FUKAYA-KATO'S RESULT ON SHARIFI'S Υ MAP

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Notations:

- $H = \varprojlim H^1(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\mathrm{ord}}_{\mathfrak{m}}$, where $\mathfrak{m}$ is an Eisenstein maximal ideal $\mathfrak{m} \supset I$ of $\mathbb{T}^0$, the ordinary cuspidal Hecke algebra. Whenever there is a tilde $\tilde{}$ it is something about modular forms (as opposed to cusp forms).
- $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$.
- ξ is the p-adic Riemann zeta function. In particular $\mathbb{T}^0/I = \Lambda/\xi$.
- $\Gamma = \mathrm{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$. We denote $K = \mathbb{Q}(\zeta_{p^\infty})$.
- $X = \varprojlim \mathrm{Cl}(\mathbb{Q}(\zeta_{p^r}))[p^\infty] \cong \mathrm{Gal}(L/K)$ where L is the maximal unramified pro-p abelian extension of K.
- κ is the cyclotomic character.
- We replace $\mathbb{T}^0$ by $\mathbb{T}^0_{\mathfrak{m}}$.
- $\Delta = \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$.
- $\theta : (\mathbb{Z}/p\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ is an even character.

This note covers [FuKa, Chapters 6.3, 6.4, 9.1, 9.2, 9.3, 9.4, 9.5].

1. “Theorem A”

Recall the “Theorem A” of Preston’s talk, which is the commutativity of the diagram

$$\begin{tikzcd}
\text{“}\varprojlim K_2(X_1(p^r))\text{”} \arrow[r, "\text{H-S}"] \arrow[dr, "\text{eval. at } \infty"'] & H^1(\mathbb{Z}[1/p], H(2)) \arrow[r, "\mathrm{res}_p"] \arrow[d, "\text{eval. at } \infty"] & H^1(\mathbb{Q}_p, H_{\mathrm{quo}}(2)) \arrow[r, "\log"] \arrow[d, "\cup(1-p^{-1})\log\kappa"'] & D(H_{\mathrm{quo}}(1)) = S_\Lambda \arrow[dl, "\mathrm{mod}\, I"] \\
& X \arrow[r, "\xi'\Upsilon"'] & P &
\end{tikzcd}$$

Going through the top horizontal row and to P, we get ξ', and going through the diagonal arrow and the bottom row to P, we get $\xi'\Upsilon\varpi$. Thus “Theorem A” is really the proof of

Theorem 1.1 (Fukaya-Kato)**.** $\xi'\Upsilon\varpi = \xi'$,

which is a very strong evidence of Sharifi’s conjecture.

We will talk first about the construction of Υ and then focus on the commutativity of the middle square. After that, we will talk a bit about geometry and the commutativity of the left triangle. Namely, we will try to cover every non-analytic part of the commutativity.

The correct diagram, minus the right triangle, is as follows.

Theorem 1.2. *The following diagram is commutative.*

$$\begin{tikzcd}
\varprojlim_r H^2_{\text{ét}}(X_1(p^r)_{\mathbb{Z}[1/p]}, \mathbb{Z}_p(2)) \arrow[r, "\text{Hoch-Serre}"] \arrow[dr, "\text{pullback through } \infty\text{-cusp}"'] & H^1(\mathbb{Z}[1/p], H(2)) \arrow[r, "\mathrm{res}_p"] \arrow[d, "\text{eval. at } \infty"] & H^1(\mathbb{Q}_p, P(2)) \arrow[d, "\cup(1-p^{-1})\log\kappa"] \\
& X \arrow[r, "\xi'\Upsilon"'] & P
\end{tikzcd}$$

All the maps will be justified some time later in this note. In particular, the middle vertical arrow is $H^1(\mathbb{Z}[1/p], H(2)) \to H^1(\mathbb{Z}[1/p], Q(2)) \cong X(1)$.

We will see many maps as connecting maps arising from some long exact sequence. Something like the following will eventually be shown.

- Evaluation at ∞ map is a connecting map.
- Υ is a connecting map.
- $\cup(1-p^{-1})\log(\kappa)$ is ξ' times a connecting map (!).

For a cleaner treatment, we fix a Δ-character θ so that we study a θ-isotypic part, i.e. we take θ-component of everything, where for a Λ-module M, $M_\theta = M \otimes_{\mathbb{Z}_p[(\mathbb{Z}/p\mathbb{Z})^\times]} \mathcal{O}_\theta$, where $\mathcal{O}_\theta = \mathbb{Z}_p[\theta]$. We will omit θ in expressions though, as everything will be taken its θ-part. Note that our discussion is meaningful only when the θ-isotypic part has an Eisenstein component, which in this normalization is the same as $\theta = \omega^{2-k}$ for (p,k) an irregular pair.

2. Structure of H/IH and the construction of Υ

We first exhibit how to construct Υ. Recall that we had an exact sequence

$$0 \to P \to H/IH \to Q \to 0,$$

where P is the minus part of H/IH. We will show that this is an exact sequence of $G_{\mathbb{Q}}$-modules with very specific Galois actions on P and Q.

The Υ map is then first defined as a map

$$\Upsilon : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{Hom}_{\mathbb{T}^0}(Q, P),$$

just given by

$$\Upsilon(\sigma)(x) = (\sigma - 1)x,$$

for $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and $x \bmod P \in (H/IH)/P = Q$. For this to be well-defined, $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ should act trivially on P and Q. This will follow from Galois module structures of P and Q. Also, from this, it is evident that the Galois action is unramified away from p. The map Υ will really become a map $X \to P$ when we show the following.

(A) As we said we need to show that $0 \to P \to H/IH \to Q \to 0$ is an exact sequence of $\mathbb{T}^0[G_{\mathbb{Q}}]$-modules with very specific Galois structure.
(B) The $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-action on H/IH should be unramified at every prime of K. As the Galois action on H is unramified away from p, we only need to consider primes above p.
(C) Q is "canonically isomorphic" to $\mathbb{T}^0/I \cong \Lambda/\xi$, so that $\mathrm{Hom}_{\mathbb{T}^0}(Q, P) = Q$.

Really then Υ factors through X and we evaluate at the canonical basis "1" of Q and we get the desired Υ. We will make many identifications of P and Q with some other things we know.

Remark 2.1. The definition of Υ illustrates why we expect Υ to be a "connecting map."

2.1. **Ingredients.** Recall that from Giada's talk that we have a Λ-**adic Poincaré pairing**

$$(-,-) : H \times H \to \Lambda,$$

which is some specific normalization of the usual Poincaré duality that has many nice properties, e.g.:

(1) For any $a \in \mathbb{T}^0$, $(ax, y) = (x, ay)$.
(2) For any $a \in \mathbb{Z}_p^\times$, $(\langle a \rangle x, y) = (x, \langle a \rangle y) = [a](x, y)$. Here, $\langle a \rangle \in \mathbb{T}^0$ is the diamond operator corresponding to a, and $[a] \in \Lambda$ is the group element.
(3) For any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, we have $(\sigma x, \sigma y) = \kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}(x, y)$. Here, $\langle \sigma \rangle = \langle a \rangle \in \mathbb{T}^0$ where $a \in \mathbb{Z}_p^\times$ is such that $\sigma(\zeta_{p^r}) = \zeta_{p^r}^a$ for all $r \geq 1$.
(4) We have an isomorphism $H \xrightarrow{\sim} \mathrm{Hom}_\Lambda(H, \Lambda)$, $x \mapsto (y \mapsto (x, y))$.

In Preston's talk, we were introduced to the Drinfeld-Manin modification, which for example gave an exact sequence of $\mathbb{T}^0[G_{\mathbb{Q}}]$-modules

$$0 \to H \to \widetilde{H}_{\mathrm{DM}} \to \Lambda/\xi \to 0,$$

where the surjective map is $\{0, \infty\}_{\mathrm{DM}} \mapsto -1$.

What is the Galois action on Λ/ξ? Ohta showed in [Oht] that $(\Lambda/\xi)(1)$, as the quotient of $H(1) \to \widetilde{H}_{\mathrm{DM}}(1)$, is naturally identified with some module generated by cusps, and in particular every cusp appearing in the expression is a "0-cusp", i.e. cusp corresponding to $a/c \in \mathbb{P}^1(\mathbb{Q})$ such that c is not divisible by p. Every such cusp is defined over $\mathbb{Q}$, so the $G_{\mathbb{Q}}$-action is trivial on the cuspidal group. Thus, the Λ/ξ has the Galois action by $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts via $\kappa(\sigma)^{-1}$.

