§4. The binary expansion: subdominant terms.

The general form of the binary expansion was given in (1.7), but with the coefficients altogether unspecified. Upon the application of a projection operator P for which $P\vec{\theta}_2(0) = P\partial_{jk}\vec{\theta}_2(0) = 0$ the first nontrivial term involves either $P\vec{\alpha}^{0;1;1}_{\lambda;j}$ or $P\vec{\alpha}^{0;0;2}_{j_1 j_2}$, and has the rather simple form described in Theorem 5 or its corollary. It is of course a consequence of this observation that the remaining coefficients in (1.7) are all vectors lying in the subspace spanned by the vectors $P\vec{\theta}_2(0)$ and $P\partial_{jk}\vec{\theta}_2(0)$, so can be expressed essentially uniquely as linear combinations of the latter vectors. It is actually not hard to write out these linear relations quite explicitly, by arguing much as in the preceding sections; the result is the following rather more explicit version of the general binary expansion.

THEOREM 6. *For any points $z_1, z_2, a_1, a_2 \in \widetilde{M}$*

$$\vec{\theta}_2(w(z_1 + z_2 - a_1 - a_2))q(z_1, z_2)^2 q(a_1, a_2)^2 \Big[\prod_{\mu,\nu=1}^{2} q(z_\mu, a_\nu)^{-2} \Big]$$

$$= \vec{\theta}_2(0) \Big\{ w'_{a_1}(z_1) w'_{a_2}(z_2) + w'_{a_1}(z_2) w'_{a_2}(z_1)$$

$$- 2w'_{a_1}(a_2) w'_{a_1,a_2}(z_1) w'_{a_1,a_2}(z_2) \Big\}$$

$$+ \frac{1}{2} \sum_{jk} \partial_{jk} \vec{\theta}_2(0) \Big\{ w'_{a_1}(z_1) w'_j(z_2) w'_k(a_2) + w'_{a_1}(z_2) w'_j(z_1) w'_k(a_2)$$

$$+ w'_{a_2}(z_1) w'_j(z_2) w'_k(a_1) + w'_{a_2}(z_2) w'_j(z_1) w'_k(a_1)$$

$$- 2w'_j(a_1) w'_k(a_2) w'_{a_1,a_2}(z_1) w'_{a_1,a_2}(z_2) \Big\}$$

$$+ \sum_{j} \vec{\alpha}_j^{0;0;1}(a_1, a_2) \{ w'_{a_1,a_2}(z_1) w'_j(z_2) + w'_{a_1,a_2}(z_2) w'_j(z_1) \}$$

$$+ \sum_{jk} \vec{\alpha}_{jk}^{0;0;2}(a_1, a_2) w'_j(z_1) w'_k(z_2).$$

Here $\vec{\alpha}_j^{0;1;1}(a_1, a_2)$ and $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ are meromorphic functions on $\widetilde{M} \times \widetilde{M}$, with the only singularities being at most double poles along the subvarieties $a_1 = Ta_2$ for all $T \in \Gamma$; actually $\vec{\alpha}_j^{0;1;1}(a_1, a_2)$ has at most a simple pole along the particular subvariety $a_1 = a_2$, while $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ is regular there. Moreover $\vec{\alpha}_j^{0;1;1}(a_1, a_2)$ is skew-symmetric in the variables a_1, a_2, while $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ is symmetric in these variables.

Proof. The first step is to multiply (1.7) by $q(z_2, a_2)^2$ and take the limit as z_2 tends to a_2. The only nontrivial terms that can arise on the right-hand side are those involving the meromorphic Abelian differential $w'_{a_2}(z_2)$, which has a double pole at $z_2 = a_2$ to cancel out the double zero of the factor $q(z_2, a_2)^2$. The left-hand side can be calculated quite easily, and the result is readily seen to be

$$\vec{\theta}_2(w(z_1 - a_1))q(z_1, a_1)^{-2}$$

$$= \vec{\alpha}_{12}^{2;0;0} w'_{a_1}(z_1) + \vec{\alpha}_{22}^{2;0;0} w'_{a_2}(z_1) + \vec{\alpha}_{21}^{1;1;0} w'_{a_1,a_2}(z_1)$$

$$+ \sum_j \vec{\alpha}_{2j}^{1;0;1} w'_j(z_1).$$

Comparing this and the primary expansion (3.1) shows that

$$\vec{\alpha}_{12}^{2;0;0}(a_1, a_2) = \vec{\theta}_2(0), \quad \vec{\alpha}_{22}^{2;0;0}(a_1, a_2) = 0,$$

$$\vec{\alpha}_{21}^{1;1;0}(a_1, a_2) = 0, \quad \vec{\alpha}_{2j}^{1;0;1}(a_1, a_2) = \frac{1}{2} \sum_k \partial_{jk} \vec{\theta}_2(0) w'_k(a_1),$$

and similarly or from the obvious symmetries

$$\vec{\alpha}_{11}^{2;0;0}(a_1, a_2) = 0, \quad \vec{\alpha}_{11}^{1;1;0}(a_1, a_2) = 0,$$

$$\vec{\alpha}_{1j}^{1;0;1}(a_1, a_2) = \frac{1}{2} \sum_k \partial_{jk} \vec{\theta}_2(0) w'_k(a_2).$$

This determines all the coefficients $\vec{\alpha}_{\mu_1\mu_2}^{2;0;0}$, $\vec{\alpha}_{\mu\lambda}^{1;1;0}$, and $\vec{\alpha}_{\mu j}^{1;0;1}$.

Next multiply (1.7) by $q(z_2, a_2)^2$, but this time apply the differential operator $\partial/\partial z_2$ before taking the limit as z_2 tends to a_2. It was shown in

D(5.3) that

$$q(z_2,a_2)^2 w'_{a_2}(z_2) = 1 + O(z_2 - a_2)^4,$$

so the derivative of this term with respect to z_2 tends to 0 as z_2 tends to a_2. It is clear that

$$q(z_2,a_2)^2 w'_{a_1,a_2}(z_2) = -(z_2 - a_2) + O(z_2 - a_2)^2,$$

so the derivative of this term with respect to z_2 tends to -1 as z_2 tends to a_2. Finally

$$q(z_2,a_2)^2 w'_j(z_2) = O(z_2 - a_2)^2,$$

so the derivative of this term with respect to z_2 tends to 0 as z_2 tends to a_2. Thus the only nontrivial terms that can arise on the right-hand side are those involving the function $w'_{a_1,a_2}(z_2) = u'_1(z_2)$. On the left-hand side the variable z_2 appears both in the theta function and in some of the prime function factors. As far as the latter are concerned note that

$$\frac{\partial}{\partial z_2}\frac{q(z_1,z_2)^2}{q(a_1,z_2)^2} = \frac{q(z_1,z_2)^2}{q(a_1,z_2)^2}\frac{\partial}{\partial z_2}\log\frac{q(z_1,z_2)^2}{q(a_1,z_2)^2}$$

$$= 2\,\frac{q(z_1,z_2)^2}{q(a_1,z_2)^2}\, w'_{z_1,a_1}(z_2)$$

by Theorem B10. Altogether it is readily verified that the result is

(1)

$$\sum_j \partial_j \vec{\theta}_2(w(z_1 - a_1)) w'_j(a_2) q(z_1,a_1)^{-2} + 2\vec{\theta}_2(w(z_1 - a_1)) q(z_1,a_1)^{-2} w'_{z_1,a_1}(a_2)$$

$$= -\vec{\alpha}_{11}^{0;2;0} w'_{a_1,a_2}(z_1) - \sum_j \vec{\alpha}_j^{0;1;1} w'_j(z_1),$$

since $\vec{\alpha}_{\mu 1}^{1;1;0} = 0$ as has already been shown. Now multiply this formula by $q(z_1, a_2)$ and take the limit as z_1 tends to a_2, and observe that

$$-2\vec{\theta}_2(w(a_2 - a_1))q(a_2, a_1)^{-2} = \vec{\alpha}_{11}^{0;2;0}(a_1, a_2) \ .$$

From the primary expansion formula (3.1) it then follows that

$$\vec{\alpha}_{11}^{0;2;0}(a_1, a_2) = -2\vec{\theta}_2(0)w'_{a_1}(a_2) - \sum_{jk} \partial_{jk}\vec{\theta}_2(0)w'_j(a_1)w'_k(a_2),$$

thereby determining this coefficient. These results when substituted into (1.7) give the desired explicit expansion.

Finally from the general observations of Theorem 1 it follows that the coefficients $\vec{\alpha}_j^{0;1;1}(a_1, a_2)$ and $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ are meromorphic functions on $\widetilde{M} \times \widetilde{M}$ with singularities at most along the subvarieties $a_1 = Ta_2$ for $T \in \Gamma$. To be somewhat more precise, note that by using the explicit form just obtained for the coefficients $\vec{\alpha}_{11}^{0;2;0}(a_1, a_2)$ the expansion (1) can be rewritten

(2)
$$\sum_j \vec{\alpha}_j^{0;1;1}(a_1, a_2)w'_j(z_1)q(z_1, a_1)^2 q(z_1, a_2)^2$$

$$= 2\vec{\theta}_2(w(a_1 - a_2))w'_{a_1, a_2}(z_1)q(a_1, a_2)^{-2}q(z_1, a_1)^2 q(z_1, a_2)^2$$

$$- 2\vec{\theta}_2(w(z_1 - a_1))w'_{z_1, a_1}(a_2)q(z_1, a_2)^2$$

$$- \sum_j \partial_j\vec{\theta}_2(w(z_1 - a_1))w'_j(a_2)q(z_1, a_2)^2.$$

The right-hand side here is evidently a meromorphic function of $(z_1, a_1, a_2) \in \widetilde{M}^3$ with singularities at most double poles along the subvarieties $a_1 = Ta_2$, and actually at most a simple pole along the subvariety $a_1 = a_2$ since the function $w'_{a_1,a_2}(z_1)$ vanishes along that subvariety; since the functions $w'_j(z_1)q(z_1,a_1)^2 q(z_1,a_2)^2$ of z_1 are linearly independent the coefficients $\vec{\alpha}_j^{0;1;1}$ must consequently have at most the same singularities, as asserted. Then in the expansion formula of the present theorem all the terms appearing except for those involving the functions $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ are meromorphic functions of $(z_1, z_2, a_1, a_2) \in \widetilde{M}^4$ with singularities at most double poles along the subvarieties $a_1 = Ta_2$, $z_i = Ta_j$, and since the functions $w'_j(z_1)w'_k(z_2)$ are linearly independent it follows that $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ have singularities at most double poles along the subvarieties $a_1 = Ta_2$; actually all the singularities along the subvariety $a_1 = a_2$ are cancelled by the zeros of the functions $w'_{a_1,a_2}(z_j)$ along that subvariety, so that $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ are even holomorphic along that subvariety as asserted. Moreover the left-hand side and the first two terms on the right-hand side of this expansion formula are symmetric in the variables a_1 and a_2, while the coefficient of $\vec{\alpha}_j^{0;1;1}(a_1, a_2)$ is skew-symmetric in these variables and the coefficient of $\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ is symmetric; since these functions are uniquely determined by the expansion formula they have the asserted symmetries, and the proof is thereby concluded.

One of the auxiliary results obtained in the course of the proof of the preceding theorem is worth restating explicitly here as follows.

COROLLARY 1. *For any points* $z, a, b, \in \widetilde{M}$

$$
\begin{aligned}
& q(z,a)^{-2}\sum_j \partial_j\vec{\theta}_2(w(z-a))w'_j(b) \\
& = 2\vec{\theta}_2(0)\Big[w'_a(b)w'_{a,b}(z) + w'_a(z)w'_{a,z}(b)\Big] \\
& + \sum_{jk}\partial_{jk}\vec{\theta}_2(0)\Big[w'_{a,b}(z)w'_j(a)w'_k(b) + w'_{a,z}(b)w'_j(a)w'_k(z)\Big] \\
& - \sum_j \vec{\alpha}_j^{0;1;1}(a,b)w'_j(z),
\end{aligned}
$$

where $\vec{\alpha}_j^{0;1;1}(a,b)$ *is the function described in the theorem.*

Proof. This follows immediately from (2) in the proof of the theorem, upon using the expansion (3.1) and making the obvious change of notation.

Another useful auxiliary result that follows from the formula of the theorem itself is a set of functional equations that the coefficients in that formula must satisfy.

COROLLARY 2. *For any transformation $T \in \Gamma$*

$$\vec{\alpha}_j^{0;1;1}(Ta_1, a_2)\kappa(T, a_1)^{-1} - \vec{\alpha}_j^{0;1;1}(a_1, a_2)$$

$$= \vec{\theta}_2(0)4\pi i\beta_j(T)w'_{a_1}(a_2)$$

$$+ \sum_{\ell m} \partial_{\ell m}\vec{\theta}_2(0)2\pi i\beta_j(T)w'_\ell(a_1)w'_m(a_2),$$

$$\vec{\alpha}_{jk}^{0;0;2}(Ta_1, a_2)\kappa(T, a_1)^{-1} - \vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$$

$$= -\vec{\theta}_2(0)2(2\pi i)^2\beta_j(T)\beta_k(T)w'_{a_1}(a_2)$$

$$- \sum_{\ell m} \partial_{\ell m}\vec{\theta}_2(0)(2\pi i)^2\beta_j(T)\beta_k(T)w'_\ell(a_1)w'_m(a_2)$$

$$- 2\pi i\{\vec{\alpha}_j^{0;1;1}(a_1, a_2)\beta_k(T) + \vec{\alpha}_k^{0;1;1}(a_1, a_2)\beta_j(T)\}.$$

where $\kappa(T, a_1)$ is the canonical factor of automorphy.

Proof. In the formula of Theorem 6 replace the variable a_1 by Ta_1 throughout. On the one hand the left-hand side transforms as an Abelian differential in a_1, so the overall effect is just to multiply the entire formula by the canonical factor of automorphy $\kappa(T, a_1)$. On the other hand it is easy to determine the effect of this substitution on the individual terms on the right-hand side; $w'_j(a_1)$ and $w'_{a_1}(z) = w'_z(a_1)$ are themselves Abelian differentials, so are multiplied by $\kappa(T, a_1)$, while by Theorem B11 the differential $w'_{a_1,a_2}(z) = w_z(a_1) - w_z(a_2)$ satisfies

$$w'_{Ta_1,a_2}(z) = w'_{a_1,a_2}(z) + 2\pi i \sum_\ell \beta_\ell(T)w'_\ell(z).$$

Upon comparing the results of these two approaches on the right-hand side of the formula and cancelling the common terms it follows in a straightforward manner that

$$\kappa(T,a_1)\sum_j \vec{\alpha}_j^{0;1;1}(a_1,a_2)\Big\{w'_{a_1,a_2}(z_1)w'_j(z_2)+w'_{a_1,a_2}(z_2)w'_j(z_1)\Big\}$$

$$+\kappa(T,a_1)\sum_{jk}\vec{\alpha}_{jk}^{0;0;2}(a_1,a_2)w'_j(z_1)w'_k(z_2)$$

$$=-\vec{\theta}_2(0)\kappa(T,a_1)2w'_{a_1}(a_2)\Big\{2\pi i\sum_\ell \beta_\ell(T)w'_\ell(z_1)w'_{a_1,a_2}(z_2)$$

$$+2\pi i\sum_\ell \beta_\ell(T)w'_\ell(z_2)w'_{a_1,a_2}(z_1)+(2\pi i)^2\sum_{\ell m}\beta_\ell(T)\beta_m(T)w'_\ell(z_1)w'_m(z_2)\Big\}$$

$$-\sum_{jk}\partial_{jk}\vec{\theta}_2(0)\kappa(T,a_1)w'_j(a_1)w'_k(a_2)\Big\{2\pi i\sum_\ell \beta_\ell(T)w'_\ell(z_1)w'_{a_1,a_2}(z_2)$$

$$+2\pi i\sum_\ell \beta_\ell(T)w'_\ell(z_2)w'_{a_1,a_2}(z_1)+(2\pi i)^2\sum_{\ell m}\beta_\ell(T)\beta_m(T)w'_\ell(z_1)w'_m(z_2)\Big\}$$

$$+\sum_j \vec{\alpha}_j^{0;1;1}(Ta_1,a_2)\Big\{\big[w'_{a_1,a_2}(z_1)+2\pi i\sum_\ell \beta_\ell(T)w'_\ell(z_1)\big]w'_j(z_2)$$

$$+\big[w'_{a_1,a_2}(z_2)+2\pi i\sum_\ell \beta_\ell(T)w'_\ell(z_2)\big]w'_j(z_1)\Big\}$$

$$+\sum_{jk}\vec{\alpha}_{jk}^{0;0;2}(Ta_1,a_2)w'_j(z_1)w'_k(z_2).$$

Now this last formula is an identity among Abelian differentials in the variables z_1, z_2, involving meromorphic Abelian differentials $w'_{a_1,a_2}(z_i)$ and holomorphic Abelian differentials $w'_j(z_i)$, and since these differentials are linearly independent it amounts to an identity among the coefficients of these differentials. In particular, in view of the symmetry in the variables z_1 and z_2, the coefficients of $w'_{a_1,a_2}(z_1)w'_j(z_2) + w'_{a_1,a_2}(z_2)w'_j(z_1)$ must be the same on the two sides of the equation, and similarly for the coefficients of $w'_j(z_1)w'_k(z_2)+w'_k(z_1)w'_j(z_2)$. It is then rather simple to verify that these two identities are precisely those of the statement of the lemma, thereby concluding the proof.

