HIGHER KOECHER'S PRINCIPLE, HARMONIC HILBERT MAASS FORMS AND THEIR BORCHERDS LIFT

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Abstract. We reinterpret the notion of harmonic Maass forms in the context of coherent cohomology of modular curves. This suggests a natural generalization of weakly holomorphic modular forms and harmonic Maass forms to the case of Hilbert modular varieties. Furthermore, the theory of Fourier expansions for higher coherent cohomology classes is developed. In particular, the Borcherds lift of [Br2] is reinterpreted as being applied to harmonic Hilbert Maass forms in our sense.

Contents

1. Introduction

The theory of harmonic Maass forms was initiated by Ramanujan and its theoretical foundation was set by Zwegers in his thesis. Although harmonic Maass forms have been shown to be closely related to arithmetic, they have not been connected to the realm of arithmetic geometry or automorphic forms. Indeed, it was spectulated that harmonic Maass forms are related to a still-speculative concept of "extensions of automorphic representations" (e.g. [BK]), which does not exist in the current theory of automorphic representations.

In this manuscript, we recast the theory of harmonic Maass forms in terms of complex/arithmetic geometry of modular curves. In particular, we give a purely algebro-geometric interpretation of the *principal parts* of harmonic Maass forms using the local cohomology of modular curves at cusps[1]. Such interpretation would make sense over any base, for example $\mathbb{Z}$ or $\mathbb{F}_q$. On the other hand, harmonic Maass forms are expressed using the complex geometry of modular curves, by

[1]This is in accordance with the usual analogy between harmonic Maass forms and overconvergent modular forms. Indeed, overconvergent modular forms are recently also realized to be related with local cohomology at a cusp (see [BP], [Pa]).

using Hodge theory and the relative Dolbeault cohomology of modular curves. We believe that this re-interpretation is new, and this would help us to set up a correct picture of the theory, amenable to further development.

As an example, we develop a generalization of harmonic Maass forms in the context of Hilbert modular forms. Such generalization, or even a generalization of weakly holomorphic modular forms (namely, modular holomorphic functions on the upper half plane of at most linear exponential growth), has not been realized, due to the *Koecher's principle*. Namely, any singularity of a holomorphic function of codimension ≥ 2 can always be filled, and Hilbert modular varieties can be compactified so that the boundary is a finite collection of points, which is of codimension ≥ 2 as soon as one escapes the case of modular curves.

Under the coherent-cohomological interpretation of harmonic Maass forms developed in this paper, we realize that one can avoid Koecher's principle using higher coherent cohomology of Hilbert modular varieties. Namely, it is realized in [La2, §9] that the analogue of Koecher's principle for coherent cohomology of Shimura varieties *must fail* in a certain cohomological degree, which is usually not H^0. For example, for a Hilbert modular variety associated to a totally real field F of degree d, the failure of Koecher's principle occurs in H^{d-1} (and H^d, although the statement is trivial in that case). We define (principal parts of) *weakly holomorphic Hilbert modular forms* and *harmonic Hilbert Maass forms* using the coherent cohomology of an open Hilbert modular variety and its toroidal compactification.

To have a hold on them, we compute the space of principal parts of harmonic Hilbert Maass forms using the analogue of Fourier expansions for higher coherent cohomology classes. The computation exploits the geometry of toroidal compactification. In contrast to the q-expansion of holomorphic Hilbert modular forms, we see the following exotic features.

(1) This "higher" q-expansion is a U_F-coinvariance class of Laurent polynomials with totally negative exponents, where $U_F = \mathcal{O}_F^\times$. This is in contrast with the fact that the q-expansion of holomorphic Hilbert modular forms has only totally positive exponents.
(2) The higher q-expansion of those coming from cuspidal holomorphic Hilbert modular forms is zero. Thus, the higher q-expansion map exactly detects the principal parts of weakly holomorphic Hilbert modular forms.

In [Br2], an attempt to overcome Koecher's principle for Hilbert modular forms was made. As harmonic Maass forms are spanned by non-holomorphic Poincaré series, built out of "bad" Whittaker function, *op. cit.* uses bad Whittaker functions, named *harmonic Whittaker forms*, in place of harmonic Hilbert Maass forms. In particular, the Borcherds lift of harmonic Whittaker forms is constructed, and its relation to Kudla–Millson lift is studied as in [BF].

In the final part of this paper, we observe that the standard harmonic Whittaker forms are harmonic Hilbert Maass forms with q-expansion q^ν for some $\nu < 0$. In this way, the Borcherds lift of [Br2] can be regarded as lifting harmonic Hilbert Maass forms in our sense.

1.1. **Related problems.** As the primary objective of this manuscript is to suggest a correct viewpoint, various directions can stem out of it.

(1) Clearly, one can try to generalize the concept of harmonic Maass forms to other higher-dimensional Shimura varieties. The analogues of Whittaker functions of exponential growth for Sp_{2n} is developed in [BFK], which can be useful in developing the theory for Siegel modular varieties.

(2) One can explicitly compute the Dolbeault representative of harmonic Hilbert Maass forms. This might suggest how to directly connect Whittaker functions with the harmonic Hilbert Maass forms, or to directly construct the Borcherds lift of harmonic Hilbert Maass forms using their Dolbeault representatives.
(3) It would be interesting to investigate the principal parts of harmonic Maass forms over more general base ring, say $\mathbb{Z}$ or $\mathbb{F}_q$. Such consideration can make the analogy between harmonic Maass forms and overconvergent modular forms more rigorous.
(4) It is still unclear how the full harmonic Maass forms are related to arithmetic geometry. For weakly holomorphic modular forms, one can indeed just use the cohomology of open modular curve. On the other hand, the local cohomology long exact sequence starts from the cohomology of closed modular curve, and it is not clear how to detach this term from others.

1.2. **Notation.** The upper half plane $\mathbb{H}$ is defined as $\{\tau = u + iv \in \mathbb{C} \mid v > 0\}$. Every algebraic variety is defined over $\mathbb{C}$ unless otherwise noted. Any complex analytic manifold will have an in its superscript, to distinguish it from its algebraization. For example, $H^0(\mathbb{A}^1, \mathcal{O}) = \mathbb{C}[z]$ is the algebraic coherent cohomology, whereas $H^0(\mathbb{A}^{1,\mathrm{an}}, \mathcal{O})$ is the space of all holomorphic functions on $\mathbb{C}$. Given an automorphic vector bundle $\mathcal{E}$ over an open Shimura variety M, $\mathcal{E}^{\mathrm{can}}$, $\mathcal{E}^{\mathrm{sub}}$ are the canonical and subcanonical extensions of $\mathcal{E}$ over a toroidal compactification M^{tor} (where we do not specify the choice of a datum if possible), respectively, and $\mathcal{E}$ will also denote its natural extension to the minimal compactification M^{min}. The boundary divisors $M^{\mathrm{tor}} - M$ and $M^{\mathrm{min}} - M$ are all denoted D if there is no confusion. Over M^{tor}, we also tend to use the same letter $\mathcal{E}$ for its canonical extension and $\mathcal{E}(-D)$ for its subcanonical extension.

2. Cohomological interpretation of harmonic Maass forms

We first reinterpret the classical theory of harmonic Maass forms in terms of coherent cohomology and Dolbeault cohomology.

Definition 2.1. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. A *harmonic Maass form* of level Γ and weight k is a real analytic function $f : \mathbb{H} \to \mathbb{C}$ such that
(1) for all $\gamma \in \Gamma$, $f(\gamma\tau) = j(\gamma, \tau)^k f(\tau)$, where $j(\gamma, \tau)$ is the usual factor of automorphy,
(2) f is at most of exponential growth towards $+i\infty$,
(3) and $\Delta_k f = 0$, where Δ_k is the hyperbolic Laplacian of weight k,

$$\Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v} \right).$$

The space of such forms is denoted $H_k(\Gamma)$.

Since $\Delta_k = -R_{k-2}L_k$, where R_k and L_k are the raising and lowering operators, respectively, it follows that, for a harmonic Maass form f, $L_k f$ is closely related to a (weakly) anti-holomorphic modular form.