From the above sequence, $\xi\{0, \infty\}_{\mathrm{DM}} \in H$, so we can make sense of the homomorphism

$$H \to \Lambda; x \mapsto (x, \xi\{0, \infty\}_{\mathrm{DM}}).$$

For any $a \in I$, since $\mathbb{T}^0/I = \Lambda/\xi$, we have $a\{0, \infty\} \in H$, so that $(ax, \xi\{0, \infty\}_{\mathrm{DM}}) = (x, a\xi\{0, \infty\}_{\mathrm{DM}}) = \xi(x, a\{0, \infty\}_{\mathrm{DM}}) \subset (\xi) \subset \Lambda$. Thus, the homomorphism yields a homomorphism

$$H/IH \to \Lambda/\xi.$$

Note that by the Galois/Hecke equivariance of Drinfeld-Manin modification and Λ-adic Poincaré pairing, this map is both Hecke and Galois equivariant. Here the Galois action of $\sigma \in G_{\mathbb{Q}}$ on Λ/ξ is via $\langle\sigma\rangle^{-1}$, because

$$(\sigma x, \xi\{0, \infty\}_{\mathrm{DM}}) = \kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}(x, \sigma^{-1}\xi\{0, \infty\}_{\mathrm{DM}}) = \langle\sigma\rangle^{-1}(x, \xi\{0, \infty\}_{\mathrm{DM}}),$$

as the Galois equivariance of the exact sequence $0 \to H \to \widetilde{H}_{\mathrm{DM}} \to \Lambda/\xi \to 0$ tells us that σ acts on $\{0, \infty\}_{\mathrm{DM}}$ via $\kappa(\sigma)^{-1}$.

Theorem 2.1. *The map $H/IH \to \Lambda/\xi$ is, as a homomorphism of $\mathbb{T}^0[G_{\mathbb{Q}}]$-modules, the same as the map $H/IH \twoheadrightarrow Q$. More precisely, $H/IH \to \Lambda/\xi$ is surjective, and* $\ker(H/IH \to \Lambda/\xi) \subset H/IH$ *is exactly the minus part of H/IH.*

This says that $\sigma \in G_{\mathbb{Q}}$ acts on Q via $\langle\sigma\rangle^{-1}$. Moreover, $\sigma \in G_{\mathbb{Q}}$ acts on P via $\kappa(\sigma)^{-1}$.

Note that we can expect the Galois action on P to be $\kappa(\sigma)^{-1}$ because on the rational module $H \otimes_{\mathbb{T}^0} Q(\mathbb{T}^0)$, a free rank 2 $Q(\mathbb{T}^0)$-module, the Galois action of σ is known to have determinant $\kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}$. This then covers Point A and C. In particular, modulo this, we know Υ as a map $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \to P$ is well-defined.

Recall also from Giada's talk that as local Galois representations (i.e. $\mathbb{T}^0[G_{\mathbb{Q}_p}]$-representations) we had an exact sequence

$$0 \to H_{\mathrm{sub}} \to H \to H_{\mathrm{quo}} \to 0,$$

where

- $H_{\mathrm{quo}}(1)$ is unramified,
- $H_{\mathrm{quo}}(1) \cong S_\Lambda$ as $\mathbb{T}^0$-modules,
- $H_{\mathrm{sub}}(1) \cong \mathbb{T}^0$ as $\mathbb{T}^0$-modules.

Along the way of proving Theorem 2.1, we make the following identifications which were also mentioned in Preston's talk (mod I here, though).

Theorem 2.2. *The compositions*

$$P \hookrightarrow H/IH \twoheadrightarrow H_{\mathrm{quo}}/IH_{\mathrm{quo}}, H_{\mathrm{sub}}/IH_{\mathrm{sub}} \hookrightarrow H/IH \twoheadrightarrow Q,$$

are bijections.

The idea is simple, namely $\kappa(\sigma)$ **and $\langle\sigma\rangle$ are so different that all these different rank 1 stuffs must be identified in this way.** Note that we only need to prove one of the two, because the other directly follows; indeed, $A \to B \to B/C$ is an injection means $A \cap C = 0$, which is symmetric, and $A \to B \to B/C$ is a surjection means any $b \in B$ mod C is in A, or $b = c + a$ for some $c \in C, a \in A$, which is also symmetric.

Given Theorem 2.2, we can see that $0 \to P \to H/IH \to Q \to 0$ is actually split as $G_{\mathbb{Q}_p}$-representations, as the inverse of the composition $H_{\mathrm{sub}}/IH_{\mathrm{sub}} \to H/IH \to Q$ gives an explicit splitting. Thus, Point B is covered, and the Υ map is justified.

2.2. **Actual proof.**

(Step 1) **Using P', Q' instead.** We let $P' = \ker(H/IH \to \Lambda/\xi)$ and $Q' = \Lambda/\xi$. We already know $G_{\mathbb{Q}}$-action on Q' (recall: σ acts by $\langle\sigma\rangle^{-1}$), and proving that $\sigma \in G_{\mathbb{Q}}$ acts on P' via $\kappa(\sigma)^{-1}$ will automatically prove $P' = P$ (and thus $Q' = Q$) because then it says the complex conjugation acts on P' by -1 and acts on Q' by $+1$.

(Step 2) $H/IH \xrightarrow{x \mapsto (x, \xi\{0,\infty\}_{\mathrm{DM}})} \Lambda/\xi$ **is surjective.** This is because H and $\widetilde{H}_{\mathrm{DM}}$ are both free Λ-modules, so that by the very existence of the exact sequence

$$0 \to H \to \widetilde{H}_{\mathrm{DM}} \xrightarrow{\{0,\infty\}_{\mathrm{DM}} \mapsto -1} \Lambda/\xi \to 0,$$

we see that $\xi\{0,\infty\}_{\mathrm{DM}}$ can be extended to a free basis of H; the surjectivity then follows from the perfectness of the Λ-adic Poincaré pairing.

(a) H **is free.** This is a part of Hida theory.

(b) $\widetilde{H}_{\mathrm{DM}}$ **is free.** We first see that by the Betti version of Drinfeld-Manin modification $\widetilde{H}_{\mathrm{DM}} = \mathrm{im}(\widetilde{H} \to H \otimes_\Lambda Q(\Lambda))$ is torsion-free. Thus, $f = 1 - \langle 1+p\rangle \in \Lambda$ ("weight 2 specialization"), which is a non-zerodivisor in Λ, acts as a non-zerodivisor on $\widetilde{H}_{\mathrm{DM}}$. By Nakayama it is enough to prove that $\widetilde{H}_{\mathrm{DM}}/f\widetilde{H}_{\mathrm{DM}}$ is free over Λ/f, and it is enough to prove that it is p-torsion free ((f, p) is a regular sequence of Λ). But this weight

2 specialization is by the coherent version of Drinfeld-Manin modification ($\widetilde{H}_{\mathrm{DM}} = \widetilde{H} \otimes_{\mathbb{T}} \mathbb{T}^0$) just $H^1_{\text{ét}}(Y_1(p))^{\mathrm{ord}}_{\mathrm{DM}}$ which is again by the Betti version a certain image in finitely generated $\mathbb{Q}_p$-vector space, so it is p-torsion free.

(Step 3) **Choice of $\tau \in G_{\mathbb{Q}_p}$ that distinguishes $\kappa(\tau)$ and $\langle\tau\rangle$.** To flesh out our idea, we pick one specific Galois element τ. More precisely, let $\tau \in I_{\mathbb{Q}_p}$ be an element in the inertia whose image in $\mathrm{Gal}(\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ is a generator. For simplicity we say

$$f = \kappa(\tau)^{-1} \in \mathbb{Z}_p^\times \subset \mathbb{T}^0,\ g = \langle\tau\rangle^{-1} \in \mathbb{T}^0.$$

These are very different, in a sense that they are different in the residue field of $\mathbb{T}^0$; note that $\mathbb{T}^0$ is a local ring by $\mathbb{T}^0/I \cong \Lambda/\xi$, so in particular the residue field is that of $\mathcal{O}_\theta$. Let $a \in \mathbb{Z}_p^\times$ such that $\tau(\zeta_{p^n}) = \zeta_{p^n}^a$. Then by our choice of τ, a is nonzero mod p, and f in the residue field of $\mathbb{T}^0$ is just a^{-1}. On the other hand, as we have taken θ-isotypic part, g in the residue field of $\mathbb{T}^0$ is $\theta([a])^{-1} = \theta(a)^{-1}$. Now recall that $\theta = \omega^{2-k}$ for an irregular pair (p, k). Thus, $k \neq 1$, so θ cannot be ω, and therefore a and $\theta(a)$ are different, so f and g are different.

