C. First-order Riemannian theta functions

§1. Riemann's theorem

If M is a marked Riemann surface of genus $g > 0$, the canonical Abelian integrals define a holomorphic mapping $w: \tilde{M} \to \mathbb{C}^g$ that induces the Abel-Jacobi imbedding $w: M \to J$ of M in its Jacobi variety $J = \mathbb{C}^g/L$. The theta factor of automorphy $\xi(\lambda, v)$ for the action of the lattice subgroup L on $\mathbb{C}^g$ induces the factor of automorphy $\xi(T,z) = \rho_r(T)\zeta(T,z)^g$ for the action of the covering translation group Γ on $\tilde{M}$ as discussed in §B5. Correspondingly any theta function induces a relatively automorphic function on $\tilde{M}$, and these induced functions are the next topic of discussion. To begin with the simple theta function or one of its translates, since $\theta(t+v) = \theta[0|t](v) \in \Gamma(\rho_{-t}\xi)$ as a function of v it follows that

$$\theta[0|t](w(z)) \in \Gamma(\rho_{-t}\xi) = \Gamma(\rho_{r-t}\zeta^g) \quad \text{as a function of } z \in \tilde{M}$$

where $r \in \mathbb{C}^g$ is the Riemann point; of course it is always possible that this induced function of $z \in \tilde{M}$ vanishes identically, and this possibility must be kept in mind. In some ways it is more convenient to consider instead the function

$$(1) \qquad f_t(z) = \theta[0|r-t](w(z)) \in \Gamma(\rho_t\zeta^g).$$

This function either vanishes identically, or is a nontrivial relatively automorphic function with divisor

(2) $\mathcal{J}(f_t) = z_1 + \dots + z_g$ on M

such that

(3) $w(z_1+\dots+z_g) = w(z_1)+\dots+w(z_g) = t$ in J,

where the Abel-Jacobi mapping is extended to divisors as usual. From this observation it is quite easy to establish the Riemann vanishing theorem, characterizing the theta locus $\underline{\Theta}=\{v\epsilon J:\ \theta(v)=0\}$ in the Jacobi variety J.

Theorem 1. In a Jacobi variety J,

$$\underline{\Theta}=r-W_{g-1}$$

where $r\epsilon J$ is the Riemann point.

Proof. It is enough just to show that $\underline{\Theta}\subset r-W_{g-1}$, for since W_{g-1} and hence $r-W_{g-1}$ are irreducible subvarieties of J it then follows automatically that $\underline{\Theta}=r-W_{g-1}$; equivalently then, it is only necessary to show that if $\theta(r-t)=0$ then $t\epsilon W_{g-1}$.

Consider therefore a point $t\epsilon J$ such that $\theta(r-t)=0$, and suppose at first that $f_t(z)=\theta[0|r-t](w(z))$ is not identically zero. Then $f_t(z_o)=\theta[0|r-t](0)=\theta(r-t)=0$ for the base point $z_0 \epsilon \tilde{M}$, so that $\mathcal{J}(f_t)=z_o+z_1+\dots+z_{g-1}$ for some $g-1$ other points $z_i\epsilon M$ by (2) and $t=w(z_o+z_1+\dots+z_{g-1}) = w(z_1+\dots+z_{g-1})$ by (3) and so that $t\epsilon W_{g-1}$ as desired. Next if $f_t(z)\equiv 0$ there will be some divisor $z_1+\dots+z_r$ such that

$$f_t(z)=\theta[0|r-t-w(z_1+\dots+z_r)](w(z))\not\equiv 0,$$

since the translate $\theta[0|r-t-t_1](v(z))$ will be nontrivial for some point $t_1\in J$ and $J=W_g$ so that t_1 can be written as the image of some positive divisor of degree at most g; it can be supposed that $r\geq 1$ is the least integer for which this is the case, and then

$$\mathfrak{f}_t(z_i)=\theta[0|r-t-v(z_1+\dots+z_{i-1}+z_{i+1}\dots+z_r)](v(z_o))=0$$

for $1\leq i\leq r$. Thus

$$\mathfrak{D}(\mathfrak{f}_t)=z_1+\dots z_r+z_{r+1}+\dots+z_g.$$

for some further points $z_{r+1},\dots,z_g$, and by (3)

$$v(z_1+\dots+z_g)=t+v(z_1+\dots+z_r)$$

so that $t=v(z_{r+1}+\dots+z_g)\in W_{g-r}\subseteq W_{g-1}$ as desired. That suffices to conclude the proof.

Since the theta function is an even function necessarily $r-W_{g-1}=\underline{\underline{\theta}}=-\underline{\underline{\theta}}=-r+W_{g-1}$, or since $2r=k$

(4) $k-W_{g-1}=W_{g-1}$;

but this is really just a special case of the Riemann-Roch theorem in the geometric form B(7.8). This can be used to rewrite the Riemann vanishing theorem in the form

(5) $\underline{\underline{\theta}}=r-W_{g-1}=W_{g-1}-r.$

Another direct consequence of this theorem is the following

Corollary. The function $f_t(z) = \theta[0|r-t](w(z))$ is identically zero precisely when $t\in W_g^1 = k - W_{g-2}$.

Proof. It follows from the preceding theorem that $f_t(z)=\theta(r-t+w(z))=0$ for all points $z\in M$ precisely when $r-t+w(z)\in\underline{\underline{\Theta}} = r-W_{g-1}$ for all points $z\in M$, hence precisely when $t-w(z)\in W_{g-1}$ for all points $z\in M$; but with the notation and results of §B7 this is just the condition that $t-W_1\in W_{g-1}$, hence that $t\in W_{g-1}\ominus(-W_1)=W_g^1$. The geometric form of the Riemann-Roch theorem B(7.8) shows that $W_g^1=k-W_2$; or alternatively this can be deduced by proceeding as in the first part of the proof but with the theta locus taken in the form $\underline{\underline{\Theta}}=W_{g-1}-r$.

This corollary has a very interesting interpretation, which makes it all the easier to remember. As noted in (1), the restriction $f_t(z) = \theta[0|r-t](w(z))\in\Gamma(\rho_t\zeta^g)$, while $\dim\Gamma(\rho_t\zeta^g)= \gamma(\rho_t\zeta^g)=1$ precisely when $t\notin W_g^1$, hence by the corollary precisely when $f_t(z)\not\equiv 0$. Thus the restriction $f_t(z) = \theta[0|r-t](w(z))$ is a basis for the vector space $\Gamma(\rho_t\zeta^g)$ whenever this vector space is one dimensional, while the restriction vanishes identically whenever the vector space $\Gamma(r_t\zeta^g)$ has dimension strictly greater than one so that the restriction could not possibly be a basis. There are some rather interesting questions that are suggested by this point of view.

Since the subvariety $W_{g-1}\subseteq J$ is the image of the connected and hence irreducible complex manifold M^{g-1} under the Abel-Jacobi mapping, it follows from simple general results in complex analysis that W_{g-1} is an irreducible holomorphic subvariety of J; hence from Theorem 1 the theta locus $\underset{=}{\theta}$ is an irreducible subvariety. This had, as noted in the discussion in part A, a number of very convenient consequences, all of which are thus always available in the case of Jacobi varieties.

The theta function generates the ideal of the theta locus at each of its points, so the singularities of $\underset{=}{\theta}$ are precisely the points where θ and all of its first-order derivatives vanish. To discuss these and the higher-order derivatives, it is convenient to simplify the notation by writing

$$(6) \qquad \partial_{j_1 \cdots j_\nu} \theta(v) = \frac{\partial^\nu \theta(v)}{\partial v_{j_1} \cdots \partial v_{j_\nu}}.$$

The singular locus of $\underset{=}{\theta}$ is then the subvariety

$$(7) \qquad \underset{=}{\theta}^1 = \{v \in \underset{=}{\theta} : \partial_j \theta(v) = 0, \ 1 \le j \le g\}.$$

More generally, the <u>singular points of multiplicity ν</u> on the theta locus are defined as the subvarieties

$$(8) \qquad \underset{=}{\theta}^\nu = \{v \in \underset{=}{\theta} : \partial_j \theta(v) = \partial_{j_1 j_2} \theta(v) = \cdots = \partial_{j_1 \cdots j_\nu} \theta(v) = 0, \ 1 \le j_i \le g\};$$

this provides the filtration $\underset{=}{\theta}^1 \supseteq \underset{=}{\theta}^2 \supseteq \cdots$ of the singularities of the theta locus into more and more extensive singularities. In these terms there is the following extension of Riemann's vanishing theorem; the first equality here is Riemann's singularity theorem.

<u>Theorem</u> 2. In a Jacobi variety J,

$$\underline{\underline{\theta}}^{\nu} = r - W^{\nu}_{g-1} = \{t\in J: \theta(t+w(z_1+\dots+z_\nu-a_1-\dots-a_\nu))=0, \text{ all } z_i, a_i\in \tilde{M}\}$$

<u>Proof</u>. Set $S_\nu = \{t\in J: \theta(t+w(z_1+\dots+z_\nu-a_1-\dots-a_\nu)\}=0$, all $z_i, a_i\in\tilde{M}\}$ for the course of this proof, as a notational convenience. Note first that $t\in S_\nu$ precisely when $t+W_\nu-W_\nu\subseteq \underline{\underline{\theta}} = r-W_{g-1}$, hence precisely when

$$r-t\in W_{g-1}\ominus(W_\nu-W_\nu) = W^{\nu}_{g-1}$$

by B(7.13); therefore $S_\nu = r - W^{\nu}_{g-1}$.

Next if $t\in S_\nu$ then $\theta(t+w(z_1+\dots+z_\nu-a_1-\dots-a_\nu))$ is identically zero in $z_i, a_i\in\tilde{M}$, so the result of applying the differential operator $\partial^\mu/\partial z_1\dots\partial z_\mu$ for any index μ in the range $1\le\mu\le\nu$ and setting $z_i = a_i$ for $1\le i\le\nu$ is identically zero in $a_i\in\tilde{M}$. The differentation is as usual with respect to the canonical coordinates on $\tilde{M}$, and the result of this calculation with the obvious use of the chain rule for differentiation is the identity

$$\Sigma_{i_1\dots i_\mu}\, \partial_{i_1\dots i_\mu}\theta(t)\cdot w'_{i_1}(a_1)\dots w'_{i_\mu}(a_\mu) = 0, \text{ all } a_i\in\tilde{M}.$$

The functions $w'_i(a_j)$ for $1\le i\le g$ are linearly independent functions of the variable a_j, a basis for $\Gamma(\kappa)$, so this identity can only hold when

$$\partial_{i_1\dots i_\mu}\theta(t) = 0, \quad 1\le\mu\le\nu, \quad 1\le i_1,\dots,i_\mu\le g,$$

that is, when $t\in\underline{\underline{\theta}}^{\nu}$; therefore $S_\nu\subseteq\underline{\underline{\theta}}^{\nu}$.

Finally it will be demonstrated by induction on the index ν that $\underline{\theta}^{\nu} \subseteq S_{\nu}$. The initial case $\nu=0$ is quite trivial, just the definition of the theta locus, so suppose that the assertion has already been demonstrated for all indices $<\nu$ and consider a point $t \in \underline{\theta}^{\nu}$. If the function

$$f(z)=\theta(t+v(z+z_1+\dots+z_{\nu-1}-a-a_1-\dots-a_{\nu-1}))$$

is identically zero for all $z_i, a, a_i \in \tilde{M}$ then $t \in S_{\nu}$ as desired; otherwise this function is not identically zero for some $z_i, a, a_i \in \tilde{M}$, and is a relatively automorphic function $f \in \Gamma(\rho_s \zeta^g)$ where $s = r-t-v(z_1+\dots+z_{\nu-1}-a-a_1-\dots-a_{\nu-1})$. The function

$$g(z)=\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_{\nu-1}} f(z)\Big|_{z_1=a_1,\dots,z_{\nu-1}=a_{\nu-1}}$$

then clearly satisfies func[illegible]al equations of the form

$$g(Tz) = \rho_{r-t+v(a)}(T)\zeta(T,z)^g g(z) + h(T,z)$$

where $h(T,z)$ is an expression involving at most $\nu-2$ derivatives of $f(z)$ and some derivatives of $\rho_s(T)$, which is a function of the variables z_i, a, a_i since s is. However

$$\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_{\nu-2}} f(z)\Big|_{z_1=a_1,\dots,z_{\nu-1}=a_{\nu-1}}$$

$$= \frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_{\nu-2}}\, \theta(t+v(z+z_1+\dots+z_{\nu-2}-a-a_1-\dots-a_{\nu-2}))\Big|_{z_1=a_1,\dots,z_{\nu-2}=a_{\nu-2}}$$

$$= 0$$

since $t \in \underline{\underline{\theta}}^{\nu} \subset \underline{\underline{\theta}}^{\nu-1}$ so by the inductive hypothesis $t \in S_{\nu-1}$ and

$$\theta(t+v(z+z_1+\dots+z_{\nu-2}-a-a_1-\dots-a_{\nu-2})) \equiv 0;$$

the same argument evidently applies to all similar derivatives of f of total order at most $\nu-2$, so that actually $h(T,z) \equiv 0$ and consequently $g \in \Gamma(\rho_{r-t+v(a)}\zeta^g)$. This function g is zero whenever $z = a_i$ or $z=a$, again by the inductive hypothesis since $\theta(t+v(a_i+z_1+\dots+z_{\nu-2}-a-a_1\dots-a_{\nu-2})) \equiv 0$; moreover it vanishes at least to the second order at $z=a$, for

$$\frac{\partial}{\partial z} g(z)\bigg|_{z=a} = \frac{\partial}{\partial z}\frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_{\nu-1}} \theta(t+v(z+z_1+\dots+z_{\nu-1}-a-a_1-\dots-a_{\nu-1}))\bigg|_{z=a,z_i=a_i}$$

$$= \Sigma_{i,i_1,\dots,i_{\nu-1}} \partial_{ii_1,\dots,i_{\nu-1}} \theta(t)\cdot v_i'(a)v_{i_1}'(a_1)\dots v'_{i_{\nu-1}}(a_{\nu-1})$$

$$= 0$$

since by assumption $t \in \underline{\underline{\theta}}^{\nu}$. Therefore if g is not identically zero then

$$\mathscr{D}(g) = 2a+a_1+\dots+a_{\nu-1}+z_{\nu-1}\dots+z_{g-1}$$

for some additional points $z_\nu,\dots,z_{g-2}$, and as in (3)

$$r-t+v(a) = v(2a+a_1+\dots+a_{\nu-1}+z_\nu+\dots+z_{g-2})$$

so that

(9) $\quad r-t-v(a+a_1+\dots+a_{\nu-1}) \in W_{g-\nu-1}$.

On the other hand if g is identically zero then by the definition of that function

(10) $\Sigma_{i_1,\ldots,i_{\nu-1}} \partial_{i_1\ldots i_{\nu-1}} \theta(t+v(z-a))v'_{i_1}(a_1)\ldots v'_{i_{\nu-1}}(a_{\nu-1}) = 0$
for all $z\epsilon\tilde{M}$.

Thus for any points $a,a_1,\ldots,a_{\nu-1}\epsilon\tilde{M}$, either (9) or (10) holds. Now (9) when viewed as a condition on these points describes a holomorphic subvariety of $\tilde{M}^\nu$, which is either all of $\tilde{M}^\nu$ or a proper subvariety. If it is all of $\tilde{M}^\nu$ then (9) means that $r-t\epsilon W_{g-\nu-1}\ominus(-W_\nu) = W^\nu_{g-1}$ by B(7.13), and hence $t\epsilon r-W^\nu_{g-1} = S_\nu$ by the first part of this proof. If it is a proper subvariety then on the dense open complement of this subvariety (9) fails so that (10) must hold, and it then follows from continuity that (10) must hold for all points $a,a_1,\ldots, a_{\nu-1}$; but then $\partial_{i_1\ldots i_{\nu-1}}\theta(t+v(z-a)) = 0$ for all indices $i_1,\ldots,i_{\nu-1}$ and all points $z,a\epsilon\tilde{M}$, so that $t+v(z-a)\epsilon\underline{\underline{\theta}}^{\nu-1}\subseteq S_{\nu-1}$ by the inductive hypothesis, and clearly $t+v(z-a)\epsilon S_{\nu-1}$ for all $z,a\ \tilde{M}$ implies that $t\epsilon S_\nu$. In either case $t\epsilon S_\nu$, so that $\underline{\underline{\theta}}^\nu\subseteq S_\nu$ and the proof is thereby concluded.

It is worth noting that by the Riemann-Roch theorem in the geometric form B(7.8)

$$W^\nu_{g-1} = k - W^\nu_{g-1} = 2r-W^\nu_{g-1}$$

so that Riemannn's singularity theorem can be stated in the form

(11) $\underline{\underline{\theta}}^\nu = r-W^\nu_{g-1} = W^\nu_{g-1} - r$

§2. Prime function expansion

There are two directions in which it is possible to proceed: one is to seek to express the function $f_t(z)=\theta[0|r-t](w(z))\in\Gamma(\rho_t\zeta^g)$ in terms of the previously introduced standard functions and differential forms on the Riemann surface M, the other is conversely to seek to express these standard functions and forms in terms of $f_t(z)$. To begin with the first direction, there is a simple natural expression for $f_t(z)$ that serves as well as a further normalization of the prime function $q(z,a)$. It should be recalled that the prime function and the constructions based on it, such as the canonical coordinates on $\tilde{M}$ in terms of which all differentiation is to be taken, have so far only been determined up to an arbitrary constant factor.