Definition 2.2. For $f \in H_k(\Gamma)$, let

$$\xi_k(f)(\tau) := v^{k-2}\overline{L_k f(\tau)}.$$

This is an antilinear map $\xi_k : H_k \to M^!_{2-k}$. We let H_k^+ be $\xi_k^{-1}(S_{2-k})$.

Under this definition, there is a short exact sequence

$$0 \to M_k^! \to H_k^+ \xrightarrow{\xi_k} S_{2-k} \to 0.$$

In terms of coherent cohomology of open and closed modular curves $Y(\Gamma)$ and $X(\Gamma)$, we can express two terms of the three in above:

$$0 \to H^0(Y(\Gamma), \omega^k) \to H_k^+ \to H^1(X(\Gamma), \omega^k) \to 0.$$

Here, the cohomology groups are coherent cohomology groups of complex algebraic varieties. A variant of this would be:

$$0 \to H^0(X(\Gamma), \omega^k) \to H^0(Y(\Gamma), \omega^k) \to \frac{H_k^+}{M_k} \to H^1(X(\Gamma), \omega^k) \to 0.$$

On the other hand, we can think of a local cohomology exact sequence in algebraic geometry:

$$0 \to H^0(X(\Gamma), \omega^k) \to H^0(Y(\Gamma), \omega^k) \to H^1_D(X(\Gamma), \omega^k) \to H^1(X(\Gamma), \omega^k) \to 0,$$

where $D = X(\Gamma) - Y(\Gamma)$; the first term is exact trivially, and the last term is exact as $Y(\Gamma)$ is affine.

Therefore, we can suspect that $H^1_D(X(\Gamma), \omega^k) \cong \frac{H_k^+}{M_k}$ in a natural way. This is indeed made possible via the notion of *relative Dolbeault cohomology.*

Definition 2.3 ([Suw, Definition 2.14]). Let $i : Y \hookrightarrow X$ be an open embedding of topological spaces, and let $\mathcal{F}^\bullet$ be a cohomologically graded complex of sheaves on X. Then, the *relative sheaf cohomology* $H_i^*(\mathcal{F}^\bullet)$ is the cohomology of the complex $\widetilde{\mathcal{F}}^\bullet$, where

$$\widetilde{\mathcal{F}}^q = \mathcal{F}^q(X) \oplus \mathcal{F}^{q-1}(Y), \quad d(a, b) = (da, i^*a - db).$$

If X, Y are complex manifolds and $\mathcal{F}^\bullet$ is the Dolbeault complex of a coherent sheaf $\mathcal{F}$ on X, then $H^*_{Y-X}(X, \mathcal{F}) := H_i^*(\mathcal{F}^\bullet)$ is also called the *relative Dolbeault cohomology.*

Proposition 2.4 ([Suw, Theorem 2.23]). *There is a canonical isomorphism between the relative Dolbeault cohomology and the analytic local cohomology, respecting the long exact sequences.*

This is not difficult, as $\widetilde{\mathcal{F}}^\bullet$ is just constructed as the *mapping cone* of two Dolbeault complexes. We cannot just naively use this version of relative Dolbeault cohomology as it computes *analytic* local cohomology. We therefore need to recast algebraic coherent cohomology in terms of analytic coherent cohomology. The key is that, if we denote $i : Y(\Gamma) \hookrightarrow X(\Gamma)$ be the natural open embedding, then $\mathrm{R}^a i_* \omega^k = 0$ for $a > 0$, and $i_*\omega^k = \varinjlim_{n\to\infty} \omega^k(nD)$. Thus, by Leray spectral sequence and GAGA over $X(\Gamma)$,

$$H^i(Y(\Gamma), \omega^k) = \varinjlim_{n\to\infty} H^i(X(\Gamma), \omega^k(nD)) = \varinjlim_{n\to\infty} H^i(X(\Gamma), \omega^k(nD))^{\mathrm{an}}.$$

Thus, by the mapping cone construction as above, $H^1_D(X(\Gamma), \omega^k)$ is H^1 of the complex

$$0 \to \mathcal{A}^{0,0}(X(\Gamma), \omega^k) \to \mathcal{A}^{0,1}(X(\Gamma), \omega^k) \oplus \varinjlim_{n\geq 0} \mathcal{A}^{0,0}(X(\Gamma), \omega^k(nD)) \to \varinjlim_{n\geq 0} \mathcal{A}^{0,1}(X(\Gamma), \omega^k(nD)) \to 0.$$

Since $\varinjlim_{n\geq 0} \mathcal{A}^{0,0}(X(\Gamma), \omega^k(nD))$ is precisely consisted of real analytic functions $\mathbb{H} \to \mathbb{C}$ with modular property and polar growth property, we can denote

$$\mathcal{A}^{0,0}(Y(\Gamma), \omega^k)_{\mathrm{polar}} := \varinjlim_{n\geq 0} \mathcal{A}^{0,0}(X(\Gamma), \omega^k(nD)),$$

namely the space of those of at most polar growth at boundary. Then,

$$H^1_D(X(\Gamma),\omega^k) = \frac{\{(\omega, g)\in \mathcal{A}^{0,1}(X(\Gamma),\omega^k)\oplus \mathcal{A}^{0,0}(Y(\Gamma),\omega^k)_{\mathrm{polar}} : \omega|_{Y(\Gamma)} = \overline{\partial} g\}}{\{(\overline{\partial} f, f|_{Y(\Gamma)}) : f\in \mathcal{A}^{0,0}(X(\Gamma),\omega^k)\}}.$$

Proposition 2.5. *There is a natural isomorphism*

$$j : \frac{H_k^+}{M_k} \xrightarrow{\sim} H^1_D(X(\Gamma),\omega^k),$$

given by

$$j(f) = (\overline{\partial} f, f),$$

for $f\in H_k^+$.

Proof. Certainly the above map defines a map $H_k^+\to H^1_D(X(\Gamma),\omega^k)$, because $\overline{\partial} f = \overline{\xi_k(f)} d\overline{z}$, and this, a priori only defined over $Y(\Gamma)$, extends to $X(\Gamma)$ because $\xi_k(f)$ can be extended. The kernel is precisely consisted of holomorphic f's, namely weakly holomorphic f's. This constructs an injection $j : \frac{H_k^+}{M_k} \to H^1_D(X(\Gamma),\omega^k)$. Both sit in a diagram of short exact sequences where the first and the third terms are identical, so j is automatically a surjection by the five-lemma. ☐

Thus, the space of principal parts of harmonic Maass forms fit perfectly into algebraic geometry of modular curves. In particular, it can be given a natural $\mathbb{Q}$-structure.

Remark 2.6. It is easy to see that the natural $\mathbb{Q}$-structure is consisted of harmonic Maass forms whose holomorphic parts (a *mock modular form*) have $\mathbb{Q}$-rational principal parts.

The space of harmonic Maass forms itself, H_k^+, can be also seen as follows.

