# QUANTUM COHOMOLOGY OF HOMOGENEOUS VARIETIES

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Unfortunately I do not care very much about counting curves...

## 1. Quantum cohomology primer

Let $X$ be a complex projective variety. Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be Kontsevich's moduli of stable curves $X$, for $\beta \in H_2(X,\mathbb{Z})_{free}$. It has a leg map

$$e_i : \overline{\mathcal{M}}_{g,n}(X,\beta) \to X.$$

Note $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a compact complex orbifold with dimension $\dim X + \langle c_1(T_X), \beta\rangle + 3g + n - 3$. For $\gamma \in H^*(X,\mathbb{C})$,

$$\Phi(\gamma) = \sum_{n\geq 3} \sum_{\beta \in H_2(X,\mathbb{Z})_{free}} \frac{1}{n!} \left( \int_{\overline{\mathcal{M}}_{0,n}(X,\beta)} e_1^*(\gamma) \cdots e_n^*(\gamma) \right) e^{\beta}.$$

This is a sum of Gromov–Witten invariants for genus zero. Namely, the GW invariant

$$GW_{0,n}^{X,\beta}(\gamma_1, \cdots, \gamma_n) := \int_{\overline{\mathcal{M}}_{0,n}(X,\beta)} e_1^*(\gamma_1) \cdots e_n^*(\gamma_n),$$

counts the number of stable rational curves passing through general cycles Poincare-dual to $\gamma_i$ at $i$-th marked point. This potential $\Phi$ is a $\mathbb{C}$-valued formal series on $H^*(X)$, considered as a linear space.

It satisfies a WDVV equation. What is this? We have a forgetful map $\phi : \overline{\mathcal{M}}_{0,n}(X,\beta) \to \overline{\mathcal{M}}_{0,4}$, where $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$. Note that $\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0,1,\infty\}$, and there are three reducible curves,

$$\{\{1,2\},\{3,4\}\}, \quad \{\{1,3\},\{2,4\}\}, \quad \{\{1,4\},\{2,3\}\}.$$

Note that these are all linearly equivalent (in $H^2(\overline{\mathcal{M}}_{0,4})$). This equation pulled back via $\phi$ is the WDVV equation.

Let $QH^*(X) = H^*(X,\mathbb{Z})_{free} \otimes_{\mathbb{Z}} \Lambda$, where

$$\Lambda = \{\lambda = \sum_{A \in H_2(X)_{free}} \lambda_A e^A\}.$$

Then, for $A \in H_2(X)_{free}$ and $a, b \in H^*(X)_{free}$, $(a * b)_A \in H^*(X)_{free}$ is defined as

$$\int_X (a * b)_A \cup c = GW_{0,3}^{X,A}(a, b, c).$$

Then the quantum product is defined as

$$a * b = \sum_{A \in H_2(X)_{free}} (a * b)_A \otimes e^A.$$

As $A = 0$ implies $(a * b)_0 = a \cup b$, this means quantum product, specialized at the ordinary cohomology, is the cup product. In terms of quantum product, WDVV is precisely the associativity of quantum product.

## 2. WDVV

Goal: Dubrovin, *Geometry of 2D topological field theories.*

Very generally, WDVV is something like the following. Let $F(x_1, \cdots, x_n)$ be a function in $n$ coordinates. Let $h$ be the Hessian of $F$, and let $c_j = \frac{\partial h}{\partial x_j}$. Suppose $c_1$ is invertible. Then the WDVV equation is

$$c_j c_1^{-1} c_l = c_l c_1^{-1} c_j.$$

If you write $C_j = c_1^{-1} c_j$, then this is

$$C_j C_l = C_l C_j,$$

so it's related to associativity. Sometimes one requires $\frac{\partial c_1}{\partial x_j} = 0$, sometimes not. In Dubrovin, this is asserted. Their notation is

$$c_{ijk} = (c_i)_{jk},$$

$$\eta_{ij} = c_{1ij},$$

$$\eta^{ij} = (c_1^{-1})_{ij},$$

$$c_{ij}^k = \sum_{\epsilon=1}^{n} \eta^{k\epsilon} c_{\epsilon ij}.$$

These are used to define an associative algebra $A_t$ on an $n$-dimensional space with basis $e_1, \cdots, e_n$,