(Step 4) **Using τ.** Now we know τ acts via f on H_{quo}, and as we know τ acts on $H \otimes_{\mathbb{T}^0} Q(\mathbb{T}^0)$ as an element of determinant fg, τ acts on $H_{\mathrm{sub}} \otimes_{\mathbb{T}^0} Q(\mathbb{T}^0)$ as an element of determinant g. But we know H_{sub} is free of rank 1 over $\mathbb{T}^0$, so τ acts on H_{sub} as g. Now we just define a $\mathbb{T}^0$-submodule $S \subset H$ by

$$S = \{x \in H \mid \tau x = fx\}.$$

Then by Nakayama, $H = H_{\mathrm{sub}} \oplus S$ (such decomposition holds at the residue field).

(Step 5) **Theorem 2.2 for P', Q'.** Now we prove Theorem 2.2 for P', Q'. Consider $H_{\mathrm{sub}}/IH_{\mathrm{sub}} \to H/IH \to Q'$. We know both source and target are free $\mathbb{T}^0/I$-modules of rank 1, it is sufficient to prove the surjectivity. Let cokernel of this map be denoted as C. Then $H/IH \twoheadrightarrow C$ has $H_{\mathrm{sub}}/IH_{\mathrm{sub}}$ as a kernel, so we get a natural map $H_{\mathrm{quo}}/IH_{\mathrm{quo}} \twoheadrightarrow C$, which is $\mathbb{T}^0[G_{\mathbb{Q}_p}]$-equivariant. Now τ acts on Q' by g, but τ acts on H_{quo} by f, so it contradicts Hecke equivariance. The other follows by our earlier observation.

In particular, by Theorem 2.2, we know τ acts on P' by f, so that $P' = S/IS$.

(Step 6) **$G_{\mathbb{Q}}$-action for P'.** Now we show that $\sigma \in G_{\mathbb{Q}}$ acts on P' by $\kappa(\sigma)^{-1}$. Let $\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$ be a matrix representation of $\sigma : H_{\mathrm{sub}} \oplus S \to H_{\mathrm{sub}} \oplus S$. What we want to show is that $d(\sigma)$, as an element of $\mathrm{Hom}_{\mathbb{T}^0/I}(S/IS, S/IS)$, is the same as $\kappa(\sigma)^{-1} \in \mathbb{T}^0$. What do we know now?

(a) H_{sub} is free of rank 1 over $\mathbb{T}^0$, so $a(\sigma) \in \mathbb{T}^0$.
(b) S/IS is a $G_{\mathbb{Q}}$-stable sub (!!!!), $b(\sigma) \equiv 0 (\mathrm{mod}\, I)$.
(c) As $\mathbb{T}^0$-modules, $S \cong H_{\mathrm{quo}}$, which is a dualizing module, so $d(\sigma) \in \mathrm{Hom}_{\mathbb{T}^0}(S, S) = \mathbb{T}^0$.
(d) By the determinant condition, and as everything becomes free rank 1 after $\otimes_{\mathbb{T}^0} Q(\mathbb{T}^0)$, $a(\sigma)d(\sigma) - b(\sigma)c(\sigma) = \kappa(\sigma)^{-1}\langle\sigma\rangle^{-1}$ as elements in $Q(\mathbb{T}^0)$. Note that this makes sense as $b(\sigma)c(\sigma) \in \mathrm{Hom}_{\mathbb{T}^0}(H_{\mathrm{sub}}, H_{\mathrm{sub}}) = \mathbb{T}^0$. Thus this holds in $\mathbb{T}^0$.
(e) As $b(\sigma)c(\sigma) \equiv 0 (\mathrm{mod}\, I)$, $a(\sigma)d(\sigma) \equiv \kappa(\sigma)^{-1}\langle\sigma\rangle^{-1} (\mathrm{mod}\, I)$.

Thus we are done.

3. X as Galois cohomology

Before everything we start to study X and many other modules using Galois cohomology. We learned from Leo's talk about Iwasawa cohomology that

$$X(1) \cong \varprojlim_r H^2(\mathbb{Z}[1/p, \zeta_{p^r}], \mathbb{Z}_p(2))_{\mathfrak{m},\theta} \cong H^2(\mathbb{Z}[1/p], \Lambda^\sharp(2)) (=: S),$$

where $\Lambda^\sharp$ is Λ with $G_{\mathbb{Q}}$-action as $\sigma \in G_{\mathbb{Q}}$ acts $[a]^{-1}, \cdots$.

Note that $\Lambda^\sharp/(\xi)$ is precisely the $G_{\mathbb{Q}}$-module structure of Q, so we have a short exact sequence

$$0 \to \Lambda^\sharp(2) \xrightarrow{\xi} \Lambda^\sharp(2) \to Q(2) \to 0,$$

which gives a long exact sequence

$$H^1(\mathbb{Z}[1/p], \Lambda^\sharp(2)) \to H^1(\mathbb{Z}[1/p], Q(2)) \to X(1) \xrightarrow{\xi} X(1) \to H^2(\mathbb{Z}[1/p], Q(2)) \to 0.$$

(p is odd, so H^3 and above is zero) Now by the Iwasawa Main Conjecture, X has characteristic ideal ξ, which means $X(1) \xrightarrow{\xi} X(1)$ is a zero map. Furthermore, we know from Leo's talk the leftmost Galois cohomology group is a twist of certain unit group:

$$\begin{aligned} H^1(\mathbb{Z}[1/p], \Lambda^\sharp(2)) &= \varprojlim_r H^1(\mathbb{Z}[1/p, \zeta_{p^r}], \mathbb{Z}_p(2))_{\omega^{2-k}} \\ &= \varprojlim_r H^1(\mathbb{Z}[1/p, \zeta_{p^r}], \mathbb{Z}_p(1))_{\omega^{1-k}}(1) \\ &= \varprojlim_r (\mathcal{O}^\times_{\mathbb{Q}(\zeta_{p^r}),p} \otimes_{\mathbb{Z}} \mathbb{Z}_p)_{\omega^{1-k}}(1). \end{aligned}$$

We know from classical Iwasawa theory that this unit group $\mathcal{E} = \varprojlim_r (\mathcal{O}^\times_{\mathbb{Q}(\zeta_{p^r}),p} \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ has nonzero ω^i-isotypic component for $i = 1$ ("ζ_{p^m}"'s) or i even, but $1-k$ is not 1 nor even, so $H^1(\mathbb{Z}[1/p], \Lambda^\sharp(2)) = 0$. We have thus proved the

Proposition 3.1. *The connecting map $H^1(\mathbb{Z}[1/p], Q(2)) \to X(1)$ and the natural map $X(1) \to H^2(\mathbb{Z}[1/p], Q(2))$ are isomorphisms.*

4. Poitou-Tate duality and Υ as a connecting map

4.1. **Poitou-Tate duality.** We first start with the most standard part, realizing $\Upsilon : X \to P$ as a connecting map of Galois cohomology. We already saw this is kind of plausible as the very definition of Υ was like a formula for a connecting homomorphism from H^0 to H^1. We will see that by Poitou-Tate duality, Υ, which is really a map from H^2 (by the previous section) to H^3, which is "dual to $H^0 \to H^1$," is a connecting homomorphism. To make this precise we recall the Poitou-Tate duality (although I think this form is usually referred as Artin-Verdier duality).

