A PROOF OF NÉRON–OGG–SHAFAREVICH CRITERION VIA ITS ARCHIMEDEAN ANALOGUE

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Contents

In this short note, we deduce the classical Néron–Ogg–Shafarevich criterion on good reduction of abelian varieties from its *archimedean analogue*, which will be reviewed shortly. The proof crucially uses Mumford's theory of degeneration of abelian varieties, and the work of Faltings–Chai [FC]. This proof is a very long detour; on the other hand, it is technically speaking not circular, as the ingredients do not use the Néron–Ogg–Shafarevich criterion, so this proof can be regarded as a new proof of the criterion. Moreover, from the use of degeneration of abelian varieties, we can link the Néron–Ogg–Shafarevich criterion with its archimedean analogue. The idea to prove using "degeneration of degeneration" is based on [Oda], although the current note is independent of *op. cit.*

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1. Archimedean Néron–Ogg–Shafarevich criterion

Let's first state what's the complex analytic analogue of the Néron–Ogg–Shafarevich criterion.

Proposition 1.1 (Archimedean Néron–Ogg–Shafarevich criterion)**.** *Let $f : A \to D^\times$ be a holomorphic family of principally polarized abelian varieties (ppavs in short) of dimension g over the punctured disc $D^\times$. Then, f extends to a family of ppavs over D, if and only if the monodromy representation $\rho : \pi_1(D^\times, t_0) \cong \mathbb{Z} \to \operatorname{Aut} \pi_1(A_{t_0})$ is trivial, where $t_0 \in D^\times$ is a fixed base point.*

Proof. As the Siegel upper half space $\mathbb{H}_g$ is the moduli space of ppavs of dimension g, f defines a classifying map $p : D^\times \to \mathbb{H}_g$ (in general the target is $\mathbb{H}_g$ modulo monodromy). As $\mathbb{H}_g$ is conformally equivalent to a bounded domain (the *bounded realization* of Siegel upper half space), 0 is a removable singularity of p. The other direction is immediate: the monodromy representation factors through $\pi_1(D, t_0) = 1$. □

Using this, we would like to prove

Proposition 1.2 (Néron–Ogg–Shafarevich criterion). *Let $K/\mathbb{Q}_p$ be a finite extension, and A/K be an abelian variety. Let $\rho : G_K \to \mathrm{GL}(T_\ell A)$ be the ℓ-adic monodromy representation for $\ell \neq p$. Then, A has good reduction if and only if $\rho|_{I_K}$ is trivial.*

The proof of Proposition 1.2 will be the content of the rest of the article. As a first reduction step, we realize that both sides of the statement of Proposition 1.2 are invariant under base change. Furthermore, by Zahrin's trick, one can assume that A is principally polarized. As one direction is easy, we need to prove the other nontrivial direction, assuming $\rho|_{I_K}$ is trivial.

We now use the arithmetic toroidal compactification of moduli of principally polarized abelian varieties, as in [FC]. In *op. cit.*, the following is proved:

Theorem 1.3 ([FC]). *Let $A_{g,n}$ be the stack of groupoids, where*

$$\mathrm{Ob}(A_{g,n}(S)) = \left\{ (A/S, \lambda, \alpha) : \left\{ \begin{array}{c} (A/S,\lambda) \textit{ is a ppav over } S \textit{ of relative dimension } g, \\ \textit{and } \alpha : A[n] \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^{2g} \textit{ is a symplectic} \\ \textit{isomorphism of finite locally free group schemes} \\ \textit{over } S \end{array} \right\} \right\}.$$

(1) If $n \geq 3$, $A_{g,n}$ is (represented by) a smooth quasi-projective scheme over $\mathbb{Z}[1/n]$.

(2) If $n \geq 3$, for a good enough choice of auxiliary data, there is a projective $\mathbb{Z}[1/n]$-scheme $\overline{A}_{g,n}$, which contains $A_{g,n}$ as a dense open subscheme. Furthermore, there is a semi-abelian scheme $\overline{G} \to \overline{A}_{g,n}$ extending the universal abelian scheme $G \to A_{g,n}$.

Using this, we have a partial result.

Proposition 1.4. *Let A/K be a ppav, and ρ be the ℓ-adic monodromy representation of A for $\ell \neq p$. If $\rho|_{I_K}$ is trivial, then A has at worst semistable reduction.*

Proof. By assumption, $A[\ell^n]$ is unramified for any $n \geq 1$. Take a finite unramified extension L/K over which $A[\ell^3]$ splits. Then, all $\overline{K}$-points of $A[\ell^3]$ are in fact defined over L. Pick an L-point of exact order ℓ^3. Then, (A, P) defines a point in $A_{g,\ell^3}(L) \subset \overline{A}_{g,\ell^3}(L)$. This extends to an $\mathcal{O}_L$-point over the compactification, $\overline{A}_{g,\ell^3}(\mathcal{O}_L)$ by valuative criterion for properness.