Upon applying a linear projection operator P for which

$$P\vec{\theta}_2(0) = P\partial_{jk}\vec{\theta}_2(0) = 0$$

the function $P\vec{\alpha}_j^{0;1;1}(a_1, a_2)$ takes the simple form described in Theorem 5, while upon applying a linear projection P for which

$$P\vec{\theta}_2(0) = P\partial_{jk}\vec{\theta}_2(0) = P\vec{\alpha}_j^{0;1;1}(a_1, a_2) = 0$$

the function $P\vec{\alpha}_{jk}^{0;0;2}(a_1, a_2)$ takes the simple form described in the Corollary to Theorem 5; these forms are just those of the dominant terms in the differential expansions. In both cases it is not difficult to obtain quite explicit forms for the next terms beyond the dominant terms, but these are somewhat more complicated forms and characteristically involve the quadratic period functions discussed in Section B8.

The coefficients $\vec{\alpha}_j^{0;1;1}$ and $\vec{\alpha}_{j_1 j_2}^{0;0;2}$ in the general binary expansion formula of Theorem 6 have not yet been described very explicitly, although it is evident from Theorem 5 and its corollary that they can be written

(3)

$$\vec{\alpha}_j^{0;1;1}(a_1,a_2) = \sum_k \vec{\xi}^{k_1 k_2}_{\ \ j} w'_{k_1}(a_1) w'_{k_2}(a_2) + \sum_k \partial_{k_1 k_2} \vec{\theta}_2(0) f_j^{k_1 k_2}(a_1,a_2) + \vec{\theta}_2(0) f_j(a_1,a_2) \ ,$$

(4)

$$\vec{\alpha}_{j_1 j_2}^{0;0;2}(a_1,a_2) = \sum_k \vec{\xi}^{k_1 k_2}_{j_1 j_2} w'_{k_1}(a_1) w'_{k_2}(a_2) + \sum_k \vec{\xi}^{k_2 k_3}_{\ \ k_1} f^{k_1 k_2 k_3}_{j_1 j_2}(a_1,a_2) + \sum_k \partial_{k_1 k_2} \vec{\theta}_2(0) f^{k_1 k_2}_{j_1 j_2}(a_1,a_2) + \vec{\theta}_2(0) f_{j_1 j_2}(a_1,a_2) \ ,$$

where $\vec{\xi}^{k_1 k_2}_{\ \ j}$ and $\vec{\xi}^{k_1 k_2}_{j_1 j_2}$ are some representatives in $\mathbb{C}^{2^g}$ for the corresponding dominant terms and $f(a_1,a_2)$ are some meromorphic functions on $\widetilde{M}^2$. The vectors $\vec{\xi}^{k_1 k_2}_{\ \ j}$ are uniquely determined modulo the subspace L_1 spanned by $\vec{\theta}_2(0)$ and $\partial_{jk}\vec{\theta}_2(0)$, while the vectors $\vec{\xi}^{k_1 k_2}_{j_1 j_2}$ are uniquely determined modulo the subspace $L_2{}^*$ spanned by L_1 and $\vec{\xi}^{k_1 k_2}_{\ \ j}$; it is clearly possible to choose these vectors to have the symmetry properties as in Theorem 5 and its corollary. Once the vectors $\vec{\xi}^{k_1 k_2}_{\ \ j}$ have been chosen the coefficients $f_j^{k_1 k_2}(a_1,a_2)$ and $f_j(a_1,a_2)$ in (3) are uniquely determined, under the assumption that the former coefficients are symmetric in the indices k_1, k_2, since the vectors $\partial_{k_1 k_2}\vec{\theta}_2(0)$ and $\vec{\theta}(0)$ are linearly independent modulo this symmetry.

However after these vectors and the vectors $\vec{\xi}^{k_1k_2}_{j_1j_2}$ have been chosen the coefficients in (4) may not be uniquely determined, even supposing that they have the obvious symmetries in their indices, since the vectors $\vec{\xi}^{k_2k_3}_{k_1}$, $\partial_{k_1k_2}\vec{\theta}_2(0)$,$\vec{\theta}_2(0)$ may not be linearly independent modulo their symmetries. Of course some maximal subset of the vectors $\vec{\xi}^{k_2k_3}_{k_1}$ together with $\partial_{k_1k_2}\vec{\theta}_2(0)$ and $\vec{\theta}_2(0)$ will be linearly independent modulo their symmetries, and if it is assumed that the coefficients $f^{k_1k_2k_3}_{j_1j_2}$ corresponding to all the other of these vectors are identically zero then all the coefficients in (4) are uniquely determined by these choices. At any rate it is clear that all the coefficients f can be taken to be meromorphic functions on $\widetilde{M}^2$, with the singularities as described in Theorem 6.

The first terms on the right-hand side of (3) and (4) are the dominant terms considered in the preceding sections, and the remaining terms can conveniently be described as the subdominant terms. It is possible to say something further about these remaining terms, and at the same time to be rather more explicit about the choices of the vectors $\vec{\xi}^{k_1k_2}_{j}$, $\vec{\xi}^{k_1k_2}_{j_1j_2}$ representing the dominant terms, obviously intertwined considerations. Again the first of the subdominant terms, the principal subdominant terms as it were, are the easiest to handle, and only these will really be considered here. In addition it is most reasonable and convenient just to consider these principal subdominant terms modulo the remaining subdominant terms. For (4) the result is as follows.

THEOREM 7. *If P is any projection operator such that*

$$P\vec{\theta}_2(0) = P\partial_{jk}\vec{\theta}_2(0) = 0$$

for all indices $1 \leqq j,k \leqq g$ and $\vec{\xi}^{k_1 k_2}_{\ \ j}$ are the vectors described in Theorem 5 then there are uniquely determined vectors $\vec{\xi}^{k_1 k_2}_{j_1 j_2}$ in the range of P such that

$$\vec{\xi}^{k_1 k_2}_{j_1 j_2} = \vec{\xi}^{k_1 k_2}_{j_2 j_1} = \vec{\xi}^{k_2 k_1}_{j_1 j_2} = \vec{\xi}^{j_1 j_2}_{k_1 k_2}$$

and

$$P\vec{\alpha}^{0;0;2}_{j_1 j_2}(a_1, a_2) = \sum_{k_1 k_2} \vec{\xi}^{k_1 k_2}_{j_1 j_2} w'_{k_1}(a_1) w'_{k_2}(a_2)$$

$$+ \sum_{\pi,\rho \in \mathfrak{S}_2} \sum_{k_1 k_2} \vec{\xi}^{k_2 k_3}_{\ \ k_1} \Big[w'_{k_1}(a_{\rho 1}) \varphi^{j_{\pi 1}}_{k_2}(a_{\rho 2}) \delta^{j_{\pi 2}}_{k_3}$$

$$- \pi i w'_{k_1}(a_{\rho 1}) w'_{k_2}(a_{\rho 2}) \delta^{j_{\pi 1}}_{k_2} \delta^{j_{\pi 2}}_{k_3} \Big]$$

in terms of the quadratic period functions $\varphi^j_k(a)$.

Proof. Apply the projection operator P to the second formula of Corollary 2 to Theorem 6, note that by Theorem 5

$$P\vec{\alpha}^{0;1;1}_{j}(a_1, a_2) = \sum_{\ell_1 \ell_2} \vec{\xi}^{\ell_1 \ell_2}_{\ \ j} w'_{\ell_1}(a_1) w'_{\ell_2}(a_2),$$

and use this and the Corollary to Theorem B12 to rewrite the result in terms of the quadratic period functions in the form

$$P\vec{\alpha}_{jk}^{0;0;2}(Ta_1,a_2)\kappa(T,a_1)^{-1} - P\vec{\alpha}_{jk}^{0;0;2}(a_1,a_2)$$

$$= -2\pi i \sum_{\ell_1\ell_2} \Big[\vec{\xi}^{\ell_1\ell_2}_{\;j}\beta_k(T) + \vec{\xi}^{\ell_1\ell_2}_{\;k}\beta \;\; {}_k(T) \Big] w'_{\ell_1}(a_1)w'_{\ell_2}(a_2)$$

$$= -\sum_{\ell_1\ell_2} \vec{\xi}^{\ell_1\ell_2}_{\;j} w'_{\ell_2}(a_2)\Big[\varphi^k_{\ell_1}(Ta_1;a_2)\kappa(T,a_1)^{-1} - \varphi^k_{\ell_1}(a_1;a_2) \Big]$$

$$-\sum_{\ell_1\ell_2} \vec{\xi}^{\ell_1\ell_2}_{\;k} w'_{\ell_2}(a_2)\Big[\varphi^j_{\ell_1}(Ta_1;a_2)\kappa(T,a_1)^{-1} - \varphi^j_{\ell_1}(a_1;a_2) \Big].$$

This last identity really amounts to the assertion that the expression

$$\vec{f}_{jk}(a_1,a_2) = P\vec{\alpha}_{jk}^{0;0;2}(a_1,a_2) + \sum_{\ell_1\ell_2} \Big[\vec{\xi}^{\ell_1\ell_2}_{\;j}\varphi^k_{\ell_1}(a_1;a_2) + \vec{\xi}^{\ell_1\ell_2}_{\;k}\varphi^j_{\ell_1}(a_1;a_2) \Big] w'_{\ell_2}(a_2)$$

$$= P\vec{\alpha}_{jk}^{0;0;2}(a_1,a_2) + \sum_{\ell_1\ell_2} \Big[\vec{\xi}^{\ell_1\ell_2}_{\;j}\varphi^k_{\ell_1}(a_1) + \vec{\xi}^{\ell_1\ell_2}_{\;k}\varphi^j_{\ell_1}(a_1) \Big] w'_{\ell_2}(a_2)$$

transforms as a differential form on M in the variable $a_1 \in \widetilde{M}$. It is evident from Theorem 6 that this meromorphic differential form has as singularities at most a simple pole at the point of M represented by a_2, so since the total residue is zero it is actually a holomorphic differential form. The expression

$$\vec{g}_{jk}(a_1,a_2) = \vec{f}_{jk}(a_1,a_2) + \sum_{\ell_1\ell_2} \Big[\vec{\xi}^{\ell_1\ell_2}_{\;j}\varphi^k_{\ell_1}(a_2) + \vec{\xi}^{\ell_1\ell_2}_{\;k}\varphi^j_{\ell_1}(a_2) \Big] w'_{\ell_2}(a_1)$$

differs from $\vec{f}_{jk}(a_1,a_2)$ by a holomorphic differential form in a_1 so it too must be a holomorphic differential form in a_1. On the other hand since $\vec{\alpha}_{jk}^{0;0;2}(a_1,a_2)$ is symmetric in the variables a_1,a_2 by Theorem 6 it is clear that $\vec{g}_{jk}(a_1,a_2)$ is also symmetric in these variables, hence must be a holomorphic differential form in the variable a_2 as well. There are consequently some uniquely determined vectors $\vec{\eta}_{jk}^{\ell m}$ such that

$$\vec{g}_{jk}(a_1,a_2) = \sum_{\ell m} \vec{\eta}_{jk}^{\ell m} w'_\ell(a_1) w'_m(a_2),$$

and from the observed symmetries it is apparent that these vectors are symmetric in the indices j,k and also in the indices ℓ,m. In terms of these vectors

(5)
$$\begin{aligned} P\vec{\alpha}_{jk}^{0;0;2}(a_1,a_2) &= \sum_{\ell_1\ell_2} \vec{\xi}^{\ell_1\ell_2}_{j} \Big[w'_{\ell_1}(a_1)\varphi^k_{\ell_2}(a_2) + w'_{\ell_1}(a_2)\varphi^k_{\ell_2}(a_1) \Big] \\ &+ \sum_{\ell_1\ell_2} \vec{\xi}^{\ell_1\ell_2}_{k} \Big[w'_{\ell_1}(a_1)\varphi^j_{\ell_2}(a_2) + w'_{\ell_1}(a_2)\varphi^j_{\ell_2}(a_1) \Big] \\ &+ \sum_{\ell m} \vec{\eta}_{jk}^{\ell m} w'_\ell(a_1) w'_m(a_2), \end{aligned}$$

which is much as desired except that the vectors $\vec{\eta}_{jk}^{\ell m}$ do not quite have all the symmetries desired.

To handle the further symmetries observe that substituting (5) into the formula of Theorem 6 yields

(6)

$$P\vec{\theta}_2(w(z_1+z_2-a_1-a_2))q(z_1,z_2)^2q(a_1,a_2)^2\Big[\prod_{\mu,\nu=1}^{2} q(z_\mu,a_\nu)^{-2}\Big]$$

$$= \sum_{j\ell_1\ell_2} \vec{\xi}^{\,\ell_1\ell_2}_{j} w'_{\ell_1}(a_1)w'_{\ell_2}(a_2)\Big[\, w'_{a_1,a_2}(z_1)w'_j(z_2) + w'_{a_1,a_2}(z_2)w'_j(z_1)\,\Big]$$

$$+ \sum_{jk\ell_1\ell_2} \vec{\xi}^{\,\ell_1\ell_2}_{j}\Big[\, w'_{\ell_1}(a_1)\varphi^k_{\ell_2}(a_2) + w'_{\ell_1}(a_2)\varphi^k_{\ell_2}(a_1)\,\Big]$$

$$\Big[\, w'_j(z_1)w'_k(z_2) + w'_j(z_2)w'_k(z_1)\,\Big]$$

$$+ \sum_{jk\ell m} \vec{\eta}^{\,\ell m}_{jk} w'_\ell(a_1)w'_m(a_2)w'_j(z_1)w'_k(z_2).$$

In view of the obvious symmetries on the left-hand side, subtracting from (6) the same formula but with the z's and a's interchanged readily yields the result that

(7)

$$\sum_{jk\ell m} \left(\vec{\eta}^{jk}_{\ell m} - \vec{\eta}^{\ell m}_{jk} \right) w'_\ell(a_1) w'_m(a_2) w'_j(z_1) w'_k(z_2)$$

$$= \sum_{jk\ell} \vec{\xi}^{k\ell}_{j} \Big\{ w'_k(a_1) w'_\ell(a_2) \left[w'_{a_1,a_2}(z_1) w'_j(z_2) + w'_{a_1,a_2}(z_2) w'_j(z_1) \right]$$

$$- w'_k(z_1) w'_\ell(z_2) \left[w'_{z_1,z_2}(a_1) w'_j(a_2) + w'_{z_1,z_2}(a_2) w'_j(a_1) \right]$$

$$+ \sum_m \left[w'_k(a_1) \varphi^m_\ell(a_2) + w'_k(a_2) \varphi^m_\ell(a_1) \right] \left[w'_j(z_1) w'_m(z_2) + w'_j(z_2) w'_m(z_1) \right]$$

$$- \sum_m \left[w'_k(z_1) \varphi^m_\ell(z_2) + w'_k(z_2) \varphi^m_\ell(z_1) \right] \left[w'_j(a_1) w'_m(a_2) + w'_j(a_2) w'_m(a_1) \right] \Big\}.$$