$$e_i \cdot e_j = \sum_{k=1}^{n} c_{ij}^k(t) e_k.$$

The extra $c_{1ij}$ being constant means that $e_1$ is the unity, namely $c_{1i}^j(t) = \delta_{ij}$. Associativity menas

$$\sum_{a,b} F_{ija} \eta^{ab} F_{klb} = \sum_{a,b} F_{kja} \eta^{ab} F_{ilb},$$

which is WDVV.

We further require that $F$ is a quasihomogeneous function,

$$F(c^{d_1} t_1, \cdots, c^{d_n} t_n) = c^{d_F} F(t_1, \cdots, t_n).$$

In other words, in terms of the Euler vector field

$$E = \sum_i d_i t_i \partial_i,$$

$$\mathcal{L}_E F(t) := \sum_i d_i t_i \partial_i F(t) = d_F F(t).$$

Or more generally one considers Euler vector field where $d_i t_i$ is replaced by a linear function in $t_i$'s, of form

$$E(t) = \sum_i (\sum_j q_{ij} t_i + r_i)\partial_i.$$

If $Q = (q_{ij})$ is diagonalizable, this can be reduced into, via linear change of variables,

$$E(t) = \sum_i d_i t_i \partial_i + \sum_{i, d_i = 0} r_i \partial_i.$$

These $d_i$'s are eigenvalues of $Q$. And one has an ambiguity of adding a quadratic function in $t_i$'s as we are concerned about triple derivatives, which makes

$$\mathcal{L}_E F(t) = d_F F(t) + \sum_{ij} A_{ij} t_i t_j + \sum_i B_i t_i + C.$$

We normalize so that $d_1 = 1$. Let $q_i = 1 - d_i$, $d = 3 - d_F$. If you differentiate thrice the quasihomoegenity relation, we get

$$\mathcal{L}_E \eta_{ij} = (d_F - d_1)\eta_{ij}.$$

This shows that if $Q$ has simple eigenvalues we can make linear change of variables to make $t_1$ part pretty explicit:

$$F(t) = \begin{cases} \frac{1}{2} t_1^2 t_n + \frac{1}{2} t_1 \sum_{i=2}^{n-1} t_i t_{n+1-i} + f(t_2, \cdots, t_n) & \text{if } \eta_{11} = 0 \\ \frac{c}{6} t_1^3 + \frac{1}{2} t_1 \sum_{i=1}^{n-1} t_i t_{n+1-i} + f(t_2, \cdots, t_n) & \text{if } \eta_{11} \neq 0, \end{cases}$$

$$\begin{cases} \eta_{ij} = \delta_{i+j,n+1}, q_1 = 0, q_n = d, q_i + q_{n+1-i} = d & \text{if } \eta_{11} = 0 \\ d_i + d_{n-i+1} = 2d_1 & \text{if } \eta_{11} \neq 0. \end{cases}$$

The first case, $\eta_{11} = 0$, is physically more natural somehow. For example, if $m = 3$, then $F(t) = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + f(t_2, t_3)$. In these terms, in $A_t$, $e_1 = 1, e_2, e_3$, the multiplication law is that

$$\begin{aligned} e_2^2 &= f_{xxy} e_1 + f_{xxx} e_2 + e_3, \\ e_2 e_3 &= f_{xyy} e_1 + f_{xxy} e_2, \\ e_3^2 &= f_{yyy} e_1 + f_{xyy} e_2. \end{aligned}$$

