DECOMPLETIONS

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Contents

1. Tate-Sen conditions

Sen's theory of decompletion is made to overcome the following situation. Given k a p-adic field and an infinitely ramified $\mathbb{Z}_p$-extension $k_\infty = \cup_n k_n$ of k (say, $k_n = k(\mu_{p^n})$), one wants to "descend" a $\mathbb{C}_k$-representation of G_k to a finite level, namely over k_n for some n. However, $\mathbb{C}_k^{G_{k_\infty}}$ is not k_∞, but rather $\widehat{k_\infty}$, so taking G_{k_∞}-invariants gives a functor

$$\mathrm{Rep}_{\mathbb{C}_k}(G_k) \longrightarrow \mathrm{Rep}_{\widehat{k_\infty}}(\mathrm{Gal}(k_\infty/k)).$$

At this point, it is not obvious whether a nonzero input always yields a nonzero output, or even whether the functor is an equivalence. However, due to the existence of ***Tate's normalized trace map*** in this case, there is indeed an equivalence with the representation category even over k_∞! The classical Sen theory constructs a functor

$$D_{\mathrm{Sen}} : \mathrm{Rep}_{\mathbb{C}_k}(G_k) \longrightarrow \mathrm{Rep}_{k_\infty}(\mathrm{Gal}(k_\infty/k)),$$

which is a quasi-inverse to $W \mapsto \mathbb{C}_k \otimes_{k_\infty} W$. More precisely, for each $V \in \mathrm{Rep}_{\mathbb{C}_k}(G_k)$, the Sen theory finds a unique $(\dim_{\mathbb{C}_k} V)$-dimensional G_k-stable k_∞-subspace $D_{\mathrm{Sen}}(V) \subset V$ on which G_{k_∞} acts trivially and $\mathbb{C}_k \otimes_{k_\infty} D_{\mathrm{Sen}}(V) = V$. This is extremely useful; for example, we can justify $(\mathbb{C}_p \otimes \chi)^{\mathrm{Gal}(\overline{k}/k)} = 0$, where χ is the cyclotomic character.

Proof. If not, by the Sen theory and the usual Galois descent, $(k_n \otimes \chi)^{\mathrm{Gal}(\overline{k}/k)} \neq 0$ for some n. However, this implies that $\ker \chi$ contains $\mathrm{Gal}(\overline{k}/k_n)$, so that χ has finite image, which is false. □

The aforementioned Tate's normalized trace map is defined as

$$R_{k_m/k_n} := \frac{1}{[k_m : k_n]} \mathrm{Tr}_{k_m/k_n} : k_m \longrightarrow k_n.$$

This then extends unambiguously to k_∞, which defines $R_{k_\infty/k_n} = \varinjlim R_{k_m/k_n} : k_\infty \longrightarrow k_n$. The role of normalized trace maps is that they give some control (in terms of metric) on elements of (finite extensions of) $\widehat{k_\infty}$.

The decompletion argument of Sen is further axiomatized by Colmez, who formulated the general ***Tate-Sen conditions*** which formally imply a similar consequence as that of the classical Sen theory. We first informally formulate the three Tate-Sen conditions in certain generality, and try to justify the conditions afterwards.

Definition 1.1 (Tate-Sen conditions)**.** *Let k_∞/k be an infinitely ramified extension. Let $\widetilde{\Lambda}$ be a $\mathbb{Q}_p$-algebra with a complete topology equipped by a "valuation" $v : \widetilde{\Lambda} \longrightarrow \mathbb{R} \cup \{+\infty\}$. Suppose $\widetilde{\Lambda}$ has an isometric and continuous G_k-action. The following three conditions for $\widetilde{\Lambda}$ are called the* **Tate-Sen conditions***.*

(TS1) There is a constant $c_1 \in \mathbb{R}_{>0}$ such that, for any finite Galois extensions $l'/l/k$, there exists $\alpha \in \widetilde{\Lambda}^{G_{l'_\infty}}$ satisfying $v(\alpha) > -c_1$ and $\mathrm{Tr}_{l'_\infty/l_\infty}(\alpha) = 1$.

