \frac{a_i}{b_i} D_i$ with $(a_i, b_i) = 1$. By openness of ample cone, we can perturb E a little bit to assume that $\text{char}(k)$ does not divide b_i for every i . The log structure $\mathcal{M}_{X,D}$ is locally free. To be more precise, for every geometric point $\bar{x} \rightarrow X$, there is a canonical isomorphism $\overline{\mathcal{M}}_{X,D,\bar{x}} \cong \oplus_{C(\bar{x})} \mathbb{N}$ where $C(\bar{x})$ is the set of irreducible components of the inverse image of D in $\text{Spec}(\mathcal{O}_{X,\bar{x}})$ (étale locally it is evidently true under locally free assumption because étale locally it is snc). Now let $\pi : \mathfrak{X} \rightarrow X$ be the fibered category whose fiber $\mathfrak{X}(Y)$ over $f : Y \rightarrow X$ is the groupoid of simple morphisms of log structures $\varphi : f^* \mathcal{M}_{X,D} \rightarrow \mathcal{M}$, such that for each geometric point $\bar{y} \rightarrow Y$ with $\bar{x} = f(\bar{y})$, the integer c_i associated to a component $C_i \in C(\bar{x})$ is equal to b_i . By general nonsense it is a stack with respect to étale topology.

We claim that this moduli space is a Deligne-Mumford stack. We can check this étale locally so we can assume that $X = \text{Spec } k[x_1, \dots, x_n]$ and $D_i = Z(x_i)$. It is known that the Deligne-Mumford stack

$$[\text{Spec } k[y_1, \dots, y_n]/(\mu_{b_1} \times \dots \times \mu_{b_r})],$$

where the structure map $k[x_1, \dots, x_n] \rightarrow k[y_1, \dots, y_n]$ is the map $x_1 = y_1^{b_1}, \dots, x_r = y_r^{b_r}, x_{r+1} = y_{r+1}, \dots, x_n = y_n$, and the group action is via twisting by roots of unity, is the stack which to any $f : Y \rightarrow X$ associates the groupoid of morphisms of log structure $f^* \mathcal{M}_{X,D} \rightarrow \mathcal{M}$ together with a map $\bar{\beta}' : \mathbb{N}^r \rightarrow \overline{\mathcal{M}}$ which étale locally fits into a chart, such that the diagram

$$\begin{array}{ccc} \mathbb{N}^r & \xrightarrow{\oplus(\times b_i)} & \mathbb{N}^r \\ \bar{\beta} \downarrow & & \downarrow \bar{\beta}' \\ f^{-1} \overline{\mathcal{M}}_{X,D} & \longrightarrow & \overline{\mathcal{M}} \end{array}$$

commutes, where $\bar{\beta}$ denotes the map induced by the standard chart on $\mathcal{M}_{X,D}$. Thus there is a natural morphism $F : [\text{Spec } k[y_1, \dots, y_n]/(\mu_{b_1} \times \dots \times \mu_{b_r})] \rightarrow \mathfrak{X}$ forgetting the data of $\bar{\beta}'$. This is easily seen to be fully faithful, and we know every object is étale locally in the essential image, so it is an equivalence.

This local description shows that $\pi : \mathfrak{X} \rightarrow X$ is a coarse moduli space. Also \mathfrak{X} is a tame Deligne-Mumford stack. Thus there is a quasi-isomorphism

$$\Omega_{X/k}^j(\log D) \otimes \mathcal{O}_X(-\sum_i \lfloor \frac{a_i}{b_i} \rfloor D_i) \rightarrow R\pi_*(\Omega_{\mathfrak{X}/k}^j \otimes \mathcal{O}_{\mathfrak{X}}(-\sum_i a_i \tilde{D}_i)),$$

where \tilde{D}_i 's are appropriately chosen reduced divisors (namely \tilde{D}_i is the unique reduced closed substack whose support is equal to the support of the closed substack $\pi^* \mathcal{O}_X(-D_i)$). More precisely one proves that the underived pushforward is isomorphic to the scheme part, and the derived pushforward is quasi-isomorphic to the $R^0 \pi_*$ because it is a tame Deligne-Mumford stack to its coarse moduli space.

Now the projection formula gives

$$H^j(X, \Omega_X^i(\log D) \otimes \mathcal{O}_X(-[E])) = H^j(\mathfrak{X}, \Omega_{\mathfrak{X}/k}^i \otimes \mathcal{O}_{\mathfrak{X}}(-\pi^*[E] - \sum_i a_i \tilde{D}_i)).$$

Now some power of $\mathcal{O}_{\mathfrak{X}}(\pi^*[E] + \sum_i a_i \tilde{D}_i)$ descends to an ample sheaf on X , namely its $\prod_i b_i$ -power. Thus this follows from the Kodaira vanishing for Deligne-Mumford stacks, which will be stated below. \square

Theorem 3.5 (Kodaira vanishing for Deligne-Mumford stacks). *Let \mathfrak{X}/k be a smooth proper tame Deligne-Mumford stack of dimension d with a projective coarse moduli space $\pi : \mathfrak{X} \rightarrow X$. Suppose*

π is flat. Let L be an invertible sheaf on \mathfrak{X} such that some power of L descends to an ample invertible sheaf on X . Then $H^i(\mathfrak{X}, \Omega_{\mathfrak{X}/k}^i \otimes L^{-1}) = 0$ for $i + j < d$ (and $< p$ if...)

One could prove this using Deligne-Illusie's strategy verbatim.

4. TIGHT CLOSURE AND TEST IDEALS

We now review the theory of tight closures and test ideals.