Recall that there is a semi-abelian scheme $\overline{G} \to \overline{A}_{g,\ell^3}$. After pulling back this family via the $\mathcal{O}_L$-point $\mathrm{Spec}\,\mathcal{O}_L \to \overline{A}_{g,\ell^3}$, we get a semi-abelian scheme over $\mathcal{O}_L$ where the generic fiber is A_L. This implies that A_L (thus A) has at worst semistable reduction. □

In [FC], it is proven that any semi-abelian degeneration of abelian varieties can be given by *Mumford's construction of degenerating abelian varieties*. Given a degeneration with *two-dimensional base*, we could relate the ℓ-adic monodromy representation of A with the actual geometric monodromy of the degeneration, which is exactly what we will do.

2. Warm-up: the case of elliptic curves

The aforementioned Mumford's construction in the case of elliptic curves is exactly the *Tate curve*. Before delving into the case of higher dimensional abelian varieties, we show how the proof of Proposition 1.2 should go in the case of elliptic curves.

Proof of Proposition 1.2, in the case of elliptic curves. Suppose that A has bad reduction. As we know A has at worst semistable reduction, we know that A has semistable bad reduction. We can then use that A has a Tate uniformization. Let $\mathscr{T} \to \operatorname{Spf} \mathcal{O}_{K^{\mathrm{nr}}}[[q]]$ be the Raynaud's *universal Tate elliptic curve.* As the discriminant is nonvanishing at $q \neq 0$, $\mathscr{T}$ is smooth over $\mathcal{O}_{K^{\mathrm{nr}}}((q))$. Also, by Tate uniformization, $\mathscr{T}|_{q=q(A)}$ gives a semistable model of $A_{K^{\mathrm{nr}}}$. Let

- V be the slice $\{q = q(A)\} \cong \operatorname{Spf} \mathcal{O}_{K^{\mathrm{nr}}}$,
- η be the generic point of V, $\overline{\eta}$ be the geometric point underlying it,
- $U = \{q \neq 0\} = \operatorname{Spf} \mathcal{O}_{K^{\mathrm{nr}}}((q))$,
- ι be the generic point $\operatorname{Spf} K^{\mathrm{nr}}((q)) \in U$, and $\overline{\iota}$ be the geometric generic point underlying it.

Then on $\pi_{1,\text{ét}}(A_{\overline{K^{\mathrm{nr}}}})_\ell = \pi_{1,\text{ét}}(\mathscr{T}_{\overline{\eta}})_\ell$, there are two monodromy actions you can think of:

$$\rho_{\eta,\overline{\eta}} : \pi_{1,\text{ét}}(\eta, \overline{\eta})_\ell (\cong (G_{K^{\mathrm{nr}}})_\ell = (I_K)_\ell) \to \operatorname{Aut}(\pi_{1,\text{ét}}(\mathscr{T}_{\overline{\eta}})_\ell),$$

$$\rho_{U,\overline{\eta}} : \pi_{1,\text{ét}}(U, \overline{\eta})_\ell \to \operatorname{Aut}(\pi_{1,\text{ét}}(\mathscr{T}_{\overline{\eta}})_\ell).$$

The first ($\rho_{\eta,\overline{\eta}}$) factors through the second ($\rho_{U,\overline{\eta}}$) via the natural map $\pi_{1,\text{ét}}(\eta, \overline{\eta})_\ell \to \pi_{1,\text{ét}}(U, \overline{\eta})_\ell$. By Abhyankar's lemma, both the target and the source are isomorphic to $\mathbb{Z}_\ell$; all ℓ-covers of U are Kummer covers, as is for K^{nr}. Thus, after identifying both with $\mathbb{Z}_\ell$, the natural map is multiplication by ℓ^n for some $n \geq 0$.[1] We assumed $\rho_{\eta,\overline{\eta}}$ is trivial, so $\rho_{U,\overline{\eta}}$ has finite image. Now $\mathscr{T}|_U \to U$ is formally smooth and proper, so the specialization map is an isomorphism in this case. Thus $\rho_{U,\overline{\eta}}$ is, up to conjugation, the same as

$$\rho_{U,\overline{\iota}} : \pi_{1,\text{ét}}(U, \overline{\iota})_\ell \to \operatorname{Aut}(\pi_{1,\text{ét}}(\mathscr{T}_{\overline{\iota}})_\ell)$$

