CUP PRODUCTS AND L-VALUES OF CUSP FORMS

GYUJIN OH

Warning: daggered parts of the notes (not covered in the actual talk) are left unedited.

- $\Delta = \mathrm{Gal}(F/\mathbb{Q})$
- $S = \{1-\zeta_p\}$ a singleton consisted of the unique prime above p
- ω is the Teichmuller character.
- $\mathfrak{H}, \mathfrak{h}$ in general denote modular Hecke algebra and cuspidal Hecke algebra with coefficients in $\mathbb{Z}_p$, respectively.
- $A_F = A$ is the p-part of $\mathrm{Cl}(F)$.
- Weight k level N modular symbols $\mathcal{M}_k(N, \mathbb{Z})$ is defined as the sub-$\mathbb{Z}$-module of $\mathrm{Sym}^{k-2}\,\mathbb{Z}^2 \otimes_{\mathbb{Z}[\Gamma_1(N)]} H_1(\mathbb{H}^*, \mathbb{P}^1(\mathbb{Q}), \mathbb{Z})$, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, and $\Gamma_1(N)$ acts on $\mathbb{Z}^2$ via standard representation of GL_2, generated by elements of form $P \otimes \{\alpha \to \beta\}$, where $P(X, Y)$ is a degree $(k-2)$ homogeneous polynomial in variables X, Y, and $\{\alpha \to \beta\}$ is a (geodesic) path from α to β, for $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$. Cuspidal symbols $\mathcal{S}_k(N, \mathbb{Z})$ are those that die under the boundary map $\partial : H_1(\mathbb{H}^*, \mathbb{P}^1(\mathbb{Q}), bZ) \to H_0(\mathbb{P}^1(\mathbb{Q}), \mathbb{Z})$.
  - Canonically $\mathcal{M}_2(N, \mathbb{Z}) = H_1(X_1(N), C_1(N), \mathbb{Z})$, $\mathcal{S}_2(N, \mathbb{Z}) = H_1(X_1(N), \mathbb{Z})$.
  - (Eichler-Shimura isomorphism) The integration
  $$\langle , \rangle : \mathcal{M}_k(N, \mathbb{C}) \times (S_k(N) \oplus \overline{S_k(N)}) \to \mathbb{C},$$
  defined by
  $$\langle P(X,Y)\{0 \to \infty\}, (f, g)\rangle = \int_\alpha^\beta f(z)P(z,1)dz + \int_\alpha^\beta g(\overline{z})P(\overline{z},1)d\overline{z},$$
  is perfect upon restricting the first factor to $\mathcal{S}_k(N, \mathbb{C})$.
- Generalized Bernoulli number $B_{k,\chi}$, for χ Dirichlet character of conductor f, is defined as
$$\sum_{a=1}^{f} \chi(a)\frac{te^{at}}{e^{ft}-1} = \sum_{k=0}^{\infty} B_{k,\chi}\frac{t^k}{k!}.$$
It satisfies $L(1-k,\chi) = -\frac{B_{k,\chi}}{k}$.
- Eisenstein series with nebentype χ:
$$E_{k,\chi}(z) = \frac{L(1-k,\chi)}{2} + \sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^{k-1}q^n.$$

1. L-values of cusp forms vs. cup product of cyclotomic units

I would like to first recap what has happened so far. In Eric's talk, we saw (via an example of $\Delta \equiv E_{12} (\mathrm{mod}\, 691)$) the following phenomenon. Choose an irregular pair (p, k) (i.e. $p|B_k$, p odd), and then there is a normalized cuspidal eigenform f of level 1 and weight k such that it satisfies a congruence

$$f \equiv E_k (\mathrm{mod}\, \mathfrak{p}_f),$$

where

$$E_k = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

and $\mathfrak{p}_f \subset \mathcal{O}_f$ is a fixed maximal ideal over p inside $\mathcal{O}_f$, the $\mathbb{Z}$-algebra generated by Fourier coefficients of f.

