D. Second-order Riemannian theta functions

1. The analogues of Riemann's tneorems.

The discussion of Riemann's theorems dealt with the restrictions of the first-order theta function and its translates to the Riemann surface M canonically imbedded in its Jacobi variety J. Turning next to the consideration of the restrictions of the second-order theta functions and their translates leads to somewhat different and perhaps even deeper questions. In this case it is necessary to consider a number of theta functions simultaneously, the basis $\theta_2[\nu|t](w;\Omega) \in \Gamma(\rho_{-t}\zeta^2)$ for $\nu \in \mathbb{Z}^g/2\mathbb{Z}^g$ or the vector $\vec{\theta}_2[t](w)$ composed of these 2^g basic theta functions. It is course clear that

$$\theta_2[\nu|t](w(z)) \in \Gamma(\rho_{-t}\zeta^2) = \Gamma(\rho_{2r-t}\zeta^{2g}) \text{ as a function of } z \in \tilde{M},$$

where $r \in \mathbb{C}^g$ is the Riemann point, but in parallel to the discussion of the first-order theta function it is in some ways more convenient to consider instead the functions

(1) $f_{\nu,t}(z) = \theta_2[\nu|2r-t](w(z)) \in \Gamma(\rho_t\zeta^{2g})$

for $\nu \in \mathbb{Z}^g/2\mathbb{Z}^g$. Note from the Riemann-Roch theorem that $\gamma(\rho_t\zeta^{2g}) = g+1$ for all $t \in \mathbb{C}^g$, and since $g+1 \leq 2^g$ there are certainly enough restricted functions $f_{\nu,t}$ to span the vector space $\Gamma(\rho_t\zeta^{2g})$; they actually do span that space.

Theorem 1. For any $t \in \mathbb{E}^g$ the 2^g functions $f_{\nu,t}$ for $\nu \in \mathbb{Z}^g/2\mathbb{Z}^g$ span the $(g+1)$-dimensional vector space $\Gamma(\rho_t\zeta^{2g})$.

Proof. Restricting to the subvariety $M \subseteq J$ is clearly a linear mapping from the space of relatively automorphic functions $\Gamma(J,\rho_t\zeta^2)$ on $\mathbb{E}^g$ to the space of relatively automorphic functions $\Gamma(M,\rho_t\zeta^{2g})$ on $\tilde{M}$, and the assertion of the theorem is that this mapping is surjective. Since the image is necessarily a linear subspace of $\Gamma(M, \rho_t\zeta^{2g})$, it is enough just to show that the image contains an open subset of $\Gamma(M,\rho_t\zeta^{2g})$.

For this purpose choose a basis $h_1(z),\ldots,h_{g+1}(z)$ for $\Gamma(M,\rho_t\zeta^{2g})$ and let $w_1'(z),\ldots,w_g'(z)$ be the canonical basis for $\Gamma(M,\kappa)$; then consider any points $z_1,\ldots,z_g \in \tilde{M}$ such that

(2) $\operatorname{rank}\{h_i(z_j): 1\le i\le g+1,\ 1\le j\le g\} = g$ and

(3) $\operatorname{rank}\{w'_i(z_j): 1\le i,j\le g\} = g$,

points which certainly exist since the functions $h_i(z)$ and the functions $w'_i(z)$ both form linearly independent families of functions. The set of points $(z_1,\ldots,z_g)$ satisfying these two conditions form an open subset $U \subseteq \tilde{M}^g$, the complement of a proper holomorphic subvariety of the complex manifold $\tilde{M}^g$. It is clear from (2) that there is up to a constant factor a unique function $h = \Sigma_{i=1}^{g+1} c_i h_i \in \Gamma(M,\rho_t\zeta^{2g})$ such that $h(z_j) = 0$ for $1\le j\le g$. The set of multiples of h can be viewed as a point in the g dimensional projective space $\mathbb{P}\Gamma(M,\rho_t\zeta^{2g})$ associated to the g+1 dimensional vector space $\Gamma(M,\rho_t\zeta^{2g})$, and the mapping $\phi: U \to \mathbb{P}\Gamma(M,\rho_t\zeta^{2g})$ thus defined is easily seen to be holomorphic since the coefficients c_i satisfy equations $\Sigma_i c_i h_i(z_j) = 0$ depending analytically on the points z_j. The divisor of h is of the form $\mathcal{D}(h) = z_1+\dots+z_g+ z'_1+\dots+z'_g$, and the only points of U which map to this function h must be g-tuples from along these 2g points. That shows that the inverse image under ϕ of any point is a finite point set, and it then follows from general results in function theory that ϕ is an open mapping. The image $\phi(U)$ is then an open subset of $P\Gamma(M,\rho_t\zeta^{2g})$, and the proof will be concluded by showing that any function $f \in \Gamma(M,\rho_t\zeta^{2g})$ corresponding to a point in the image of ϕ is the restriction of a function in $\Gamma(J,\rho_t\xi^2)$.

Thus suppose $(z_1,\dots,z_g) \in U$ and consider a function $h \in \Gamma(M,\rho_t\zeta^{2g})$ such that $\mathcal{D}(h) = z_1+\dots+z_g+z'_1+\dots+z'_g$. First it follows from (3) that $s=w(z_1+\dots+z_g) \notin W^1_g$, so by the Corollary to Theorem C1 the function $f_s(z)=\theta(r-s+w(z)) \in \Gamma(\rho_s\zeta^g)$ is not identically zero. Next the divisor $z'_1+\dots+z'_g$ is uniquely determined by $z_1+\dots+z_g$, so must be the unique divisor such that $s' = w(z'_1+\dots+z'_g) = t-w(z_1+\dots+z_g)$, and consequently $s' \notin W^1_g$; so again by the Corollary to Theorem C1 the function $f_{s'}(z) = \theta(r-s'+w(z)) \in \Gamma(\rho_{s'}\zeta^g)$ is not identically 0. Now $f_s(z)f_{s'}(z) = \theta(r-s+w(z))\theta(r-s'+w(z))={}^t\theta_2(\frac{s'-s}{2})\ \theta_2(w(z)+r-\frac{s+s'}{2})$ is the restriction of a second-order theta function, and since $f_s(z)f_{s'}(z) \in \Gamma(\rho_s\zeta^g \cdot \rho_{s'}\zeta^g) = \Gamma(\rho_t\zeta^{2g})$ and vanishes at $z_1+\dots+z_g$ it is a constant multiple of the function $h(z)$. That suffices to conclude the proof.

From the preceding result it is quite easy to derive the analogue of Riemann's theorems for the second-order theta functions, providing simple descriptions in terms of these functions for the subvarieties of special positive divisors. This rests on the following auxiliary description of these subvarieties. For any point $t \in \mathbb{C}^g$ choose a basis $h_1(z),\ldots,h_{g+1}(z)$ of the space $\Gamma(\rho_t \zeta^{2g})$, and let $\vec{h}(z)$ be the column vector of length $g+1$ with entries $h_i(z)$. Then for any divisor

(4) $\quad \underset{\sim}{\vartheta} = n_1 z_1 + n_2 z_2 + \ldots,\quad$ where $z_i \in \tilde{M}$ represent distinct points of M

of degree $n = \sum_j n_j$ where $z_i \in M$ represent distinct points of M introduce the $(g+1)\times n$ matrix

(5) $\quad H_t(\underset{\sim}{\vartheta}) = \{\vec{h}(z_1), \vec{h}'(z_1), \ldots, \vec{h}^{(n_1-1)}(z_1), \vec{h}(z_2), \vec{h}'(z_2), \ldots, \vec{h}^{(n_2-1)}(z_2), \ldots\};$

here differentiation can be with respect to any local coordinates whatsoever near the points z_i on $\tilde{M}$.

Lemma 1. With the notation just established, $k-t+w(\underset{\sim}{\vartheta}) \in W^{\nu}_{n-2}$ precisely when rank $H_t(\underset{\sim}{\vartheta}) < n-\nu$.

Proof. If $m = $ rank $H_t(\underset{\sim}{\vartheta})$ then clearly

$$g+1-m = \dim\{\vec{c} \in \mathbb{C}^{g+1} : {}^t\vec{c} \cdot H_t(\underset{\sim}{\vartheta}) = 0\}$$

$$= \dim\{\vec{c} \in \mathbb{C}^{g+1} : \underset{\sim}{\vartheta}(\Sigma_i c_i h_i) \geq \underset{\sim}{\vartheta}\}$$

$$= \gamma(\rho_{t-w(\underset{\sim}{\vartheta})} \zeta^{2g-n})$$

$$= \gamma(\rho_{k-t+w(\underset{\sim}{\vartheta})} \zeta^{n-2}) + g+1-n.$$

where the last equality is an application of the Riemann-Roch theorem. Therefore $k-t+w(\mathfrak{z}) \in W^{\nu}_{n-2}$ precisely when

$$\nu < \gamma(\rho_{k-t+w(\mathfrak{z})}\zeta^{n-2}) = n-m,$$

thus yielding the desired result.

To apply this to the problem at hand, in terms of the divisor $\mathfrak{z}$ as in (4) introduce the $2^g \times n$ matrix

(6) $$\underline{\underline{\theta}}(t,\mathfrak{z}) =$$
$$= \{\hat{\theta}_2[t](w(z_1)), \frac{\partial}{\partial z_1}\hat{\theta}_2[t](w(z_1)), \ldots, \frac{\partial^{n_1-1}}{\partial z_1^{n_1-1}}\hat{\theta}_2[t](w(z_1)), \hat{\theta}_2[t](w(z_2)), \ldots\},$$

where again differentiation can be with respect to any local coordinates whatsoever near the points z_i on M. In these terms the analogue of Riemann's theorem for the second-order theta functions is as follows.

<u>Theorem 2</u>. For any divisor $\mathfrak{z}$ of degree n of the form (4) and the associated matrix (6),

$$W^{\nu}_{n-2} = \{t \in J \;:\; \text{rank } \underline{\underline{\theta}}(t-w(\mathfrak{z}),\mathfrak{z}) < n-\nu\}.$$

<u>Proof</u>. For any fixed point $t \in \mathbb{C}^g$ introduce a basis $h_1(z),\ldots,h_{g+1}(z)$ of the space $\Gamma(\rho_t\zeta^{2g})$, and note that the functions $f_{\nu,t}(z) = \theta_2[\nu|2r-t](w(z)) \in \Gamma(\rho_t\zeta^{2g})$ can be written as $f_{\nu,t}(z) = \Sigma_{i=1}^{g+1} a_{\nu i}\, h_i(z)$ for some constants $a_{\nu i}$; in terms of the $2^g \times (g+1)$ matrix $A = \{a_{\nu i} : \nu \in \mathbb{Z}^g/2\mathbb{Z}^g, 1 \le i \le g+1\}$ this can be rewritten equivalently

$$\hat{\theta}_2[2r-t](w(z)) = A \cdot \hat{h}(z)$$

where $\hat{h}(z)$ is the column vector of length $g+1$ with entries $h_i(z)$. Since the functions $f_{\nu,t}(z)$ span $\Gamma(\rho_t\zeta^{2^g})$ by Theorem 1, the matrix A must be of rank $g+1$; thus the linear mapping $A : \mathbb{C}^{g+1} \to \mathbb{C}^{2^g}$ is injective. Now for the matrices (5) and (6) it is clear that

$$\underline{\underline{\theta}}(2r-t,\underset{\sim}{\mathfrak{d}}) = A \cdot H_t(\underset{\sim}{\mathfrak{d}}),$$

and since A is injective, rank $\underline{\underline{\theta}}(2r-t,\underset{\sim}{\mathfrak{d}})$ = rank $H_t(\underset{\sim}{\mathfrak{d}})$; therefore by the preceding lemma $k-t+w(\underset{\sim}{\mathfrak{d}}) \in W^{\nu}_{n-2}$ precisely when rank $\underline{\underline{\theta}}(2r-t,\underset{\sim}{\mathfrak{d}}) < n-2$. Since the canonical point k and the Riemann point r are related by $k=2r$, this establishes the desired result.

It is instructive to examine the simplest cases of the preceding theorem in more detail. First if $n=1$ the auxiliary divisor is just $\underset{\sim}{\mathfrak{d}} = 1.z_1$ and $W^0_{-1} = \emptyset$, so the conclusion is that

(7) $\quad$ rank $\hat{\theta}_2[t-w(z_1)](w(z_1)) = 1$ for all $t \in \mathbb{C}^g, z_1 \in \tilde{M}$,

a result that is hardly surprising since by Theorem A12 it is always the case that $\hat{\theta}_2(w) \neq 0$. Next if $n = 2$ there are two possibilities for the auxiliary divisor, either $\underset{\sim}{\mathfrak{d}} = 1.z_1+1.z_2$ where $z_1 \neq z_2$ on M or $\underset{\sim}{\mathfrak{d}} = 2.z_1$; and $W^0_0 = 0$, the image of the origin $0 \in \mathbb{C}^g$, while $W^1_0 = \emptyset$. The conclusion for the first auxiliary divisor is that whenever $z_1 \neq z_2$ on M then

$$0 = W^0_0 = \{t \in J:\ \text{rank}[\hat{\theta}_2[t-w(z_1+z_2)](w(z_1)),\ \hat{\theta}_2[t-w(z_1+z_2)](w(z_2))]<2\}$$

or equivalently

(8) $\vec{\theta}_2(\frac{1}{2}(t+w(z_1-z_2)))$, $\vec{\theta}_2(\frac{1}{2}(t+w(z_2-z_1)))$ are linearly dependent precisely when $t = 0 \in J$.

Again this result is hardly surprising, since by Theorem A13 the vectors $\vec{\theta}_2(t_1)$ and $\vec{\theta}_2(t_2)$ are linearly dependent precisely when $t_1 = \pm\, t_2$ in J. That these two vectors have no common zeros, corresponding to the condition that $W_0^1 = \emptyset$, is an immediate consequence of the general condition that $\vec{\theta}_2(t) \neq 0$. For the second auxiliary divisor the conclusion is that

$$0 = W_0^0 = \{t \in J;\ \text{rank}\ \{\vec{\theta}_2[t-w(2z_1)](w(z_1)),\ \Sigma_j\, \partial_j \vec{\theta}[t-w(2z_1)](w(z_1))\ w_j'(z_1)\} < 2\},$$

or equivalently

(9) $\vec{\theta}_2(\frac{1}{2}t)$, $\Sigma^g_{j=1}\partial_j \vec{\theta}_2(\frac{1}{2}t) w_j'(z_1)$ are linearly dependent precisely when $t = 0 \in J$.

That these vectors are linearly dependent when $t = 0$ is quite clear, for $\vec{\theta}_2$ is an even function so $\partial_j \vec{\theta}_2(0) = 0$; the interesting point is that these two vectors are linearly independent otherwise, but again that is a consequence of more general results already established. Indeed by the Corollary to Theorem A14 the condition that the mapping $[\vec{\theta}_2]: J \to \mathbb{P}^{2^g-1}$ be nonsingular at a point t is precisely that the $g+1$ vectors $\vec{\theta}_2(t)$, $\partial_1\vec{\theta}(t), \ldots, \partial_g\vec{\theta}(t)$ are linearly independent, and the result of Theorem A14 was that this condition holds except at the half periods, at the points $\frac{1}{2}t$ where t is a lattice vector and hence represents the point $0 \in J$. Again the two vectors in (9) have no common zeros, since $\hat{\theta}_2(t) \neq 0$.