So, $(e_2^2)e_3 = e_2(e_2 e_3)$ implies

$$f_{xxy}^2 = f_{yyy} + f_{xxx} f_{yyy}.$$

This looks quite similar to Kontsevich's WDVV for counting rational curves in $\mathbb{P}^2$. Quasi-homogeneity further implies that

| | $d \neq 1, 2, 3$ | $d = 1$ | $d = 2$ | $d = 3$ |
|---|---|---|---|---|
| Q.hom. | $(1-d/2) x f_x + (1-d) y f_y = (3-d) f$ | $x f_x/2 + r f_y = 2f$ | $r f_x - y f_y = f$ | $x f_x/2 + 2y f_y = c$ |
| $f(x,y)$ | $x^{4+q}\phi(yx^q)$ | $x^4 \phi(y - 2r\log x)$ | $y^{-1}\phi(x + r\log y)$ | $2c\log x + \phi(yx^{-4})$ |
| ODE for $\phi(z)$ | $((12+14q+4q^2)\phi' + (7q+5q^2) z\phi'' + q^2\phi''')^2 = \phi''' + ((2+q)(3+q)(4+q)\phi + q(26+27q+7q^2) z\phi' + q^2(9+6q) z^2\phi'' + q^3 z^3 \phi''')((4+3q)\phi'' + qz\phi''')$ | $\phi'''(r^3 + 2\phi' - r\phi'') - (\phi'')^2 - 6r^2\phi'' + 11r\phi' - 6\phi = 0$ | $-144(\phi')^2 + 96\phi\phi'' + 128 r\phi'\phi'' - 52r^2(\phi'')^2 + \phi''' - 48r\phi\phi''' + 8r^2\phi'\phi''' + 8r^3\phi''\phi''' = 0$ | $\phi''' = 400(\phi')^2 + 32c\phi'' + 1120 z\phi'\phi'' + 784 z^2(\phi'')^2 + 16cz\phi''' + 160z^2\phi'\phi''' + 192 z^3\phi''\phi'''$ |

This is a particular case of Painleve VI.

What is a coordinate-free way? A $\mathbb{C}$-algebra $A$ is **Frobenius algebra** if it is commutative and it has a symmetric bilinear nondegenerate inner product $\langle , \rangle : A \times A \to \mathbb{C}$ such that $\langle ab, c\rangle = \langle a, bc\rangle$ (invariance). Every Frobenius algebra without nilpotents is $\oplus\mathbb{C}$. Our $A_t$ is a Frobenius algebra, and we have a family of Frobenius algebras, $A \to M$.

**Idea**. This can be identified with $TM \to M$.

**Definition 2.1.** $M$ is a **Frobenius manifold** if $TM \to M$ is a family of Frobenius algebras. Namely,

(1) there is an invariant inner-product $\langle , \rangle$ on $T_tM$ which is flat,
(2) $\nabla e = 0$, where $\nabla$ is the Levi-Civita connection wrt $\langle , \rangle$,
(3) For a symmetric 3-tensor $c(u, v, w) = \langle uv, w\rangle$, the 4-tensor $\nabla_z c(u, v, w)$ is symmetric.
(4) The Euler vector field $E$ is determined such that $\nabla(\nabla E) = 0$ and that the corresponding one-parameter group of diffeomorphisms acts by conformal transformations of the metric $\langle , \rangle$ and by rescalings on the Frobenius algebras $T_tM$. The covariantly constant operator $Q = \nabla E(t)$ is called the grading operator.

**Lemma 2.2.** *A solution of WDVV gives a Frobenius manifold structure, and locally any Frobenius manifold structure comes from a solution of WDVV equations.*

The formulae are

$$\partial_i = \frac{\partial}{\partial t_i}, e = \partial_1,$$

$$\partial_i \partial_j = \sum_k c_{ijk}(t)\partial_k,$$

$$\langle \partial_i, \partial_j\rangle = \eta_{ij}.$$

**Example 2.3.** For $M = \{\lambda(p) = p^n + a_n p^{n-1} + \cdots + a_1 \mid a_1, \cdots, a_n \in \mathbb{C}\}$, the algebra $A_\lambda$ is given by $\mathbb{C}[p]/(\lambda'(p))$, and the invariant inner product is given by

$$\langle f, g\rangle_\lambda = \mathrm{res}_{p=\infty} \frac{f(p)g(p)}{\lambda'(p)}.$$

And we have

$$e = \frac{\partial}{\partial a_1}, E = \frac{1}{n+1}\sum_i (n-i+1)a_i \frac{\partial}{\partial a_i}.$$

Also there is an algebraic characterization of Frobenius manifolds. For $R$ a commutative algebra over a field of char $\neq 2$, we want a Frobenius algebra structure on $\mathrm{Der}(R)$, the $R$-module of $k$-derivations.