Proposition 2.7. *Let* $(\mathcal{A}^{0,*}, \overline{\partial})$ *denote the Dolbeault complex of a complex manifold, and let* $\mathscr{H}^{p,q}$ *be the space of* (p, q)*-harmonic differential forms. Then* H_k^+ *is naturally identified with*

$$H_k^+ = \overline{\partial}^{-1}\left(\mathscr{H}^{0,1}(X(\Gamma),\omega^k)\right)\subset \mathcal{A}^{0,0}(Y(\Gamma),\omega^k)_{\mathrm{polar}},$$

and the exact sequence $0\to M_k^!\to H_k^+\to S_{2-k}\to 0$ *can be naturally identified with*

$$0\to H^0(Y(\Gamma),\omega^k)\to H_k^+\to H^1(X(\Gamma),\omega^k)\to 0.$$

Proof. Because of Hodge theory, the quotient map $q : \mathcal{A}^{0,1}(X(\Gamma),\omega^k)\to H^1(X(\Gamma),\omega^k)$ from Dolbeault cohomology gives rise to an isomorphism

$$q|_{\mathscr{H}^{0,1}(X(\Gamma),\omega^k)} : \mathscr{H}^{0,1}(X(\Gamma),\omega^k)\xrightarrow{\sim} H^1(X(\Gamma),\omega^k).$$

Thus, $H^1_D(X(\Gamma),\omega^k) = \frac{H_k^+}{M_k}$ can be rewritten as

$$H^1_D(X(\Gamma),\omega^k) = \left\{(\omega, g)\in \mathscr{H}^{0,1}(X(\Gamma),\omega^k)\oplus \frac{\mathcal{A}^{0,0}(Y(\Gamma),\omega^k)_{\mathrm{polar}}}{\omega^k(Y(\Gamma))} : \omega|_{Y(\Gamma)} = \overline{\partial} g\right\},$$

and by the same reason, H_k^+ can be rewritten as

$$H_k^+ = \{(\omega, g)\in \mathscr{H}^{0,1}(X(\Gamma),\omega^k)\oplus \mathcal{A}^{0,0}(Y(\Gamma),\omega^k)_{\mathrm{polar}} : \omega|_{Y(\Gamma)} = \overline{\partial} g\}.$$

Since ω is uniquely determined by g (it is the extension of $\overline{\partial} g$, and it is unique if it exists), one can remove ω and simply identify with the preimage of $\overline{\partial}$. ☐

Thus, in some sense, weakly holomorphic modular forms and harmonic Maass forms exist *precisely because of the discrepancy between the coherent cohomology of open and closed modular curves.*

It is also notable that the space of principal parts of harmonic Maass forms is something arising from local picture around cusps. Namely, harmonic Maass forms are often regarded analogous to overconvergent modular forms, and the space of overconvergent modular forms is also the "stalk at ∞" (see [Pa, §5.2.4]).

3. Beyond Koecher's principle: weakly holomorphic Hilbert modular forms

Koecher's principle prevents naively generalizing the definition of weakly holomorphic modular forms and harmonic Maass forms to higher rank cases. On the other hand, the slogan can be extended to other Shimura varieties, thanks to the following work of Lan on *higher Koecher's principle*:

Theorem 3.1 ([La2, Theorem 2.5]). *Let M be a Shimura variety, M^{min} be its minimal compactification, and M^{tor} be a toroidal compactification of M, which sit in the diagram*

$$\begin{array}{ccc} & & M^{\mathrm{tor}} \\ & \overset{j^{\mathrm{tor}}}{\nearrow} & \downarrow \pi \\ M & \underset{j^{\mathrm{min}}}{\hookrightarrow} & M^{\mathrm{min}} \end{array}$$

Let $\mathcal{E}$ be an automorphic vector bundle, and let $\mathcal{E}^{\mathrm{can}}$ be its canonical extension over M^{tor}. Let $c_M = \operatorname{codim}(M^{\mathrm{min}} - M, M^{\mathrm{min}})$.

1. *The natural map $\mathrm{R}^i\pi_*(\mathcal{E}^{\mathrm{can}}) \to \mathrm{R}^i j^{\mathrm{min}}_*\mathcal{E}$ induced by j^{tor} is an isomorphism for $i < c_M - 1$ and is an injection for $i = c_M - 1$.*
2. *The natural restriction map $H^i(M^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \to H^i(M, \mathcal{E})$ is an isomorphism for $i < c_M - 1$, and is an injection but not an isomorphism for $i = c_M - 1$.*

We record the following facts used in the course of proof of Theorem 3.1.

Proposition 3.2. *Retaining the notations of Theorem 3.1, we have the following.*

1. *([La2, (3.1)]) $j^{\mathrm{tor}}_*\mathcal{E} = \varinjlim_{n\geq 0} \mathcal{E}^{\mathrm{can}}(nD)$ and $\mathrm{R}^a j^{\mathrm{tor}}_*\mathcal{E} = 0$ for $a > 0$, where $D = M^{\mathrm{tor}} - M$.*
2. *([La2, (3.2)]) $\mathrm{R}^a j^{\mathrm{min}}_*\mathcal{E} \cong \varinjlim_{n\geq 0} \mathrm{R}^a\pi_*(\mathcal{E}^{\mathrm{can}}(nD))$ for $a \geq 0$.*
3. *([La2, Theorem 3.9]) $\mathrm{R}^a\pi_*\mathcal{E}^{\mathrm{sub}}(-nD) = 0$ for any $n \geq 0$.*

Proof. The only difference between this statement and that in [La2] is that D' is used in place of D in *op. cit.*, but one can obtain this statement as there is $\ell \in \mathbb{N}$ such that $D \leq D' \leq \ell D$. $\square$

Let us restrict to the case of Hilbert modular forms. Let F be a totally real field of degree $d > 1$, and let $\Gamma \leq \mathrm{SL}_2(\mathcal{O}_F)$ be a congruence subgroup. Then, the Hilbert modular variety $M = Y(\Gamma)$ has $c_M = d - 1$. Thus, $H^i(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}})$ and $H^i(Y(\Gamma), \mathcal{E})$ can only differ in degrees $i = d - 1$ and d:

$$H^i(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \xrightarrow{\sim} H^i(Y(\Gamma), \mathcal{E}), \quad i \leq d - 2,$$
$$H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \subsetneq H^i(Y(\Gamma), \mathcal{E}),$$
$$H^d(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \twoheadrightarrow H^d(Y(\Gamma), \mathcal{E}) = 0.$$

Using the facts we have stated so far, we obtain the following

Proposition 3.3.

(1) *We have* $H^i_D(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) = 0$ *for* $0 \le d-1$, *and an exact sequence*

$$0 \to H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \to H^{d-1}(Y(\Gamma), \mathcal{E}) \to H^d_D(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \to H^d(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \to 0.$$

(2) *We have*

$$H^d_D(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) = H^{d-1}\left(X(\Gamma)^{\mathrm{tor}}, \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right) = \frac{H^0(X(\Gamma)^{\mathrm{min}}, \mathrm{R}^{d-1} j^{\mathrm{min}}_* \mathcal{E})}{H^0(X(\Gamma)^{\mathrm{min}}, \mathrm{R}^{d-1}\pi_*\mathcal{E}^{\mathrm{can}})} = \frac{H^d_D(X(\Gamma)^{\mathrm{min}}, \mathcal{E})}{H^d_\pi(X(\Gamma)^{\mathrm{min}}, \mathcal{E})}.$$

Proof. The first part is a part of the local cohomology exact sequence for $Y(\Gamma) \hookrightarrow X(\Gamma)^{\mathrm{tor}}$. The sequence breaks into parts because of higher Koecher's principle, and the exact sequence in (1) is the end piece of the long exact sequence. By Proposition 3.1(1), the local cohomology exact sequence is also the same as long exact sequence of

$$0 \to \mathcal{E}^{\mathrm{can}} \to j^{\mathrm{tor}}_* \mathcal{E} \to \frac{j^{\mathrm{tor}}_*\mathcal{E}}{\mathcal{E}^{\mathrm{can}}} \to 0,$$

because of Leray spectral sequence. This implies $H^d_D(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) = H^{d-1}\left(X(\Gamma)^{\mathrm{tor}}, \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right)$. By Leray spectral sequence for $\pi : X(\Gamma)^{\mathrm{tor}} \to X(\Gamma)^{\mathrm{min}}$, we have

$$H^a\left(X(\Gamma)^{\mathrm{min}}, \mathrm{R}^b\pi_* \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right) \Rightarrow H^{a+b}\left(X(\Gamma)^{\mathrm{tor}}, \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right).$$

We have a long exact sequence

$$\cdots \to \mathrm{R}^i\pi_*\mathcal{E}^{\mathrm{can}} \to \mathrm{R}^i\pi_*(\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)) \to \mathrm{R}^i\pi_* \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}} \to \cdots.$$

