DEFORMATION THEORY LEARNING SEMINAR

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We would like to study deformation theory, and particularly want to understand how to use cotangent complexes without knowing too much about them. After a short series of general theory, we would let people talk about anything they would want to talk about (hopefully related to deformation theory).

Talk 1. Formal deformation theory.
First-order deformations, ([Har, Sections 1-4]). Motivation for formal moduli ([Har, Section 14]). Tangent-obstruction theory. Versal family, pro-representable hull and Schlessinger's criterion ([Har, Sections 15-16], [FGA, Chapter 6]). A word about a proof ([Oss]).

Talk 2. Examples of deformation functors.
Deformation of schemes, coherent sheaves, morphisms; obstruction space, cases of having pro-representable hull or versal family. Cases of having "nice" deformation theory: smooth, locally complete intersection, Cohen-Macaulay in codimension 2. References are [Har], [Ser], [Vis] (for lci).

Talk 3. Applications of elementary deformation theory.
One should talk about some neat theorems one can prove using formal deformation theory, and the content is not restricted to any specific application. Some possible examples are the following: Lefschetz hyperplane theorem, deformation of Calabi-Yau is Calabi-Yau; Tian-Todorov theorem, deformations of abelian varieties are unobstructed, construction of Hilbert/Quot/Picard schemes, versal deformation of hyperelliptic curves. Various examples can be found in e.g. [Litt], [FGA].

Talk 4. Motivating the cotangent complex.
Lichtenbaum-Schlessinger ([Har, Sections 3,10], [LS]). Several exact sequences that may indicate these are coming from some long exact sequences or spectral sequences. Expected properties of (co)homology theory of rings ([Iye, Sections 1,2,8,9], [Qui1, Sections 1-3],[Litt, Section 9]).

Talk 5. The cotangent complex I.
Simplicial preliminaries; Dold-Kan equivalence ([Iye, Sections 3-4]). Construction of cotangent complex for commutative rings, basic properties; Quillen spectral sequence, comparison with Lichtenbaum-Schlessinger, Atiyah class ([Qui2], [Qui1], [Iye, Sections 5-7], [Stacks, The Cotangent Complex, Sections 11, 12, 16]). How does it globalize ([Stacks, The Cotangent Complex], [Ill1], [Ill2]).

Talk 6. The cotangent complex II.
How to use the cotangent complexes. Relation between DGLA and cotangent complexes in characteristic zero. How to find a finite type resolution for a finite type morphism to locally Noetherian scheme. References: [Litt, Section 9], [Man], [Ols, Section 8], [Stacks, The Cotangent Complex, Sections 18-27]

Talk 7 and onwards. My favorite topic/paper.

Based on general theory we have developed so far, each talk will be about a paper, or more generally a topic, that the presenter can choose according to his/her own interest. The papers/topics should have something to do with deformation theory. Each topic is expected to occupy 1∼2 talks (one talk is $\leq$ 2 hrs), so the presenter may assume relevant specialized backgrounds and proceed swiftly in presenting the topic/paper. In this way, for each talk only interested participants can choose to come, so that we could keep a wide variety of overall participants for the seminar. The following are examples of suggested topics/papers.

Possible topics:

- **Construction of virtual fundamental class.**
  (Behrend-Fantechi, *The intrinsic normal cone*, Inventiones 1997. Behrend, *Gromov-Witten invariants in algebraic geometry*, Inventiones 1997.)
  - Given an embedding X into a smooth space Y via a map i, the normal cone $N_X Y$ has a natural action of i^*T_Y. The quotient $[N_X Y/i^*T_Y]$ is somehow "intrinsic" to X, definable by the cotangent complex of X. It sits inside "[obstruction/tangent]" of a perfect obstruction theory of X. Applying this to a reasonable moduli space gives a construction of a virtual fundamental class, used in e.g. Gromov-Witten theory.
- **Galois deformation theory, deformation of finite flat group schemes;**
  (Böckle, *Deformations of Galois representations.* Kisin, *Moduli of finite flat group schemes, and modularity*, Annals 2009.)
  - The study of lifts of residual Galois representation is crucial in modern algebraic number theory, particularly in proving various *modularity lifting theorems.* To allow a more relaxed deformation condition at p, one is led to study a *flat deformation functor*, which roughly only captures deformations that arise from finite flat group schemes.

» **its derived variants.**
  (Khare-Thorne, *Potential automorphy and the Leopoldt conjecture*, AJM 2018. Galatius-Venkatesh, *Derived Galois deformation rings*, Adv. Math. 2018. Wang-Erickson, *Deformations of residually reducible Galois representations via A_∞-algebra structure on Galois cohomology.*)
  - There have been several more recent attempts to understand arithmetic/Galois cohomology groups in a derived setting, yielding more refined information on these. These include a graded action of Tor-algebra on the arithmetic homology via using (pro)simplicial Galois deformation ring and presentation of universal pseudodeformation ring using homotopy algebra structure on $H^*(G, \mathrm{End}(\rho))$ for a residual Galois representation ρ.

