# F. Differential expansions of Riemannian theta functions

## §1. The general expansion formula.

The model for the differential expansion of the second order theta functions is the formula of Theorem D6, which can be written

$$(1)\quad \vec{\theta}_2(w(z-a))\ q(z,a)^{-2} = $$
$$= \vec{\theta}_2(0)\cdot w_a'(z) + 1/2 \sum_{j_1 j_2} \partial_{j_1 j_2}\vec{\theta}_2(0)\cdot w_{j_1}'(z) w_{j_2}'(a).$$

Each component on the left-hand side of (1), when viewed as a function of $z \in \tilde{M}$ for a fixed point $a \in \tilde{M}$, is a meromorphic relatively automorphic function for the canonical factor of automorphy $\kappa$, or equivalently a meromorphic Abelian differential on M, with at most double poles at the points Ta; that function can be written uniquely as some linear combination of the canonical Abelian differential of the second kind $w_a'(z)$ and the ordinary canonical Abelian differentials $w_j'(z)$, and (1) describes this expansion quite explicitly. This result was derived from the trisecant formula by a limiting process, but can just as easily be obtained directly. There is always some expansion in terms of the canonical Abelian differentials, hence some formula of the form

$$\vec{\theta}_2(w(z-a)) = \vec{\alpha}\ q(z,a)^2\ w_a'(z) + \textstyle\sum_j \vec{\alpha}_j q(z,a)^2 w_j'(z)$$

where $\vec{\alpha}, \vec{\alpha}_j \in \mathbb{C}^{2^g}$ are vectors that depend on $a$ but not on $z$, and it is merely a matter of determining these vectors explicitly. That can be done by much the same sort of limiting process that was earlier applied to the trisecant formula; the argument will be used repeatedly in the sequel, in more general circumstances, so will not be discussed further here.

The general differential expansion arises as the analogous treatment of the function $\vec{\theta}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))$. This expansion can be developed through appropriate limiting cases of the multisecant formula, but the direct approach seems to yield a clearer overall view of the structure of the formulas. The general case is somewhat more complicated than the special case (1) though; with more poles involved, meromorphic differentials of both second and third kinds are involved, and the formulas depend critically on more detailed properties of the Riemann surface.

Consider then the theta function

$$(2)\quad \vec{\theta}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n)) = \\ = \vec{\theta}_2[2w(z_2+\cdots+z_n-a_1-\cdots-a_n)]\ (w(z_1))$$

as a function of the variable $z_1$ alone. As in the earlier discussion, each component of (2) is a relatively automorphic function for the factor of automorphy

$$\rho_{2w(a_1+\cdots+a_n-z_2-\cdots-z_n)}\,\zeta^2 = \rho_{k+2w(a_1+\cdots+a_n-z_2-\cdots-z_n)}\,\zeta^{2g} = \\ = \kappa\zeta^2_{a_1}\cdots\zeta^2_{a_n}\zeta^{-2}_{z_2}\cdots\zeta^{-2}_{z_n})$$

where as usual $\zeta_z = \rho_{w(z)}\zeta$ is the standard factor of automorphy associated to the divisor $1\cdot z$. To obtain functions of $z_1$ that transform by the canonical factor of automorphy $\kappa$ alone, the function (2) can merely be multiplied by the meromorphic factor $[\Pi_{j=2}^{n} q(z_1,z_j)^2][\Pi_{k=1}^{n} q(z_1,a_k)^{-2}]$; the result will then be a meromorphic function, with singularities at most those of the divisor $-2(a_1+\cdots+a_n)$ on M. For symmetry however it is better to consider instead the function

$$(3)\quad \vec{f} = \vec{\theta}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))[\prod_{\mu<\nu} q(z_\mu,z_\nu)^2 q(a_\mu,a_\nu)^2][\prod_{\mu,\nu} q(z_\mu,a_\nu)^{-2}];$$

the terms in this multiplicative factor that involve the variable $z_1$ are just as considered above, but the factor is symmetric in all the variables $z_1,\ldots,z_n$, as well as in the variables $a_1,\ldots,a_n$ and under the exchange of z and a. The function f is meromorphic in all its variables, with the singularities in each variable corresponding to those in the variable $z_1$. It is sometimes more convenient to consider instead the holomorphic function

$$(4)\quad \vec{f}^* = \vec{\theta}_2(w(z_1+\cdots+z_n-a_1\cdots-a_n))[\prod_{\mu<\nu} q(z_\mu,z_\nu)^2 q(a_\mu,a_\nu)^2].$$

The components of $\vec{f}^*$ as functions of any variable $z_i$ belong to the space of relatively automorphic functions $\Gamma(\kappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2)$.

It follows immediately from the Riemann-Roch theorem that

$$\gamma(\kappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2) = g + 2n - 1$$

whenever $n \geq 1$. If the points $a_i \in \tilde{M}$ represent distinct points of M it is possible to write down a convenient explicit basis for the vector space $\Gamma(\kappa\zeta_{a_1}^2\cdots\zeta_{a_n}^2)$ in terms of the canonical Abelian differentials $w_j'(z)$ and the canonical meromorphic differentials $w_{a_\mu}'(z)$, $w_{a_\lambda,a_\nu}'(z)$ of the second and third kinds. To simplify the notation somewhat, set

$$(5)\quad \begin{aligned} u_\lambda'(z) &= w_{a_\lambda,a_n}'(z) \quad \text{for } \lambda = 1, \cdots, n-1, \text{ and} \\ v_\mu'(z) &= w_{a_\mu}'(z) \quad \text{for } \mu=1, \cdots, n. \end{aligned}$$

In these terms the functions

$$
\begin{aligned}
&w_j'(z) \cdot \prod_{\nu=1}^{n} q(z,a_\nu)^2 \quad \text{for } j=1, \ldots, g, \\
(6)\quad &u_\lambda'(z) \cdot \prod_{\nu=1}^{n} q(z,a_\nu)^2 \quad \text{for } \lambda=1, \ldots, n-1, \\
&v_\mu'(z) \cdot \prod_{\nu=1}^{n} q(z,a_\nu)^2 \quad \text{for } \mu=1, \ldots, n
\end{aligned}
$$

clearly belong to the space $\Gamma\left(\kappa\zeta_{a_1}^2 \cdots \zeta_{a_n}^2\right)$, and so long as $a_1, \ldots, a_n$ represent distinct points of M they are also linearly independent. Indeed the g functions on the first line are linearly independent, since the Abelian differentials $w_j'(z)$ are, and all vanish to at least the second order at each of the points $a_i$; the $\lambda$-th function on the second line on the other hand has a simple zero at $a_\lambda$ and $a_n$ but double zeros at the other points $a_i$, while the $\mu$-th function on the last line is nonzero at $a_\mu$ but has double zeros at the other points $a_i$, so all these functions are linearly independent as desired. This provides a convenient basis for the space $\Gamma\left(\kappa\zeta_{a_1}^2 \cdots \zeta_{a_n}^2\right)$, a basis that will be used consistently throughout the discussion here. A slight disadvantage is that the use of the functions $u_\lambda'(z)$ implies that the point $a_n$ is singled out, thus destroying the complete symmetry of the basis in the variables $a_1, \ldots, a_n$; that is not a serious matter though, provided that it is kept in mind, and is more than counterbalanced by the simplicity of this basis. It must also be kept in mind that these functions are no longer linearly independent if there are coincidences among the points of M represented by $a_i$; a basis then requires the introduction of meromorphic differentials having yet higher order singularities.

Before turning to the differential expansion itself, it is convenient to introduce one further bit of notation. Let $\mathfrak{S}_n$ be the symmetric group on n letters, viewed as the group of permutations of the n indices 1,2, ..., n. If k, $\ell$, m are any nonnegative integers such that $k + \ell + m = n$, let $\mathfrak{S}(m,\ell,k)$ denote the subset of $\mathfrak{S}_n$ consisting of those permutations $\pi \in \mathfrak{S}_n$ such that

$$\pi(1) < \pi(2) < \cdots < \pi(m),$$

$$\pi(m+1) < \pi(m+2) < \cdots \pi(m+\ell),$$

$$\pi(m+\ell+1) < \pi(m+\ell+2) < \cdots < \pi(n).$$

Clearly $\mathfrak{S}(m,\ell,k)$ is just a particular set of coset representatives of the subgroup $\mathfrak{S}_m \times \mathfrak{S}_\ell \times \mathfrak{S}_k \subseteq \mathfrak{S}_n$ acting in the natural way on this index set. In these terms, the general form of the differential expansion is as follows.

<u>Theorem</u> 1. There are uniquely determined meromorphic vector-valued functions

$$\vec{\alpha}^{m;\ell;k}_{\mu;\lambda;j} = \vec{\alpha}^{m;\ell;k}_{\mu_1,\ldots,\mu_m;\ \lambda_1,\ldots,\lambda_\ell;\ j_1,\ldots,j_k}$$

of the points $a_1,\ldots, a_n \in \tilde{M}$, indexed by nonnegative integers $k,\ell,m$ such that $k + \ell + m = n$ and for any such triple by m indices $\mu_1,\ldots,\mu_m \in [1,n]$, by $\ell$ indices $\lambda_1,\ldots,\lambda_\ell \in [1,n-1]$, and by k indices $j_1,\ldots,j_k \in [1,g]$, with the following properties:

(i) the only singularities of these functions lie on the subvarieties $a_i = Ta_j$ of $\tilde{M}$, for $i \neq j$ and $T \in \Gamma$;

(ii) these functions are symmetric in the m indices $\mu_i$, in the $\ell$ indices $\lambda_i$, and in the k indices $j_i$;

(iii) with the notation (3), there is the identity

$$\vec{f}(z_1, \ldots, z_n; a_1, \ldots, a_n) =$$
$$= \sum_{k+\ell+m=n} \sum_{\pi\in\mathfrak{S}(m,\ell,k)} \sum_{\mu,\lambda,j} \vec{\alpha}^{m;\ell;k}_{\mu;\lambda;j} v'_{\mu_1}(z_{\pi(1)}) \cdots v'_{\mu_m}(z_{\pi(m)}) \cdot$$
$$u'_{\lambda_1}(z_{\pi(m+1)}) \cdots u'_{\lambda_\ell}(z_{\pi(m+\ell)}) w'_{j_1}(z_{\pi(m+\ell+1)}) \cdots w'_{j_k}(z_{\pi(n)})$$

that holds for all points $a_i, z_i \in \tilde{M}$.