The right-hand side of (7) can be rewritten as the sum

(8) $$\sum_{j'<k'<\ell'} \vec{\xi}^{k'\ell'}_{j'} \sum_{jk\ell} \delta^{j'k'\ell'}_{j\,k\,\ell} f_{jk\ell}(z_1, z_2; a_1, a_2),$$

where $\delta^{j'k'\ell'}_{j\,k\,\ell}$ is zero unless (j, k, ℓ) is a permutation of the indices (j', k', ℓ') and is then the sign of the permutation , and $f_{jk\ell}(z_1, z_2; a_1, a_2)$ is the entire expression in braces. Formally $f_{jk\ell}(z_1, z_2; a_1, a_2)$ is a meromorphic function on $\widetilde{M}^4$, with singularities at most simple poles along the subvarieties $z_j = Ta_k$ for $T \in \Gamma$, and is symmetric in the variables z_1, z_2, in the variables a_1, a_2, and under the interchange of the z's and the a's. It follows from (7) that the double sum (8) is really a holomorphic differential form in each variable; it is actually the case that the inner sum in (8) is also a holomorphic differential form in each variable, a result that does not at all involve the auxiliary vectors $\vec{\xi}^{k\ell}_{j}$. Indeed it follows readily from Theorem B11 and its Corollary together with Theorem B12 that

$$f_{jk\ell}(Tz_1, z_2; a_1, a_2)\,\kappa(T, z_1)^{-1} - f_{jk\ell}(z_1, z_2; a_1, a_2)$$

$$= -2\pi i \sum_m \beta_m(T)\Big[w'_j(a_1)w'_m(a_2) + w'_j(a_2)w'_m(a_1)\Big]\bullet$$

$$\bullet\Big[w'_k(z_1)w'_\ell(z_2) + w'_k(z_2)w'_\ell(z_1)\Big]$$

$$+\frac{2\pi i}{g}\Big[\sum_m \beta_m(T)w'_m(z_1)w'_k(z_2)\Big]\Big[w'_j(a_1)w'_\ell(a_2) + w'_j(a_2)w'_\ell(a_1)\Big];$$

here the first line on the right-hand side is symmetric in the indices k and ℓ while the second line is symmetric in the indices j and ℓ, so multiplying by $\delta^{j'k'\ell'}_{j\,k\,\ell}$ and summing over the indices j,k,ℓ yields zero. This shows that the sum transforms as a differential form on M in the variable z_1, and in view of the symmetries it has the same property in the other variables. Next note that

$$\lim_{z_1\to a_1} q(z_1, a_1)f_{jk\ell}(z_1, z_2; a_1, a_2) = w'_k(a_1)w'_\ell(a_2)w'_j(z_2)+w'_k(a_1)w'_j(a_2)w'_\ell(z_2);$$

multiplying this by $\delta^{j'k'\ell'}_{j\,k\,\ell}$ and summing over the indices j,k,ℓ again yields zero, since in the two terms on the right-hand side the indices j, k, ℓ appear with permutations of opposite parity. The sum as a meromorphic differential form on M in the variable z_1 is therefore regular at a_1, and since its total residue is zero it must also be regular at the only possible remaining pole a_2; it is thus a holomorphic differential form in z_1, and in view of the symmetries it is the same in the other variables. Therefore altogether for any indices $j' < k' < \ell'$

$$\text{(9)} \qquad \sum_{jk\ell} \delta^{j'k'\ell'}_{j\ k\ \ell} f_{jk\ell}(z_1, z_2; a_1, a_2)$$

$$= \sum_{m_1 m_2 n_1 n_2} c^{j'k'\ell'}_{m_1 m_2; n_1 n_2} w'_{m_1}(z_1) w'_{m_2}(z_2) w'_{n_1}(a_1) w'_{n_2}(a_2)$$

for some uniquely determined constants c; from the symmetries of the functions $f_{jk\ell}(z_1, z_2; a_1, a_2)$ it is evident that in their lower indices these coefficients have the symmetries

$$c_{m_1 m_2; n_1 n_2} = c_{m_2 m_1; n_1 n_2} = c_{m_1 m_2; n_2 n_1} = -c_{n_1 n_2; m_1 m_2}.$$

To calculate these coefficients note first from the period normalization of the Abelian differentials that integrating (9) yields

$$\text{(10)} \qquad \int_{z_1 \in \alpha_{m_1}} \int_{z_2 \in \alpha_{m_2}} \sum_{jk\ell} \delta^{j'k'\ell'}_{j\ k\ \ell} f_{jk\ell}(z_1, z_2; a_1, a_2) dz_1 dz_2$$

$$= \sum_{n_1 n_2} c^{j'k'\ell'}_{m_1 m_2; n_1 n_2} w'_{n_1}(a_1) w'_{n_2}(a_2).$$

If a_1, a_2 are distinct points in the fundamental polygon $\Delta \subset \widetilde{M}$ then from the period relations of Theorems B11 and B12 together with the defintion of the quadratic period functions it follows that

$$\int_{z_1\in\alpha_{m_1}}\int_{z_2\in\alpha_{m_2}} f_{jk\ell}(z_1,z_2;a_1,a_2)dz_1dz_2$$

$$= -\int_{z_1\in\alpha_{m_1}}\int_{z_2\in\alpha_{m_2}} \Big\{w'_k(z_1)w'_\ell(z_2)\bullet$$

$$\bullet\Big[w'_j(a_2)\big(w_{a_1}(z_1)-w_{a_1}(z_2)\big)+w'_j(a_1)\big(w_{a_2}(z_1)-w_{a_2}(z_2)\big)\Big]\Big\}$$

$$+\sum_m \big(w'_k(a_1)\varphi^m_\ell(a_2)+w'_k(a_2)\varphi^m_\ell(a_1)\big(\delta^{m_1}_j\delta^{m_2}_m+\delta^{m_2}_j\delta^{m_1}_m\big)$$

$$-\sum_m \big(w'_j(a_1)w'_m(a_2)+w'_j(a_2)w'_m(a_1)\big)\bullet$$

$$\bullet\int_{z_1\in\alpha_{m_1}}\int_{z_2\in\alpha_{m_2}}\big(\omega_k(z_1)\Phi^m_\ell(z_2)+\omega_k(z_2)\Phi^m_\ell(z_1)\big)$$

$$= -w'_j(a_2)\varphi^{m_1}_k(a_1)\delta^{m_2}_\ell + w'_j(a_2)\varphi^{m_2}_\ell(a_1)\delta^{m_1}_k$$

$$- w'_j(a_1)\varphi^{m_1}_k(a_2)\delta^{m_2}_\ell + w'_j(a_1)\varphi^{m_2}_\ell(a_2)\delta^{m_1}_k$$

$$+ w'_k(a_1)\varphi^{m_2}_\ell(a_2)\delta^{m_1}_j + w'_k(a_1)\varphi^{m_1}_\ell(a_2)\delta^{m_2}_j$$

$$+ w'_k(a_2)\varphi^{m_2}_\ell(a_1)\delta^{m_1}_j + w'_k(a_2)\varphi^{m_1}_\ell(a_1)\delta^{m_2}_j$$

$$-2\pi i\sum_m \big(w'_j(a_1)w'_m(a_2)+w'_j(a_2)w'_m(a_1)\big)$$

$$\big(\delta^{m_1}_k\delta^{m_2}_\ell\delta^{m_2}_m+\delta^{m_2}_k\delta^{m_1}_\ell\delta^{m_1}_m\big).$$

Upon multiplying this identity by $\delta^{j'k'\ell'}_{j\,k\,\ell}$ and summing over the indices j,k,ℓ

it follows readily that the terms in the first two lines on the right-hand side cancel in pairs; for instance the first and last terms are the same except for sign and the order of the lower indices, which differ by an even permutation, so these two terms cancel, and similarly for the others. Thus (10) becomes

$$\sum_{n_1 n_2} c^{j'k'\ell'}_{m_1 m_2;n_1 n_2} w'_{n_1}(a_1) w'_{n_2}(a_2)$$

$$= -2\pi i \sum_{jk\ell m} \left(w'_j(a_1) w'_m(a_2) + w'_j(a_2) w'_m(a_1)\right) \bullet$$

$$\bullet\, \delta^{j'k'\ell'}_{j\,k\,\ell} \left(\delta^{m_1}_k \delta^{m_2}_\ell \delta^{m_2}_m + \delta^{m_2}_k \delta^{m_1}_\ell \delta^{m_1}_m\right)$$

$$= -2\pi i \sum_j \Big[\delta^{j'k'\ell'}_{j m_1 m_2} \left(w'_j(a_1) w'_{m_2}(a_2) + w'_j(a_2) w'_{m_2}(a_1)\right)$$

$$+ \delta^{j'k'\ell'}_{j m_2 m_1} \left(w'_j(a_1) w'_{m_1}(a_2) + w'_j(a_2) w'_{m_1}(a_1)\right)\Big],$$

and since it holds for all points $a_1, a_2 \in \Delta$ and the Abelian differentials are linearly independent it follows readily that

$$(11) \qquad c^{jk\ell}_{m_1 m_2;n_1 n_2} = -2\pi i \Big(\delta^{j\ \ k\ \ \ell}_{n_1 m_1 m_2} \delta^{m_2}_{n_2} + \delta^{j\ \ k\ \ \ell}_{n_2 m_1 m_2} \delta^{m_2}_{n_1}$$

$$+ \delta^{j\ \ k\ \ \ell}_{n_1 m_2 m_1} \delta^{m_1}_{n_2} + \delta^{j\ \ k\ \ \ell}_{n_2 m_2 m_1} \delta^{m_1}_{n_1} \Big).$$

This expression is clearly symmetric in the indices m_1, m_2 and in the indices n_1, n_2; moreover it is zero unless $m_i = n_j$ for some particular indices i, j, and if for instance $m_2 = n_2 = p$ then

$$c^{jk\ell}_{mp;np} = 2\pi i \delta^{jk\ell}_{mnp} \ , \tag{12}$$

and this is skew-symmetric in the indices m, n. Then substituting (8), (9), and (11) into (7) shows that

$$\sum_{jk\ell m} \left(\vec{\eta}^{jk}_{\ell m} - \vec{\eta}^{\ell m}_{jk} \right) w'_j(z_1) w'_k(z_2) w'_\ell(a_1) w'_m(a_2)$$

$$= -2\pi i \sum_{\substack{j<k<\ell \\ m_1 m_2 n_1 n_2}} \vec{\xi}^{k\ell}_{j} \Big(\delta^{j\ k\ \ell}_{n_1 m_1 m_2} \delta^{m_2}_{n_2} + \delta^{j\ k\ \ell}_{n_2 m_1 m_2} \delta^{m_2}_{n_1} + \delta^{j\ k\ \ell}_{n_1 m_2 m_1} \delta^{m_1}_{n_2}$$

$$+ \delta^{j\ k\ \ell}_{n_2 m_2 m_1} \delta^{m_1}_{n_1} \Big) w'_{m_1}(z_1) w'_{m_2}(z_2) w'_{n_1}(a_1) w'_{n_2}(a_2),$$

and since the Abelian differentials are linearly independent it follows that

$$\vec{\eta}^{jk}_{\ell m} - \vec{\eta}^{\ell m}_{jk} = -2\pi i \Big(\vec{\xi}^{jk}_{\ell} \delta^{k}_{m} + \vec{\xi}^{jk}_{m} \delta^{k}_{\ell} + \vec{\xi}^{kj}_{\ell} \delta^{j}_{m} + \vec{\xi}^{kj}_{m} \delta^{j}_{\ell} \Big). \tag{13}$$

The right-hand side of (13) vanishes unless at least one of the indices j, k is equal to one of the indices ℓ, m, and if $m = j$ for instance then (13) reduces to the simpler formula

$$\vec{\eta}^{jk}_{j\ell} - \vec{\eta}^{j\ell}_{jk} = 2\pi i \vec{\xi}^{jk}_{\ell}. \tag{14}$$

The right-hand side of (13) is formally symmetric in the indices j, k and also in the indices ℓ, m, and if nonzero then reduces to the right-hand side of (14) so is formally skew-symmetric under interchanging the pairs (j,k) and (ℓ,m). The expression

$$(15) \qquad \vec{\xi}^{jk}_{\ell m} = \vec{\eta}^{jk}_{\ell m} + \pi i\Big(\vec{\xi}^{jk}_{\ \ell}\delta^k_m + \vec{\xi}^{jk}_m\delta^k_\ell + \vec{\xi}^{kj}_{\ \ell}\delta^j_m + \vec{\xi}^{kj}_m\delta^j_\ell\Big)$$

therefore satsifies the desired symmetry conditions, and in terms of these vectors

$$\sum_{\ell m}\vec{\eta}^{\ell m}_{jk}w'_\ell(a_1)w'_m(a_2)$$

$$= \sum_{\ell m}\Big\{\vec{\xi}^{\ell m}_{jk} - \pi i\big[\vec{\xi}^{\ell m}_{\ \ j}\delta^m_k + \vec{\xi}^{\ell m}_{\ \ k}\delta^m_j$$

$$+ \vec{\xi}^{m\ell}_{\ \ j}\delta^\ell_k + \vec{\xi}^{m\ell}_{\ \ k}\delta^\ell_j\big]\Big\}w'_\ell(a_1)w'_m(a_2)$$

$$= \sum_{\ell m}\vec{\xi}^{\ell m}_{jk}w'_\ell(a_1)w'_m(a_2)$$

$$- \pi i\sum_{\ell m}\Big(\vec{\xi}^{\ell m}_{\ \ j}\delta^m_k + \vec{\xi}^{\ell m}_{\ \ k}\delta^m_j\Big)\Big(w'_\ell(a_1)w'_m(a_2) + w'_\ell(a_2)w'_m(a_1)\Big).$$

Substituting this last formula into (5) yields the desired result and concludes the proof.

COROLLARY 1. *With the hypotheses and notation as in the theorem,*

$$P\vec{\theta}_2\big(w(z_1+z_2-a_1-a_2)\big)q(z_1,z_2)^2q(a_1,a_2)^2\left[\prod_{j_1k=1}^{2}q(z_j,a_k)^{-2}\right]$$

$$=\sum_{jk_1k_2}\vec{\xi}^{k_1k_2}_{j}w'_{k_1}(a_1)w'_{k_2}(a_2)\Big[w'_{a_1,a_2}(z_1)w'_j(z_2)+w'_{a_1,a_2}(z_2)w'_j(z_1)\Big]$$

$$+\sum_{jk_1k_2\ell}\vec{\xi}^{k_1k_2}_{j}\Big[w'_{k_1}(a_1)\varphi^{\ell}_{k_2}(a_2)+w'_{k_1}(a_2)\varphi^{\ell}_{k_2}(a_1)\Big]\bullet$$

$$\bullet\Big[w'_j(z_1)w'_\ell(z_2)+w'_j(z_2)w'_\ell(z_1)\Big]$$

$$-\pi i\sum_{jk_1k_2}\vec{\xi}^{k_1k_2}_{j}\Big[w'_{k_1}(a_1)w'_{k_2}(a_2)+w'_{k_1}(a_2)w'_{k_2}(a_1)\Big]\bullet$$

$$\bullet\Big[w'_j(z_1)w'_{k_2}(z_2)+w'_j(z_2)w'_{k_2}(z_1)\Big]$$

$$+\sum_{j_1j_2k_1k_2}\vec{\xi}^{k_1k_2}_{j_1j_2}w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{j_1}(z_1)w'_{j_2}(z_2).$$

Proof. This follows directly upon substituting the formula for $\beta_{jk}(a_1,a_2)=P\vec{\alpha}^{0;0;2}_{jk}(a_1,a_2)$ from Theorem 7 into the expansion of Theorem 5.

Corollary 2. *With the hypotheses and notation as in the theorem,*

$$\sum_{k_1 k_2} \vec{\xi}^{k_1 k_2}_{j_1 j_2} w'_{k_1}(z) w'_{k_2}(z)$$

$$= 2 \sum_{k_1 k_2} \left(\vec{\xi}^{k_1 k_2}_{\ \ j_1} \varphi^{j_2}_{k_1}(z) + \vec{\xi}^{k_1 k_2}_{\ \ j_2} \varphi^{j_1}_{k_1}(z) \right) w'_{k_2}(z)$$

$$+ 2\pi i \sum_{k} \vec{\xi}^{j_1 j_2}_{\ \ k} \left(w'_{j_1}(z) - w'_{j_2}(z)\right) w'_k(z)$$

for all indices j_1, j_2.