Remark 1.1. Note that this is always possible as

(1) the mod p Hecke eigenclass $\phi : \mathfrak{H}_k \to \mathbb{F}_p$, $\phi(T_\ell) = 1 + \ell^{k-1}$, factors through the cuspidal Hecke algebra $\phi : \mathfrak{h}_k \to \mathbb{F}_p$, as every Hecke action computing the constant term vanishes,
(2) the Hecke algebra $\mathfrak{h}_k$ is finite flat over $\mathbb{Z}_p$ (certainly p-torsion-free, if you think about it), so that $\mathbb{Z}_p \subset \mathfrak{h}_k$ has Going-down property, so that one could find $\mathfrak{q} \subset \ker \phi \subset \mathfrak{h}_k$ lying over $0 \subset (p) \subset \mathbb{Z}_p$,
(3) and then $\mathfrak{h}_k \twoheadrightarrow \mathfrak{h}_k/\mathfrak{q}$ gives a p-integral Hecke eigenclass ($\mathfrak{h}_k/\mathfrak{q}$ is a finite free $\mathbb{Z}_p$-algebra), i.e. a normalized cuspidal eigenform.

Then the ratios between odd periods of f in the critical strip should match the ratios of cup products of cyclotomic units. Recall that the group of cyclotomic p-units, denoted as $C \subset \mathbb{Z}[\mu_p, \frac{1}{p}]^\times$, is the group generated by roots of unity and $1 - \zeta_p^i$, $1 \le i \le p-1$. Let $\mathcal{C}$ be its p-completion (i.e. $C \otimes_{\mathbb{Z}} \mathbb{Z}_p$). It has a natural action by $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$, and as $|\Delta|$ is p-invertible, we can decompose $\mathcal{C}$ into eigenspaces using idempotents

$$\pi_i = \frac{1}{p-1} \sum_{\sigma \in (\mathbb{Z}/p\mathbb{Z})^\times} \omega^{-i}(\sigma)[\sigma] \in \mathbb{Z}_p[\Delta],$$

which projects to the ω^i-eigenspace. We define, for odd $3 \le i \le k-2$, $\eta_i = \pi_{1-i}(1-\zeta_p) \in \mathcal{C}^{(1-i)}$. Modulo p-th powers, this is the same as

$$\eta_i \equiv \prod_{u=1}^{p-1} (1-\zeta_p^u)^{u^{i-1}} (\bmod\, \mathcal{O}_{F,S}^{\times p}).$$

Using the cup product pairing $(,) : \mathcal{O}_{F,S}^\times \times \mathcal{O}_{F,S}^\times \to A_F \otimes \mu_p$ (justified in Leo's talk), restricted to C and extended to $\mathcal{C}$ by linearity, we can define

$$e_{i,k} := (\eta_i, \eta_{k-i}) \in A \otimes_{\mathbb{Z}} \mu_p.$$

Note that by considering the Δ-action, $e_{i,k} \in (A \otimes_{\mathbb{Z}} \mu_p)^{(2-k)} = A^{(1-k)} \otimes_{\mathbb{Z}} \mu_p$.

A "numerically verifiable" conjecture then was that the "ratios" of $e_{i,k}$ should match the ratios of (normalized) L-values of cusp forms. A priori $A^{(1-k)} \otimes_{\mathbb{Z}} \mu_p$ is some random space, but we have seen in Leo's talk that under Vandiver's conjecture,

- $A = \bigoplus_{3 \le j \le p-2, k \text{ odd}} A^{(j)}$, i.e. $A^{(\text{even})} = 0$,
- $A^{(j)}$ is cyclic (thus $\cong \mathbb{Z}_p/L(0,\omega^{-j})\mathbb{Z}_p = \mathbb{Z}_p/B_{1,\omega^{-j}}\mathbb{Z}_p$ by IMC) for $j = 3, 5, \cdots, p-2$ (proof by Kummer reflection principle; see [Wa]).

Thus, under the Vandiver's conjecture, $A^{(1-k)} \otimes_{\mathbb{Z}} \mu_p \cong \mathbb{Z}/p\mathbb{Z}(1-k)$ (nonzero because irregular), and we can really talk about ratios. We can for example formulate the conjecture in the following way.