(TS2) There is a constant $c_2 \in \mathbb{R}_{>0}$ such that, for any finite Galois extension l/k, there exist an increasing sequence of closed $\mathbb{Q}_p$-subalgebras $\Lambda_{l,n} \subset \widetilde{\Lambda}^{G_{l_\infty}}$ and normalized trace maps $R_{l,n} : \widetilde{\Lambda}^{G_{l_\infty}} \longrightarrow \Lambda_{l,n}$ for large enough n's. The normalized trace maps have some "uniformly controlled behavior" in terms of c_2; in particular, $v(R_{l,n}(x)) \geq v(x) - c_2$ and $\lim_{n\to\infty} R_{l,n}(x) = x$.

(TS3) There is a constant $c_3 \in \mathbb{R}_{>0}$ such that, for any finite Galois extension l/k, $\gamma \in \mathrm{Gal}(l_\infty/k)$ and n large enough (depending on l and γ; the closer γ is to 1, the larger n needs to be), $(\gamma - 1)$ is invertible on $X_{l,n} := \ker(R_{l,n})$ and $v(x) \geq v((\gamma-1)x) - c_3$ for all $x \in X_{l,n}$.

In the Sen theory, $\widetilde{\Lambda} = \mathbb{C}_k$. The above definition is by no means precise, and the actual definition of Tate-Sen conditions is more involved; see for example [BrCo, 14.1], [Ber]. A typical consequence of the Tate-Sen formalism is as follows.

Theorem 1.1. *If $\widetilde{\Lambda}$ satisfies the Tate-Sen conditions, then the natural map*

$$\varinjlim_{l/k \text{ Galois}} \varinjlim_{n} H^1(\mathrm{Gal}(l_\infty/k), \mathrm{GL}_d(\Lambda_{l,n})) \longrightarrow H^1(G_k, \mathrm{GL}_d(\widetilde{\Lambda})),$$

is an isomorphism.

In terms of G_k-representations over $\widetilde{\Lambda}$, we have the following theorem, which says that we can ***uniquely***, in a strong sense, find a descent submodule inside the given representation.

Theorem 1.2. *Let W be a free rank d G_k-representation over $\widetilde{\Lambda}$. Then, there exists a finite extension l/k and a finite free rank d $\Lambda_{l,\infty} := \varinjlim_n \Lambda_{l,n}$-submodule $W' \subset W$ such that W' is a descent of W as a $\mathrm{Gal}(l_\infty/k)$-representation (i.e. $W = W' \otimes_{\Lambda_{l,\infty}} \widetilde{\Lambda}$ as $\widetilde{\Lambda}[G_k]$-modules), and is unique in a certain sense.*

If one believes the above assertions and that the Tate-Sen conditions are indeed satisfied for $\widetilde{\Lambda} = \mathbb{C}_k$ (with Tate's normalized trace maps), then we could deduce the following

Theorem 1.3. *Given a p-adic representation V of G_k, for any large enough n, there exists a Galois stable k_n-vector space $W_n \subset (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{G_{k_\infty}}$ such that the inclusion induces an isomorphism $W_n \otimes_{k_n} \widehat{k_\infty} \xrightarrow{\sim} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{G_{k_\infty}}$.*

Remark 1.1. Theorem 1.3 implies that, for a d-dimensional p-adic G_k-representation V, $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{G_{k_\infty}}$ is d-dimensional over $\widehat{k_\infty}$. On the other hand, this conclusion can be alternatively achieved from the so-called ***overconvergence of p-adic representations***. More precisely, Fontaine's theory of (φ, Γ)-modules relates an étale (φ, Γ)-module (over some period ring "$\mathbb{B}_k$") to V, and a theorem of Cherbonnier-Colmez implies that an étale (φ, Γ)-module can be descended down to be over the overconvergent period ring ("$\mathbb{B}_k^\dagger$"), from which we can explicitly find a basis of $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{G_{k_\infty}}$ by using comparison isomorphism and a θ-like map; note that θ does not extend to the whole $W(\mathbb{C}_k^\flat)$ as the sum may not converge, but it extends to a subring where the formal series gives a convergent sum.