<u>Theorem 3</u>. It is possible to normalize the prime function $q(z,a)$ in such a manner that

$$\theta(r+w(z-z_1-\dots-z_g))\cdot\prod_{1\leq j<k\leq g} q(z_j,z_k)$$

$$=\det\{w'_j(z_k)\ :\ 1\leq j,k\leq g\}\cdot\prod_{1\leq i\leq g} g(z,z_i)$$

for all points $z,z_1,\dots,z_g\in\tilde{M}$; the prime function is then determined uniquely up to a factor ε where $\varepsilon^{\binom{g}{2}}=1$.

<u>Proof.</u> For any $t\in E^g$ that does not represent a point of W^1_g the function $f_t(z)=\theta[0|r-t](w(z))\in\Gamma(\rho_t\zeta^g)$ is not identically zero, by the Corollary to Theorem 1. This function then has a well defined divisor $z_1+\dots+z_g$ on M for which $w(z_1+\dots+z_g)=t$ in J; actually it is clearly possible to choose representative points $z_i\in\tilde{M}$ such that $w(z_1+\dots+z_g)=t$ in E^g rather than just in $J=E^g/L$. With these choices, the function

$\Pi_{i=1}^{g}\ q(z,z_i)$ is a relatively automorphic function for the same factor of automorphy $\rho_t\zeta^g = \rho_{w(z_1+\ldots+z_g)}\zeta^g$ as $f_t(z)$ and has the same divisor as $f_t(z)$, so must be a constant multiple of $f_t(z)$; thus

$$(1) \qquad \theta(r+w(z-z_1-\ldots z_g)) = c\cdot\ \Pi_{i=1}^{g}\ q(z,z_i)$$

for some complex number c that is independent of z. This holds for any t outside W_g^1, hence for any divisor $z_1+\ldots+z_g$ for which $w(z_1+\ldots+z_g)\notin W_g^1$, but the constant c of course depends on the divisor so should be viewed as a function $c(z_1,\ldots,z_g)$ of the points $z_i\in\tilde{M}^g$. The identity (1) really defines c as a meromorphic function of $(z,z_1,\ldots,z_g)\in\tilde{M}^{g+1}$ with at most simple poles at the subvarieties $z = Tz_i$ for $T\in\Gamma$, $1\leq i\leq g$; but since c is independent of z on a dense open subset of $\tilde{M}^{g+1}$ it must be altogether independent of z by the identity theorem for meromorphic functions, and is therefore necessarily a holomorphic function of $(z_1,\ldots,z_g)\in\tilde{M}^g$.

From the known transformation properties of the functions $\theta(-r+w(z_1+\ldots+z_g-z)) = \theta[0|r-(k-w(z_2+\ldots+z_g-z))](w(z_1))$ and $q(z_1,z)$ it follows readily that replacing z_1 by Tz_1 in (1) yields the identity

$$c(Tz_1,z_2,\ldots,z_g) = \rho_{k-w(z_2+\ldots+z_g)}(T)\quad \zeta(T,z_1)^{g-1}\ c(z_1,z_2,\ldots,z_g)$$

$$= \kappa(T,z_1)\ \zeta_{z_2}(T,z_1)^{-1}\cdots\zeta_{z_g}(T,z_1)^{-1}\ \ c(z_1,z_2,\ldots,z_g).$$

Therefore as a function of z_1 alone

$$c(z_1,z_2,\ldots,z_g)\cdot\Pi_{i=2}^{g}\,q(z_1,z_i)\in\Gamma(\kappa),$$

and this function vanishes at the divisor $z_2+\ldots+z_g$. Whenever $w(z_2+\ldots+z_g)\notin W^1_{g-1}$ there is up to a constant factor a unique section of $\Gamma(\kappa)$ that vanishes at this divisor; if moreover the points $z_2,\ldots,z_g$ are distinct then in terms of the canonical basis $w'_i(z)$ for $\Gamma(\kappa)$ one such section is clearly the determinant

$$W(z,z_2,\ldots,z_g)=\det\{\vec{w}'(z),\vec{w}'(z_2),\ldots,\vec{w}'(z_g)\}$$

where $\vec{w}'(z)$ denotes the column vector of length g having entries $w'_i(z)$. In this case, after introducing the additional nonzero factor $\Pi_{2\leq j<k\leq g}\,q(z_j,z_k)$, it follows that

$$c(z_1,z_2,\ldots,z_g)\cdot\prod_{1\leq j<k\leq g}q(z_j,z_k)=c_1(z_2,\ldots,z_g)\cdot W(z_1,z_2,\ldots,z_g),$$

where $c_1(z_2,\ldots,z_g)$ is independent of z_1. Now this holds in a dense open subset of M^g. The function $c(z_1,z_2,\ldots,z_g)$ is clearly symmetric in the variables z_i, since it only depends on the divisor $z_1+\ldots+z_g$, so the left-hand side of this identity is skew symmetric in the variables z_i; the function $W(z_1,z_2,\ldots z_g)$ is also skew symmetric, so $c_1(z_2,\ldots,z_g)$ must be a symmetric function of all the variables $z_1,z_2,\ldots,z_g$, hence must be independent of all these variables. This yields the desired formula except for the undetermined constant factor c. However the prime function can be multiplied by an arbitrary constant ε; this changes the canonical coordinate by the same factor ε hence changes the derivatives $v_i'(z_j)$ by the factor ε^{-1}, so altogether changes the factor c_1 into $c_1\varepsilon^{\binom{g}{2}}$. It is in particular possible to choose ε so that $c_1\varepsilon^{\binom{g}{2}} = 1$, leaving ε determined up to a root of unity as asserted, to conclude the proof.

Henceforth it will be assumed, generally without any further explicit notice, that the prime function has been normalized as in the preceding theorem; that still leaves open the possibility of a further normalization by an appropriate root of unity. The formula of the preceding theorem can be rewritten as follows.

<u>Corollary</u>. Whenever a point $t \in \mathbb{C}^g$ is written as the image $t = w(z_1+\dots+z_g)$ for some points $z_i \in \tilde{M}$ then

$$f_t(z) = \theta[0|r-t](w(z)) = Q(z_1,z_2,\dots,z_g) \prod_{i=1}^{g} q(z,z_i),$$

where

$$Q(z_1,z_2,\dots z_g) = \Big[\prod_{1\leq j<k\leq g} q(z_j,z_k)^{-1}\Big] \det\{v_j'(z_k) : 1 \leq j,k \leq g\}$$

is a holomorphic function on the complex manifold $\tilde{M}^g$ and is symmetric in the variables z_j so can be viewed alternatively as a holomorphic function on the complex manifold $\tilde{M}^{(g)}$.

<u>Proof</u>. The only point not evident from the preceding theorem is that $Q(z_1,z_2,\ldots,z_g)$ is a holomorphic function. It is clearly a meromorphic function, with at most simple poles along the subvarieties $z_j=Tz_k$ for $T \in \Gamma$; but the determinant vanishes along these same subvarieties, to cancel the poles from the factors $q(z_j,z_k)$ in the denominator, so the function is actually holomorphic as asserted.

It is an immediate consequence of the corollaries to Theorems 1 and 3 that

(2) $Q(z_1,\ldots,z_g) = 0$ precisely when $w(z_1+\ldots+z_g) \in W_g^1$,

hence alternatively that $Q(z_1,\ldots,z_g) = 0$ precisely when the divisor $z_1+\ldots+z_g$ lies in the subvariety $G_g^1 \subset M^{(g)}$. It may be worth noting here that the formula for the function $Q(z_1,z_2,\ldots,z_g)$ in the preceding corollary really defines that function outside the subvarieties $z_j=Tz_k$ for $T \in \Gamma$, and the values at points of these subvarieties are then determined by continuity. The limiting values can of course be obtained explicitly quite easily. For example when $g = 2$

$$Q(z_1,z_2) = q(z_1,z_2)^{-1} \det \begin{pmatrix} w_1'(z) & w_1'(z_2) \\ w_2'(z_1) & w_2'(z_2) \end{pmatrix}$$

so in the limit as z_2 appraoches z_1

(2) $$Q(z_1,z_1) = \det \begin{pmatrix} w_1''(z_1) & w_1'(z_1) \\ w_2''(z_1) & w_2'(z_1) \end{pmatrix} .$$

There are of course similar results in general, but they will be treated later as the need arises. The preceding theorem and its corollary can also be used to obtain similar expressions for the partial derivatives of the theta function function, as follows.

<u>Theorem 4</u>. For any points $z, z_1, \ldots, z_{g-1} \in \tilde{M}$,

$$\Big[\prod_{1\leq j<k\leq g-1} q(z_j,z_k)\Big] \cdot \sum_{i=1}^{g} \partial_i \theta(r - v(z_1+\ldots+z_{g-1}))v'_i(z)$$
$$= \det\{\vec{v}'(z), \vec{v}'(z_1), \ldots, \vec{v}'(z_{g-1})\}$$

where $\vec{v}'(z)$ denotes the column vector of length g with entries $v'_i(z)$.

<u>Proof</u>. Differentiate the formula of the preceding theorem with respect to the variable z and then set $z_g = z$. The left-hand side clearly becomes

$$\Big[\prod_{1\leq j<k\leq g-1} q(z_j,z_k)\Big]\Big[\prod_{1\leq j\leq g-1} q(z_j,z)\Big] \cdot \sum_{i=1}^{g} \partial_i \theta(r - v(z_1+\ldots+z_{g-1}))v'_i(z).$$

On the right-hand side the factor $q(z,z_g)$ vanishes to first order when $z = z_g$, so the only nontrivial terms that can arise when $z = z_g$ come from differentiating this factor; the right-hand side thus becomes

$$\Big[\prod_{1\leq \ell\leq g-1} q(z,z_\ell)\Big] \cdot \det\{\vec{v}'(z_1), \vec{v}'(z_2), \ldots, \vec{v}'(z_{g-1}), \vec{v}'(z)\}$$

and the desired result follows immediately.

<u>Corollary 1</u>. Consider a divisor $\underline{\vartheta} = z_1 + \ldots + z_{g-1} \in G_{g-1} \sim G^1_{g-1}$, so that $t = r - v(\underline{\vartheta}) \in \underline{\theta} \sim \underline{\theta}^1$. There is up to a constant factor a unique nontrivial Abelian differential that vanishes at $\underline{\vartheta}$, and it is given by

$$v'_t(z) = \sum_{i=1}^{g-1} \partial_i \theta(t)\, v'_i(z)$$
$$= \Big\{\prod_{1\leq j<k\leq g-1} q(z_j,z_k)^{-1}\Big\} \cdot \det\{\vec{v}'(z), \vec{v}'(z_1), \ldots, \vec{v}'(z_{g-1})\}$$
$$= \Big\{\prod_{j=1}^{g-1} q(z,z_j)\Big\} \cdot Q(z, z_1, \ldots, z_{g-1}).$$

<u>Proof</u>. That there is up to a constant factor a unique such differential is just the assertion that $\gamma(\kappa\zeta_{\underline{\vartheta}}^{-1}) = 1$, and since $\gamma(\kappa\zeta_{\underline{\vartheta}}^{-1}) = \gamma(\zeta_{\underline{\vartheta}})$ by the Riemann-Roch theorem, this condition is precisely

equivalent to the hypothesis that $r-t = w(\mathcal{D}) \in W_{g-1} - W_{g-1}^1$. If $z_1, \ldots, z_{g-1}$ represent distinct points on M this uniqueness clearly means that there is up to a constant factor a unique vector $\vec{c} \in \mathbb{C}^g$ such that ${}^t\vec{c} \cdot \vec{w}'(z_j) = 0$ for $j = 1, \ldots, g-1$, since any Abelian differential can be written $\vec{c} \cdot \vec{w}'(z)$ for some vector $\vec{c} \in \mathbb{C}^g$; thus the vectors $\vec{w}'(z_1), \ldots, \vec{w}'(z_{g-1})$ are linearly independent, and $\det\{\vec{w}'(z), \vec{w}'(z_1), \ldots, \vec{w}'(z_{g-1})\}$ gives the desired differential form. The equality of the first two lines of the formula of the corollary is just Theorem 4, and that formula thus has the desired property whenever $z_1, \ldots, z_{g-1}$ represent distinct points of M; but the first line is a nontrivial form even if there are coincidences among the points $z_1, \ldots, z_{g-1}$, so long as $t \notin \underline{\underline{\theta}}^1$, and the second line vanishes identically at $z = z_i$ for all parameter values $z_1, \ldots, z_{g-1}$, so $w'_t(z)$ is the desired form even if there are coincidences. The equality of the last two lines of the formula of the corollary follows immediately from the definition of the function $Q(z_1, \ldots, z_g)$.

The partial derivatives of the theta functions appearing in the preceding corollary can also be expressed explicitly as follows.

<u>Corollary 2</u> Whenever a point $t \in \underline{\underline{\theta}}$ is written as $t = r - w(z_1 + \ldots + z_{g-1})$ for some points $z_i \in \tilde{M}$ then

$$\partial_i \theta(t) = Q_i(z_1, z_2, \ldots, z_{g-1})$$

where

$$Q_i(z_1, z_2, \ldots, z_{g-1}) = = (-1)^{i-1} \Big\{ \prod_{1 \leq j < k \leq g-1} q(z_j, z_k)^{-1} \Big\} \det \begin{Bmatrix} w_1'(z_1) & w_1'(z_2) & \ldots & w_1'(z_{g-1}) \\ & \text{(omit row } i) & & \\ w_g'(z_1) & w_g'(z_2) & \ldots & w_g'(z_{g-1}) \end{Bmatrix}$$

is a holomorphic function on M^{g-1} and is symmetric in the variables z_j so can be viewed alternatively as a holomorphic function on $M^{(g-1)}$.

<u>Proof</u>. That $\partial_i\theta(t)$ has the explicit form given follows immediately from the equality of the first two lines in the formula of the preceding corollary, upon expanding the determinant in the second line by minors of the first column and noting that the functions $w_i'(z)$ are linearly independent so that their coefficients must be equal in the resulting identity. Again the functions $Q_i(z_1,\ldots,z_{g-1})$ are formally meromorphic functions with at most simple poles along the subvarieties $z_j = Tz_k$ for $T \in \Gamma$, but the determinant vanishes along these subvarieties to cancel the singularity of the other factor.

As with the function $Q(z_1,\ldots,z_g)$, the values of the function $Q_i(z_1\ldots,z_{g-1})$ are formally given by the defining formula only when all the points z_j are distinct, and are otherwise obtained by continuity with easily determined limiting values. For instance when $g = 3$

$$Q_1(z_1,z_2) = q(z_1,z_2)^{-1} \det \begin{pmatrix} w_2'(z_1) & w_2'(z_2) \\ w_3'(z_1) & w_3'(z_2) \end{pmatrix}$$

so in the limit as z_2 approaches z_1

$$Q_1(z_1,z_1) = \det \begin{pmatrix} w_2''(z_1) & w_2'(z_1) \\ w_2''(z_1) & w_3'(z_1) \end{pmatrix} ,$$

and similarly in general. It is clear from the statement of the corollary that $Q_i(z_1,z_2,\ldots,z_{g-1})$ only depends on the image $t=r-w(z_1+\ldots+z_{g-1}) \in \mathbb{C}^g$. The function $\theta(t)$ is the defining equation for the theta locus $\underline{\underline{\Theta}}$, so the partial derivatives $\partial_i\theta(t)$ define the singular locus $\underline{\underline{\Theta}}^1 \subset \underline{\underline{\Theta}}$; by Riemann's theorem . $\underline{\underline{G}}^1 = r - W^1_{g-1}$, hence

(3) $W^1_{g-1} = \{w(z_1+\dots+z_{g-1}) : Q_i(z_1,z_2,\dots,z_{g-1}) = 0 \text{ for } 1 < i < g\}$.

Finally note that the definition of the function $Q(z_1,z_2,\dots,z_g)$ can be rewritten as

$$Q(z_1,z_2,\dots,z_g) \prod_{j=1}^{g-1} q(z_j,z_g)$$
$$= \left[\prod_{1 \leq j < k \leq g-1} q(z_j,z_k)^{-1}\right] \det \{w'_j(z_k) : 1 < j,\ k < g\} .$$

Upon expanding the determinant by minors of the last column and recalling the definition of the functions $Q_i(z_1,z_2,\dots,z_{g-1})$ it follows readily that

(4) $$Q(z_1,z_2,\dots,z_g) \cdot \prod_{j=1}^{g-1} q(z_g,z_j)$$
$$= \sum_{i=1}^{g} Q_i(z_1,z_2,\dots,z_{g-1})\, w'_i(z_g);$$

the reversal of the order of the factors $q(z_j,z_g)$ cancels the sign $(-1)^{g-1}$arising from the expansion of the determinant by minors of its g-th column. Similarly upon expanding the determinant by minors of row i it follows that

$$Q(z_1,z_2,\dots,z_g) \prod_{1\leq k<\ell\leq g} q(z_k,z_\ell) = \det \{w'_k(z_\ell) : 1 \leq k,\ \ell \leq g\}$$

$$= \sum_{j=1}^{g} (-1)^{i+j}\, w'_i(z_j) \det \{w'_k(z_\ell) : 1 \leq k,\ \ell \leq g,\ k \neq i,\ \ell \neq j\}$$

$$= \sum_{j=1}^{g} (-1)^{j-1}\, w'_i(z_j) \left[\prod_{\substack{1 \leq k \leq \ell \leq g \\ k,\ell\neq j}} q(z_k,z_\ell)\right] Q_i(z_1,\dots,z_{j-1},z_{j+1},\dots,z_g)$$

$$= \left[\prod_{1\leq k<\ell\leq g} q(z_k,z_\ell)\right] \sum_{j=1}^{g} \left[\prod_{\substack{1\leq k\leq g \\ k\neq j}} q(z_j,z_k)^{-1}\right] Q_i(z_1,\dots,z_{j-1},z_{j+1},\dots,z_g)\, w'_i(z_j).$$

On the other hand if the matrix $\{w'_k(z_\ell) : 1 \leq k, \ell \leq g\}$ is changed by replacing row i by row ℓ for some $\ell \neq i$ the resulting determinant is zero, and the same calculation shows that the last line above vanishes if $w'_i(z_j)$ is replaced by $w'_\ell(z_j)$ for all j. Altogether therefore

(5) $$\delta^i_\ell \, Q(z_1,z_2,\ldots,z_g) = \sum_{j=1}^{g} \left[\prod_{\substack{1 \leq k \leq g \\ k \neq j}} q(z_j,z_k)^{-1} \right] Q_i(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_g) \, w'_\ell(z_j).$$

The relations between the auxiliary functions $Q(z_1,z_2,\ldots,z_g)$ and $Q_i(z_1,z_2,\ldots,z_{g-1})$ expressed by (4) and (5) will be used at various subsequent points.

