# BORCHERDS PRODUCTS

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## 1. Basic theory (Bruinier's book)

Here we give a crash course on the Borcherds products

$$\left\{\begin{array}{c}\text{Weakly holomorphic}\\ \text{modular forms of weight}\\ 1-l/2 \text{ and level } 1\end{array}\right\} \to \left\{\begin{array}{c}\text{Meromorphic modular forms}\\ \text{on } \mathrm{O}(2,l)\end{array}\right\},$$

for $l \geq 3$, as in Bruinier's textbook.

1.1. **Weakly holomorphic modular forms.** (=whmf). Here $1-l/2<0$, so there is no nonzero holomorphic cusp form of weight $k=1-l/2$. So the principal part determines the meromorphic modular form. We start with a vector-valued whmf and want to obtain a meromorphic modular form on $\mathrm{O}(2,l)$ with level a finite index subgroup of $\mathrm{O}(L)$ for an even lattice $L$ of signature $(2,l)$. You start with a $\mathbb{C}[L'/L]$-valued whmf, with Fourier expansion at $\infty$

$$f(\tau)=\sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}-q(\gamma),n\gg-\infty}c(\gamma,n)e(n\tau)[\gamma],$$

where $q$ is the quadratic form $q(x)=(x,x)/2$ and $L'$ is the dual lattice of $L$.

Recall that cusp forms of weight $\kappa$ are spanned by Poincare series,

$$P_{\beta,m}(\tau)=\frac{1}{2}\sum_{A\in\widetilde{\Gamma}_\infty\backslash\widetilde{\mathrm{Sp}_2}(\mathbb{Z})}e(m\tau)[\beta]|_\kappa^* A,$$

where $\kappa=1+l/2$. This is the dual (wrt Petersson inner product) of the linear functional $f\mapsto c(\beta,m)$ (up to scalar).

We want to achieve similar thing for negative weight whmf. Recall that (harmonic, resp.) Maass wave forms of weight $k$ are eigenfunctions of $\Delta_k$ (with eigenvalue $0$, resp.), the hyperbolic Laplacian of weight $k$. Good thing about this is that one can use bad Whittaker functions, which have exponential growth toward cusps, which is exactly we would want to capture poles at cusps. Namely, the Whittaker differential equation has two linearly independent solutions, $W$ and $M$, where $W$ is the usual "good Whittaker function", namely that of rapid decay towards $z\to+\infty$ (this characterizes $W$ up to scalar because the Whittaker differential equation has irregular singularity at infinity), and $M$ is the "bad Whittaker function", namely that of exponential growth towards $z\to+\infty$ plus a prescribed local behavior around $0$, another singularity which is regular. Letting $\mathcal{M}_s(y)=y^{-k/2}M_{-k/2,s-1/2}(y)$, $\mathcal{M}_s(4\pi|m|y)e(mx)$ is an eigenfunction of $\Delta_k$ with eigenvalue $s(1-s)+(k^2-2k)/4$. So

$$F_{\beta,m}(\tau,s)=\frac{1}{2\Gamma(2s)}\sum_{A\in\widetilde{\Gamma}_\infty\backslash\widetilde{\mathrm{Sp}_2}(\mathbb{Z})}\mathcal{M}_s(4\pi|m|y)e(mx)[\beta]|_k A,$$

is an $\widetilde{\mathrm{Sp}}_2(\mathbb{Z})$-invariant function which is an eigenfunction of $\Delta_k$ with eigenvalue $s(1-s)+(k^2-2k)/4$ (if $\operatorname{Re} s > 1$). In particular, $F_{\beta,m}(\tau, 1-k/2)$ is "harmonic Maass", and blows up like $e(m\tau)[\beta] + e(-m\tau)[-\beta]$. So $f$ is determined by its principal part,

$$f(\tau) = \frac{1}{2} \sum_{\gamma \in L'/L} \sum_{n \in \mathbb{Z}+q(\gamma), n<0} c(\gamma, n) F_{\gamma,n}(\tau, 1-k/2),$$

because the RHS expression picks up all poles correctly. Moreover, the principal part gives rise to an actual whmf iff it pairs trivially with all cusp forms with weight $\kappa$. This is because the functional gives an element in the dual of $S_\kappa$, and by Serre duality this is isomorphic to $H^1(\omega_k(-D))[L'/L]$, and using the affine cover $\{\text{open modular curve}, \text{small ball around cusp}\}$, this is the same as $\frac{\text{all possible principal parts}}{\text{actual principal parts}}$.