Proof First suppose that $a_1, \ldots, a_n$ are fixed points of $\tilde{M}$ that represent distinct points of M, and to simplify the notation let $f_1, \ldots, f_{g+2n-1}$ denote the $g+2n-1$ basis functions (6). Each component of the function $\vec{f}^*$ defined by (4) belongs to the space $\Gamma(\kappa\zeta^2_{a_1}\cdots\zeta^2_{a_n})$ when viewed as a function of $z_1 \in \tilde{M}$ for any fixed points $z_2, \ldots, z_n$, so there are uniquely determined vectors $\vec{\alpha}_{i_1}$ such that

$$\vec{f}^*(z_1) = \sum_{i_1} \vec{\alpha}^*_{i_1} f_{i_1}(z_1).$$

Each component of the vector $\vec{\alpha}_{i_1}$ is evidently a holomorphic function of the variables $z_2, \ldots, z_n$, and when viewed as a function of $z_2 \in \tilde{M}$ for any fixed points $z_3, \ldots, z_n$ belongs to $\Gamma(\kappa\zeta^2_{a_1}\cdots\zeta^2_{a_n})$; hence there are uniquely determined vectors $\vec{\alpha}_{i_1 i_2}$ such that

$$\vec{\alpha}_{i_1}(z_2) = \sum_{i_2} \vec{\alpha}_{i_1 i_2} f_{i_2}(z_2).$$

The process can evidently be continued, and there results an expansion

$$\vec{f}^*(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n} \vec{\alpha}_{i_1 \ldots i_n} f_{i_1}(z_1) \cdots f_{i_n}(z_n)$$

where the vectors $\vec{\alpha}_{i_1 \ldots i_n}$ are uniquely determined and are independent of $z_1, \ldots, z_n$. These coefficients of course depend on the points $a_1, \ldots, a_n$, and can be calculated by Cramer's formula in the usual manner; when viewed as functions of the points $a_i \in \tilde{M}$ they are consequently holomorphic so long as the functions (6) are linearly independent, but at least extend to meromorphic functions on $\tilde{M}^n$. Note that since $\vec{f}^*$ is symmetric in the variables $z_1, \ldots, z_n$ the coefficients $\vec{\alpha}_{i_1 \ldots i_n}$ must be symmetric in the indices $i_1, \ldots, i_n$.

This is of course easy and straightforward; the only complication arises in taking into account quite explicitly the fact that the functions $f_1, \ldots, f_{g+2n-1}$ of the basis (6) really naturally split into three distinct types, which must be considered somewhat separately. That corresponds to a decomposition of the index set $I = [1, g+2n-1]$ as the disjoint union of three subsets $I' = [1, g]$, $I'' = [g+1, g+n-1]$, and $I''' = [g+n, \ldots, g+2n-1]$. If in a typical term

$$\vec{\alpha}_{i_1 \ldots i_n} f_{i_1}(z_1) \cdots f_{i_n}(z_n)$$

there are $k$ of the indices $i_1, \ldots, i_n$ in the range $I'$, $\ell$ in the range $I''$, and $m$ in the range $I'''$, then there is a unique rearrangement of these indices grouping together first the $m$ indices in the range $I'''$, next the $\ell$ indices in the range $I''$, and finally the $k$ indices in the range $I'$, and in each case leaving the indices in each range in the same order as that in which they appeared initially. This rearrangement will have the form

$$\vec{\alpha}_{i_{\pi(1)} \ldots i_{\pi(n)}} f_{i_{\pi(1)}}(z_{\pi(1)}) \cdots f_{i_{\pi(n)}}(z_{\pi(n)})$$

for some element $\pi \in \mathfrak{S}(m,\ell,k)$, where $i_{\pi(1)},\dots,i_{\pi(m)}$ are in I''', $i_{\pi(m+1)},\dots,i_{\pi(m+\ell)}$ are in I'', and $i_{\pi(m+\ell+1)},\dots i_{\pi(n)}$ are in I'; it is thus in the form as in the statement of the theorem. As the indices $i_1,\dots, i_n$ vary but $k,\ell,m$ remain fixed, the positions of those indices that are in I''', I'', or I' will vary over all possibilities, hence $\pi$ will range over the full subset $\mathfrak{S}(m,\ell,k)$, while for each rearrangement $\pi$ the individual indices will range over all possible values. The result is precisely as in the statement of the theorem, after dividing throughout by $\Pi_{\mu,\nu}q(z_\mu,a_\nu)^{-2}$.

The formula derived above seems rather complicated, but is actually quite natural; it is just that there are really quite a number of distinct forms for the terms in this differential expansion. This can possibly best be illustrated by writing out quite explicitly the first nontrivial case, that in which n = 2. The result is the formula

$$\text{(7)}\quad \vec{\theta}_2(w(z_1+z_2-a_1-a_2))\; q(z_1,z_2)^2\; q(a_1,a_2)^2\; [\prod_{\mu,\nu=1}^{2} q(z_\mu,a_\nu)^{-2}] =$$

$$= \sum_\mu \vec{\alpha}^{2;0;0}_{\mu_1\mu_2}\, v'_{\mu_1}(z_1)v'_{\mu_2}(z_2) + \sum_{\mu,\lambda}\vec{\alpha}^{1;1;0}_{\mu;\lambda}[v'_\mu(z_1)u'_\lambda(z_2) + v'_\mu(z_2)u'_\lambda(z_1)] +$$

$$+ \sum_{\mu,j}\vec{\alpha}^{1;0;1}_{\mu;j}[v'_\mu(z_1)w'_j(z_2) + v'_\mu(z_2)w'_j(z_1)] + \sum_\lambda \vec{\alpha}^{0;2;0}_{\lambda_1\lambda_2}u'_{\lambda_1}(z_1)u'_{\lambda_2}(z_2) +$$

$$+ \sum_{\lambda,j}\vec{\alpha}^{0;1;1}_{\lambda;j}[u'_\lambda(z_1)w'_j(z_2) + u'_\lambda(z_2)w'_j(z_1)] + \sum_j \vec{\alpha}^{0;0;2}_{j_1j_2}w'_{j_1}(z_1)w'_{j_2}(z_2).$$

The instances in which a nontrivial permutation $\pi \in \mathfrak{S}(m,\ell,k)$ enters are written out in square brackets. The expansion has much the same form as the decomposition of a homogeneous quadratic polynomial when the variables are of three classes, corresponding to a dissection of the matrix representing that polynomial into three pieces on each side. For n = 3 the formula is akin to the corresponding homogeneous cubic polynomial or three dimensional matrix, and so on.

The expansion formula of Theorem 1 is of course a purely formal result, but is a convenient basis for a more detailed and substantive analysis. There is not a simple explicit expression for the coefficient vectors in that expansion for the cases $n > 1$, no real parallel to the simple universal formula (1) in the case $n = 1$; but there are various universal restrictions on these coefficients. The discussion of the general form of the formulas can be considerably simplified by the judicious use of suitable projection operators in the vector space $\mathbb{C}^{2^g}$. It may be helpful first to see another instance in which such projections can be used, to clarify their role here. The KP equation in the form of Theorem D8 asserts that for any point $z \in \tilde{M}$ the vector

$$\sum_{j_1\cdots j_4} \partial_{j_1\cdots j_4} \vec{\theta}_2(0)\ w'_{j_1}(z)\cdots w'_{j_4}(z)$$

lies in the subspace of $\mathbb{C}^{2^g}$ spanned by the vectors $\vec{\theta}_2(0)$ and $\partial_{j_1 j_2} \vec{\theta}_2(0)$. It actually says much more, indeed gives quite explicitly the coefficients of the linear relation among these vectors. From some points of view the most interesting thing about this equation is just that there is such a linear relation, not what it is explicitly. The mere existence of such a relation can be handled quite simply in terms of a suitable projection operator; for it amounts to the condition that if $P:\mathbb{C}^{2^g} \to \mathbb{C}^n$ is any linear mapping such that $P\vec{\theta}_2(0) = P\partial_{j_1 j_2}\vec{\theta}_2(0) = 0$ for all indices $j_1, j_2$, then

$$\sum_{j_1\cdots j_4} P\partial_{j_1\cdots j_4} \vec{\theta}_2(0)\ w'_{j_1}(z)\cdots w'_{j_4}(z) = 0$$

for all points $z \in \tilde{M}$. The same sort of simplification can be used to handle the differential expansions of the second order theta functions. The idea is to assume that something is known about the expansion formula of type n-1, and then to choose a projection operator that ignores altogether the formula of type n-1 but indicates what additional terms appear in the expansion formula of type n.

Theorem 2 Let $P:\mathbb{C}^{2^g} \to \mathbb{C}^n$ be any linear mapping such that

$$P\vec{\theta}_2(w(z_1+\cdots+z_{n-1}-a_1-\cdots-a_{n-1})) = 0$$

for all points $z_i, a_i \in \tilde{M}$. When this projection is applied to the vectors appearing in the expansion formula of the preceding theorem:

(i) $P\vec{\alpha}^{m;\ell;k}_{\mu;\lambda;j} = 0$ whenever $m > 0$;

(ii) $P\vec{\alpha}^{0;\ell;k}_{\lambda_1,\ldots,\lambda_\ell;j_1,\ldots,j_k} = 0$ whenever $\lambda_{i_1} = \lambda_{i_2}$ for $i_1 \neq i_2$.