- Let R be a characteristic p noetherian ring. Let $I \subset R$ be an ideal. The **tight closure** I^* of I is consisted of elements $x \in R$ such that there exists $c \in R^\circ$, i.e. c is not contained in any minimal prime ideal of R , such that for $e \gg 0$, $cx^{p^e} \in I^{[p^e]}$, where $I^{[p^e]}$ is the ideal generated by p^e -th powers of elements of I .
- Clearly $I^{**} = I^* \supset I$, so it is a closure operation.
- $x \in I^*$ iff for all minimal primes \mathfrak{p} of R , image of x modulo \mathfrak{p} is in the tight closure of IR/\mathfrak{p} . Thus we can do things in integral domain.
- $I \subset I^* \subset \bar{I}$, where \bar{I} is the integral closure.
- If I is tightly closed, so it $(I : J)$.
- If R is regular, every ideal is tightly closed.
- If R is a normal domain, every ideal is tightly closed.
- Already we can prove Briancon-Skoda theorem in characteristic p :

Theorem 4.1 (Briancon-Skoda, Hochster-Huneke). *For a noetherian R of characteristic p and $I \subset R$ generated by at most n elements, for all $m \geq 0$, $\overline{I^{m+n}} \subset (I^{m+1})^*$.*

Indeed, we can assume R is a domain and $n > 0$. For $x \in \overline{I^{m+n}}$, there is some l such that for all $k \geq 1$, $(I^{m+n} + xR)^{k+l} = I^{(m+n)k}(I^{m+n} + xR)^l \subset I^{(m+n)k}$. If $I = (a_1, \dots, a_n)$, then $I^{(m+n)k} \subset (a_1^k, \dots, a_n^k)^{m+1}$. Thus the tight closure condition is then satisfied with $c = x^l$, since $cx^{p^e} \in (a_1^{p^e}, \dots, a_n^{p^e})^{m+1} = (I^{[p^e]})^{m+1} = (I^{m+1})^{[p^e]}$.

- From this we know that in an integrally closed domain, every principal ideal is tightly closed.
- If every ideal in R is tightly closed, we say R is **weakly F -regular**. If every localization of R is weakly F -regular, then we say R is **F -regular**. It is suspected that tight closure commutes with localization.
- Weakly F -regular \implies reduced normal.
- Gorenstein local \implies (weakly F -regular \iff every parameter ideal is tightly closed \iff one parameter ideal is tightly closed). In general we say that a local Cohen-Macaulay equicharacteristic p has **F -rational singularity** if some (any) parameter ideal is tightly closed.
- If A is a direct summand of R , $A^\circ \subset R^\circ$, and R is (weakly) F -regular, then A is also (weakly) F -regular.
- A direct summand of regular ring is Cohen-Macaulay. This follows from **colon capturing**, as follows. For a noetherian equidimensional local ring R of characteristic p which is a homomorphic image of a local Cohen-Macaulay ring S , for any system of parameters $x_1, \dots, x_n, (x_1, \dots, x_{n-1}) :_R x_n \subset (x_1, \dots, x_{n-1})^*$. For example this shows that the ring of invariants of an action of a linear reductive algebraic group on a regular algebra over a field is Cohen-Macaulay.
- If R is reduced we define $R^{1/q}$ to be the set of all elements in an algebraic closure of the total field of fractions of R whose q -th power is in R .

- A ring is ***F*-finite** if R is module-finite over R^p . An *F*-finite reduced ring R is **strongly *F*-regular** if for every $c \in R^\circ$ there is an integer e such that the R -linear map $R \rightarrow R^{1/q}$ mapping 1 to $c^{1/q}$ splits as a map of R -modules.
- Strong *F*-regularity is a stalk-local property. Strong *F*-regular locus is open.
- Reduced strongly *F*-regular \Rightarrow *F*-regular.
- Gorenstein + weakly *F*-regular \Rightarrow *F*-regular.
- Gorenstein + weakly *F*-regular + *F*-finite \Rightarrow strongly *F*-regular.
- We can define the notion of tight closure for modules. Namely via power of Frobenius we can see R itself as an R -algebra. Tensoring with this R -algebra we call this functor as $F^e(M)$, if we use e -th iterate of Frobenius map. Then given $N \subset M$, $N_M^{[q]} = \ker(F^e(M) \rightarrow F^e(M/N)) = \text{im}(F^e(N) \rightarrow F^e(M))$. This coincides with $I^{[q]} = I_R^{[q]}$. Thus we can make the analogous definition of tight closure N_M^* of N in M . If the ambient module is understood we often drop M .
- Tight closure has another very important property, **persistence**, which says that for any ring map $R \rightarrow S$, $I^*S \subset (IS)^*$. This is probably the most difficult basic property of tight closure, and it needs some auxiliary definitions. An element $c \in R$ is in the **test ideal** $\tau(R)$ of R if for all I and $z \in I^*$, $cz^{p^e} \in I^{[p^e]}$ for all e . An element c is a **test element** if it is in the test ideal but not in any minimal prime ideal of R . If Frobenius is finite for R , then for any $c \in R$ with R_c regular, a power of c is a test element. Thus the test ideal contains an ideal defining the nonregular locus of $\text{Spec } R$. It is conjecture that the test ideal defines the non-*F*-regular locus.
- For a locally excellent domain of prime characteristic R , $I^* = \bar{I}$.

TBA, like test ideal = multiplier ideals

5. *F*-SINGULARITIES AND SINGULARITIES IN CHARACTERISTIC ZERO

TBA, like *F*-rational = rational. Also introduce MMP singularities (klt etc etc) and compare with du Bois, \mathbb{Q} -Gorenstein etc etc

6. PERFECTOID MULTIPLIER IDEALS AND TEST IDEALS

TBA

7. MIXED CHARACTERISTIC SINGULARITIES

TBA. In particular need to introduce big Cohen-Macaulay algebras