The Cherbonnier-Colmez theorem is usually proved also by using a Tate-Sen formalism [BeCo, 4.2]. On the other hand, if one's objective is to just prove that $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_p)^{G_{k_\infty}}$ is of right dimension, then one can work over the perfectoid field $K = \widehat{k_\infty}$, and the theory becomes much simpler. We sketch the argument briefly here. Take a G_K-stable lattice $T \subset V$. Fontaine's theory gives an equivalence of categories (where the argument is an easy application of Galois descent; see the proof of [Ked, Theorem 2.3.5])

$$\left\{ \begin{matrix} \text{Free finite rank} \\ \mathbb{Z}_p\text{-representations of} \\ G_K \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Étale } \varphi\text{-modules} \\ \text{over } W(K^\flat) \end{matrix} \right\},$$

where an ***étale* φ*-module*** over $W(K^\flat)$ is a finite rank free $W(K^\flat)$-module with a semilinear Frobenius action such that the Frobenius action takes a basis to a basis. On the other hand, the overconvergence of étale φ-module in this case (cf. [Ked, Lemma 2.4.4]) implies that the base change functor

$$\left\{ \begin{array}{c} \text{Étale } \varphi\text{-modules} \\ \text{over } W^{(0,1]}(K^\flat) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Étale } \varphi\text{-modules} \\ \text{over } W(K^\flat) \end{array} \right\},$$

is an equivalence, where

$$W^{(0,r]}(K^\flat) = \left\{ \sum_{i\geq 0} p^i[a_i] \;:\; \lim_{i\to\infty} \left(v(a_i) + \frac{i}{r} \right) = +\infty \right\}.$$

What this says is that there is a d-dimensional étale φ-module $D^{(0,1]}(T)$ over $W^{(0,1]}(K^\flat)$ such that there is a comparison isomorphism

$$D^{(0,1]}(T) \otimes_{W^{(0,1]}(K^\flat)} W^{(0,1]}(\mathbb{C}_K^\flat) \xrightarrow{\sim} T \otimes_{\mathbb{Z}_p} W^{(0,1]}(\mathbb{C}_K^\flat),$$

respecting Galois and Frobenius action on both sides. What is really proved is that one finds a (unique) $W^{(0,1]}(\mathbb{C}_K^\flat)$-basis of $T \otimes_{\mathbb{Z}_p} W^{(0,1]}(\mathbb{C}_K^\flat)$ such that the Galois action has matrix entries in $W^{(0,1]}(K^\flat)$. The upshot here is that the θ map now linearly extends to $\theta \,:\, W^{(0,1]}(K^\flat) \longrightarrow K$. Thus, applying θ to the comparison isomorphism, we find a basis of $T \otimes_{\mathbb{Z}_p} \mathbb{C}_K$ whose Galois actions have matrix entries contained in K. The K-module generated by the basis, which is d-dimensional, is contained in $(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K)^{G_K}$, so we get the right dimension.

2. Example: Sen theory

We finish the discussion of Tate-Sen formalism by justifying some conditions in the classical setting of Sen, namely $\widetilde{\Lambda} = \mathbb{C}_k$. Firstly, the first condition (TS1) is a direct consequence of the almost purity result of Tate:

Theorem 2.1 (Tate)**.** *Let k_∞/k be an infinitely ramified Galois extension, whose Galois group is locally isomorphic to $\mathbb{Z}_p$. For any finite extension M/k_∞, the image of $\mathrm{Tr}_{M/k_\infty} : \mathcal{O}_M \longrightarrow \mathcal{O}_{k_\infty}$ contains $\mathfrak{m}_{k_\infty}$.*

From this, we can see that, in the classical Sen theory, (TS1) is satisfied for any choice of $c_1 > 0$. Namely, Tate's almost purity says that $\mathfrak{m}_{l_\infty} \subset \mathrm{Tr}_{l'_\infty/l_\infty}(\mathcal{O}_{l'_\infty})$, and l_∞/l is infinitely ramified, we can choose $a \in \mathfrak{m}_{l_\infty}$ such that $v(a) < c_1$. Thus, $1 \in \mathrm{Tr}_{l'_\infty/l_\infty}(a^{-1}\mathcal{O}_{l'_\infty})$, and every element in $a^{-1}\mathcal{O}_{l'_\infty}$ has valuation bounded below by $-v(a) > -c_1$.