Proof (i) The hypothesis clearly implies that each component of the vector $P\vec{\theta}_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))$ when viewed as a holomorphic function of the variables $z_i \in \tilde{M}$ vanishes at the points $z_i = a_j$. That means that each component of the vector $P\vec{f}(z_1,\ldots,z_n;\ a_1,\ldots,a_n)$ has as singularities at most simple poles at the points $z_i = a_j$, so that the differential expansion does not actually involve any meromorphic differentials of the second kind, or equivalently, any function $v'_\mu(z_i)$. The assertion that none of these terms appear is precisely (i).

(ii) First multiply the expansion formula of Theorem 1 by $\Pi_{\mu,\nu} q(z_\mu, a_\nu)^2$, so that all the functions that are involved are holomorphic, and then apply the projection operator P; the result is an expansion of the holomorphic function $\overset{+}{P}f^*$ in terms of the holomorphic functions (6). Then apply the differential operator $\partial/\partial z_1$ and set $z_1 = a_1$. Since the hypothesis on P implies that the function $\overset{+}{P}\theta_2(w(z_1+\cdots+z_n-a_1-\cdots-a_n))$ has a zero at $z_1 = a_1$, the only possible nontrivial term that can arise on the left-hand side is that in which the differentiation is applied to this theta function rather than to the other factors in $\overset{+}{P}f^*$; the result on the left-hand side thus becomes

$$(8)\quad \Sigma_j w_j'(a_1)\,\partial_j \overset{+}{P}\theta_2(w(z_2+\cdots+z_n-a_2-\cdots-a_n))\cdot \cdot\Big[\prod_{1<\nu} q(z_\nu,a_1)^2\Big]\Big[\prod_{1<\mu<\nu} q(z_\mu,z_\nu)^2\Big]\Big[\prod_{\mu<\nu} q(a_\mu,a_\nu)^2\Big].$$

On the right-hand side the variable $z_1$ appears in the factors $q(z_1,a_\nu)^2$, and either in $u_\lambda'(z_1)$ for some $\lambda$ or in $w_j'(z_1)$ for some j, since from part (i) of the proof there are no terms $v_\mu'(z_1)$ appearing in the expansion. The only possible nontrivial terms that can arise are those in which the double zero of the factor $q(z_1,a_1)^2$ is cancelled, and since $n > 1$ that can only happen by differentiating one factor and multiplying the other by $u_1'(z_1) = w'_{a_1,a_n}(z_1)$ to cancel the zero by a simple pole; these are the terms for which $\ell \geq 1$, $\pi(1) = 1$, $\lambda_1 = 1$, and in view of the known local form of these functions the right hand side becomes

$$(9)\quad \sum_{\substack{\ell,k\\ \ell\geq 1}}\ \sum_{\substack{\pi\in\mathfrak{S}(0,\ell,k)\\ \pi(1)=1}}\ \sum_{\substack{\lambda,j\\ \lambda_1=1}} \overset{+}{P}\lambda^{0;\ell;k}_{\lambda;j}\, u'_{\lambda_2}(z_{\pi(2)})\cdots w'_{j_k}(z_{\pi(n)})\cdot \Big[\prod_{\nu>1} q(a_1,a_\nu)^2\Big]\Big[\prod_{\substack{\mu,\nu\\ \mu>1}} q(z_\mu,a_\nu)^2\Big],$$

for in canonical local coordinates near $z = a$ it has been observed that $q(z,a)w'_{a,b}(z) = [(z-a) + O(z-a)^3][\frac{1}{z-a} + O(1)] = 1+O(z-a)$. Then to the identity (8) = (9) apply the differential operator $\partial/\partial z_2$ and set $z_2 = a_1$. The left-hand side vanishes, since the factor $q(z_2,a_1)^2$ has a double zero at $z_2 = a_1$. On the right-hand side the variable $z_2$ appears in the factors $q(z_2,a_\nu)^2$, and in either $u'_\lambda(z_2)$ for some $\lambda$ or $w'_j(z_2)$ for some $j$. The only possible nontrivial terms that can appear are those in which the double zero of the factor $q(z_2,a_1)^2$ is cancelled, and that can only happen by differentiating one factor and multiplying the other factor by $u'_1(z_2) = w'_{a_1,a_n}(z_2)$ to cancel the other zero by a simple pole; these are the terms for which $\ell \geqq 2$, $\pi(2) = 2$, $\lambda_2 = 1$, and in view of the known local forms again the whole result is the identity

$$0 = \sum_{\substack{\ell,k \\ \ell \geqq 2}} \sum_{\substack{\pi\in\mathfrak{S}(0,\ell,k) \\ \pi(1)=1,\pi(2)=2}} \sum_{\substack{\lambda,j \\ \lambda_1=\lambda_2=1}} \overset{+}{P}\lambda^{0;\ell;k}_{\lambda;j} u'_{\lambda_3}(z_{\pi(3)})\cdots w'_{j_k}(z_{\pi(n)}) \cdot \Big[\prod_{\nu>1} q(a_1,a_\nu)^4\Big]\Big[\prod_{\substack{\mu,\nu \\ \mu>2}} q(z_\mu,a_\nu)^2\Big].$$

The expansions of this sort are unique, since the differentials involved are linearly independent, so all the coefficients $\overset{+}{P}\alpha^{0;\ell;k}_{\lambda;j}$ appearing here must vanish; thus $\overset{+}{P}\alpha^{0;\ell;k}_{11\lambda_3\cdots\lambda_\ell;j_{\ell+1}\cdots j_n} = 0$ for all $\ell \geqq 2$, $\lambda_i$, $j_i$. The coefficients are symmetric in the indices $\lambda_1,\ldots,\lambda_\ell$, and the same argument can of course be applied when setting $z_1 = z_2 = a_i$ for $i = 2,\ldots,n-1$, and that suffices for the proof.

## §2. Dominant terms in the expansion.

Let $P : \mathbb{E}^{2g} \to \mathbb{E}^n$ be any linear mapping such that

(1) $\quad P\overset{\to}{\theta}(w(z_1+\cdots+z_{n-1} - a_1 \cdots - a_{n-1})) = 0$

for all points $z_i, a_i \in \tilde{M}$, and consider the result of applying this operator to the expansion of Theorem 1; the result is the formula

(2) $\quad P\overset{\to}{\theta}(w(z_1+\cdots+z_n-a_1-\cdots-a_n)) \cdot [\prod_{\mu<\nu} q(z_\mu,z_\nu)^2 q(a_\mu,a_\nu)^2][\prod_{\mu,\nu} q(z_\mu,a_\nu)^{-2}] =$

$$= \sum_{k+\ell=n} \sum_{\pi\in\mathfrak{E}(0;\ell;k)} \sum_{\lambda,j} P\overset{\to}{\alpha}^{0;\ell;k}_{\lambda;j} u'_{\lambda_1}(z_{\pi(1)}) \cdots w'_{j_k}(z_{\pi(n)}),$$

in view of the result of Theorem 2. The number $\ell$ of singular differentials appearing in any term on the right hand side of (2) will be called the <u>order</u> of that term, and the terms of maximal order actually appearing nontrivially in (1) will be called the <u>dominant terms</u>; their common order will be called the <u>dominant order</u>. Thus the dominant order L is given by

(3) $\quad L = \max\{ \ell : P\overset{\to}{\alpha}^{0;\ell;k}_{\lambda;j} \neq 0 \text{ for some indices } \lambda, j\}.$

It should be observed that the dominant order and the set of dominant terms do depend on the particular projection operator P being considered. It was shown in Theorem 2 that $P\overset{\to}{\alpha}^{0;\ell;k}_{\lambda;j} = 0$ unless all the indices $\lambda_i$ are distinct, so since $1 \leq \lambda_i \leq n-1$ it is clear that $P\overset{\to}{\alpha}^{0;n;0}_{\lambda} = 0$ for all indices $\lambda_i$; thus the dominant order L must lie in the range $0 \leq L \leq n-1$. The coefficients of the dominant terms have a particularly simple form. Before discussing that, it is convenient to establish the following auxiliary combinatorial identity.

Lemma 1. If $x_{i;j} = x_{i_1,\dots i_n;j_1,\dots,j_n}$ are expressions that are symmetric in the indices i and in the indices j separately and that satisfy

(4) $\sum_{\lambda=1}^{n+1} x_{i_1,\dots,i_{n-1},j_\lambda;j_1,\dots,j_{\lambda-1},j_{\lambda+1},\dots,j_{n+1}} = 0$

for all indices $i_1,\dots,i_{n-1},j_1,\dots,j_{n+1}$, then

(5) $x_{i;j} = (-1)^n x_{j;i}.$

Proof To simplify the notation set

$$(1,2,\dots,n;n+1,\dots,2n) = x_{i_1,\dots,i_n;i_{n+1},\dots,i_{2n}}.$$

It will be demonstrated by induction on m that

(6) $(1,\dots,n;n+1,\dots,2n) =$

$$= (-1)^m \sum_J (1,\dots,n-m,j_1,\dots,j_m;n-m+1,\dots,n,j_{m+1},\dots,j_n)$$

where the summation is extended over all those permutations $j_1,\dots,j_n$ of the indices $n+1,\dots,2n$ such that $j_1<\dots<j_m$ and $j_{m+1}<\dots<j_n$, or equivalently, is extended over the $\binom{n}{m}$ distinct ways of splitting the indices $n+1,\dots,2n$ into two subsets, one consisting of m indices $j_1,\dots,j_m$ and the other of the remaining n-m indices $j_{m+1},\dots,j_n$. The case m=0 is of course trivial, while the case m=1 is the hypothesis (4); on the other hand the case m=n is the desired result (5).