§3. The Gauss mapping.

From the functional equation for the theta function in the form $\theta(t+\lambda) = \xi(\lambda,t)\ \theta(t)$ it follows immediately that the partial derivatives $\partial_i\theta(t) = \partial\theta(t)/\partial t_i$ satisfy the slightly more complicated functional equations

(1) $\quad \partial_i\theta(t+\lambda) = \xi(\lambda,t)\,\partial_i\theta(t) + \partial_i\xi(\lambda,\tau)\,\theta(t)$

for all lattice vectors $\lambda \in \mathcal{L}$. For points $t \in \mathbb{C}^g$ representing points in the theta locus $\underline{\underline{\theta}} \subseteq J$ the last term in (1) vanishes, so that the functional equation reduces to the same form as that for the theta function; the functions $\partial_i\theta|\underline{\underline{\theta}}$ can thus be viewed as sections of the restriction $\xi|\underline{\underline{\theta}}$ of the line bundle defined by the theta factor of automorphy. It is sometimes more convenient to introduce the vector

(2) $\quad \vec{\partial}\theta(t) = \{\partial_i\theta(t) : 1 \leq i \leq g\}$

with these partial derivatives as components; this vector then satisfies $\vec{\partial}\theta(t+\lambda) = \xi(\lambda,t)\cdot\vec{\partial}\theta(t)$ whenever $t \in \underline{\underline{\theta}} \subseteq \mathbb{C}^g$. Outside the singular locus $\underline{\underline{\theta}}^1 \subseteq \underline{\underline{\theta}}$ the vector $\vec{\partial}\theta(t)$ is nonzero so represents a well defined point $[\vec{\partial}\theta(t)] \in \mathbb{P}^{g-1}$, and the functional equation implies that $[\vec{\partial}\theta(t+\lambda)] = [\vec{\partial}\theta(t)]$. There is thus a well defined mapping

(3) $\quad [\vec{\partial}\theta] : \underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1 \to \mathbb{P}^{g-1}$

from the (g-1)-dimensional complex manifold $\underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1 \subseteq J$ to the (g-1)-dimensional complex manifold $\mathbb{P}^{g-1}$, which is called the <u>Gauss mapping</u> defined by the theta function. To describe this mapping in more detail, it may be recalled that a holomorphing mapping between two complex manifolds is said to be <u>finite</u> if the inverse image of each point in the range is a finite set of points in the domain.

<u>Theorem</u> 5. The Gauss mapping $[\partial\theta] : \underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1 \to \mathbb{P}^{g-1}$ is a finite holomorphic mapping.

<u>Proof</u>. To any point $t \in \underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1$ there corresponds the Abelian differential $w_t'(z) = \sum_i \partial_i \theta(z)\, w'_i(z)$ of Corollary 2 of the preceding theorem, a well defined point in the g-dimensional vector space $\Gamma(\kappa)$ of Abelian differentials; in these terms the Gauss mapping can be described as associating to t the set of all constant multiples of $w_t'(z)$, viewed as a point in the associated projective space $\mathbb{P}\Gamma(\kappa)$. By Corollary 2 this set of Abelian differentials is characterized as those vanishing at the divisor $z_1 + \dots + z_{g-1}$, where $r-t \in W_{g-1} \sim W_{g-1}^1$ is written as $r-t = w(z_1 + \dots + z_{g-1})$. The divisor of this differential $w_t'(z)$ is $\mathfrak{I}(w_t') = z_1 + \dots + z_{g-1} + z_g + \dots + z_{2g-2}$ for some further $g-1$ points $z_g, \dots, z_{2g-2}$ of M. Now if $t' \in \underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1$ has the same image as t under the Gauss mapping then it will determine the same Abelian differential, up to a nonzero constant factor; thus if $r-t' = w(z_1' + \dots + z_{g-1}')$ then $z_1', \dots, z_{g-1}'$ must be some set of $g-1$ points from among the $2g-2$ points $z_1, \dots, z_{2g-2}$ describing the divisor of $w_t'(z)$. It is evident from these observations that there are at most $\binom{2g-2}{g-1}$ distinct points of $\underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1$ that have the same image under the Gauss mapping. -

It is a general result in complex analysis that a finite holomorphic mapping is an open mapping; this is the natural generalization of the familiar theorem that any nontrivial holomorphic function of one complex variable is an open mapping, since such a function is always at least locally a finite mapping. A consequence of this is that the image of the Gauss mapping is an open subst of $\mathbb{P}^{g-1}$, so in particular is not contained within any proper holomorphic subvariety of $\mathbb{P}^{g-1}$; this observation will be used later.

A more detailed analysis of the Gauss mapping is quite rewarding. An obvious first question is what happens to this mapping near points of the subvariety $\underline{\underline{\Theta}}^1 \subseteq \underline{\underline{\Theta}}$, and there is a standard approach that nicely handles this question. Consider the holomorphic subvariety $Y \subseteq \underline{\underline{\Theta}} \times \mathbb{P}^{g-1}$ defined by

(4) $\quad Y = \{(t,[v]) \in \underline{\underline{\Theta}} \times \mathbb{P}^{g-1} : \partial\theta(t) \text{ and } v \text{ are linearly dependent}\}$,

where $[v] \in \mathbb{P}^{g-1}$ is the equivalence class represented by the nonzero vector $v \in \mathbb{C}^g$. The restriction of this subvariety Y to the open subset $(\underline{\underline{\Theta}} - \underline{\underline{\Theta}}^1) \times \mathbb{P}^{g-1} \subset \underline{\underline{\Theta}} \times \mathbb{P}^{g-1}$ is just the graph of the Gauss mapping. Since this restriction can be viewed as the complement in Y of the subvariety $Y \cap (\underline{\underline{\Theta}}^1 \times \mathbb{P}^{g-1}) \subset Y$, it follows readily that its point set closure is a holomorphic subvariety $Y_o \subseteq Y$, indeed a union of irreducible components of Y. Thus at least the graph of the Gauss mapping naturally extends to a holomorphic subvariety $Y_o \subseteq \underline{\underline{\Theta}} \times \mathbb{P}^{g-1}$; but this subvariety may have the property that over a point of $\underline{\underline{\Theta}}^1$ there lie many points of Y_o, indeed a whole subvariety of positive dimension in Y_o, so that the subvariety fails there to be the graph of a well defined mapping from $\underline{\underline{\Theta}}$ to $\mathbb{P}^{g-1}$. This is a quite familiar situation in algebraic or analytic geometry in several dimensions.

The Gauss mapping can of course be defined for the theta function associated to an arbitrary period matrix $\Omega \in \mathfrak{H}_g$, not necessarily the period matrix of the Jacobi variety of a Riemann surface, and its graph extends to all of $\underline{\underline{\Theta}}$ as above. For a general period matrix $\Omega \in \mathfrak{H}_g$ it was noted earlier that $\underline{\underline{\Theta}}^1$ is empty, so the Gauss mapping is defined in all of $\underline{\underline{\Theta}}$ and $Y = Y_o$ is precisely its graph. On the other hand for

Jacobi varieties the situation is somewhat different. The subvariety $\underline{\underline{\Theta}}$ is irreducible, so that $\underline{\underline{\Theta}} - \underline{\underline{\Theta}}^1$ is a connected and hence irreducible complex manifold of dimension $g-1$, and since the graph of the Gauss mapping is a one-sheeted cover of $\underline{\underline{\Theta}} - \underline{\underline{\Theta}}^1$ it too is an irreducible subvariety of dimension $g-1$; its closure Y_o is consequently an irreducible subvariety of dimension $g-1$ as well. If $t \in \underline{\underline{\Theta}}^1$ then $\vartheta\,\theta(t) = 0$ so that $t \times P^{g-1} \subseteq Y$; thus $\underline{\underline{\Theta}}^1 \times P^{g-1} \subseteq Y$. It is known that $\dim \underline{\underline{\Theta}}^1 \geq g-4$ so if $g > 4$ then $\dim (\underline{\underline{\Theta}}^1 \times P^{g-1}) = \dim \underline{\underline{\Theta}}^1 + \dim P^{g-1} > g-1 = \dim Y_o$ and it follows that Y is necessarily reducible, with Y_o as one component but with at least another component having dimension greater than that of Y_o.

For Jacobi varieties this analysis can be carried further. It is convenient to identify P^{g-1} with the projective space $P\Gamma(\kappa)$ associated to the space $\Gamma(\kappa)$ of Abelian differentials on M, and hence as in the proof of Theorem 5 to view the Gauss mapping as associating to any point $t \in \underline{\underline{\Theta}} - \underline{\underline{\Theta}}^1$ the point $[w_t'(z)] \in P\Gamma(\kappa)$ represented by the function $w_t'(z) \in \Gamma(\kappa)$. Introduce then the holomorphic subvariety $\tilde{X} \subseteq M^{(g-1)} \times P\Gamma(\kappa)$ defined by

$$(5)\quad \tilde{X} = \{(z_1 + \dots + z_{g-1}, [v'(z)]) \in M^{(g-1)} \times P\Gamma(\kappa) : \mathcal{D}(v') \geq z_1 + \dots + z_{g-1}\}.$$

The modified Abel-Jacobi mapping

$$z_1 + \dots + z_{g-1} \in M^{(g-1)} \to r - w(z_1 + \dots + z_{g-1}) \in r - W_{g-1} = \underline{\underline{\Theta}} \subset J$$

induces a proper holomorphic mapping

$$w_o : M^{(g-1)} \times P\Gamma(\kappa) \to \underline{\underline{\Theta}} \times P(\kappa),$$

and by Remmert's proper mapping theorem the image of $\tilde{X}$ under this mapping will be a well defined holomorphic subvariety $X \subseteq \underline{\underline{\Theta}} \times P\Gamma(\kappa)$. This subvariety can then be described alternatively as

(6) $X = \{(t,[v'(z)]) \in \underline{\Theta} \times \mathbb{P}\Gamma(\kappa) :$ there is a divisor $z_1 + \dots + z_{g-1} \in M^{(g-1)}$ such that $r-t=w(z_1 + \dots + z_{g-1})$ and $\underline{\mathcal{D}}(v') \geq z_1 + \dots + z_{g-1}\}$.

It is perhaps not quite so obvious from this characterization that X is a holomorphic subvariety of $\underline{\Theta} \times \mathbb{P}\Gamma(\kappa)$; but it is clear that $X \subseteq Y$ and that the restriction of X to the open subset $(\underline{\Theta} - \underline{\Theta}^1) \times \mathbb{P}^{g-1}$ is again the Gauss mapping, so the closure of the Gauss mapping is also an irreducible subvariety $X_o = Y_o \subseteq X$.

The natural projection mapping $\pi_2 : X \to \mathbb{P}\Gamma(\kappa)$ is clearly surjective, since any nontrivial function $v'(z) \in \Gamma(\kappa)$ has a well defined divisor $\underline{\mathcal{D}}(v'(z)) = z_1 + \dots + z_{2g-2} \in M^{(2g-2)}$ and $(r-w(z_{i_1} + \dots + z_{i_{g-1}})$, $[v'(z)]) \in X$ for any choice of $g-1$ points $z_{i_1}, \dots, z_{i_{g-1}}$ of the divisor $\underline{\mathcal{D}}(v'(z))$. Indeed the inverse image $\pi_2^{-1}([v'(z)])$ is not only nonempty but finite, consisting of at most $\binom{2g-2}{g-1}$ points corresponding to the possible choices of $g-1$ points from the divisor $\underline{\mathcal{D}}(v'(z))$. It is known that for any nonhyperelliptic Riemann surface of genus $g > 0$ there exists an Abelian differential $v'(z) \in \Gamma(\kappa)$ with a divisor consisting of $2g-2$ distinct points such that no two different subsets of $g-1$ of these points represent linearly equivalent divisors; for such a differential the inverse image $\pi_2^{-1}([v'(z)])$ consists of the maximum possible number $\binom{2g-2}{g-1}$ of points, and the same is clearly the case for all points of $\mathbb{P}\Gamma(\kappa)$ in an open neighborhood of $[v'(z)]$. It follows from general results in complex analysis that the restriction of π_2 to any irreducible component of X is what is known as a finite branched holomorphic covering of its image. In an open neighborhood of any point covered by the maximum number of points the covering is a simple unbranched covering, so each component of X locally covers an open subset of $\mathbb{P}\Gamma(\kappa)$. Therefore each irreducible component of X is of dimension $g-1$ and is exhibited by π_2 as a finite branched holomorphic covering over all of $\mathbb{P}\Gamma(\kappa)$; that is in particular true of the component X_o, the closure of the graph of the Gauss mapping. It should be noted that if X is reducible then each irreducible component arises from a particular subcollection of the $(g-1)$-tuples of points from the divisor of any Abelian differential, a subcollecton that must be defined intrinsically and consistently over all the Abelian differentials; there must then be at least two distinct ways of choosing $(g-1)$-tuples of points from the the divisor of the Abelian differentials for all such differentials, a rather peculiar possibility that does actually occur. for hyperelliptic surfaces.

The natural projection mapping $\pi_1 : X \to \underline{\underline{\theta}}$ is also clearly surjective, since any point $t \in \underline{\underline{\theta}} = r-W_{g-1}$ can be written $t = r-w(z_1 +\dots+ z_{g-1})$ for some divisor $z_1 +\dots+ z_{g-1} \in M^{(g-1)}$ and there will always be at least one Abelian differential that vanishes at this divisor. If $t \notin \underline{\underline{\theta}}^1 = r - W^1_{g-1}$ this divisor is unique, and $w'_t(z)$ is the unique Abelian differential that vanishes at it; that is just the observation that over $\underline{\underline{\theta}} - \underline{\underline{\theta}}^1$ the Gauss mapping assigns to any $t \in \underline{\underline{\theta}} - \underline{\underline{\theta}}^1$ the point $\pi_2(\pi_1^{-1}(t)) \in \mathbb{P}\Gamma(\kappa)$. If X is reducible, one irreducible component X_o is the closure of the graph of the Gauss mapping; any other irreducible component X_1 must then have the property that $\pi_1(X_1) \subseteq \underline{\underline{\theta}}^1$. Recall from the preceding paragraph that both X_o and X_1 are necessarily $(g-1)$-dimensional holomorphic subvarieties. Using some general properties of the subvarieties W^{ν}_{g-1} leads rather easily to the result that there can only be an irreducible component of X other than X_o when the Riemann surface M is hyperelliptic. The basic property needed is a special case of Martens's extension of Clifford's theorem, the result that $\dim W^{\nu}_{g-1} \leq g-1-2\nu$ with equality only when the Riemann surface M is hyperelliptic. In addition there is the observation that over the subset $W^{\nu}_{g-1} \sim W^{\nu+1}_{g+1}$ the Abel-Jacobi mapping $G^{\nu}_{g-1} \sim G^{\nu+1}_{g-1} \to W^{\nu}_{g-1} \sim W^{\nu+1}_{g-1}$ is a holomorphic fibre bundle with fibre P^{ν}, so that $\dim (G^{\nu}_{g-1} \sim G^{\nu+1}_{g-1}) \leq g-1-\nu$ with equality only when M is hyperelliptic. Still further there is the observation that over $G^{\nu}_{g-1} \sim G^{\nu+1}_{g-1}$

$$X^{(\nu)} = \bigcup_{z_1+\cdots+z_{g-1} \in G^{\nu}_{g-1} \sim G^{\nu+1}_{g-1}} P\Gamma(\kappa\zeta^{-1}_{z_1}\cdots\zeta^{-1}_{z_{g-1}})$$

is also a holomorphic fibre bundle with fibre P^{ν}, so that $\dim X^{(\nu)} \leq g-1$ with equality only when M is hyperelliptic. Now $X^{(\nu)} = \pi_1^{-1}(\underline{\underline{\theta}}^{\nu} \sim \underline{\underline{\theta}}^{\nu+1})$, so it is evident from this that $\pi_1^{-1}(\underline{\underline{\theta}}^{1}) = X^{(1)} \cup X^{(2)} \ldots$ can only be of dimension $g-1$ when M is hyperelliptic. What has thus been demonstrated is that <u>if</u> M <u>is not hyperelliptic the subvariety</u> $X \subseteq \underline{\underline{\theta}} \times P\Gamma(\kappa)$ <u>defined by</u> (6) <u>is irreducible, and is the closure of the graph of the Gauss mapping</u>. In the hyperelliptic case on the other hand the subvarieties $X^{(\nu)}$ are all of dimension $g-1$ when nonempty, so do represent other irreducible components of the subvariety X; in this case therefore it must be possible to choose $(g-1)$ - tuples of points from the divisors of Abelian differentials in several essentially distinct ways. It is worth noting explicitly that the graph of the Gauss mapping distinguishes between nonhyperelliptic and hyperelliptic surfaces M since for the former the finite branched holomorphic mapping $\pi_2 : X_0 \to P\Gamma(\kappa)$ is of order $\binom{2g-2}{g-1}$ while for the latter the order is strictly less.