Proof. The left-hand side of the formula of Corollary 1 vanishes when $a_1 = a_2$ because of the presence of the factor $q(a_1, a_2)^2$, and the first line on the right-hand side vanishes when $a_1 = a_2$ because of the presence of the factors $w'_{a_1, a_2}(z)$. Thus for $a_1 = a_2 = z$ that formula reduces to

$$0 = 2 \sum \vec{\xi}^{k_1 k_2}_{\ \ j} w'_{k_1}(z) \varphi^{\ell}_{k_2}(z) \Big(w'_j(z_1) w'_\ell(z_2) + w'_j(z_2) w'_\ell(z_1) \Big)$$

$$- 2\pi i \sum \vec{\xi}^{k_1 k_2}_{\ \ j} w'_{k_1}(z) w'_{k_2}(z) \Big(w'_j(z_1) w'_{k_2}(z_2) + w'_j(z_2) w'_{k_2}(z_1) \Big)$$

$$+ \sum \vec{\xi}^{k_1 k_2}_{j_1 j_2} w'_{k_1}(z) w'_{k_2}(z) w'_{j_1}(z_1) w'_{j_2}(z_2).$$

Comparing the coefficients of the linearly independent functions $w'_{j_1}(z_1) w'_{j_2}(z_2)$ in this last formula yields the desired result.

The formula of Corollary 2 is just the more detailed version of (3.6), allowing for nontrivial vectors $\vec{\xi}^{k_1 k_2}_{\ \ j}$. There is of course an alternative version of the formula of this corollary, obtained by setting $z_1 = z_2$ in the formula of Corollary 1; that these two versions are equivalent follows from the symmetry properties that have been established for the vectors $\vec{\xi}^{k_1k_2}_{j_1j_2}$, and the version stated here is the simpler one. There remains the question whether the vectors $\vec{\xi}^{k_1k_2}_{j_1j_2}$ can be modified by further linear combinations of the vectors $\vec{\xi}^{k_1k_2}_{j}$ so that in addition to all the properties already established the stronger version of Corollary 2 amounting to the identity $\sum_k \vec{\xi}^{k_1k_2}_{j_1j_2} w'_{k_1}(z) w'_{k_2}(z) = 0$ actually holds. The right-hand side of the formula of Corollary 2 can be rewritten as the sum

$$\sum_{j'<k'<\ell'} \xi^{k'\ell'}_{j} \Big\{ 2\sum_{k_1k_2} \Big(\delta^{j'k'\ell'}_{j_1k_1k_2} \varphi^{j_2}_{k_1}(z) + \delta^{j'k'\ell'}_{j_2k_1k_2} \varphi^{j_1}_{k_1}(z) \Big) w'_{k_2}(z)$$

$$+ 2\pi i \sum_{k} \delta^{j'k'\ell'}_{j_1j_2k} \Big(w'_{j_1}(z) - w'_{j_2}(z_1) \Big) w'_k(z) \Big\}$$

in terms of the skew-symmetric Kronecker symbols considered earlier. It is a straightforward consequence of Theorem B12 that the expression in braces is a quadratic differential on M, so if M is not hyperelliptic it can be written as a quadratic polynomial in the Abelian differentials. However it appears rather difficult to obtain reasonably explicit expressions, so proceeding further in analogy with the argument in the last part of the proof of the theorem is probably not yet worth the effort.

To turn next to the prinicpal subdominant term in (3), the analogue of Theorem 7 is as follows.

THEOREM 8. *If P is any projection operator such that $P\vec{\theta}_2(0) = 0$ then there are uniquely determined vectors $\vec{\xi}^{ij}_{k}$ in the range of P such that*

$$P\vec{\alpha}_j^{0;1;1}(a_1,a_2) = \sum_{k\ell} P\partial_{k\ell}\vec{\theta}_2(0)\Big[\varphi_k^j(a_1)w'_\ell(a_2) - \varphi_k^j(a_2)w'_\ell(a_1)\Big]$$

$$+ \sum_k P\partial_{jk}\vec{\theta}_2(0)\Big[g_k(a_1,a_2)$$

$$- \pi i\big(w'_j(a_1)w'_k(a_2) - w'_j(a_2)w'_k(a_1)\big)\Big]$$

$$+ \sum_{k\ell} \vec{\xi}^{k\ell}_{j} w'_k(a_1)w'_\ell(a_2)$$

and

$$q(z,a)^{-2}P\partial_j\vec{\theta}_2\big(w(z-a)\big)$$

$$= -\sum_{k\ell} P\partial_{k\ell}\vec{\theta}_2(0)\Big[\varphi_k^j(z)w'_\ell(a) - \varphi_k^j(a)w'_\ell(z)\Big]$$

$$- \sum_k P\partial_{jk}\vec{\theta}_2(0)\Big[g_k(z,a) - \pi i\big(w'_j(z)w'_k(a) - w'_j(a)w'_k(z)\big)\Big]$$

$$- \sum_{k\ell} \vec{\xi}^{k\ell}_{j} w'_k(z)w'_\ell(a).$$

Here the vectors $\vec{\xi}^{ij}_{k}$ are skew-symmetric in all three indices, and

$$g_k(z,a) = \pi i w'_k(z)w'_k(a) + w_a(z)w'_k(z) - \sum_\ell \varphi_k^\ell(z)w'_\ell(a) \tag{16}$$

are meromorphic functions on $\widetilde{M} \times \widetilde{M}$ that are skew-symmetric in the variables z and a.

Proof. Apply the projection operator P to the first formula of Corollary 2 to Theorem 6 and use the Corollary of Theorem B12 to rewrite the result in terms of the quadratic period functions in the form

$$P\vec{\alpha}_j^{0;1;1}(Ta_1,a_2)\kappa(T,a_1)^{-1} - P\vec{\alpha}_j^{0;1;1}(a_1,a_2)$$

$$= \sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)2\pi i\beta_j(T)w'_\ell(a_1)w'_m(a_2)$$

$$= \sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)w'_m(a_2)\Big[\varphi^j_\ell(Ta_1;a_2)\kappa(T,a_1)^{-1} - \varphi^j_\ell(a_1;a_2)\Big].$$

This last identity really amounts to the assertion that the expression

$$\vec{f}_j(a_1,a_2) = P\vec{\alpha}_j^{0;1;1}(a_1,a_2) - \sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)\varphi^j_\ell(a_1;a_2)w'_m(a_2)$$

transforms as a differential form on M in the variable $a_1 \in \widetilde{M}$. It is evident from Theorem 6 and the definition of the functions $\varphi^j_\ell(a_1;a_2)$ that this meromorphic differential form has as singularities at most a simple pole at the point of M represented by a_2, so since the total residue is zero it is actually a holomorphic differential form; therefore

$$\vec{f}_j(a_1;a_2) = \sum_{\ell} \vec{\eta}^{\ell}_j(a_2)w'_\ell(a_1)$$

for some uniquely determined values $\vec{\eta}_j^\ell(a_2)$, which must consequently be holomorphic functions of $a_2 \in \widetilde{M}$. To investigate these functions note that upon replacing a_2 by Ta_2 and multiplying by $\kappa(T,a_2)^{-1}$ it follows from the transformational formulas of Corollary 2 to Theorem 6 and of Theorem B11 that

$$\sum_\ell \vec{\eta}_j^\ell(Ta_2)\kappa(T,a_2)^{-1}w'_\ell(a_1)$$

$$= P\vec{\alpha}_j^{0;1;1}(a_1,a_2) - \sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)2\pi i\beta_j(T)w'_\ell(a_1)w'_m(a_2)$$

$$- \sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)w'_m(a_2)\Big[\varphi_\ell^j(a_1;a_2) - \delta_\ell^j 2\pi i\sum_k \beta_k(T)w'_k(a_1)\Big]$$

$$= \sum_\ell \vec{\eta}_j^\ell(a_2)w'_\ell(a_1)$$

$$+ 2\pi i\sum_{\ell m}\Big[P\partial_{jm}\vec{\theta}_2(0)\beta_\ell(T) - P\partial_{\ell m}\vec{\theta}_2(0)\beta_j(T)\Big]w'_\ell(a_1)w'_m(a_2).$$

The functions $w'_\ell(a_1)$ are linearly independent so their coefficients in the preceding formula must also be equal, and from the Corollary to Theorem B12 again this can be rewritten in the form

$$\vec{\eta}_j^\ell(Ta_2)\kappa(T,a_2)^{-1} - \vec{\eta}_j^\ell(a_2)$$

$$= \sum_m P\partial_{jm}\vec{\theta}_2(0)\Big[\varphi_m^\ell(Ta_2;a_1)\kappa(T,a_2)^{-1} - \varphi_m^\ell(a_2;a_1)\Big]$$

$$- \sum_m P\partial_{\ell m}\vec{\theta}_2(0)\Big[\varphi_m^j(Ta_2;a_1)\kappa(T,a_2)^{-1} - \varphi_m^j(a_2;a_1)\Big].$$

This amounts to the condition that the expression

$$\vec{\eta}_j^{\ell}(a_2) + \sum_m P\partial_{\ell m}\vec{\theta}_2(0)\varphi_m^j(a_2;a_1) - \sum_m P\partial_{jm}\vec{\theta}_2(0)\varphi_m^{\ell}(a_2;a_1)$$

$$= \vec{\eta}_j^{\ell}(a_2) + \sum_m \Big[P\partial_{\ell m}\vec{\theta}_2(0)\varphi_m^j(a_2) - P\partial_{jm}\vec{\theta}_2(0)\varphi_m^{\ell}(a_2)\Big]$$

is a holomorphic differential form on M, so it can be written as some linear combination $\sum_m \vec{\eta}_j^{\ell m} w_m'(a_2)$ of the canonical Abelian differentials. Altogether then

(17)

$$P\vec{\alpha}_j^{0;1;1}(a_1,a_2) = \sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)\Big[\varphi_\ell^j(a_1)w_m'(a_2) - \varphi_\ell^j(a_2)w_m'(a_1)\Big]$$

$$+ \sum_m P\partial_{jm}\vec{\theta}_2(0)\Big[\sum_\ell \varphi_m^\ell(a_2)w_\ell'(a_1) - w_{a_1}(a_2)w_m'(a_2)\Big]$$

$$+ \sum_{\ell m} \vec{\eta}_j^{\ell m} w_\ell'(a_1)w_m'(a_2)$$

for some uniquely determined vectors $\vec{\eta}_j^{\ell m}$. This already gives much of the first assertion of the present theorem, although the vector $\vec{\eta}_j^{\ell m}$ is not necessarily skew-symmetric in its indices.

To introduce the further symmetries desired, note that the function $\vec{\alpha}_j^{0;1;1}(a_1,a_2)$ is skew-symmetric in the variables a_1 and a_2 so

$$0 = P\vec{\alpha}_j^{0;1;1}(a_1,a_2) + P\vec{\alpha}_j^{0;1;1}(a_2,a_1)$$

$$= \sum_m P\partial_{jm}\vec{\theta}_2(0)f_m(a_1,a_2) + \sum_{\ell m}\left(\vec{\eta}_j^{\ell m} + \vec{\eta}_j^{m\ell}\right)w_\ell'(a_1)w_m'(a_2)$$

where

$$f_m(a_1, a_2) = \sum_{\ell} \varphi_m^{\ell}(a_1) w'_{\ell}(a_2) + \sum_{\ell} \varphi_m^{\ell}(a_2) w'_{\ell}(a_1)$$
$$- w_{a_1}(a_2) w'_m(a_2) - w_{a_2}(a_1) w'_m(a_1)$$

is a meromorphic function on $\widetilde{M} \times \widetilde{M}$ with at most simple poles along the subvarieties $a_1 = Ta_2$ as singularities and is symmetric in the variables a_1 and a_2. It is a straightforward matter to verify from the functional equations for the various terms appearing in its defining formula that $f_m(a_1, a_2)$ transforms as a differential form on M in each variable, so must actually be holomorphic and expressible as

$$f_m(a_1, a_2) = \sum_{k\ell} c_m^{k\ell} w'_k(a_1) w'_{\ell}(a_2)$$

for some uniquely determined constants $c_m^{k\ell} = c_m^{\ell k}$. These constants can be evaluated quite readily by using the integral formulas of Theorem B11 and

B12; for any fixed point $a_1 \in \Delta$ and indices ℓ, m

$$\sum_k c_m^{k\ell} w'_k(a_1) = \int_{a_2 \in \alpha_\ell} f_m(a_1, a_2) da_2$$

$$= \sum_k \varphi_m^k(a_1) \int_{a_2 \in \alpha_\ell} w'_k(a_2) da_2$$

$$+ \sum_k w'_k(a_1) \int_{a_2 \in \alpha_\ell} \varphi_m^k(a_2) da_2$$

$$- \int_{a_2 \in \alpha_\ell} w_{a_1}(a_2) w'_m(a_2) da_2 - w'_m(a_1) \int_{a_2 \in \alpha_\ell} w_{a_2}(a_1) da_2$$

$$= \varphi_m^\ell(a_1) + 2\pi i \delta_m^\ell w'_\ell(a_1) - \varphi_m^\ell(a_1) - 0,$$

so that

$$c_m^{k\ell} = 2\pi i \delta_m^k \delta_m^\ell$$

and consequently

$$f_m(a_1, a_2) = 2\pi i w'_m(a_1) w'_m(a_2).$$

Upon substituting this into the skew-symmetry condition it follows that

$$\sum_{\ell m} \left(\vec{\eta}_j^{\ell m} + \vec{\eta}_j^{m\ell} \right) w'_\ell(a_1) w'_m(a_2)$$

$$= -2\pi i \sum_m P \partial_{jm} \vec{\theta}_2(0) w'_m(a_1) w'_m(a_2),$$

hence that

$$\vec{\eta}_j^{\ell m} + \vec{\eta}_j^{m\ell} = -2\pi i \delta_m^\ell P \partial_{jm} \vec{\theta}_2(0).$$

The vectors $\vec{\zeta}_j^{\ell m}$ defined by

$$\vec{\zeta}_j^{\ell m} = \begin{cases} \vec{\eta}_j^{\ell m} & \text{if} \quad \ell \neq m \\ 0 & \text{if} \quad \ell = m \end{cases}$$

are consequently skew-symmetric in the indices ℓ, m, and

$$\vec{\eta}_j^{\ell m} = \vec{\zeta}_j^{\ell m} - \pi i \delta_m^\ell P \partial_{jm} \vec{\theta}_2(0);$$

and in terms of these vectors (17) can be rewritten

(18)

$$\begin{aligned} P\vec{\alpha}_j^{0;1;1}(a_1, a_2) = &\sum_{\ell m} P\partial_{\ell m}\vec{\theta}_2(0)\Big[\varphi_\ell^j(a_1)w_m'(a_2) - \varphi_\ell^j(a_2)w_m'(a_1)\Big] \\ &+ \sum_m P\partial_{jm}\vec{\theta}_2(0) g_m(a_1, a_2) \\ &+ \sum_{\ell m} \vec{\zeta}_j^{\ell m} w_\ell'(a_1) w_m'(a_2) \end{aligned}$$

where

$$g_m(a_1, a_2) = \sum_\ell \varphi_m^\ell(a_2) w_\ell'(a_1) - w_{a_1}(a_2) w_m'(a_2) - \pi i w_m'(a_1) w_m'(a_2).$$

Since all the other terms in (18) are clearly skew-symmetric in the variables a_1 and a_2, the functions $g_m(a_1, a_2)$ must be also; that can be verified alternatively by comparing the two expressions just obtained for the functions $f_k(a_1, a_2)$.