Remark 2.1. Theorem 2.1 is usually proven by studying ramification carefully, and intermediately one proves that, given a finite Galois extension k'/k, $v_p(\mathfrak{D}_{k'_n/k_n}) \longrightarrow 0$ as $n \longrightarrow \infty$, where $\mathfrak{D}$ denotes the different. This statement is literally the almost purity; recall that, in a perfectoid setting, almost purity is the following assertion: *for a perfectoid algebra A* (over some perfectoid field, say)*, any finite étale A-algebra B is perfectoid, and B°/A° is almost finite étale.* As the different is the annihilator of Ω^1, that $\lim_{n\to\infty} v_p(\mathfrak{D}_{k'_n/k_n}) = 0$ implies that $\Omega^1_{\mathcal{O}_{k'_\infty}/\mathcal{O}_{k_\infty}}$ is annihilated by an element of arbitrarily small valuation, thus annihilated by $\mathfrak{m}_{k'_\infty}$, or that $\Omega^1_{\mathcal{O}_{k'_\infty}/\mathcal{O}_{k_\infty}}$ is "almost zero," so that $\mathrm{Tr}_{k'_\infty/k_\infty} : \mathcal{O}_{k'_\infty} \longrightarrow \mathcal{O}_{k_\infty}$ is almost étale.

The proof of Theorem 2.1 indicated above inherently uses an "integral model," and the same is true when deriving consequences formally from the Tate-Sen conditions. However, almost purity can be proved by exploting tilting equivalence of the generic fiber. Thus, one may think that the whole process of decompletion may be done without a control of ramification of integral model. This is the idea that will be demonstrated through the formalism of ***decompletion systems*** (after Kedlaya-Liu).

The conditions of (TS2) can be summarized as follows.

- $R_{l,n}$'s are Galois equivariant and compatible with respect to l and n.
- $R_{l,n}$ restricted to $\Lambda_{l,n}$ is the identity map.

- For n large enough depending on l, $v(R_{l,n}(x)) \geq v(x) - c_2$.
- $\lim_{n\to\infty} R_{l,n}(x) = x$.

We justify some of the above aspects for the Tate's normalized trace map $R_{k_\infty/k_n} : k_\infty \longrightarrow k_n$.

- First of all, we need to extend the Tate's normalized trace to $\widehat{k_\infty}$, which follows from the continuity of them, a direct consequence of Theorem 2.1.
- The ramification theory says that there is a constant $n(k)$, c and a bounded sequence $\{a_n\}_{n\geq n(k)}$ such that $v(\mathfrak{D}_{k_n/k_{n(k)}}) = n + c + p^{-n}a_n$ for all $n \geq n(k)$ (cf. [BrCo, Proposition 13.1.9]). Using this we easily get that, for any $c_2 > 0$, $v(R_{l,n}(x)) \geq v(x) - c_2$ for large n, for $x \in l_\infty$. By continuity we can extend this to $x \in \widehat{l_\infty}$. That $\lim_{n\to\infty} R_{l,n}(x) = x$ also follows from this estimate.

We finally justify (TS3). As $\ker(\gamma - 1) \subset l_n$ for some large n and $X_{l,m} \cap l_n = 0$ for large enough m, for $m \geq n \gg 0$, $(\gamma - 1)$ induces a k-linear injection on $X_{l,n} \cap l_m$. As this space is finite-dimensional, $(\gamma - 1)$ is a bijection. Taking a limit $m \longrightarrow 0$, we get that $(\gamma - 1)$ induces a bijection on $X_{l,n} \cap l_\infty$. Provided that we have a bounded inverse for $(\gamma - 1)$ on $X_{l,n} \cap l_\infty$, we can extend the invertibility to the whole $X_{l,n} (= X_{l,n} \cap \widehat{l_\infty})$. For the bound $v(x) \geq v((\gamma - 1)x) - c_3$ for $x \in X_{l,n} \cap l_\infty$, we can try to bound $v(x - R_{l_\infty/l_n}(x))$ instead. One can see that any $c_3 > 1$ works in this case, by noticing that all elements of $\mathrm{Gal}(l_k/l_{k-1})$ can be expressed as a power of γ for large enough k (so that $x - R_{l_\infty/l_n}(x) = p^{-m}P(\gamma)(1-\gamma)x$ for some $P(X) \in \mathbb{Z}[X]$ and $m \in \mathbb{N}$), and that $v(\mathrm{Tr}_{l_m/l_n}(x)) \geq v(x) + v(\mathfrak{D}_{l_m/l_n})$ (so that, for large n, $v(R_{l_m/l_n}(x)) \geq v(x) + p^{-m}a_m - p^{-n}a_n$).

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