- There is a symmetric inner product $\langle , \rangle : \mathrm{Der}(R) \times \mathrm{Der}(R) \to R$ (namely $\mathrm{Hom}_R(\mathrm{Der}(R), R) \xrightarrow{\sim} \mathrm{Der}(R)$).
- For $u, v \in \mathrm{Der}(R)$, the covariant derivative $\nabla_u v \in \mathrm{Der}(R)$ is defined by

$$\langle \nabla_u v, w\rangle = \frac{1}{2}\left(u\langle v, w\rangle + v\langle w, u\rangle - w\langle u, v\rangle + \langle [u, v], w\rangle + \langle [w, u], v\rangle + \langle [w, v], u\rangle\right).$$

- We want $\nabla_u e = 0$. Moreover, for $\widetilde{\nabla}_u(\lambda)v := \nabla_u v + \lambda u \cdot v$, we have
$$\widetilde{\nabla}_u(\lambda)\widetilde{\nabla}_v(\lambda) - \widetilde{\nabla}_v(\lambda)\widetilde{\nabla}_u(\lambda) = \widetilde{\nabla}_{[u,v]}(\lambda).$$
In other words, $\widetilde{\nabla}_u(\lambda)$ defines a flat connection. The flatness of this deformed connection detects the associativity condition (thus WDVV).

Natural gradings are
$$\deg t_i = 1 - q_i, \quad \deg \partial_i = q_i.$$
A Frobenius manifold is special in that there is actually a pencil of flat metrics. What I mean by it is this. For a Frobenius manifold $M$, one can define a new metric $(,)$ on $T^*M$ defined by
$$(\omega_1, \omega_2) = (\omega_1 \cdot \omega_2)(E),$$
where $\cdot$ on $T_t^*M$ is defined by transporting the product on $T_tM$ via $T_tM \cong T_t^*M$ via $\langle , \rangle$.

You can transport this back to $T_tM$ as another inner product. So we have an **old product** $\langle , \rangle$ and a **new product** $(,)$. These are related via
$$(E \cdot u, v) = \langle u, v \rangle.$$
This metric itself is well-defined for $t$ such that $E(t)$ is invertible. The two metrics form a **flat pencil**:

**Theorem 2.4.** *Two metrics $(,)_1$ and $(,)_2$ form a* ***flat pencil*** *if, for all $\lambda \in \mathbb{R}$,*
$$(,)_\lambda = (,)_1 + \lambda(,)_2,$$
*is flat, and the Levi-Civita connection $\nabla_\lambda$ for $(,)_\lambda$ satisfies*
$$\nabla_\lambda = \nabla_1 + \lambda \nabla_2,$$
*where $\nabla_i$ is the Levi-Civita connection for $(,)_i$.*

The new metric $(,)$ is called the **intersection form**.

**Definition 2.5.** A point $t \in M$ of a Frobenius manifold is **semisimple** if $T_tM$ is semisimple (has no nilpotents).

Semisimplicity is an open property. A Frobenius manifold is semisimple if its generic point is semisimple. I don't know why but one calls such Frobenius manifold massive...

**Lemma 2.6.** *Around a semisimple point, there are local coordinates $u_1, \cdots, u_n$ such that*
$$\partial_i \cdot \partial_j = \delta_{ij}\partial_i.$$

This is because one has a parallel transported idempotents to get $\partial_i \cdot \partial_j = \delta_{ij}\partial_i$ as vector fields, and these form local coordinates by showing that $[\partial_i, \partial_j] = 0$. This can be reformulated as follows.

**Lemma 2.7.** *For an $n$-dimensional massive Frobenius manifold $M$, the group of algebraic symmetries $G(M)$, a finite-dimensional Lie group, has connected component of the identity an $n$-dimensional commutative Lie group acting locally transitively on $M$.*

Here, $f : M \to M$ is an algebraic symmetry if $f$ is a diffeomorphism such that it preserves multiplication laws.