Now $\pi_{1,\text{ét}}(\mathcal{O}_{K^{\mathrm{nr}}}) = 1$, so $\rho_{U,\overline{\eta}}$ will stay the same even if we base change $\mathcal{O}_{K^{\mathrm{nr}}}$ to an algebraically closed field, say $\mathbb{C}$! So this is the same as

$$\rho_{U,\overline{\iota}_{\mathbb{C}}} : \pi_{1,\text{ét}}(\iota_{\mathbb{C}}, \overline{\iota}_{\mathbb{C}})_\ell \to \operatorname{Aut}(\pi_{1,\text{ét}}(\mathscr{T}_{\overline{\iota}_{\mathbb{C}}})_\ell),$$

or

$$(G_{\mathbb{C}((q))})_\ell \to \operatorname{Aut}(\pi_{1,\text{ét}}(\mathscr{T}_{\overline{\mathbb{C}((q))}})_\ell)$$

Now this is over $\mathbb{C}$, so this is the ℓ-completion of the topological monodromy of the family $\mathscr{T}$ **seen as a family over a small puncutured disc** which is the analytification of the formal family we have. We know that the analytification of $\mathscr{T}$ makes sense over a unit punctured disc $\{0 < |q| < 1\}$ in the complex numbers, so we can consider the topological monodromy of this. As $\pi_1(\{0 < |q| < 1\}) \cong \mathbb{Z}$, we see that the topological monodromy also has to be of finite image. We can see in many ways that we have a contradiction here.

[1] This was pointed out to us by John Halliday.

- One can see explicitly how cycles go under monodromy using that $\mathscr{T} \cong \mathbb{C}^\times/q^{\mathbb{Z}}$ as complex torus. One sees that upon a choice of appropriate basis, the monodromy representation sends 1 to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
- One can argue abstractly using basic Hodge theory, making use of the archimedean Néron–Ogg–Shafarevich. Since the monodromy is quasi-unipotent, we know that over a cyclic cover (say over an n-cover) the monodromy becomes unipotent. Then the monodromy up above should be trivial because there is no nontrivial unipotent matrix of finite order. Now the whole family pulled back via $\{|q| < 1\} \xrightarrow{q \mapsto q^n} \{|q| < 1\}$ realizes not only the family over the cyclic cover of the punctured disc but also its singular fiber, so the family over the cyclic cover has trivial monodromy which can also be extended to a family over the whole disc with singular fiber at 0. There is only one way of extending a family of elliptic curves (due to Deligne, on the level of variation of Hodge structures, and for elliptic curves it's equivalent to thinking in terms of variation of Hodge structures), so it contradicts the archimedean Néron–Ogg–Shafarevich criterion, Proposition 1.1.

□

3. A proof of Néron–Ogg–Shafarevich criterion

For higher-dimensional abelian varieties, there are analogous construction of Mumford.

Theorem 3.1 (Mumford; see [FC, §III]). *Let $S = \operatorname{Spf} R$, where R is an I-adic completion of a normal excellent ring, and let η be its generic point. Let G/S be a semi-abelian scheme. Then, it can be associated with the following data:*

- *an extension $0 \to T \to \widetilde{G} \xrightarrow{\pi} A \to 0$, where A/S is an abelian scheme and T/S is a torus;*
- *a homomorphism $\iota : \underline{Y} \to \widetilde{G} \otimes \operatorname{Frac} R$, where $\underline{Y}$ is an etale sheaf on S whose fibers are free abelian groups of rank $\dim_S T$;*
- *a cubical invertible sheaf $\widetilde{\mathcal{L}}$ on $\widetilde{G}$ induced from a cubical invertible sheaf on A;*
- *an action $\underline{Y}$ on $\widetilde{Y}_\eta$ satisfying some positivity condition.*

Furthermore, G arises as a "quotient" of $\widetilde{G}$ by Y in a suitable sense.

Now we can use Mumford's construction to prove Proposition 1.2 in the general case.

Proof of Proposition 1.2. In the proof of the case of elliptic curves, the key was that there is a *one-dimensional degeneration* of degenerating elliptic curves, where the singular fiber is not a degeneration of abelian varieties. Using Mumford's construction, we can similarly form a *one-dimensional degeneration of degenerating abelian varieties* over $\operatorname{Spf} \mathcal{O}_{K^{\mathrm{nr}}}[[q]]$. Suppose that A has semistable bad reduction. We get degeneration data out of the semi-abelian scheme A we have, and in particular $\dim_S T > 0$ as we have bad reduction. Now consider a family of degenerating abelian varieties over $\operatorname{Spf} \mathcal{O}_{K^{\mathrm{nr}}}[[q]]$ formed by the same degeneration data but $\iota = q\iota_A$, the ι of

A scaled by q. This is a nonconstant one-dimensional family of general degenerating abelian varieties where the singular fiber at $q = 0$ is not actually a degenerating abelian variety. We can then use the proof of the case of elliptic curves word by word; now the last part about topological monodromy will need the archimedean Néron–Ogg–Shafarevich criterion, Proposition 1.1. $\square$

References

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