- **Kollár-Shepherd-Barron-Alexeev approach to compactification of moduli spaces of varieties of general type.**
  (Kollár-Shepherd-Barron, *Threefolds and deformations of surface singularities*, Inventiones 1988. Hacking, *Compact moduli of plane curves*, Duke 2004. Kollár, *Moduli of Varieties of General Type*, and *Hulls and Husks.* Bhatt-Ho-Patakfalvi-Schnell, *Moduli of products of stable varieties*, Compositio 2013.)
  - *Kollár-Shepherd-Barron-Alexeev (KSBA)* compactification refers to a way of compactifying moduli of higher dimensional varieties by finding an appropriate notion of "stable varieties" similar to the spirit of Deligne-Mumford compactification of moduli of curves. By work of many people evolving for long time, and due to the recent development of minimal model program, many "correct" notions (e.g. "objects", "family of objects", $\cdots$) are found for varieties of *general type.* Obviously many deformation theoretic considerations have been made for the development of such correct notion.
- **Deformation of abelian varieties, geometry of Shimura varieties**
  (Katz, *Serre-Tate local moduli*, in LNM 868. Moonen, *Serre-Tate theory for moduli spaces of PEL type*, ASÉNS 2004, and *A dimension formula for Ekedahl-Oort strata*, Annales Fourier 2004. Chai-Conrad-Oort,

Complex multiplication and lifting problems. Kisin, *Integral models for Shimura varieties of abelian type*, JAMS 2010, and *Mod p points of Shimura varieties of abelian type*, JAMS 2017. Kisin-Madapusi Pera-Shin, *Honda-Tate theory for Shimura varieties.*)

  – There are many classical results about deformation and lifting of abelian varieties and equivalences of categories which recast arithmetic objects into linear algebraic data. For example, Serre-Tate theory enables us to study local description of an ordinary locus of $\mathscr{A}_g$, or more generally Ekedahl-Oort strata of PEL type Shimura varieties; Honda-Tate theory gives us control of special fiber of Shimura variety in terms of CM points, which can be used to e.g. make Lefschetz trace formula explicit so that it can be compared to the Arthur-Selberg trace formula ("Langlands-Kottwitz method"), or the closure relation/nonemptiness of Newton strata.

- **DGLA/A_∞ perspective on deformation theory.**
  (Manetti, *Deformation theory via differential graded Lie algebras*, and *Lectures on deformations of complex manifolds.* Kontsevich-Soibelman, *Deformation theory I.* Fukaya, *Deformation theory, homological algebra and mirror symmetry.* Nagai-Sato, *Deformation of a smooth Deligne-Mumford stack via differential graded Lie algebra*, J. Algebra 2008.)
  – A work of Kuranishi ("Newlander-Nirenberg theorem") says that deformations of a compact complex manifold are the same as solutions of Maurer-Cartan equation (modulo gauge equivalence), or deformations of "Kodaira-Spencer DGLA" (higher version of "Lie algebras = formal groups"). This led people to believe that, over a field of characteristic zero, every deformation problem is controlled by some differential graded Lie algebra.
- **Gabber's works on almost mathematics, desingularization and local uniformization of quasi-excellent schemes;**
  (Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, ASÉNS 1973. Gabber-Ramero, *Almost ring theory.* Illusie-Laszlo-Orgogozo, *Travaux de Gabber.* Temkin, *Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case*, Duke 2012.)
  – Using the cotangent complex, Elkik defines the "Jacobian ideal" of a morphism of formal schemes, which measures the degree of non-smoothness of the morphism. This idea is continued in Gabber-Ramero along the lines of understanding Faltings' *almost purity* results. By keeping track of the degree of non-smoothness carefully, one gets algebraization and lifting results. This enables us to extend in a controlled manner what's happening in the generic fiber, where theorems (e.g. resolution of singularities) on varieties of characteristic zero can be applied. This yields many powerful consequences, e.g. local uniformization, refined desingularization and alteration theorems.

» **tilting equivalence for perfectoid algebras.**
  (Scholze, *Perfectoid spaces*, Publ. IHÉS, 2012.)
  – As an example application, in proving the tilting equivalence for perfectoid algebras over a perfectoid field K, one shows that a perfectoid $K^{\circ a}/\varpi$-algebra can be uniquely deformed into a perfectoid $K^{\circ a}$-algebra, which uses vanishing of cotangent complex of a perfectoid algebra.