- The canonical coordinates are given by roots of the characteristic polynomial
$$\det(g^{ij}(t) - u\eta^{ij}) = 0.$$

- That this polynomial has only simple roots is equivalent to $t$ being semisimple.
- Around a semisimple point, canonical coordinate can be chosen in the way that

$$E = \sum u_i \partial_i.$$

- $\langle , \rangle$ is diagonal wrt canonical coordinate ("curvilinear").

From these, we can totally classify how massive Frobenius manifolds can look like locally.

(1) Consider the **rotation coefficients**

$$\gamma_{ij}(u) = \frac{\partial_j \sqrt{\eta_{ii}(u)}}{\sqrt{\eta_{jj}(u)}} \qquad i \neq j.$$

These are symmetric, namely $\gamma_{ij} = \gamma_{ji}$, and $\sum_k \partial_k \eta_{ii}(u) = 0$. The functions $\eta_{ii}(u)$ and $\gamma_{ij}(u)$ are homogeneous functions of the canonical coordinates of the degrees $-d$ and $-1$, respectively.

(2) The rotation coefficients satisfy

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj},$$

for $i, j, k$ distinct,

$$\sum_{k=1}^{n} \partial_k \gamma_{ij} = 0,$$

$$\sum_{k=1}^{n} u_k \partial_k \gamma_{ij} = -\gamma_{ij} \qquad \text{(Scaling homogeneity)}.$$

(3) These set of equations conversely characterize massive local Frobenius manifold, under the extra assumption that $V(u) = [(\gamma_{ij}(u)), \operatorname{diag}(u_1, \cdots, u_n)]$ is diagonalizable (genericity assumption). This has a property that the eigenvalues of $V(u)$ do not depend on $u$.

**Example 2.8.** For $n = 2$, the equations are $\gamma_{12} = \gamma_{21}$ and

$$\partial_1 \gamma + \partial_2 \gamma = 0,$$

$$u_1 \partial_1 \gamma + u_2 \partial_2 \gamma = -\gamma.$$

So this is linear. If you solve it you get

$$\gamma = \frac{C}{u_1 - u_2}.$$

In this case $V(u) = \left(\begin{smallmatrix} 0 & -C \\ C & 0 \end{smallmatrix}\right)$.

For $n = 3$, WDVV reduces to Painleve VI. Well I can only say that it comes from a time-dependent Hamiltonian system

$$\frac{dy}{dz} = \frac{\partial H}{\partial p}, \frac{dp}{dz} = -\frac{\partial H}{\partial y},$$

$$H = \frac{y(y-1)(y-z)p^2 + (y-1)(y-(R+\frac{1}{2})(y-z))p - \frac{1}{2}R(y-z)}{z(z-1)}.$$

Don't know QFT.. We are more interested in the relationship with isomonodromic deformations. Consider $\Lambda = \frac{d}{dz} - U - \frac{1}{z}V$, where $U, V$ are constant $n \times n$ matrices, and $U$ is diagonal with pairwise distinct diagonal entries $\mathrm{diag}(u_1, \cdots, u_n)$, and $V(u)$ is as given. Then the equations for the rotation coefficients are those coming from the equations

$$\partial_k \psi_i = \gamma_{ik}\psi_k, i \neq k$$
$$\sum_{k=1}^{n} \partial_k \psi_i = z\psi_i,$$
$$\Lambda\psi = 0.$$

Namely these equations will yield the rotation coefficient equations, e.g. $\partial_j \partial_k \psi_i = \partial_k \partial_j \psi_i$.

The first two equations are about how things change wrt $u$, and the last equation is the real ODE happening over a geometric space we are interested in. So we are deforming "differential equations in $z$" in $u$. We would like to show that this $u$-deformation is **isomonodromic**. This operator $\Lambda$ has singularities at $0$ and $\infty$.