The first case in which something interesting happens is $n = 3$, and what happens then is very interesting indeed. There are three possibilities for the auxiliary divisor in this case, either $\underset{\sim}{\mathfrak{z}} = 1.z_1+1.z_2+1.z_3$ where z_1,z_2,z_3 are distinct points of M, or $\underset{\sim}{\mathfrak{z}} = 2.z_1+1.z_2$ where z_1,z_2 are distinct points of M, or $\underset{\sim}{\mathfrak{z}} = 3.z_1$; and W_1^0 is the image of the Riemann surface M canonically imbedded in J, while $W_1^1=W_1^2=\emptyset$. The conclusion for the first auxiliary divisor is that whenever z_1,z_2,z_3 are distinct points of M then

$$W_1 = \{t\in J:\ \mathrm{rank}\{\hat{\theta}_2[t-w(\underset{\sim}{\mathfrak{z}})](w(z_1)),\ \hat{\theta}_2[t-w(\underset{\sim}{\mathfrak{z}})](w(z_2)),\ \hat{\theta}_2[t-w(\underset{\sim}{\mathfrak{z}})](w(z_3))\}<3\}$$

where $\underset{\sim}{\mathfrak{z}} = 1.z_1+1.z_2+1.z_3$, or equivalently

(10) $\hat{\theta}_2(\frac{1}{2}(t+w(z_1-z_2-z_3))),\ \hat{\theta}_2(\frac{1}{2}(t+w(z_2-z_1-z_3))),\ \hat{\theta}_2(\frac{1}{2}(t+w(z_3-z_1-z_2)))$
are linearly dependent precisely when $t\in W_1$,

while since $W_1^1 = \emptyset$ these three vectors always span at least a two-dimensional subspace of $\mathbb{C}^{2^g}$ so describe a three-dimensional linear subspace for $t \notin W_1$ and a two-dimensional linear subspace for $t\in W_1$. This now is something quite special to the complex tori that arise as the Jacobi varieties of Riemann surfaces. For the second auxiliary divisor the conclusion is that whenever z_1,z_2 are distinct points of M then

$$W_1 = \{t\in J:\ \mathrm{rank}\{\hat{\theta}_2[t-w(\underset{\sim}{\mathfrak{z}})](w(z_1)),\ \Sigma_j\, \partial_j \hat{\theta}_2[t-w(\underset{\sim}{\mathfrak{z}})](w(z_1))\ w'_j(z_1),\ \hat{\theta}_2[t-w(\underset{\sim}{\mathfrak{z}})](w(z_2))<3\}$$

where $\mathfrak{y} = 2\cdot z_1 + 1\cdot z_2$, or equivalently

(11) $\hat{\theta}_2(\frac{1}{2}(t-w(z_2)))$, $\Sigma_{j=1}^{g} \partial_j \hat{\theta}_2(\frac{1}{2}(t-w(z_2)))w_j'(z_1)$, $\hat{\theta}_2(\frac{1}{2}(t+w(z_2-2z_1)))$ are linearly dependent precisely when $t \in W_1$,

while again the three vectors always span a linear subspace of dimension at least two. Finally for the divisor $\mathfrak{y} = 3\cdot z_1$ the conclusion is that

$$W_1 = \{t\in J:\ \mathrm{rank}\{\hat{\theta}_2[t-3w(z_1)](w(z_1)),\ \Sigma_j \partial_j \hat{\theta}_2[t-3w(z_1)](w(z_1))\cdot w'_j(z_1),$$
$$\Sigma_{j,k} \partial_{jk} \hat{\theta}_2[t-3w(z_1)](w(z_1))w_j'(z_1)w'_k(z_1) + \Sigma_j \partial_j \hat{\theta}_2[t-3w(z_1)](w(z_1))w''_j(z_1)\} < 2\}$$

or equivalently

(12) $\hat{\theta}_2(\frac{1}{2}(t-w(z_1)))$, $\Sigma_j \partial_j \hat{\theta}_2(\frac{1}{2}(t-w(z_1)))w_j'(z_1)$,
$\Sigma_{j,k=1}^{g} \partial_{jk} \hat{\theta}_2(\frac{1}{2}(t-w(z_1)))w_j'(z_1)w_k'(z_1) + \Sigma_{j=1}^{g} \partial_j \hat{\theta}_2(\frac{1}{2}(t-w(z_1)))w_j''(z_1)$
are linearly dependent precisely when $t \in W_1$,

and again these three vectors always span a linear subspace of dimension at least two. These last two formulas are perhaps more appealing when rewritten as the assertions that

(11') $\hat{\theta}_2(\frac{1}{2}t)$, $\Sigma_{j=1}^{g} \partial_j \hat{\theta}_2(\frac{1}{2}t)w_j'(z_1)$, $\hat{\theta}_2(\frac{1}{2}t+w(z_2-z_1))$ are linearly dependent precisely when $t \in W_1 - w(z_2)$, provided that $z_1 \neq z_2$ on M, and

(12') $\hat{\theta}_2(\frac{1}{2}t)$, $\Sigma_{j=1}^{g} \partial_j \hat{\theta}_2(\frac{1}{2}t)w_j'(z_1)$,
$\Sigma_{j,k=1}^{g} \partial_{jk} \hat{\theta}_2(\frac{1}{2}t)w_j'(z_1)w_k'(z_1) + \Sigma_{j=1}^{g} \partial_j \hat{\theta}_2(\frac{1}{2}t)w_j''(z_1)$
are linearly dependent precisely when $t \in W_1 - w(z_1)$.

§2. Translation by half periods.

The description of the subvarieties of special positive divisors provided by Theorem 2 in the simplest case, that in which the auxiliary divisor is formed from distinct points, is in terms of some functions $\vartheta_2[t](\alpha_i) = \vartheta_2[2\alpha_i](\frac{1}{2}t)$ of the variable $t \in \mathbb{C}^g$, where $\alpha_i \in \mathbb{C}^g$ are points determined by the auxiliary divisor. The loci so described are subvarieties of $\mathbb{C}^g$ invariant under the lattice subgroup $\underline{L}$, but the defining functions $\theta_2[\nu|2\alpha_i](\frac{1}{2}t;\Omega)$ are not relatively automorphic functions for this lattice. Their behavior under $\underline{L}$ reflects the behavior of the second-order theta functions under translation by half periods, a slightly more complicated matter than the scalar automorphy that has thus far been considered. It is useful to examine this behavior in some detail here.

For this purpose introduce the function $\eta(\lambda,t)$ that associates to any lattice vector $\lambda = p+\Omega q \in \underline{L}$ and any point $t \in \mathbb{C}^g$ the value

(1) $\eta(\lambda,t) = \exp -\pi i\, {}^t q(t+\frac{1}{2}\Omega q).$

This is just a square root of the theta factor of automorphy $\xi(\lambda,t)$ as defined in (A3.6), in the sense that $\eta(\lambda,t)^2 = \xi(\lambda,t)$, and it is clear from this that $\eta(\lambda,t)$ is itself a factor of automorphy for the action of the lattice group $\underline{L}$ on $\mathbb{C}^g$ up to sign. More precisely if $\lambda_1 = p_1+\Omega q_1$ and $\lambda_2 = p_2+\Omega q_2$ then

$$\eta(\lambda_1+\lambda_2,t) = \exp - \pi i\, {}^t(q_1+q_2)(t+\tfrac{1}{2}\Omega(q_1+q_2))$$

$$= \exp - \pi i[{}^t q_1(t+p_2+\Omega q_2+\tfrac{1}{2}\Omega q_1) + {}^t q_2\,(t+\tfrac{1}{2}\Omega q_2) - {}^t q_1 p_2]$$

and consequently

$$(2)\quad \eta(\lambda_1+\lambda_2,t) = (-1)^{{}^tq_1\cdot p_2}\eta(\lambda_1,\ t+\lambda_2)\ \eta(\lambda_2,t).$$

In terms of this almost factor of automorphy the second-order theta functions transform under half periods as follows.

Theorem 3. To every $\lambda\in L$ there corresponds a unique $2^g\times 2^g$ matrix $\chi(\lambda)$ such that

$$\vec{\theta}_2[\alpha](\tfrac{1}{2}(t+\lambda)) = \rho_{-\frac{1}{2}\alpha}(\lambda)\ \eta(\lambda,t)\ \chi(\lambda)\ \vec{\theta}_2[\alpha](\tfrac{1}{2}t).$$

Proof. If $\lambda=p+\Omega q\in L$ then

$\theta_2[\nu|\alpha](\frac{1}{2}(t+\lambda);\Omega)$

$$= \theta[\tfrac{\nu}{2}|\alpha](t+\lambda;2\Omega) \qquad \text{by definition (A5.1)}$$

$$= \theta[\tfrac{\nu}{2}|\alpha](t+\Omega q;2\Omega)\ \exp \pi i\, {}^tp\cdot\nu \qquad \text{by Theorem A1}$$

$$= \theta[\tfrac{\nu}{2}+\tfrac{q}{2}|\alpha](t;2\Omega)\cdot \exp \pi i[{}^tp\cdot\nu-{}^tq\cdot(t+\alpha+\tfrac{1}{2}\Omega q)] \qquad \text{by (A2.3)}$$

$$= \theta_2[\nu+q|\alpha](\tfrac{1}{2}t;\Omega)\cdot\eta(\lambda,t)\ \exp \pi i[{}^tp\cdot\nu-{}^tq\cdot\alpha]$$

$$= \rho_{-\frac{1}{2}\alpha}(\lambda)\ \eta(\lambda,t)\ \sum_{\mu}\chi_{\nu\mu}(\lambda)\ \theta_2[\mu|\alpha](\tfrac{1}{2}t;\Omega)$$

where this last summation is extended over $\mathbb{Z}^g/2\,\mathbb{Z}^g$ and

$$(3)\quad \chi_{\nu\mu}(\lambda) = \chi_{\nu\mu}(p+\Omega q) = \delta_{\mu}^{\nu+q} \exp \pi i\, {}^tp\cdot\nu.$$

When written out in vector notation this is just the desired formula, for the $2^g \times 2^g$ matrix $\chi(\lambda)$ given by (3), and this matrix is uniquely determined since the component functions making up the vector $\vec{\theta}_2[\alpha](t)$ are linearly independent functions of t.

A number of properties of the matrix function $\chi(\lambda)$ follow immediately from this theorem and the defining formula (3). First since $\chi(\lambda)$ is completely determined by the functional equation of Theorem 3 it is easy to see that the expression $\rho_{-\frac{1}{2}\alpha}(\lambda)\ \eta(\lambda,t)\ \chi(\lambda)$ must define a factor of automorphy for the action of the lattice group $\underline{L}$ on $\mathbb{C}^g$, a matricial factor of automorphy of rank 2^g; the functional equation of Theorem 3 can then be interpreted as the assertion that $\vec{\theta}_2\,[\alpha](\frac{1}{2}t)$ as a function of t is a relatively automorphic function for this factor of automorphy, or

$$\vec{\theta}_2[\alpha](\tfrac{1}{2}t) \in \Gamma(\rho_{-\frac{1}{2}\alpha}\ \eta\ \chi).$$

The representation $\rho_{-\frac{1}{2}\alpha} \in \mathrm{Hom}\,(\underline{L},\mathbb{C}^*)$ is itself a factor of automorphy, while the function $\eta(\lambda,t)$ is a factor of automorphy up to sign in the sense of (2), and consequently it is evident that

$$(4)\quad \chi(\lambda_1+\lambda_2) = (-1)^{{}^tq_1\cdot p_2}\ \chi(\lambda_1)\chi(\lambda_2) = (-1)^{{}^tq_2\cdot p_1}\chi(\lambda_2)\ \chi(\lambda_1).$$

This can also be deduced directly from the defining formula (3) by a straightforward calculation. Since $\chi(0)=I$ it follows from this that the matrices $\chi(\lambda)$ are all nonsingular, indeed that $1 = \det(-1)^{{}^t q\cdot p}\chi(\lambda)\chi(\lambda) = (\det \chi(\lambda))^2$ so that $\det \chi(\lambda) = \pm 1$ for any $\lambda \in L$. It is further clear from (3) that the entries in $\chi(\lambda)$ are all integers, indeed are only 0 or ± 1, and that the value $\chi(\lambda)$ depends only on the residue class of λ in $L/2L$. Thus the function χ can be viewed as a mapping

$$\text{(5)} \quad \chi:\ L/2L \to \{X \in GL(2^g,\mathbb{Z}):\ x_{\mu\nu} = 0,\ +1,\ \text{or} -1\}$$

that satisfies (4), and this essentially determines the mapping uniquely. Mappings of this sort have arisen in a number of quite similar contexts, in which a square root can only be defined by passing from a scalar to a matrix quantity; the spinor representation is in some sense of this sort. In the present case, the square root of the theta factor of automorphy does not exist as a scalar factor of automorphy but does as a matrix factor of automorphy. Another useful property of the mapping χ is as follows.

Lemma 2. The mapping $\chi: L \to GL(2^g, \mathbb{Z})$ is irreducible, in the sense that if A is any $2^g \times 2^g$ complex matrix such that $A\cdot\chi(\lambda)=\chi(\lambda)\cdot A$ whenever $\lambda \in L$ then necessarily $A = aI$ for some $a \in \mathbb{C}$.

Proof. This can be deduced quite easily directly from the defining equation (3) for the matrix $\chi(\lambda)$. Indeed by using (3) the identity

$$\Sigma_\sigma A_{\nu\sigma}\chi_{\sigma\mu}(\lambda) = \Sigma_\sigma \chi_{\nu\sigma}(\lambda)A_{\sigma\mu}$$

for a lattice vector $\lambda = p + \Omega q \in L$ can be written out as

$$\Sigma_\sigma A_{\nu\sigma}\delta_\mu^{\sigma+q} \exp \pi i\, {}^t p.\sigma = \Sigma_\sigma \delta_\sigma^{\nu+q} A_{\sigma\mu} \exp \pi i\, {}^t p\cdot\nu$$

or equivalently

$$A_{\nu,\mu-q} = A_{\nu+q,\mu} \exp \pi i\, {}^t p\cdot(\nu-\mu+q);$$

the indices of entries in the matrix range over $\mathbb{Z}^g/2\mathbb{Z}^g$ while p,q range over $\mathbb{Z}^g$, but that is clearly not a problem. In particular for q=0

$$A_{\nu,\mu} = A_{\nu,\mu} \cdot \exp \pi i\, {}^t p\cdot(\nu-\mu) \quad \text{for all } p\in\mathbb{Z}^g,$$

so that $A_{\nu,\mu} = 0$ unless $\nu-\mu \in 2\mathbb{Z}^g$, that is, unless $\nu = \mu$ in $\mathbb{Z}^g/2\mathbb{Z}^g$; thus the matrix A must be a diagonal matrix. Then for $p = 0$ and $q = \mu - \nu$

$$A_{\nu,\nu} = A_{\mu,\mu}$$

so that A is a scalar matrix as asserted.

<u>Theorem 4</u>. For any $a\in\mathbb{C}^g$ the vector space $\Gamma(\rho_{\frac{1}{2}a}\ \eta\ \chi)$ of vector valued relatively automorphic functions for this matricial factor of automorphy is one dimensional, and $\vec{\theta}_2[a](\frac{1}{2}t)$ is a basis.