Next apply the projection operator P to the formula of Corollary 1 to Theorem 6 and use (18) to rewrite the result in the form

(19)

$$q(z,a_1)^{-2}\sum_j P\partial_j\vec{\theta}_2\big(w(z-a_1)\big)w_j'(a_2)$$

$$=\sum_{jk} P\partial_{jk}\vec{\theta}_2(0)\big[w'_{a_1,a_2}(z)w_j'(a_1)w_k'(a_2)+w'_{a_1,z}(a_2)w_j'(a_1)w_k'(z)\big]$$

$$-\sum_j P\vec{\alpha}_j^{0;1;1}(a_1,a_2)w_j'(z)$$

$$=\sum_{k\ell} P\partial_{k\ell}\vec{\theta}_2(0)\Big\{\sum_m\big[\varphi_k^m(a_1)w_\ell'(z)-\varphi_k^m(z)w_\ell'(a_1)\big]w_m'(a_2)-g_k(z,a_1)w_\ell'(a_2)$$

$$+\pi i w_\ell'(a_2)\big[2w_k'(a_1)w_\ell'(z)-w_k'(z)w_\ell'(a_1)-w_k'(z)w_k'(a_1)\big]\Big\}$$

$$-\sum_{jk\ell}\vec{\zeta}_j^{k\ell}w_j'(z)w_k'(a_1)w_\ell'(a_2).$$

The left-hand side of the preceding identity is skew-symmetric in the variables z and a_1, as is the first line on the last version of the right-hand side; consequently

$$\sum_{jk\ell}(\vec{\zeta}_j^{k\ell}+\vec{\zeta}_k^{j\ell})w_j'(z)w_k'(a_1)w_\ell'(a_2)$$

$$=\pi i\sum_{k\ell} P\partial_{k\ell}\vec{\theta}_2(0)w_\ell'(a_2)\big[w_k'(a_1)w_\ell'(z)+w_k'(z)w_\ell'(a_1)-2w_k'(z)w_k'(a_1)\big],$$

or equivalently

$$\vec{\zeta}_j^{k\ell}+\vec{\zeta}_k^{j\ell}=\pi i(\delta_\ell^j+\delta_\ell^k)P\partial_{jk}\vec{\theta}_2(0)-2\pi i\delta_k^j P\partial_{k\ell}\vec{\theta}_2(0).$$

Since the vectors $\vec{\zeta}_j^{k\ell}$ are skew-symmetric in the indices k and ℓ it follows readily that

$$\vec{\zeta}_j^{k\ell} = -\vec{\zeta}_k^{j\ell} \qquad \text{if} \quad j, k, \ell \text{ are distinct}$$

while

$$\vec{\zeta}_j^{kj} = \pi i P \partial_{jk} \vec{\theta}_2(0) \qquad \text{if} \quad j, k \text{ are distinct.}$$

The vectors $\vec{\xi}^{k\ell}_{\ j}$ defined by

$$\vec{\xi}^{k\ell}_{\ j} = \begin{cases} \vec{\zeta}_j^{k\ell} & \text{if } j, k, \ell \text{ are distinct} \\ 0 & \text{otherwise} \end{cases}$$

are consequently skew-symmetric in the indices j, k, ℓ, and it is a straightforward matter to verify that

$$\sum_{\ell m} \vec{\zeta}_j^{\ell m} w'_\ell(a_1) w'_m(a_2)$$

$$= \sum_{\ell m} \vec{\xi}^{\ell m}_{\ j} w'_\ell(a_1) w'_m(a_2) - \pi i \sum_k P \partial_{jk} \vec{\theta}_2(0) \left[w'_j(a_1) w'_k(a_2) - w'_j(a_2) w'_k(a_1) \right].$$

Substituting this into (18) yields the first formula of the present theorem;

substituting it into (19) yields

$$q(z,a_1)^{-2}\sum_j P\partial_j\vec{\theta}_2\big(w(z-a_1)\big)w_j'(a_2)$$

$$=\sum_{k\ell} P\partial_{k\ell}\vec{\theta}_2(0)\Big\{\sum_m \left[\varphi_k^m(a_1)w_\ell'(z)-\varphi_k^m(z)w_\ell'(a_1)\right]w_m'(a_2)-g_k(z,a_1)w_\ell'(a_2)$$

$$+\pi i w_\ell'(a_2)\big[2w_k'(a_1)w_\ell'(z)-w_k'(z)w_\ell'(a_1)-w_k'(z)w_k'(a_1)\big]$$

$$-\pi i w_\ell'(z)\big[w_k'(a_1)w_\ell'(a_2)-w_k'(a_2)w_\ell'(a_1)\big]\Big\}$$

$$-\sum_{jk\ell}\vec{\xi}^{\ell m}_{\,j} w_j'(z)w_\ell'(a_1)w_m'(a_2),$$

and comparing the coefficients of the linearly independent functions $w_j'(a_2)$ leads readily to the second formula of the present theorem. That suffices to conclude the proof.

§5. The ternary expansion.

In discussing more general differential expansion formulas it is convenient to use the auxiliary meromorphic functions introduced in D(6.10), namely the functions

(1)

$$Q(z_1,\ldots,z_n;a_1,\ldots,a_n)=\Big[\prod_{1\leqq j,k\leqq n} q(z_j,a_k)\Big]\Big/\Big[\prod_{1\leqq j<k\leqq n} q(z_j,z_k)q(a_j,a_k)\Big]$$

with the understanding that $Q(z_1;a_1)=q(z_1,a_1)$. These functions in general have simple poles along the subvarieties $z_j=Tz_k$ and $a_j=Ta_k$ for $j\neq k$ and simple zeros along the subvarieties $z_j=Ta_k$ for all j,k, and for all $T\in\Gamma$. Note that for $n>1$

$$\lim_{z_n\to a_n} q(z_n,a_n)^{-1}Q(z_1,\ldots,z_n;a_1,\ldots,a_n) = (-1)^{n-1}Q(z_1,\ldots,z_{n-1};a_1,\ldots,a_{n-1}), \tag{2}$$

by a straightforward calculation.

To consider the dominant terms in the ternary differential expansion choose a linear projection mapping $P:\mathbb{C}^{2^g}\to\mathbb{C}^n$ such that

$$P\vec{\theta}_2\big(w(z_1+z_2-a_1-a_2)\big)=0 \qquad \text{for all } z_i,a_i\in\widetilde{M}, \tag{3}$$

on equivalently such that

$$P\vec{\theta}_2(0)=P\partial_{jk}\vec{\theta}_2(0)=P\vec{\xi}^{k_1k_2}_{j}=P\vec{\xi}^{k_1k_2}_{j_1j_2}=0 \tag{4}$$

for all indices, where $\vec{\xi}^{k_1k_2}_{j}$ and $\vec{\xi}^{k_1k_2}_{j_1j_2}$ are any vectors in $\mathbb{C}^{2^g}$ representing those invariants of the binary expansion.

THEOREM 9. *If P is any linear projection mapping satisfying (3) then there are uniquely determined constant vectors $\vec{\xi}^{k_1k_2k_3}_{j_2j_3}$ in the range of P and holomorphic functions $\vec{\beta}_{j_1j_2j_3}(a_1,a_2,a_3)$ on $\widetilde{M}^3$ with values in the range of P such that*

(5)

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3-a_1-a_2-a_3)\big)Q(z_1,z_2,z_3;a_1,a_2,a_3)^{-2}$$

$$= \sum_{\pi\in\mathfrak{S}(0,1,2)}\sum_{jk}\vec{\xi}^{k_1k_2k_3}_{j_2j_3}w'_{j_2}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{k_3}(a_3)\bullet$$

$$\bullet\Big[w'_{a_1,a_3}(z_{\pi 1})w'_{k_1}(a_1)w'_{k_2}(a_2)+w'_{a_2,a_3}(z_{\pi 1})w'_{k_1}(a_2)w'_{k_2}(a_1)\Big]$$

$$+\sum_{j}\vec{\beta}_{j_1j_2j_3}(a_1,a_2,a_3)w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3).$$

Here the vectors $\vec{\xi}^{i_1j_2j_3}_{k_2k_3}$ are symmetric in the indices j_2,j_3 and in the indices k_2,k_3, and further satisfy

(6) $$\vec{\xi}^{i_1j_2j_3}_{k_2k_3}=-\vec{\xi}^{i_1k_2k_3}_{j_2j_3},\qquad \vec{\xi}^{i_1i_2i_3}_{k_2k_3}+\vec{\xi}^{i_2i_1i_3}_{k_2k_3}+\vec{\xi}^{i_3i_1i_2}_{k_2k_3}=0\ ;$$

they are determined uniquely by the identity

(7) $$\partial_i P\vec{\theta}_2\big(w(z_1+z_2-a_1-a_2)\big)Q(z_1,z_2;a_1,a_2)^{-2}$$

$$=\sum_{jk}\vec{\xi}^{ij_1j_2}_{k_1k_2}w'_{j_1}(a_1)w'_{j_2}(a_2)w'_{k_1}(z_1)w'_{k_2}(z_2).$$

The vectors $\vec{\beta}_{j_1j_2j_3}(a_1, a_2, a_3)$ are symmetric in the indices j_1, j_2, j_3.

Proof. The result of applying the projection mapping P to the general expansion of Theorem 1 is (2.2), and by Theorem 3 for $n = 3$ the dominant order satisfies $2L \leqq 3$ or $L \leqq 1$; thus there are only two sorts of nontrivial terms in the expansion, namely $P\vec{\alpha}^{0;1;2}_{\lambda j_2 j_3}$ and $P\vec{\alpha}^{0;0;3}_{j_1 j_2 j_3}$ where $1 \leqq \lambda \leqq 2$ and $1 \leqq j_i \leqq g$. The first sort have the explicit form given in Theorem 3, and their contribution to the expansion is

$$\sum_{\pi\in\mathfrak{S}(0,1,2)} \sum_{\lambda j_2 j_3} P\vec{\alpha}^{0;1;2}_{\lambda j_2 j_3} u'_\lambda(z_{\pi 1}) w'_{j_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3})$$

$$= \sum_{\pi\in\mathfrak{S}(0;1;2)} \sum_{jk} \left[\vec{\xi}^{k_1k_2k_3}_{j_2j_3} u'_1(z_{\pi 1}) + \vec{\xi}^{k_2k_1k_3}_{j_2j_3} u'_2(z_{\pi 1})\right] \bullet$$

$$\bullet\, w'_{j_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3}) w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3)$$

where $u'_1(z) = w'_{a_1,a_3}(z)$ and $u'_2(z) = w'_{a_2,a_3}(z)$; when rearranged by grouping together those terms with the same vectors $\vec{\xi}$ this evidently yields the first part of the desired formula. The contribution of the second sort of nontrivial terms is

$$\sum_{\pi\in\mathfrak{S}(0;0;3)} \sum_{j_1 j_2 j_3} P\vec{\alpha}^{0;0;3}_{j_1 j_2 j_3} w'_{j_1}(z_{\pi 1}) w'_{j_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3})$$

$$= \sum_{j} P\vec{\alpha}^{0;0;3}_{j_1 j_2 j_3} w'_{j_1}(z_1) w'_{j_2}(z_2) w'_{j_3}(z_3),$$

and that yields the second part of the desired formula with $\vec{\beta}_{j_1j_2j_3} = P\vec{\alpha}^{0;0;3}_{j_1j_2j_3}$. The coefficient vectors have the desired symmetries in their indices by Theorems 1 and 3. The second formula in the statement of the theorem determines these vectors by the Corollary to Theorem 3. The general results from Theorem 1 only imply that the functions $\vec{\beta}$ are meromorphic, with singularities at most along the subvarieties $a_i = Ta_j$; but none of the other terms in the expansion have such singularities and the functions $w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)$ are linearly independent, so the functions $\vec{\beta}$ are clearly actually holomorphic.

The asymmetry arising from the choice of the basis $w'_{a_1,a_3}(z)$,$w'_{a_2,a_3}(z)$ for the Abelian differentials of the third kind is more apparent in this case than in that of the binary expansion, since in the latter case the basis consisted just of $w'_{a_1,a_2}(z)$ which is at least skew symmetric in the variables a_1, a_2. This asymmetry can be alleviated by using the Abelian integrals of the second kind rather than the Abelian differentials of the third kind, since $w'_{a,b}(z) = w_z(a) - w_z(b)$. There are g rather than $g-1$ such integrals, but the resulting expansions are still unique since the integrals $w'_z(a_i)$ transform by the rather complicated formulas of Theorem B11 so only special linear combinations of them transform as Abelian differentials. In these terms the ternary expansion can be rewritten as follows.

COROLLARY 1. *With the hypotheses and notation as in the theorem*

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3-a_1-a_2-a_3)\big)Q(z_1,z_2,z_3;a_1,a_2,a_3)^{-2}$$

$$= \sum_{\pi,\rho\in\mathfrak{S}(0,1,2)} \sum_{jk} \vec{\xi}^{k_1k_2k_3}_{j_2j_3} w'_{k_1}(a_{\rho 1})w'_{k_2}(a_{\rho 2})w'_{k_3}(a_{\rho 3})w_{z_{\pi 1}}(a_{\rho 1})w'_{j_2}(z_{\pi 2})w'_{j_3}(z_{\pi 3})$$

$$+\sum_j \vec{\beta}_{j_1j_2j_3}(a_1,a_2,a_3)w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3),$$

and the functions $\vec{\beta}_{j_1j_2j_3}(a_1,a_2,a_3)$ *are also symmetric in the variables* a_1,a_2,a_3.

Proof. In the first line on the right-hand side of the principal formula of Theorem 9 the product of the expression in brackets with $w'_{k_3}(a_3)$ can evidently be rewritten

$$w_{z_{\pi 1}}(a_1)w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)+w_{z_{\pi 1}}(a_2)w'_{k_1}(a_2)w'_{k_2}(a_1)w'_{k_3}(a_3)$$

$$+w_{z_{\pi 1}}(a_3)w'_{k_1}(a_3)w'_{k_2}(a_1)w'_{k_3}(a_2)-w_{z_{\pi 1}}(a_3)\Big[w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)$$

$$+w'_{k_1}(a_2)w'_{k_2}(a_1)w'_{k_3}(a_3)+w'_{k_1}(a_3)w'_{k_2}(a_1)w'_{k_3}(a_2)\Big],$$

and

$$\sum_k \vec{\xi}^{k_1k_2k_3}_{j_2j_3}\Big[w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3) + w'_{k_1}(a_2)w'_{k_2}(a_1)w'_{k_3}(a_3)$$

$$+ w'_{k_1}(a_3)w'_{k_2}(a_1)w'_{k_3}(a_2)\Big]$$

$$= \sum_k \Big[\vec{\xi}^{k_1k_2k_3}_{j_2j_3} + \vec{\xi}^{k_2k_1k_3}_{j_2j_3} + \vec{\xi}^{k_3k_1k_2}_{j_2j_3}\Big] w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)$$

$$= 0$$

as a consequence of the symmetry conditions satisfied by the vectors $\vec{\xi}^{k_1k_2k_3}_{j_2j_3}$. That yields the formula of the corollary quite easily. In that formula the left-hand side and the first line on the right-hand side are evidently symmetric functions of the variables a_1, a_2, a_3, so the second line on the right-hand side must also be symmetric; since the functions $w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)$ are linearly independent the individual coefficients must actually be symmetric as well, and that suffices to conclude the proof.

If the projection mapping P is one for which $P\vec{\alpha}^{0;1;2}_{\lambda j_2j_3} = 0$ then the leading terms in the expansion of the preceding theorem are trivial and the dominant terms are $P\vec{\alpha}^{0;0;3}_{j_1j_2j_3}$. In that case the theorem can be modified as follows.

Corollary2. *With the hypotheses and notation as in the theorem, if all* *the vectors $\vec{\xi}^{k_1 k_2 k_3}_{\ \ j_2 j_3}$ vanish then*

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3-a_1-a_2-a_3)Q(z_1,z_2,z_3;a_1,a_2,a_3)^{-2}$$

$$=\sum_{jk}\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)$$

for some uniquely determined vectors $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$ in the range of P; these vectors are symmetric in the indices $j_1j_2j_3$, and in the indices $k_1k_2k_3$, and under the interchange of the j's and the k's.

Proof. If all the vectors $\vec{\xi}^{k_1 k_2 k_3}_{\ \ j_2 j_3}$ vanish then the dominant terms in the differential expansion have the form

$$P\vec{\alpha}^{0;0;3}_{j_1j_2j_3}=\sum_{k}\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}w'_{k_1}(a_1)w'_{k_2}(a_2)w'_{k_3}(a_3)$$

as in Theorem 3, where the vectors $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$ have the asserted symmetries; that is the desired result.