Let F be a number field, and $U \subset \operatorname{Spec} \mathcal{O}_F[1/p]$ be a Zariski open dense subscheme. For a finite abelian group T with p-power order with an action of $\pi_{1,\text{ét}}(U) =: G$, note that there is an exact sequence

$$\cdots \to H^i_c(U, T) \to H^i_{\text{ét}}(U, T) \to \oplus_{v \notin U} H^i(F_v, T) \to H^{i+1}_c(U, T) \to \cdots,$$

where the sum runs over all places (including Archimedean places) of F, and H^i_c is the **cohomology with compact support**. The definition of such functor is just the formal definition that should follow from that the functor sits in the desired position of the long exact sequence. Namely, it is just a cohomology group of a complex which is the mapping fiber (or any "homotopy fiber") of $C(G, T) \to \oplus_{v \notin U} C(F_v, T)$ where C's are the complexes of continuous cochains. If we remove Archimedean places (in particular real places) from the long exact sequence, then the one that should sit instead at the same position as $H^i_c(U, T)$ is $H^i_{\text{ét}}(\operatorname{Spec} \mathcal{O}_F, j_! T)$, where $j : U \hookrightarrow \operatorname{Spec} \mathcal{O}_F$, so we know that $H^i_c(U, T)$ and $H^i_{\text{ét}}(\operatorname{Spec} \mathcal{O}_F, j_! T)$ differ by a group killed by 2. As our p is odd this difference thus will be not a problem, so we will ignore this issue from now on.

Then, the Poitou-Tate (or Artin-Verdier) duality can be stated as follows.

Theorem 4.1 (Poitou-Tate duality, e.g. [Mil, Cor. III.3.3(b)]). *Let $T^D = \operatorname{Hom}(T, \mathbb{G}_m) = \operatorname{Hom}(T, \mathbb{Q}/\mathbb{Z})(1)$ be the Cartier dual of T (in $U_{\text{ét}}$). Then the cup product pairing $H^i_c(U, T) \times$*

$H^{3-i}_{\text{ét}}(U,T^D) \to H^3_c(U,\mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$ *is a perfect pairing of finite abelian groups. In particular, if* T *is killed by* p^n*, then the cup product becomes*

$$H^i_c(U,T) \times H^{3-i}_{\text{ét}}(U,T^\vee(1)) \to H^3_c(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \cong \mathbb{Z}/p^n\mathbb{Z},$$

where $T^\vee = \operatorname{Hom}(T,\mathbb{Q}/\mathbb{Z})$ *is the Pontryagin dual.*

The proof is quite involved, so let me instead justify why this precisely encodes (the p-part of) global class field theory when $T = \mathbb{Z}/p^n\mathbb{Z}(1)$ and $i = 1$.

(1) **Calculation of $H^3_c(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \cong \mathbb{Z}/p^n\mathbb{Z}$.**
The long exact sequence for compactly supported cohomology firstly gives

$$0 \to H^2_c(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to H^2(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to \oplus_{v\notin U} \operatorname{Br}(F_v)/p^r \operatorname{Br}(F_v)$$
$$\to H^3_c(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to H^3(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to 0,$$

(uses Hilbert 90 for the leftmost injectivity) but on the other hand the long exact sequence of étale sheaves

$$0 \to \mu_{p^n} \to g_*\mu_{p^n,\eta} \to \operatorname{Div}_U/p^n \operatorname{Div}_U \to 0,$$

for $g : \eta \hookrightarrow U$ the inclusion of the generic point and $\operatorname{Div}_U = \oplus_{v\in U^0} i_{v*}\mathbb{Z}$, where U^0 is the set of closed points of U and $i_v : v \mapsto U$, we get an exact sequence

$$0 \to H^2(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to \operatorname{Br}(F)/p^n \operatorname{Br}(F) \to \oplus_{v\in U^0} \operatorname{Br}(F_v)/p^n \operatorname{Br}(F_v) \to H^3(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to 0,$$

which uses $H^3(F,\mathbb{G}_m) = 0$. The global class field theory says that there is an exact sequence

$$0 \to \operatorname{Br}(F) \to \oplus_{\text{all } v} \operatorname{Br}(F_v) \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

we can see that the above exact sequence can be replaced with

$$0 \to H^2(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to \oplus_{v\notin U} \operatorname{Br}(F_v)/p^n \operatorname{Br}(F_v) \xrightarrow{\sum \operatorname{inv}_v} \mathbb{Z}/p^n\mathbb{Z} \to H^3(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) \to 0,$$

which gives the desired conclusion (as well as $H^2_c(U,(\mathbb{Z}/p^n\mathbb{Z})(1)) = 0$).

(2) **Artin reciprocity as a duality of class formation.** Recall that the p-primary part of the reciprocity map of global class field theory, in the language of class formulation, can be thought as a pairing

$$\langle , \rangle : C_S^{G_S}(p) \times H^2(G_S,\mathbb{Z})(p) \to \mathbb{Q}_p/\mathbb{Z}_p,$$

where (p) means the p-primary part, $C_S = \varinjlim_{F\subset K\subset F_S} C_{K,S}$ is the limit of S-idèle class groups (i.e. $\prod'_{w\in S} K_w^\times/\mathcal{O}_{K,S}^\times$), and $C_S^{G_S} = C_F/U_{F,S}$ (**ray class group of modulus "$\mathfrak{m}_S^\infty$"**) where C_F is the idèle class group and

$$U_{F,S} = \prod_{w\notin S} \widehat{\mathcal{O}}_w^\times.$$

Note that the reciprocity map $\operatorname{rec} : C_S^{G_S} \to G_S^{\text{ab}}$ is then obtained by realizing $H^2(G_S,\mathbb{Z}) \xleftarrow{\sim} H^1(G_S,\mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}_{\text{cts}}(G_S,\mathbb{Q}/\mathbb{Z})$. Also, $C_S^{G_S} = C_F/U_{F,S}$ sits inside a natural exact sequence

$$1 \to C_{F,S} \to C_F/U_{F,S} \to \operatorname{Cl}(\mathcal{O}_{F,S}) \to 1,$$

where Cl is the ideal class group.

(3) $H^1_c(U,(\mathbb{Z}/p^n\mathbb{Z})(1))$ **is the idèle class group.** On the other hand we have an exact sequence like

$$0 \to H^0_c(U,\mathbb{Z}/p^n\mathbb{Z}(1)) \to \Gamma(U,\mathcal{O}_U)^\times/\Gamma(U,\mathcal{O}_U)^{\times p^n}$$
$$\to \oplus_{v\notin U} F_v^\times/F_v^{\times p^n} \to H^1_c(U,\mathbb{Z}/p^n\mathbb{Z}(1)) \to \operatorname{Pic}(U)/p^n \operatorname{Pic}(U) \to 0.$$

In particular we see that $H^1_c(U, \mathbb{Z}/p^n\mathbb{Z}(1))$ also sits inside a short exact sequence like

$$1 \to C_{F,S}[p^n] \to H^1_c(U, \mathbb{Z}/p^n\mathbb{Z}(1)) \to \mathrm{Cl}(\mathcal{O}_{F,S})[p^n] \to 1,$$

where S is the set of primes not in U. In particular, we see here that the Artin-Verdier duality for $T = \mathbb{Z}/p^n\mathbb{Z}(1)$ and $i = 1$, which says $H^1_c(U, \mathbb{Z}/p^n\mathbb{Z}(1)) \cong C_S^{G_S}$ is perfectly paired with $H^2(U, \mathbb{Z}/p^n\mathbb{Z})$, is precisely the p^n-torsion part of the reciprocity map of global class field theory.

4.2. **Υ as a connecting map.** Now we apply this to see Υ as a connecting homomorphism.

Proposition 4.1. *The composition $H^2_c(\mathbb{Z}[1/p], Q(2)) \to H^2(\mathbb{Z}[1/p], Q(2)) \cong X \xrightarrow{\Upsilon} P$ is the same as the* ***minus of the connecting map***

$$H^2_c(\mathbb{Z}[1/p], Q(2)) \to H^3_c(\mathbb{Z}[1/p], P(2)) \cong P(1) \cong P,$$

from the exact sequence

$$0 \to P(2) \to H/IH(2) \to Q(2) \to 0.$$

Inherent in the statement is that by the Poitou-Tate duality $H^3_c(U, T) \cong T(-1)_G$, the G-coinvariants in $T(-1)$, and that $P(1)$ is a trivial $G_{\mathbb{Q}}$-module.