It is then just a matter of proving the inductive step, so assume that (6) holds for some index m in the range $1\leq m\leq n-1$. For any index $\lambda$ in the range $n-m\leq\lambda\leq n$ interchange the indices n-m and $\lambda$ in (6), and note that by the assumed symmetry the result can be rewritten

$(1,\dots,n;n+1,\dots,2n) =$

$$= (-1)^m \sum_J (1,\dots,n-m-1,\lambda,j_1,\dots,j_m;n-m,\dots,\hat{\lambda},\dots,n,j_{m+1},\dots,j_n)$$

where $\hat{\lambda}$ indicates that $\lambda$ is to be deleted. Add these equations for the m+1 possible values of $\lambda$ to obtain on the one hand the result that

$$(m+1)(1,\dots,n;n+1,\dots,2n) =$$
$$= (-1)^m \sum_J \sum_{\lambda=n-m}^{n} (1,\dots,n-m-1,\lambda,j_1,\dots,j_m;n-m,\dots,\hat{\lambda},\dots,n,j_{m+1},\dots,j_n),$$

and on the other hand note as a consequence of the hypothesis (4) applied to the string of indices $(1,\dots,n-m-1,j_1,\dots,j_m;n-m,\dots,n,j_{m+1},\dots,j_n)$ that

$$0 = \sum_{\lambda=n-m}^{n} (1,\dots,n-m-1,\lambda,j_1,\dots,j_m;n-m,\dots\hat{\lambda},\dots,n,j_{m+1},\dots,j_n)$$
$$+ \sum_{\nu=m+1}^{n} (1,\dots,n-m-1,j_1,\dots,j_m,j_\nu;n-m,\dots,n,j_{m+1},\dots,\hat{j}_\nu,\dots,j_n);$$

combining these two observations yields the formula

$$(m+1)(1,\dots,n;n+1,\dots,2n) =$$
$$= (-1)^{m+1} \sum_J \sum_{\nu=m+1}^{n} (1,\dots,n-m-1,j_1,\dots,j_m,j_\nu;n-m,\dots,n,j_{n+1},\dots,j_\nu,\dots,j_n).$$

There are $(n-m)\binom{n}{m} = (m+1)\binom{n}{m+1}$ summands on the right-hand side of this last identity, corresponding to m+1 copies of the $\binom{n}{m+1}$ ways of splitting the indices $n+1,\dots,2n$ into two subsets, one consisting of m+1 indices and the other of the remaining n-m-1 of them. Thus aside from the factor m+1 this is precisely the case m+1 of the formula (6), thereby establishing the inductive step and completing the proof of the theorem.

With this result in hand, the dominant terms in the expansion (2) can be described as follows:

<u>Theorem 3.</u> If P is any projection operator for which (1) holds then the dominant order L of the expansion (2) satisfies $2L \leq n$. There are constant vectors

$$\vec{\xi}^{\,i;j}_{k} = \vec{\xi}^{\,i_1,\dots,i_L;j_{L+1},\dots,j_n}_{k_{L+1},\dots,k_n} \in P\mathbb{C}^{2^g}$$

with indices in the range $1 \le i,j,k \le g$ such that the coefficients of the dominant terms in (2) have the form

$$\vec{P}\alpha^{0;L;n-L}_{\lambda;j} = \sum_k \vec{\xi}^{\,k_{\lambda_1},\dots,k_{\lambda_L};k_{\mu_{L+1}},\dots,k_{\mu_n}}_{j_{L+1},\dots,j_n}\, w'_{k_1}(a_1)\cdots w'_{k_n}(a_n),$$

where $\lambda_1,\dots,\lambda_L$ are distinct integers in the range $1 \le \lambda \le n-1$ and $\mu_{L+1},\dots,\mu_n$ are the ordered complementary integers in the range $1 \le \mu \le n$, so that $\lambda_1,\dots,\lambda_L,\mu_{L+1},\dots,\mu_n$ is a permutation of the integers $1,\dots,n$ and the $\mu_i$ are uniquely determined. The vectors $\vec{\xi}^{\,i;j}_{k}$ are symmetric in the L indices i, in the n-L indices j, and in the n-L indices k separately; moreover they satisfy the further symmetry conditions

$$\xi^{i;j}_{k} = (-1)^L \xi^{i;k}_{j},$$

$$\sum_{\nu=L}^{n} \xi^{i_1,\dots,i_{L-1},i_\nu;i_L,\dots,i_{\nu-1},i_{\nu+1},\dots,i_n}_{k_{L+1},\dots\dots,k_n} = 0,$$

and in the extreme case that $2L = n$ these vectors are symmetric under the interchange of the sets i,j,k if L is even and skew symmetric if L is odd.

<u>Proof</u> Choose any L indices $1 \le \sigma_1 < \cdots \sigma_L \le n$ and let $1 \le \tau_{L+1} < \cdots < \tau_L \le n$ be the complementary set of indices, so that the σ's and τ's together comprise the full set of indices $1,\dots,n$. Multiply the expansion (2) by $\prod_{\mu,\nu} q(z_\mu, z_\nu)^2$ so that all the terms appearing there can be considered as holomorphic functions; then apply the differential operator $\partial^L/\partial z_{\sigma_1}\cdots\partial z_{\sigma_L}$ to the result and set $z_{\sigma_i} = a_{\sigma_i}$. The hypothesis for the projection operator implies that $\vec{P}\theta\big(w(z_1+\cdots+z_n-a_1-\cdots-a_n)\big)$ vanishes whenever $z_{\sigma_i} = a_{\sigma_i}$ for any i, so the only possible nontrivial terms that can arise on the left-hand side are those in which all the differentiation

is applied to that function; the result on the left-hand side thus becomes

$$\sum_j w'_{j_1}(a_{\sigma_1})\cdots w'_{j_L}(a_{\sigma_L})\ \partial_{j_1\cdots j_L} P\vec{\theta}_2(w(z_{\tau_{L+1}}+\cdots+z_{\tau_n}-a_{\tau_{L+1}}-\cdots-a_{\tau_n}))\ \cdot$$
$$\cdot\ [\prod_{j<k} q(a_{\sigma_j},a_{\sigma_k})^4][\prod_{j,k} q(a_{\sigma_j},z_{\tau_k})^2 q(a_{\sigma_j},a_{\tau_k})^2]\ \cdot$$
$$\cdot\ [\prod_{j<k} q(z_{\tau_j},z_{\tau_k})^2 q(a_{\tau_j},a_{\tau_k})^2].$$

The only possible nontrivial terms that can appear on the right-hand side are those in which the double zeros of the factors $q(z_{\sigma_i},a_{\sigma_i})^2$ are cancelled, and that can only happen by differentiating one factor and multiplying the other by a suitable differential $u'_\lambda(z_{\sigma_i})$ to cancel the other zero by a simple pole. All except the dominant terms then vanish, since the other terms all have strictly fewer than L meromorphic differentials $u'_\lambda(z_{\sigma_i})$, and among the dominant terms the only ones that can possibly be nontrivial are those for which $\pi(1)=\sigma_1,\ldots,\pi(L)=\sigma_L$, $\pi(L+1)=\tau_{L+1},\ldots,\pi(n)=\tau_n$. There are however two cases that must be considered separately. (i) If $\sigma_L<n$ then $u'_{\lambda_i}(z_{\sigma_i})$ has a pole at $z_{\sigma_i}=a_{\sigma_i}$ only when $\lambda_i=\sigma_i$ for each i. In that case the right-hand side becomes

$$\sum_j P\vec{\alpha}^{0;L;n-L}_{\sigma;j} w'_{j_1}(z_{\tau_{L+1}})\cdots w'_{j_{n-L}}(z_{\tau_L})\ \cdot$$
$$\cdot\ [\prod_{j\neq k} q(a_{\sigma_j},a_{\sigma_k})^2][\prod_{j,k} q(z_{\tau_j},a_{\tau_k})^2]\ \cdot$$
$$\cdot\ [\prod_{j,k} q(a_{\sigma_j},a_{\tau_k})^2 q(a_{\sigma_j},z_{\tau_k})^2].$$

(ii) If $\sigma_L=n$ then for $1\le i\le L-1$ as before $u'_{\lambda_i}(z_{\sigma_i})$ has a pole at $z_{\sigma_i}=a_{\sigma_i}$ only when $\lambda_i=\sigma_i$; but $u'_\lambda(z_n) = w'_{a_\lambda,a_n}(z_n)$ has a pole at $z_n=a_n$ for any value of $\lambda$, although with residue $-1$. Thus in this case the right-hand side becomes

$$- \sum_{\lambda,j} P\overset{+}{\alpha}^{0;L;n-L}_{\sigma_1\cdots\sigma_{L-1}\lambda;j_1\cdots j_{n-L}} w'_{j_1}(z_{\tau_{L+1}})\cdots w'_{j_{n-L}}(z_{\tau_n}) \cdot$$
$$\cdot \Big[\prod_{j\neq k} q(a_{\sigma_j},a_{\sigma_k})^2\Big]\Big[\prod_{j,k} q(z_{\tau_j},a_{\tau_k})^2\Big] \cdot$$
$$\cdot \Big[\prod_{j,k} q(a_{\sigma_j},a_{\tau_k})^2 q(a_{\sigma_j},z_{\tau_k})^2\Big].$$