Since $\mathrm{R}^i\pi_* \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}$ is supported on $X(\Gamma)^{\mathrm{min}} - Y(\Gamma)$, $H^a\left(X(\Gamma)^{\mathrm{min}}, \mathrm{R}^b\pi_* \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right) = 0$ for $a > \dim(X(\Gamma)^{\mathrm{min}} - Y(\Gamma)) = 0$. Thus,

$$H^{d-1}\left(X(\Gamma)^{\mathrm{tor}}, \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right) = H^0\left(X(\Gamma)^{\mathrm{min}}, \mathrm{R}^{d-1}\pi_* \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}\right).$$

Also, since $\mathrm{R}^i\pi_*(\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)) = \mathrm{R}^i j^{\mathrm{min}}_* \mathcal{E}$ by Proposition 3.1(2), by the higher Koecher's principle the long exact sequence of higher direct image sheaves breaks into pieces, and the end piece is

$$0 \to \mathrm{R}^{d-1}\pi_*\mathcal{E}^{\mathrm{can}} \to \mathrm{R}^{d-1} j^{\mathrm{min}}_* \mathcal{E} \to \mathrm{R}^{d-1}\pi_* \frac{\varinjlim_{n\ge 0} \mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}} \to 0,$$

from which we get the rest of (2). □

The situation is entirely analogous to that of harmonic Maass forms. Thus, we are led to the following

Definition 3.4. The space of *weakly holomorphic Hilbert modular forms* $M^!_k(\Gamma)$ is defined to be

$$M^!_k(\Gamma) := H^{d-1}(Y(\Gamma), \omega^k).$$

The space of *principal parts* of weakly holomorphic Hilbert modular forms is defined to be

$$\frac{M_k^!(\Gamma)}{M_k(\Gamma)} := \frac{H^{d-1}(Y(\Gamma), \omega^k)}{H^{d-1}(X(\Gamma)^{\text{tor}}, \omega^k)}.$$

Finally, the space of *principal parts of harmonic Hilbert Maass forms* is defined to be

$$\frac{H_k^+(\Gamma)}{M_k(\Gamma)} := H_D^d(X(\Gamma)^{\text{tor}}, \omega^k),$$

so that they fit into the exact sequence

$$0 \to \frac{M_k^!(\Gamma)}{M_k(\Gamma)} \to \frac{H_k^+(\Gamma)}{M_k(\Gamma)} \to H^d(X(\Gamma)^{\text{tor}}, \omega^k) \to 0.$$

Remark 3.5. The final part $H^d(X(\Gamma)^{\text{tor}}, \omega^k)$ can be regarded as S_{2-k} by Serre duality, so the situation is entirely analogous to the case of harmonic Maass forms.

We now compute this using relative Dolbeault cohomology. As in the case of harmonic Maass forms, the algebraic local cohomology is not the same as the analytic local cohomology, and we would instead have to use $H_D^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) = H^{d-1}\left(X(\Gamma)^{\text{tor}}, \frac{\varinjlim_{n\geq 0} \mathcal{E}^{\text{can}}(nD)}{\mathcal{E}^{\text{can}}}\right)$ and use the mapping cone construction over $X(\Gamma)^{\text{tor}}$ where now GAGA can be used. Namely, $H_D^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}})$ would be the H^d of the complex

$$0 \to \mathcal{A}^{0,0}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \to \mathcal{A}^{0,1}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \oplus \varinjlim_{n\geq 0} \mathcal{A}^{0,0}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD)) \to$$

$$\cdots \to \mathcal{A}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \oplus \varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD)) \to \varinjlim_{n} \mathcal{A}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD)) \to 0.$$

Namely,

$$H_D^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) =$$

$$\frac{\{(\omega_1, \omega_2) \in \mathcal{A}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \oplus \varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD)) : \omega_1 = \overline{\partial}\omega_2\}}{\{(\overline{\partial}\nu_1, \nu_1 - \overline{\partial}\nu_2) : (\nu_1, \nu_2) \in \mathcal{A}^{0,d-1}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \oplus \varinjlim_{n\geq 0} \mathcal{A}^{0,d-2}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD))\}}.$$

To obtain the whole space of harmonic Maass forms, we use the Hodge theory of $X(\Gamma)^{\text{tor}}$. Namely, as $X(\Gamma)^{\text{tor}}$ is a compact Kähler manifold, the quotient map for Dolbeault cohomology

$$\mathcal{A}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \twoheadrightarrow H^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}),$$

restricts to an isomorphism

$$\mathscr{H}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \xrightarrow{\sim} H^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}),$$

by Hodge theory, where $\mathscr{H}$ is the space of harmonic forms. This means that any form in $\mathcal{A}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}})$ can be modified into $\mathscr{H}^{0,d}$ via adding an appropriate $\overline{\partial}$ of a $(0, d-1)$-form. Thus, $H_D^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}})$ has the following description,

$$H_D^d(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) =$$

$$\left\{(\omega_1, \omega_2) \in \mathscr{H}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \oplus \frac{\varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD))}{\ker(\mathcal{A}^{0,d-1}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,d}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}})) + \overline{\partial}\left(\varinjlim_{n\geq 0} \mathcal{A}^{0,d-2}(X(\Gamma)^{\text{tor}}, \mathcal{E}^{\text{can}}(nD))\right)} : \omega_1 = \overline{\partial}\omega_2\right\} =$$

$$\left\{ \omega \in \frac{\varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}(nD))}{\ker(\mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}})) + \overline{\partial}\left(\varinjlim_{n\geq 0} \mathcal{A}^{0,d-2}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}(nD))\right)} : \overline{\partial}\omega \in \mathscr{H}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}^{\mathrm{can}}) \right\}.$$

Removing the part in the denominator on cohomology of $\mathcal{E}^{\mathrm{can}}$, we arrive at the following definition.

Definition 3.6. Let the space of *harmonic Hilbert Maass forms* $H_k^+(\Gamma)$ be defined as

$$H_k^+(\Gamma) = \frac{\ker \overline{\partial}^{-1}\left(\mathscr{H}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k)\right)}{\mathrm{im}\, \overline{\partial}} \subset \frac{\varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD))}{\mathrm{im}\, \overline{\partial}}.$$

Proposition 3.7. *The space of harmonic Hilbert Maass forms $H_k^+(\Gamma)$ sits in a short exact sequence*

$$0 \to M_k^!(\Gamma) \to H_k^+(\Gamma) \to H^d(X(\Gamma)^{\mathrm{tor}}, \omega^k) \to 0.$$

Proof. This is proven similarly as in the case of harmonic Maass forms. Namely, consider the diagram of Dolbeault complexes for $\omega^k \subset \omega^k(\infty D)$ over $X(\Gamma)^{\mathrm{tor}}$:

$$\begin{array}{ccccccc}
\mathcal{A}^{0,0}(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} & \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \xrightarrow{\overline{\partial}} & \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k) \\
\downarrow & & \downarrow & & \downarrow \\
\varinjlim_{n\geq 0} \mathcal{A}^{0,0}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD)) & \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} & \varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD)) & \xrightarrow{\overline{\partial}} & \varinjlim_{n\geq 0} \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD))
\end{array}$$

Since $\varinjlim_{n\geq 0} H^d(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD)) = H^d(Y(\Gamma), \omega^k) = 0$, we have a diagram of exact sequences

$$\begin{array}{ccccccccc}
0 \to & \ker \overline{\partial} & \to & \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \xrightarrow{\overline{\partial}} & \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \to & H^d(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \ker \overline{\partial} & \to & \varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD)) & \xrightarrow{\overline{\partial}} & \varinjlim_{n\geq 0} \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD)) & \to & 0 &
\end{array}$$

Quotienting out by $\mathrm{im}\, \overline{\partial}$, we have

$$0 \to H^{d-1}(Y(\Gamma), \omega^k) \to \frac{\varinjlim_{n\geq 0} \mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD))}{\mathrm{im}\, \overline{\partial}} \to \varinjlim_{n\geq 0} \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD)) \to 0.$$

By taking the preimage of the image of $\mathscr{H}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k)$ in $\varinjlim_{n\geq 0} \mathcal{A}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k(nD))$, we get the desired result. □

We now explain elementarily what a harmonic Hilbert Maass form is. Note that the Dolbeault complex for a Shimura variety is identified with the Chevalley–Eilenberg complex for $(\mathfrak{p}, K)$-cohomology of certain functions on adelic quotient of G (cf. [Su]). The harmonic forms $\mathscr{H}^{0,d}(X(\Gamma)^{\mathrm{tor}}, \omega^k)$ are, by Serre duality, all of form $f \wedge_{i=1}^d d\overline{z}_i$, for a Hilbert modular cusp form f of parallel weight $2-k$, antiholomorphic in all variables. Under the same choice of $d\overline{z}_1, \cdots, d\overline{z}_d$, an element of $\mathcal{A}^{0,d-1}(Y(\Gamma), \omega^k)$ is of form

$$\sum_{i=1}^d f_i d\overline{z}_1 \wedge \cdots \wedge \widehat{d\overline{z}_i} \wedge \cdots \wedge d\overline{z}_d,$$

where hat means the entry is missing, and f_i is a smooth function over $Y(\Gamma)$ that transforms like a modular form of weight $(2-k, \cdots, k, \cdots, 2-k)$ (k at the i-th entry, $2-k$ at the other entries).