1.2. **Theta lift of Poincare series.** We want to "theta lift" whmfs, and we saw it is sufficient to theta-lift Poincare series $F_{\beta,m}$. There is a certain theta function corresponding to $L \otimes \Lambda$, where $\Lambda$ is $\mathbb{Z}^2$ equipped with the standard symplectic form, $\theta_L(\tau, v)$, where $\tau \in \mathbb{H}$ and $v \in \mathrm{Gr}(L)$, the Grassmannian of $L$, which means $v \subset L$ is a 2-dimensional positive-definite subspace. There are some variants, like $\theta_\gamma(\tau, v; r, t)$ for $\gamma \in L'/L$, $r, t \in V := L \otimes \mathbb{R}$, and $\Theta_L(\tau, v; r, t)$, a $\mathbb{C}[L'/L]$-valued function on $\mathbb{H} \times \mathrm{Gr}(L)$, whose definition we won't recall. We just recall that

$$\theta_L(\tau, v) = \sum_{\lambda \in L} e(\tau q(\lambda_v) + \overline{\tau} q(\lambda_{v^\perp})),$$

where $\lambda = \lambda_v + \lambda_{v^\perp}$ is the decomposition correpsonding to $V = v \oplus v^\perp$. We want to theta-lift $F_{\beta,m}$, namely want to compute

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \langle F_{\beta,m}(\tau), \Theta_L(\tau, v)\rangle \frac{dxdy}{y}.$$

At least it is formally $\mathrm{O}(L)$-invariant. But this is divergent as $F_{\beta,m}$ blows up as you go towards the cusp. The idea is to regularize this by first integrating over $dx$ and then integrating over $dy$. To do this, one has to know the Fourier-expansion of $\Theta_L(\tau, v)$ around the cusp, and you can expect that what appears are theta functions over $\mathrm{O}(1, l-1)$ (basically you're computing Fourier–Jacobi expansion of some sort). Anyways, using it, a precise form one proves is that

$$\int_{u\to\infty} \int_{\mathcal{F}\cap\{y\le u\}} \langle F_{\beta,m}(\tau), \Theta_L(\tau, v)\rangle \frac{dxdy}{y},$$

defines a real analytic function on $\mathrm{Gr}(L)$ with log-singularities along $H(\beta, m) = \bigcup_{\lambda \in \beta+L, q(\lambda)=m} \lambda^\perp$, if $\operatorname{Re} s > (l+2)/4 = 1-k/2$. This can be meromorphically continued to $\operatorname{Re} s > 1$ with a simple pole at $s = (l+2)/4$. One defines the theta lift at $s = 1-k/2$, $\Phi_{\beta,m}(v)$, to be the constant term of the Laurent expanstion at $s = 1-k/2$ of the analytic continuation of the integral.

1.3. **Borcherds products.** Through explicit inductive formulae (inducting over lower-rank orthogonal groups), one defines a decomposition

$$\Phi_{\beta,m} = \psi_{\beta,m} + \xi_{\beta,m}.$$

It has the following properties.

- $\xi_{\beta,m}$ is real-analytic over the whole symmetric domain.