For the remainder of this section only nonhyperelliptic Riemann surfaces M will be considered. It is perhaps worth repeating for emphasis that then the irreducible subvariety $X \subseteq \underline{\underline{\Theta}} \times \mathbb{P}\Gamma(\kappa)$ is the closure of the graph of the Gauss mapping, so is described completely by the theta function alone without any reference to the underlying Riemann surface M; but of course $\bar{X}$ on the other hand involves the Riemann surface M quite explicitly in its description. There are three auxiliary subvarieties of $\mathbb{P}\Gamma(\kappa)$ that arise quite naturally and merit some further attention. The first is the subvariety

(7) $A = \pi_2(\bar{\pi}_1^{-1}(\underline{\underline{\Theta}})) \subseteq \mathbb{P}\Gamma(\kappa)$,

the image under π_2 of the complement of the graph of the Gauss mapping in its closure X. Equivalently A can be described as the holomorphic subvariety of $\mathbb{P}\Gamma(\kappa)$ consisting of the equivalence classes of those differential forms that vanish on divisors lying in G^1_{g-1}. It follows as in the argument in the preceding paragraph that $\dim A = g-2$. Indeed $W^1_{g-1} \sim W^2_{g-1}$ is known to be of dimension $g-4$, the Abel-Jacobi mapping $G^1_{g-1} \sim G^2_{g-1} \to W^1_{g-1} \sim W^2_{g-1}$ is a holomorphic fibre bundle with fibre $\mathbb{P}^1$ so that $\dim G^1_{g-1} \sim G^2_{g-1} = g-3$, and

$$A^{(1)} = \bigcup_{z_1+\dots+z_{g-1} \in G^1_{g-1} \sim G^2_{g-1}} \mathbb{P}\Gamma(\kappa\zeta^{-1}_{z_1} \cdots \zeta^{-1}_{z_{g-1}}) \subseteq A$$

is also a homomorphic fibre bundle with fibre $\mathbb{P}^1$ so that $\dim A \geq \dim A^{(1)} = g-2$; on the other hand since A is a proper holomorphic subvariety of $\mathbb{P}\Gamma(\kappa) = \mathbb{P}^{g-1}$ necessarily $\dim A < g-1$. Each irreducible component of $\underline{\underline{\Theta}}^1 \sim \underline{\underline{\Theta}}^2 = r-(W^1_{g-1} \sim W^2_{g-1})$ yields an irreducible component of dimension $g-2$ of A, and similarly each irreducible component of $\underline{\underline{\theta}}^{\nu} - \underline{\underline{\theta}}^{\nu+1} = r(W^{\nu}_{g-1} - W^{\nu+1}_{g-1})$ of dimension $g-2-2\nu$ yields another irreducible component of dimension $g-2$ of A. For $\nu > 1$ there are only quite special Riemann surfaces for which $\dim(\underline{\underline{\theta}}^{\nu} - \underline{\underline{\theta}}^{\nu+1})$ attains this maximal value; otherwise there result subvarieties of A of dimension less than $g-2$, either separate components or merely subvarieties of other components.

Next it has been observed that the mapping $\pi_2 : X \to P\Gamma(\kappa)$ is a finite branched holomorphic covering of order $\binom{2g-2}{g-1}$, and similarly the mapping $\tilde{\pi}_2 = \pi_2 \circ \pi_0 : \tilde{X} \to P\Gamma(\kappa)$ is also a finite branched holomorphic covering of order $\binom{2g-2}{g-1}$. Any such mapping is a locally biholomorphic mapping over all points outside a holomorphic subvariety of pure dimension $g-2$ in $P\Gamma(\kappa)$; this subvariety, called the <u>branch locus</u>, consists precisely of those points of $P\Gamma(\kappa)$ over which there are fewer than $\binom{2g-2}{g-1}$ distict points under the branched holomorphic covering mapping. Thus there naturally arise the branch loci $B \subset P\Gamma(\kappa)$ of the mapping π_2 and $\tilde{B} \subset P\Gamma(\kappa)$ of the mapping $\tilde{\pi}_2 = \pi_2 \circ \pi_0$, and it is clear that $\tilde{B} \subseteq B$. The subvariety $\tilde{B} \subseteq P\Gamma(\kappa)$ can be characterized alternatively as consisting of the equivalence classes of those differential forms $\bar{v}'(z)$ such that not all divisors of order $g-1$ contained in the divisor of $v'(z)$ are distinct, or what is obviously the same thing, such that the divisor of $v'(z)$ does not consist of distinct points; the subvariety $B \subseteq P\Gamma(\kappa)$ can be characterized correspondingly as consisting of the equivalence classes of those differential forms $v'(z)$ such that not all divisors of order $g-1$ contained in the divisor of $v'(z)$ are linearly inequivalent.

To describe these branch loci in more detail, consider the subvariety $Z \subseteq M^{(g-1)}$ consisting of divisors of the form $2z_1 + z_2 + \dots + z_{g-2}$; since Z is the image of the obvious holomorphic mapping $M \times M^{(g-3)} \to M^{(g-1)}$ it is evident that Z is an irreducible holomorphic subvariety of $M^{(g-1)}$ of dimension $g-2$. Since $\dim G^1_{g-1} = g-3$ the intersection $Z \cap G^1_{g-1}$ is a proper subvariety of Z, hence $Z \sim Z \cap G^1_{g-1}$ is an irreducible variety as well, a dense open subset of Z. Now to every point $\mathfrak{z} \in Z$ there corresponds a differential form $v'(z) \in \Gamma(\kappa)$, unique up to a constant factor, such that $\mathfrak{z}(v') \geq \mathfrak{z}$; that evidently yields a holomorphic mapping $\tilde{\sigma} : Z \sim Z \cap G^1_{g-1} \to \tilde{X}$ by defining

$$\tilde{\sigma}(\mathfrak{z}) = (\mathfrak{z}, [v'(z)]) \in \tilde{X} \subseteq M^{(g-1)} \times \mathbb{P}\Gamma(\kappa),$$

and the composite $\pi_0 \circ \tilde{\sigma} = \sigma : Z \sim Z \cap G^1_{g-1} \to X$ is the holomorphic mapping given by

$$\sigma(\mathfrak{z}) = (r-u(\mathfrak{z}), [v'(z)]) \in X \subset W_{g-1} \times \mathbb{P}\Gamma(\kappa).$$

Again although these mappings do not necessarily extend to holomorphic mappings at points of $Z \cap G^1_{g-1}$ their graphs extend to holomorphic subvarieties of $Z \times \tilde{X}$ and $Z \times X$; in each case the closure of the graph is a well defined irreducible holomorphic subvariety of dimension $g-2$, by an argument so much like that used earlier that further details can be omitted here. The images of the closures of these graphs are clearly the same holomorphic subvariety $B_0 \subseteq \mathbb{P}\Gamma(\kappa)$, an irreducible holomorphic subvariety of dimension $g-2$ contained in $\mathbb{P}\Gamma(\kappa)$. It is also clear from the construction that

(8) $\quad \tilde{B} \sim \tilde{B} \cap A = B \sim B \cap A = B_0 \sim B \cap A;$

therefore B_0 <u>is an irreducible component of</u> B <u>and of</u> $\tilde{B}$, <u>and any other</u>

<u>irreducible components of</u> B <u>or of</u> $\tilde{B}$ <u>are necessarily irreducible components of</u> A.

For a general Riemann surface the subvarieties B and $\tilde{B}$ are irreducible and consequently coincide with B_o. The determination of those special Riemann surfaces for which at least B has other irreducible components lying in A is an interesting problem, but leads too far afield to be pursued further here. However it might be worthwhile including an example of a Riemann surface for which the locus B is reducible. There are Riemann surfaces M of genus 4 for which W_3^1 consists of a single point e; these surfaces are three-sheeted branched coverings of the Riemann sphere P^1, and the divisors in G_3^1 are precisely the inverse images of points of P^1 under this branched covering. For any divisor $x_1 + x_2 + x_3 \in G_3^1$ there is a two-dimensional space of Abelian differentials vanishing at that divisor, and the equivalence classes of these differentials in $P\Gamma(\kappa)$ comprise the subvariety A; thus A is an irreducible two-dimensional subvariety of $P\Gamma(\kappa)$, a P^1 bundle over P^1 given by this construction. Now if $\underset{\sim}{\vartheta}(v') = x_1 + x_2 + x_3 \in G_3^1$ then $\underset{\sim}{\vartheta}(v') = x_1 + x_2 + x_3 + y_1 + y_2 + y_3$ where $y_1 + y_2 + y_3 \in k - G_3^1 = G_3^1$, and $w(x_1 + x_2 + x_3) = w(y_1 + y_2 + y_3) = e = W_3^1$. That means that $[v'(z)] \in \bar{B}$, and thus shows that $A \subseteq B$. On the other hand the covering $M \to P^1$ has three distinct points lying over a general point of P^1, so that a general point of A does not lie in the subvariety B_o. Thus for such a Riemann surface M the subvarieties A and B_o are distinct irreducible subvarieties of $P\Gamma(\kappa)$ and $B = A \cup B_o$. The subvariety $\tilde{B}$ is irreducible though, so that $\tilde{B} = B_o$.

The most interesting associated subvarieties of $\mathbb{P}\Gamma(\kappa)$ are thus really A and B_o, and these are fully determined by the theta function itself without any reference to the underlying Riemann surface. The most convenient explicit descriptions do involve the Riemann surface though; it would be interesting to have a more explicit description in terms of the theta functions. It may be recalled that A consists of those equivalence classes of Abelian differentials that vanish on divisors $z_1 + z_2 + \dots + z_{g-1} \in G^1_{g-1}$, while B_o is the closure of the set consisting of those Abelian differentials that vanish on divisors $2z_1 + z_2 + \dots + z_{g-2} \notin G^1_{g-1}$. It should be noted that B_o can be described equivalently in terms of tangent hyperplanes to the canonical curve; this description will not be pursued further here though.

Some observations in a slightly different direction arise as consequences of the occasionally useful result that <u>if</u> $w'(z) \in \Gamma(\kappa)$ <u>is an abelian differential with divisor</u> $\mathfrak{d}(w'(z)) = z_1 + z_2 + \dots + z_{2g-2}$ <u>consisting of distinct points on</u> M <u>then not all</u> $(g-1)$-<u>tuples of points from this divisor can represent a divisor in</u> G^1_{g-1}. To demonstrate this auxiliary result, suppose to the contrary that $w'(z) \in \Gamma(\kappa)$ has a divisor $\mathfrak{d}(w'(z)) = z_1 + z_2 + \dots + z_{2g-2}$ of distinct points z_i on M and that $z_{i_1} + \dots + z_{i_{g-1}} \in G^1_{g-1}$ for all indices $1 \leq i_1 < \dots < i_{g-1} \leq 2g-2$. Thus for any such divisor $\mathfrak{d} = z_{i_1} + \dots + z_{i_{g-1}}$ it follows from the Riemann-Roch theorem that $\gamma(\kappa\zeta_{\mathfrak{d}}^{-1}) = \gamma(\zeta_{\mathfrak{d}}) > 1$; since $\Gamma(\kappa\zeta_{\mathfrak{d}}^{-1})$ can as usual be identified with the space of Abelian differentials that vanish at $\mathfrak{d}$, there are consequently at least two linearly independent Abelian differentials that vanish at $\mathfrak{d}$. Now if $\vec{w}'(z)$ denotes the vector with entries $w'_i(z)$, a general Abelian differential can be written ${}^t\vec{c} \cdot \vec{w}'(z)$ for a vector $\vec{c} \in \mathbb{C}^g$. The condition that this differential vanishes at $\mathcal{D}$ is just that ${}^t\vec{c} \cdot \vec{w}'(z_{i_1}) = \dots = {}^t\vec{c} \cdot \vec{w}'(z_{i_{g-1}}) = 0$, so there must be at least two linearly independent vectors $\vec{c}$ such that ${}^t\vec{c}\{\vec{w}'(z_{i_1}),\dots,\vec{w}'(z_{i_{g-1}})\} = 0$; that means that the $g \times g-1$ matrix $\{\vec{w}'(z_{i_1}),\dots,\vec{w}'(z_{i_{g-1}})\}$ has rank at most $g-2$, and since that is the case for all such indices $i_1,\dots,i_{g-1}$ it follows that the $g \times 2g-2$ matrix $\{\vec{w}'(z_1), \vec{w}'(z_2), \dots, \vec{w}'(z_{2g-2})\}$ must have rank at most $g-2$ as well. The $2g-2$ points $\vec{w}'(z_i) \in \mathbb{C}^g$ thus lie in a common linear subspace $L \subseteq \mathbb{C}^g$ of dimension at most $g-2$, and upon passing to the associated projective space it follows that the $2g-2$ points $[\vec{w}'(z_i)] \in \mathbb{P}^{g-1}$ lie in a common linear subspace $[L] \subseteq \mathbb{P}^{g-1}$ with $\dim [L] \leq g-3$. The $2g-2$ points $[\vec{w}'(z_i)]$ then lie on the intersection of the linear subspace $[L]$ with the canonical curve $[w'(M)] \subseteq \mathbb{P}^{g-1}$ associated to the Riemann surface M. It follows readily from this that the canonical curve must actually be contained entirely within L; for if $[\vec{w}'(z_0)]$ is any point in the canonical curve not in $[L]$ then that point and $[L]$ span a linear hyperplane $[H] \subset \mathbb{P}^{g-1}$, this hyperplane necessarily meets the canonical curve in precisely $2g-2$ points since the degree of the canonical curve is $2g-2$, and that is impossible since $[H]$ meets the canonical curve in the point $[\vec{w}'(z_0)]$ together with the $2g-2$ other points $[L] \cap [w'(M)]$. On the other hand the canonical curve cannot lie in any proper linear subspace of $\mathbb{P}^{g-1}$ since the canonical Abelian differentials are linearly independent, and this contradiction demonstrates the truth of the original observation. This argument of course only works when $g \geq 3$, but when $g < 3$ the set G^1_{g-1} is empty and there is really nothing to prove. It should be observed that the argument fails when not all of the points z_i are distinct on M; what happens in such a case is perhaps worth further examination.

It follows from this auxiliary result that <u>the image of the Gauss mapping</u> $[\vec{\partial}\,\theta] : \underset{\approx}{\theta} - \underset{\approx}{\theta}^1 \to \mathbb{P}^{g-1}$ <u>contains the complement of the branch locus</u> $\tilde{B} \subset \mathbb{P}^{g-1}$. Indeed if $[v'(z)] \in \mathbb{P}\Gamma(\kappa) = \mathbb{P}^{g-1}$ is not contained in the branch locus $\tilde{B}$ then the divisor $\underset{\approx}{\theta}(v'(z))$ consists of distinct points, so for some $g-1$ of them the divisor $z_1+\dots+z_{g-1} \notin G^1_{g-1}$; the image $r-t=v(z_1+\dots+z_{g-1}) \in W_{g-1} - W^1_{g-1}$ and then $[v'(z)] = [\vec{\partial}\,\theta(t)]$ as desired.

§4. Semicanonical functions.

To turn next to the problem of expressing the canonical functions and differential forms on the Riemann surface M in terms of the Riemannian theta functions, the converse of what was done in section 2, consider the function

$$\theta\big(t+w(z-a)\big) = f_{r-t+w(a)}(z) \in \Gamma\left(\rho_{r-t+w(a)}\zeta^{g}\right) = \Gamma\left(\rho_{-t}\,\zeta_a\,\sigma\right);$$

here $\sigma = \rho_r \zeta^{g-1}$ is the semicanonical factor of automorphy associated to the Riemann point r, so that $\sigma^2 = \rho_k \zeta^{2g-2} = \kappa$ is the canonical factor of automorphy. It follows from Theorem 2 that $\theta\big(t+w(z-a)\big) = 0$ for all points $z, a \in \tilde{M}$ precisely when $t \in \underline{\underline{\theta}}^1$; so to avoid triviality it will generally be supposed henceforth that $t \notin \underline{\underline{\theta}}^1$. It may still be the case though that $f_{r-t+w(a)}(z) \equiv 0$ in z for some fixed point $a \in \tilde{M}$; by the Corollary to Theorem 1 that happens precisely when $r-t+w(a) \in k-W_{g-2}$, hence when $t \in W_{g-2} + w(a)-r \subseteq W_{g-1}-r = \underline{\underline{\theta}}$. Therefore

(1) $\quad t \notin \underline{\underline{\theta}} \Rightarrow f_{r-t+w(a)} \not\equiv 0$ for arbitrary $a \in \tilde{M}$.

On the other hand if $t \in \underline{\underline{\theta}} - \underline{\underline{\theta}}^1$ then $r+t \in W_{g-1} - W^1_{g-1}$ so that $r+t = w(a_1 +\dots+ a_{g-1})$ for a unique divisor $a_1 +\dots+ a_{g-1}$ on M; therefore $t \in W_{g-2} + w(a)-r$ precisely when a represents one of the points a_i, so

(2) $\quad t \in \underline{\underline{\theta}} - \underline{\underline{\theta}}^1 \Rightarrow r+t = w(a_1 +\dots+ a_{g-1})$ for a unique divisor $a_1 +\dots+ a_{g-1}$ on M, and $f_{r-t+w(a)} \equiv 0$ precisely when a represents one of the points of this divisor.