Similarly, an element of $\varinjlim_{n\geq 0}\mathcal{A}^{0,d-1}(X(\Gamma)^{\mathrm{tor}},\omega^k(nD))$ is of the same form, where f_i now has at most linear exponential growth towards cusps (when regarded as a function over $\mathbb{H}^d$).

4. Fourier expansions of higher coherent cohomology classes

We develop the theory of Fourier expansion for higher coherent cohomology classes of Hilbert modular variety, by computing its local cohomology. This will enable us to relate our definition of harmonic Hilbert Maass forms and weakly holomorphic Hilbert modular forms with the work of [Br2]. Our method of computation of local cohomology using explicit structure of the boundary has its origin in [Fr]. The scope of the theory developed here is only for Hilbert modular varieties, but a similar strategy would work for more general Shimura varieties.

4.1. **Analytic q-expansions.** In this section, we work with the analytic category. Analytically, the Fourier expansion would be cxomputed after restricting a Dolbeault class to a neighborhood of a cusp. For simplicity, let us assume that $\Gamma=\mathrm{SL}_2(\mathcal{O}_F)$ and $h_F=1$, so that there is only one cusp, ∞. The isotropy group of ∞ in Γ is

$$C_\infty := \left\{\begin{pmatrix}\varepsilon & \mu\\ 0 & \varepsilon^{-1}\end{pmatrix} : \varepsilon\in U_F, \mu\in\mathcal{O}_F\right\}/\{\pm 1\},$$

where U_F is the group of units. More neatly, one can express this as

$$\left\{\begin{pmatrix}\varepsilon & \mu\\ 0 & 1\end{pmatrix} : \varepsilon\in U_F^2, \mu\in\mathcal{O}_F\right\}.$$

This contains a subgroup $\Gamma_\infty=\{\left(\begin{smallmatrix}1&\mu\\0&1\end{smallmatrix}\right):\mu\in\mathcal{O}_F\}$, and $C_\infty/\Gamma_\infty\cong U_F^2$, which is by Dirichlet's unit theorem isomorphic to $\mathbb{Z}^{d-1}$. A base of open neighborhoods of ∞ in $X(\Gamma)^{\min}$ is given by $\{C_\infty\backslash V_R\cup\{\infty\}:R\gg 0\}$, where

$$V_R=\{(z_1,\cdots,z_d)\in\mathbb{H}^d:\prod_{i=1}^d \mathrm{Im}(z_i)>R\}.$$

The q-expansion of a classical Hilbert modular form is obtained by taking the Fourier expansion of the restriction of the Hilbert modular form to $C_\infty\backslash V_R$ for a large enough R, or more precisely its pullback to $C_\infty\backslash V_R$. To deal with higher cohomology of $C_\infty\backslash V_R$, we first prove

Lemma 4.1 ([Fr, Hilfssatz 5.1]). *For $R\geq 0$, $C_\infty\backslash V_R\cup\{\infty\}$ is Stein.*

Proof. Note that as Γ_∞ acts via translation by real vectors, $\mathrm{Im}(z_i):V_R\to\mathbb{R}^+$ descends to a function $\Gamma_\infty\backslash V_R\to\mathbb{R}^+$. Furthermore, $U_F^2=C_\infty/\Gamma_\infty$ acts so that $\prod_{i=1}^d\mathrm{Im}(z_i)$ is unchanged. It is clear that

$$\frac{1}{\prod_{i=1}^d\mathrm{Im}(z_i)}:\Gamma_\infty\backslash V_R\cup\{\infty\}\to\mathbb{R}^+,$$

is a smooth, strictly plurisubharmonic, exhaustive function on $\Gamma_\infty\backslash V_R\cup\{\infty\}$, where the function is defined to be zero at $\{\infty\}$. Thus, by Oka's theorem, it is Stein. □

Now the local cohomology $H^i_{\{\infty\}}(C_\infty\backslash V_R\cup\{\infty\},\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ fits into the local cohomology long exact sequence

$$\cdots\to H^i_{\{\infty\}}(C_\infty\backslash V_R\cup\{\infty\},\mathcal{F})\to H^i(C_\infty\backslash V_R\cup\{\infty\},\mathcal{F})\to H^i(C_\infty\backslash V_R,\mathcal{F})\to\cdots,$$

and by Lemma 4.1, we thus know that

$$H^i_{\{\infty\}}(C_\infty\backslash V_R \cup \{\infty\}, \mathcal{F}) \cong H^{i-1}(C_\infty\backslash V_R, \mathcal{F}),$$

for any $i > 1$. Now the excision says that the left hand side is the same as $H^i_{\{\infty\}}(X(\Gamma)^{\min}, \mathcal{F})$, if $\mathcal{F}$ came from a coherent sheaf over $X(\Gamma)^{\min}$ denoted by the same letter $\mathcal{F}$.

Thus, as in [Fr, §5], the Grothendieck spectral sequence degenerates and yields

$$H^{i-1}(C_\infty\backslash V_R, \mathcal{F}) = H^{i-1}(U_F^2, H^0(\Gamma_\infty\backslash V_R, \mathcal{F})),$$

for any coherent sheaf $\mathcal{F}$ on $C_\infty\backslash V_R$ (here, the same letter $\mathcal{F}$ is used for its pullback to $\Gamma_\infty\backslash V_R$). In practice, $\mathcal{F}$ would all come from automorphic vector bundles of the Hilbert modular varieties. There is a canoncial section of $\mathcal{F}$ over $\Gamma_\infty\backslash V_R$ coming from the canonical differential of relative sheaf of differentials on the universal abelian scheme, which enables us to regard a holomorphic section of $\mathcal{F}$ as a holomorphic function. For such functions, we can take Fourier expansion with respect to Γ_∞-invariance,

$$H^0(\Gamma_\infty\backslash V_R, \mathcal{F}) \to \mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee},$$

where q^ν stands for $\exp(2\pi i \operatorname{Tr} \nu z)$, $\langle -\rangle$ means the formal Laurent series is the Fourier expansion of a germ of a meromorphic function, and $\mathcal{O}_F^\vee = \{\lambda \in F : \operatorname{Tr} \lambda\mu \in \mathbb{Z}\ \forall \mu \in \mathcal{O}_F\}$. Thus, we have a q-expansion map for higher coherent cohomology of Hilbert modular varieties,

$$q\exp^i : H^i(Y(\Gamma), \omega^k) \to H^{i+1}_{\{\infty\}}(X(\Gamma)^{\min}, \omega^k) \xleftarrow{\sim} H^i(C_\infty\backslash V_R, \omega^k) \xrightarrow{\sim} H^i(\mathbb{Z}^{d-1}, H^0(\Gamma_\infty\backslash V_R, \omega^k)) \to H^i(\mathbb{Z}^{d-1}, \mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee}).$$

We note that the action of $U_F^2 = C_\infty/\Gamma_\infty \cong \mathbb{Z}^{d-1}$ on $\mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee}$ is, for $\mu \in U_F^2$, just $\mu \cdot q^\nu = q^{\mu\nu}$. In this optic, we can recast the (failure of) Koecher's principle as follows.