After cancelling out those factors that appear on both sides of the equation, in case (i) the identity reduces to

$$(7) \quad \sum_j w'_{j_1}(a_{\sigma_1})\cdots w'_{j_L}(a_{\sigma_L})\, \partial_{j_1\cdots j_L} P\overset{+}{\theta}_2(w(z_{\tau_{L+1}}+\cdots+z_{\tau_n}-a_{\tau_{L+1}}-\cdots-a_{\tau_n}))\cdot$$
$$\cdot \Big[\prod_{j<k} q(z_{\tau_j},z_{\tau_k})^2 q(a_{\tau_j},a_{\tau_k})^2\Big] =$$
$$= \sum_j P\overset{+}{\alpha}^{0;L;n-L}_{\sigma;j} w'_{j_1}(z_{\tau_{L+1}})\cdots w'_{j_{n-L}}(z_{\tau_{n-L}}) \cdot \Big[\prod_{j,k} q(z_{\tau_j},a_{\tau_k})^2\Big],$$

while in case (ii) it becomes

$$(8) \quad \sum_j w'_{j_1}(a_{\sigma_1})\cdots w'_{j_L}(a_{\sigma_L})\partial_{j_1\cdots j_L} P\overset{+}{\theta}(w(z_{\tau_{L+1}}+\cdots+z_{\tau_n}-a_{\tau_{L+1}}-\cdots-a_{\tau_n}))\cdot$$
$$\cdot \Big[\prod_{j<k} q(z_{\tau_j},z_{\tau_k})^2 q(a_{\tau_j},a_{\tau_k})^2\Big] =$$
$$= -\sum_{\lambda,j} P\overset{+}{\alpha}^{0;L;n-L}_{\sigma_1\cdots\sigma_{L-1}\lambda;j_1\cdots j_{n-L}} w'_{j_1}(z_{\tau_{L+1}})\cdots w'_{j_{n-L}}(z_{\tau_n})\cdot\Big[\prod_{j,k} q(z_{\tau_j},a_{\tau_k})^2\Big].$$

First consider the identity (7) in more detail. The differentials $w'_j(z)$ are linearly independent, so this identity determines the coefficients $P\overset{+}{\alpha}^{0;L;n-L}_{\sigma;j}$ uniquely. It is then clear that these coefficients are holomorphic functions of the variables $a_1,\dots,a_n \in \tilde{M}$. It is also clear that each component of any of these coefficients belongs to $\Gamma(\kappa)$ as a function of the variables $a_{\sigma_1},\dots,a_{\sigma_L}$; the same is true as a function of the remaining variables $a_{\tau_{L+1}},\ldots,a_{\tau_n}$, but that is not quite so obvious so will be demonstrated. For this purpose note that the result of applying the differential operator $\partial^L/\partial w_{j_1}\cdots\partial w_{j_L}$ to the functional equation satisfied by $\vec{\theta}_2[\tau](w)$ when the variable $w$ is translated by some lattice vector in $L$ is a rather complicated formula, but one that can be viewed as the assertion that $\partial_{j_1\ldots j_L}\vec{\theta}_2[\tau](w)$ satisfies the same functional equation as $\vec{\theta}_2[\tau](w)$ modulo lower order derivatives; consequently each component of $\partial_{j_1\ldots j_L}\vec{\theta}_2[\tau](w(z))$ belongs to $\Gamma(\rho_{k-\tau}\zeta^{2g})$ modulo lower order derivatives. It then follows as in the discussion at the beginning of the preceding section that each component of the left-hand side of (7) belongs to $\Gamma(\kappa\zeta^2_{z_{\tau_{L+1}}}\cdots\zeta^2_{z_{\tau_n}})$ as a function of each variable $a_{\tau_i}$, since the lower order derivatives all vanish upon applying the projection operator P. The coefficients of $P\vec{\alpha}^{0;L;n-L}_{\sigma;j}$ then evidently belong to $\Gamma(\kappa)$ as a function of each variable $a_{\tau_i}$ as asserted.

Altogether then it is evident that there are some constant vectors

$$\vec{n}^{k}_{\sigma;j} = \vec{n}^{k_1,\ldots,k_L;k_{L+1},\ldots,k_n}_{\sigma_1,\ldots,\sigma_L;j_1,\ldots,j_{n-L}} \in P\mathbb{C}^{2^g}$$

such that

$$(9)\quad P\vec{\alpha}^{0;L;n-L}_{\sigma;j} = \textstyle\sum_k \vec{n}^{k}_{\sigma;j} w'_{k_1}(a_1)\cdots w'_{k_n}(a_n).$$

Upon substituting this into the right-hand side of (7), replacing $j_i$ by $k_{\sigma_i}$ on the left-hand side, comparing the coefficients of the Abelian differentials $w'_{k_{\sigma_1}}(a_{\sigma_1})\cdots w'_{k_{\sigma_L}}(a_{\sigma_L})$ on the two sides of the equation, and then writing $z_i$ in place of $z_{\tau_i}$ and $a_i$ in place of $a_{\tau_i}$, there results the identity

$$(10)\quad \partial_{k_{\sigma_1}\cdots k_{\sigma_L}} P\vec{\theta}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))\cdot[\textstyle\prod_{j<k} q(z_j,z_k)^2 q(a_j,a_k)^2]=$$
$$=\sum_{j,k_{\tau_i}} \vec{\eta}^{\,k}_{\sigma;j}\, w'_{j_1}(z_{L+1})\cdots w'_{j_{n-L}}(z_n)\, w'_{k_{\tau_{L+1}}}(a_{L+1})\cdots w'_{k_{\tau_n}}(a_n)\cdot$$
$$\cdot\,[\prod_{j,k} q(z_j,a_k)^2];$$

this holds for any fixed indices $k_{\sigma_1},\ldots,k_{\sigma_L}$ for which $\sigma_L<n$. The right-hand side here contains both these fixed indices and the indices of summation $k_{\tau_{L+1}},\ldots,k_{\tau_n}$, a notation that is convenient and should not really be confusing; it clearly indicates which among the n upper indices in the vectors $\vec{\eta}^{\,k}_{\sigma;j}$ are variable and which are fixed, an important point since the extent to which these vectors are symmetric in their indices has not yet been considered. The situation is particularly simple in the special case that $\sigma_i=i$ and hence $\tau_i=i$ as well; set

$$(11)\quad \vec{\xi}^{\,k_1\cdots k_L;k_{L+1}\cdots k_n}_{j_{L+1}\cdots j_n}=\vec{\eta}^{\,k_1\cdots k_L k_{L+1}\cdots k_n}_{1\cdots L;j_{L+1}\cdots j_n},$$

and note that (10) then takes the form

$$(12)\quad \partial_{k_1\cdots k_L} P\vec{\theta}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))\cdot[\prod_{j<k} q(z_j,z_k)^2 q(a_j,a_k)^2] =$$
$$= \sum_{i,j} \vec{\xi}^{\,k_1\cdots k_L;i_{L+1}\cdots i_n}_{\,j_{L+1}\cdots j_n}\, w'_{j_{L+1}}(z_{L+1})\cdots w'_{j_n}(z_n) w'_{i_{L+1}}(a_{L+1})\cdots w'_{i_n}(a_n)\cdot$$
$$\cdot [\prod_{j,k} q(z_j,a_k)^2].$$

It is clear from the symmetries on the left-hand side of (12) that the vectors $\vec{\xi}^{\,k;i}_{\,j}$ are symmetric in the indices i, in the indices j, and in the indices k separately. Furthermore interchanging the variables z and a has the effects of multiplying the left-hand side of (12) by $(-1)^L$, since $\vec{\theta}_2(w)$ is an even function of w, and interchanging the indices i and j on the right-hand side; consequently

$$\vec{\xi}^{\,k;i}_{\,j} = (-1)^L \vec{\xi}^{\,k;j}_{\,i}.$$

Comparing (10) and (12) and keeping these symmetries in mind shows finally that

$$(13)\quad \vec{\eta}^{\,k_1\cdots k_L k_{L+1}\cdots k_n}_{\,\sigma_1\cdots\sigma_L;j_{L+1}\cdots j_n} = \vec{\xi}^{\,k_{\sigma_1}\cdots k_{\sigma_L};k_{\tau_{L+1}}\cdots k_{\tau_n}}_{\,j_{L+1}\cdots j_n}$$

whenever $\sigma_L<n$.

With these results, a great deal of the desired theorem has essentially been established. In particular a comparison of (9) and (13) shows that

$$P\vec{\alpha}^{\,0;L;n-L}_{\,\sigma;j} = \sum_k \vec{\xi}^{\,k_{\sigma_1}\cdots k_{\sigma_L};k_{\tau_{L+1}}\cdots k_{\tau_n}}_{\,j_{L+1}\cdots j_n}\, w'_{k_1}(a_1)\cdots w'_{k_n}(a_n),$$

thus demonstrating that these coefficients have an expansion of the form asserted in terms of the vectors $\vec{\xi}^{\,k;i}_{\,j}$. Moreover these vectors have been shown to be symmetric in the separate sets of indices i,j,and k, and to behave in the asserted manner upon the interchange of the sets of indices i and j. There remain only the last symmetry conditions and the inequality $2L \leq n$ yet to be proved, and these follow from an analysis of the case (ii) paralleling the preceding analysis of case (i).