These two cases are rather different in may ways, and the second is the more useful for the purposes at hand.

Whenever $f_{r-t+w(a)} \not\equiv 0$ then as observed earlier this function is a basis for the vector space $\Gamma(\rho_{-t}\zeta_a\sigma)$, so that $\gamma(\rho_{-t}\zeta_a\sigma) = 1$ and hence $\gamma(\rho_{-t}\sigma \leq 1)$; moreover $\mathcal{D}(f_{r-t+w(a)}) = z_1 + \ldots + z_g$ is the unique positive divisor of degree g on M such that $r-t+w(a) = w(z_1 + \ldots + z_g)$. If $\gamma(\rho_{-t}\sigma) = 1$ then multiplying any nontrivial function in $\Gamma(\rho_{-t}\sigma)$ by $q(z,a)$ yields an element of $\Gamma(\rho_{-t}\zeta_a\sigma)$, hence a multiple of $f_{r-t+w(a)}$, and since this function vanishes at a necessarily $0 = f_{r-t+w(a)}(a) = \theta(t)$ so that $t \in \underline{\theta}$. Conversely if $t \in \underline{\theta}$ then $0 = \theta(t) = f_{r-t+w(a)}(a)$, so that a must be one of the points of the divisor $z_1 + \ldots + z_{g-1}$ and hence $q(z,a)^{-1}f_{r-t+w(a)}(z) \in \Gamma(\rho_{-t}\sigma)$. Thus for $t \in \underline{\theta} \sim \underline{\theta}^1$ and a any point not in the divisor (2) the function $q(z,a)^{-1}f_{r-t+w(a)}(z) = q(z,a)^{-1}\theta(t+w(z-a))$ is a basis for the one-dimensional vector space $\Gamma(\rho_{-t}\sigma)$, and as a varies this function can only change by a constant multiple; when a belongs to the divisor (2) this multiple is zero. The functions in $\Gamma(\rho_{-t}\sigma)$, which will be called the <u>semicanonical functions</u>, are thus essentially described by the Riemannian theta functions, and in a rather interesting way. Indeed

(3) $\quad t \in \underline{\theta} \sim \underline{\theta}^1 \Rightarrow \gamma(\rho_{-t}\sigma) = 1$ and $q(z,a)^{-1}\,\theta(t + w(z-a)) \in \Gamma(\rho_{-t}\sigma)$

as a function of z for all points $a \in M$.

These functions then have a variety of applications, such as the following.

<u>Theorem</u> 6. For any fixed point $t \in \underline{\theta} \sim \underline{\theta}^1$ the cross-ratio function on M can be written.

$$p(z_1,z_2,a_1,a_2) = \frac{\theta(t+w(z_1-a_1))\quad \theta(t+w(z_2-a_2))}{\theta(t+w(z_1-a_2))\quad \theta(t+w(z_2-a_1))}$$

Proof Whenever a_1, a_2 do not represent points of the divisor (2) associated to t then the functions $q(z,a_i)^{-1}\theta(t+v(z-a_i)) \in \Gamma(\rho_{-t}\sigma)$ are not identically zero, so each is a basis for $\Gamma(\rho_{-t}\sigma)$ and their quotient is therefore independent of z; thus

$$\frac{q(z_1,a_1)^{-1}\theta(t+v(z_1-a_1))}{q(z_1,a_2)^{-1}\theta(t+v(z_1-a_2))} = \frac{q(z_2,a_1)^{-1}\theta(t+v(z_2-a_1))}{q(z_2,a_2)^{-1}\theta(t+v(z_2-a_2))}$$

or equivalently

$$(4) \qquad \frac{\theta(t+v(z_1-a_1))\ \theta(t+v(z_2-a_2))}{\theta(t+v(z_1-a_2))\ \theta(t+v(z_2-a_1))} = \frac{q(z_1,a_1)\ q(z_2,a_2)}{q(z_1,a_2)\ q(z_2,a_1)}.$$

The right-hand side of this last equation is equal to the cross-ratio function $p(z_1,z_2,a_1,a_2)$ by formula B(6.2) and that establishes the desired result for all points a_1, a_2 outside the divisor (2); that result then holds by the identity theorem for meromorphic functions.

Corollary 1. For any fixed point $t \in \underline{\underline{\theta}} \sim \underline{\underline{\theta}}^1$ the canonical meromorphic differentials of the second and third kinds can be written

$$w_a'(z) = \frac{\partial^2}{\partial z\,\partial a} \log \theta(t+v(z-a)) ,$$

$$w_{a_1,a_2}'(z) = \frac{\partial}{\partial z} \log \frac{\theta(t+v(z-a_1))}{\theta(t+v(z-a_2))}.$$

Proof. Formula (4) in the proof of the preceding theorem can be used to write $q(z_1,a_1)$ in terms of the other expressions appearing there; that leads first to the result that

$$\frac{\partial^2}{\partial z_1 \partial a_1} \log q(z_1,a_1) = \frac{\partial^2}{\partial z_1 \partial a_1} \log \theta(t+v(z_1-a_1)),$$

since these are the only terms that involve both z_1 and a_1, and by

Theorem B10 that yields the first assertion. Then from the theorem itself

$$\frac{\partial}{\partial z_1}\log p(z_1,z_2,a_1,a_2)=\frac{\partial}{\partial z_1}\log\frac{\theta(t+v(z_1-a_1))}{\theta(t+v(z_1-a_2))},$$

and in view of B(3.4) that yields the second assertion.

It should be noted that the formulas of this corollary must be interpreted as identities between meromorphic functions of the variables z,a,a_1,a_2; for some fixed values of a the function $\theta(t+v(z-a))$ vanishes identically in z and the logarithmic terms are really undefined, but the formula remains true by analytic continuation. It is interesting to note that the formulas of this corollary can be rewritten equivalently in the form

(5) $$\theta(t+v(z-a)]^2\, v'_a(z)$$
$$=\sum_{j,k}[\partial_j\theta(t+v(z-a))\partial_k\theta(t+v(z-a))-\theta(t+v(z-a)\partial_{jk}\theta(t+v(z-a))]\, v'_j(z)\, v'_k(z)$$

and

(6) $$v'_{a_1,a_2}(z)=\sum_j[\theta(t+v(z-a_1)]^{-1}\partial_j\theta(t+v(z-a_1)) - \theta(t+v(z-a_2)]^{-1}\partial_j\theta(t+v(z-a_2))]\, v'_j(z)\ .$$

Furthermore from the first formula of the corollary it is clear that the difference

$$v_a(z)-\frac{\partial}{\partial a}\log\theta(t+v(z-a))=c_t(a)$$

is independent of the variable z; thus

$$\frac{\partial}{\partial a}\log\theta(t+v(z-a))=v_a(z)-c_t(a)$$

is really just another normalization of the canonical meromorphic Abelian integral of the second kind. This observation can be rewritten by explicitly evaluating the expression $c_t(a)$ as follows.

Corollary 2. For any fixed point $t \in \underline{\Theta} - \underline{\Theta}^1$ the canonical meromorphic integral of the second kind can be written

$$w_a(z) = \frac{\partial}{\partial a} \log \theta\left(t+w(z-a)\right) + \frac{1}{2} \frac{\sum_{jk} \partial_{jk}\theta(t)\, w'_j(a)\, w'_k(a) - \sum_j \partial_j \theta(t)\, w''_j(a)}{\sum_j \partial_j\theta(t)\, w'_j(a)}$$

Proof. Using the explicit expression B(6.9) for the canonical integral of the second kind shows that

$$c_t(a) = \frac{\partial}{\partial a} \log \frac{q(z,a)}{\theta\left(t+w(z-a)\right)}$$

$$= \frac{q(z,a)}{\theta\left(t+w(z-a)\right)} \cdot \frac{\theta\left(t+w(z-a)\right)\frac{\partial}{\partial a}q(z,a) + q(z,a) \sum_j \partial_j \theta\left(t+w(z-a)\right) w'_j(a)}{q(z,a)^2} .$$

This expression is independent of z, and can be evaluated explicitly by taking the limit as z tends to a. The functions $q(z,a)$ and $\theta\left(t+w(z-a)\right)$ both have simple zeros at $z=a$, and the first factor above clearly tends to $\left[\sum \partial_j \theta(t)\, w'_j(a)\right]^{-1} = w'_t(a)^{-1}$; the second factor is possibly most easily evaluated by calculating the Taylor expansions of the separate terms in powers of the canonical coordinate $z-a$, and since

$$\theta\left(t+w(z-a)\right) = w'_t(a)\ (z-a) + 1/2 \left[\sum_{jk} \partial_{jk}\theta(t) w'_j(a) w'_k(a) + \sum_j \partial_j \theta(t) w''_j(a)\right](z-a)^2 + O(z-a)^3$$

and

$$q(z,a) = (z-a) + O(z-a)^3$$

the asserted result follows from a straightforward calculation.

Now in a slightly different direction, for any points $t \in \underline{\Theta}$ and $a,b \in \tilde{M}$ the product of the semicanonical functions $q(z,a)^{-1}\theta\left(t+w(z-a)\right) \in \Gamma(\rho_{-t}\sigma)$ and $q(z,b)^{-1}\theta\left(-t+w(z-b)\right) \in \Gamma(\rho_t\sigma)$ will be a holomorphic function belonging to the space $\Gamma(\rho_{-t}\sigma \cdot \rho_t\sigma) = \Gamma(\sigma^2) = \Gamma(\kappa)$, hence will be an ordinary Abelian differential. Just what this Abelian differential is can quite easily be determined as follows.

<u>Theorem</u> 7. For any points $t \in \underline{\underline{\Theta}} \subseteq \mathbb{C}^g$ and $a,b \in \tilde{M}$,

$$q(z,a)^{-1}q(z,b)^{-1}\ \theta[t+w(z-a)]\ \theta[-t+w(z-b)] = q(a,b)^{-1}\ \theta[-t+w(a-b)]\ w'_t(z)$$

where as before

$$w'_t(z) = \sum_{j=1}^{g} \partial_j\theta(t)\ w'_j(z).$$

<u>Proof</u> Consider a fixed point $t \in \underline{\underline{\Theta}} - \underline{\Theta}^1 \subseteq \mathbb{C}^g$, and recall that the divisors of the semicanonical functions $q(z,a)^{-1}\theta[t+w(z-a)]$ and $q(z,b)^{-1}\theta[-t+w(z-b)]$ are independent of a and b; the product of these functions will then be a differential form with a divisor that is independent of a and b. Choose a differential form $\sum_j c_j w'_j(z)$ with this divisor, and note that

$$(7)\qquad q(z,a)^{-1}q(z,b)^{-1}\ \theta[t+w(z-a)]\ \theta[-t+w(z-b)] = f(a,b)\ \sum_j c_j w'_j(z)$$

for a well defined holomorphic function $f(a,b)$ on $\tilde{M}$. It is clear that $f(a,b) \in \Gamma(\rho_t\sigma)$ as a function of a, while $f(a,b) \in \Gamma(\rho_{-t}\sigma)$ as a function of b; since $q(a,b)^{-1}\ \theta[-t+w(a-b)]$ has the same properties and $\gamma(\rho_t\sigma) = 1$ it must be the case that

$$f(a,b) = c\ q(a,b)^{-1}\ \theta[-t+w(a-b)]$$

where c is independent of a and b. After replacing the coefficients c_j of the differential form by cc_j, the identity (7) can be rewritten

$$q(z,a)^{-1}q(z,b)^{-1}\theta[t+w(z-a)]\theta[-t+w(z-b)] = q(a,b)^{-1}\theta[-t+w(a-b)]\ \sum_j c_j w'_j(z).$$

Letting b approach a here and observing that

$$\lim_{b\to a} q(a,b)^{-1}\ \theta[-t+w(a-b)] = -\sum_k \partial_k\theta(t)\ w'_k(a)$$

yields the result that

$$q(z,a)^{-2}\ \theta[t+w(z-a)]\ \theta[-t+w(z-a)]$$
$$= -\ \{\sum_j c_j\ w'_j(z)\}\ \{\sum_k \partial_k\theta(t)\ w'_k(a)\}.$$

Since the left-hand side is clearly symmetric in z and a the same must be true for the right-hand side, so that

$$\textstyle\sum_j c_j\, w'_j(z) = c_o \sum_j \partial_j \theta(t)\, w'_j(z)$$

for some constant c_o, and consequently

$$q(z,a)^{-2}\, \theta(t+w(z-a))\, \theta(-t+w(z-a))$$
$$= -c_o \left(\textstyle\sum_j \partial_j \theta(t)\, w'_j(z)\right) \left(\sum_k \partial_k \theta(t)\, w'_k(a)\right).$$

Letting z approach a here and observing that

$$\lim_{z\to a} q(z,a)^{-1}\, \theta(t+w(z-a)) \cdot q(z,a)^{-1}\, \theta(-t+w(z-a))$$
$$= -\left(\textstyle\sum_j \partial_j \theta(t)\, w'_j(a)\right) \left(\sum_k \partial_k \theta(t)\, w'_k(a)\right)$$

shows that $c_o = 1$ and thereby concludes the proof of the theorem.

<u>Corollary</u> 1. For any points $t \in \underline{\underline{\Theta}} \subset \mathbb{C}^g$ and $a \in \tilde{M}$,

$$q(z,a)^{-2}\, \theta(t+w(z-a))\, \theta(-t+w(z-a)) = -w'_t(z)\, w'_t(a).$$

<u>Proof</u> This follows immediately from the preceding theorem upon taking the limit as b approaches a, with the calculation as in the proof of that theorem.

This result is particularly symmetrical when t is an odd half period, that is, when $2t = p + \Omega q \in \underline{L}$ where ${}^t q \cdot p$ is an odd integer. By Theorem A4 the function $\theta[0|t](w)$ is then an odd function of the variable w, so that $0 = \theta[0|t](0) = \theta(0)$ and necessarily $t \in \underline{\underline{\Theta}}$.

<u>Corollary</u> 2. If $t \in \mathbb{C}^g$ is an odd half-period then $t \in \underline{\underline{\Theta}}$ and

$$q(z,a)^2\, w'_t(z)\, w'_t(a) = e^{\pi i\, {}^t q \cdot w(z-a)} \left[\theta(t+w(z-a))\right]^2$$

where $2t = p + \Omega q \in \underline{L}$.

Proof If $2t = p + \Omega q \in L$ then from the functional equation of Theorem A1 it follows that

$$\theta(-t+v(z-a)) = \theta(t+v(z-a) - 2t)$$
$$= \theta(t+v(z-a)) \exp 2\pi i\, {}^t q \cdot (t+v(z-a) - \tfrac{1}{2}\Omega q)$$
$$= \theta(t+v(z-a)) \exp 2\pi i\, {}^t q \cdot (v(z-a) + \tfrac{1}{2} p),$$

and if ${}^t q \cdot p$ is odd then $\exp \pi i\, {}^t q \cdot p = -1$. Substituting this into the preceding corollary yields the desired result.

It t is an odd half period and $t \notin \mathcal{G}^1$ then $v'_t(z)$ is a nontrivial function in $\Gamma(\kappa)$, and the preceding corollary shows that all of its zeros are of even order; it is therefore possible to choose a single-valued branch of $\sqrt{v'_t(z)}$ over $\tilde{M}$, yielding a well defined holomorphic function on $\tilde{M}$ with a Γ-invariant divisor. Actually if a_o is a point of $\tilde{M}$ at which $v'_t(a_o) \neq 0$ then for any choice of $\sqrt{v'_t(a_o)}$ it follows from Corollary 2 that it is possible to take

$$(8) \qquad \sqrt{v'_t(z)} = \frac{1}{\sqrt{v'_t(a_o)}}\, e^{\pi i\, {}^t q \cdot v(z-a_o)}\, q(z,a_o)^{-1}\, \theta(t+v(z-a_o));$$

the two choices of $\sqrt{v'_t(a_o)}$ determine the two possible choices of $\sqrt{v'_t(z)}$. The function $f(z) = \exp \pi i\, {}^t q \cdot v(z)$ satisfies

$$f(A_j z) = e^{\pi i q_j} f(z)$$
$$f(B_j z) = e^{\pi i\, {}^t \delta_j \Omega q} f(z) = e^{\pi i (2t_j - p_j)} f(z),$$

hence with the notation as in A(3.3) can be viewed as a function

$$f \in \Gamma(\sigma_{\frac{1}{2}q}\, \rho_{t-\frac{1}{2}p});$$

consequently

$$\sqrt{v'_t(z)} \in \Gamma(\sigma_{\frac{1}{2}q}\, \rho_{-\frac{1}{2}p}\, \sigma),$$

where $\sigma_s \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ must be distinguished from the semicanonical factor of automorphy $\sigma = \rho_T\, t^{g-1}$. Thus $w'_t(z)$ transforms by a factor of automorphy of the form $\pm\sigma(T,Z)$ for a suitable choice of sign. Note that Corollary 2 readily yields the formula

$$q(z,a)\sqrt{w'_t(z)}\,\sqrt{w'_a(z)} = \pm\, e^{\pi i\, {}^t q\cdot w(z-a)}\,\theta(t+w(z-a))$$

for some choice of sign, a choice that is uniquely determined whenever $\sqrt{w'_t(z)}$ and $\sqrt{w'_t(a)}$ are the same square root function evaluated at the points z and a, and that can be viewed as yielding an explicit expression for the prime function. The choice of sign is readily determined, as follows.