Proposition 4.2. *For $i = d-1$, the image of $q\exp^{d-1}$, which a priori lies in the space of U_F^2-coinvariants $(\mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee})_{U_F^2}$, lies in $(\mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee,\nu\le 0})_{U_F^2}$, where $\nu \le 0$ means ν is either 0 or totally negative. Furthermore, $\operatorname{im}(H^{d-1}(X^{\min}, \omega^k) \to H^{d-1}(Y, \omega^k)) = \ker(q\exp^{d-1})$.*

Proof. The second statement is precisely the consquence of local cohomology long exact sequence. We now prove the first statement. it is sufficient to prove that any $\sum_{\nu\in\mathcal{O}_F^\vee,\nu\not\le 0} a_\nu q^\nu \in \mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\varepsilon}$ is in $(U_F^2 - 1)\mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee}$. Let $\sigma_1, \cdots, \sigma_d$ be d real embeddings of F, and for $I \subset \{1, \cdots, d\}$ and $x \in F$, we define the expression $x \sim (-1)^I$ to mean that $\sigma_i(x) < 0$ if $i \in I$ and $\sigma_i(x) > 0$ if $i \notin I$. Then, it is sufficient to prove the same for the Laurent series of form $g = \sum_{\nu\in\mathcal{O}_F^\vee,\nu\sim(-1)^I} a_\nu q^\nu$ for $I \neq \{1, \cdots, d\}$.

We can find a unit $\mu_I \in U_F^2$ such that $\sigma_i(\mu_I) < 1$ for $i \in I$, as $I \neq \{1, \cdots, d\}$. Then, consider

$$f(z) := \sum_{\nu\in\mathcal{O}_F^\vee,\nu\sim(-1)^I} a_\nu \left(\sum_{n=1}^\infty q^{\mu_I^n \nu}\right).$$

The infinite sum is uniformly convergent, and defines a germ of a meromorphic function with d variables. Furthermore, by uniform convergence,

$$\mu_I^{-1} \cdot f = \sum_{\nu\in\mathcal{O}_F^\vee,\nu\sim(-1)^I} a_\nu \left(\sum_{n=0}^\infty q^{\mu_I^n \nu}\right),$$

so $(\mu_I^{-1} - 1)f = g$, as desired. □

We would like to say that the q-expansion map precisely detects the failure of Koecher's principle. Proposition 4.1 is very close to what we would like, except that the computation relates $H^{d-1}(X(\Gamma)^{\min}, \omega^k)$ instead of $H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k)$. These two spaces are indeed different, but they are not too different: the difference is precisely due to the Eisenstein series (which is one-dimensional in our case, as we assumed that there is only one cusp for simplicity).

Proposition 4.3. *The cokernel of the natural map* $H^{d-1}(X(\Gamma)^{\min}, \omega^k) \to H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k)$ *is naturally identified with the dual of the space of Eisenstein series of weight* $2-k$.

Proof. Note that there is a long exact sequence of relative sheaf cohomology

$$\cdots \to H^{d-1}(X^{\min}, \omega^k) \to H^{d-1}(X^{\mathrm{tor}}, \omega^k) \to H^d_\pi(X^{\min}, \omega^k) \to H^d(X^{\min}, \omega^k) \to H^d(X^{\mathrm{tor}}, \omega^k) \to 0.$$

Thus,

$$\begin{aligned}
\operatorname{coker}(H^{d-1}(X^{\min}, \omega^k) \to H^{d-1}(X^{\mathrm{tor}}, \omega^k)) &= \frac{H^{d-1}(X^{\mathrm{tor}}, \omega^k)}{\operatorname{im}(H^{d-1}(X^{\min}, \omega^k) \to H^{d-1}(X^{\mathrm{tor}}, \omega^k))} \\
&= \frac{H^{d-1}(X^{\mathrm{tor}}, \omega^k)}{\ker(H^{d-1}(X^{\mathrm{tor}}, \omega^k) \to H^d_\pi(X^{\min}, \omega^k))} \\
&= \operatorname{im}(H^{d-1}(X^{\mathrm{tor}}, \omega^k) \to H^d_\pi(X^{\min}, \omega^k)) \\
&= \ker(H^d(X^{\min}, \omega^k) \to H^d(X^{\mathrm{tor}}, \omega^k)) \\
&= \operatorname{coker}(H^d(X^{\mathrm{tor}}, \omega^k)^* \to H^d(X^{\min}, \omega^k)^*)^*,
\end{aligned}$$

where $(-)^*$ is the $\mathbb{C}$-linear dual. By Kodaira–Spencer isomorphism and (classical) Koecher's principle,

$$\begin{aligned}
\operatorname{coker}(H^d(X^{\mathrm{tor}}, \omega^k)^* \to H^d(X^{\min}, \omega^k)^*) &= \operatorname{coker}(H^0(X^{\mathrm{tor}}, \omega^{2-k,\mathrm{sub}}) \to H^0(X^{\min}, \omega^{2-k})) \\
&= \operatorname{coker}(H^0(X^{\mathrm{tor}}, \omega^{2-k,\mathrm{sub}}) \to H^0(X^{\mathrm{tor}}, \omega^{2-k})),
\end{aligned}$$

as desired. □

4.2. **Algebraic q-expansions, using geometry of toroidal compactifications.** We now compute the local cohomology group using the geometry of compactification, as in [La2]. This will help us identify the image of the q-expansion map.

By Proposition 3.2(2), we would like to compute

$$\varinjlim_{n \geq 0} H^0\left(X(\Gamma)^{\min}, \frac{\mathrm{R}^{d-1}\pi_*(\mathcal{E}^{\mathrm{can}}(nD))}{\mathrm{R}^{d-1}\pi_*\mathcal{E}^{\mathrm{can}}}\right) = \varinjlim_{n \geq 0} H^0(X(\Gamma)^{\min}, \mathrm{R}^{d-1}\pi_*\mathcal{Q}_n),$$

where $\mathcal{Q}_n = \frac{\mathcal{E}^{\mathrm{can}}(nD)}{\mathcal{E}^{\mathrm{can}}}$. Since $\mathrm{R}^{d-1}\pi_*\mathcal{Q}_n$ is supported at $\{\infty\}$, we are to compute

$$\varinjlim_{n \geq 0} H^{d-1}((X(\Gamma)^{\mathrm{tor}})^\wedge_\infty, (\mathcal{Q}_n)^\wedge_\infty),$$

where $(X(\Gamma)^{\mathrm{tor}})^\wedge_\infty$ is the pullback of $X(\Gamma)^{\mathrm{tor}}$ under the strict localization at ∞ (regarded as a geometric point), $(X(\Gamma)^{\min})^\wedge_\infty \to X(\Gamma)^{\min}$. This is because $(X(\Gamma)^{\min})^\wedge_\infty$ is affine. Since $\mathcal{Q}_n$ is supported at $D^{\mathrm{tor}} = X(\Gamma)^{\mathrm{tor}} - Y(\Gamma)$, we are to compute

$$\varinjlim_{n \geq 0} H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \mathcal{Q}_n).$$

We use Serre duality, noting that by Kodaira–Spencer, $\omega^{2,\mathrm{sub}}$ is the dualizing sheaf of $X(\Gamma)^{\mathrm{tor}}$:

$$H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \mathcal{Q}_n) = H^0(X(\Gamma)^{\mathrm{tor}}, \mathcal{E}xt^1(\mathcal{Q}_n, \omega^{2,\mathrm{sub}}))^* = \left(H^0\left(X(\Gamma)^{\mathrm{tor}}, \frac{\mathcal{H}om(\mathcal{E}^{\mathrm{can}}, \omega^{2,\mathrm{sub}})}{\mathcal{H}om(\mathcal{E}^{\mathrm{can}}, \omega^{2,\mathrm{sub}})(-nD)}\right)\right)^*.$$

In particular, if $\mathcal{E} = \omega^k$, we have

$$H^d_D(X(\Gamma)^{\mathrm{tor}}, \omega^k) = \varinjlim_{n\geq 0} \left(H^0\left(X(\Gamma)^{\mathrm{tor}}, \frac{\omega^{2-k,\mathrm{sub}}}{\omega^{2-k,\mathrm{sub}}(-nD)}\right)\right)^*.$$

Thus, we would like to compute what $H^0\left(X(\Gamma)^{\mathrm{tor}}, \frac{\omega^{2-k,\mathrm{sub}}}{\omega^{2-k,\mathrm{sub}}(-nD)}\right)$ is.