- A linear transformation of $\psi_{\beta,m}$ is a log of a holomorphic function on the symmetric domain, denoted $\Psi_{\beta,m}$ (more precisely, $\log|\Psi_{\beta,m}(Z)| = \frac{1}{4}(C_{\beta,m} - \psi_{\beta,m}(Z))$). It satisfies $\operatorname{div}\Psi_{\beta,m} = H(\beta,m)$. So $e^{C_{\beta,m}/4-\Phi_{\beta,m}(Z)/4} = |\Psi_{\beta,m}(Z)|e^{-\xi_{\beta,m}(Z)/4}$ is $\Gamma(L)$-invariant.
- $\frac{1}{4}\partial\overline{\partial}\xi_{\beta,m}(Z)$ is a harmonic $L^2$-representative of the Heegner divisor $H(\beta,m)$, because the above expression implies that $e^{\xi_{\beta,m}(Z)/4}$ defines a Hermitian metric on the sheaf $\mathcal{O}(H(\beta,m))$. This means that $\Phi_m(Z)$ is a Green current for the divisor $H(m)$.
- Somehow, when trying to theta-lift whmf $f = \sum_{\gamma\in L'/L}\sum_{n\in\mathbb{Z}+q(\gamma),n<0} c(\gamma,n)F_{\gamma,n}(\tau,1-k/2)$ of weight $k$ by lifting each Poincare series, the contribution of $\xi$-part cancels away! More precisely,
$$\sum_{\beta\in L'/L}\sum_{m\in\mathbb{Z}+q(\beta),m<0} c(\beta,m)\xi_{\beta,m}(Z) = C\log(Y^2),$$
where $Y = \operatorname{Re} Z$ and $Y^2 = (Y,\overline{Y})$. So,
$$\Psi(Z) = \prod_{\beta\in L'/L}\prod_{m\in\mathbb{Z}+q(\beta),m<0} \Psi_{\beta,m}(Z)^{c(\beta,m)/2},$$
is a meromorphic modular form of weight $c(0,0)/2$ (weight comes from this remaining $\log(Y^2)$ term). This is the **Borcherds product**.
- The existence of Borcherds products plus cohomological interpretation implies that
$$\Omega(Z,\tau) = \frac{1}{4}\partial\overline{\partial}\log(Y^2)[0] + \frac{1}{2}\sum_{\beta\in L'/L}\sum_{m\in\mathbb{Z}-q(\beta),m>0}\left(\frac{1}{4}\partial\overline{\partial}\xi_{\beta,-m}(Z)\right)e(m\tau)[\beta],$$
defines a harmonic $(1,1)$-form with values in $M_{\kappa,L}$.

1.4. **Converse theorem.** One can ask what modular forms on $\mathrm{O}(2,l)$ arise as Borcherds products. Bruinier proves that **every meromorphic modular form with divisor equal to Heegner divisor is a Borcherds lift of a Maass wave form (which can blow up at cusps)**. This is proven by constructing an invariant Laplacian on the symmetric space and using the harmonicity (like harmonic maximum principle). One actually expects that it's weakly holomorphic.

## 2. Kudla–Millson lift

Kudla–Millson lift is at least cosmetically a geometric theta lift going to the opposite direction, from comological classes on a locally symmetric space to modular forms. We summarize its theory as follows. Let $V$ be an orthogonal space over a totally real field of signature $(p,q)$ at one archimedean component and positive definite at other archimedean component. Let $V_{\mathbb{R}}$ be the basechange at the archimedean place where it has signature $(p,q)$. Let $G = \mathrm{SO}(V_{\mathbb{R}})$, and $D = G/K$, $\Gamma\subset G$ be a torsion-free congruence subgroup of $\mathrm{GL}(V)$ preserving a lattice $L$. For $U\subset V$ a subspace where $(,)|_U$ is nondegenerate, one can construct a cycle $C_U\subset\Gamma\backslash D$ as a sub-Shimura-variety corresponding to $\mathrm{Sh}_U\times\mathrm{Sh}_{U^\perp}$. If $U$ is positive or negative definite, then we say the cycle is of definite type.

These can be packaged in a slightly different way. Let $\beta$ be an integral symmertic $n\times n$ matrix and $Q_\beta = \{(x_1,\cdots,x_n)\in V^n : (x_i,x_j) = \beta_{ij}\}$. There is a single closed $G$-orbit $Q_\beta^c$, consisted of $(x_1,\cdots x_n)$ such that they span a vector space of dimension $\operatorname{rank}\beta$ and $(,)$ restricts to a nondegenerate bilinear form. The intersection $Q_\beta^c\cap L^n$ has finitely many $\Gamma$-orbits, represented by $Y_1, \cdots, Y_l$, and letting $U_j = \operatorname{span} Y_j$, we define $C_\beta = \sum C_{U_j}$. Each irreducible component of $C_\beta$ has dimension $(p-t)q$. The slogan is that

$$\sum_{\beta^*=\beta, \beta\geq 0} C_\beta q^\beta \text{ is a Siegel modular form.}$$