Conjecture 1.1 (Conjecture A, with Vandiver)**.** *Assume Vandiver's conjecture. For $1 \le j \le k-1$, let $r_j(f) = \frac{(j-1)!}{(-2\pi i)^j} L(f,j)$ be the normalized L-value of f. For $j = 3, 5, \cdots, k-3$, let $p_j = \frac{r_j(f)}{r_1(f)} \in K_f$ (seen in Eric's talk). Then, the ratio $[p_3 : \cdots : p_{k-3}]$ modulo $\mathfrak{p}_f$ matches with $[e_{3,k} : \cdots : e_{k-3,k}]$.*

Here, the ratio $[p_3 : \cdots : p_{k-3}]$ is understood as a corresponding point in $\mathbb{P}^{(k-6)/2}(\mathcal{O}_f)$ with at least one entry not contained in $\mathfrak{p}_f$ so that one can compare ratios mod $\mathfrak{p}_f$. Indeed by the congruence $f \equiv E_k (\operatorname{mod} \mathfrak{p}_f)$, $\mathcal{O}_f/\mathfrak{p}_f = \mathbb{F}_p$, so such comparison is valid.

Now if we do not assume Vandiver's conjecture, we can formulate Conjecture A in the following somewhat artificial way.

Conjecture 1.2 (Conjecture A, without Vandiver, [ShCup, Conjecture 3])**.** *Let H_f be the $\mathcal{O}_f$-submodule of K_f spanned by $p_3, \cdots, p_{k-3}$. Then, for any $\mathbb{F}_p$-linear functional $\psi : H_f/\mathfrak{p}_f H_f \to \mathbb{F}_p$, there exists an $\mathbb{F}_p$-linear functional $\phi : A_F^{(1-k)} \otimes \mu_p \to \mathbb{F}_p$ such that*

$$(\phi(e_{3,k}), \phi(e_{5,k}), \cdots, \phi(e_{k-3,k})) = \left(\psi\left(\frac{r_3(f)}{r_1(f)}\right), \psi\left(\frac{r_5(f)}{r_1(f)}\right), \cdots, \psi\left(\frac{r_{k-3}(f)}{r_1(f)}\right)\right),$$

inside $\mathbb{F}_p^{k/2-2}$.

Note that, with a view towards cyclicity of $A_F^{(1-k)} \otimes \mu_p$, McCallum-Sharifi in [McSh] conjectured that $e_{i,k}$'s generate $A^- \otimes \mu_p$.

Conjecture 1.3 (McCallum-Sharifi, [ShAWS, Conjecture 5.1.12])**.** *The elements for odd i and even k generate $A^- \otimes_{\mathbb{Z}} \mu_p$. In other words, $\mathcal{C} \otimes_{\mathbb{Z}_p} \mathcal{C} \xrightarrow{(,)} A^- \otimes_{\mathbb{Z}} \mu_p$ is surjective.*

2. Modular symbols vs. cup product of cyclotomic units

We have also seen in Eric's talk that special L-values of f in the critical strip can be encoded in terms of **modular symbols**. We briefly recall the definition of it.

Weight k modular symbols $\mathcal{M}_k(\mathbb{Z})$ is defined as the torsion-free quotient of the $\mathbb{Z}$-submodule of $\operatorname{Sym}^{k-2} \mathbb{Z}^2 \otimes_{\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]} H_1(\mathbb{H}^*, \mathbb{P}^1(\mathbb{Q}), \mathbb{Z})$, where $\mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$, and $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{Z}^2$ via standard representation of GL_2, generated by elements of form $P \otimes \{\alpha \to \beta\}$, where $P(X, Y)$ is a degree $(k-2)$ homogeneous polynomial in variables X, Y, and $\{\alpha \to \beta\}$ is a (geodesic) path from α to β, for $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$. In our case, any modular symbol can be written as $P \otimes 0, \infty\}$ for P a homogeneous degree $(k-2)$ polynomial in variables X, Y, coefficients in $\mathbb{Z}$.