These vectors satisfy some further conditions, in addition to the symmetries that have already been noted. First since $Q(z_1,z_2;a_1,a_2)^{-2}=0$ whenever $z_1=z_2$ and the functions $w'_{j_1}(a_1)w'_{j_2}(a_2)$ are linearly independent it follows directly from (7) of Theorem 9 that

$$\sum_{k_1k_2}\vec{\xi}^{ij_1j_2}_{\ k_1k_2}w'_{k_1}(z)w'_{k_2}(z)=0 \tag{8}$$

for all points $z \in \widetilde{M}$ and all indices i, j_1, j_2. By the symmetries of these vectors this can be rewritten as

$$(9) \quad \sum_{k_1 k_2 k_3} \vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2} w'_{k_1}(z_1) w'_{k_2}(z_2) w'_{k_3}(z_3) = 0 \qquad \text{if } z_{i_1} = z_{i_2} \quad \text{for } i_1 < i_2$$

for any points $z_i \in \widetilde{M}$ and any indices i_1, i_2, j_1, j_2; for the case that $z_2 = z_3$ that is immediately clear from (8) and the first symmetry (6), while

$$\sum_{k_1 k_2} \vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2} w'_{k_1}(z) w'_{k_2}(z) = -\frac{1}{2} \sum_{k_1 k_2} \vec{\xi}^{k_3 k_1 k_2}_{j_1 j_2} w'_{k_1}(z) w'_{k_2}(z)$$

from the second symmetry (6). Next *if* $\vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2} = 0$ *for all indices* then since $Q(z_1, z_2, z_3; a_1, a_2, a_3)^{-2} = 0$ whenever $z_1 = z_2$ and $w'_{j_1}(a_1) w'_{j_2}(a_2) w'_{j_3}(a_3) w'_{k_3}(z_3)$ are linearly independent it follows directly from Corollary 2 that

$$(10) \qquad \sum_{k_1 k_2} \vec{\xi}^{j_1 j_2 j_3}_{i k_1 k_2} w'_{k_1}(z) w'_{k_2}(z) = 0$$

for all points $z \in \widetilde{M}$ and all indices i, j_1, j_2, j_3. Again by the symmetries of these vectors this can obviously be rewritten as

$$(11) \quad \sum_{k_1 k_2 k_3} \vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2 j_3} w'_{k_1}(z_1) w'_{k_2}(z_2) w'_{k_3}(z_3) = 0 \qquad \text{if } z_{i_1} = z_{i_2} \text{ for } i_1 < i_2,$$

for any points $z_i \in \widetilde{M}$ and any indices i_1, i_2, j_1, j_2, j_3, and correspondingly upon interchanging the upper and lower indices of the vectors.

As for the significance of these conditions, clearly (8) means that for any fixed indices i, j_1, j_2 each component of the vector $\vec{\xi}^{ij_1j_2}_{k_1k_2}$ must lie in the Petri space $\mathcal{P}_2$ of quadratic forms vanishing on the canonical curve, and by symmetry the same must be the case for any fixed indices i, k_1, k_2. Thus if $p_i(x) = \sum_{j_1j_2} \; p^i_{j_1j_2}x_{j_1}x_{j_2}$ is any chosen basis for $\mathcal{P}_2$ as before then evidently

$$\vec{\xi}^{ij_1j_2}_{k_1k_2} = \sum_{\ell_1\ell_2} \vec{\eta}^{\,i\ell_1}_{\ \ell_2} p^{\ell_1}_{j_1j_2} p^{\ell_2}_{k_1k_2} \tag{12}$$

for some uniquely determined vectors $\vec{\eta}^{i\ell_1}_{\ell_2}$ in the range of P. The first symmetry condition (6) is just that

$$\vec{\eta}^{i\ell_1}_{\ell_2} = -\vec{\eta}^{i\ell_2}_{\ell_1}. \tag{13}$$

The second symmetry condition (6) though is that

$$\sum_{\ell_1\ell_2} \left(\vec{\eta}^{j_1\ell_1}_{\ell_2} p^{\ell_1}_{j_2j_3} + \vec{\eta}^{j_2\ell_1}_{\ell_2} p^{\ell_1}_{j_1j_3} + \vec{\eta}^{j_3\ell_1}_{\ell_2} p^{\ell_1}_{j_1j_2} \right) p^{\ell_2}_{k_1k_2} = 0,$$

which since the forms p_i are linearly independent is equivalently that

$$\sum_{\ell} \left(\vec{\eta}^{j_1\ell}_{k} p^{\ell}_{j_2j_3} + \vec{\eta}^{j_2\ell}_{k} p^{\ell}_{j_1j_3} + \vec{\eta}^{j_3\ell}_{k} p^{\ell}_{j_1j_2} \right) = 0. \tag{14}$$

Thus beyond the simple skew-symmetry condition (13) these auxiliary vectors $\vec{\eta}^{j\ell}_{k}$ are subject to a system of linear equations that depends to a nontrivial extent on the structure of the Petri space rather than just on its dimension. It is not difficult to find examples of spaces of polynomials $p_i(x)$ of the same dimension for which the sets of vectors $\vec{\eta}_k^{j\ell}$ satisfying (13) and (14) have different dimensions; the question whether that is also the case for spaces $p_i(x)$ which are possible bases for the Petri spaces of Riemann surfaces is another matter of course. Next if $\vec{\xi}^{k_1k_2k_3}_{j_1j_2} = 0$ for all indices then condition (10) applies to the vectors $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$. This is a somewhat stronger condition than that each component of this vector belongs to the Petri space $\mathcal{P}_3$ of cubic polynomials vanishing on the canonical curve, in either upper or lower indices. Formally let $\mathcal{P}_n^*$ be the space of homogeneous polynomials $p(x) = \sum_j p_{j_1} \cdots_{j_n} x_{j_1} \cdots x_{j_n}$ such that

$$\sum_j p_{j_1} \cdots_{j_n} w'_{j_1}(z) w'_{j_2}(z) w'_{j_3}(z_3) \ldots w'_{j_n}(z_n) = 0$$

for all points $z, z_3, \ldots, z_n \in \widetilde{M}$, the modified Petri space $\mathcal{P}_n^* \subseteq \mathcal{P}_n$. If $p_i^*(x) = p^{*i}_{j_1j_2j_3}$ is a basis for $\mathcal{P}_3^*$ then (10) means that

$$\vec{\xi}^{j_1j_2j_3}_{k_1k_2k_3} = \sum_{\ell_1\ell_2} \vec{\eta}^{\ell_1}_{\ell_2} p^{*\ell_1}_{j_1j_2j_3} p^{*\ell_2}_{k_1k_2k_3} \tag{15}$$

for some uniquely determined vectors $\vec{\eta}^{\ell_1}_{\ell_2}$ in the range of P, and the symmetry condition of Corollary 2 is just that

$$\vec{\eta}^{\ell_1}_{\ell_2} = \vec{\eta}^{\ell_2}_{\ell_1}. \tag{16}$$

This is more like the corresponding situation for the binary expansion.

For the case of the natural projection mapping

$$P_2 : \mathbb{C}^{2^g} \rightarrow \mathbb{C}^{2^g}/L_2, \tag{17}$$

the minimal projection satisfying (2), there result canonically defined vectors $\vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2}$ in the range of P_2, spanning an intrinsically determined linear subspace $L_3' \subseteq$ range P_2. Then for the further natural projection

$$P_3' : (\text{range } P_2) \rightarrow (\text{range } P_2)/L_3' \tag{18}$$

there result canonically defined vectors $\vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2 j_3}$ in the range of P_3'; these vectors span the space L_3/L_3', where L_3 is the span of all vectors $\vec{\theta}_2\big(w(z_1 + z_2 + z_3 - a_1 - a_2 - a_3)\big)$ for arbitrary points $z_i, a_i \in \widetilde{M}$. As in the case of the binary expansion this situation can be summarized in the following diagram, in which $L_3^* = P_2^{-1}(L_2')$.

$$\begin{array}{ccccccc}
 & L_2 & \subseteq & L_3^* & \subseteq & L_3 & \\
P_2 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \subseteq & L_3' = L_3^*/L_2 & \subseteq & L_3/L_2 & (L_3^*/L_2 = \text{span } \vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2}) \\
P_2' & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \subseteq & 0 & \subseteq & L_3/L_3^* & (L_3/L_3^* = \text{span } \vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2 j_3})
\end{array} \tag{19}$$

It is of course possible to describe this situation by using the vectors $\vec{\eta}^{jk}_{\ell}$ in place of $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$ and $\vec{\eta}^{k}_{\ell}$ in place of $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$, although this requires a choice of bases of the Petri spaces $\mathcal{P}_2$ and $\mathcal{P}_3^*$.

It is again possible to say something about the subdominant terms in this expansion without much difficulty. As a first observation, the coefficients $\vec{\beta}_{j_1j_2j_3}$ in the formula of Theorem 9 admit an explicit expression in much the same form as in Corollary 2, even without any further assumptions about the projection operator P, as follows.

COROLLARY 3. *With the hypotheses and notation as in the theorem, there are uniquely determined vectors $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$ in the range P such that*

$$\vec{\beta}_{j_1j_2j_3}(a_1,a_2,a_3) = \sum_{\pi,\rho\in\mathfrak{S}(0,1,2)} \sum_{k} \vec{\xi}^{k_1k_2\ k_3}_{\ \ \ j_{\pi2}j_{\pi3}} \varphi^{j_{\pi1}}_{k_1}(a_{\rho1}) w'_{k_2}(a_{\rho2}) w'_{k_3}(a_{\rho3})$$

$$+ \sum_{k} \vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3} w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3);$$

these vectors are symmetric in the indices j_1, j_2, j_3 and in the indices k_1, k_2, k_3.

Proof. In the formula of Corollary 1 replace a_1 by Ta_1 for some $T \in \Gamma$. The left-hand side is a relatively automorphic function for the canonical factor of automorphy $\kappa(T, a_1)$, as is each term $w'_k(a_1)$, while by Theorem B11

$$w_z(Ta_1) = w_z(a_1) + 2\pi i \sum_{\ell} \beta_\ell(T) w'_\ell(z).$$

It then follows from a straightforward calculation that

$$\sum_{\pi\in\mathfrak{S}(0,1,2)} \sum_{jk\ell} \vec{\xi}^{k_1 k_2 k_3}_{\ \ j_2 j_3} 2\pi i \beta_\ell(T) w'_\ell(z_{\pi 1}) w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3) w'_{j_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3})$$

$$+ \kappa(T,a_1)^{-1} \sum_j \vec{\beta}_j(Ta_1,a_2,a_3) w'_{j_1}(z_1) w'_{j_2}(z_2) w'_{j_3}(z_3)$$

$$= \sum_j \vec{\beta}_j(a_1,a_2,a_3) w'_{j_1}(z_1) w'_{j_2}(z_2) w'_{j_3}(z_3),$$

or upon comparing coefficients of the linearly independent functions $w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)$ that equivalently

$$\vec{\beta}_{j_1 j_2 j_3}(Ta_1,a_2,a_3)\kappa(T,a_1)^{-1} - \vec{\beta}_{j_1 j_2 j_3}(a_1,a_2,a_3)$$

$$= 2\pi i \sum_k \Big[\vec{\xi}^{k_1 k_2 k_3}_{\ \ j_2 j_3} \beta_{j_1}(T) + \vec{\xi}^{k_1 k_2 k_3}_{\ \ j_1 j_3} \beta_{j_2}(T)$$

$$+ \vec{\xi}^{k_1 k_2 k_3}_{\ \ j_1 j_2} \beta_{j_3}(T) \Big] w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3)$$

$$= \sum_{\pi\in\mathfrak{S}(0,1,2)} \sum_k \vec{\xi}^{k_1 k_2\ k_3}_{\ \ j_{\pi 2} j_{\pi 3}} w'_{k_2}(a_2) w'_{k_3}(a_3) \bullet$$

$$\bullet \Big[\varphi^{j_{\pi 1}}_{k_1}(Ta_1;a_0)\kappa(T,a_1)^{-1} - \varphi^{j_{\pi 1}}_{k_1}(a_1;a_0) \Big],$$

where the last equality is an application of the Corollary to Theorem B12 for any fixed $a_0 \in \widetilde{M}$. This really amounts to the condition that

$$\vec{f}_{j_1 j_2 j_3}(a_1,a_2,a_3) = \vec{\beta}_{j_1 j_2 j_3}(a_1,a_2,a_3)$$

$$- \sum_{\pi\in\mathfrak{S}(0,1,2)} \sum_k \vec{\xi}^{k_1 k_2\ k_3}_{\ \ j_{\pi 2} j_{\pi 3}} \varphi^{j_{\pi 1}}_{k_1}(a_1;a_0) w'_{k_2}(a_2) w'_{k_3}(a_3)$$

transforms as a differential form on M as a function of the variable $a_1 \in \widetilde{M}$.

Since

$$\sum_{\pi\in\mathfrak{S}(0,1,2)} \sum_{k_1} \vec{\xi}^{k_1 k_2 \ k_3}_{\ \ \ j_{\pi 2} j_{\pi}} \delta^{j_{\pi 1}}_{k_1} = - \sum_{\pi\in\mathfrak{S}(0,1,2)} \vec{\xi}^{j_{\pi 1} j_{\pi 2} j_{\pi 3}}_{\ \ \ \ k_2 \ k_3}$$

$$= -\vec{\xi}^{j_1 j_2 j_3}_{\ \ \ k_2 k_3} - \vec{\xi}^{j_2 j_1 j_3}_{\ \ \ k_2 k_3} - \vec{\xi}^{j_3 j_1 j_2}_{\ \ \ k_2 k_3} = 0$$

in view of the symmetries of these vectors then actually

$$\vec{f}_{j_1 j_2 j_3}(a_1, a_2, a_3) = \vec{\beta}_{j_1 j_2 j_3}(a_1, a_2, a_3)$$

$$- \sum_{\pi\in\mathfrak{S}(0,1,2)} \sum_{k} \vec{\xi}^{k_1 k_2 \ k_3}_{\ \ \ j_{\pi 2} j_{\pi 3}} \varphi^{j_{\pi 1}}_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3)$$

so these functions do not depend on the point a_0; they are indeed holomorphic functions of the variables $a_i \in \widetilde{M}$, and a holomorphic differential form on M in the variable a_1. The functions

$$\vec{g}_{j_1 j_2 j_3}(a_1, a_2, a_3) = \vec{\beta}_{j_1 j_2 j_3}(a_1, a_2, a_3)$$

$$- \sum_{\pi,\rho\in\mathfrak{S}(0,1,2)} \sum_{k} \vec{\xi}^{k_1 k_2 \ k_3}_{\ \ \ j_{\pi 2} j_{\pi 3}} \varphi^{j_{\pi 1}}_{k_1}(a_{\rho 1}) w'_{k_2}(a_{\rho 2}) w'_{k_3}(a_{\rho 3})$$

differ from $\vec{f}_{j_1 j_2 j_3}(a_1, a_2, a_3)$ by holomorphic functions of a_i that are also holomorphic differential forms on M in the variable a_1, and are symmetric in the variables a_1, a_2, a_3; they are consequently holomorphic differential forms on M in each variable a_i, so can be written

$$\vec{g}_{j_1 j_2 j_3}(a_1, a_2, a_3) = \sum_{k} \vec{\xi}^{k_1 k_2 k_3}_{j_1 j_2 j_3} w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3)$$

for some uniquely determined vectors $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$ that are symmetric in their upper indices and also in their lower indices. The desired result follows directly from these observations.

The vectors $\vec{\xi}^{k_1k_2k_3}_{j_1j_2j_3}$ of Corollary 3 can be viewed as representatives of the correspondingly denoted vectors of Corollary 2, where the latter are viewed as lying in the quotient space of the range of P modulo the subspace spanned by the vectors $\vec{\xi}^{k_1}{}^{k_2k_3}_{j_2j_3}$. The vectors of Corollary 3 consequently satisfy the further symmetry condition that interchanging the j's and k's changes them only by vectors lying in the subspace spanned by the vectors $\vec{\xi}^{k_1}{}^{k_2k_3}_{j_2j_3}$. It is actually possible to choose representatives that are really fully symmetric under interchanging the j's and k's, by arguing as in the second part of the proof of Theorem 7; the result and proof are sufficiently complicated that it seems unnecessary to provide the details here.

§6. The quaternary expansion.

To describe the quaternary expansion consider a linear projection mapping $P : \mathbb{C}^{2g} \to \mathbb{C}^n$ such that

$$(1) \quad P\vec{\theta}_2\big(w(z_1 + z_2 + z_3 - a_1 - a_2 - a_3)\big) = 0 \qquad \text{for all } z_i, a_i \in \widetilde{M}.$$

The dominant terms are described as follows.