<u>Proof</u>. The result of Theorem 3 is that $\vec{\theta}_2[a](\frac{1}{2}t)\epsilon\Gamma(\rho_{\frac{1}{2}a}\ \eta\ \chi)$, so it is only necessary to show that every other relatively automorphic function is a scalar multiple of $\vec{\theta}_2[a](\frac{1}{2}t)$. If $\vec{f}(t)\epsilon\Gamma(\rho_{\frac{1}{2}a}\ \eta\ \chi)$ then set $\vec{h}(t) = \vec{f}(2t)$ and note that for any lattice vector $\lambda=p+\Omega q \in L$

$$\vec{h}(t+\lambda) = \vec{f}(2t+2\lambda)$$

$$= \rho_{-\frac{1}{2}\alpha}(2\lambda)\ \eta(2\lambda,2t)\ \chi(2\lambda)\ \vec{h}(t)$$

$$= \rho_{-\alpha}(\lambda)\ \xi(\lambda,t)^2\ \vec{h}(t),$$

since it is clear from (1) that $\eta(2\lambda,2t) = \xi(\lambda,t)^2$ and was already noted in (5) that $\chi(2\lambda)=I$. The entries in the vector $\vec{h}(t)$ are thus relatively automorphic functions for the scalar factor of automorphy $\rho_{-\alpha}\xi^2$, so by Theorem A6 must be linear combinations of the second-order theta functions $\theta_2[\nu|\alpha](t)$; in matrix notation $\vec{h}(t)=A\cdot\vec{\theta}_2[\alpha](t)$ for some $2^g\times 2^g$ complex matrix A, or equivalently $\vec{f}(t)=A\cdot\vec{\theta}_2[\alpha](\frac{1}{2}t)$. Now replacing t in this last equation by $t+\lambda$ and using the conditions that both $\vec{f}(t)$ and $\vec{\theta}_2[\alpha](\frac{1}{2}t)$ are elements of $\Gamma(\rho_{-\frac{1}{2}\alpha}\eta\chi)$ leads easily to the condition that $\chi(\lambda)A = A\chi(\lambda)$ for all $\lambda \in L$. The preceeding lemma then shows that $A=aI$, so that $\vec{f}(t)=a\cdot\vec{\theta}_2[\alpha](\frac{1}{2}t)$ as desired.

§3. Fay's trisecant identity.

The description of the subvariety $W_1 \subseteq J$ in terms of second-order theta functions provided by (1.10) can be viewed as an identity among theta functions, the assertion that the three theta functions involved are linearly dependent vectors whenever $t = w(z) \in M$; a somewhat more precise description of this linear dependence relation is the trisecant identity that was implicit in earlier works but made explicit and used quite effectively by J. Fay, (Theta Functions on Riemann Surfaces, Springer-Verlag 1973).

Theorem 5. For all points $z, z_1, z_2, z_3, \in \tilde{M}$

$$0 = q(z,z_1)q(z_2,z_3)\ \vec{\theta}_2(\tfrac{1}{2}w(z+z_1-z_2-z_3))$$

$$+q(z,z_2)q(z_3,z_1)\vec{\theta}_2(\tfrac{1}{2}w(z+z_2-z_1-z_3))$$

$$+q(z,z_3)q(z_1,z_2)\vec{\theta}_2(\tfrac{1}{2}w(z+z_3-z_1-z_2)).$$

Proof. It follows from (1.10) that the three vectors of second-order theta functions in the statement of the theorem are linearly dependent whenever z_1, z_2, z_3 are distinct points of M, and by continuity they must of course be linearly dependent for arbitrary points $z, z_1, z_2, z_3 \in \tilde{M}$. On the other hand it follows readily from (1.8) that the last two vectors of second-order theta functions are linearly independent whenever $z \neq z_1$ and $z_2 \neq z_3$ on M, and then the first vector is a uniquely determined linear combination of them. Thus for all points $z, z_1, z_2, z_3 \in \tilde{M}$ for which $z \neq z_1$ on M and $z_2 \neq z_3$ on M there are uniquely determined complex values $f_2 = f_2(z, z_1, z_2, z_3)$ and $f_3 = f_3(z, z_1, z_2, z_3)$ such that

$$\hat{\theta}_2(\tfrac{1}{2}v(z+z_1-z_2-z_3))$$

$$(1) \qquad = f_2\hat{\theta}_2(\tfrac{1}{2}v(z+z_2-z_3-z_1))+ f_3\hat{\theta}_2(\tfrac{1}{2}v(z+z_3-z_1-z_2)).$$

The values f_2, f_3 are clearly holomorphic functions whenever well defined, and extend to meromorphic functions on $\tilde{M}^4$ with singularities at most along the subvarieties $z = Tz_1$ and $z_2 = Tz_3$ for all $T \in \Gamma$; the problem is now just to determine these functions explicitly.

Note first of all that the left-hand side of (2) is symmetric in z_2 and z_3, while the two theta functions on the right-hand side of (1) are interchanged when z_2 and z_3 are interchanged; consequently

$$f_3(z,z_1,z_2,z_3) = f_2(z,z_1,z_3,z_2),$$

so it is really only necessary to examine one of these two functions. Next interchanging z with z_2 and z_1 with z_3 simultaneously leaves all three theta functions unchanged, since $\hat{\theta}_2(w)$ is an even function, so that

$$(2) \quad f_2(z, z_1,z_2,z_3) = f_2(z_2,z_3,z,z_1).$$

On the other hand interchanging z with z_3 leads to the relation

$$\hat{\theta}_2(\tfrac{1}{2}v(z+z_2-z_1-z_3)) = f_2(z_3,z_1,z_2,z)\ \hat{\theta}_2(\tfrac{1}{2}v(z+z_1-z_2-z_3))$$

$$+f_3(z_3,z_1,z_2,z)\ \hat{\theta}_2(\tfrac{1}{2}v(z+z_3-z_1-z_2)),$$

and comparison of this with (1) shows that

$$(3)\quad f_2(z_3,z_1,z_2,z) = f_2(z,z_1,z_2,z_3)^{-1}.$$

Similarly interchanging z_1 with z_2 shows that

$$(4)\quad f_2(z,z_2,z_1,z_3) = f_2(z,z_1,z_2,z_3)^{-1}.$$

When z is replaced by Tz the arguments in the theta functions are translated by the half-period $\frac{1}{2}\omega(T)$, and the behavior of the theta functions is as described in Theorem 3; if $\frac{1}{2}w(z)$ is viewed as the variable and the expressions $\alpha = w(\pm z_1 \pm z_2 \pm z_3)$ with appropriate signs are viewed as characteristics, then these three theta functions are multiplied by the common factor $\eta(\omega(T),w(z))\ \chi(\omega(T))$ and in addition by the appropriate factor $\rho_{-\frac{1}{2}\alpha}(T)$, so that (1) becomes

$$\rho_{-\frac{1}{2}w(z_1-z_2-z_3)}(T)\quad \hat{\theta}_2(\tfrac{1}{2}w(z+z_1-z_2-z_3))$$

$$= f_2(Tz,z_1,z_2,z_3)\quad \rho_{-\frac{1}{2}w(z_2-z_3-z_1)}(T)\quad \hat{\theta}_2(\tfrac{1}{2}w(z+z_1-z_2-z_3))$$

$$+ f_3(Tz,z_1,z_2,z_3)\quad \rho_{-\frac{1}{2}w(z_3-z_1-z_2)}(T)\quad \hat{\theta}_2(\tfrac{1}{2}w(z+z_3-z_1-z_2)).$$

Comparing this with (1) shows that

$$(5)\quad f_2(Tz,z_1,z_2,z_3) = \rho_{v(z_2-z_1)}(T)\ \ f_2(z,z_1,z_2,z_3).$$

The symmetries described by (2), (3), (4) easily imply that the function f_2 satisfies corresponding functional equations under the action of the group Γ on the other three variables; the divisor of f_2 is consequently invariant under the action of Γ^4 on $\tilde{M}^4$, so really can be viewed as a divisor on M^4. As far as this divisor is concerned, it was already noted that f_2 has as singularities at most poles along the subvarieties $z = z_1$, and $z_2 = z_3$ in $\tilde{M}^4$. On the other hand if $f_2(z,z_1,z_2,z_3) = 0$ at some particular point $(z,z_1,z_2,z_3) \in \tilde{M}^4$ then from (1) it follows that the vectors $\hat{\theta}_2(\frac{1}{2}v(z+z_1-z_2-z_3))$ and $f_3(z,z_1,z_2,z_3)\ \hat{\theta}_2(\frac{1}{2}v(z+z_3-z_1-z_2))$ are linearly dependent; but from (1.7) it is clear that $f_3(z,z_1,z_2,z_3) \neq 0$, and from (1.8) as before it follows easily that either $z = z_2$ or $z_1 = z_3$ on M. Thus f_2 has zeros at most along the subvarieties $z = z_2$ and $z_1 = z_3$ on M^4. If z_1,z_2,z_3 represent any distinct points of M then when f_2 is viewed as a function of the variable z alone its divisor must be of the form $\mathcal{D}(f_2) = \nu_2 \cdot z_2 - \nu_1 \cdot z_1$ for some nonnegative integers ν_1, ν_2. Now from (5) it is evident that $\rho_{v(z_2-z_1)}$ is a factor of automorphy associated to this divisor, while from the Corollary to Theorem B6 it follows that

$\rho_{w(\nu_2 z_2 - \nu_1 z_1)} \zeta^{\nu_2 - \nu_1}$ is also a factor of automorphy associated to this divisor; these two factors of automorphy must therefore be analytically equivalent for arbitrary distinct points z_1, z_2, and from Theorems B4 and B5 it follows readily that necessarily $\nu_1 = \nu_2 = 1$, hence that f_2 has first-order zeros and poles. The symmetries (2), (3), (4) readily show that the same is true when f_2 is viewed as a function of z_3, so that altogether

(6) $f_2(z, z_1, z_2, z_3)$ has simple zeros along the subvarieties $z = Tz_2$, $z_3 = Tz_1$ and simple poles along the subvarieties $z = Tz_1, z_3 = Tz_2$ on $\tilde{M}$, and no other zeros or singularities.

Finally note that when $z = z_3$ then $f_3(z, z_1, z_2, z) = f_2(z, z_1, z, z_2) = 0$ so that (1) becomes

$$\hat{\theta}_2(\tfrac{1}{2}w(z_1 - z_2)) = f_2(z, z_1, z_2, z)\ \hat{\theta}_2(\tfrac{1}{2}w(z_2 - z_1)),$$

and since $\hat{\theta}_2$ is an even function clearly

(7) $f_2(z, z_1, z_2, z) = 1.$

Now it follows immediately from Theorem B3 that properties (2) through (7) characterize the cross-ratio function $p(z, z_3, z_2, z_1)$, and consequently

$$f_2(z, z_1, z_2, z_3) = f_3(z, z_1, z_3, z_2) = p(z, z_3, z_2, z_1);$$

equivalently when the cross-ratio function is expressed in terms of the prime function

$$f_2(z,z_1,z_2,z_3) = f_3(z,z_1,z_3,z_2) = \frac{q(z-z_2)}{q(z-z_1)} \frac{q(z_3-z_1)}{q(z_3-z_2)},$$

and with this (1) reduces precisely to the desired result, thereby concluding the proof.

For some purposes it is more convenient to rewrite the trisecant formula in a slightly different notation , using z_1, z_2, a_1, a_2, in place of z, z_1, z_2, z_3 to designate points of $\tilde{M}$; the result is then

$$(8)\quad q(z_1,z_2)\, q(a_1,a_2)\, \vartheta_2(\tfrac{1}{2}w(z_1+z_2-a_1-a_2))$$

$$= q(z_1,a_1)\, q(z_2,a_2)\, \vartheta_2(\tfrac{1}{2}w(z_1+a_1-z_2-a_2))$$

$$-q(z_1,a_2)q(z_2,a_1)\, \vartheta_2(\tfrac{1}{2}w(z_1+a_2-z_2-a_1)).$$

A rather less trivial alteration is to rewrite this formula in terms of first-order theta functions, using the addition theorem for theta functions; the result in this case is as follows.

<u>Corollary</u>. For any point $t \in \Gamma^g$ and any points $z_1, z_2, a_1, a_2 \in \tilde{M}$,

$$- \theta(t)\ \theta(t+w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)q(a_1,a_2)\Big[\prod_{j,k=1}^{2} q(z_j,z_k)^{-1}\Big]$$

$$= \det \begin{Bmatrix} \dfrac{\theta(t+w(z_1-a_1))}{q(z_1,a_1)} & \dfrac{\theta(t+w(z_1-a_2))}{q(z_1,a_2)} \\ \\ \dfrac{\theta(t+w(z_2-a_1))}{q(z_2,a_1)} & \dfrac{\theta(t+w(z_2-a_2))}{q(z_2,a_2)} \end{Bmatrix} .$$

<u>Proof</u>. Multiplying (1) by ${}^t\theta_2(t+\frac{1}{2}w(z_1+z_2-a_1-a_2))$ and using Theorem A7 lead immediately to the result that

$$q(z_1,z_2)q(a_1,a_2)\ \theta(t)\ \theta(t+w(z_1+z_2-a_1-a_2))$$

$$= q(z_1,a_1)q(z_2,a_2)\ \theta(t+w(z_1-a_2))\ \theta(t+w(z_2-a_1))$$

$$-q(z_1,a_2)q(z_2,a_1)\ \theta(t+w(z_1-a_1))\ \theta(t+w(z_2-a_2))$$

$$= \Big[\prod_{j,k=1}^{2} q(z_j,a_k)\Big] \left\{ \frac{\theta(t+w(z_1-a_2))}{q(z_1,a_2)}\ \frac{\theta(t+w(z_2-a_1))}{q(z_2,a_1)} - \frac{\theta(t+w(z_1-a_1))}{q(z_1,a_1)}\ \frac{\theta(t+w(z_2-a_2))}{q(z_2,a_2)} \right\} .$$

and the assertion of the corollary is an immediate consequence of this.

If $t \in \underline{\underline{\Theta}}$ then the preceeding corollary reduces to the assertion that the determinant is identically zero. The functions $q(z,a)^{-1}\theta(t+w(z-a)) \in \Gamma(\rho_{-t}\sigma)$ in that determinant are the semicanonical functions as discussed in section C4, and the determinant condition is easily verified to reduce to the formula for the cross-ratio function given in Theorem C6; thus in a sense this last corollary is an extension of Theorem C6 to general points $t \in \mathbb{C}^g$.

§4. Limiting forms of the trisecant identity.

It is interesting to view the trisecant identity (3.8) as a relation between holomorphic functions of the variables $z_1, z_2 \in \tilde{M}$ for fixed points $a_1, a_2 \in \tilde{M}$, and to examine this identity more closely for values z_j near a_j. If $z_j = a_j$ this identity is easily seen to reduce to a triviality; but the relations that arise between the coefficients in the Taylor expansion in powers of $z_j - a_j$ are quite nontrivial. These can be approached most directly by differentiating (3.8) repeatedly with respect to the variables z_j, always in terms of the canonical coordinates on $\tilde{M}$, and then setting $z_j = a_j$. The first nontrivial results arise from applying the differential operator $\partial^2/\partial z_1 \partial z_2$.