Cuspidal symbols $\mathcal{S}_k(N, \mathbb{Z})$ are those that die under the boundary map $\partial : H_1(\mathbb{H}^*, \mathbb{P}^1(\mathbb{Q}), bZ) \to H_0(\mathbb{P}^1(\mathbb{Q}), \mathbb{Z})$ (we let $\mathcal{B}_k(\mathbb{Z})$ to be the same thing with $H^1(\mathbb{H}^*, \mathbb{P}^1(\mathbb{Q}), \mathbb{Z})$ replaced with $H_0(\mathbb{P}^1(\mathbb{Q}), \mathbb{Z})$). Let $\mathcal{M}_k(R) = \mathcal{M}_k(\mathbb{Z}) \otimes_{\mathbb{Z}} R$, and similarly define $\mathcal{S}_k(R)$.

Remark 2.1. The action of a matrix $\gamma \in \mathrm{GL}_2(\mathbb{Z})^+$ on $P\{\alpha \to \beta\}$ is given by $(P\gamma^{-1})\{\gamma\alpha \to \gamma\beta\}$. Through this we can define a $\mathfrak{h}_k$-action on $\mathcal{S}_k(\mathbb{Z}_p)$ and $\mathfrak{H}_k$-action on $\mathcal{M}_k(\mathbb{Z}_p)$.

Then, as $\mathcal{B}_k(R)$ (space of boundary symbols) can be shown to be generated by $X^{k-2}\{0, \infty\}$, and $\mathcal{S}_k(R)$ is generated by $X^{i-1}Y^{k-1-i}\{0, \infty\}$ with $2 \leq i \leq k-2$.

We can talk about $\pm$-parts of it under the invloution by

$$P(X, Y)\{0, \infty\} \mapsto P(-X, Y)\{0, \infty\},$$

which corresponds to $z \mapsto -\overline{z}$ in $\mathbb{H}^*$. Then $\mathcal{S}_k(R)^+$ has a basis of $X^{i-1}Y^{k-i-1}\{0, \infty\}$ for $i = 3, 5, \cdots, k-3$. Here we know the relation because $\gamma \cdot 0 = \infty$ means $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so that we know the only new thing introduced in $\mathcal{B}_k(R)$ is the flipping of X and Y (up to sign).

Recall also that the L-values of f are related to modular symbols, as follows. Recall that we had a Hecke-equivariant pairing

$$\langle , \rangle : \mathcal{M}_k(\mathbb{C}) \times S_k(\mathbb{C}) \to \mathbb{C},$$

$$\langle P\{0 \to \infty\}, g\rangle_k = \int_0^\infty g(z) P(z, 1) dz.$$

Remark 2.2. It is Hecke equivariant in a sense that the Hecke algebra action on $\mathcal{M}_k(\mathbb{C})$ is actually via the adjoint Hecke algebra. This is quite expected because modular symbols are "homology classes," whereas modular forms are "cohomology classes," so really modular symbols are something dual to modular forms (of course one can use a dual form of modular symbols to match with modular forms; both approaches are common).

In particular,

$$r_j(f) = \frac{(j-1)!}{(-2\pi i)^j} L(f,j) = i^{j-1}\int_0^\infty f(iy)y^{j-1}dz = \int_0^{i\infty} f(z)z^{j-1}dz = \langle X^{j-1}Y^{k-j-1}\{0\to\infty\}, f\rangle.$$

Now we can formulate a variant of Conjecture A which fits better in a general picture of Sharifi's conjectures.

Conjecture 2.1 (Conjecture B, [ShCup, Conjecture 6])**.** *Consider a map $\Pi_k : \mathcal{S}_k(\mathbb{Z}_p)^+ \to A_F^{(1-k)} \otimes \mu_p$ defined by $\Pi_k(X^{i-1}Y^{k-i-1}\{0,\infty\}) = e_{i,k}$ (which is well-defined as we know the basis). Then, this factors through the Eisenstein quotient*

$$\Pi_k : \mathcal{S}_k(\mathbb{Z}_p)^+/\mathfrak{m}\mathcal{S}_k(\mathbb{Z}_p)^+ \xrightarrow{\sim} A^{(1-k)} \otimes \mu_p,$$

where $\mathfrak{m} = (p, I_k)$ is the maximal ideal containing the Eisenstein ideal $I_k = \langle T_\ell - 1 - \ell^{k-1}\rangle_\ell \subset \mathfrak{h}_k$, and this is moreover an isomorphism.