Proof. Let $\mathfrak{X} = \mathrm{Gal}(M/K)$, where M is the maximal pro-p abelian extension of $K = \mathbb{Q}(\zeta_{p^\infty})$ unramified away from p (standard notation in Iwasawa theory). Then by restriction of crossed homomorphism one sees that $\mathfrak{X}$ is the Pontryagin dual of $H^1(\mathbb{Z}[1/p, \zeta_{p^\infty}], \mathbb{Q}_p/\mathbb{Z}_p)$, or by Poitou-Tate duality,

$$\mathfrak{X} \cong \varprojlim_r H^2_c(\mathbb{Z}[1/p, \zeta_{p^r}], \mathbb{Z}_p(1)) \cong H^2_c(\mathbb{Z}[1/p], \Lambda^\sharp(2)).$$

Now the dual of $\mathfrak{X} \to H^2_c(\mathbb{Z}[1/p], Q(2))$ is $H^1(\mathbb{Z}[1/p], Q^\vee(-1)) \to H^1(\mathbb{Z}[1/p], (\Lambda^\sharp)^\vee(-1))$, which is injective as $H^0(\mathbb{Z}[1/p], (\Lambda^\sharp)^\vee(-1)) = 0$, so $\mathfrak{X} \to H^2_c(\mathbb{Z}[1/p], Q(2))$ is surjective. Thus it is enough to show the Proposition with $H^2_c(\mathbb{Z}[1/p], Q(2))$ replaced with $\mathfrak{X}$. As $\mathfrak{X} \twoheadrightarrow X$ is just the quotient as Galois group, $\mathfrak{X} \twoheadrightarrow X \xrightarrow{\Upsilon} P$ also literally has the same formula $\sigma \mapsto (\sigma - 1)\widetilde{1}$ where $\widetilde{1}$ is a lift of canonical $1 \in Q \cong \Lambda/\xi$.

Now we prove the Proposition by writing everything down explicitly. You can see this is plausible by taking the Pontryagin dual of everything. The Pontryagin dual of the composition map is

$$P^\vee \cong H^0(\mathbb{Z}[1/p], P^\vee(-1)) \to H^1(\mathbb{Z}[1/p], Q^\vee(-1)) \to H^1(\mathbb{Z}[1/p, \zeta_{p^\infty}], Q^\vee) \cong \mathrm{Hom}_{\mathrm{cts}}(\mathfrak{X}, Q^\vee) \to \mathfrak{X}^\vee,$$

where the first map is the **dual of the connecting map** $H^2_c \to H^3_c$, which is actually **the minus** of the connecting map of

$$0 \to Q^\vee(-1) \to (H/IH)^\vee(-1) \to P^\vee(-1) \to 0.$$

What is this composition? For $f \in P^\vee$, you take a lift $\widetilde{f} \in (H/IH)^\vee$, and the image as a crossed homomorphism in $H^1(\mathbb{Z}[1/p], Q^\vee(-1))$ is $\sigma \mapsto -(\sigma - 1)\widetilde{f}$. Now the rest maps this to

$$\text{“}\sigma \mapsto -(\sigma - 1)\widetilde{f}(\widetilde{1})\text{”} \in \mathfrak{X}^\vee,$$

which is the image of $\widetilde{f}$ via the minus of the dual of $\mathfrak{X} \twoheadrightarrow X \xrightarrow{\Upsilon} P$. $\square$

5. Cup product map is ξ' times a connecting map from Galois cohomology

Now here comes the most interesting part, which explains why ξ' appears in the square.

Theorem 5.1. *Let the composition $H^1(\mathbb{Z}[1/p], Q(2)) \xrightarrow{\sim} X(1) \xrightarrow{\sim} H^2(\mathbb{Z}[1/p], Q(2))$ be denoted as a. Then, $\xi' a$ is the same as the cup product $\cup(1-p^{-1})\log(\kappa) : H^1(\mathbb{Z}[1/p], Q(2)) \to H^2(\mathbb{Z}[1/p], Q(2))$, where $(1-p^{-1})\log(\kappa) : \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \xrightarrow{\sim} \mathbb{Z}_p$ which gives rise to a cocycle in $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p)$.*

To be precise, $\xi' = td\xi/dt$ where the derivation is done after we make an identifiaction $\Lambda_\theta \cong \mathcal{O}_\theta[[T]]$ by picking $t \in \Gamma$ with $(1-p^{-1})\log(\kappa(t)) = 1$. Here $\kappa : \pi_{1,\text{ét}}(\mathbb{Z}[1/p]) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to \mathbb{Z}_p^\times$ and $(1-p^{-1})\log : \mathbb{Z}_p^\times \to \mathbb{Z}_p$

Proof. We will abbreviate $(1-p^{-1})\log(\kappa)$ as $l_t : \Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \to \mathbb{Z}_p$. As we have done in Preston's talk, we make somewhat unnecessary distinction between $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p]]$ and $\mathbb{Z}_p[[\Gamma]]$. In particular we use $\mathbb{Z}_p[[\Gamma]]^\sharp$ for $\mathbb{Z}_p[[\Gamma]]$ with Galois action $\sigma \in G_\mathbb{Q}$ by $[\sigma]^{-1}$.

(Step 1) **$\cup l_t$ as connecting map.** Namely we realize $\cup l_t$ also as something that "forgets the multiplication map," just as in the definition of a. To be more precise, we claim that, for any pro-p abelian group A with continuous $\pi_{1,\text{ét}}(\mathbb{Z}[1/p])$-action, $-l_t\cup : H^i(\mathbb{Z}[1/p], A) \to H^{i+1}(\mathbb{Z}[1/p], A)$ is the same as the composition

$$H^i(\mathbb{Z}[1/p], A) \to H^{i+1}(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} A) \to H^{i+1}(\mathbb{Z}[1/p], A),$$

which comes from the long exact sequence of group cohomology of

$$0 \to \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} A \xrightarrow{t\otimes 1-1} \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} A \to A \to 0,$$

by "omitting the multiplication by $(t \otimes 1 - 1)$ map." Why? By Yoneda embedding, the composition is the same as cupping with the image of 1 under the composition $H^0(\mathbb{Z}[1/p], \mathbb{Z}_p) \to H^1(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp) \to H^1(\mathbb{Z}[1/p], \mathbb{Z}_p)$, so we only need to show this for $i = 0, A = \mathbb{Z}_p$. This is some explicit calculation; the cocycle in $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp)$ corresponding to the image of 1 by the connecting map is $g \mapsto r(g)$ where $g^{-1} - 1 = (t-1)r(g)$. In particular t^n maps to $-(t^{-n} + \cdots + t^{-1})$. Thus this maps to $t^n \mapsto -n$ in $H^1(\mathbb{Z}[1/p], \mathbb{Z}_p)$, which is precisely $-l_t$.

(Step 2) **Double complex picture.** Now we have to compare two connecting maps "omitting multiplication by sth map." Here comes the general slogan.

If you have two natural maps $X \to Y$ from homological algebra, they'd better be the same (up to sign).

An application of this in our situation is the following: if there is a diagram of complexes

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P' & \to & Q' & \to & R' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & Q & \to & R & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P'' & \to & Q'' & \to & R'' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns, then if $H^j(Q') \to H^j(R')$ is surjective and $H^j(P) \to H^j(Q)$ is injective, there are two kinds of connecting maps $H^{j-1}(R'') \to H^j(P'')$: one coming from the long exact sequence

$$\cdots \to H^{j-1}(P'') \to H^{j-1}(Q'') \to H^{j-1}(R'') \to H^j(P'') \to H^j(Q'') \to H^j(R'') \to \cdots ,$$

and one coming from the snake lemma of the diagram

$$\begin{tikzcd}
 & & & \ker(H^j(R') \to H^j(R)) \arrow[d] & \\
 & H^j(P') \arrow[r] \arrow[d] & H^j(Q') \arrow[r] \arrow[d] & H^j(R') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^j(P) \arrow[r] \arrow[d] & H^j(Q) \arrow[r] & H^j(R) & \\
 & \mathrm{coker}(H^j(P') \to H^j(P)) & & &
\end{tikzcd}$$

namely

$$H^{j-1}(R'') \twoheadrightarrow \ker(H^j(R') \to H^j(R)) \xrightarrow{\delta} \mathrm{coker}(H^j(P') \to H^j(P)) = \mathrm{im}(H^j(P) \to H^j(P'')) \hookrightarrow H^j(P'').$$

Then, **the two connecting maps are minuses to each other.** This is an easy diagram chase so I will not prove it but this seems certainly plausible.