To turn then to a corresponding detailed analysis of (8), the preceding considerations have at least established that the coefficients $P\vec{\alpha}^{0;L;n-L}_{\sigma,\lambda;j}$ appearing there have the form (9). It should be noted in view of the result of Theorem 2 that the index $\lambda$ can be restricted to vary over the indices $\tau_{L+1},\ldots,\tau_n$. To proceed as in the preceding case then, upon substituting (9) into (8), replacing j by $k_{\sigma_i}$ on the left hand side, comparing the coefficients of the Abelian differentials $w'_{k_{\sigma_1}}(a_{\sigma_1})\cdots w'_{k_{\sigma_L}}(a_{\sigma_L})$ on the two sides of the equation, and then writing $z_i$ in place of $z_{\tau_i}$ and $a_i$ in place of $a_{\tau_i}$, it follows that

$$a_{k_{\sigma_1}\cdots k_{\sigma_L}} P\vec{\theta}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))[\prod_{j<k} q(z_j,z_k)^2 q(a_j,a_k)^2]$$
$$= -\sum_{\lambda,j,k_{\tau_i}} \vec{\eta}^{\,k_1\ \cdot\ \cdot\ \cdot\ k_n}_{\sigma_i\cdots\sigma_{L-1}\lambda;j_1\cdots j_{n-L}} w'_{j_1}(z_{L+1})\cdots w'_{j_{n-L}}(z_n)\ \cdot$$
$$\cdot\ w'_{k_{\tau_{L+1}}}(a_{L+1})\cdots w'_{k_{\tau_n}}(a_n)[\prod_{j,k} q(z_j,a_k)^2].$$

Here $\sigma_L=n$, and for the special values $\sigma_1=1,\ldots,\sigma_{L-1}=L-1$, $\tau_{L+1}=L,\ldots,\tau_n=n-1$ this reduces using (13) to

$$a_{k_1\cdots k_{L-1}k_n} P\vec{\theta}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))[\prod_{j<k} q(z_j,z_k)^2 q(a_j,a_k)^2]$$
$$= -\sum_{\lambda=L}^{n-1}\ \sum_{jk_L\cdots k_{n-1}} \vec{\xi}^{\,k_1\cdots k_{L-1}k_\lambda;k_L\cdots k_{\lambda-1}k_{\lambda+1}\cdots k_n}_{j_{L+1}\ \cdot\ \cdot\ \cdot\ j_n} \cdot w'_{j_{L+1}}(z_{L+1})\cdots w'_{j_n}(z_n)$$
$$w'_{k_L}(a_{L+1})\cdots w'_{k_{n-1}}(a_n)[\prod_{j,k} q(z_j,a_k)^2].$$

From a comparison of this with (12) it follows that

$$(14)\quad \overset{+}{\xi}{}^{k_1\cdots k_{L-1}k_n;k_L\cdots k_{n-1}}_{j_{L+1}\cdots j_n} = \sum_{\lambda=L}^{n-1} \overset{+}{\xi}{}^{k_1\cdots k_{L-1}k_\lambda;k_L\cdots k_{\lambda-1}k_{\lambda+1}\cdots k_n}_{j_{L+1}\cdot\quad\cdot\quad\cdot\quad\cdot j_n},$$

which is another of the symmetry conditions on the vectors $\xi^{k;i}_{j}$.

The final assertions of the theorem follow quite easily from (14) and Lemma 1. First if 2L = n then the sets of indices i,j,k in the vectors $\overset{+}{\xi}{}^{k;i}_{j}$ contain the same number L = n-L of indices. It was already noted that $\overset{+}{\xi}{}^{k;i}_{j} = (-1)^L\overset{+}{\xi}{}^{k;j}_{i}$, and it follows from (14) and Lemma 1 that $\overset{+}{\xi}{}^{k;i}_{j} = (-1)^L\overset{+}{\xi}{}^{i;k}_{j}$; thus these vectors are symmetric or skew-symmetric in the sets i,j,k according to the parity of L, as desired. Finally if 2L>n then write $k=(k_1,\ldots,k_L) = (k',k'')$, where k' consists of the first 2L-n indices and k" of the last n-L indices of k. It is clear that the vectors $\overset{+}{\xi}{}^{k;i}_{j}$ satisfy the hypothesis of Lemma 1 in the indices k",i for any fixed indices k',j, and it therefore follows from that lemma that

$$\overset{+}{\xi}{}^{k;i}_{j} = (-1)^{n-L}\overset{+}{\xi}{}^{k'i;k''}_{j};$$

consequently

$$\begin{aligned}
&\overset{+}{\xi}{}^{k_1\cdots k_L;i_{L+1}\cdots i_n}_{j_{L+1}\cdots j_n}\\
&= (-1)^{n-L}\overset{+}{\xi}{}^{k_1\cdots k_{2L-n}i_{L+1}\cdots i_n;k_{2L-n+1}\cdots k_L}_{j_{L+1}\cdot\cdot\cdot j_n}\\
&= (-1)^{n-L}\overset{+}{\xi}{}^{k_2\cdots k_{2L-n}i_{L+1}\cdots i_nk_1;k_{2L-n+1}\cdots k_L}_{j_{L+1}\cdot\cdot\cdot j_n}\\
&= \overset{+}{\xi}{}^{k_2\cdots k_{2L-n}i_{L+1}k_{2L-n+1}\cdots k_L;i_{L+2}\cdots i_nk_1}_{j_{L+1}\cdots j_n}\\
&= \overset{+}{\xi}{}^{i_{L+1}k_2\cdots k_L;k_1i_{L+2}\cdots i_n}_{j_{L+1}\cdot\cdot\cdot j_n}.
\end{aligned}$$

The vectors $\vec{\xi}^{k;i}_{j}$ are thus symmetric in $k_1$ and $i_{L+1}$, and since they are symmetric separately in the indices k and i they must actually be fully symmetric in the n indices k,i together. However in that case it is clear from (14) that $\vec{\xi}^{k;i}_{j} = 0$ for all indices i,j,k, contradicting the assumption that L is the dominant order. That serves to conclude the proof.

At least one of the auxiliary results obtained in the course of the proof of the preceding theorem merits some special attention.

Corollary. If P is any projection operator for which (1) holds and L is the dominant order then the coefficients of the dominant term are characterized by the condition that

$$\partial_{k_1 \cdots k_L} P\vec{\theta}_2(w(z_{L+1}+\cdots+z_n-a_{L+1}-\cdots-a_n))[\prod_{j<k} q(z_j,z_k)^2 q(a_j,a_k)^2][\prod_{j,k} q(z_j,a_k)^{-2}]$$
$$= \sum_{i,j} \vec{\xi}^{\,k_1\cdots k_L;i_{L+1}\cdots i_n}_{j_{L+1}\cdots j_n}\, w'_{j_{L+1}}(z_{L+1})\cdots w'_{j_n}(z_n) w'_{i_{L+1}}(a_{L+1})\cdots w'_{i_n}(a_n).$$

Proof The assertion of the corollary is just formula (12) in the proof of the preceding theorem. Since the differential forms $w'_j(z)$ and $w'_i(a)$ are linearly independent functions, it is clear that this formula determines the vectors $\vec{\xi}^{k;i}_{j}$ uniquely, as desired.

It is possibly worth pointing out explicitly that the preceding results hold as stated even in the special case that the dominant order is L=0; in that case the expansion involves merely the ordinary Abelian differentials, and corollary 1 is just a restatement of the expansion formula.

## §3. The primary and binary expansions: dominant terms.

As already noted, in the special case $n = 1$ the differential expansion considered in the preceding sections takes the particularly simple form

$$(1)\quad \vec{\theta}_2(w(z-a))q(z,a)^{-2} = \vec{\theta}_2(0)\; w_a'(z) + \tfrac{1}{2}\sum_{jk} \partial_{jk}\vec{\theta}_2(0)\; w_j'(z)\; w_k'(a).$$

The vectors $\partial_{jk}\vec{\theta}(0)$ are symmetric in the indices $j,k$, but satisfy no other linear relations; indeed the following holds.

Theorem 4. For any Riemann surface of genus $g > 1$ the $1 + \binom{g+1}{2}$ vectors $\vec{\theta}_2(0)$, $\partial_{jk}\vec{\theta}_2(0)$ for $1 \leqq j \leqq k \leqq g$ are linearly independent.

Proof. Suppose to the contrary that there are some constants $c, c_{jk} = c_{kj}$, not all of which are zero, such that

$$0 = c\; \vec{\theta}_2(0) + \sum_{jk} c_{jk}\; \partial_{jk}\vec{\theta}_2(0).$$

Since $\vec{\theta}_2(0) \neq 0$ not all of the constants $c_{jk}$ can be zero. Multiplying this identity by ${}^t\vec{\theta}_2(t)$ and using the result of Lemma D3 shows that

$$0 = c\; \theta(t)^2 + 2\sum_{jk} c_{jk}[\theta(t)\; \partial_{jk}\; \theta(t) - \partial_j\theta(t)\; \partial_k(t)]$$

for all points $t \in \mathbb{C}^g$; so in particular

$$(2)\quad \sum_{jk} c_{jk}\; \partial_j\theta(t)\; \partial_k\theta(t) = 0 \quad \text{whenever } \theta(t) = 0.$$

Now whenever $t$ represents a point in $\underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1$ the vector $\vec{\partial}\; \theta(t) = \{\partial_j\theta(t) : 1 \leqq j \leqq g\} \in \mathbb{C}^g$ is nonzero so represents a point $[\vec{\partial}\; \theta(t)] \in \mathbb{P}^{g-1}$, and the mapping from $\underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1$ to $\mathbb{P}^{g-1}$ thus defined is the Gauss mapping discussed in section C. The main result established there was that the Gauss mapping is finite, hence that its image is a nonempty open set in $\mathbb{P}^{g-1}$. However (2) implies that the image of the Gauss mapping lies in a nontrivial quadratic cone in $\mathbb{P}^{g-1}$, so cannot be an open subset of $\mathbb{P}^{g-1}$; that contradiction establishes the theorem.

The next case n = 2 is much more complicated and better illustrates the general differential expansion, so is worth examining in some detail. The general form of this expansion as given in Theorem 1 was written out quite explicitly in equation (1.7), and can be simplified as in Theorems 2 and 3 by the use of a suitable projection operator. Consider then a linear mapping

$$P : \mathbb{C}^{2^g} \to \mathbb{C}^n$$

such that

(3) $P\vec{\theta}_2(0) = P\partial_{jk}\vec{\theta}_2(0) = 0$ for all $1 \leqq j, k \leqq g$;

the kernel of P must thus contain the linear subspace of dimension $1 + \binom{g+1}{2}$ spanned by these vectors, and by (1) and Theorem 4 this can be restated equivalently as the condition that

(4) $P\vec{\theta}_2(w(z-a)) = 0$ for all points $z, a \in \tilde{M}$.