This seems much more similar to Sharifi's other forms of conjectures. Conjecture A indeed looks like a shadow of Conjecture B. We will in fact later show that Conjecture A is implied by Conjecture B.

3. Examples

We briefly recall the example of $\Delta \equiv E_{12} \pmod{691}$ seen in Eric's talk. After exploiting the symmetry $r_j(f) = (-1)^{j+1}r_{k-j}(f)$, we saw that $p_3(\Delta) = -\frac{691}{2^2\times 3^4\times 5}$ and $p_5 = \frac{691}{2^3\times 3^2\times 5\times 7}$, so that $[p_3 : p_5] = [152 : 1] \pmod{691}$. On the other hand, a table that can be found in Sharifi's webpage, which computes the ratio $[e_{1,k} : e_{3,k} : \cdots : e_{p-2,k}]$, says that for $(p,k) = (691, 12)$, the list goes

$$[1 : 222 : 647 : 44 : 469 : \cdots],$$

where one checks that $[222 : 647] = [152 : 1] \pmod{691}$.

Remark 3.1. Note that most of Sharifi's table are conjectural, as we will see below how he might have computed the table (although the case of $(691, 12)$ is computed unconditionally using very technical results in [McSh]).

Well, how does one calculate $e_{i,k}$ in general? We have seen in Leo's talk that it is somewhat gruesome to compute $e_{i,k}$'s. I would like to say how one might write a program computing a cup product pairing $\mathcal{C} \times \mathcal{C} \to A^{(1-k)} \otimes \mu_p$ assuming Vandiver's conjecture and nonvanishing of cup product pairing for irregular pairs (i.e. McCallum-Sharifi's conjecture, Conjecture 1.3). It turns out that a few formal relations that we know about the pairing are quite strong.

(1) We have seen also in Leo's talk that, whenever $a, 1-a \in \mathcal{O}_{F,S}^\times$, $(a, 1-a) = 0$.
(2) Thus, $(\zeta_p, 1-\zeta_p^i) = \frac{1}{i}(\zeta_p^i, 1-\zeta_p^i) = 0$.
(3) Also, $(,)$ is alternating (it is a cup product).
(4) We now abbreviate $u_i = 1-\zeta_p^i$ and $(i,j) = (u_i, u_j)$ for simplicity. Then we have things like
  - $(i,j) = -(j,i)$,
  - $(i,j) = (i, p-j)$, because $u_{p-j} = -\zeta_p^{-j}u_j$.

(5) Now we have some funny relations. Namely, we have identities

$$\frac{1-x^a}{1-x^{a+b}} + x^a\frac{1-x^b}{1-x^{a+b}} = 1,$$

and

$$\frac{(1-x^{2a})(1-x^{a+b})}{(1-x^a)(1-x^{2(a+b)})} - x^a\frac{(1-x^b)(1-x^{a+b})}{1-x^{2(a+b)}} = 1.$$

(6) In terms of (i,j)'s, this translates into $(a,b)-(a,a+b)-(a+b,b)=0$, and $(2a,b)+(2a,a+b)-(2a,2(a+b))+(a+b,b)-(a+b,2(a+b))-(a,b)-(a,a+b)+(a,2(a+b))-(2(a+b),b)-(2(a+b),a+b)=0$, whenever all entries are nonzero modulo p. In the latter expression, $(a+b,2(a+b))+(2(a+b),a+b)=0$, so that we have

$$(a,b)=(a,a+b)+(a+b,b),$$

$$(2a,b)+(2a,a+b)-(2a,2(a+b))+(a+b,b)-(a,b)-(a,a+b)+(a,2(a+b))-(2(a+b),b)=0.$$

In fact, McCallum-Sharifi computed in [McSh] that for $p<25000$ and k forming irregular pair with p, there is a unique (up to scalar) nontrivial antisymmetric Galois-equivariant pairing $\mathcal{C}\times\mathcal{C}\to\mathbb{F}_p(2-k)$ satisfying just the longer relation of the above two.