Now note that $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ acts on $\mathscr{S}(V^n)$ via the Weil representation, and therefore on $H^i(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n))$. Let $\mathrm{MU}(n) \subset \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ be the maximal compact subgroup lying over $\mathrm{U}(n)$ (sorry for a bad notation), and let $D'$ be the (Hermitian) symmetric space of $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$. Then, differentiating, $\mathfrak{p}'$ acts on $H^i(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n))$. We say a cohomology class $c \in H^*(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n))$ is **holomorphic** if $\mathfrak{p}'^-$ annihilates it.

We can relate this with relative Lie algebra cohomology; since $D$ is diffeomorphic to a Euclidean space which also admits a continuous $\mathrm{SO}(p,q)^+$-action, $0 \to \mathscr{S}(V^n) \to \mathscr{A}^0(D) \otimes \mathscr{S}(V^n) \to \mathscr{A}^1(D) \otimes \mathscr{S}(V^n) \to \cdots$ is a resolution of $\mathscr{S}(V^n)$ as an $\mathrm{SO}(p,q)^+$-representation. Here, $\mathscr{A}^i(D)$ is the space of smooth differential $i$-forms on $D$. So,

$$H^i(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n)) = H^i(C^\bullet) = H^i(\mathfrak{g}, K; \mathscr{S}(V^n)),$$

where $C^i = (\mathscr{A}^i(D) \otimes \mathscr{S}(V^n))^{\mathrm{SO}(p,q)^+}$. So, a continuous cohomology class can be thought as a closed differential form on $D$ with values in $\mathscr{S}(V^n)$. The differential operators coming from $\mathfrak{p}'^-$ is most naturally thought as taking $\overline{\partial}$ on $D'$, so the holomorphicity condition is most natural if the cohomology class is **regarded as a function on** $D'$; namely, take a closed differential $i$-form $\varphi \in (\mathscr{A}^i(D) \otimes \mathscr{S}(V^n))^{\mathrm{SO}(p,q)^+}$, and take a function on $D'$ valued in $(\mathscr{A}^i(D) \otimes \mathscr{S}(V^n))^{\mathrm{SO}(p,q)^+}$ which is $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$-equivariant and takes the chosen value at a reference point of $D'$. It is however not known to be $\mathrm{MU}(n)$-invariant, so rather it is a section of some homogeneous bundle over $D'$, provided that $\varphi$ is $\mathrm{MU}(n)$-finite, which we assume. These can be packaged in a concise way as follows: let $E$ be the $\mathrm{MU}(n)$-representation of $\mathrm{MU}(n)$-finite vectors in $H^*(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n))$; then,

- a continuous cohomology class $[\varphi] \in H^i(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n))$ is thought as an element $\varphi \in C^{i,0}$, where $C^{p,q}$ is the double complex

$$C^{p,q} = (\mathscr{A}^p(D) \otimes \mathscr{A}^{0,q}(D') \otimes E \otimes \mathscr{S}(V^n))^{\mathrm{SO}(p,q)^+ \times \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})},$$

  with the differentials $d$ (on $D$) and $\overline{\partial}$ (on $D'$; recall $D'$ is Hermitian symmetric), such that $d\varphi = 0$; or, in other words, it is an element of $E_1^{i,0}$ for the double complex coming from the filtration by

$$F^q C^n = \bigoplus_{p+q'=n, q'\geq q} C^{p,q'};$$

- and, $\varphi$ is holomorphic iff $\varphi$ is a cocycle as an element of $E_1^{i,0}$.

Note that $\varphi$ being holomorphic thus means $\overline{\partial}\varphi = d\psi$ for some $\psi \in C^{i-1,1}$. Also, in the above construction, one could've taken a much smaller $E$ just corresponding to the $\mathrm{MU}(n)$-representation spanned by $\varphi$.