- The singularity at $z = 0$ is regular, so $\psi(z) = z^V \psi_0$ for a vector $\psi_0$. So monodromy is quite simple.. It is given by $\mathrm{diag}(e^{2\pi i \mu_1}, \cdots, e^{2\pi i \mu_n})$, where $\mu_1, \cdots, \mu_n$ are eigenvalues of $V(u)$. Thus this is **isomonodromic**!
- The singularity at $z = \infty$ is irregular, so the monodromy is really given by Stokes data. We would like to show that the **Stokes data also do not depend on** $u$.

**Theorem 2.9.** *On the space of all such operators with fixed monodromy around $z = 0$, there is a natural Frobenius structure. Conversely, any Frobenius manifold satisfying a semisimplicity assumption can be obtained by such a construction.*

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## 3. Quantum cohomology of $\mathbb{P}^2$

Goal: Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$.

The actual equation that will be used is

$$X'' = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)(X')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)X' + \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

Consider the Legendre family $E \to B = \mathbb{P}^1 - \{0, 1, \infty\}$, $Y^2 = X(X-1)(X-t)$. It is pretty well-known that the "periods of the Legendre family" are solutions to the Gauss hypergeometric differential equation

$$L_t\phi = 0, \quad L_t = t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}.$$

Here, a period is an integral of the invariant holomorphic 1-form $\frac{dx}{y}$ over a "parallel" family of 1-cycles. The reason why this holds is because of the following calculation. We consider $x$ and $t$ being "independent". Then, $y$ depends on $t$, in that

$$2y\frac{dy}{dt} = -x(x-1).$$

So,

$$\frac{d}{dt}\left(\frac{dx}{y}\right) = -\frac{\frac{dy}{dt}}{y^2}dx = \frac{dx}{2y(x-t)},$$

$$\frac{d^2}{dt^2}\left(\frac{dx}{y}\right) = \frac{3dx}{4y(x-t)^2}.$$

So,

$$L_t \frac{dx}{y} = \frac{-x^2 + x(2-2t) + t}{4y(x-t)^2} dx.$$

The RHS is in fact $\frac{1}{2} d\left(\frac{y}{(x-t)^2}\right)$ if you expand. Anyways, so it is exact, so $L_t \int_\gamma \frac{dx}{y} = 0$ for a 1-cycle $\gamma$.

So what about $L_t \int_\infty^{(X(t),Y(t))} \frac{dx}{y}$? This involves chain rule.

$$L_t \int_\infty^{(X,Y)} \frac{dx}{y} = \int_\infty^{(X,Y)} L_t \frac{dx}{y} + (1-2t)\frac{X'}{Y} + t(1-t)\left(\frac{d}{dt}\left(\frac{X'}{Y}\right) + \frac{d}{dt}\int_\infty^{(X,Y)} \frac{d}{dt}\frac{dx}{y} - \int_\infty^{(X,Y)} \frac{d^2}{dt^2}\frac{dx}{y}\right)$$

$$= \frac{1}{2}\frac{y}{(x-t)^2}|_\infty^{(X,Y)} + (1-2t)\frac{X'}{Y} + t(1-t)\left(\frac{X''}{Y} - \frac{X'Y'}{Y^2} + \frac{X'}{2Y(X-t)}\right)$$

$$= Y\left(\frac{1}{2(X-t)^2} + (1-2t)\frac{X'}{Y^2} + t(1-t)\left(\frac{X''}{Y^2} - \frac{X'Y'}{Y^3} + \frac{X'}{2Y^2(X-t)}\right)\right).$$

Now differentiating $Y^2 = X(X-1)(X-t)$ wrt $t$, we get

$$\begin{aligned} 2YY' &= X'(X-1)(X-t) + XX'(X-t) + X(X-1)(X'-1) \\ &= X'((X-1)(X-t) + X(X-t) + X(X-1)) - X(X-1). \end{aligned}$$

So

$$\frac{Y'}{Y} = \frac{1}{2}X'\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right) - \frac{1}{2(X-t)}.$$

So after applying Painleve VI,

$$X'' - \frac{X'Y'}{Y} + \frac{X'}{2(X-t)} = -\left(\frac{1}{t} + \frac{1}{t-1}\right)X' + \frac{Y^2}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