Theorem 6. For any points $z, a \in \tilde{M}$,

$$\hat{\theta}_2(w(z-a)) = q(z,a)^2 w_a'(z)\ \hat{\theta}_2(0) + \frac{1}{2} q(z,a)^2 \sum_{j_1 j_2} \partial_{j_1 j_2} \hat{\theta}_2(0) w_{j_1}'(z) w_{j_2}'(a) .$$

Proof. The proof is a straightforward calculation, and amounts to applying the differential operator $\partial^2/\partial z_1 \partial z_2$ to (3.8) and then setting $z_j = a_j$; but a few preliminary observations can simplify matters somewhat. In the first line the differential operators $\partial/\partial z_1$ and $\partial/\partial z_2$ can be applied either to the prime function $q(z_1, z_2)$ or to the theta function. Since $\hat{\theta}_2$ is an even function $\partial_j \hat{\theta}_2(0) = 0$, so the only nontrivial results arise from applying either both or neither of these two differential operators to the theta function. Set $\hat{\theta}_2(\frac{1}{2} w(z_1 + z_2 - a_1 - a_2)) = \hat{\theta}_2$ as a purely temporary convenience, and note that

$$\frac{\partial}{\partial z_1}\hat{\theta}_2 = \frac{1}{2}\sum_j \partial_j\hat{\theta}_2\, v_j'(z_1)$$
$$\frac{\partial^2}{\partial z_1\partial z_2}\hat{\theta}_2 = \frac{1}{4}\sum_j \partial_{j_1 j_2}\hat{\theta}_2\, v_{j_1}'(z_1)v_{j_2}'(z_2)\ .$$

In the second line since $q(a,a) = 0$ the only nontrivial results arise from applying $\partial/\partial z_j$ to $q(z_j,a_j)$; recall also that $\partial_1 q(a,a) = \partial q(z,a)/\partial z|_{z=a} = 1$, where differentiation is with respect to the canonical local coordinate on $\tilde{M}$. The third line is much like the first line, except that there are two prime functions with variable arguments and the argument of the theta function involves $-v(z_2)$ rather than $+v(z_2)$ so that differentiation with respect to z_2 leads to a minus sign. Altogether then

$$\begin{aligned} q\,\partial_1\partial_2 q\,\hat{\theta}_2(0) &+ \frac{1}{4}q^2\sum_{j_1 j_2}\partial_{j_1 j_2}\hat{\theta}_2(0)\, v_{j_1}'(a_1)v_{j_2}'(a_2) \\ &= \hat{\theta}_2(v(a_1 - a_2)) + \partial_1 q\,\partial_2 q\,\hat{\theta}_2(0) \\ &\quad - \frac{1}{4}q^2\sum_{j_1 j_2}\partial_{j_1 j_2}\hat{\theta}_2(0)\, v_{j_1}'(a_1)v_{j_2}'(a_2)\ , \end{aligned}$$

where $q = q(a_1,a_2)$, $\partial_1 q = \partial q(a_1,a_2)/\partial a_1$, and so on. The coefficient of $\hat{\theta}_2(0)$ can be simplified by recalling from B6(11) that

$$v_{a_2}'(a_1) = \frac{\partial^2}{\partial a_1\partial a_2}\log q = q^{-1}\partial_1\partial_2 q - q^{-2}\partial_1 q\,\partial_2 q\ .$$

Replacing a_1 by z and a_2 by a then yields the desired result.

This is a very interesting result indeed, and should be examined more closely before passing on to some other limiting forms of the trisecant identity. First note that the components of the vector-valued function

$\hat{\theta}_2(w|z-a))$ are the 2^g functions

$$\theta_2[\nu|0](w(z-a)) = \theta_2[\nu|-2w(a)](w(z)) \in \Gamma(\rho_{2w(a)+2r}\zeta^{2g}) = \Gamma(\zeta_a^2\kappa) .$$

The Riemann-Roch theorem shows that $\gamma(\zeta_a^2\kappa) = g+1$; the functions $q(z,a)^2 w_j'(z)$ are g linearly independent elements of $\Gamma(\zeta_a^2\kappa)$ that vanish at the point a , while $q(z,a)^2 w_a'(z)$ is an element of $\Gamma(\zeta_a^2\kappa)$ that takes the value 1 at the point a , and consequently

(1) the $g+1$ functions $q(z,a)^2 w_1'(z),\dots,q(z,a)^2 w_g'(z)$, $q(z,a)^2 w_a'(z)$ are a basis for the vector space $\Gamma(\zeta_a^2\kappa)$.

The result of Theorem 6 can be viewed as yielding an explicit expression for the functions $\theta_2[\nu|0](w(z-a))$ in terms of this canonical basis. It follows from Theorem 1 that the restricted theta functions necessarily span the space $\Gamma(\zeta_a^2\kappa)$; it is an easy consequence of this that the $g+1$ vectors $\hat{\theta}_2(0)$, $\sum_j \partial_{1j}\hat{\theta}_2(0)w_j'(a),\dots,\sum_j \partial_{gj}\hat{\theta}_2(0)w_j'(a)$ are linearly independent for any point $a \in \tilde{M}$.

To translate this identity into one expressed in terms of first-order theta functions, it is convenient to have the following auxiliary result.

<u>Lemma 3</u>. (a) ${}^t\hat{\theta}_2(t)\cdot\hat{\theta}_2(0) = \theta(t)^2$,

$$\text{(b)}\quad {}^t\hat{\theta}_2(t)\cdot\partial_{j_1j_2}\hat{\theta}_2(0) = 2\theta(t)^2\frac{\partial^2}{\partial t_{j_1}\partial t_{j_2}}\log\theta(t)$$
$$= 2\theta(t)\partial_{j_1j_2}\theta(t) - 2\partial_{j_1}\theta(t)\partial_{j_2}\theta(t).$$

<u>Proof</u>. The addition theorem for theta functions is that ${}^t\hat{\theta}_2(t)\cdot\hat{\theta}_2(s) = \theta(t+s)\,\theta(t-s)$, and for the special case $s=0$ this is just the assertion (a). Applying the differential operator $\partial^2/\partial s_{j_1}\partial s_{j_2}$ to the addition theorem yields

$$ {}^t\hat\theta_2(t) \cdot \partial_{j_1 j_2}\hat\theta_2(s) = \partial_{j_1 j_2}\theta(t+s)\ \theta(t-s) - \partial_{j_1}\theta(t+s)\ \partial_{j_2}\theta(t-s) $$
$$ - \partial_{j_2}\theta(t+s)\ \partial_{j_1}\theta(t-s) + \theta(t+s)\ \partial_{j_1 j_2}\theta(t-s) , $$

and for the special case $s = 0$ this is just assertion (b) since

$$ \frac{\partial^2}{\partial t_{j_1} \partial t_{j_2}} \log \theta(t) = \theta(t)^{-1} \partial_{j_1 j_2}\theta(t) - \theta(t)^{-2}\ \partial_{j_1}\theta(t)\ \partial_{j_2}\theta(t) . $$

Corollary. For any points $t \in \mathbb{C}^g$ and $z,a \in \tilde{M}$,

$$ q(z,a)^{-2}\theta(t)^{-2}\theta(t + w(z-a))\ \theta(t - w(z-a)) = $$
$$ = w_a'(z) + \sum_{j_1 j_2} [\partial^2 \log \theta(t)/\partial t_{j_1} \partial t_{j_2}]\ w_{j_1}'(z)\ w_{j_2}'(a) . $$

Proof. This follows immediately from the preceding theorem by multiplying the formula of that theorem by ${}^t\hat\theta_2(t)$, using the addition theorem to rewrite the left-hand side, and applying Lemma 3 to rewrite the right-hand side.

After multiplying the formula of the preceding Corollary by $\theta(t)^2$ and taking the limit as t approaches a point in the theta locus, there results the formula of Corollary 1 to Theorem C7; thus the present result can be viewed as an extension of the preceding formula to general points $t \in \mathbb{C}^g$, beyond those in the theta locus. For these general points the sort of expansions discussed in section C4 involve not just the Abelian differentials $w_j'(z)$ but also the meromorphic Abelian differentials $w_a'(z)$, as in this Corollary; these results can be obtained by extending the discussion in section C4, as well as by specializing the trisecant identity.

This procedure can be extended indefinitely, to yield a whole hierarchy of results analogous to the expansion given in Theorem 6. The calculations rapidly become very complicated, and the general structure of the formulas that arise is not terribly clear from the direct calculation. An alternative approach to these formulas, by expansions in terms of canonical holomorphic and meromorphic differentials as mentioned briefly in the preceding paragraph, seems better adapted to studying the general structure of the formulas; this method will be developed in detail later in section F. For the present though it is perhaps more interesting to examine some alternative limiting cases of the trisecant identity, letting points coincide in other patterns.

Theorem 7. For any points $z, a_1, a_2 \in \tilde{M}$

$$\begin{aligned} q(z,a_1)\, \hat{\theta}_2\!\left(\tfrac{1}{2} w(z+a_1) - w(a_2)\right) &= \left(q(a_1,a_2)\, \partial_2 q(z,a_2) - q(z,a_2)\, \partial_2 q(a_1,a_2)\right) \hat{\theta}_2\!\left(\tfrac{1}{2} w(z-a_1)\right) \\ &\quad + q(a_1,a_2)\, q(z,a_2) \textstyle\sum_j \partial_j \hat{\theta}_2\!\left(\tfrac{1}{2} w(z-a_1)\right) w_j'(a_2) \,, \\ 0 &= \textstyle\sum_{j_1 j_2} \partial_{j_1 j_2} \hat{\theta}_2\!\left(\tfrac{1}{2} w(z-a)\right) w_{j_1}'(a)\, w_{j_2}'(a) \\ &\quad + \textstyle\sum_j \partial_j \hat{\theta}_2\!\left(\tfrac{1}{2} w(z-a)\right)\left(w_j''(a) - 2w_a(z)\, w_j'(a)\right) \\ &\quad + \left(q(z,a)^{-1} \partial_2^2 q(z,a) - 6q_3(a,a)\right) \hat{\theta}_2\!\left(\tfrac{1}{2} w(z-a)\right) , \end{aligned}$$

where the prime function is written in canonical coordinates in the form

$$q(z,a) = (z-a) + (z-a)^3\, q_3(z,a) \,.$$

Proof. Return to the trisecant identity as in Theorem 5, and consider the limiting values as z_3 tends towards z_2. The first nontrivial result arises from applying the differential operator $\partial/\partial z_3$ and then setting $z_3 = z_2$, and is

$$\begin{aligned} 0 = &-q(z,z_1)\ \hat{\theta}_2(\tfrac{1}{2}\, v(z + z_1 - 2z_2)) \\ &+ q(z,z_2)\ \partial_1 q(z_2,z_1)\ \hat{\theta}_2(\tfrac{1}{2}\, v(z - z_1)) \\ &- \tfrac{1}{2}\, q(z,z_2)\ q(z_2,z_1)\ \textstyle\sum_j \partial_j \hat{\theta}_2(\tfrac{1}{2}\, v(z - z_1))\ v_j'(z_2) \\ &+ \partial_2 q(z,z_2)\ q(z_1,z_2)\ \hat{\theta}_2(\tfrac{1}{2}\, v(z - z_1)) \\ &+ \tfrac{1}{2}\, q(z,z_2)\ q(z_1,z_2)\ \textstyle\sum_j \partial_j\ \hat{\theta}_2(\tfrac{1}{2}\, v(z - z_1))\ v_j'(z_2)\ ; \end{aligned}$$

this yields the first formula in the statement of the theorem. Next consider the limiting values in this formula as a_2 tends to a_1 ; the first nontrivial result arises from applying the differential operator $\partial^2/\partial a_2^2$ and then setting $a_2 = a_1$, although the calculation is a bit more involved than in the derivation of the first formula. As a preliminary recall the expansion B(6.3) of the prime functions in canonical coordinates in the form

$$q(a_1,a_2) = (a_1 - a_2) + (a_1 - a_2)^3\ q_3(a_1,a_2)\ ;$$

it is clear from this that

$$\partial_2 q(a_1,a_2)|_{a_2=a_1} = -1\ ,\quad \partial_2^2 q(a_1,a_2)|_{a_2=a_1} = 0\ ,\quad \partial_2^3 q(a_1,a_2)|_{a_2=a_1} = -6q_3(a_1,a_1)$$

and consequently that

$$\frac{\partial^2}{\partial a_2^2}\{q(a_1,a_2)\ \partial_2 q(z,a_2) - q(z,a_2)\ \partial_2 q(a_1,a_2)\}|_{a_2=a_1}$$
$$= 6q_3(a_1,a_1)\ q(z,a_1) - \partial_2^2 q(z,a_1)\ .$$

With this out of the way the rest of the calculation is fairly straightforward, leading to the result that

$$q(z,a_1)\{\sum \partial_{j_1j_2}\ \hat{\theta}_2(\tfrac{1}{2}\,w(z-a_1))\ w'_{j_1}(a_1)\ w'_{j_2}(a_1) - \sum \partial_j \hat{\theta}_2(\tfrac{1}{2}\,w(z-a_1))\ w''_j(a_1)\}$$
$$= (6q_3(a_1,a_1)\ q(z,a_1) - \partial_2^2 q(z,a_1)\ \hat{\theta}_2(\tfrac{1}{2}\,w(z-a_1))$$
$$- 2\partial_2 q(z,a_1)\ \textstyle\sum_j \partial_j \hat{\theta}_2(\tfrac{1}{2}\,w(z-a_1))\ w'_j(a_1)$$
$$- 2q(z,a_1)\ \textstyle\sum_j \partial_j \hat{\theta}_2(\tfrac{1}{2}\,w(z-a_1))\ w''_j(a_1)\ ,$$

which yields the second formula in the statement of the theorem since $q(z,a)^{-1}\partial_2(z,a) = \partial \log q(z,a)/\partial a = w_a(z)$.

The interest in these formulas is that they are in a sense the analogues of the trisecant formula of Theorem 5 for the case of coincidences among the points. More precisely, the trisecant formula arose from viewing (2.10) as asserting that the three vectors of second-order theta functions appearing in Theorem 5 are always linearly dependent, and then merely evaluating the coefficents of this linear dependence. Equations (2.11) and (2.12) can be viewed similarly, and Theorem 7 describes the corresponding coefficient functions. It is interesting that these coefficient functions become more complicated as the points in the divisors involved in the description of $W_1 \subseteq J$ have more coincidences.

§5. A special limiting form: the KP equation

The general discussion of limiting forms of the trisecant identity will not be pursued further just here; indeed as noted there are other approaches to the formulas arising in the limit, leading to a clearer understanding of their general structure. However there is one further special case that should be considered in some detail, in a sense the next case in which all four points are coincident. It can be approached by applying to the formula of Theorem 6 the same analysis that led to its derivation from the trisecant identity, namely by viewing it as an identity for functions of z and comparing coefficients in the Taylor expansion in powers of $z - a$. When $z = a$ that formula reduces to a triviality. The first interesting case arises from applying the differential operator $\partial^4 / \partial z^4$ and setting $z = a$.