Proposition 4.4. *The two Serre duality pairings are compatible with each other, namely*

$$\begin{array}{ccccc} H^d(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \times & H^0(X(\Gamma)^{\mathrm{tor}}, \omega^{2-k,\mathrm{sub}}) & \longrightarrow & \mathbb{C} \\ \uparrow & & \downarrow & & \\ H^d_D(X(\Gamma)^{\mathrm{tor}}, \omega^k) & \times & H^0\left(X(\Gamma)^{\mathrm{tor}}, \frac{\omega^{2-k,\mathrm{sub}}}{\omega^{2-k,\mathrm{sub}}(-nD)}\right) & \longrightarrow & \mathbb{C} \end{array}$$

commutes.

Proof. By the setup of coherent duality, namely that the coherent duality is a consequence of the construction of the trace 2-functor as in [Ha, Corollary VII.3.4], the Serre duality is compatible with long exact sequences. □

We now invoke more specific geometry of compactifications, as recalled in [La2, §4].

(1) There is a split torus $\Xi = \mathrm{Spec}\left(\bigoplus_{\ell\in\mathbb{S}} \Psi(\ell)\right)$ over ∞, with character group $\mathbb{S}$.
(2) Each cone $\tau \subset \mathbb{S}^\vee_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{S}, \mathbb{R})$ defines a toroidal embedding $\Xi \hookrightarrow \Xi(\tau) = \mathrm{Spec}\left(\bigoplus_{\ell\in\tau^\vee} \Psi(\ell)\right)$, where $\tau^\vee = \{\ell \in \mathbb{S} : \langle \ell, y\rangle \geq 0\ \forall y \in \tau\}$.
(3) Let $\mathfrak{U}_\tau$ be the formal completion of $\Sigma(\tau)$ along $\Sigma(\tau) - \Sigma$. These, for $\tau \in \Sigma^+$ for some rational polyhedral cone decomposition Σ^+, glue to form a formal scheme $\mathfrak{X}$ over ∞. Let $\widetilde{\mathfrak{N}}$ be the nerve of the covering induced by the closures of the cones in Σ^+.
(4) For each $\gamma \in U^2_F$, there is a canonical isomorphism $\gamma : \mathfrak{U}_\sigma \xrightarrow{\sim} \mathfrak{U}_{\gamma\sigma}$ over ∞, induced by the isomorphisms $\gamma^*\Psi(\gamma\ell) \xrightarrow{\sim} \Psi(\ell)$. These isomorphisms give an action of U^2_F on $\mathfrak{X}$, whence a local isomorphism $\mathfrak{X} \to \mathfrak{X}/U^2_F$.
(5) Now, $\mathfrak{X}/U^2_F \cong (X(\Gamma)^{\mathrm{tor}})^\wedge_\infty$.
(6) Let $\mathscr{E}^{(n)}$ be the pullback of $\mathcal{E}(nD)$ to $\mathfrak{X}/U^2_F$, and $\mathfrak{N} = \widetilde{\mathfrak{N}}/U^2_F$. Let $\underline{\mathscr{H}}^d(\mathscr{M})$, for a quasicoherent sheaf $\mathscr{M}$ on $\mathfrak{X}/U^2_F$, be the constructible sheaf over $\mathfrak{N}$ which has stalks $H^d(\mathfrak{U}_\sigma, \mathscr{M}|_{\mathfrak{U}_\sigma})$ over σ. There is a *nerve spectral sequence*

$$H^i(\mathfrak{N}, \underline{\mathscr{H}}^j(\mathscr{E}^{(n)})) \Rightarrow H^{i+j}(\mathfrak{X}/U^2_F, \mathscr{E}^{(n)}).$$

Each term of spectral sequence also admits a Hochschild–Serre spectral sequence

$$H^a(U^2_F, H^b(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^c(\mathscr{E}^{(n)}))) \Rightarrow H^{a+b}(\mathfrak{N}, \underline{\mathscr{H}}^c(\mathscr{E}^{(n)})).$$

(7) There exists an $\mathbb{R}_{>0}$-valued *polarization function* pol on the positive cone $\mathbb{P}$ which is piecewise linear on each cone $\sigma \in \Sigma$.
(8) For $\mathcal{E} = \omega^k$, we define $\mathrm{FJ}^{0,(\ell)}(\mathscr{E}) = H^0(\infty, \Psi(\ell))$, which coincides with [La2, Corollary 5.9], as $\mathscr{E}_0 = \mathcal{O}_\infty$ for $\mathcal{E} = \omega^k$.

Aruging similarly to [La2], we compute $\varinjlim_{n\geq 0} H^{d-1}(\mathfrak{X}/U_F^2, \mathscr{E}^{(n)})$.

Proposition 4.5. *Let $n \geq 0$.*

(1) The sheaf $\underline{\mathscr{H}}^c(\mathscr{E}^{(n)}) = 0$ for $c > 0$.

(2) The natural map

$$H^{d-1}(\mathfrak{X}/U_F^2, \mathscr{E}^{(n)}) \twoheadrightarrow H^{d-1}(U_F^2, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)}))),$$

becomes an isomorphism after taking the limit,

$$\varinjlim_{n\geq 0} H^{d-1}(\mathfrak{X}/U_F^2, \mathscr{E}^{(n)}) \xrightarrow{\sim} \varinjlim_{n\geq 0} H^{d-1}(U_F^2, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)}))).$$

(3) If $\mathcal{E} = \omega^k$, we have

$$\varinjlim_{n\geq 0} H^{d-1}(U_F^2, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)})) \cong H^{d-1}\left(U_F^2, \bigoplus_{\ell\in -\mathbb{P}^{\vee,+}} \mathrm{FJ}^{0,(\ell)}(\mathscr{E})\right).$$

Proof. (1) follows from the fact that $\mathfrak{U}_\sigma$'s are affine in our case. To prove (2), we need to argue similarly as [La2, Proposition 7.8]. That the natural map is surjective comes from the spectral sequence. To prove the full statement, it is sufficient to prove that $\varinjlim_{n\geq 0} H^i(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)})) = 0$ for $i > 0$. We will prove that the direct system $\{H^i(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)}))\}$ is eventually zero. Now the cohomology of simplicial complex $H^i(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)}))$ is identified with the local cohomology with support at $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(n)}$. As $\varinjlim_{n\geq 0} \widetilde{\mathfrak{N}}^{\ell,(n)} = \widetilde{\mathfrak{N}}$, any cohomology class over $\widetilde{\mathfrak{N}}$ would have vanishing image when restricted to local cohomology with support in $\widetilde{\mathfrak{N}} - \widetilde{\mathfrak{N}}^{\ell,(n)}$ for $n \gg 0$. To prove (3), first we note that $U_F^2 \cong \mathbb{Z}^{d-1}$, and one can take a basis $\epsilon_1, \cdots, \epsilon_{d-1}$ such that $\sigma_i(\epsilon_i) > 1$ and $\sigma_j(\epsilon_i) < 1$ for $j \neq i$. If $\ell_i > 0$, then $H^1(\epsilon_i^{\mathbb{Z}}, \prod_{\ell' \in \epsilon_i^{\mathbb{Z}}\cdot \ell} \mathrm{FJ}^{0,(\ell')}(\mathscr{E})) = 0$ as in the proof of [La2, Proposition 7.16] by Shapiro's lemma. Now

$$\varinjlim_{n\geq 0} H^{d-1}(U_F^2, H^0(\widetilde{\mathfrak{N}}, \underline{\mathscr{H}}^0(\mathscr{E}^{(n)}))) = H^{d-1}\left(U_F^2, \widehat{\bigoplus_{\ell\in\mathbb{S}}} \mathrm{FJ}^{0,(\ell)}(\mathscr{E})\right),$$