With this operator the expansion takes the following form.

Theorem 5. If P is any projection operator satisfying (3) then there are uniquely determined constant vectors $\vec{\xi}\,^{ij}_{k}$ in the range of P and holomorphic functions $\vec{\beta}_{jk}(a_1, a_2)$ on $\tilde{M} \times \tilde{M}$ with values in the range of P such that

$$P\,\vec{\theta}_2(w(z_1 + z_2 - a_1 - a_2))\, q(z_1, z_2)^2\, q(a_1, a_2)^2 \left[\prod_{j,k=1}^{2} q(z_j, a_k)^{-2}\right]$$

$$= \sum_{jk_1k_2} \vec{\xi}\,^{k_1k_2}_{j}\, w'_{k_1}(a_1) w'_{k_2}(a_2)\, [w'_{a_1,a_2}(z_1) w'_j(z_2) + w'_{a_1,a_2}(z_2)\, w'_j(z_1)]$$

$$+ \sum_{jk} \vec{\beta}_{jk}\,(a_1, a_2)\, w'_j(z_1)\, w'_k(z_2).$$

Here $\vec{\xi}\,^{ij}_{k}$ are skew-symmetric in the indices i,j,k and are determined by the condition that

$$\partial_k\, P\theta_2(w(z-a))\, q(z,a)^{-2} = -\sum_{ij} \vec{\xi}\,^{ij}_{k}\, w'_i(z)\, w'_j(a);$$

the functions $\mathfrak{B}_{jk}(a_1,a_2)$ are symmetric in the indices $j,k$ and in the variables $a_1,a_2$.

Proof. It follows from Theorems 2 and 3 that if $n = 2$ and $P$ is a projection operator satisfying (4) then for the coefficients in the expansion (2.2) necessarily $P\vec{\alpha}\,{}^{m;\ell;k}_{\mu;\lambda;j} = 0$ whenever $m > 0$ or $\ell > 1$; thus the only non-trivial terms are $P\,\vec{\alpha}\,{}^{0;1;1}_{\lambda\;j}$ and $P\,\vec{\alpha}\,{}^{0;0;2}_{j_1j_2}$. For the first of these the index $\lambda$ just takes the value $\lambda = 1$, so can really be ignored, and by Theorem 3 again

$$P\,\vec{\alpha}\,{}^{0;1;1}_{j} = \sum_{k_1k_2} \vec{\xi}\,{}^{k_1k_2}_{j}\, w'_{k_1}(a_1)\, w'_{k_2}(a_2)$$

where the vectors $\vec{\xi}\,{}^{k_1k_2}_{j}$ are skew symmetric in the three indices $j,k_1,k_2$. Setting $P\,\vec{\alpha}\,{}^{0;0;2}_{j_1j_2} = \vec{\mathfrak{B}}_{j_1j_2}$ then yields an expansion of the general form desired, although one for which it is as yet only known that the coefficients $\vec{\mathfrak{B}}_{j_1j_2}(a_1,a_2)$ are meromorphic functions of the variables $a_1,a_2 \in \tilde{M}$, although symmetric in the indices $j_1,j_2$ and in the variables $a_1,a_2$. However on the left-hand side and in the first line on the right-hand side of this expansion the only singularities are at most simple poles along the subvarieties $z_j = Ta_k$, since the theta function term vanishes whenever $z_j = Ta_k$ as a consequence of the hypothesis (4); the same must be the case for the second line on the right-hand side, and it is evident from that that $\vec{\mathfrak{B}}_{j_1j_2}(a_1,a_2)$ is actually holomorphic. Finally by the Corollary to Theorem 3

$$\partial_k P\,\vec{\mathfrak{B}}_2(w(z-a))q(z,a)^{-2} = \sum_{i,j} \vec{\xi}\,{}^{ki}_{j}\, w'_j(z)\, w'_i(a),$$

and since $\vec{\xi}\,{}^{ki}_{j} = \vec{\xi}\,{}^{ij}_{k}$ this gives the final result desired and thereby concludes the proof.

It may well be the case that $\xi_j^{k_1k_2} = 0$ for all indices $j, k_1, k_2$, as can always be achieved by applying a further projection operator with all these vectors in its kernel; in view of the last assertion of the preceding theorem that amounts to the condition that

(5) $\partial_j P\vartheta_2(w(z-a)) = 0$ for all points $z, a \in \tilde{M}$ and indices $j$.

That theorem as stated then becomes somewhat less interesting, but can be analyzed further to yield the following.

Corollary. If P is any projection operator satisfying (4) and (5) then there are uniquely determined vectors $\xi_{j_1j_2}^{k_1k_2}$ in the range of P such that

$$P\vartheta_2(w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)^2\ q(a_1,a_2)^2\ [\prod_{j,k=1}^{2} q(z_j,a_k)^{-2}]$$

$$= \sum_{j_1j_2k_1k_2} \xi_{j_1j_2}^{k_1k_2}\ w'_{k_1}(a_1)\ w'_{k_2}(a_2)\ w'_{j_1}(z_1)\ w'_{j_2}(z_2).$$

These vectors satisfy the symmetry conditions

$$\xi_{j_1j_2}^{k_1k_2} = \xi_{j_2j_1}^{k_1k_2} = \xi_{j_1j_2}^{k_2k_1} = \xi_{k_1k_2}^{j_1j_2},$$

and it is further the case that

$$\sum_{k_1k_2} \partial_{k_1k_2} P\vartheta_2(w(z-a))\ q(z,a)^{-2}\ w'_{k_1}(b)\ w'_{k_2}(b)$$
$$= \sum_{j_1j_2k_1k_2} 2\ \xi_{j_2k_2}^{j_1k_1}\ w'_{j_1}(z)\ w'_{j_2}(a)\ w'_{k_1}(b)\ w'_{k_2}(b)$$

for any points $z,a,b \in \tilde{M}$.

Proof. In view of the results of the preceding theorem, it is clear that the additional hypothesis (5) implies that the dominant order of the differential expansion is $L = 0$; the explicit form of the expansion and the symmetry conditions satisfied by the coefficient vectors are then an immediate consequence of Theorem 3. In this case the Corollary to Theorem 3 reduces just to the differential expansion itself, rather than the final assertion of the present corollary; but the latter result can be derived directly and easily. Multiply the differential expansion formula just obtained by $q(z_2,a_2)^2$, apply the differential operator $\partial^2/\partial z_2^2$, and take the limit as $z_2$ tends to $a_2$; the left-hand side reduces to that of the final assertion as desired after the obvious change of notation, as an immediate consequence of the assumptions (4) and (5), while differentiating $q(z_2,a_2)^2$ introduces the factor 2 on the right-hand side. That suffices to conclude the proof.

The corollary can of course be reformulated as the assertion that in the formula of the theorem if $\xi_{j}^{k_1k_2} = 0$ for all indices $j,k_1,k_2$ then

$$\beta_{j_1j_2}(a_1,a_2) = \sum_{k_1k_2} \xi_{j_1j_2}^{k_1k_2}\, w'_{k_1}(a_1)\, w'_{k_2}(a_2)$$

for some uniquely determined vectors $\xi_{j_1j_2}^{k_2k_2}$ with the symmetries as stated. Now in addition to these symmetries the vectors $\xi_{j_1j_2}^{k_1k_2}$ are subject to some further linear constraints that follow almost immediately from the formulas of the corollary. In view of the presence of the factor $q(a_1,a_2)^2$, the left-hand side vanishes identically in $z_1,z_2$ whenever $a_1 = a_2$; since the functions $w'_{j_1}(z_1)\, w'_{j_2}(z_2)$ are linearly independent it follows that

(6) $$\sum_{k_1k_2} \xi_{j_1j_2}^{k_1k_2}\, w'_{k_1}(a)\, w'_{k_2}(a) = 0 \quad \text{for all } a \in \tilde{M} \text{ and all } j_1,j_2.$$

For any fixed indices $j_1, j_2$ each component of the vector $\xi^{k_1k_2}_{j_1j_2}$ must consequently lie in the Petri space $\mathcal{P}_2$ of quadratic forms vanishing on the canonical curve. To make this explicit choose a basis $p_i(x) = \sum_{j_1j_2} p^{\,i}_{j_1j_2} x_{j_1} x_{j_2}$ for the vector space $\mathcal{P}_2$; the coefficients $p^{\,i}_{j_1j_2}$ are to be symmetric in the indices $j_1, j_2$, and $p_i(x)$ form a basis for the space of quadratic polynomials $p(x)$ for which $p(w'(z)) = \sum_{j_1j_2} p_{j_1j_2} w'_{j_1}(z)\, w'_{j_2}(z) = 0$ for all points $z \in \tilde{M}$. It may be recalled from the discussion in section B10 that

$$(7) \quad d_2 = \dim \mathcal{P}_2 = \begin{cases} \binom{g-1}{2} & \text{if } M \text{ is hyperelliptic,} \\ \binom{g-2}{2} & \text{if } M \text{ is nonhyperelliptic.} \end{cases}$$