So

$$t(1-t)\left(\frac{X''}{Y^2} - \frac{X'Y'}{Y^3} + \frac{X'}{2Y^2(X-t)}\right) = \frac{(2t-1)X'}{Y^2} + \frac{1}{t(1-t)}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

So you get

$$L_t \int_\infty^{(X,Y)} \frac{dx}{y} = \frac{Y}{t(1-t)}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \left(\delta - \frac{1}{2}\right)\frac{t(t-1)}{(X-t)^2}\right).$$

Phew! F

We saw in the calculations that somehow $L_t \int_\infty^{(X,Y)} \frac{dx}{y}$ introduces a lot of cancellations, but not enough. What does the RHS mean? Manin says this fits in the context of $\mu$-equations.

**Definition 3.1.** A **$\mu$-equation** is

$$\sum_{i=1}^n L_i^{(j)} \int_0^s \omega_i = s^*(\Phi^{(j)}), \quad j = 1, \cdots, N,$$

where the symbols mean:

- $\pi : A \to B$ is a family of abelian varieties and a section $s$, over a small enough $B$ such that $\pi_*(\Omega^1_{A/B})$ and $\mathcal{D}_B$ are $\mathcal{O}_B$-free,

- $\omega_1, \cdots, \omega_n \in \Gamma(B, \pi_*(\Omega^1_{A/B}))$ is an $\mathcal{O}_B$-basis of vertical 1-forms,
- $\sum_{i=1}^n L_i^{(j)} \int_\gamma \omega_i = 0$, for $j = 1, \cdots, N$, is a system of generators of the $\mathcal{D}_B$-module of the Picard–Fuchs equations, where $\gamma$ runs over families of closed paths in the fibers spanning $H_1(B_t)$,
- $\Phi^{(j)}$, $j = 1, \cdots, N$, are families of meromorphic functions on $A$.

Now let's consider $\pi : E \to B$ case. For any symbol of order two $\sigma \in S^2(T_B)$ and $\omega$ a generator of $\pi_*(\Omega^1_{E/B})$, there is the Picard–Fuchs operator $L_{\sigma,\omega}$ where 1. its principal symbol is $\sigma$ and 2. it annihilates all periods of $\omega$.

Note that, for $f$ a function on $B$, $L_{f\sigma,\omega} = f L_{\sigma,\omega}$, and $L_{\sigma,g\omega} = g L_{\sigma,\omega} \circ g^{-1}$. So $L_{\sigma,\omega} \int_0^s \omega$ is $\mathcal{O}_B$-bilinear in $\sigma$ and $\omega$. So,

$$\mu := \left( L_{\sigma,\omega} \int_0^s \omega \right) \otimes \sigma^{-1} \otimes \omega^{-1} \in S^2(\Omega^1_B) \otimes (\pi_* \Omega^1_{E/B})^{-1},$$

depends only on $s$ and not on $\sigma$ or $\omega$. So $\Phi^{(j)}$'s should really be regarded as meromorphic sections of $\pi^*(S^2(\Omega^1_B) \otimes (\pi_*\Omega^1_{E/B})^{-1})$.

Now one goes to the uniformization. The Legendre family and the uniformized family $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \mapsto \tau \in \mathbb{H}$ is related via

$$\begin{gathered} (z, \tau) \mapsto (X, Y, t), \\ X = \frac{\wp(z,\tau) - e_1}{e_2 - e_1}, \\ Y = \frac{\wp_z(z,\tau)}{2(e_2 - e_1)^{3/2}}, \\ t = \frac{e_3 - e_1}{e_2 - e_1}. \end{gathered}$$

$$e_i(\tau) = \wp(P_i, \tau) \qquad (P_0 = 0, P_1 = \frac{1}{2}, P_2 = \frac{\tau}{2}, P_3 = \frac{1+\tau}{2}),$$

where $\wp$ is the Weierstrass $\wp$-function.

## 4. Quantum cohomology of $G/B$

Goal: Kim, *Quantum cohomology of flag manifolds $G/B$ and quantum Toda lattices.*

## 5. Whittaker functions

Moral: Whittaker functions are solutions of quantum $D$-modules of flag varieties.