<u>Theorem</u> 8 For any point $z \in \tilde{M}$,

$$\begin{aligned}
&\sum_{j_1 j_2 j_3 j_4} \partial_{j_1 j_2 j_3 j_4} \vec{\theta}_2(0) w'_{j_1}(z) w'_{j_2}(z) w'_{j_3}(z) w'_{j_4}(z) \\
&= \sum_{j_1 j_2} \partial_{j_1 j_2} \vec{\theta}_2(0) \left(2 w'''_{j_1}(z) w'_{j_2}(z) - 3 w''_{j_1}(z) w''_{j_2}(z) + 24 q_3(z,z) w'_{j_1}(z) w'_{j_2}(z)\right) \\
&\quad + \vec{\theta}_2(0)\ (4!)\ \left(3 \partial_1 \partial_2 q_3(z,z) - 2 \partial_1^2 q_3(z,z) + 3 q_3(z,z)^2\right),
\end{aligned}$$

when the prime function is written in canonical coordinates in the form

$$q(z,a) = (z - a) + (z-a)^3\, q_3(z,a).$$

Proof The desired result is a straightforward calculation, and amounts to applying the differential operator $\partial^4/\partial z^4$ to the formula of Theorem 6 and then setting $z = a$. On the left-hand side it follows from the chain rule for differentation that

$$\partial^4 \vec{\theta}_2(w(z-a))/\partial z^4 =$$
$$= \sum_j \partial_{j_1 j_2 j_3 j_4} \vec{\theta}_2(w(z-a))w'_{j_1}(z)w'_{j_2}(z)w'_{j_3}(z)w'_{j_4}(z)$$
$$+ 6 \sum_j \partial_{j_1 j_2 j_3} \vec{\theta}_2(w(z-a)) \; w''_{j_1}(z)w'_{j_2}(z)w'_{j_3}(z)$$
$$+ \sum_j \partial_{j_1 j_2} \vec{\theta}_2(w(z-a)) \; [4w'''_{j_1}(z)w'_{j_2}(z) + 3w''_{j_1}(z)w''_{j_2}(z)]$$
$$+ \sum_j \partial_j \vec{\theta}_2(w(z-a))w''''_j(z),$$

and $\partial_j \vec{\theta}_2(0) = \partial_{j_1 j_2 j_3} \vec{\theta}(0) = 0$ since $\vec{\theta}_2(w)$ is an even function of w. On the right-hand side note that from B(6.3) the prime function has the expansion

$$q(z,a)^2 = (z-a)^2 + 2(z-a)^4 q_3(z,a) + (z-a)^6 q_3(z,a)^2$$

in cannonical coordinates near $z=a$, and consequently

$$\frac{\partial}{\partial z} q(z,a)^2 = 2(z-a) + 8(z-a)^3 q_3(z,a) + O(z-a)^4,$$

$$\frac{\partial^2}{\partial z^2} q(z,a)^2 = 2 + 24(z-a)^2 q_3(z,a) + O(z-a)^3,$$

$$\frac{\partial^3}{\partial z^3} q(z,a)^2 = 48(z-a)q_3(z,a) + O(z-a)^2,$$

$$\frac{\partial^4}{\partial z^4} q(z,a)^2 = 48 q_3(z,a) + O(z-a);$$

therefore the only nontrivial terms in the second part arise from applying the operators $\partial^2/\partial z^2$ or $\partial^4/\partial z^4$ to the factor $q(z,a)^2$ and the remaining differentiation if any to the remaining factor. On the other hand from the expansion B(6.3) of the prime function note that

$$\partial_1 q(z,a) = 1 + 3(z-a)^2 q_3(z,a) + (z-a)^3 \partial_1 q_3(z,a),$$
$$\partial_2 q(z,a) = -1 - 3(z-a)^2 q_3(z,a) + (z-a)^3 \partial_2 q_3(z,a)$$
$$\partial_1 \partial_2 q(z,a) = -6(z-a) q_3(z,a) + 3(z-a)^2 \; [\partial_2 q_3(z,a) - \partial_1 q_3(z,a)] + (z-a)^3 \partial_1 \partial_2 q_3(z,a),$$

and consequently that

$$\begin{aligned} q(z,a)^2 w_a'(z) &= q(z,a)^2 \frac{\partial^2}{\partial z \, \partial a} \log q(z,a) \\ &= q(z,a) \; \partial_1 \partial_2 q(z,a) - \partial_1 q(z,a) \; \partial_2 q(z,a) \\ &= 1 + 2(z-a)^3 \left(\partial_2 q_3(z,a) - \partial_1 q_3(z,a)\right) + \\ &\quad + (z-a)^4 \left(\partial_1 \partial_2 q_3(z,a) + 3 q_3(z,a)^2\right) + O(z-a)^5. \end{aligned}$$

To simplify this last expression recall that $q_3(z,a) = q_3(a,z)$, and observe upon applying the operator $\partial/\partial z$ and setting $z=a$ that

(1) $\quad \partial_1 q_3(a,a) = \partial_2 q_3(a,a).$

It therefore follows that

$$\partial_2 q_3(z,a) - \partial_1 q_3(z,a) = (z-a) \; \Psi(z,a)$$

for some holomorphic function $\Psi(z,a)$ near $z=a$. After differentiating this equality first with respect to z then with respect to a and setting $z=a$ it further follows that

$$\partial_1 \partial_2 q_3(a,a) - \partial_1^2 q_3(a,a) = \Psi(a,a)$$
$$\partial_2^2 q_3(a,a) - \partial_1 \partial_2 q_3(a,a) = -\Psi(a,a);$$

consequently

(2) $\partial_1^2 q_3(a,a) = \partial_2^2 q_3(a,a)$

and

$$\partial_2 q_3(z,a) - \partial_1 q_3(z,a) = (z-a)\ \left(\partial_1\partial_2 q_3(a,a) - \partial_1^2 q_3(a,a)\right) + O(z-a)^2 .$$

Using this observation leads to the result that

(3) $$q(z,a)^2 w_a'(z) = 1 + (z-a)^4 \left(3\partial_1\partial_2 q_3(a,a) - 2\partial_1^2 q_3(a,a) + 3q_3(a,a)^2\right) + O(z-a)^5 .$$

Then combining all these results leads in turn to the formula

$$\begin{aligned}
&\textstyle\sum_j \partial_{j_1j_2j_3j_4} \vec{\theta}_2(0) w_{j_1}'(a) w_{j_2}'(a) w_{j_3}'(a) w_{j_4}'(a) \\
&\quad + \textstyle\sum_j \partial_{j_1j_2} \vec{\theta}_2(0)\ \left(4w_{j_1}'''(a) w_{j_2}'(a) + 3w_{j_1}''(a) w_{j_2}''(a)\right) \\
&= 4!\ \ \vec{\theta}_2(0)\ \left(3\partial_1\partial_2 q_3(a,a) - 2\partial_1^2 q_3(a,a) + 3q_3(a,a)^2\right) \\
&\quad + \textstyle\sum_j \partial_{j_1j_2} \vec{\theta}_2(0)\ \left(6w_{j_1}'''(a) w_{j_2}'(a) + 24 q_3(a,a) w_{j_1}'(a) w_{j_1}'(a)\right),
\end{aligned}$$

which yields the desired theorem.

As will be demonstrated later, the only linear relations among the vectors $\vec{\theta}_2(0)$, $\partial_{j_1j_2}\ \vec{\theta}_2(0)$ are the obvious symmetries $\partial_{j_1j_2}\vec{\theta}(0) = \partial_{j_2j_1}\vec{\theta}_2(0)$; thus the $1 + \binom{g+1}{2}$ vectors $\vec{\theta}_2(0)$ and $\partial_{j_1j_2}\vec{\theta}(0)$ for $1 \leq j_1 \leq j_2 \leq g$ are linearly independent. A first observation arising from Theorem 8 is that the situation is quite different for higher derivatives; the formula of that theorem is a nontrivial linear relation among the derivatives of order four or less. Indeed on a nonhyperelliptic Riemann surface there are $7(g-1)$ linearly independent functions among the products $w_{j_1}'(z)w_{j_2}'(z)w_{j_3}'(z)w_{j_4}'(z)$ so this formula amounts to $7(g-1)$ independent linear combinations of the fourth-order theta derivatives at the origin expressible as linear combinations of lower-order derivatives;

on a hyperelliptic Riemann surface there are only 4g-3 linearly independent functions among those products. Alternatively the formula of Theorem 8 can be viewed as expressing some functions in $\Gamma(\kappa^4)$, some explicit fourth degree homogeneous polynomials in the Abelian differentials $w_i'(z)$, in terms of other explicitly given functions in $\Gamma(\kappa^4)$. In particular it must be the case, in view of the linear independence already noted, that

$$(4)\quad \begin{aligned} &w_{j_1}'''(z)w_{j_2}'(z) + w_{j_2}'''(z)w_{j_1}'(z) - 3w_{j_1}''(z)w_{j_2}''(z) \\ &\qquad + 24\, q_3(z,z)w_{j_1}'(z)w_{j_2}'(z) \in \Gamma(\kappa^2) \end{aligned}$$

and

$$(5)\quad 3\partial_1\partial_2 q_3(z,z) - 2\partial_1^3 q_3(z,z) + 3q_3(z,z)^2 \in \Gamma(\kappa^2).$$

These can be rewritten in terms of the prime function itself upon observing that

$$(6)\quad \partial_1^3 q(z,z) = -\partial_1^2\partial_2 q(z,z) = \partial_1\partial_2^2 q(z,z) = \partial_2^3 q(z,z) = 6q_3(z,z),$$

as a simple consequence of the expansion of the prime function used earlier.

The differential expression (4) is a rather familiar one in complex analysis, for in the special case $j_1 = j_2$ it really reduces to the Schwarzian differential operator. It may be recalled that the Schwarzian differential operator D_2 is the nonlinear operator defined by

$$(7)\quad (D_2 w)(z) = \frac{2w'(z)w'''(z) - 3w''(z)^2}{w'(z)^2} = -2\, w'(z)^{1/2}\frac{d^2}{dz^2}\, w'(z)^{-1/2}.$$

When applied to the composition $(v \circ w)(z) = v(w(z))$ of two functions this operator satisfies the characteristic property

$$(8)\quad (D_2(v \circ w))(z) = (D_2 v)(w(z)) \cdot w'(z)^2 + (D_2 w)(z).$$

The set of those functions w such that $D_2 w = o$ is therefore closed under composition, so forms a pseudogroup of mappings; indeed it consists precisely of the linear fractional mappings $w(z) = \frac{az + b}{cz + d}$ where a,b,c,d are constants. In these terms, (4) in the special case that $j_1 = j_2$ can be rewritten

(9) $\quad w_j'(z)^2 \{24 q_3(z,z) + (D_2 w_j)(z)\} \in \Gamma(\kappa^4).$

This in turn yields a rather easy way in which to derive the formula for the effect of a transformation $T \in \Gamma$ on the function $q_3(z,z)$; the direct way is just to use (6) and the functional equation for the prime function, but is a more elaborate calculation. Thus set $f_j = D_2 w_j$ as a convenient abbreviation and note from (9) since $w_j'(z)^2 \in \Gamma(\kappa^2)$ that

$$24 q_3(Tz,Tz) + f_j(Tz) = \kappa(T,z)^2 \{24 q_3(z,z) + f_j(z)\}.$$

Now the Abelian integral $w_j(z)$ satisfies the periodicity condition $w_j(Tz) = w_j(z) + \omega_j(T)$, or equivalently $w_j \circ T = T_j \circ w_j$ where T_j denotes the translation by $\omega_j(T)$, a special linear fractional transformation; therefore by (8)

$$f_j(Tz) \cdot T'(z)^2 + (D_2 T)(z) = f_j(z),$$

where $T'(z) = \kappa(T,z)^{-1}$. Combining these last two displayed formulas yields

(10) $\quad 24 q_3(Tz,Tz)\ \kappa(T,z)^{-2} = 24 q_3(z,z) - (D_2 T)(z).$

A function $q_3(z)$ satisfying a functional equation of this sort is called a projective connection on the Riemann surface M, the adjective projective referring to the fact that the Schwarzian differential operator defines the linear fractional or projective transformations; such a connection transforms in such a manner that when added to $f_j = D_2 w_j$ the result transforms by the factor of automorphy $\kappa(T,z)^2$. Condition (4) shows that there is a similar result for the modified Schwarzian differential operator applied to a pair of functions. It would be interesting to have a more explicit description of the projective structure on M determined by the projective connection $q_3(z,z)$.

To rewrite this another way, recall from B(6.7) that the canonical coordinates were chosen so that $T'(z) = \kappa(T,z)^{-1} = \sigma(T,z)^{-2}$, where $\sigma(T,z) = \rho_r(T)\ \zeta(T,z)^{g-1}$ is the standard semicanonical factor of automorphy; consequently using the second form of the defining equation (7) for the Schwarzian derivative yields

$$(D_2T)\ (z) = -2\ \sigma(T,z)^{-1} \frac{d^2}{dz^2}\ \sigma(T,z).$$

This yields a more natural version of (10) in the present context.

This too can be expressed as an identity in terms of first-order theta functions, and for that purpose the obvious extension of Lemma 3 is required. The formula is somewhat more complicated than the ones involved in Lemma 3, and to handle it more readily it is convenient to introduce some further formal notation. If $\xi_{j_1\cdots j_n}$ is a tensor with some symmetries, and if it can be made fully symmetric in its indices by summing over some m permutations of its indices, let $\mathcal{S}_m[\xi_{j_1\cdots j_n}]$ denote the result of this symmetrization. This description is somewhat vague, but in all instances in which the notation will be applied here it will be quite evident what is meant. For instance, if $\xi_{j_1 j_2 j_3} = \alpha_{j_1 j_2}\beta_{j_3}$ where $\alpha_{j_1 j_2} = \alpha_{j_2 j_1}$ then the symmetrization involved is evidently given by

$$\mathcal{S}_3[\alpha_{j_1 j_2}\beta_{j_3}] = \alpha_{j_1 j_2}\beta_{j_3} + \alpha_{j_1 j_3}\beta_{j_2} + \alpha_{j_2 j_3}\beta_{j_1}.$$

With this convention, the desired result is as follows:

Lemma 4. For any point $t \in \mathbb{C}^g$,

$$
\begin{aligned}
{}^t\vec{\theta}_2(t)\cdot \partial_{j_1j_2j_3j_4}\vec{\theta}_2(0) = 2\theta(t)\ &\partial_{j_1j_2j_3j_4}\theta(t) \\
&-2\ \underset{\approx}{S}_4[\partial_{j_1}\theta(t)\ \partial_{j_2j_3j_4}\theta(t)] \\
&+2\ \underset{\approx}{S}_3[\partial_{j_1j_2}\theta(t)\ \partial_{j_3j_4}\theta(t)].
\end{aligned}
$$

Proof. Again this is a straightforward consequence of differentiating the addition theorem for theta functions in the form ${}^t\vec{\theta}_2(t)\cdot\vec{\theta}_2(s) = \theta(s+t)\,\theta(s-t)$; it is merely a matter of applying the differential operator $\partial_J = \partial_{j_1j_2j_3j_4} = \partial^4/\partial s_{j_1}\partial s_{j_2}\partial s_{j_3}\partial s_{j_4}$ to this identity and setting $s = 0$. The product rule for differentiation can be written as

$$\partial_J[\theta(s+t)\ \theta(s-t)] = \sum_{J'\cup J''=J} \partial_{J'}\theta(s+t)\cdot\partial_{J''}\theta(s-t),$$

where the summation is extended over all ways of decomposing the set of indices $J = \{j_1j_2j_3j_4\}$ as a union of two subsets. The sum over those terms in which $J' = \emptyset$ consists just of the single term $\theta(s+t)\ \partial_J\theta(s-t)$. There are 4 terms in which J' has one element, and the corresponding sum can be written $\underset{\approx}{S}_4[\partial_{j_1}\theta(s+t)\ \ \partial_{j_2j_3j_4}\theta(s-t)]$. There are $\binom{4}{2} = 6$ terms in which J' has two elements, and the corresponding sum can be written $2\underset{\approx}{S}_3[\partial_{j_1j_2}\theta(s+t)\ \partial_{j_3j_4}\theta(s-t)]$ since $\partial_{j_1j_2}\theta(s+t)$ and $\partial_{j_3j_4}\theta(s-t)$ are both symmetric expressions in their indices. There are 4 more terms in which J' has three elements, and a final term in which J' has four elements, and they can be handled in the same way. When $s = 0$ the terms in which J' has no elements or 4 elements reduce to the same form, as do the terms in which J' has either 1 or 3 elements; for $\partial_{J''}\theta(-t) = \pm\,\partial_{J''}\theta(t)$, the sign being + if J'' has an even number of elements and - otherwise. That yields the desired result.