In terms of this basis it must then be the case that $\xi^{k_1k_2}_{j_1j_2} = \sum_i \vec{\eta}^{\,i}_{j_1j_2}\, p^{\,i}_{k_1k_2}$ for some uniquely determined vectors $\vec{\eta}^{\,i}_{j_1j_2}$. It follows from the symmetry properties of the vectors $\xi^{k_1k_2}_{j_1j_2}$ that

$$\sum_{ij_1j_2} \vec{\eta}^{\,i}_{j_1j_2}\, p^{\,i}_{k_1k_2}\, w'_{j_1}(a)\, w'_{j_2}(a) = \sum_{j_1j_2} \xi^{k_1k_2}_{j_1j_2}\, w'_{j_1}(a)\, w'_{j_2}(a)$$
$$= \sum_{j_1j_2} \xi^{j_1j_2}_{k_1k_2}\, w'_{j_1}(a)\, w'_{j_2}(a) = 0$$

for all $a \in \tilde{M}$ and all $k_1, k_2$, and since the polynomials $p_i(x)$ are linearly independent

$$\sum_{j_1j_2} \vec{\eta}^{\,i}_{j_1j_2}\, w'_{j_1}(a)\, w'_{j_2}(a) = 0 \quad \text{for all } a \in \tilde{M} \text{ and all } i.$$

It must consequently also be the case that $\vec{\eta}^{\,i}_{j_1j_2} = \sum_j \vec{\eta}^{\,i}_{j}\, p^{\,j}_{j_1j_2}$ for some uniquely determined vectors $\vec{\eta}^{\,i}_{j}$. Altogether therefore

(8) $\vec{\xi}\,{}^{k_1k_2}_{j_1j_2} = \sum_{i_1i_2} \vec{\eta}\,{}^{i_2}_{i_1}\; p^{i_1}_{j_1j_2}\; p^{i_2}_{k_1k_2}$

for some uniquely determined vectors $\vec{\eta}\,{}^{i_2}_{i_1}$ in the range of $P$, and from the symmetry properties of the vectors $\vec{\xi}\,{}^{k_1k_2}_{j_1j_2}$ it is evident that

(9) $\vec{\eta}\,{}^{i_2}_{i_1} = \vec{\eta}\,{}^{i_1}_{i_2}.$

This cuts down considerably on the possibilities for these vectors $\vec{\xi}\,{}^{k_1k_2}_{j_1j_2}$.

It is worth pausing here to examine in some detail the significance of the preceding results, and in the process to establish some further useful notation. The vectors $\vec{\theta}_2(0)$ and $\partial_{jk}\vec{\theta}_2(0)$ taken together span a linear subspace $L_1 \subseteq \mathbb{C}^{2^g}$, where $\delta_1 = \dim L_1 = 1 + \binom{g+1}{2}$ as a consequence of Theorem 4. The identity (1) easily implies that $L_1$ can be described equivalently as the span of the vectors $\vec{\theta}_2(w(z-a))$ as $z$ and $a$ vary throughout $\tilde{M}$. From a function-theoretic point of view this means that there are precisely $\delta_1 = 1 + \binom{g+1}{2}$ linearly independent functions among the restricted second-order theta functions $\theta_2[\nu|0](w(z-a))$ viewed as functions of the two variables $(z,a) \in \tilde{M} \times \tilde{M}$ for $\nu \in \mathbb{Z}^g/2\mathbb{Z}^g$. On the other hand from a geometric point of view, in terms of the two-sheeted mapping $\vec{\theta}_2 : J \to K$ from the Jacobi variety $J$ to its associated Wirtinger variety $K \subseteq \mathbb{P}^{2^g-1}$ defined by the second-order theta functions, this means that $\vec{\theta}_2(W_1-W_1) \subseteq K \cap [L_1]$, where $[L_1] \subseteq \mathbb{P}^{2^g-1}$ is the projective linear subspace associated to the vector subspace $L_1 \subseteq \mathbb{C}^{2^g}$; moreover $[L_1]$ is the smallest linear subspace with this property, since it is the linear subspace spanned by all the points of the subvariety $\vec{\theta}_2(W_1-W_1) \subseteq \mathbb{P}^{2^g-1}$.

The analogous results for the binary differential expansion are somewhat more complicated and consequently somewhat more interesting. Let $L_2$ be the span of the vectors $\vec{\theta}_2(w(z_1+z_2 - a_1-a_2))$ as $z_j$ and $a_k$ vary throughout $\tilde{M}$, and set $\delta_2 = \dim L_2$. Thus there are precisely $\delta_2$ linearly independent functions among the restricted second-order theta functions $\theta_2[\nu|0](w(z_1+z_2-a_1-a_2))$ viewed as functions of the four variables $(z_1,z_2,a_1,a_2) \in \tilde{M}^4$ for $\nu \in \mathbb{Z}^g/2\mathbb{Z}^g$. On the other hand geometrically $\vec{\theta}_2(W_2-W_2) \subseteq K \cap [L_2]$, where $[L_2] \subseteq \mathbb{P}^{2^g-1}$ is the projective linear subspace associated to the vector subspace $L_2 \subseteq \mathbb{C}^{2^g}$, and $[L_2]$ is the smallest linear subspace with this property since it is the linear subspace spanned by all the points of the subvariety $\vec{\theta}_2(W_2-W_2) \subseteq \mathbb{P}^{2^g-1}$.

The complications arise in attempting to describe $L_2$ in terms of some natural basis, or even in attempting to determine the dimension $\delta_2$ explicitly. Of course $L_1 \subseteq L_2$, so that $L_2$ contains at least the $\delta_1$ linearly independent vectors $\vec{\theta}_2(0)$ and $\partial_{jk}\vec{\theta}_2(0)$ for $j \leqq k$. One approach to the description of $L_2$ is to factor out the subspace $L_1$ and to see what can be said about the quotient space. Introduce therefore the natural linear projection

$$(10) \qquad P_1 : \mathbb{C}^{2^g} \to \mathbb{C}^{2^g}/L_1$$

having kernel precisely $L_1$. It then follows from Theorem 5 that $P_1(L_2)$ is spanned by the vectors $\vec{\xi}_j^{\,k_1k_2}$ together with the vectors $\vec{\beta}_{jk}(a_1,a_2)$ for all points $a_i \in \tilde{M}$. The vectors $\vec{\xi}_j^{\,k_1k_2}$ are quite canonically determined, and since they are skew-symmetric in their indices there are at most $\binom{g}{3}$ linearly independent vectors among them; thus if $L_2' \subseteq$ range $P_1$ is the linear subspace spanned by these vectors then $\dim L_2' \leqq \binom{g}{3}$. There remains the question of the extent to which the vectors $\vec{\beta}_{jk}(a_1,a_2)$ span something beyond the space $L_2'$, and that can be approached by factoring out the subspace $L_2'$ and examining the quotient space. Introduce therefore the further natural linear projection

(11) $\qquad P_2' : (\text{range } P_1) \to (\text{range } P_1) \;/\; L_2'$

having kernel precisely $L_2'$. It follows from the Corollary to Theorem 5 that $P_2'(P_1(L_2))$ is spanned by the vectors $\vec{\xi}\,{}^{k_1k_2}_{j_1j_2}$; these vectors are subject to the symmetries as noted, and also to a further set of linear constraints, which together amount to the fact that they can be expressed in terms of the more primitive vectors $\vec{\eta}\,{}^{i}_{j}$. These latter vectors are also intrinsically determined after a choice for the basis of the Prym space $\underline{\mathcal{P}}_2$ has been made, and are symmetric in their indices. Since these indices are in the range $1 \leqq i, j \leqq d_2$, where $d_2 = \dim \underline{\mathcal{P}}_2$ is given explicitly in (7), it follows that there are at most $\binom{d_2+1}{2}$ linearly independent vectors among them; thus $\dim P_2'(P_1(L_2)) \leqq \binom{d_2+1}{2}$. The whole situation can be summed up reasonably perspicuously in the following diagram, in which $L_2^{*} = P_1^{-1}(L'_2)$.

(12)

$$\begin{array}{ccccccl}
 & L_1 & \subseteq & L_2^{*} & \subseteq & L_2 & \\
P_1 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \subseteq & L_2' = L_2^{*}/L_1 & \subseteq & L_2/L_1 & \quad (L_2^{*}/L_1 = \text{span } \vec{\xi}\,{}^{k_1k_2}_{j}) \\
P_2' & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \subseteq & 0 & \subseteq & L_2/L_2^{*} & \quad (L_2/L_2^{*} = \text{span } \vec{\eta}\,{}^{i}_{j})
\end{array}$$

It has been noted that $\dim L_2^{*}/L_1 \leqq \binom{g}{3}$ and $\dim L_2/L_2^{*} \leqq \binom{d_2+1}{2}$, so that

(13) $$\delta_2 \leqq \delta_1 + \binom{g}{3} + \binom{d_2+1}{2}.$$

The splitting of the extension from $L_1$ to $L_2$ through the intermediate space $L_2^*$ is an additional intrinsic structure. To describe the space $L_2$ in much the same way that the space $L_1$ was described initially simply choose some vectors in $L_2$ that have as their images under the appropriate projections the vectors $\vec{\xi}^{k_1k_2}_{j}$ and $\vec{\eta}^{i}_{j}$; there are of course many ways in which such choices can be made, and more will be said about that later. If these choices are also denoted by $\vec{\xi}^{k_1k_2}_{j}$ and $\vec{\eta}^{i}_{j}$, which is really not confusing since it is usually clear from the context what is meant, then

$$L_1 = \text{span}\ \{\ \vec{\theta}_2(0),\ \partial_{jk}\vec{\theta}_2(0)\ \}$$

$$(14) \qquad L_2^* = \text{span}\ \{\ \vec{\theta}_2(0),\ \partial_{jk}\vec{\theta}_2(0),\ \vec{\xi}^{k_1k_1}_{j}\ \}$$

$$L_2 = \text{span}\ \{\ \vec{\theta}_2(0),\ \partial_{jk}\vec{\theta}_2(0),\ \vec{\xi}^{k_1k_2}_{j},\ \vec{\eta}^{i}_{j}\ \}.$$

These vectors are sometimes, but not always, linearly independent modulo the symmetries as noted.