In particular, the next step shows that the above lemma is applicable in our situation via the double complex of Galois cochains of the following double complex of Galois modules:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \arrow[r, "t\otimes 1-1"] \arrow[d, "1\otimes\xi"'] & \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \arrow[r] \arrow[d, "1\otimes\xi"'] & \mathbb{Z}_p[[\Gamma]]^\sharp(2) \arrow[r] \arrow[d, "\xi"'] & 0 \\
0 \arrow[r] & \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \arrow[r, "t\otimes 1-1"] \arrow[d] & \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \arrow[r] \arrow[d] & \mathbb{Z}_p[[\Gamma]]^\sharp(2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_p[[\Gamma]]^\sharp \otimes Q(2) \arrow[r, "t\otimes 1-1"] \arrow[d] & \mathbb{Z}_p[[\Gamma]]^\sharp \otimes Q(2) \arrow[r] \arrow[d] & Q(2) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

(Step 3) **The top two row short exact sequences are easy.** Namely, the exact sequence

$$0 \to \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \xrightarrow{t\otimes 1-1} \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \to \mathbb{Z}_p[[\Gamma]]^\sharp(2) \to 0,$$

is quite simple. As Galois modules,

$$\mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]]^\sharp \xrightarrow{t_1\otimes t_2 \mapsto t_1 \otimes t_1^{-1} t_2} \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]],$$

is an isomorphism; here the absence of $\sharp$ means $G_{\mathbb{Q}}$ acts trivially on that coordinate. Then the above exact sequence becomes

$$0 \to \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]](2) \xrightarrow{t\otimes t^{-1}-1} \mathbb{Z}_p[[\Gamma]]^\sharp \widehat{\otimes} \mathbb{Z}_p[[\Gamma]](2) \xrightarrow{x\otimes y \mapsto xy} \mathbb{Z}_p[[\Gamma]]^\sharp(2) \to 0,$$

which is evidently split. Thus, one can really apply the above lemma and thus we have

$$-\cup l_t = l_t\cup = -\text{connecting map of bottom row and project} = \text{connecting map of snake lemma and project}.$$

The advantage of this approach is that **both connecting map of snake lemma and connecting map of the rightmost column start with the same connecting map**, so that we can ignore the effect of a connecting map. Then what remains happens at the same cohomological degree so that we could easily diagram chase.

The diagram for snake lemma is as follows.

$$\begin{tikzcd}
 & & & H^1(\mathbb{Z}[1/p], Q(2)) \arrow[d, "\partial"'] & \\
0 \arrow[r] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2))\widehat{\otimes}\mathbb{Z}_p[[\Gamma]] \arrow[r, "t\otimes t^{-1}-1"] \arrow[d, "1\otimes\xi"'] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2))\widehat{\otimes}\mathbb{Z}_p[[\Gamma]] \arrow[r, "x\otimes y\mapsto xy"] \arrow[d, "1\otimes\xi"'] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2)) \arrow[r] \arrow[d, "\xi"'] & 0 \\
0 \arrow[r] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2))\widehat{\otimes}\mathbb{Z}_p[[\Gamma]] \arrow[r, "t\otimes t^{-1}-1"] \arrow[d] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2))\widehat{\otimes}\mathbb{Z}_p[[\Gamma]] \arrow[r, "x\otimes y\mapsto xy"] & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2)) \arrow[r] & 0 \\
 & H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp\widehat{\otimes}Q(2)) & & &
\end{tikzcd}$$

Let $x \in H^2(\mathbb{Z}[1/p], Q(2))$. Then $\partial(x) = y \in H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2))$ is killed by ξ, so its lift $y \otimes 1 \in H^2(\mathbb{Z}[1/p], \mathbb{Z}_p[[\Gamma]]^\sharp(2))\widehat{\otimes}\mathbb{Z}_p[[\Gamma]]$ is killed by $\xi \otimes 1$. Thus, $(1 \otimes \xi)(y \otimes 1) = (1 \otimes \xi - \xi \otimes 1)(y \otimes 1)$. Now definitely $(1 \otimes \xi - \xi \otimes 1)$ is a multiply of $(t \otimes t^{-1} - 1)$, so let $g = \frac{1\otimes\xi-\xi\otimes 1}{t\otimes t1-1} \in \mathbb{Z}_p[[\Gamma]]\widehat{\otimes}\mathbb{Z}_p[[\Gamma]]$. Thus it lands at the leftmost nonzero part of the bottom exact row as gy. Now from this when we project down to $H^2(\mathbb{Z}[1/p], Q(2))$, if you think about it, it goes to $p(g)y$, where

$$p : \mathbb{Z}_p[[\Gamma]]\widehat{\otimes}\mathbb{Z}_p[[\Gamma]] \xrightarrow{u\otimes v\mapsto uv} \mathbb{Z}_p[[\Gamma]].$$

Now what we want to prove is that this operation

$$\xi \mapsto 1 \otimes \xi - \xi \otimes 1 \mapsto g \mapsto p(g),$$

yields us $-td\xi/dt = -\xi'$. Indeed,

$$1 \otimes t^n - t^n \otimes 1 = (t \otimes t^{-1} - 1)(-t^{n-1} \otimes t - t^{n-2} \otimes t^2 - \cdots - 1 \otimes t^n),$$

so this eventually maps to $-nt^n$, as desired.

□

6. Commutativitiy of the middle square

Now we can prove the commutativitiy of the middle square.

Theorem 6.1. *The following diagram is commutative.*

$$\begin{tikzcd}
H^1(\mathbb{Z}[1/p], H(2)) \arrow[r, "\mathrm{res}_p"] \arrow[d] & H^1(\mathbb{Q}_p, P(2)) \arrow[d, "\cup(1-p^{-1})\log\kappa"] \\
X \arrow[r, "\xi'\Upsilon"'] & P
\end{tikzcd}$$

Here, the upper horizontal arrow uses

$$H \twoheadrightarrow H_{\mathrm{quo}}/IH_{\mathrm{quo}} \cong P,$$

and the right vertical map goes to $H^2(\mathbb{Q}_p, P(2)) = H^2(\mathbb{Q}_p, \mathbb{Z}_p(1)) \otimes P = P$, by the unramifiedness of $P(1)$.

Proof. We can take $\cup(1 - p^{-1}) \log \kappa$ first and take res_p later, so we can use the following square instead:

$$\begin{tikzcd}
H^1(\mathbb{Z}[1/p], H(2)) \arrow[r, "\cup(1-p^{-1}) \log \kappa"'] \arrow[d] & H^2(\mathbb{Z}[1/p], H(2)) \arrow[d, "\mathrm{res}_p"] \\
X \arrow[r, "\xi' \Upsilon"'] & P
\end{tikzcd}$$

We also know from cup $= \xi'$ times omission so that we know that the left block of

$$\begin{tikzcd}
H^1(\mathbb{Z}[1/p], H(2)) \arrow[rr, "\cup(1-p^{-1}) \log \kappa"] \arrow[d] & & H^2(\mathbb{Z}[1/p], H(2)) \arrow[dl] \arrow[d, "\mathrm{res}_p"] \\
X \arrow[r, "\xi'"'] & X \arrow[r, "\Upsilon"'] & P
\end{tikzcd}$$

commutes! So what remains is the commutativity of the right triangle. This follows from Υ as a connecting map: Υ is minus the connecting map of $H^2_c(\mathrm{global}, Q(2)) \to H^3_c(\mathrm{global}, P(2))$ of $0 \to P(2) \to H/IH(2) \to Q(2) \to 0$. As you could have guessed it from the minus, this also fits in the double complex picture as follows:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C_c(\mathbb{Z}[1/p], P(2)) \arrow[r] \arrow[d] & C_c(\mathbb{Z}[1/p], H/IH(2)) \arrow[r] \arrow[d] & C_c(\mathbb{Z}[1/p], Q(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(\mathbb{Z}[1/p], P(2)) \arrow[r] \arrow[d] & C(\mathbb{Z}[1/p], H/IH(2)) \arrow[r] \arrow[d] & C(\mathbb{Z}[1/p], Q(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C(\mathbb{Q}_p, P(2)) \arrow[r] \arrow[d] & C(\mathbb{Q}_p, H/IH(2)) \arrow[r] \arrow[d] & C(\mathbb{Q}_p, Q(2)) \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