Corollary 3. If $t \in \mathbb{C}^g$ is an odd half-period and $t \notin \underline{\underline{\theta}}^1$ then for any choice of a branch of the function $\sqrt{w'_t(z)}$

$$q(z,a) = e^{\pi i\, {}^t q\cdot w(z-a)}\,\theta(t+w(z-a)) \Big/ \sqrt{w'_t(z)}\,\sqrt{w'_t(a)}$$

where $2t = p + \Omega q \in \underline{L}$.

Proof. It follows from (8) that

$$\sqrt{w'_t(z)}\,\sqrt{w'_t(a)} = w'_t(a_o)^{-1}\, q(z,a_o)^{-1}\, q(a,a_o)^{-1}\, e^{\pi i\, {}^t q\cdot w(z+a-2a_o)}$$
$$\theta(t+w(z-a_o))\,\theta(t+w(a-a_o)).$$

Using the argument as in the proof of Corollary 2 and then the result of Theorem 7 shows that

$$\theta(t+w(z-a_o))\,\theta(t+w(a-a_o)) = \theta(-t+w(a_o-z))\,\theta(-t+w(a_o-a))$$
$$= \theta(-t+w(a_o-z))\,\theta(t+w(a_o-a))\, e^{\pi i\, {}^t q[w(2z-2a)+p]}$$
$$= q(a_o,a)\, q(a_o,z)\, q(a,z)^{-1}\,\theta(-t+w(a-z))\, w'_t(a_o)\, e^{\pi i\, {}^t q[w(2a_o-2a)+p]}.$$

Substituting this into the preceding formula then yields the desired result since $\exp \pi i\, {}^t q\cdot p = -1$ for the odd half-period t.

If it is demonstrated that there always exist nonsingular odd half periods then the formula of the preceding corollary can be used as an explicit expression for the prime function. It should be noted though that the prime function is implicitly invoked in that formula to the extent that the derivatives $w'_t(z)$ are taken with respect to the canonical coordinates. The result is that this does not really determine the normalization of the prime function, which is still only settled up to a factor of a $\binom{g}{2}$- th root of unity. On the other hand Klein's prime form can be written correspondingly in the form

$$E(z,a) = e^{\pi i\, {}^t q\cdot w(z-a)}\,\theta(t+w(z-a))\Big/\sqrt{\omega_t(z)}\ \ \sqrt{\omega_t(z)}$$

where $\sqrt{\omega_t(z)} = \sqrt{\omega'_t(z)}\ \sqrt{dz}$ is at least to some extent an intrinsically defined quantity, a half-order differential.

The preceding results arose from a consideration of the product $\theta(t+w(z-a))\theta(t-w(z-a))$, and something can also be said about the quotient $f(z,a) = \theta(t-w(z-a))/\theta(t+w(z-a))$. If $t \in \underline{\underline{\Theta}} \sim \underline{\underline{\Theta}}^1$ then

$$r-t = w(x_1+\ldots+x_{g-1}) \quad\text{and}\quad r+t = w(y_1+\ldots+y_{g-1})$$

for some uniquely determined divisors $x_1+\ldots+x_{g-1}$ and $y_1+\ldots+y_{g-1}$ on M . It is clear from the discussion in the first part of this section that for a general value of a the function $f(z,a)$ is a well defined meromorphic relatively automorphic function for the factor of the factor of automorphy ρ_{2t} with divisor $y_1+\ldots+y_{g-1}-x_1-\ldots-x_{g-1}$, and since clearly $f(z,a) = 1/f(a,z)$ this function has the corresponding properties as a function of a for a general fixed value of z . To be more precise note from the Corollary to Theorem 3 and Corollary 1 to Theorem 4 that

$$(9) \qquad \theta(t+w(z-a)) = Q(a,x_1,\dots,x_{g-1})q(z,a)\prod_{i=1}^{g-1} q(z,x_i)$$

$$= w_t'(a)q(z,a)\prod_{i=1}^{g-1}\frac{q(z,x_i)}{q(a,x_i)} .$$

Replacing t by $-t$ has the effect of interchanging the divisors $x_1+\dots+x_{g-1}$ and $y_1+\dots+y_{g-1}$, so since $w_{-t}'(a) = -w_t'(a)$ it follows that

$$(10) \qquad \theta(-t+w(z-a)) = -w_t'(a)q(z,a)\prod_{i=1}^{g-1}\frac{q(z,y_i)}{q(a,y_i)} .$$

From these two formulas it follows immediately that

$$(11) \quad \frac{\theta(t-w(z-a))}{\theta(t+w(z-a))} = -\prod_{i=1}^{g-1}\frac{q(z,y_i)\ q(a,x_i)}{q(z,x_i)\ q(a,y_i)} = -\prod_{i=1}^{g-1} p(z,a,y_i,x_i) ,$$

since the cross-ratio function satisfies B(6.2). On the other hand interchanging z and a in the first formula yields

$$(12) \qquad \theta(t-w(z-a)) = w_t'(z)q(a,z)\prod_{i=1}^{g-1}\frac{q(a,x_i)}{q(z,x_i)} ,$$

so that alternatively

$$(13) \qquad \frac{\theta(t-w(z-a))}{\theta(t+w(z-a))} = -\frac{w_t'(z)}{w_t'(a)}\prod_{i=1}^{g-1}\frac{q(a,x_i)^2}{q(z,x_i)^2} .$$

In both cases the quotient is written as the quotient of a function of z by the same function of a, aside from the sign.

§5. Prime function expansion for theta derivatives.

There are various possible extensions of the prime function expansion discussed in section 2, one of which is an analogue of the formula of Theorem 3 and its corollary but for the first partial derivatives of the theta function rather than for the theta function itself. While this expansion is just a simple consequence of that of Theorem 3, the explicit formulas are necessarily slightly more complicated; so it may be clearer first to discuss the general form of such an expansion, and then to deduce the expansion quite independently.

For this purpose consider again the functional equation (3.1) satisfied by the first partial derivatives of the theta function, and note that after multiplying by $\theta(t+\lambda)^{-1} = \xi(\lambda,t)^{-1}\ \theta(t)^{-1}$ that equation can be written

(1) $\theta(t+\lambda)^{-1}\ \partial_j\,\theta(t+\lambda) = \theta(t)^{-1}\partial_j\,\theta(t) + \dfrac{\partial}{\partial t_j}\log\ \xi(\lambda,t).$

For a lattice vector $\lambda = p+\Omega\, q \in \underline{\underline{L}}$ it is clear from A(3.6) that

(2) $\dfrac{\partial}{\partial t_j}\log\ \xi(\lambda,t) = -2\pi q_j = -2\pi i\ \beta_j(\lambda),$

where $\beta_j \in \mathrm{Hom}\,(\underline{\underline{L}},\mathbb{Z})$ is naturally induced from the homomorphism $\beta_j \in \mathrm{Hom}\,(\Gamma,\mathbb{Z})$ of B(7.4) when $\underline{\underline{L}}$ is viewed as the abelianization of Γ, and in these terms (1) can be rewritten

(3) $\theta(t+\lambda)^{-1}\partial_j\,\theta(t+\lambda) = \theta(t)^{-1}\partial_j\,\theta(t) - 2\pi i\,\beta_j(\lambda).$

The meromorphic differential form $\theta(t)^{-1}\partial_j\,\theta(t)$ on $\mathbb{C}^g$ thus transforms as an Abelian integral on $J = \mathbb{C}^g/\underline{\underline{L}}$ under the action of $\underline{\underline{L}}$; its restriction to any translate of the Riemann surface M imbedded in J if well defined must therefore be a meromorphic Abelian integral on M. As before the most natural restriction to consider is as in $\theta\big(r-t+w(z)\big)$. The Corollary to

Theorem 1 shows that $\theta(r-t+w(z)) \equiv 0$ in z precisely when $t \in W_g^1$; so whenever $t \notin W_g^1$ the expression $\theta(r-t+w(z))^{-1}\,\partial_j\theta(r-t+w(z))$ is a well defined meromorphic Abelian integral on M. Moreover whenever $t \notin W_g^1$ then $t = w(z_1+\dots+z_g)$ for a unique divisor $z_1+\dots+z_g$ on M, and this is precisely the divisor of the function $\theta(r-t+w(z))$; if the points $z_1,\dots,z_g$ on M are distinct the meromorphic Abelian integral of interest must have as singularities at most simple poles at these g points on M. There must thus be an expansion of the general form

(4) $$\theta(r+w(z-z_1-\dots-z_g))^{-1}\,\partial_j\theta(r+w(z-z_1-\dots-z_g))$$
$$= \sum_{k=1}^{g} f_{jk}(z_1,\dots,z_g)\, w_{z_k}(z) + \sum_{k=1}^{g} g_{jk}(z_1,\dots,z_g)\, w_k(z) + c_j(z_1,\dots,z_g)$$

where f_{jk}, g_{jk}, c_j are well defined meromorphic functions on $\tilde{M}^g$ with singularities at most at those points $(z_1,\dots,z_g) \in \tilde{M}^g$ for which either $w(z_1+\dots+z_g) \in W_g^1$ or $z_\ell = Tz_m$ for some $T \in \Gamma$ and $\ell \neq m$. There are various ways of evaluating these coefficients, leading to the following result.

<u>Theorem 8</u>. For any points $z,z_1,\dots,z_g \in \tilde{M}$

$$\theta(r+w(z-z_1-\dots-z_g))^{-1}\,\partial_j\theta(r+w(z-z_1-\dots-z_g))$$

$$= \quad -\sum_{k=1}^{g} w_{z_k}(z)\,\frac{Q_j(z_1,\dots,z_{k-1},z_{k+1},\dots,z_g)}{Q(z_1,\dots,z_g)}\prod_{\substack{i=1\\ i\neq k}}^{g} q(z_k,z_i)^{-1}$$

$$-\sum_{k=1}^{g}\frac{Q_j(z_1,\dots,z_{k-1},z_{k+1},\dots,z_g)}{Q(z_1,\dots,z_g)^2}\,\frac{\partial Q(z_1,\dots,z_g)}{\partial z_k}\prod_{\substack{i=1\\ i\neq k}}^{g} q(z_k,z_i)^{-1}$$

where $Q(z_1,\dots,z_g)$ and $Q_j(z_1,\dots,z_{g-1})$ are the holomorphic functions on $\tilde{M}^{(g)}$ and $\tilde{M}^{(g-1)}$ described in the Corollary to Theorems 3 and 4.

Proof. The Corollary to Theorem 3 asserts that

$$\theta(r+w(z-z_1-\dots-z_g)) = Q(z_1,\dots,z_g)\prod_{i=1}^{g} q(z,z_i),$$

and upon taking the logarithmic derivative of this formula with respect to the variable z_k and recalling Theorem B11 it follows readily that

$$-\theta(r+w(z-z_1-\dots-z_g))^{-1}\sum_{\ell}\partial_\ell\theta(r+w(z-z_1-\dots-z_g))\,w'_\ell(z_k)$$
$$= w_{z_k}(z) + \frac{\partial}{\partial z_k}\log Q(z_1,\dots,z_g).$$

Multiply this by $Q_j(z_1,\dots,z_{k-1},z_{k+1},\dots,z_g)\prod_{\substack{i=1\\ i\neq k}}^{g} q(z_k,z_i)^{-1}$ and sum the resulting formula over all values $1 \leq k \leq g$; it then follows readily from (2.5) that

$$-Q(z_1,\dots,z_g)\ \theta(r+w(z-z_1-\dots-z_g))^{-1}\partial_j\theta(r+w(z-z_1-\dots-z_g))$$
$$= \sum_{k=1}^{g}\left[w_{z_k}(z) + \frac{\partial}{\partial z_k}\log Q(z_1,\dots,z_g)\right] Q_j(z_1,\dots,z_{k-1},z_{k+1}\dots,z_g)\sum_{\substack{i=1\\ i\neq k}}^{g} q(z_k,z_i)^{-1},$$

from which the desired result is an immediate consequence.

The formulas as given in the preceding theorem exhibit quite explicitly the extent to which the coefficients become singular either when $w(z_1+\dots+z_g)\in W_g^1$, in which case $Q(z_1,\dots,z_g)=0$, or when $z_\ell = Tz_m$, in which case $q(z_\ell,z_m)=0$. It is interesting to note that the coefficients g_{jk} in (4) must all vanish identically, since no ordinary Abelian integrals actually appear in the expansion formula of the theorem. That can also be seen quite directly, by examining the periods of all the Abelian integrals appearing in the expansion (4) under the transformations $A_1,\dots,A_g$; for the left-hand side of (4) has all these periods zero by (3), the canonical Abelian integrals $w_{z_k}(z)$ have all these periods zero from the normalization chosen, but no nontrivial linear combination of ordinary Abelian integrals $w_k(z)$ can have all these periods zero. After this observation has been made, a similar argument can be used to determine the coefficients f_{jk} directly. The period of the left-hand side of (4) under the transformation B_ℓ is give in (3), while the periods of the canonical Abelian integrals $w_{z_k}(z)$ were determined in Theorem B11, and it follows readily that

$$(5) \quad -\delta^j_\ell = \sum_k f_{jk}(z_1,\ldots,z_g)\; w'_\ell(z_k).$$

This system of linear equations in the unknown functions $f_{jk}(z_1,\ldots,z_g)$ can be solved explicitly by using (2.4), to yield the same expressions for these coefficients as can be read from the formula of the theorem. Alternatively these coefficients can be determed directly merely by examining the residues of the meromorphic functions appearing in (4) at the various poles z_k; for upon multiplying (4) by $q(z,z_k)$ and taking the limit as z tends to z_k it follows easily that

$$(6) \quad \frac{\partial_j \theta(r-w(z_1+\cdots+z_{k-1}+z_{k+1}+\cdots+z_g))}{\sum_\ell \partial_\ell \theta(r-w(z_1+\cdots+z_{k-1}+z_{k+1}+\cdots+z_g))\, w'_\ell(z_k)} = -f_{jk}(z_1,\ldots,z_g).$$

That this is the same expression for these coefficients as given in the formula of the theorem is evident from the Corollaries to Theorem 4. All the terms in the expansion (4) except for the constant term $c_j(z_1,\ldots,z_g)$ have thus been calculated directly in another way; but for the constant term the easiest approach seems to be through the expansion formula of Theorem 3 as in the preceding proof.

For some purposes it is convenient to eliminate the singularities appearing in the formula of the preceding theorem and to rewrite that formula as follows.

Corollary. For any points $z, z_1, \ldots, z_g \in \tilde{M}$

$$\partial_j \, \theta(r + w(z - z_1 - \cdots - z_g)) = \sum_{k=1}^{g} Q_j(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_g)$$

$$= \left[\prod_{\substack{i=1 \\ i \neq k}}^{g} \frac{q(z, z_i)}{q(z_k, z_i)} \right] q(z, z_k) \left[w_{z_k}(z) + \frac{\partial}{\partial z_k} \log Q(z_1, \ldots, z_g) \right] .$$

Proof. This follows immediately from the formula of the preceding theorem upon multiplying throughout by

$$\theta(r + w(z - z_1 - \cdots - z_g)) = Q(z_1, \ldots, z_g) \prod_{i=1}^{g} q(z, z_i)$$

and regrouping the terms.

The preceding theorem and its corollary provide a prime function expansion for the first partial derivatives of the theta function analogous in some ways to the expansion of the theta function itself given in Theorem 3 and its corollary, although somewhat more complicated. From this result it is possible to obtain some useful information about the second partial derivatives, in much the same way that Theorem 4 and its corollaries were derived from Theorem 3.

Theorem 9. For any points $z, z_1, \ldots, z_{g-1} \in \tilde{M}$,

$$\sum_{k=1}^{g} \partial_{ik} \theta(r - w(z_1 + \cdots + z_{g-1})) \, w'_k(z) =$$

$$= \sum_{k=1}^{g-1} Q_i(z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_{g-1}, z) \left[\prod_{\substack{\ell=1 \\ \ell \neq k}}^{g-1} \frac{q(z, z_\ell)}{q(z_k, z_\ell)} \right] \cdot$$

$$\cdot \left[w_{z_k}(z) + \frac{\partial}{\partial z_k} \log Q(z_1, \ldots, z_{g-1}) \right]$$

$$+ Q_i(z_1, \ldots, z_{g-1}) \left[\sum_{\ell=1}^{g-1} w_z(z_\ell) - \frac{\partial}{\partial z} \log Q(z_1, \ldots, z_{g-1}, z) \right]$$

where $Q(z_1, \ldots, z_g)$ and $Q_i(z_1, \ldots, z_{g-1})$ are the auxiliary functions already considered.