Corollary 1. For any points $t \in \mathbb{C}^g$, $z \in \tilde{M}$,

$$\sum_{j_1 j_2 j_3 j_4} \{\theta(t)\partial_{j_1 \ldots j_4}\theta(t) - 4\,\partial_{j_1}\theta(t)\partial_{j_2 \ldots j_4}\theta(t) + 3\partial_{j_1 j_2}\theta(t)\partial_{j_3 j_4}\theta(t)\}\, w'_{j_1}(z)\ldots w'_{j_4}(z)$$
$$= \sum_{j_1 j_2} \{\theta(t)\partial_{j_1 j_2}\theta(t) - \partial_{j_1}\theta(t)\partial_{j_2}\theta(t)\}\ \{2w'''_{j_1}(z)w'_{j_2}(z) - 3w''_{j_1}(z)w''_{j_2}(z)$$
$$+ 24q_3(z,z)w'_{j_1}(z)w'_{j_2}(z)\} + \psi(z)\ \theta(t)^2$$

where

$$\psi(z) = 1/2\ (4!)\ \{3\partial_1\partial_2 q_3(z,z) - 2\partial_1^{\,2} q_3(z,z) + 3q_3(z,z)^2\}.$$

Proof. This follows directly from the preceding theorem by multiplying the formula of that theorem by ${}^t\hat{e}_2(t)$. The left-hand side is evaluated by using Lemma 4; it is only necessary to observe that

$$\sum_j S_4[\partial_{j_1}\theta(t)\partial_{j_2 j_3 j_4}\theta(t)]\ w'_{j_1}(z)w'_{j_2}(z)w'_{j_3}(z)w'_{j_4}(z)$$
$$= 4\sum_j \partial_{j_1}\theta(t)\partial_{j_2 j_3 j_4}\theta(t)w'_{j_1}(z)w'_{j_2}(z)w'_{j_3}(z)w'_{j_4}(z),$$

and similarly for the other term. The right-hand side is evaluated by using Lemma 3. Dividing throughout by a common factor 2 yields the stated result.

To rewrite this corollary in another way, choose fixed points $z \in \tilde{M}$, $Z \in \mathbb{C}^g$, and introduce the points $U, V, W \in \mathbb{C}^g$ defined by

(11) $U_j = w'_j(z)$, $V_j = w''_j(z)$, $W_j = 1/2\ (w'''_j(z) + 12q_3(z,z)w'_j(z))$.

In terms of auxiliary variables $x, y, t \in \mathbb{C}^g$ consider the holomorphic function f on $\mathbb{C}^3$ defined by

$$f(x,y,t) = \theta(Ux + Vy + Wt + Z).$$

It is clear that

$$\partial^n f/\partial x^n = \sum_{j_1 \ldots j_n} \partial_{j_1 \ldots j_n} \theta \cdot U_{j_1} \cdots U_{j_n},$$

and similarly for the other partial derivatives; thus the formula of the preceding corollary can be rewritten in terms of the function f, and upon setting $\partial f/\partial x = f_x$, $\partial^2 f/\partial x^2 = f_{xx}$, and so on, takes the following form.

Corollary 2. With the notation as above, the function f satisfies the partial differential equation

$$0 = f\, f_{xxxx} - 4f_x f_{xxx} + 3f^2_{xx} + 4f_x f_t - 4f\, f_{xt} + 3f\, f_{yy} - 3\, f_y^{\,2} - \gamma f^2.$$

Proof. This is a straightforward translation of the equation of the preceding corollary, with W_j replacing the terms involving w'''_j and $q_3 w'_j$ as indicated.

This result can be rewritten yet again in terms of one of the standard nonlinear partial differential equations, the Kadomcev-Petviašvili or KP equation, as follows

Corollary 3. For any point $z \in \tilde{M}$ and with the vectors $U, V, W \in \mathbb{C}^g$ as in (11) the function

$$u(x,y,t) = 2 \frac{\partial^2}{\partial x^2} \log \theta(xU + yV + t\, W + Z)$$

is a solution of the partial differential equation

$$3\, u_{yy} = \frac{\partial}{\partial x} \left(4u_t - 6uu_x - u_{xxx}\right).$$

Proof. It is a straightforward calculation to verify that this equation reduces to that of the preceding corollary.

6. Multisecant identities

The argument used to prove the trisecant identity extends quite easily to yield an analogous identity derived from the description in terms of the second-order theta functions of the subvarieties W_r for $r > 2$. Explicitly the result is as follows.

Theorem 9. If $3 \leqq n \leqq g+2$ and $x_1,\ldots,x_{n-2},z_1\ldots,z_n$ are any points of $\tilde{M}$ then

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n-2} q(z_i,x_j)}{\prod_{\substack{k=1\\k\neq i}}^{n} q(z_i,z_k)} \vec{\theta}_2[t](w(z_i)) = 0$$

where $t = w(x_1+\ldots+x_{n-2}-z_1-\ldots-z_n)$.

Proof If $z_1,\ldots,z_n \in \tilde{M}$ represent distinct points of M and $\mathfrak{z} = z_1+\ldots+z_n$ then by Theorem 2 the n vectors $\vec{\theta}_2[s-w(\mathfrak{z})](w(z_i))$ for $1 \leqq i \leqq n$ are linearly dependent precisely when $s \in W_{n-2}$. Thus if $s=w(x_1+\ldots+x_{n-2}) \in W_{n-2}$ and $t = s-w(\mathfrak{z})$ as in the statement of the theorem then the n vectors $\vec{\theta}_2[t](w(z_i))$ for $1 \leqq i \leqq n$ are linearly dependent; by continuity they remain linearly dependent for arbitrary points $z_i,x_j \in \tilde{M}$, whether there are coincidences or not. On the other hand, with the same notation, if the first n-1 points $z_1,\ldots z_{n-1} \in \tilde{M}$ represent distinct points of M and $\mathfrak{z}' = z_1+\ldots+z_{n-1}$ then the n-1 vectors $\vec{\theta}_2[t](w(z_i)) = \vec{\theta}_2[s-w(z_n)-w(\mathfrak{z}')](w(z_i))$ for $1 \leqq i \leqq n-1$ are linearly dependent preceisely when $s-w(z_n) \in W_{n-3}$;

thus if

$$X = \{(z,x) = (z_1,\dots,z_n,x_1,\dots x_{n-2}) \in \tilde{M}^{2n-2} : \text{either } z_i = z_j \text{ on } M \text{ for } 1 \leq i < j \leq n-1 \text{ or } w(x_1+\dots+x_{n-2}-z_n) \in W_{n-3}\}$$

then X is a holomorphic subvariety of $\tilde{M}^{2n-2}$ and the $n-1$ vectors $\vec{\theta}_2[t](w(z_i))$ are linearly independent whenever $(z,x) \in \tilde{M}^{2n-2} \sim X$. It follows that there are uniquely determined holomorphic functions $f_i(z,x)$ on $\tilde{M}^{2n-2} \sim X$ such that

$$(1) \quad \vec{\theta}_2[t](w(z_n)) = \sum_{i=1}^{n-1} f_i(z,x)\ \vec{\theta}_2[t](w(z_i)),$$

where again t is as in the statement of the theorem, and since these functions $f_i(x,t)$ are determined in terms of the theta functions by Cramer's rule they evidently extend to meromorphic functions on $\tilde{M}^{2n-2}$ with singularities at most along the subvariety X, provided of course that X is a proper subvariety of $\tilde{M}^{2n-2}$.

To examine this subvariety X in more detail note first that if $n \geq g+3$ then $W_{n-3} = J$ and hence $X = \tilde{M}^{2n-2}$; in this case the preceding analysis is vacuous. On the other hand if $3 \leq n \leq g+2$ then W_{n-3} is a proper subvariety of J, and clearly $w(x_1+\dots+x_{n-2}-z_n) \in W_{n-3}$ precisely when $w(x_1+\dots+x_{n-2}) = w(z_n+y_1+\dots+y_{n-3})$ for some points $y_i \in \tilde{M}$; this last condition in turn means that either $w(x_1+\dots+x_{n-2}) \in W^1_{n-2}$ or $w(x_1+\dots+x_{n-2}) \in W_{n-2} \sim W^1_{n-2}$ and $z_n = x_i$ for some index i. Thus if $3 \leq n \leq g+2$ the subvariety X can be written as the union $X = X_1 \cup X_2 \cup X_3$

of three proper subvarieties of $\tilde{M}^{2n-2}$, where

$$X_1 = \{(z,x) \in \tilde{M}^{2n-2} : z_i = z_j \text{ for } 1 \leqq i < j \leqq n-1\}$$

$$X_2 = \{(z,x) \in \tilde{M}^{2n-2} : z_n = x_i \text{ for } 1 \leqq i \leqq n-2\}$$

$$X_3 = \{(z,x) \in \tilde{M}^{2n-2} : w(x_1+\dots+x_{n-2}) \in W^1_{n-2}\}.$$

Here X_1 and X_2 are subvarieties of pure codimension one, with the obvious irreducible components. As far as X_3 is concerned, the situation is somewhat more complicated and requires some further general knowledge of Riemann surfaces. In the special case n = 3 considered in the two preceding sections this complication does not arise, since then $W^1_{n-2} = W^1_1 = \emptyset$. In the special case $n = g+2$ on the other hand $W^1_{n-2} = W^1_g = k - W_{g-2}$ so that W^1_{n-2} is of pure dimension $g-2 = n-4$ in all cases. In the middle range $4 \leqq n \leqq g+1$, or equivalently $2 \leqq n-2 \leqq g-1$, it is a result of H. H. Martens (On the varieties of special divisors on a curve, J. reine angew. Math. 227 (1967), 111-120) that $\dim W^1_{n-2} \leqq n-4$ with equality holding precisely when M is a hyperelliptic Riemann surface. Now X_3 is just the product $X_3 = \tilde{M}^n \times \tilde{G}^1_{n-2}$ where $\tilde{G}^1_{n-2} = \{x_1 + \dots + x_{n-2} \in \tilde{M}^{(n-2)} : w(x_1 + \dots + x_{n-2}) \in W^1_{n-2}\}$. The Abel-Jacobi mapping $w: \tilde{G}^1_{n-2} \to W^1_{n-2}$ over the dense open subset $W^1_{n-2} \sim W^2_{n-2}$ is a holomorphic fibration with fibre $\mathbb{P}^1$, so that part of $\tilde{G}^1_{n-2}$ has dimension equal to $1 + \dim W^1_{n-2}$, and the part over W^2_{n-2} is a proper subvariety so is of still smaller dimension; thus altogether $\dim X_3 = n+1 + \dim W^1_{n-2} \leqq 2n-3$, with equality only when $n=g+2$ or $4 \leqq n \leqq g+1$ and M is hyperelliptic. In general therefore X_3 has codimension at least 2, so is a removable singularity set for analytic functions and hence cannot really be part of the singular locus for a meromorphic function; the only cases in which X_3 has codimension 1 and hence can be part of the singular locus are $n=g+2$ or $4\leq n\leq g+1$ and M is hyperelliptic.

In these exceptional cases it is necessary to describe the subvariety X_3 in a bit more detail; actually it is enough just to describe for general points $x_2,\ldots,x_{n-2}$ those points x_1 such that $x_1+x_2+\ldots+x_{n-2}\in\tilde{G}^1_{n-2}$. In case that $n=g+2$ note that $w(x_1+\ldots+x_g)\in W^1_g=k-W_{g-2}$ precisely when there are some points $y_1,\ldots,y_{g-2}$ such that $w(x_1+\ldots+x_g+y_1+\ldots+y_{g-2})=k$, or equivalently such that $x_1+\ldots+x_g+y_1+\ldots+y_{g-2}$ is the divisor of an Abelian differential. If $x_2,\ldots,x_g$ are general, in the sense that $w(x_2+\ldots+x_g)\notin W^1_{g-1}$, there is a unique such divisor $x_2+\ldots+x_g+x'_1+\ldots+x'_{g-1}$, and x_1 can only be one of these other $g-1$ points. If $4\leq n\leq g+1$ and M is hyperelliptic with hyperelliptic involution $E: M\to M$ then it is known that $x_1+x_2+\ldots+x_{n-2}\in\tilde{G}^1_{n-2}$ precisely when $x_i=Ex_j$ for some $i\neq j$; thus if $x_2,\ldots,x_{n-2}$ are general, in the sense that $x_i\neq Ex_j$ for $2\leq i<j\leq n-2$, then $x_1+x_2+\ldots+x_{n-2}\in\tilde{G}^1_{n-2}$ precisely when $x_1=Ex_i$ for some index in the range $2\leq i\leq n-2$.

With these observations out of the way, it is a reasonably straightforward matter to determine quite explicitly the functions $f_i(z,x)$ in (1). First it is clear that interchanging the variables z_1 and z_i has the effect of interchanging the functions f_1 and f_i, in the sense that

(2) $f_i(z_1,\ldots,z_n,x_1,\ldots,x_{n-2})=f_1(z_i,z_2,\ldots,z_{i-1},z_1,z_{i+1},\ldots,z_n,x_1,\ldots,x_{n-2})$;

thus it is really enough just to determine the function f_1. It is also clear from (1) that the function f_1 is completely symmetric in the variables $x_1,\ldots,x_{n-2}$ and also in the variables $z_2,\ldots,z_{n-1}$; so to a considerable extent it is enough just to examine f_1 as a function of say the variables x_1,z_1,z_2,z_n. Furthermore interchanging the variables z_1 and z_n has the effect of interchanging the function f_1 and $1/f_1$, in the sense that

(3) $1/f_1(z_1,\dots,z_n x_1,\dots,x_{n-2}) = f_1(z_n,z_2,\dots,z_{n-1},z_1,x_1,\dots,x_{n-2})$.

Next, as in the proof of Theorem 5, the effect of a transformation $T \varepsilon \Gamma$ acting on any of the variables appearing in the function f_1 can be deduced quite easily from an application of the results of Theorem 3 on the effect of translating a theta function by a half-period. To consider the variable z_1 first, set $t = t' - w(z_1)$ and rewrite (1) as

$$\vec{\theta}_2[-t'-2w(z_n)]\,(\,w(z_1)) = f_1(z_1)\,\vec{\theta}_2\,[t']\,(\,w(z_1)) + \cdots .$$

Replacing z_1 by Tz_1 in $\vec{\theta}_2\,[\alpha]\,(\,w(z_1))$ transforms that theta function into

$$\vec{\theta}_2[\alpha]\,(\,w(z_1) + \omega(T)) =$$

$$= \rho_{-\frac{1}{2}\alpha}(T)\;\eta(\omega(T),\,w(z_1))\;\chi(\omega(T))\;\vec{\theta}_2[\alpha]\,(\,w(z_1));$$

the common factors in all these transformations can be divided out, and it follows readily that

(4) $f_1(Tz_1) = \rho_{t'+w(z_n)}(T)\;f_1(z_1) = \rho_{t+w(z_1+z_n)}(T)\;f_1(z_1)$.