THEOREM 11. *If P is any linear projection mapping satisfying (1) then there are uniquely determined constant vectors $\vec{\xi}^{k_1k_2k_3k_4}_{j_3j_4}$ in the range of P and holomorphic functions $\vec{\beta}_{\lambda j_2j_3j_4}(a_1,a_2,a_3,a_4)$ and $\vec{\beta}_{j_1j_2j_3j_4}(a_1,a_2,a_3,a_4)$ on $\widetilde{M}^4$ with values in the range of P such that*

(2)

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3+z_4-a_1-a_2-a_3-a_4)\big)Q(z_1,z_2,z_3,z_4;a_1,a_2,a_3,a_4)^{-2}$$

$$= \sum_{\substack{\pi\in\mathfrak{S}(0,2,2)\\ \rho\in\mathfrak{S}(2,1)}} \sum_{j,k} \vec{\xi}^{k_1k_2k_3k_4}_{j_3j_4} w'_{k_1}(a_{\rho 1})w'_{k_2}(a_{\rho 2})w'_{k_3}(a_{\rho 3})w'_{k_4}(a_4)\bullet$$

$$\bullet\, w'_{a_{\rho 1},a_4}(z_{\pi 1})w'_{a_{\rho 2},a_4}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

$$+ \sum_{\pi\in\mathfrak{S}(0,1,3)} \sum_{\lambda,j} \vec{\beta}_{\lambda j_2j_3j_4}(a_1,a_2,a_3,a_4)\bullet$$

$$\bullet\, w'_{a_\lambda,a_4}(z_{\pi 1})w'_{j_2}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

$$+\sum_j \vec{\beta}_{j_1j_2j_3j_4}(a_1,a_2,a_3,a_4)w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)w'_{j_4}(z_4).$$

Here the vectors $\vec{\xi}^{i_1i_2j_1j_2}_{k_1k_2}$ are symmetric in the indices i_1,i_2, in the indices j_1,j_2, and in the indices k_1,k_2, and under any interchange of the three sets *of indices i, j, k; moreover they satisfy the further symmetry condition that*

(3) $$\vec{\xi}^{i_1 i_2 i_3 i_4}_{j_3 j_4} + \vec{\xi}^{i_1 i_3 i_2 i_4}_{j_3 j_4} + \vec{\xi}^{i_1 i_4 i_2 i_3}_{j_3 j_4} = 0,$$

and are determined uniquely by the identity

(4) $$\partial_{i_1 i_2} P\vec{\theta}_2\big(w(z_1 + z_2 - a_1 - a_2)\big) Q(z_1, z_2; a_1, a_2)^{-2}$$

$$= \sum_{jk} \vec{\xi}^{i_1 i_2 j_1 j_2}_{k_1 k_2} w'_{j_1}(z_1) w'_{j_2}(z_2) w'_{k_1}(a_1) w'_{k_2}(a_2).$$

The vectors $\vec{\beta}_{\lambda j_2 j_3 j_4}(a_1, a_2, a_3, a_4)$ are symmetric in the indices j_2, j_3, j_4, while $\vec{\beta}_{j_1 j_2 j_3 j_4}(a_1, a_2, a_3, a_4)$ are symmetric in the indices j_1, j_2, j_3, j_4.

Proof. The result of applying P to the general expansion of Theorem 1 is (2.2), and by Theorem 3 for $n = 4$ the dominant order satisfies $2L \leqq 4$ or $L \leqq 2$; thus there are only three sorts of nontrivial terms in the expansion, namely $P\vec{\alpha}^{0;2;2}_{\lambda_1 \lambda_2 j_3 j_4}$, $P\vec{\alpha}^{0;1;3}_{\lambda j_2 j_3 j_4}$, and $P\vec{\alpha}^{0;0;4}_{j_1 j_2 j_3 j_4}$. The first sort have the explicit form given in Theorem 3, and their contribution to the expansion is

$$\sum_{\pi \in \mathfrak{S}(0,2,2)} \sum_{\lambda, j} P\vec{\alpha}^{0;2;2}_{\lambda_1 \lambda_2 j_3 j_4} u'_{\lambda_1}(z_{\pi 1}) u'_{\lambda_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3}) w'_{j_4}(z_{\pi 4})$$

$$= \sum_{\pi \in \mathfrak{S}(0,2,2)} \sum_{jk} \Big[\vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4} u'_1(z_{\pi 1}) u'_2(z_{\pi 2}) + \vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4} u'_1(z_{\pi 1}) u'_3(z_{\pi 2})$$

$$+ \vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4} u'_2(z_{\pi 1}) u'_3(z_{\pi 2}) \Big] \bullet$$

$$\bullet\, w'_{j_3}(z_{\pi 3}) w'_{j_4}(z_{\pi 4}) w'_{k_1}(a_1) \ldots w'_{k_4}(a_4),$$

where $u_1'(z) = w'_{a_1,a_4}(z)$, $u_2'(z) = w'_{a_2,a_4}(z)$, and $u_3'(z) = w'_{a_3,a_4}(z)$; when rearranged by grouping together those terms with the same vectors $\vec{\xi}$, which is effected by permuting the variables a_1, a_2, a_3 by the permutations $\rho \in \mathfrak{S}(2,1) \subset \mathfrak{S}_3$, this yields the first part of the desired formula. The last two terms arise from (2.2) by setting $\vec{\beta}_{\lambda j_2 j_3 j_4} = P\vec{\alpha}^{0;1;3}_{\lambda j_2 j_3 j_4}$ and $\vec{\beta}_{j_1 j_2 j_3 j_4} = P\vec{\alpha}^{0;0;4}_{j_1 j_2 j_3 j_4}$. The coefficient vectors have the desired symmetries by Theorems 1 and 3. The second formula of the theorem determines the vectors involved by the Corollary to Theorem 3.

The general results from Theorem 1 only imply that the functions $\vec{\beta}$ are meromorphic, with singularities at most along the subvarieties $a_i = Ta_j$. They will be shown to be holomorphic by using an auxiliary formula somewhat analogous to (4); the discussion is a bit more extensive than necessary just to complete the proof, since this formula is rather interesting by itself. To derive it, multiply the expansion (2) by $q(z_\nu, a_\nu)^2$, apply the differential operator $\partial/\partial z_\nu$, and take the limit as $z_\nu \to a_\nu$. On the left-hand side the product $q(z_\nu, a_\nu)^2 Q(z_1, \ldots, a_4)^{-2}$ is holomorphic at $z_\nu = a_\nu$, with the limiting value obtained from (5.2), while the term $P\vec{\theta}_2\big(w(z_1 + \cdots - a_4)\big)$ vanishes at $z_\nu = a_\nu$ as a consequence of the assumption (1); thus the only nontrivial terms arise from differentiating the theta function. Each separate term on the right-hand side will have a double zero at $z_\nu = a_\nu$ coming from the factor $q(z_\nu, a_\nu)^2$, and hence will contribute nothing to the final result, except for those terms involving a meromorphic factor $w'_{a_\nu,b}(z_\nu)$ for some point b, and for the latter

$$\lim_{z_\nu\to a_\nu}\frac{\partial}{\partial z_\nu}\,q(z_\nu,a_\nu)^2 w'_{a_\nu,b}(z_\nu)=1.$$

Thus if $\nu=1$

(5)

$$\sum_i \partial_i P\vec{\theta}_2\big(w(z_2+z_3+z_4-a_2-a_3-a_4)\big)w'_i(a_1)Q(z_2,z_3,z_4;a_2,a_3,a_4)^{-2}$$

$$=\sum_{\substack{\pi,\rho\\ \pi 1=\rho 1=1}}\sum_{j,k}\vec{\xi}^{k_1k_2k_3k_4}_{j_3j_4}w'_{k_1}(a_1)w'_{k_2}(a_{\rho 2})w'_{k_3}(a_{\rho 3})w'_{k_4}(a_4)\bullet$$

$$\bullet\, w'_{a_{\rho 2},a_4}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

$$+\sum_j \vec{\beta}_{1j_2j_3j_4}(a_1,a_2,a_3,a_4)w'_{j_2}(z_2)w'_{j_3}(z_3)w'_{j_4}(z_4),$$

while if $\nu=4$

(6)

$$\sum_i \partial_i P\vec{\theta}_2\big(w(z_1+z_2+z_3-a_1-a_2-a_3)\big)w'_i(a_4)Q(z_1,z_2,z_3;a_1,a_2,a_3)^{-2}$$

$$=-\sum_{\substack{\pi,\rho\\ \pi 2=4}}\sum_{j,k}\vec{\xi}^{k_1k_2k_3k_4}_{j_3j_4}w'_{k_1}(a_{\rho 1})w'_{k_2}(a_{\rho 2})w'_{k_3}(a_{\rho 3})w'_{k_4}(a_4)\bullet$$

$$\bullet\, w'_{a_{\rho 1},a_4}(z_{\pi 1})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

$$-\sum_{\lambda,j}\vec{\beta}_{\lambda j_2j_3j_4}(a_1,a_2,a_3,a_4)w'_{j_2}(z_1)w'_{j_3}(z_2)w'_{j_4}(z_3);$$

in these formulas $\pi \in \mathfrak{S}(0,2,2) \subset \mathfrak{S}_4$ and $\rho \in \mathfrak{S}(2,1) \subset \mathfrak{S}_3$, with the further restrictions as noted. Now in (5) the only singularities on the left-hand side and in the first line on the right-hand side are poles of order at most two along the subvarieties $z_i = Ta_j$; that must consequently be the case for the second line on the right-hand side also, and it is then clear that the functions $\vec{\beta}_{1j_2j_3j_4}$ are holomorphic. The cases $\nu = 2,3$ yield corresponding formulas, so that $\vec{\beta}_{\nu j_2j_3j_4}$ are holomorphic for $\nu = 1,2,3$, while from (6) it follows that $\sum_\lambda \vec{\beta}_{\lambda j_2j_3j_4}$ are holomorphic and consequently so are $\vec{\beta}_{4j_2j_3j_4}$. Thus all the terms $\vec{\beta}_{\nu j_2j_3j_4}$ are holomorphic, and it then follows immediately from the main expansion formula (2) that the terms $\vec{\beta}_{j_1j_2j_3j_4}$ must be as well, to conclude the proof.

It may be worth noting in passing that the reason for the extra step in showing that the functions $\vec{\beta}$ are holomorphic is that the product $\vec{\beta}_{\lambda j_2j_3j_4}(a_1,a_2,a_3,a_4)w'_{a_\lambda,a_4}(z_{\pi 1})$ is holomorphic even when $\vec{\beta}_{\lambda j_2j_3j_4}(a_1,a_2,a_3,a_4)$ has a simple pole along the subvariety $a_\lambda = a_4$. Just as for the ternary expansion the asymmetries here in the variables a_λ can be alleviated by rewriting the formula in terms of the Abelian integrals of the second kind as follows.

COROLLARY 1. *With the hypotheses and notation as in the theorem*

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3+z_4-a_1-a_2-a_3-a_4)\big)Q(z_1,z_2,z_3,z_4;a_1,a_2,a_3,a_4)^{-2}$$

$$= \sum_{\pi,\sigma\in\mathfrak{S}(2,2)} \sum_{j,k} \vec{\xi}^{k_1k_2k_3k_4}_{j_3j_4} w'_{k_1}(a_{\sigma1})w'_{k_2}(a_{\sigma2})w'_{k_3}(a_{\sigma3})w'_{k_4}(a_{\sigma4}) \bullet$$

$$\bullet\, w_{z_{\pi1}}(a_{\sigma1})w_{z_{\pi2}}(a_{\sigma2})w_{j_3}(z_{\pi3})w_{j_4}(z_{\pi4})$$

$$+ \sum_{\pi\in\mathfrak{S}(1,3)} \sum_{\nu=1}^{4}\sum_{j} \vec{\beta}^*_{\nu j_2j_3j_4}(a_1,a_2,a_3,a_4)w_{z_{\pi1}}(a_\nu)w'_{j_2}(z_{\pi2})w'_{j_3}(z_{\pi3})w'_{j_4}(z_{\pi4})$$

$$+ \sum_{j} \vec{\beta}_{j_1j_2j_3j_4}(a_1,a_2,a_3,a_4)w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)w'_{j_4}(z_4)$$

for uniquely determined holomorphic functions $\vec{\beta}^*_{\nu j_2j_3j_4}$ and $\vec{\beta}_{j_1j_2j_3j_4}$ on $\widetilde{M}^4$ *with values in the range of* P. *Here*

$$\sum_{\nu=1}^{4} \vec{\beta}^*_{\nu j_2j_3j_4}(a_1,a_2,a_3,a_4) = 0 \tag{7}$$

and

$$\vec{\beta}^*_{(\sigma^{-1}\nu)j_2j_3j_4}(a_{\sigma1},a_{\sigma2},a_{\sigma3},a_{\sigma4}) = \vec{\beta}^*_{\nu j_2j_3j_4}(a_1,a_2,a_3,a_4) \tag{8}$$

for any permutation $\sigma\in\mathfrak{S}_4$; *moreover* $\vec{\beta}_{j_1j_2j_3j_4}(a_1,a_2,a_3,a_4)$ *is a symmetric function of the variables* a_i, *and both sets of functions are symmetric in the indices* j.

Proof. In the first line on the right-hand side of (2) make the substitution $w'_{a,b}(z) = w_z(a) - w_{\dot{z}}(b)$. With the shorthand notation

$$(i_1 i_2; i_3 i_4 i_5 i_6) = w_{z_{\pi 1}}(a_{i_1}) w_{z_{\pi 2}}(a_{i_2}) w'_{k_1}(a_{i_3}) w'_{k_2}(a_{i_4}) w'_{k_3}(a_{i_5}) w'_{k_4}(a_{i_6})$$

the coefficient of $\vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4} w'_{j_3}(z_{\pi 3}) w'_{j_4}(z_{\pi 4})$ in that line for any fixed permutation π and indices j, k can be written

$$(9) \qquad \sum_{\rho \in \mathfrak{S}(2,1)} \Big\{ (\rho 1\, \rho 2; \rho 1\, \rho 2\, \rho 3\, 4) - (\rho 1\, 4; \rho 1\, \rho 2\, \rho 3\, 4) - (4\, \rho 2; \rho 1\, \rho 2\, \rho 3\, 4) + (4\, 4; \rho 1\, \rho 2\, \rho 3\, 4) \Big\}$$

$$\begin{aligned}
= \quad & (12;1234) - (14;1234) - (42;1234) + (44;1234) \\
& + (13;1324) - (14;1324) - (43;1324) + (44;1324) \\
& + (23;2314) - (24;2314) - (43;2314) + (44;2314) \,.
\end{aligned}$$

Here observe that

$$\begin{aligned}
(10) \qquad & \sum_k \vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4} \Big\{ (44;1234) + (44;1324) + (44;2314) \Big\} \\
& = \sum_k w_{z_{\pi 1}}(a_4) w_{z_{\pi 2}}(a_4) w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3) w'_{k_4}(a_4) \bullet \\
& \qquad \bullet \Big\{ \vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4} + \vec{\xi}^{k_1 k_3 k_2 k_4}_{j_3 j_4} + \vec{\xi}^{k_2 k_3 k_1 k_4}_{j_3 j_4} \Big\} \\
& = 0
\end{aligned}$$

as a consequence of the symmetries of the vectors $\vec{\xi}$, so the last column on the right-hand side of (9) will contribute nothing to the final sum and can therefore be dropped. Then add and subtract the sum $(14;1423)+(24;2413)+(34;3412)$ on the right-hand side of (9). The negative terms there can be regrouped and rewritten as the sum

$$-\sum_{\nu=1}^{3}\Big\{(\nu 4;1234)+(\nu 4;1324)+(\nu 4;1423)\Big\},$$

and just as in (10) this will contribute nothing to the final result. The positive terms are then the only ones that need be considered, and they can be rewritten as the sum

$$\sum_{\sigma\in\mathfrak{S}(2;2)}(\sigma 1\,\sigma 2;\sigma 1\,\sigma 2\,\sigma 3\,\sigma 4),$$

yielding the first line on the right-hand side of the formula of the corollary. The second line on the right-hand side of (2) can be rewritten correspondingly in the form

$$\sum_{\pi,\lambda,j}\vec{\beta}_{\lambda j_2 j_3 j_4}\big(w_{z_{\pi 1}}(a_\lambda)-w_{z_{\pi 1}}(a_4)\big)w'_{j_2}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

$$=\sum_{\pi,j}\sum_{\nu=1}^{4}\vec{\beta}^{*}_{\nu j_2 j_3 j_4}w_{z_{\pi 1}}(a_\nu)w'_{j_2}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

where

$$(11)\qquad \vec{\beta}^*_{\nu j_2 j_3 j_4} = \begin{cases} \vec{\beta}_{\nu j_2 j_3 j_4} & \text{for} \quad \nu = 1,2,3, \\ -\sum_{\lambda=1}^{3} \vec{\beta}_{\lambda j_2 j_3 j_4} & \text{for} \quad \nu = 4; \end{cases}$$

these are consequently uniquely determined holomorphic functions on $\widetilde{M}$, and it is evident from (11) that they satisfy (7) as desired.