- **Geometry of (truncated) Hitchin fibration.**
  (Laumon, *Fibres de Springer et jacobiennes compactifiées.* Ngô, *Fibration de Hitchin et endoscopie*, Inventiones 2006, and *Le lemme fondamental pour les algèbres de Lie*, Publ. IHÉS 2010. Chadouard-Laumon, *Le lemme fondamental pondéré II. Énoncés cohomologiques*, Annals 2012.)
  – In the course of using the "support theorem," which is crucial in Ngô's proof of fundamental lemma, one needs that a given abelian fibration is *δ-regular*. Unlike the characteristic zero case, this is rather a delicate issue; Chadouard-Laumon extended Ngô's work so that the Hitchin fibration is shown to be δ-regular even at some parts

outside the elliptic part of the Hitchin base. This amounts to dimension calculation using deformation theory.

- **Derived de Rham complex, de Rham-Witt complex, p-adic Hodge theory.**
(Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, ASÉNS 1979. Langer-Zink, *De Rham-Witt cohomology for a proper and smooth morphism*, Jussieu 2004. Beilinson, *p-adic periods and derived de Rham cohomology*, JAMS 2012, and *On the crystalline period map*, Camb. J. Math. 2013. Nekovář-Nizioł, *Syntomic cohomology and regulators for varieties over p-adic fields*, ANT 2016. Bhatt-Morrow-Scholze, *Integral p-adic Hodge theory.* Bhatt-Lurie-Mathew, *Revisiting the de Rham-Witt complex.*)
  - The de Rham-Witt complex is constructed so that it computes crystalline cohomology in the same way that the usual de Rham complex computes the de Rham cohomology for complex varieties. Beilinson showed that the crystalline comparison theorem (originally proved by somewhat different methods) can be indeed achieved by relating the de Rham-Witt complex to the derived de Rham complex $L\Omega^\bullet_{B/A}$, something very close to the cotangent complex $L_{B/A}$ ("$\mathrm{gr}^i\, L\widehat{\Omega}^\bullet_{B/A} \cong (L\Lambda^i_B(L_{B/A}))[-i]$"). Construction of relative de Rham-Witt complex as well as its connection to derived de Rham complex has several applications, including construction of cohomology theory valued in Breuil-Kisin modules (Bhatt-Morrow-Scholze) and syntomic regulators (Nekovář-Nizioł).
- **Deformation of p-divisible groups, Grothendieck-Messing theory;**
(de Jong, *Crystalline Dieudonné module theory via formal and rigid geometry*, Publ. IHÉS 1995. Rapoport-Zink, *Period spaces for p-divisible groups.* Faltings, *Integral crystalline cohomology over very ramified valuation rings*, JAMS 1999. Kisin, *Crystalline representations and F-crystals.* Scholze-Weinstein, *Moduli of p-divisible groups*, Camb. J. Math. 2013.)
  - The crystalline Dieudonné theory (or Zink's theory of Dieudonné windows/displays) enables us to interpret deformation theory of p-divisible groups in terms of geometric/linear algebraic datum, which gives e.g. classification of p-divisible groups and construction of *Rapoport-Zink spaces.*

» **Harder-Narasimhan filtration, canonical filtrations.**
(Fargues, *La Filtration de Hader-Narasimhan des schémas en groupes finis et plats*, Crelle 2010, and *La filtration canonique des points de torsion des groupes p-divisibles*, ASÉNS 2011. Hernandez, *La filtration canonique des $\mathcal{O}$-modules p-divisibles*, Annalen 2018.)
  - Results of similar flavor are about Harder-Narasimhan filtrations/canonical subgroups (filtrations more generally) of finite flat group schemes/p-divisible groups under great generality (e.g. over more general base; thus we need cotangent complex here as well). This has been extremely crucial in construction of several p-adic geometric objects, e.g. adic eigenvarieties and perfectoid Shimura varieties.

- **Donaldson-Thomas theory; stable pairs ("Pandharipande-Thomas theory").**
(Huybrechts-Thomas, *Deformation-obstruction theory for complexes via Atiyah and Kodaira-Spencer classes*, Annalen 2010, Pandharipande-Thomas, *Curve counting via stable pairs in the derived category*, Inventiones 2009, Pandharipande-Thomas, *Stable pairs and BPS invariants*, JAMS 2010.)
  - Moduli space of curves inside a 3-fold, seen as a moduli space of sheaves, admits a perfect obstruction theory, giving a virtual fundamental class. One sees this by studying deformation-obstruction theory of objects of the derived category of coherent sheaves over a smooth projective family. See e.g. Pandharipande-Thomas, $\frac{13}{2}$ *ways of counting curves* for comparing various ways of counting curves.
- Other possible topics include (algebraicity of) moduli stack of stable curves, topological modular forms, geometry of moduli of p-covers of curves, e.g. Bertin-Mézard, *Déformations formelles des revêtements sauvagement ramifiés de courbes algèbriques*, Inventiones 2000. Another very interesting application can be found in Berthelot-Esnault-Rülling, *Rational*

points over finite fields for regular models of algebraic varieties of Hodge type ≥ 1, Annals 2012.

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