Here C means the natural complex that calculates the corresponding cohomology groups. This gives a snake lemma diagram

$$\begin{tikzcd}
& & & H^2_c(\mathbb{Z}[1/p], Q(2)) \arrow[d] & \\
& H^2(\mathbb{Z}[1/p], P(2)) \arrow[r] \arrow[d] & H^2(\mathbb{Z}[1/p], H/IH(2)) \arrow[r] \arrow[d] & H^2(\mathbb{Z}[1/p], Q(2)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^2(\mathbb{Q}_p, P(2)) \arrow[r] \arrow[d] & H^2(\mathbb{Q}_p, H/IH(2)) \arrow[r] & H^2(\mathbb{Q}_p, Q(2)) \arrow[r] & 0 \\
& P & & &
\end{tikzcd}$$

The bottom row comes from that $0 \to P \to H/IH \to Q \to 0$ is split as $\mathbb{Q}_p$-representations. Now the Υ map is the connecting homomorphism of the snake lemma, and that the class is from $H^2(\mathbb{Z}[1/p], H(2))$ is something like we know the class at

$$\begin{array}{ccccc} & & & - & \\ & - & * & - & 0 \\ 0 & - & - & - & 0 \\ & - & & & \end{array}$$

Now the connecting homomorphism of snake lemma is something like you go down the trajectory

$$\begin{array}{ccccc} & & & - & \\ & - & * & - & 0 \\ 0 & \leftarrow & \downarrow & - & 0 \\ & \downarrow & & & \end{array}$$

but the only nontrivial part of this trajectory, namely the lift of a class in $H^2(\mathbb{Q}_p, H/IH(2))$ to $H^2(\mathbb{Q}_p, P(2))$, which uses the exactness and that the class maps to 0 in $H^2(\mathbb{Q}_p, Q(2))$, can be replaced with a natural map using the unique splitting of $0 \to P \to H/IH \to Q \to 0$ as **local representations**. Thus the triangle commutes. $\square$

7. Evaluation at ∞

7.1. **Modular curve.** We now explain the left triangle, which involves geometry. Here I would like to correct a mistake I said in the first talk. I said something like "$Q(1)$ is generated by 0-cusps which correspond to $0 \in \mathbb{P}^1(\mathbb{Q})$," but this is not quite correct. Rather, in our terminology, 0-cusps correspond to $a/c \in \mathbb{P}^1(\mathbb{Q})$ where $(c, \text{level}) = 1$ (so $(c,p) = 1$), and ∞-cusps correspond to $a/c \in \mathbb{P}^1(\mathbb{Q})$ where the level divides c (so $p^r \mid c$, and it includes $\infty =$"$1/0$"). Each cusp gives a $\mathbb{Z}[1/p, \zeta_{p^r}]$-valued point of $X_1(p^r)$. In this talk all modular curves are by default over $\mathbb{Z}[1/p]$ unless otherwise noted.

A brief explanation why $Q(1)$ is generated by 0-cusps is as follows.

- Cusps are classified by pairs $(c,d) \in \mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z}$ where either c or d is coprime to p, modulo certain equivalence relation ($(a,b) \sim (a',b')$ if $a' \equiv \pm a$, $b' \equiv \pm b$ mod a, respecting signs on both sides). In our earlier notation $a/c \in \mathbb{P}^1(\mathbb{Q})$ corresponds to (c,d) via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.
- The action of $T^*(p)$ on (c,d) for c coprime to p (i.e. a 0-cusp) gives (c,d') for some different d'. In particular a combination of 0-cusps can be fixed up to unit by $T^*(p)$, thus surviving after taking ordinary projector.
- The action of $T^*(p)$ on (c,d) for $p|c$ gives a linear combination of $(c/p, d')$'s. Thus any cusp that is not a 0-cusp does not survive after taking the orindary part.

In particular, the surjective map in the exact sequence

$$0 \to H \to \widetilde{H} \to \Lambda \to 0,$$

is the inverse limit of **boundary maps to 0-cusps**,

$$H^1_{\text{ét}}(Y_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)(1) \xrightarrow{\partial} \mathbb{Z}_p[\{0\text{-cusps}\}] = \mathbb{Z}_p[(\mathbb{Z}/p^r\mathbb{Z})^\times/\{\pm 1\}].$$

Do not forget that we are taking θ-isotypic components of everything, which eliminates prime-to-p part of $(\mathbb{Z}/p^r\mathbb{Z})^\times/\{\pm 1\}$. Moreover, by this explicit map, $\{0, \infty\}$ is sent to -1, not 1.[1]

[1]This is already corrected in the notes.

7.2. **Commutativity of the left triangle.** We state the theorem we want to prove.

Theorem 7.1. *The evaluation at the ∞-cusp*

$$\varprojlim_r H^2_{\text{ét}}(X_1(p^r), \mathbb{Z}_p(2)) \to \varprojlim_r H^2(\mathbb{Z}[1/p, \zeta_{p^r}], \mathbb{Z}_p(2)),$$

coincides with the composition

$$\varprojlim_r H^2_{\text{ét}}(X_1(p^r), \mathbb{Z}_p(2)) \xrightarrow{\text{Hochschild-Serre}} H^1(\mathbb{Z}[1/p], H(2))$$

$$\to H^1(\mathbb{Z}[1/p], Q(2)) \cong X(1).$$

Some explanations.

- The "∞-cusp" is the cusp $\infty(0,1) \in X_1(p^r)(\mathbb{Z}[1/p, \zeta_{p^r}]^+)$ corresponding to $1/0 = \infty \in \mathbb{P}^1(\mathbb{Q})$. Again remember that we suppressed θ from our notation.
- The first map of the composition is really a consequence of the Hochschild-Serre spectral sequence
$$E_2^{i,j} = H^i(\mathbb{Z}[1/p], H^j_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2)) \Rightarrow E_\infty^{i+j} = H^{i+j}(X_1(p^r), \mathbb{Z}/p^n\mathbb{Z}(2)),$$
because $H^0(\mathbb{Z}[1/p], H^2_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2)) = 0$; recall that a low-degree exact sequence of a cohomological convergent spectral sequence can be read off as
$$E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to \ker(E_\infty^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}.$$

Proof. Let's prove.

(Step 1) **Removing Q.** We reinterpret $H^1(\mathbb{Z}[1/p], H(2)) \to H^1(\mathbb{Z}[1/p], Q(2))$, the connecting homomorphism of
$$0 \to \Lambda^\sharp(2) \xrightarrow{\xi} \Lambda^\sharp(2) \to Q(2) \to 0,$$
as something to do with H. Namely, the exact sequence, which is something about "constant terms of modular forms", is actually an avatar of a sequence of spaces of modular forms as follows.
$$\begin{tikzcd} 0 \arrow[r] & \Lambda^\sharp(2) \arrow[r] \arrow[d, equal] & \widetilde{H}_c(2) \arrow[r] \arrow[d, "{(-,[g])}"] & H(2) \arrow[r] \arrow[d, "{(-,\xi\{0,\infty\}_{\mathrm{DM}})}"] & 0 \\ 0 \arrow[r] & \Lambda^\sharp(2) \arrow[r] & \Lambda^\sharp(2) \arrow[r] & Q(2) \arrow[r] & 0 \end{tikzcd}$$
Here, $\widetilde{H}_c = \varprojlim_r H^1_{\text{ét},c}(Y_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p)^{\text{ord}}$, which is paired with $\widetilde{H}$ similarly via Λ-adic Poincaré pairing, and $[g] \in \widetilde{H}(1)$ is the sequence of **Siegel units** $(\{g_{0,1/p^r}\}) \in \widetilde{H}(1)$, which, a priori a compatible collection of global sections of $Y_1(p^r)$'s, are sent to $\widetilde{H}(1)$ via Kummer theory (or the exact sequence $0 \to \mathbb{Z}/p^n\mathbb{Z}(1) \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ in the étale site of $Y_1(p^r)$). Also, the top horizontal row is the Λ-dual of the exact sequence
$$0 \to H \to \widetilde{H} \to \Lambda \to 0,$$
via the Poincaré pairing. As I have explained earlier, the surjection is the boundary map at 0-cusp, so by using the explicit formula for the q-expansions of Siegel units (at 0), we check that $[g]$ maps to ξ via the surjection. This explains the commutativity of the left square.