Proof. Differentiate the formula of the preceding corollary with respect to z and set $z_g = z$. The left-hand side clearly becomes the left-hand side of the desired formula. On the right-hand side for $1 \le k \le g-1$ the corresponding term is the product of $g(z,z_g)$ and a function that is holomorphic in z and z_g; the only contribution to the final result arises from differentiating the factor $q(z,z_g)$, since $q(z,z) = 0$ while $\partial_1 q(z,z) = 1$, and that readily yields the first terms on the right-hand side of the desired formula. On the other hand for $k = g$ the term has the form

$$-Q_i(z_1,\ldots,z_{g-1}) \left[\prod_{\ell=1}^{g-1} \frac{q(z,z_\ell)}{q(z_g,z_\ell)}\right] \left[q(z,z_g)\; w_{z_g}(z) + q(z,z_g)\frac{\partial}{\partial z_g} \log Q(z_1,\ldots,z_g)\right].$$

Here

$$q(z,z_g)\; w_{z_g}(z) = q(z,z_g)\frac{\partial}{\partial z_g} \log q(z,z_g) = \frac{\partial}{\partial z_g} q(z,z_g)$$

$$= -1 - 3\,(z-z_g)^2\, q_3(z,z_g) + (z-z_g)^3\, \partial_2 q_3(z,z_g),$$

in view of the expansion B(6.9) of the prime function in terms of canonical coordinates on $\tilde{M}$; thus the zero of the factor $q(z,z_g)$ is cancelled by the simple pole of the function $w_{z_g}(z)$ at $z = z_g$, and the product takes the value -1 at $z = z_g$, but the derivative with respect to z vanishes at $z = z_g$. Therefore for the first part of this term the contribution to the final result arises from differentiating the remaining factors, and

$$\frac{\partial}{\partial z} \prod_{\ell=1}^{g-1} \frac{q(z,z_\ell)}{q(z_g,z_\ell)} = \left[\prod_{\ell=1}^{g-1} \frac{q(z,z_\ell)}{q(z_g,z_\ell)}\right] \frac{\partial}{\partial z} \log \prod_{\ell=1}^{g-1} \frac{q(z,z_\ell)}{q(z_g,z_\ell)}$$

$$= \left[\prod_{\ell=1}^{g-1} \frac{q(z,z_\ell)}{q(z_g,z_\ell)}\right] \left[\sum_{\ell=1}^{g-1} w_z(z_\ell)\right]$$

by Theorem B11. For the second part of this term the contribution to the final result arises from differentiating the factor $q(z,z_g)$, and with that the desired formula arises as stated.

Corollary 1. For any points $z,z_1,\dots,z_{g-1} \in \tilde{M}$ let $t = r-w(z_1+\dots+z_g) \in \underline{\underline{\Theta}}$. Then

$$\sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j(z)\, w'_k(z) =$$

$$= -\left[\prod_{\ell=1}^{g-1} q(z,z_\ell)^2\right] \frac{\partial}{\partial z}\left[Q(z,z_1,\dots,z_{g-1}) \prod_{\ell=1}^{g-1} q(z,z_\ell)^{-1}\right]$$

$$= -\left[\prod_{\ell=1}^{g-1} q(z,z_\ell)^2\right] \frac{\partial}{\partial z}\left[w'_t(z) \prod_{\ell=1}^{g-1} q(z,z_\ell)^{-2}\right]$$

$$= -w_t''(z) + 2\, w_t'(z) \sum_{\ell=1}^{g-1} w_z(z_\ell),$$

where $w_t'(z)$ is the differential form of Corollary 1 to Theorem 4.

Proof. To obtain the first of these equalities, multiply the formula of the theorm by $w'_i(z)$ and sum the result over the range $1 \leqq i \leqq g$. Note from (2.4) that for $1 \leqq k \leqq g-1$

$$\sum_{i=1}^{g} Q_i(z_1,\dots,z_{k-1},z_{k+1},\dots,z_{g-1},z)\, w'_i(z)$$

$$= Q(z_1,\dots,z_{k-1},z_{k+1},\dots,z_{g-1},z,z)\left[\prod_{\substack{\ell=1\\ \ell\neq k}}^{g-1} q(z,z_\ell)\right] q(z,z)$$

$$= 0$$

since $q(z,z) = 0$, while

$$\sum_{i=1}^{g} Q_i(z_1,\dots,z_{g-1})\, w'_i(z) = Q(z_1,\dots,z_{g-1},z) \prod_{\ell=1}^{g-1} q(z,z_\ell);$$

therefore

$$\sum_{i,k=1}^{g} \partial_{ik}\theta(t)\, w'_i(z)\, w'_k(z)$$

$$= Q(z_1,\ldots,z_{g-1},z)\left[\prod_{\ell=1}^{g-1} q(z,z_\ell)\right]\left[\sum_{\ell=1}^{g-1} w_z(z_\ell) - \frac{\partial}{\partial z}\log Q(z_1,\ldots,z_{g-1},z)\right]$$

$$= Q(z_1,\ldots,z_{g-1},z)\left[\prod_{\ell=1}^{g-1} q(z,z_\ell)\right] \cdot$$

$$\cdot \frac{\partial}{\partial z}\left[\log \prod_{\ell=1}^{g-1} q(z,z_\ell) - \log Q(z_1,\ldots,z_{g-1},z)\right],$$

which evidently yields the first equality. It follows from Corollary 1 to Theorem 4 that

$$Q(z,z_1,\ldots,z_{g-1}) \cdot \prod_{\ell=1}^{g-1} q(z,z_\ell)^{-1} = w'_t(z) \cdot \prod_{\ell=1}^{g-1} q(z,z_\ell)^{-2},$$

which immediately yields the second equality, and

$$\frac{\partial}{\partial z}\, w'_t\,(z) \cdot \prod_{\ell=1}^{g-1} q(z,z_\ell)^{-2}$$

$$= \left[\prod_{\ell=1}^{g-1} q(z,z_\ell)^{-2}\right]\left[\frac{\partial}{\partial z}\, w'_t(z) + w'_t(z)\frac{\partial}{\partial z}\log \prod_{\ell=1}^{g-1} q(z,z_\ell)^{-2}\right]$$

then clearly yields the third equality and concludes the proof.

<u>Corollary 2</u>. For any points $z,z_1,\ldots,z_{g-1} \in \tilde{M}$ let $t = r - w(z_1+\ldots+z_{g-1})$. Then for $1 \leq i \leq g-1$

$$\sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j(z_i)\, w'_k(z)$$

$$= -\frac{\partial}{\partial z_i}\left[Q(z_1,\ldots,z_{g-1},z)\prod_{\ell=1}^{g-1} q(z,z_\ell)\right].$$

Proof. In this case, multiply the formula of the theorem by $w'_i(z_m)$ and sum the result over the range $1 \leqq i \leqq g$. Note from (2.4) again that for $1 \leqq k \leqq g-1$

$$\sum_{i=1}^{g} Q_i(z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_{g-1},z) \quad w'_i(z_m)$$
$$= Q(z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_{g-1},z,z_m) \left[\prod_{\substack{\ell=1 \\ \ell\neq k}}^{g-1} q(z_m,z_\ell)\right] \quad q(z_m,z)$$

$$= -\delta_k^m \, Q(z_1,\ldots,z_{g-1},z) \; q(z,z_k) \left[\prod_{\substack{\ell=1 \\ \ell\neq k}}^{g-1} q(z_k,z_\ell)\right]$$

while

$$\sum_{i=1}^{g} Q_i(z_1,\ldots,z_{g-1}) \; w'_i(z_m)$$
$$= Q(z_1,\ldots,z_{g-1},z_m) \prod_{\ell=1}^{g-1} q(z_m,z_\ell) = 0;$$

therefore

$$\sum_{i,k=1}^{g} \partial_{ik}\theta(t) \; w'_i(z_m) =$$
$$= - Q(z_1,\ldots,z_{g-1},z)\left[\prod_{\ell=1}^{g-1} q(z,z_\ell)\right] \left[w_{z_m}(z) + \frac{\partial}{\partial z_m} \log Q(z_1,\ldots,z_{g-1},z)\right]$$
$$= - Q(z_1,\ldots,z_{g-1},z) \left[\prod_{\ell=1}^{g-1} q(z,z_\ell)\right] \frac{\partial}{\partial z_m} \log \left[q(z,z_m) \; Q(z_1,\ldots,z_{g-1},z)\right]$$

from which the desired result follows immediately.

It should be noted here that by using the formula of Corollary 1 to Theorem 4 the preceding result can be rewritten

$$(7) \qquad \sum_{j,k=1}^{g} \partial_{jk}\theta(t) \; w'_j(z_i) \; w'_k(z)$$
$$= - \frac{\partial}{\partial z_i} \; w'_t(z);$$

actually since by that same corollary

$$w'_t(z) = \sum_{k=1}^{g} \partial_k \theta(r-w(z_1 +\dots+ z_{g-1}))\, w'_k(z)$$

the preceding result in the form (7) can be derived alternatively merely by differentiating this expression for the function $w'_t(z)$ with respect to the variable z_i.

Corollary 3. For any points $z_1,\dots,z_{g-1} \in \tilde{M}$ let $t=r - w(z_1 +\dots+ z_{g-1})$. Then for $1 \leqq i_1, i_2 \leqq g-1$

$$\sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j(z_{i_1})\, w'_k(z_{i_2})$$

$$= \delta_{i_2}^{i_1} Q(z_1,\dots,z_{g-1},z_{i_1}) \prod_{\substack{\ell=1\\ \ell\neq i_1}}^{g-1} q(z_{i_1},z_\ell)$$

$$= \delta_{i_2}^{i_1} \sum_{j=1}^{g} Q_j(z_1,\dots,z_{g-1})\, w''_j(z_{i_1})$$

$$= \delta_{i_2}^{i_1} w''_t(z_{i_1}).$$

Proof. Taking the limit as z approaches z_m of the formula of the preceding corollary shows that

$$\sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j(z_i)\, w'_k(z_m)$$

$$= -\lim_{z\to z_m} \frac{\partial}{\partial z_i}\left[Q(z_1,\dots,z_{g-1},z)\cdot q(z,z_m)\cdot \prod_{\substack{\ell=1\\ \ell\neq m}}^{g-1} q(z,z_\ell)\right]$$

$$= \delta_m^i Q(z_1,\dots,z_{g-1},z_m) \prod_{\substack{\ell=1\\ \ell\neq m}}^{g-1} q(z_m,z_\ell).$$

For the special case that i = m this can also be derived by taking the limit as z approaches z_m of the formula of Corollary 1. This is the first desired equality, and the remainder follows by obtaining explicit expansions for the function $Q(z_1,\ldots,z_{g-1},z_m)$. Note from (2.4) that

$$0 = \sum_{j=1}^{g} Q_j(z_1,\ldots,z_{g-1})\, w'_j(z_m)$$

and hence

$$(8)\quad Q(z_1,\ldots,z_{g-1},z_m) \prod_{\substack{\ell=1\\ \ell\neq m}}^{g-1} q(z_m,z_\ell)$$

$$= \lim_{z_g\to z_m} q(z_g,z_m)^{-1}\, Q_j(z_1,\ldots,z_{g-1})\, w'_j(z_g)$$

$$= \sum_{j=1}^{g} Q_j(z_1,\ldots,z_{g-1})\, w''_j(z_m).$$

The second desired equality follows immediately from the first upon using (8), and the third follows in turn upon using the results given in the Corollaries to Theorem 4.

§6. The second-order Gauss mapping

The observations made in the preceding section can be combined to yield some interesting properties of the second derivatives of the theta functions. The first of these properties can be described quite conveniently by introducing in analogy with the differential form $w'_t(z) = \sum_{j=1}^{g} \partial_j\theta(t)\, w'_j(z)$ the double differential form

$$(1)\quad w'_t(z_1,z_2) = \sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j(z_1)\, w'_k(z_2)\ ;$$

for any fixed point $t \in \mathbb{C}^g$ this is a well defined holomorphic function of the variables $z_1, z_2 \in \tilde{M}$, is symmetric in these variables, and is a holomorphic differential form on M in each variable z_i separately for a fixed value of the other variable. Although both $w'_t(z)$ and $w'_t(z_1,z_2)$ are well defined for an arbitrary point $t \in \mathbb{C}^g$, they are particularly simple and interesting for special values of t; for instance $w'_t(z)$ was characterized alternatively for points $t \in \underline{\underline{\theta}} \subset \mathbb{C}^g$ in Corollary 1 to Theorem 4, and an analogue for the double differential $w'_t(z_1,z_2)$ is as follows.

Theorem 10. For a point $t \in \underline{\underline{\theta}} \subseteq \mathbb{C}^g$ the double differential $w'_t(z_1,z_2)$ has the following properties:

(i) $w'_t(z,z) = 0$ for all $z \in \tilde{M}$ precisely when $t \in \underline{\underline{\theta}}^1 \subset \underline{\underline{\theta}}$;

(ii) if $t \in \underline{\underline{\theta}}^1 \subset \underline{\underline{\theta}}$ then $w'_t(z,a) = 0$ for all $z \in \tilde{M}$ and some fixed $a \in \tilde{M}$ precisely when $r-w(a) \in (W^1_{g-2} + t) \cup (W^1_{g-2}-t)$.

(iii) if $t \in \underline{\underline{\theta}}^1 \subset \underline{\underline{\theta}}$ and $a \in \tilde{M}$ is a point for which $r-w(a) \notin (W^1_{g-2} + t) \cup (W^1_{g-2} -t)$ then $r-t-w(a) = w(a_1+\dots+a_{g-2})$ for a unique divisor $a_1+\dots+a_{g-2}$; moreover there is up to a constant factor a unique nontrivial Abelian differential vanishing at the divisor $2a + a_1+\dots+a_{g-2}$ and it is just $w'_t(z,a)$.

Proof. (i) If $t \in \underline{\underline{\theta}} = r-W_{g-1}$ then $r-t=w(a_1 +\dots+ a_{g-1})$ for some points $a_\ell \in \tilde{M}$, and by Corollary 1 to Theorem 9

$$(2) \quad - w'_t(z,z) = \left[\prod_{\ell=1}^{g-1} g(z,a_\ell)^2\right] \frac{\partial}{\partial z} \left[w'_t(z) \prod_{\ell=1}^{g-1} q(z,a_\ell)^{-2}\right] .$$

If $w'_t(z,z) = 0$ identically in z then clearly

$$(3) \quad w'_t(z) = c \prod_{\ell=1}^{g-1} q(z,a_\ell)^2$$

for some constant $c = c(a_1,\dots,a_{g-1})$, not involving the variable z. When z is replaced by Tz for any $T \in \Gamma$ then the left-hand side of (3) is multiplied by the canonical factor of automorphy $\rho_k(T)\ \zeta(T,z)^{2g-2}$ while the right-hand side is multiplied by the factor of automorphy $\rho_{2w(a_1+\dots+a_{g-1})}\zeta(T,z)^{2g-2} = \rho_{k-2t}(T)\ \zeta(T,z)^{2g-2}$; clearly then (3) can only hold when either $c = 0$ or $\rho_{2t}(T) = 1$ for all $T \in \Gamma$. In the first case, that in which $c = 0$, it follows from (3) that $0 = w'_t(z) = \sum_j \partial_j\theta(t)\ w'_j(z) = 0$ identically in z; so since the Abelian differentials $w'_j(z)$ are linearly independent necessarily $\partial_j\theta(t) =$ for all j, and hence $t \in \underline{\underline{\theta}}^1 \subset \underline{\underline{\theta}}$. In the second case in particular $\exp 2\pi i \cdot 2t_j = \rho_{2t}(B_j) = 1$, so that $2t = n \in \mathbb{Z}^g$. Now from the functional equation for the theta function it is evident that $\partial_j\theta(s-n) = \partial_j\theta(s)$ for all $s \in \mathbb{C}^g$, so in particular $\partial_j\theta(t) = \partial_j\theta(\frac{1}{2}n) = \partial_j\theta(\frac{1}{2}n-n) = \partial_j\theta(-\frac{1}{2}n) = \partial_j\theta(-t) = -\partial_j\theta(t)$, the theta function itself being even; thus again $\partial_j\theta(t) = 0$ for all j and hence $t \in \underline{\underline{\theta}}^1 \subset \underline{\underline{\theta}}$. This argument is really a reprise of one used earlier, for the conclusion is a consequence of the fact that the half-period t is an even half-period. Altogether these remarks show that if $t \in \underline{\theta}$ and $w'_t(z,z) = 0$ identically in z then actually $t \in \underline{\underline{\theta}}^1$. On the other hand if $t \in \theta^1$ then of course $w'_t(z) = 0$ identically in z and from (2) it follows that $w'_t(z,z) = 0$ identically in z.

(ii) If $t \in \underline{\underline{\theta}}^1 = r - W^1_{g-1}$ then for any specified point $a \in \tilde{M}$ there will be some divisor $a_1+\dots+a_{g-2}$ on $\tilde{M}$ for which $r-t = w(a+a+\dots+a_{g-2})$. By Corollary 2 to Theorem 9

$$-w'_t(z,a) = \frac{\partial}{\partial a}\ Q(a,a_1,\dots,a_{g-2},z)\ q(z,a)\prod_{\ell=1}^{g-2} q(z,a_\ell)$$

and by the Corollaries to Theorem 4 this can be rewritten

$$-w'_t(z,a) = \frac{\partial}{\partial a} \sum_{j=1}^{g} Q_j(a,a_1,\ldots,a_{g-2})\, w'_j(z).$$

Since the functions $w'_j(z)$ are linearly independent, it is clear that $w'_t(z,a)$ is identically zero in z precisely when $\partial Q_j(a,a_1,\ldots,a_{g-2})/\partial a = 0$ for $\leq j \leq g$.