For example, just using the two relations, we can show that any pairing for $p=7$ must be zero (which is a priori obvious, as 7 is a regular prime). Indeed, by symmetry we only need to take care of $(1,2),(1,3),(2,3)$, and putting $a=1,b=2$ in the shorter relation we get

$$(1,2)=(1,3)-(2,3).$$

Putting $a=b=1$ in the longer relation, we have

$$-3(1,2)-(2,3)+2(1,3)=0.$$

Combining two, we have $(1,3)=2(2,3)$ and $(1,2)=-(2,3)$. Now putting $a=1,b=2$ in the longer relation, we get

$$(1,2)+(1,3)=0,$$

showing that all symbols are zero.

4. Tying up loose ends

4.1. Conjecture B implies Conjecture A. First we show that Conjecture B implies Conjecture A (no Vandiver version).

Proof. Suppose we are given with $\psi : H_f/\mathfrak{p}_f H_f \to \mathbb{F}_p$. Then, define $\phi : \mathcal{S}_k(\mathbb{Z})^+ \to \mathbb{F}_p$ as

$$\phi(P\{0\to\infty\}) = \psi\left(\frac{\langle P\{0\to\infty\}, f\rangle}{r_1(f)}\right).$$

This surely extends to $\mathcal{S}_k(\mathbb{Z}_p)^+$, and it is obviously killed by $p\mathfrak{h}_k$. Also,

$$\phi\circ(T_\ell-1-\ell^k)(s) = \psi\left(\frac{\langle (T_\ell-1-\ell^k)s, f\rangle}{r_1(f)}\right) = \psi\left(\frac{\langle s, (T_\ell-1-\ell^k)f\rangle}{r_1(f)}\right) = \psi\left(p\frac{\langle s, \frac{a_\ell-1-\ell^k}{p}f\rangle}{r_1(f)}\right) = 0.$$

Here we used that f is eigenform an $\langle,\rangle$ pairs Hecke operator and its adjoint equivariantly. Thus, ϕ factors through the Eisenstein component. We then see that $\phi\circ\Pi_k^{-1} : A_F^{(1-k)}\otimes\mu_p\to\mathbb{F}_p$ satisfies the desired condition. ∎

4.2. † **"Mod p multiplicity one" principle.** We have the following interesting consequence of Vandiver's conjecture.

Proposition 4.1. *If we assume Vandiver's conjecture (plus some extra conditions to be precise, "(Good Eisen)" of* [BePo]*), then normalized f satisfying $f \equiv E_k (\mathrm{mod}\, \mathfrak{p}_f)$ is unique for an irregular pair (p, k).*

Proof. (1) Kurihara [Ku] showed that $A^{(1-k)}$ is cyclic if and only if $(\mathfrak{h}_k)_{\mathfrak{m}}$ is Gorenstein.

(2) One crucial property of Gorenstein-ness is that $\mathrm{Hom}((\mathfrak{h}_k)_{\mathfrak{m}}, \mathbb{Z}_p)$ is, as an $(\mathfrak{h}_k)_{\mathfrak{m}}$-module, isomorphic to $(\mathfrak{h}_k)_{\mathfrak{m}}$.

(3) Ohta proved that $\mathcal{S}_k(\mathbb{Z}_p)^+_{\mathfrak{m}} \cong \mathrm{Hom}((\mathfrak{h}_k)_{\mathfrak{m}}, \mathbb{Z}_p)$.
   - First $\mathcal{S}_k(\mathbb{Z}_p)$ can be endowed with Galois action by considering it as (étale) parabolic cohomology. Then, one could show that the decomposition of $\mathcal{S}_k(\mathbb{Z}_p)_{\mathfrak{m}}$ using "ordinarity" (by Ohta), which splits it into free $(\mathfrak{h}_k)_{\mathfrak{m}}$-module and dualizing $(\mathfrak{h}_k)_{\mathfrak{m}}$-module, matches with the $\pm$-decomposition.