Similarly for the other variables it follows that

(5) $f_1(Tz_2) = \rho_{w(z_n-z_1)}(T)\;f_1(z_2)$,

(6) $f_1(Tz_n) = \rho_{-t-w(z_1+z_n)}(T)\;f_1(z_n)$,

(7) $f_1(Tx_1) = \rho_{w(z_1-z_n)}(T) f_1\,(z_1)$,

for any transformation $T \varepsilon \Gamma$.

The singularities of the function $f_1(z,x)$ must be contained within those components of X of pure codimension 1, and in consequence of (3) the zeros must be contained within those components of pure codimension 1 in the subvariety that arises from X upon interchanging z_1 and z_n. Holding all variables except z_1 fixed but in general position thus yields a well defined meromorphic function $f_1(z_1)$ with poles at most at the points $z_2,\ldots,z_{n-1}$ and zeros at most at the points $x_1,\ldots,x_{n-2}$; when $x_1,\ldots,x_{n-2}$ are fixed and $x_1+\ldots+x_{n-2} \notin \tilde{G}^1_{n-2}$ the component X_3 plays no role in f_1 as a function of z_1. This function must then have a divisor of the form

$$\mathcal{D}(f_1(z_1)) = \mu_1 \cdot x_1 + \ldots + \mu_{n-2} \cdot x_{n-2} - \nu_2 \cdot z_2 - \ldots - \nu_{n-1} \cdot z_{n-1}$$

for some integers $\mu_i \geqq 0$, $\nu_i \geqq 0$. It follows from (4) that the factor of automorphy $\rho_{t+w(z_1+z_n)}(T)$ represents this divisor, and hence that

$$\mu_1 w(x_1) + \ldots + \mu_{n-2} w(x_{n-2}) - \nu_2 w(z_2) - \ldots - \nu_{n-1} w(z_{n-1}) =$$
$$= w(x_1) + \ldots + w(x_{n-2}) - w(z_2) - \ldots - w(z_{n-1}) \text{ in } J;$$

but since this holds for arbitrary general points x_i, z_j it must be the case that $\mu_i = \nu_i = 1$, hence that

$$\mathcal{D}(f_1(z_1)) = x_1 + \ldots + x_{n-2} - z_2 - \ldots - z_{n-1}.$$

Similarly

$$\mathcal{D}(f_1(z_2)) = z_n - z_1,$$
$$\mathcal{D}(f_1(z_n)) = z_2 + \ldots + z_{n-1} - x_1 - \ldots - x_{n-2}.$$

When all the variables except x_1 are held constant there is a slight difference, since the portion of X_3 that is of codimension 1 must also be taken into account. If $n=g+2$ and $w(x_2 + \ldots + x_{n-2}) \notin W^1_{g-1}$ there is up to a constant factor a unique Abelian differential vanishing at $x_2 + \ldots + x_{n-2}$; if its divisor is $x_2 + \ldots + x_{n-2} + y_1 + \ldots + y_{g-1}$ then the points $x_1 = y_i$ can be either poles or zeros of the function $f_1(x_1)$. If $4 \leqq n \leqq g+1$ and M is hyperelliptic with hyperelliptic involution E then the points $x_1 = Ex_i$ for $2 \leqq i \leqq n-2$ can also be either poles or zeros of $f_1(x_1)$. Thus this function has the divisor

$$\mathcal{D}(f_1(x_1)) = \nu_1 \cdot z_1 - \nu_2 \cdot z_n + \sum_{i=1}^{g-1} \alpha_i y_i + \sum_{i=2}^{n-2} \beta_i \cdot Ex_i$$

where $\nu_1 \geqq 1$, $\nu_2 \geqq 1$, $\alpha_i = 0$ unless $n=g+2$ and then is some integer of any sign, $\beta_i=0$ unless $4 \leqq n \leqq g+1$ and M is hyperelliptic and is then an integer of any sign. However from (7) and the argument as above it actually follows that

$$\mathcal{J}(f_1(x_1)) = z_1 - z_n.$$

Combining these observations then evidently yields the result that $f_1(z,x)$ as a function on $\tilde{M}^{2n-2}$ has simple zeros along the subvarieties $z_1 = Tx_i$ for $1 \leqq i \leqq n-2$ and the subvarieties $z_n = Tz_i$ for $2 \leqq i \leqq n-1$, and has simple poles along the subvarieties $z_1 = Tz_i$ for $2 \leqq i \leqq n-1$ and the subvarieties $z_n = Tx_i$ for $1 \leqq i \leqq n-2$.

Now the meromorphic function

$$f_1^*(z,x) = \left[\prod_{j=1}^{n-2} \frac{q(z_1,x_j)}{q(z_n,x_j)}\right] \left[\prod_{k=2}^{n-1} \frac{q(z_n,z_k)}{q(z_1,z_k)}\right]$$

has the same divisor on $\tilde{M}^{2n-2}$ as does the function $f_1(z,x)$, and a comparison of the transformational properties of the prime function under Γ with equations (4) through (7) shows that $f_1^*(z,x)$ and $f_1(z,x)$ transform the same way under Γ. It therefore follows that $f_1(z,x) = c\, f_1^*(z,x)$ for some constant $c \neq 0$, and it then follows from (2) that

$$f_i(z,x) = c \cdot \left[\prod_{j=1}^{n-2} \frac{q(z_i,x_j)}{q(z_n,x_j)}\right] \left[\prod_{\substack{k=1\\ k\neq i}}^{n-1} \frac{q(z_n,z_k)}{q(z_i,z_k)}\right] .$$

Substituting this into (1) and multiplying by the obvious factor then yields

$$\vec{\theta}_2[t]\,(w(z_n)) \cdot \left[\prod_{j=1}^{n-2} q(z_n,x_j)\right]\left[\prod_{\substack{k=1\\ k\neq n}}^{n} q(z_n,z_k)^{-1}\right]$$

$$= - \sum_{i=1}^{n-1} c\, \vec{\theta}_2\,[t]\,(w(z_i))\left[\prod_{j=1}^{n-2} q(z_i,x_j)\right]\left[\prod_{\substack{k=1\\ k\neq i}}^{n-1} q(z_i,z_k)^{-1}\right] .$$

It is clear by symmetry that $c = 1$, and that yields the desired result and concludes the proof.

The preceding theorem of course includes the trisecant identity, Theorem 5, as a special case; the actual formula of Theorem 5 arises by multiplying the formula of this theorem by $q(z_1,z_2)\ q(z_2,z_3)\ q(z_3,z_1)$ to reduce all the coefficients to holomorphic functions. It may be helpful to see the next case, the quadrisecant formula, more explicitly as well:

$$
\begin{aligned}
(8)\quad 0 = {} & \frac{q(z_1,x_1)\ q(z_1,x_2)}{q(z_1,z_2)\ q(z_1,z_3)\ q(z_1,z_4)}\ \vec{\theta}_2\left(\tfrac{1}{2}w(x_1+x_2+z_1-z_2-z_3-z_4)\right) \\
& + \frac{q(z_2,x_1)\ q(z_2,x_2)}{q(z_2,z_1)\ q(z_2,z_3)\ q(z_2,z_4)}\ \vec{\theta}_2\left(\tfrac{1}{2}w(x_1+x_2+z_2-z_1-z_3-z_4)\right) \\
& + \frac{q(z_3,x_1)\ q(z_3,x_2)}{q(z_3,z_1)\ q(z_3,z_2)\ q(z_3,z_4)}\ \vec{\theta}_2\left(\tfrac{1}{2}w(x_1+x_2+z_3-z_1-z_2-z_4)\right) \\
& + \frac{q(z_4,x_1)\ q(z_4,x_2)}{q(z_4,z_1)\ q(z_4,z_3)\ q(z_4,z_2)}\ \vec{\theta}_2\left(\tfrac{1}{2}w(x_1+x_2+z_4-z_1-z_2-z_3)\right) .
\end{aligned}
$$

This too can be rewritten with purely holomorphic coefficients by multiplying through by $\Pi_{j<k}\ q(z_j,z_k)$. Doing so and changing notation slightly leads to the equivalent version:

$$
\begin{aligned}
(9)\quad & q(z_1,z_3)\ q(z_2,z_3)\ q(a_1,a_2)\ q(a_1,a_3)\ q(a_2,a_3)\ \vec{\theta}_2\left(\tfrac{1}{2}w(z_1+z_2+z_3-a_1-a_2-a_3)\right) \\
& = q(z_1,a_1)\ q(z_2,a_1)\ q(z_3,a_2)\ q(z_3,a_3)\ q(a_2,a_3)\ \vec{\theta}_2\left(\tfrac{1}{2}w(z_1+z_2-z_3+a_1-a_2-a_3)\right) \\
& - q(z_1,a_2)\ q(z_2,a_2)\ q(z_3,a_1)\ q(z_3,a_3)\ q(a_1,a_3)\ \vec{\theta}_2\left(\tfrac{1}{2}w(z_1+z_2-z_3-a_1+a_2-a_3)\right) \\
& + q(z_1,a_3)\ q(z_2,a_3)\ q(z_3,a_1)\ q(z_3,a_2)\ q(a_1,a_2)\ \vec{\theta}_2\left(\tfrac{1}{2}w(z_1+z_2-z_3-a_1-a_2+a_3)\right)
\end{aligned}
$$

The trisecant formula is just the one extreme case $n=3$ of Theorem 9. The other extreme case $n=g+2$ of that theorem can be rewritten in an interesting alternative manner, effectively replacing the individual points $x_1,\ldots,x_g \in \tilde{M}$ by the single parameter $w(x_1+\ldots+x_g) \in \mathbb{C}^g$, an arbitary point in $\mathbb{C}^g$; the result is as follows

Corollary 1. For arbitrary points $t \in \mathbb{C}^g$ and $z_1, \ldots, z_{g+2} \in \tilde{M}$

$$\sum_{i=1}^{g+2} \frac{\theta(r-t-w(z_1+\ldots+z_{i-1}+z_{i+1}+\ldots+z_{g+2}))}{\prod_{\substack{k=1\\k\neq i}}^{g+2} q(z_i,z_k)} \vec{\theta}_2\,[t]\,(w(z_i)) = 0.$$

Proof. From the Corollary to Theorem C3 it follows that

$$Q(x_1,x_2,\ldots,x_g) \prod_{j=1}^{g} q(z_i,x_j) = \theta(r+w(z_i) - w(x_1 + \ldots + x_g))$$
$$= \theta(r-t+w(z_i) - w(z_1 + \ldots + z_{g+2}))$$

where $t = w(x_1 + \ldots + x_g - z_1 - \ldots - z_{g+2})$. Multiplying the formula of Theorem 9 by $Q(x_1,x_2,\ldots,x_g)$ and using this observation leads to the desired formula. Note that this formula only involves the points $x_1,\ldots,x_g$ in the variable t, and by suitable choice of the points $x_1,\ldots,x_g$ the variable t can be given any desired value in $\mathbb{C}^g$.

For any fixed point $t \in \mathbb{C}^g$ the vectors $\vec{\theta}\,[t]\,(w(z))$ span a linear space $L_t \subseteq \mathbb{C}^{2^g}$ as the point z varies throughout $\tilde{M}$, and it follows from Corollary 1 that $\dim L_t = g+1$. Indeed for any fixed values $z_1,\ldots,z_{g+1} \in \tilde{M}$ representing distinct points of M an appliction of Theorem 2 shows as usual that the vectors $\vec{\theta}_2[t]\,(w(z_1)),\ldots, \vec{\theta}_2[t]\,(w(z_{g+1}))$ are linearly independent whenever $t + w(z_1 + \ldots + z_{g+1})) \in W_{g-1}$, that is, whenever $\theta(r-t-w(z_1 + \ldots + z_{g+1})) \neq 0$; if that is the case the formula of the Corollary shows that $\vec{\theta}[t]\,(w(z))$ is expressible as a linear combination of these g+1 vectors, whence the assertion as desired. Alternatively of course the components of $\vec{\theta}_2[t]\,(w(z))$ as functions of $z \in \tilde{M}$ are relatively automorphic functions for the factor of automorphy $\rho_{k-t}\zeta^{2g}$, indeed by Theorem 1 span the (g+1)-dimensional space of such relatively automorphic functions, and it is reasonably evident that this too shows that $\dim L_t = g+1$ for all points $t \in \mathbb{C}^g$. These vector spaces can be viewed as determining a vector bundle of rank g+1 over the Jacobi variety; that leads to a number of interesting questions, some of which I have discussed elsewhere in terms of generalized theta functions, but they are in a rather different direction from that being pursued at present so will not be considered much further here. What is more relevant here is the significance of these observations to the question of how Theorem 9 extends to indices $n > g+2$.

As was clear from the proof, the formula of Theorem 9 is really uniquely determined aside from an arbitrary common factor in the range $3 \leqq n \leqq g+2$. On the other hand for $n > g+2$ there are linear relations among the vectors $\vec{\theta}_2[t](w(z_1)),\ldots, \vec{\theta}_2[t](w(z_n))$ for arbitrary parameters $t \in \mathbb{C}^g$ and $z_i \in \tilde{M}$, but there are $n-(g+1)$ linearly independent such relations; indeed for each index $i > g+1$ the formula of Corollary 1 provides a linear relation among the vectors $\vec{\theta}_2[t](w(z_1)),\ldots, \vec{\theta}_2[t](w(z_{g+1})), \vec{\theta}_2[t](w(z_i))$, and these are a basis for the space of these linear relations. It is possibly worth observing for emphasis that in these assertions $t, z_1,\ldots,z_n$ are viewed as separate parameters and the identities as functional identities; for some fixed values of these parameters there are of course further linear relations among the corresponding vectors.

It is interesting and useful to note that among those various linear relations in cases where $n > g+2$ is precisely the relation of the form in Theorem 9 for that value of n. The easiest way to demonstrate that is by a direct inductive argument; that incidentally provides another proof of the multisecnt identity, really deriving the general case inductively beginning with the trisecant formula that had earlier been proved. This alternative proof is actually simpler than the other proof used here; but it is in some ways less satisfactory, since it does not show the uniqueness for cases $n \leqq g+2$, nor does it really show so clearly just why the formula necessarily has the form it does have. Which proof serves better is somewhat a matter of personal preference though, and there are possible advantages in having various proofs available.

Corollary 2. The formula of Theorem 9 holds for all indices $n \geqq 3$.

Proof. For the case n=3 the formula is the trisecant formula demonstrated in Theorem 5. That case will thus be as given, and the formula will be proved by induction on the index n. For that purpose consider for any index $n \geqq 3$ the function

$$\vec{f}(x,z) = \sum_{i=1}^{n} \frac{q(z_i,x_1) \cdots q(z_i,x_{n-2})}{q(z_i,z_1) \cdots \hat{i} \cdots q(z_i, z_n)} \vec{\theta}_2[t](w(z_i)),$$

where $\hat{i}$ denotes the omission of the i-th term in the product and $t=w(x_1 +\dots+ x_{n-2} - z_1 -\dots- z_n)$. This is a well defined meromorphic function of the variables $x_1,\dots,x_{n-2}\ z_1,\dots, z_n \in \tilde{M}$, and the desired result is the assertion that it vanishes identically; that is the case for n=3, and will be demonstrated for $n \geqq 3$ by induction on n.