Now we see that by that $\{0, \infty\} \mapsto -1$, $[g] \mapsto \xi$, we know $\xi\{0, \infty\} + [g] \in H \subset \widetilde{H}$. But $[g]_{\mathrm{DM}} = 0$, because the elements of $\widetilde{H}$ coming from $\mathcal{O}(Y(p^r)_{\mathbb{Q}})^\times \otimes \mathbb{Z}/p^n\mathbb{Z}$ form a $\mathbb{T}$-submodule which has no intersection with H (by q-expansion principle that module into the quotient $\widetilde{H}/H$ is an injection), so the whole module is killed when we take $\otimes_{\mathbb{T}} \mathbb{T}^0$. This means $\xi\{0,\infty\}_{\mathrm{DM}} = \xi\{0,\infty\} + [g]$ by taking DM! Now note that in $(\,,\,) : \widetilde{H}_c \times \widetilde{H} \to \Lambda$, we can pull ξ out without any difficulty, so the right commutativity is established.

Therefore, the connecting map $H^1(\mathbb{Z}[1/p], H(2)) \to H^1(\mathbb{Z}[1/p], Q(2)) \xrightarrow{\sim} H^2(\mathbb{Z}[1/p], \Lambda^\sharp(2))$ coincides with the connecting map $H^1(\mathbb{Z}[1/p], H(2)) \to H^2(\mathbb{Z}[1/p], \Lambda^\sharp(2))$ of the upper row.

(Step 2) **Geometric meaning of $0 \to \Lambda^\sharp(2) \to \widetilde{H}_c(2) \to H(2) \to 0$.** The Λ-adic Poincaré pairing involves Atkin-Lehner involution, so only ∞-cusps contribute $0 \to \Lambda^\sharp(2) \to \widetilde{H}_c(2) \to H(2) \to 0$. To be more precise, what we said about nonzero cusps being killed by taking ordinary parts means that $H \to \widetilde{H}$, which is by definition

$$\varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})^{\mathrm{ord}} \to \varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, j_*(\mathbb{Z}/p^n\mathbb{Z}))^{\mathrm{ord}},$$

for $j : Y_1(p^r) \hookrightarrow X_1(p^r)$, is actually identified with

$$\varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})^{\mathrm{ord}} \to \varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, j'_*(\mathbb{Z}/p^n\mathbb{Z}))^{\mathrm{ord}},$$

where $j' : X_1(p^r) - \{0\text{-cusps}\} \hookrightarrow X_1(p^r)$. Thus, the dual of this map via the Poincaré pairing, which is by definition

$$\varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, j_!(\mathbb{Z}/p^n\mathbb{Z}))^{\mathrm{ord}} \to \varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})^{\mathrm{ord}},$$

is actually identified with

$$\varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, j''_!(\mathbb{Z}/p^n\mathbb{Z}))^{\mathrm{ord}} \to \varinjlim_{n,r} H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})^{\mathrm{ord}},$$

where $j'' : X_1(p^r) - \{\infty\text{-cusps}\} \hookrightarrow X_1(p^r)$.

This means that the dual exact sequence

$$0 \to \Lambda^\sharp(2) \to \widetilde{H}_c(2) \to H(2) \to 0,$$

is the $(\theta, \mathfrak{m}$-component of the) inverse limit of

$$0 \to T \to H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, F') \to H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, F) \to 0,$$

where $F = \mathbb{Z}/p^n\mathbb{Z}(2)$ on $(X_1(p^r)_{\overline{\mathbb{Q}}})_{\text{ét}}$, $F'' = i_*(\mathbb{Z}/p^n\mathbb{Z})(2)$ for $i : \{\infty\text{-cusps}\} \to X_1(p^r)_{\overline{\mathbb{Q}}}$, $F' = \ker(F \to F'')$ and

$$T = \mathrm{coker}(H^0_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2) \to H^0_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, F'')).$$

More precisely, i is the map $i : \mathrm{Spec}\,\mathbb{Z}[1/p, \zeta_{p^r}]^+ \otimes_{\mathbb{Z}[1/p]} \overline{\mathbb{Q}} \to X_1(p^r)_{\overline{\mathbb{Q}}}$, thus

$$T = \mathrm{coker}(\mathbb{Z}/p^n\mathbb{Z}(2) \to \mathbb{Z}/p^n\mathbb{Z}[(\mathbb{Z}/p^r\mathbb{Z})^\times/\{\pm 1\}](2)).$$

Therefore, after taking θ-isotypic component, there is actually no difference between T and $H^0_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, F'')$ ($\theta = \omega^{2-k}$ is not ω^2).

(Step 3) **Maps between Hochschild-Serre spectral sequences.** It is now sufficient to prove that

$$\begin{CD}
H^2_{\text{ét}}(X_1(p^r)_{\mathbb{Z}[1/p]}, \mathbb{Z}/p^n\mathbb{Z}(2)) @>{\text{ev. at }\infty}>> H^2_{\text{ét}}(\infty, \mathbb{Z}/p^n\mathbb{Z}(2)) \\
@V{\text{Hochschild-Serre}}VV @VVV \\
H^1_{\text{ét}}(\mathbb{Z}[1/p], H^1_{\text{ét}}(X_1(p^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^n\mathbb{Z})(2)) @>>> H^2_{\text{ét}}(\mathbb{Z}[1/p], T)
\end{CD}$$

is commutative, because we have observed that the right vertical map is an isomorphism after taking θ-component. In terms of étale sheaves, this is

$$\begin{CD}
H^2(X, F) @>>> H^2(X, F'') \\
@VVV @VVV \\
H^1(\mathbb{Z}[1/p], R^1 f F) @>>> H^2(\mathbb{Z}[1/p], T)
\end{CD}$$

where we suppressed ét, $X = X_1(p^r)_{\mathbb{Z}[1/p]}$, $f : X_{\text{ét}} \to (\operatorname{Spec}\mathbb{Z}[1/p])_{\text{ét}}$, the right vertical map comes from $H^2(X, F'') = H^2(\mathbb{Z}[1/p], fF'') \to H^2(\mathbb{Z}[1/p], T)$ ($R^{>0} fF'' = 0$), and the bottom horizontal map comes from the connecting map of $0 \to T \to R^1 fF' \to R^1 fF \to 0$. One can really check that given such an abstract situation we have a commutative square. To be more precise, the "abstract situation" we are in is as follows.

Lemma 7.1. *Let $C_1 \xrightarrow{f} C_2 \xrightarrow{g} C_3$ be left-exact functors of nice enough additive abelian categories (enough injectives is sufficient I guess) so that there is a composition-of-functors spectral sequence $E_2^{p,q}(F) = R^p g_* R^q f_* F \Rightarrow R^{p+q}(gf)F$ for any $F \in C_1$. Suppose that we are given with an exact sequence $0 \to F' \to F \to F'' \to 0$ in C_1 so that the following conditions are satisfied.*

- *$R^q f_* F'' = 0$ for all $q > 0$.*
- *$0 \to fF \to fF'' \to T \to 0$, which also fits into $0 \to T \to Rf^1 F' \to Rf^1 F \to 0$.*
- *$g(R^2 fF) = 0$.*

Then the analogous square

$$\begin{CD}
R^2(gf)F @>>> R^2(gf)F'' = R^2 g(fF'') \\
@VVV @VVV \\
R^1 g(R^1 fF) @>>> R^2 g(T)
\end{CD}$$

commutes.

This can really be proved by using a nice enough injective resolution of the double complex computing the composition-of-functors spectral sequence, where "nice enough" means that you can use the same injective resolution even after taking $R^i f$. Then really you are using the same resolution to compute maps in two different ways. This again should be believable under our general slogan, repeated below.

If you have two natural maps $X \to Y$ from homological algebra, they'd better be the same (up to sign).

□

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