where the completion is taken with respect to $\bigoplus_{\ell\in\mathbb{P}^{\vee,+}} \mathrm{FJ}^{0,(\ell)}(\mathscr{E})$. Thus, the sub

$$H^{d-1}\left(U_F^2, \widehat{\bigoplus_{\ell\in\mathbb{S}-(-\mathbb{P}^{\vee,+})}} \mathrm{FJ}^{0,(\ell)}(\mathscr{E})\right) \subset H^{d-1}\left(U_F^2, \widehat{\bigoplus_{\ell\in\mathbb{S}}} \mathrm{FJ}^{0,(\ell)}(\mathscr{E})\right),$$

has consituents of the form $H^{d-1}(U_F^2, \widehat{\bigoplus_{\ell\in U_F^2\cdot\ell_0}} \mathrm{FJ}^{0,(\ell)}(\mathscr{E}))$ for $\ell_0 \notin -\mathbb{P}^{\vee,+}$, and this is isomorphic to $\prod_{i=1}^{d-1} H^1(\epsilon_i^{\mathbb{Z}}, \widehat{\bigoplus_{\ell\in\epsilon_i^{\mathbb{Z}}\cdot\ell_0}} \mathrm{FJ}^{0,(\ell')}(\mathscr{E}))$. As one of the multiplicands is zero, this is zero. Thus, the result follows. □

As algebraic and analytic Fourier–Jacobi coefficients coincide [La1], we see the following.

Proposition 4.6. *The image of $H^{d-1}(X(\Gamma)^{\mathrm{tor}}, \omega^k) \subset H^{d-1}(Y(\Gamma), \omega^k)$ under $q\exp^{d-1}$ is identified with the image of $\mathbb{C}$ in $(\mathbb{C}\langle q^\nu\rangle_{\nu\in\mathcal{O}_F^\vee})_{U_F^2}$. The space of principal parts of harmonic Hilbert Maass forms, $\frac{H_k^+}{M_k}$, is identified with the totally negative Laurent polynomials, up to U_F^2-coinvariance.*

Example 4.7 (Harmonic Maass forms)**.** As always, we would like to see how this is analogous to an existing feature of harmonic Maass forms. These correspond to the q-expansions of the principal parts of harmonic Maass forms. Recall that a harmonic Maass form f has a decomposition $f = f^+ + f^-$, where f^+ is the holomorphic part (*mock modular form*), and f^- is the non-holomorphic part. Furthermore, the datum of the shadow $\xi_k(f)$ is equivalent to the datum of f^-.

Now the key is that f^- is determined by f^+, or even better, the principal parts of f^+, by [DL, Proposition 2.6]. Namely, the principal parts define a linear functional on S_{2-k}, and this by Serre duality pins down the (complex conjugate of the) shadow.

Example 4.8. In the case of Hilbert modular surfaces, Proposition 4.5 implies that, given $\nu < 0$, there exists a nonholomorphic function $f_{\gamma,\nu}$, for each $\gamma \in U_F^2$, so that $\gamma \cdot f_\gamma - f_\gamma = q^\nu$. One would hope that $f_\gamma = \sum_{n=1}^{\infty} \gamma^{-n} \cdot q^\nu$ would formally work, but this is actually a divergent series! This is one of the reasons why the naïve generalization of harmonic Maass forms does not work.

5. Borcherds lift of harmonic Hilbert Maass forms

In [Br2], in a view towards constructing Borcherds lift for Hilbert modular forms, a way to avoid Koecher's principle for Hilbert modular forms was suggested. This starts by realizing that the usual Borcherds lift is constructed for each non-holomorphic Poincaré series in the sense of [Br1], as the space of harmonic Maass forms is spanned by such forms. In the regularization process for singular theta integrals, one realizes that it is rather a Whittaker function that is used in the integral, which is the analogue of exponential function for the usual Poincaré series. Now, [Br2] uses the fact that the analogue of Whittaker functions has no Koecher's principle, and one can think such Whittaker functions (named *harmonic Whittaker forms*) as alternatives for then non-existent harmonic Hilbert Maass forms.

Using our new definition of harmonic Hilbert Maass forms and their q-expansions, we can now identify harmonic Whittaker forms with principal parts of harmonic Hilbert Maass forms. Namely,

$$\left\{ \begin{array}{c} \text{Principal parts of harmonic Hilbert} \\ \text{Maass forms} \end{array} \right\} = \left\{ \begin{array}{c} \text{Laurent polynomials, with totally} \\ \text{negative exponents} \end{array} \right\} / U_F^2.$$

Definition 5.1 (Scalar-valued version of [Br2, Definition 4.1]) **.** A *harmonic Whittaker form* of weight k is a linear combination of functions of form

$$f_m(\tau) = \frac{(4\pi m_2)^{k-1} \cdots (4\pi m_d)^{k-1}}{\Gamma(1-k)\Gamma(k-1)^{d-1}} (\Gamma(1-k) - \Gamma(1-k, 4\pi m_1 v_1)) e^{4\pi m_1 v_1} e(\mathrm{tr}(-m\overline{\tau})),$$

where $m \in \mathcal{O}_F^\vee$, $m \gg 0$, $\tau = (\tau_1, \cdots, \tau_d)$, $\tau_a = u_a + iv_a$, and $\Gamma(s, v)$ is the incomplete Gamma function.

It is computed that

$$v_1^{k-2} \overline{L_k^{(1)} f_m(\tau)} = \frac{(4\pi m_1)^{1-k} (4\pi m_2)^{k-1} \cdots (4\pi m_d)^{k-1}}{\Gamma(1-k)\Gamma(k-1)^{d-1}} e(\mathrm{tr}(m\tau)),$$

where $L_k^{(1)}$ is the lowering operator in variable τ_1.

Using this, we define the non-holomorphic Hilbert Poincaré series.

Definition 5.2 (Non-holomorphic Hilbert Poincaré series)**.** We define a *non-holomorphic Hilbert Poincaré series* F_m, for $m \in \mathcal{O}_F^\vee$ with $m \gg 0$, to be a harmonic Hilbert Maass form whose q-expansion is q^{-m}.

Note that this only defines a class in H_k^+/M_k, but this is enough as an input for Borcherds lift.

Definition 5.3 (Borcherds lift of harmonic Hilbert Maass forms)**.** Let (V, Q) be a quadratic space over F of dimension $n+2$ and signature $((n, 2), (n+2, 0), \cdots, (n+2, 0))$. Let $H = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GSpin}(V)$, $K \subset H(\mathbb{A})$ be an open compact subgroup and X_K be the Shimura variety associated with H and level K. For $f \in \frac{H_k^+}{M_k}$, the Borcherds lift $\Phi(f)$ is defined to be

$$\Phi(f) = \sum_{m \in \mathcal{O}_F^\vee, m \ll 0} c_m \Phi_{-m}(z, h, 1-k),$$

where the q-expansion of f (see Proposition 4.5) is

$$q \exp^{d-1}(f) = \sum_{m \in \mathcal{O}_F^\vee, m \ll 0} c_m q^m,$$

and $\Phi_{-m}(z, h, 1-k)$ is as in [Br2, (5.2)].

Proposition 5.4. *Our definition of non-holomorphic Poincaré series coincides with the non-holomorphic Poincaré series in the case of elliptic modular forms. In particular, our Borcherds lift of harmonic Hilbert Maass forms coincides with the usual Borcherds lift in the case of harmonic Maass forms.*

Proof. This follows from the computation of Fourier expansion of non-holomorphic Poincaré series in the context of harmonic Maass forms. Indeed, [Br1, Proposition 1.0] computes that the principal part of holomorphic part of F_m is e^{-m}. Furthermore, using the formula of *loc. cit.*, the shadow of F_m is the holomorphic Poincaré series associated to $\frac{(4\pi m)^{1-k}}{\Gamma(1-k)} e(m\tau)$, which coincides with the normalization of Definition 5.2. □

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