If $x_2,\dots, x_{n-2}, z_1,\dots, z_n$ are fixed points on $\tilde{M}$ then $\vec{f}(x,z) = \vec{f}(x_1)$ is evidently a well defined holomorphic function of the single variable $x_1 \in \tilde{M}$; in the defining formula $t = s + w(x_1)$ where $s \in \mathbb{C}^g$ is a constant insofar as the variable x_1 is concerned. Note that replacing x_1 by Tx_1 for some $T \in \Gamma$

has the effect of replacing $t = s + w(x_1)$ by $t + \lambda = s + w(x_1) + \lambda$ where $\lambda = \omega(T)$, and by Theorem 3

$$\vec{\theta}_2[t+\lambda](w(z_1)) = \vec{\theta}_2[s+2w(z_1)]((\tfrac{1}{2}w(x_1) + \lambda))$$
$$= \rho_{-\frac{1}{2}s-w(z_1)}(T)\ \eta(\lambda,\ w(x_1))\ \chi(\lambda)\ \vec{\theta}_2[t](w(z_1));$$

since $q(z_i, Tx_1) = q(z_i,x_1)\ \zeta(T,x_1)\ \rho_{w(z_i)}(T)$, it follows readily that each separate summand in the formula defining the function $\vec{f}$, and consequently that function itself, transforms by

$$\vec{f}(Tx_1) = \rho_{-\frac{1}{2}s}(T)\ \zeta(T,x_1)\ \eta(T,x_1)\ \chi(T)\ \vec{f}(x_1),$$

where $\eta(T,x_1) = \eta(\lambda,w(x_1))$ and $\chi(T) = \chi(\lambda)$ for $\lambda = \omega(T)$.

Now if $x_1 = z_1$ the first summand in the formula defining the function $\vec{f}$ vanishes, and the remainder of that formula is easily seen to reduce precisely to the corresponding formula in the case n-1 for the variables $x_2,\dots, x_{n-2}$, $z_2,\dots, z_n$; but then by the induction hypothesis that remainder vanishes, so the function $\vec{f}(x_1)$ has a zero at the point $x_1 = z_1$, and by symmetry must also have zeros at the points $x_1 = z_2,\dots, x_1 = z_n$. The function

$$\vec{g}(x_1) = q(x_1,z_1)^{-1} \dots q(x_1,z_n)^{-1}\ \vec{f}(x_1)$$

is therefore also a holomorphic function of the variable x_1, and it evidently transforms by

$$\vec{g}(Tx_1) = \rho_{-w(z_1+\dots+z_n)}(T)\ \zeta(T,x_1)^{-n} \cdot \rho_{-\frac{1}{2}s}(T)\zeta(T,x_1)\eta(T,x_1)\chi(T) \cdot \vec{g}\,(x_1)$$
$$= \rho_{-\frac{1}{2}\alpha}(T)\ \zeta(T,x_1)^{1-n}\ \eta(T,x_1)\ \chi(T)\ \vec{g}(x_1),$$

where $\alpha = w(x_2 + \ldots + x_{n-2} + z_1 + \ldots + z_n)$. The proof will be concluded by showing that any holomorphic function $\vec{g}$ satisfying this last condition must vanish identically, and that can possibly most easily be accomplished by restricting the transformations to the subgroup $\Gamma_o = \{T \in \Gamma : \chi(T) = I\}$ for which the condition reduces to a scalar condition. It is evident from (2.5) that the values $\chi(T)$ for $T \in \Gamma$ lie in a finite group of matrices, so that Γ_o is a normal subgroup of finite index in Γ; the precise value of this index is unimportant, but if it is say m then the quotient space $M_o = \tilde{M}/\Gamma_o$ is a Riemann surface that naturally appears as an m-sheeted unbranched covering space over M. The Riemann-Hurwitz formula shows incidentally that M_o has genus $g_o = g + (m-1)(g-1)$. Since $\chi(T) = 1$ whenever $T \in \Gamma_o$ it is evident that each component $g_i(x_1)$ of the vector $\vec{g}(x_1)$ satisfies

$$g_i(Tx_1) = \rho_{-\frac{1}{2}\alpha}(T)\ \zeta(T,x_1)^{1-n}\ \eta(T,x_1)\ g_i(x_1)$$

for all $T \in \Gamma_o$, hence is a holomorphic relatively automorphic function for the factor of automorphy $\rho_{-\frac{1}{2}\alpha}\,\zeta^{1-n}\,\eta$; η must thus be a well defined factor of automorphy for Γ_o. The representation ρ of course describes a flat factor of automorphy, one of Chern class zero. The factor of automorphy ζ for the full group Γ corresponds to the divisor $1.z_o$ on M, so admits a relatively automorphic function with that divisor on M; the same function can of course be viewed as a relatively automorphic function for ζ considered as a factor of automorphy for the subgroup $\Gamma_o \subset \Gamma$, and then clearly corresponds to the divisor $(\Gamma/\Gamma).z_o$ on M_o. Thus ζ as a factor of automorphy for Γ_o has Chern class m. Since η is a factor of automorphy for Γ_o and $\eta^2 = \zeta$ it is clear that η has Chern class $\frac{1}{2}m$; as a minor sidelight, it follows that m is even. Altogether the factor of automorphy $\rho_{-\frac{1}{2}\alpha}\zeta^{1-n}\eta$ has Chern class $(1-n)m + \frac{1}{2}m = -(n-\frac{3}{2})m < 0$, so cannot admit any nontrivial holomorphic relatively automorphic function; thus each component g_i vanishes identically, hence $\vec{g} \equiv 0$ and the proof is thereby concluded.

The multisecant identity can also be rewritten in terms of first-order theta functions. To simplify the notation introduce the auxiliary meromorphic functions

(10) $$Q(z_1,\ldots,z_n;\, a_1,\ldots,a_n) = \left[\prod_{1\leq j,k\leq m} q(z_j,a_k)\right] \Big/ \left[\prod_{1\leq j<k\leq m} q(z_j,z_k)\, q(a_j,a_k)\right]$$

of the variables z_j, $a_k \in \tilde{M}$ for $n=1, 2, \ldots$. In case $n=1$ this is to be interpreted as meaning that $Q(z_1;a_1) = q(z_1,a_1)$, so the function is then actually holomorphic; but for $n > 1$ it is meromorphic, the singularities being simple poles along the subvarieties $z_j = Tz_k$ and $a_j = Ta_k$ for $j \neq k$ and $T \in \Gamma$. Note that for any fixed indices ℓ, m this function can be decomposed into the product

(11) $$Q(z_1,\ldots,z_n\,;\, a_1,\ldots,a_n) = \frac{\left[\prod\limits_{\substack{1\leq j,k\leq n\\ j\neq\ell,k\neq m}} q(z_j,a_k)\right]\left[\prod\limits_{\substack{1\leq j\leq n\\ j\neq\ell}} q(z_j,a_m)\right]\left[\prod\limits_{\substack{1\leq k\leq n\\ k\neq m}} q(z_\ell,a_k)\right] q(z_\ell,a_m)}{\left[\prod\limits_{\substack{1\leq j<k\leq n\\ j,\,k\neq\ell}} q(z_j,z_k)\right]\left[\prod\limits_{\substack{1\leq j<k\leq n\\ j,\,k\neq m}} q(a_j,a_k)\right](-1)^{\ell+m}\left[\prod\limits_{\substack{1\leq j\leq n\\ j\neq\ell}} q(z_\ell,z_j)\right]\left[\prod\limits_{\substack{1\leq k\leq n\\ k\neq m}} q(a_m,a_k)\right]}$$

$$= (-1)^{\ell+m+n-1}\, Q(z_\ell;a_m)\, Q(z_1,\ldots,z_{\ell-1}, z_{\ell+1},\ldots,z_n;\, a_1,\ldots,a_{m-1},a_{m+1},\ldots,a_m)\cdot$$

$$\left[\prod_{\substack{1\leq j\leq n\\ j\neq\ell}} \frac{q(z_j,a_m)}{q(z_j,z_\ell)}\right]\left[\prod_{\substack{1\leq k\leq n\\ k\neq m}} \frac{q(z_\ell,a_k)}{q(a_m,a_k)}\right].$$

In these terms the multisecant formula is equivalent to the following result.

Corollary 3. If $n \geqq 3$ and $x_1,\dots, x_{n-1}, a_1,\dots, a_{n-1}$ are arbitrary points of $\tilde{M}$ while s is any point of $\mathbb{C}^g$ then

$$\theta(s)\ \theta(s+w(z_1+\dots+z_{n-1}-a_1-\dots-a_{n-1}))\ /\ Q(z_1,\dots,z_{n-1};a_1,\dots,a_{n-1}) =$$
$$= \sum_{i=1}^{n-1}(-1)^{i-1}\frac{\theta(s+w(z_{n-1}-a_i))}{q(z_{n-1};a_i)}\ \frac{\theta(s+w(z_1+\dots+z_{n-2}-a_1-\dots-\hat{a}_i-\dots-a_{n-1}))}{Q(z_1,\dots,z_{n-2};\hat{a}_1,\dots,z_i,\dots,a_{n-1})},$$

where $\hat{a}_i$ denotes the omission of the term a_i.

Proof. Multiply the formula of Theorem 9 by $t^{+}_{\theta_2}[t](s+w(z_n)) = t^{+}_{\theta_2}(\tfrac{1}{2}t+s+w(z_n))$ and use Theorem A7 to rewrite the result in the form

$$0 = \sum_{i=1}^{n}\frac{\Pi_j q(z_i,x_j)}{\Pi_{k\neq i}q(z_i,z_k)}\quad \theta(s+w(z_n-z_i))\ \theta(t+s+w(z_n+z_i))$$

$$= \sum_{i-1}^{n-1}\frac{\Pi_j q(z_i,x_j)}{\Pi_{k\neq i}q(z_i,z_k)}\quad \theta(s+w(z_n-z_i))\ \theta(s+w(x_1+\dots+x_{n-2}-z_1-\dots-\hat{z}_i-\dots-z_{n-1}))$$

$$+\ \frac{\Pi_j q(z_n,x_j)}{\Pi_{k\neq n}q(z_n,z_k)}\quad \theta(s)\ \theta(s+w(x_1+\dots+x_{n-2}+z_n-z_1-\dots-z_{n-1}))$$

where $\hat{z}_i$ denotes the omission of the term z_i. Now change the notation, replacing x_i by z_i for $1\leqq i\leqq n-2$, z_i by a_i for $1\leqq i\leqq n-1$, and z_n by z_{n-1}, so that the preceding formula can be rewritten.

$$\theta(s)\ \theta\big(s+w(z_1+\dots+z_{n-1}-a_1-\dots-a_{n-1})\big) =$$

$$= \sum_{i=1}^{n-1} \frac{\left[\prod_{1\leq j\leq n-2} q(a_i,z_j)\right]\ \left[\prod_{1\leq k\leq n-1} q(z_{n-1},a_k)\right]}{\left[\prod_{\substack{1\leq k\leq n-1\\ k\neq i}} q(a_i,a_k)\right]\ \left[\prod_{1\leq j\leq n-2} q(z_{n-1},z_j)\right] q(z_{n-1},a_i)}$$

$$\cdot\ \theta\big(s+w(z_{n-1}-a_i)\big)\ \theta\big(s+w(z_1+\dots+z_{n-2}-a_1-\dots-\hat{a}_i-\dots-a_{n-1})\big)$$

$$= \sum_{i=1}^{n-1} (-1)^{i-1}\, Q(z_1,\dots,z_{n-1};a_1,\dots,a_{n-1})\cdot \frac{\theta\big(s+w(z_{n-1}-a_i)\big)}{Q(z_{n-1};a_i)}$$

$$\frac{\theta\big(s+w(z_1+\dots+z_{n-2}-a_1-\dots-\hat{a}_i-\dots-a_{n-1})\big)}{Q(z_1,\dots,z_{n-2};a_1,\dots,\hat{a}_i,\dots,a_{n-1})}$$

after an application of (11). That yields the desired result.

The formula of the preceding corollary is rather like that for the expansion of a determinant by cofactors of one row or column, which suggests that the expressions involved can be written as suitable determinants; that is indeed the case, as follows:

Corollary 4. If $n \geq 1$ and $z_1,\dots, z_n, a_1,\dots, a_n$ are arbitrary points of $\tilde{M}$ while s is any point of $\mathbb{C}^g$ then

$$(-1)^{\frac{n(n-1)}{2}}\ \theta(s)^{n-1}\ \theta\big(s+w(z_1+\dots+z_n-a_1-\dots-a_n)\big) / Q(z_1,\dots,z_n;a_1,\dots,a_n)$$

$$= \det \left\{ \frac{\theta\big(s+w(z_j-a_k)\big)}{q(z_j,a_k)} \right\}_{j,k=1,\dots,n.}$$

Proof. The proof will be by induction on the index n. The result is quite trivially true in the initial case $n=1$, so it is only necessary to demonstrate the inductive step. Assume therefore that the corollary holds for the case $n-1$ where $2\leq n\leq g+1$. If $D(z_1,\dots, z_n; a_1,\dots, a_n)$ denotes the determinant on the right hand side of the formula to be demonstrated in the case n, the expansion of this determinant by minors of row $j=n$ has the form

$$D(z_1,\ldots, z_n; a_1,\ldots, a_n) =$$
$$= \sum_{i=1}^{n} (-1)^{n+i} \frac{\theta(s+w(z_n-a_i))}{q(z_n,a_i)} D(z_1,\ldots, z_{n-1}; a_1,\ldots, \hat{a}_i,\ldots, a_n).$$

By the inductive hypothesis

$$D(z_1,\ldots, z_{n-1}; a_1,\ldots, \hat{a}_i,\ldots, a_n) =$$
$$= (-1)^{\frac{(n-1)(n-2)}{2}} \theta(s)^{n-2} \frac{\theta(s+w(z+\cdots+z_{n-1}-a_1-\cdots-\hat{a}_i-\cdots-a_n))}{Q(z_1,\ldots,z_{n-1};a_1,\ldots,\hat{a}_i,\ldots,a_n)},$$

and substituting this into the preceding formula yields the result that

$$D(z_1,\ldots, z_n; a_1,\ldots, a_n) =$$
$$= (-1)^{\frac{n(n-1)}{2}} \theta(s)^{n-2} \sum_{i=1}^{n} (-1)^{i-1} \frac{\theta(s+w(z_n-a_i))}{q(z_n,a_i)} \cdot \frac{\theta(s+w(z_1+\cdots+z_{n-1}-a_1-\cdots-\hat{a}_i-\cdots-a_n))}{Q(z_1,\ldots,z_{n-1}; a_1,\ldots,\hat{a}_i,\ldots,a_n))}.$$

A comparison of this last formula with the formula of Corollary 3, which is valid whenever $n \geq 2$, shows that $D(z_1,\ldots, z_n; a_1,\ldots,a_n)$ has the form asserted for the case n and thereby concludes the proof. In the special case $n = 2$ the preceding result gives the expression of the original trisecant formula in terms of the first-order theta functions, as had been derived earlier in the Corollary to Theorem 5. It is worth restating here in terms of first-order theta functions the observation made earlier in terms of second-order theta functions, namely that this formula is essentially unique in the range $1 \leq n \leq g+1$, while for $n > g+1$ there are additional formulas reflecting further linear relations among the vectors of second-order theta-functions.