(4) ... See [BePo, Theorem 3.11].

□

Note that such mod p multiplicity one results are also related to Gorenstein-ness of localized Hecke algebra for residually irreducible cases (which was known from the time of proof of modularity lifting).

4.3. † **Weight k level 1 vs. weight 2 level N.** An analogous conjecture in the setting of weight 2, varying level (cf. [Sh2pL]) is that

$$(\eta_i, \eta_{k-i}) \leftrightarrow \overline{L_p(g, \omega^{i-1}, 1)},$$

where g is a level 2, level p, Nebentypus ω^{p-2} cuspidal eigenform congruent to $g \equiv E_{2,\omega^{k-2}} (\mathrm{mod}\, \mathfrak{p})$ (which is just congruent mod p to E_k); note that $E_{2,\omega^{k-2}} \equiv E_k (\mathrm{mod}\, \mathfrak{p})$. Now Birch's lemma shows that a character twist of a modular form can be written as a linear combination as

$$f_{\overline{\chi}}(z) = \frac{1}{\tau(x)} \sum_{a (\mathrm{mod}\, m)} \chi(a) f(z + \frac{a}{m}),$$

where χ is a Dirichlet character of conductor m and

$$\tau(x) = \sum_{a (\mathrm{mod}\, m)} \chi(a) e^{2\pi i a/m},$$

is the Gauss sum. Thus, an L-value of $L(g, \omega^{i-1}, 1)$ can be expressed as, some constant times a pairing between g and a modular symbol

$$\sum_{a=1}^{p-1} \omega^{1-i}(a) \{\infty \to \frac{a}{p}\}.$$

Note that in Eric's talk another form of part of Sharifi's conjecture is that

$$\Pi_N^\circ : \mathcal{S}_2^\circ(N)^+ \to K_2(\mathbb{Z}[1/N, \mu_N])^+,$$

$$[u : v]_N^* \mapsto \{1 - \zeta_N^u, 1 - \zeta_N^v\}^+,$$

should factor through the Eisenstein quotient and the resulting map, when sent to infinity Np^r, $r \to \infty$ by taking inverse limit with respect to norm maps, we should get an isomorphism. Thus that these conjectures can be formulated in p-adic families means that an appropriate congruence relations must hold, even on the formal level. For example as

$$1 - \zeta_p^u = \prod \omega^{1-i}(u) \eta_i,$$

the congruence relation like

$$[u:v]_p^* \equiv \sum_{i \text{ odd}, 1\le i\le p-2} \sum_{a=1}^{p-1} u^{1-i}v^{1-(k-i)}a^{i-1}\{\infty \to \frac{a}{p}\} (\operatorname{mod} p),$$

must hold. Indeed by switching i and a, the RHS can be written as

$$\sum_{a=1}^{p-1}\sum_{i=0}^{(p-3)/2} v^{2-k}(va/u)^{2i}\{\infty \to \frac{a}{p}\},$$

and the summand is zero unless $va/u = \pm 1$, which means it reduces to

$$[u:v]_p^* \equiv \frac{p-1}{2} v^{2-k}\left(\{\infty \to \frac{u/v}{p}\} + \{\infty \to \frac{-u/v}{p}\}\right) (\operatorname{mod} p).$$

This looks promising as $[u:v]_p^* = \frac{1}{2}w_p([u:v]_p + [-u:v]_p)$, although to be honest I got confused when I was trying to verify the relation by hand. The above relation as stated is wrong but something almost like this must hold on the level of modular symbols.

References

[BePo] Bellaïche, Pollack, Congruences with Eisenstein series and μ-invariants. Compositio.

[Ku] Kurihara, Ideal class groups of cyclotomic fields and modular forms of level 1. JNT.

[McSh] McCallum, Sharifi. Duke.

[Sh2pL] Sharifi, A reciprocity map and the two variable p-adic L-function. Annals.

[ShAWS] Sharifi, Notes for 2018 AWS.

[ShCup] Sharifi, Cup products and L-values of cusp forms. Pure and Applied Math. Quarterly.

[Wa] Washington, Introduction to cyclotomic fields. Springer.