We will describe how each holomorphic class can give rise to a theta corrsepondence. Let $\Theta \in \mathrm{Hom}_\Gamma(\mathscr{S}(V^n), \mathbb{C})$ be the theta distribution, the sum of Dirac delta distributions centered at points of $L^n$. We define

$$\theta_\varphi(g', g) = \Theta(g' \cdot \varphi(g^{-1}x)),$$

which defines a right $\mathrm{MU}(n)$-finite map $\widetilde{\mathrm{Sp}}_{2n}(\mathbb{R}) \times \mathrm{SO}(p,q)^+ \to H^i(\Gamma, \mathbb{C})$. One can show that there is an arithmetic subgroup $\Gamma' \subset \mathrm{Sp}_{2n}(\mathbb{R})$ which comes with a splitting $s : \Gamma' \to \widetilde{\mathrm{Sp}}_{2n}(\mathbb{R})$ such that $\Theta$ is invariant under $s(\Gamma')$. This implies that $\theta_\varphi$ is $\Gamma'$-invariant. Now let $\eta$ be a closed rapidly decreasing differential $\dim D - i$-form on $\Gamma\backslash D$. We define the Kudla–Millson lift to be

$$\theta_\varphi(\eta) := \int_{\Gamma\backslash D} \eta \wedge \theta_\varphi,$$

which is a $\Gamma'$-invariant function on $G'$.

**Theorem 2.1.** *If $\varphi$ is holomorphic, then $\theta_\varphi(\eta)$ descends to a holomorphic section of the automorphic vector bundle on $\Gamma'\backslash D'$ corresponding to $E$.*

*Proof.* That it transforms as $E$ under $\mathrm{MU}(n)$-action is easy. Pick $\psi \in C^{i-1,1}$ such that $\overline{\partial}\varphi = d\psi$. Then, $\overline{\partial}\theta_\varphi = d\theta_\psi$. This implies that

$$\overline{\partial}\theta_\varphi(\eta) = \overline{\partial}\left(\int_{\Gamma\backslash D} \eta \wedge \theta_\varphi\right) = \int_{\Gamma\backslash D} \eta \wedge \overline{\partial}\theta_\varphi = \int_{\Gamma\backslash D} \eta \wedge d\theta_\psi = \int_{\Gamma\backslash D} d(\eta \wedge \theta_\psi) = 0.$$

Here we used the fact that $\eta$ is rapidly decreasing to use a version of Stokes' theorem. □

More generally, the following holds. Consider the Fourier expansion of $\theta_\varphi(\eta)$ with respect to the Siegel parabolic subgroup. Namely, let $N$ be the unipotent radical of the Siegel parabolic, which is identified with the space of $n \times n$ real symmetric matrices. Let $L = N \cap \Gamma' \subset N$ be the lattice, and let $L^*$ be the dual lattice for the bilinear form $(X, Y) = \mathrm{tr}(XY)$. Then $\theta_\varphi(\eta)$ has a Fourier expansion with respect to the abelian subgroup $N$,

$$\theta_\varphi(\eta)(u + iv) = \sum_{\beta \in L^*} a_\beta(v) e(2\beta u).$$

Then for any closed $\varphi$, $a_\beta(v) = w_\beta(v) \int_{C_{2\beta}} \alpha \wedge \eta$ for nondegenerate $\beta$, where $w_\beta$ is a Whittaker function and $\alpha$ is some cohomology class on $C_{2\beta}$.

Now one defines a cohomology class $\varphi^+_{nq} \in H^{nq}(\mathrm{SO}(p,q)^+, \mathscr{S}(V^n))$ as follows. This is the $n$-th exterior product of $\varphi^+_q \in H^q(\mathrm{SO}(p,q)^+, \mathscr{S}(V))$, where the corresponding $\mathrm{MU}(n)$-representation is $\det^{(p+q)/2}$. One can define $\varphi^+_q$ that gives rise to the information indicated before.

## 3. Borcherds lift and Kudla–Millson lift

As observed by Bruinier–Funke, they are "adjoint to each other". Since there is a serious obstacle to extend Borcherds products to higher rank groups, we restrict to the case of $n = 1$, namely where the Kudla–Millson lift lands on modular forms. Then the Borcherds lift can be generalized so that the end-product of lift is a $q$-form on $\mathrm{SO}(p,q)$-locally symmetric space.