Now the first line on the right-hand side of (2) has been rewritten so as to exhibit it as a symmetric function of the variables a_i, while the left-hand side is obviously symmetric, so the remaining terms on the right-hand side are also symmetric. It is evident from this that for any permutation $\sigma \in \mathfrak{S}_4$

$$\begin{aligned} &\sum_{\pi,\nu,j} \vec{\beta}^*_{\nu j_2 j_3 j_4}(a_1, a_2, a_3, a_4) w_{z_{\pi 1}}(a_\nu) w'_{j_2}(z_{\pi 2}) \dots w'_{j_4}(z_{\pi 4}) \\ &\qquad + \sum_{j} \vec{\beta}_{j_1 j_2 j_3 j_4}(a_1, a_2, a_3, a_4) w'_{j_1}(z_1) \dots w'_{j_4}(z_4) \\ &= \sum_{\pi,\nu,j} \vec{\beta}^*_{\nu j_2 j_3 j_4}(a_{\sigma 1}, a_{\sigma 2}, a_{\sigma 3}, a_{\sigma 4}) w_{z_{\pi 1}}(a_{\sigma\nu}) w'_{j_2}(z_{\pi 2}) \dots w'_{j_4}(z_{\pi 4}) \\ &\qquad + \sum_{j} \vec{\beta}_{j_1 j_2 j_3 j_4}(a_{\sigma 1}, a_{\sigma 2}, a_{\sigma 3}, a_{\sigma 4}) w'_{j_1}(z_1) \dots w'_{j_4}(z_4) \end{aligned}$$

Comparing the residues at the point $z_m = a_n$ on the two sides of this equation shows that

$$\sum_{\substack{j \\ \pi 1 = m}} \vec{\beta}^*_{n j_2 j_3 j_4}(a_1, a_2, a_3, a_4) w'_{j_2}(z_{\pi 2}) \dots w'_{j_4}(z_{\pi 4})$$

$$= \sum_{\substack{j \\ \pi 1 = m}} \vec{\beta}^*_{(\sigma^{-1} n) j_2 j_3 j_4}(a_{\sigma 1}, a_{\sigma 2}, a_{\sigma 3}, a_{\sigma 4}) w'_{j_2}(z_{\pi 2}) \dots w'_{j_4}(z_{\pi 4}),$$

hence that these functions satisfy (8) as desired. From this it is easy to see that the second line on the right-hand side of (2) is also symmetric in the variables a_i, particularly when rewritten as in the formula of the corollary. The last line and hence each separate coefficient $\vec{\beta}_{j_1 j_2 j_3 j_4}$ must then be symmetric as well, to conclude the proof.

If the projection mapping P is one for which $P\vec{\alpha}^{0;2;2}_{\lambda_1 \lambda_2 j_3 j_4} = 0$ then the dominant terms are $P\vec{\alpha}^{0;1;3}_{\lambda j_2 j_3 j_4}$ and the preceding expansion can be rewritten as follows.

Corollary 2. *With the hypotheses and notation as in the theorem, if all* *the vectors* $\vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4}$ *vanish then*

(12)

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3+z_4-a_1-a_2-a_3-a_4)\big)Q(z_1,z_2,z_3,z_4;a_1,a_2,a_3,a_4)^{-2}$$

$$= \sum_{\pi,\rho\in\mathfrak{S}(0,1,3)} \sum_{j,k} \vec{\xi}^{k_1 k_2 k_3 k_4}_{j_2 j_3 j_4} w'_{k_1}(a_{\rho 1})w'_{k_2}(a_{\rho 2})w'_{k_3}(a_{\rho 3})w'_{k_4}(a_4) \bullet$$

$$\bullet\, w'_{a_{\rho 1},a_4}(z_{\pi 1})w'_{j_2}(z_{\pi 2})w'_{j_3}(z_{\pi 3})w'_{j_4}(z_{\pi 4})$$

$$+\sum_j \vec{\beta}_{j_1 j_2 j_3 j_4}(a_1,a_2,a_3,a_4)w'_{j_1}(z_1)w'_{j_2}(z_2)w'_{j_3}(z_3)w'_{j_4}(z_4)$$

for some uniquely determined vectors $\vec{\xi}^{k_1 k_2 k_3 k_4}_{j_2 j_3 j_4}$ *in the range of* P *and holomorphic functions* $\vec{\beta}_{j_1 j_2 j_3 j_4}$ *on* $\widetilde{M}^4$ *with values in the range of* P. *The vectors* $\vec{\xi}^{i_1 j_2 j_3 j_4}_{k_2 k_3 k_4}$ *are symmetric in the indices* j_2, j_3, j_4 *and in the indices* k_2, k_3, k_4, *and further satisfy*

$$(13)\quad \vec{\xi}^{i_1 j_2 j_3 j_4}_{k_2 k_3 k_4} = -\vec{\xi}^{i_1 k_2 k_3 k_4}_{j_2 j_3 j_4},\ \vec{\xi}^{i_1 i_2 i_3 i_4}_{k_2 k_3 k_4} + \vec{\xi}^{i_2 i_1 i_3 i_4}_{k_2 k_3 k_4} + \vec{\xi}^{i_3 i_1 i_2 i_4}_{k_2 k_3 k_4} + \vec{\xi}^{i_4 i_1 i_2 i_3}_{k_2 k_3 k_4} = 0;$$

they are determined uniquely by the identity

(14)

$$\partial_i P\vec{\theta}_2\big(w(z_1+z_2+z_3-a_1-a_2-a_3)\big)Q(z_1,z_2,z_3;a_1,a_2,a_3)^{-2}$$

$$= \sum_{j,k} \vec{\xi}^{i j_1 j_2 j_3}_{k_1 k_2 k_3} w'_{j_1}(a_1)w'_{j_2}(a_2)w'_{j_3}(a_3)w'_{k_1}(z_1)w'_{k_2}(z_2)w'_{k_3}(z_3).$$

The vectors $\vec{\beta}_{j_1 j_2 j_3 j_4}(a_1,a_2,a_3,a_4)$ *are symmetric in the indices* j_1, j_2, j_3, j_4.

Proof. If all the vectors $\vec{\xi}^{k_1 k_2 k_3 k_4}_{j_3 j_4}$ vanish then the dominant terms in the expansion are $P\vec{\alpha}^{0;1;3}_{\lambda j_2 j_3 j_4}$, which have the explicit form given in Theorem 3;

their contribution to the expansion is

$$\sum_{\pi\in\mathfrak{S}(0,1,3)}\sum_{\lambda,j} P\vec{\alpha}^{0;1;3}_{\lambda j_2 j_3 j_4} u'_\lambda(z_{\pi 1}) w'_{j_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3}) w'_{j_4}(z_{\pi 4})$$

$$= \sum_{\pi\in\mathfrak{S}(0,1,3)}\sum_{j,k}\Big[\vec{\xi}^{k_1}{}^{k_2 k_3 k_4}_{j_2 j_3 j_4} u'_1(z_{\pi 1}) + \vec{\xi}^{k_2}{}^{k_1 k_3 k_4}_{j_2 j_3 j_4} u'_2(z_{\pi 1}) + \vec{\xi}^{k_3}{}^{k_1 k_2 k_4}_{j_2 j_3 j_4} u'_3(z_{\pi 1})\Big] \bullet$$

$$\bullet\, w'_{k_1}(a_1) w'_{k_2}(a_2) w'_{k_3}(a_3) w'_{k_4}(a_4) w'_{j_2}(z_{\pi 2}) w'_{j_3}(z_{\pi 3}) w'_{j_4}(z_{\pi 4}).$$

When rearranged by grouping together those terms with the same vectors $\vec{\xi}$ this yields the first part of the desired formula. The coefficients have the desired symmetries by Theorems 1 and 3. The second formula determines the vectors involved by the Corollary to Theorem 3. The functions $\vec{\beta}_{j_1 j_2 j_3 j_4}$ are just those of the theorem itself so are holomorphic as desired, to conclude the proof.

COROLLARY 3. *With the hypotheses and notation as in the theorem, if both the vectors* $\vec{\xi}^{k_1}{}^{k_2 k_3 k_4}_{j_3 j_4}$ *and* $\vec{\xi}^{k_1}{}^{k_2 k_3 k_4}_{j_2 j_3 j_4}$ *all vanish then*

$$P\vec{\theta}_2\big(w(z_1+z_2+z_3+z_4-a_1-a_2-a_3-a_4)\big)Q(z_1,z_2,z_3,z_4;a_1,a_2,a_3,a_4)^{-2}$$

$$= \sum_{j,k} \vec{\xi}^{k_1 k_2 k_3 k_4}_{j_1 j_2 j_3 j_4} w'_{k_1}(a_1)\dots w'_{k_4}(a_4) w'_{j_1}(z_1)\dots w'_{j_4}(z_4)$$

for some uniquely determined vectors $\vec{\xi}^{k_1 k_2 k_3 k_4}_{j_1 j_2 j_3 j_4}$ *in the range of* P*; these vectors are symmetric in the indices* $j_1 j_2 j_3 j_4$ *and in the indices* $k_1 k_2 k_3 k_4$*, as well as under the interchange of the* j*'s and the* k*'s.*

Proof. Under the hypotheses of this corollary the dominant terms are $P\vec{\alpha}^{0;0;4}_{j_1j_2j_3j_4}$; the expansion as asserted follows immediately from Theorem 3, as do the symmetries of the vectors involved.

Just as for the ternary expansion so also in the quaternary expansion do the auxiliary vectors involved satisfy some further conditions beyond the symmetries already noted. First it follows immediately from (4) that

$$\sum_{k_1k_2} \vec{\xi}^{\,i_1i_2j_1j_2}_{\;k_1k_2} w'_{k_1}(z)w'_{k_2}(z) = 0 \tag{15}$$

for all points $z \in \widetilde{M}$ and all indices i, j. Thus if $p_i(x) = \sum_{j_1j_2} p^i_{j_1j_2} x_{j_1} x_{j_2}$ is a basis for the Petri space $\mathcal{P}_2$ then

$$\vec{\xi}^{\,i_1i_2j_1j_2}_{\;k_1k_2} = \sum \vec{\eta}^{\,\ell_1\ell_2}_{\;\ell_3} p^{\ell_1}_{i_1i_2} p^{\ell_2}_{j_1j_2} p^{\ell_3}_{k_1k_2} \tag{16}$$

for some uniquely determined vectors $\vec{\eta}^{\,\ell_1\ell_2}_{\;\ell_3}$ in the range of P; these vectors are symmetric in the three indices ℓ_1, ℓ_2, ℓ_3, and in addition as a consequence of (3) satisfy

$$\sum_{\ell_1\ell_2} \vec{\eta}^{\,\ell_1\ell_2}_{\;j} \left(p^{\ell_1}_{i_1i_2} p^{\ell_2}_{i_3i_4} + p^{\ell_1}_{i_1i_3} p^{\ell_2}_{i_2i_4} + p^{\ell_1}_{i_1i_4} p^{\ell_2}_{i_2i_3} \right) = 0 \tag{17}$$

for all indices i_1, i_2, i_3, i_4, j. Next *if* $\vec{\xi}^{\,i_1i_2j_1j_2}_{\;k_1k_2} = 0$ *for all indices* i, j, k then it follows immediately from (14) that

$$\sum_{k_1k_2} \vec{\xi}^{\,i_1i_2i_3i_4}_{\;j\,k_1k_2} w'_{k_1}(z)w'_{k_2}(z) = 0 \tag{18}$$

for all points $z \in \widetilde{M}$ and all indices i, j. Thus if

$$p_i(x) = \sum_{j_1 j_2 j_3} p^i_{j_1 j_2 j_3} x_{j_1} x_{j_2} x_{j_3}$$

is a basis for the modified Petri space $\mathcal{P}_3^*$ then

$$\vec{\xi}^{i j_1 \, j_2 \, j_3}_{\;\; k_1 k_2 k_3} = \sum_{\ell_1 \ell_2} \vec{\eta}^{\,i \ell_1}_{\;\;\ell_2} p^{\ell_1}_{j_1 j_2 j_3} p^{\ell_2}_{k_1 k_2 k_3} \tag{19}$$

for some uniquely determined vectors $\vec{\eta}^{\,i\ell_1}_{\;\;\ell_2}$ in the range of P; these vectors are skew-symmetric in the indices ℓ_1, ℓ_2, and in addition by (13) satisfy

$$\sum_{\ell} \left(\vec{\eta}^{\,i_1 \ell}_{\;\;j} p^{\ell}_{i_2 i_3 i_4} + \vec{\eta}^{\,i_2 \ell}_{\;\;j} p^{\ell}_{i_1 i_3 i_4} + \vec{\eta}^{\,i_3 \ell}_{\;\;j} p^{\ell}_{i_1 i_2 i_4} + \vec{\eta}^{\,i_4 \ell}_{\;\;j} p^{\ell}_{i_1 i_2 i_3} \right) = 0 \tag{20}$$

for all indices i_1, i_2, i_3, i_4, j. Finally *if* $\vec{\xi}^{i_1 i_2 j_1 \, j_2}_{\;\;\; k_1 k_2} = 0$ *and* $\vec{\xi}^{i_1 j_1 j_2 \, j_3}_{\;\;\; k_1 k_2 k_3} = 0$ *for all indices* i, j, k then it follows immediately from Corollary 3 that

$$\sum_{k_1 k_2} \vec{\xi}^{\,i_1 \, i_2 \, i_3 \, i_4}_{j_1 j_2 \, k_1 k_2} w'_{k_1}(z) w'_{k_2}(z) = 0 \tag{21}$$

for all points $z \in \widetilde{M}$ and all indices i, j. Thus if

$$p_i(x) = \sum_{j_1 j_2 j_3 j_4} p^i_{j_1 \ldots j_4} x_{j_1} \ldots x_{j_4}$$

is a basis for the modified Petri space $\mathcal{P}_4^*$ then

$$
\vec{\xi}^{\,i_1\,i_2\,i_3\,i_4}_{j_1 j_2\,j_3 j_4} = \sum_{k_1 k_2} \vec{\eta}^{\,\ell_1}_{\ell_2} p^{\ell_1}_{i_1 i_2 i_3 i_4} p^{\ell_2}_{j_1 j_2 j_3 j_4} \tag{22}
$$

for some uniquely determined vectors $\vec{\eta}^{\,\ell_1}_{\ell_2}$ in the range of P, and these vectors are symmetric in the indices k_1, k_2.

In this case the extension from L_3 to L_4, where of course L_4 is the span of all vectors $\vec{\theta}_2\big(w(z_1+\cdots+z_4-a_1-\cdots-a_4)\big)$ for arbitrary points $z_i,\ a_i \in \widetilde{M}$, naturally splits into three steps as in the following diagram

$$
\begin{array}{rccccccc}
 & L_3 & \subseteq & L_4^* & \subseteq & L_4^{**} & \subseteq & L_4 \\
P_3 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \subseteq & L_4' = L_4^*/L_3 & \subseteq & L_4^{**}/L_3 & \subseteq & L_4/L_3 \\
P_3' & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \subseteq & 0 & \subseteq & L_4'' = L_4^{**}/L_4' & \subseteq & L_4/L_4^* \\
P_3'' & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \subseteq & 0 & \subseteq & 0 & \subseteq & L_4/L_4^{**}
\end{array} \tag{23}
$$

where $L_4' =$ span $\vec{\xi}^{\,k_1 k_2 k_3 k_4}_{j_1\,j_2}$, $L_4'' =$ span $\vec{\xi}^{\,k_1 k_2 k_3 k_4}_{j_1\,j_2\,j_3}$, and

$L_4/L_4^{**} =$ span $\vec{\xi}^{\,k_1 k_2 k_3 k_4}_{j_1\,j_2\,j_3\,j_4}$; the projections P_3, P_3', P_3'' are the minimal possible ones with the indicated kernels. It is again possible to describe this situation by using the auxiliary vectors $\vec{\eta}^{\,\ell_1 \ell_2}_{\ell_3}$, $\vec{\eta}^{\,i\ell_1}_{\ell_2}$, $\vec{\eta}^{\,\ell_1}_{\ell_2}$ to describe the subspaces involved, although this requires a choice of bases for the Petri spaces $\mathcal{P}_2$, $\mathcal{P}_3^*$, $\mathcal{P}_4^*$.