To analyze this last condition, it may be clearest to consider first the case in which $a,a_1,\ldots,a_{g-2}$ represent distinct points in M; the definition in Corollary 2 to Theorem 4 then has the form

$$(4) \quad Q_j(a,a_1,\ldots,a_{g-2}) = c_j \left[\prod_{\ell=1}^{g-2} q(a,a_\ell)^{-1}\right] \det \begin{Bmatrix} w'_1(a) & w'_1(a_1) & \cdots & w'_1(a_{g-2}) \\ & \text{(omit row } j) & & \\ w'_g(a) & w'_g(a_1) & \cdots & w'_g(a_{g-2}) \end{Bmatrix}$$

where $c_j \in \mathbb{C}$ is a nonzero value depending on $a_1,\ldots, a_{g-2}$ but not on a. The condition that $r-t = w(a+a_1 +\ldots+ a_{g-2}) \in W^1_{g-2} = k - W^1_{g-1}$ means that $\gamma(\kappa\zeta_a^{-1}\zeta_{a_1}^{-1} \cdots \zeta_{a_{g-1}}^{-1}) > 1$, so there are at least two linearly independent Abelian differentials that vanish on the divisor $a + a_1 +\ldots+ a_{g-2}$; consequently the vectors $\vec{w}'(a), \vec{w}'(a_1),\ldots, \vec{w}'(a_{g-2})$ must be linearly dependent. The determinants in (4) then all vanish, so the only nontrivial terms that arise upon differentiating (4) are those for which the determinant is differentiated; thus

$$(5) \quad \frac{\partial}{\partial a} Q_j(a,a_1,\ldots,a_{g-2}) = c_j \left[\prod_{\ell=1}^{g-2} q(a,a_\ell)^{-1}\right] \det \begin{Bmatrix} w''_1(a) & w'_1(a_1) & \cdots & w'(a_{g-}) \\ & \text{(omit row } j) & & \\ w''_g(a) & w''_g(a_1) & \cdots & w''_g(a_g) \end{Bmatrix}.$$

As a consequence of this, $w'_t(z,a) = 0$ identically in z precisely when the determinants in (5) vanish for $1 \leq j \leq g$, which is in turn equivalent to the condition that the vectors $\vec{w}''(a), \vec{w}'(a_1),\ldots, \vec{w}'(a_{g-1})$ are linearly dependent. Now there are two distinct subcases that must be considered.

The first is the case in which the vectors $\vec{w}'(a_1),\ldots, \vec{w}'(a_{g-2})$ are themselves linearly dependent, so $2 < \gamma(\kappa\zeta_{a_1}^{-1}\cdots\zeta_{a_{g-2}}^{-1}) = \gamma(\zeta_{a_1}\cdots\zeta_{a_{g-2}}) + 1$, or equivalently $r-t - w(a) = w(a_1 +\ldots+ a_{g-2}) \in W_{g-2}^1$. In this case the vectors $\vec{w}''(a)$, $\vec{w}'(a_1),\ldots, \vec{w}'(a_{g-2})$ are of course linearly dependent, so that $w'_t(z,a) = 0$ identically in z. The second is the case in which the vectors $\vec{w}'(a_1),\ldots, \vec{w}'(a_{g-2})$ are linearly independent. There are then two linearly independent Abelian differentials that vanish at the divisor $a_1 +\ldots+ a_{g-2}$, and both vanish as well at the point a since $\vec{w}'(a)$ is linearly dependent on the vectors $\vec{w}'(a_1),\ldots, \vec{w}'(a_{g-2})$. The vector $\vec{w}''(a)$ is also linearly dependent on the vectors $\vec{w}'(a_1),\ldots, \vec{w}'(a_{g-2})$ precisely when these two differentials actually vanish to second order at the point a, so when $2 = \gamma(\kappa^{-1}\zeta_a^{-2}\zeta_{a_1}^{-1}\cdots\zeta_{a_{g-2}}^{-1}) = \gamma(\rho_{k-w(a)-r+t}\zeta^{g-2}) = \gamma(\rho_{r+t-w(a)}\zeta^{g-2})$, or equivalently when $r+t-w(a) \in W_{g-2}^1$. Thus $w'_t(z,a) = 0$ identically in z precisely when either $r-t-w(a) \in W_{g-2}^1$ or $r+t-w(a) \in W_{g-2}^1$, corresponding to the two possible subcases.

In general write $a+a_1 +\ldots+ a_{g-2} = (\nu+1)b + \nu_1 b_1 +\ldots+ \nu_r b_r$, where $b=a$ and $b, b_1,\ldots, b_r$ represent distinct points of M. The value $Q_j(a,a_1,\ldots,a_{g-2})$ is then defined as the appropriate limit of the formal expression given in Corollary 2 to Theorem 4, as discussed earlier. Since the derivatives with respect to a are also involved, though, it is better to view a itself as variable and to write $a + a_1 +\ldots+ a_{g-2} = a + \nu b + \nu_1 b_1 +\ldots+ \nu_r b_r$, keeping in mind that what is of interest is the limit as a tends to b.

In these terms then it is a straightforward matter to see that

$$Q_j(a,a_1,\dots,a_{g-2}) = c_j\left[\prod_{\ell=0}^{r} q(a,b_\ell)^{-\nu_\ell}\right] \det\begin{Bmatrix} w_1'(a) & w_1'(b)\dots w_1^{(\nu)}(b) & w_1'(b_1)\dots \\ & \text{(omit row } j) & \\ w_g'(a) & w_g'(b)\dots w_g^{(\nu)}(b) & w_g'(b_1)\dots \end{Bmatrix}$$

where $b_o = b$, $\nu_o = \nu$, and $c_j \in \mathbb{C}$ is a nonzero value that is independent of a. In this formula it is possible to expand the functions $w'_i(a)$ in power series in the canonical coordinates near b; the initial terms in this expansion will lead to nothing in the determinant, and the result is easily seen to be expressible in the form

$$Q_j(a,a_1,\dots,a_{g-2}) = \left[\frac{c_j}{\nu!}\prod_{\ell=1}^{r} q(a,b_\ell)^{-\nu_\ell}\right]\frac{(a-b)^\nu}{q(a,b)^\nu}\cdot$$

$$\cdot \det\begin{Bmatrix} w_1^{(\nu+1)}(b) + 0(a-b) & w'_1(b) \dots w_1^{(\nu)}(b) & w'_1(b_1)\dots \\ w_g^{(\nu+1)}(b) + 0(a-b) & w'_g(b) \dots w_g^{(\nu)}(b) & w'_g(b_1)\dots \end{Bmatrix}.$$

The hypothesis that $r-t = w((\nu+1)\,b + \nu_1 b_1 + \dots + \nu_r b_r) \in W^1_{g-1}$ means that the vectors $\vec{w}'(b),\dots, \vec{w}^{(\nu+1)}(b)$, $\vec{w}'(b_1),\dots, \vec{w}^{(\nu_r)}(b_r)$ are linearly dependent, so that the preceding formula can actually be rewritten as

$$Q_j(a,a_1,\dots,a_{g-2}) = \frac{c_j}{(\nu+1)!}\left[\prod_{\ell=1}^{r} q(a,b_\ell)^{-\nu_\ell}\right]\frac{(a-b)^\nu}{q(a,b)^\nu}\cdot$$

$$\cdot \det\begin{Bmatrix} w_1^{(\nu+2)}(b)(a-b)+0(a-b)^2 & w_1'(b) \dots w_1^{(\nu)}(b) & w_1'(b_1)\dots \\ & \text{(omit row } j) & \\ w_g^{(\nu+2)}(b)(a-b)+0(a-b)^2 & w_g'(b) \dots w_g^{(\nu)}(b) & w_g'(b_1)\dots \end{Bmatrix}$$

This is a well defined holomorphic function of the variable a near the point b, since the factor $(a-b)^{\nu}q(a,b)^{-\nu}$ is, and vanishes at $a = b$; the derivative $\partial Q_j(a,a_1,\ldots,a_{g-2})/\partial a$ in the limit as a tends to b is obviously obtained by replacing the factor $(a-b)^{\nu}q(a,b)^{-\nu}$ by its limiting value 1 and replacing the first column of the matrix by $\vec{w}^{(\nu+2)}(b)$. With these observations having been made, the proof of the desired result proceeds just as in the preceding special case. The two subcases are those in which the g-2 vectors $\vec{w}'(b),\ldots, \vec{w}^{(\nu)}(b)\ \vec{w}'(b_1),\ldots, \vec{w}^{(\nu_r)}(b_n)$ are or are not linearly dependent, with the same arguments and results as in the special case, so nothing further needs to be added.

(iii) Suppose that $t \in \underline{\underline{\theta}}^1 = r-W^1_{g-1}$, and that $a \in \tilde{M}$ is a point for which $r-w(a) \notin (W^1_{g-2} + t) \cup (W^1_{g-2} -t)$. Since $r-t \in W^1_{g-1}$ then $r-t-w(a) \in W_{g-2}$, so it is possible to write $r-t-w(a) = w(a_1 +\ldots+ a_{g-2})$ for some divisor $a_1 +\ldots+ a_{g-2}$, and since $r-t-w(a) \notin W^1_{g-2}$ this divisor is uniquely determined. Now by the Riemann-Roch theorem $\gamma(\kappa\zeta_a^{-1}\zeta_{a_1}^{-1}\cdots\zeta_{a_{g-2}}^{-1}) = \gamma(\zeta_a\zeta_a,\cdots\zeta_{a_{g-2}}) = \gamma(\rho_{r-t}\zeta^{g-1}) > 1$, so there are at least two linearly independent Abelian differentials that vanish at the divisor $a+a_1+\ldots+a_{g-2}$; some nontrivial linear combination of these two will vanish to the second order at a, so there must exist at least one nontrivial Abelian differential vanishing at the divisor $2a + a_1 +\ldots+ a_{g-2}$. Since $r+t-w(a) \notin W^1_{g-2}$ necessarily $\gamma(\kappa\zeta_a^{-2}\zeta_{a_1}^{-1} \cdots \zeta_{a_{g-2}}^{-1}) = \gamma(\rho_\kappa\zeta^{2g-2} \cdot \zeta_a^{-1} \cdot \rho_{-r+t}\zeta^{-g+1}) =$ $= \gamma(\rho_{r+t-w(a)}\zeta^{g-2}) \leqq 1$, so that this last Abelian differential is uniquely determined up to a constant factor. For this particular point $a \in \tilde{M}$

the differential form $w'_t(z,a)$ in the variable z is nontrivial by part (ii) of the present theorem, as just demonstrated. On the other hand since $t \in \underline{\underline{\theta}}^1$ then it follows from part (i) that

$$w'_t(z,z) = \sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j(z)\, w'_k(z) = 0 \text{ for all } z \in \tilde{M},$$

and upon differenting this identity in z it further follows that

$$\sum_{j,k=1}^{g} \partial_{jk}\theta(t)\, w'_j{}'(z)\, w'_k(z) = 0 \quad \text{for all } z \in \tilde{M}\,;$$

consequently $w'_t(z,a)$ vanishes to at least the second order at the point $z=a$. It follows from Corollary 3 to Theorem 9 that $w'_t(a_j,a) = 0$ for $1 \le j \le g-2$ as well, so if $a, a_1, \ldots, a_{g-2}$ represent distinct points of M the desired result has been demonstrated.

With the proof in this special case in mind, the extension to the general case is reasonably clear and straightforward. Write $a + a_1 + \ldots + a_{g-2}$ $(\nu+1)\, b + \nu_1 b_1 + \ldots + \nu_r b_r$ where $b = a$ and $b, b_1, \ldots, b_r$ represent distinct points of M. By Corollary 2 to Theorem 9

$$\begin{aligned}\left[\prod_{\ell=1}^{g-2} q(z,a_\ell)^{-1}\right] & w'_t(z,a) \\ &= -\left[\prod_{\ell=1}^{g-2} q(z,a_\ell)^{-1}\right] \frac{\partial}{\partial a}\left[Q(a,a_1,\ldots,a_{g-2},z)\, q(z,a) \prod_{\ell=1}^{g-2} q(z,a_\ell)\right] \\ &= -\frac{\partial}{\partial a}\left[Q(a,a_1,\ldots,a_{g-2},z)\, q(z,a)\right].\end{aligned}$$

This is a holomorphic function of z, since $Q(a,a_1,\ldots,a_{g-2},z)$ and $q(z,a)$ are holomorphic in all their variables, so clearly the differential $w'_t(z,a)$ in z vanishes at the divisor $a_1 + \ldots + a_{g-2} = \nu b + \nu_1 b_1 + \ldots + \nu_r b_r$. To examine its further zeros, recall that the value of the function $Q(a,a_1,\ldots,$ $a_{g-2},z)$ when there are coincidences among the points of M represented by its arguments is determined by taking the appropriate limit of the expression given in the Corollary to Theorem 3. For the present suppose that a and z are independent variables and that $a_1 + \dots + a_{g-2} = \nu b + \nu_1 b_1 + \dots + \nu_r b_r$ as before; a straightforward calculation shows that

$$Q(a,a_1,\dots, a_{g-2},z) = c\ q(z,a)^{-1} \left[q(z,b)^{-\nu}\ q(a,b)^{-\nu} \prod_{j=1}^{r} q(z,b_j)^{-\nu_j} q(a,b_j)^{-\nu_j}\right] .$$

(6)

$$\cdot \det \{\vec{w}'(z),\ \vec{w}'(a),\ \vec{w}'(b),\dots,\ \vec{w}^{(\nu)}(b),\ \vec{w}'(b_1),\dots,\ \vec{w}^{(\nu_r)}(b_r)\}$$

where $c \in \mathbb{C}$ is independent of z and a but does of course depend on $a_1,\dots, a_{g-1}$. The hypothesis that $r-t=w((\nu+1)\ b+\nu_1 b_1 + \dots + \nu_r b_r) \in W^1_{g-1}$ implies as usual that the vectors $\vec{w}'(b),\dots, \vec{w}^{(\nu+1)}(b),\ \vec{w}'(b_1),\dots,\vec{w}^{(\nu_r)}(b_r)$ are linearly dependent, while the hypothesis that $r-t-w(a) = w(\nu b+\nu_1 b_1+\dots+ \nu_r b_r) \notin W^1_{g-2}$ implies correspondingly that the vectors $\vec{w}'(b),\dots,\vec{w}^{(\nu+1)}(b)$, $\vec{w}'(b_1),\dots,\ \vec{w}^{(\nu_r)}(b_r)$ are linearly independent; thus $\vec{w}^{(\nu+1)}(b)$ must lie in the (g-2)-dimensional span of the g-2 vectors $\vec{w}'(b),\dots,\ \vec{w}^{(\nu)}(b),\ \vec{w}'(b_1),\dots,$ $\vec{w}^{(\nu_r)}(b_r)$. Now in the determinant appearing in (6) expand the functions $\vec{w}'(z)$ and $\vec{w}'(a)$ in power series in the variables z-b and a-b respectively, in terms of the canonical local coordinate near the point b. In these expansions the initial terms, those for which the coefficients involve $\vec{w}'(b),\dots,\ \vec{w}^{(\nu+1)}(b)$, will contribute nothing to the determinant, since these vectors lie in the span of the remaining columns; thus (6) can be rewritten

$$Q(a,a_1,\ldots,a_{g-2},z) = \frac{c}{((\nu+1)!)^2}\left[\prod_{j=1}^{r} q(z,b_j)^{-\nu_j}\right] q(a,b_j)^{-\nu_j} \frac{(z-b)^{\nu+1}\,(a-b)^{\nu+1}}{q(z,a)\;q(z,b)^{\nu} q(a,b)^{\nu}}$$

$$\cdot \det\left\{\begin{array}{l} \vec{w}^{(\nu+2)}(b) + \vec{w}^{(\nu+3)}(b)\,\frac{z-b}{\nu+2} + O(z-b)^2, \\ \vec{w}^{(\nu+2)}(b) + \vec{w}^{(\nu+3)}(b)\,\frac{a-b}{\nu+2} + O(a-b)^2, \\ \vec{w}'(b),\ldots,\vec{w}^{(\nu)}(b),\ \vec{w}'(b_1),\ldots,\vec{w}^{(\nu_r)}(b_r) \end{array}\right\}.$$

Next multiply this by $q(z,a)$, apply the operator $\partial/\partial a$ to the result, and take the limit as a tends to b; since the expression $(a-b)^{\nu+1}q(a,b)^{-\nu}$ has a simple zero at $a=b$ the only nontrivial result arises from the differentiation of this term, and the result is that

$$-\left[\prod_{\ell=1}^{g-2} q(z,a_\ell)^{-1}\right] w_t'(z,a) =$$

$$= \frac{c}{((\nu+1)!)^2}\left[\prod_{j=1}^{r} q(z,b_j)^{-\nu_j} q(b,b_j)^{-\nu_j}\right] \frac{(z-b)^{\nu+1}}{q(z,b)^{\nu}}$$

$$\cdot \det\left\{\begin{array}{l} \vec{w}^{(\nu+2)}(b) + \vec{w}^{(\nu+3)}(b)\,\frac{z-b}{\nu+2} + O(z-b)^2,\ \vec{w}^{(\nu+2)}(b), \\ \vec{w}'(b),\ldots,\vec{w}^{(\nu)}(b),\ \vec{w}'(b_1),\ldots,\vec{w}^{(\nu_r)}(b_r) \end{array}\right\},$$

where the left-hand side is written out in terms of the original divisor $a + a_1 + \ldots + a_{g-2}$ while the right-hand side is rewritten in terms of the distinct points $b, b_1, \ldots, b_r$ of this divisor and their multiplicities. The determinant clearly vanishes at $z=b$, since its first two columns then coincide, and the factor $(z-b)^{\nu+1}q(z,b)^{-\nu}$ also vanishes at $z=b$; consequently the entire expression has at least a double zero at the point $z=b=a$, so that $w_t'(z,a)$ vanishes at the divisor $2a + a_1 + \ldots + a_{g-2}$ as desired and the proof is thereby concluded.

In general a <u>double differential</u> $w'(z,a)$ is a holomorphic function on $\tilde{M} \times \tilde{M}$ that is an Abelian differential on M in each variable separately and that is symmetric in the variables z and a. Any such double differential can of course be written out quite explicitly in terms of the canonical Abelian differentials in the form

$$(7) \qquad w'(z,a) = \sum_{jk} p_{jk}\, w_j'(z) w_k'(a)$$

for some uniquely determined complex constants, and the symmetry of the double differential is equivalent to the symmetry condition $p_{jk} = p_{kj}$. The constants p_{jk} can be viewed as describing a quadratic form $\sum_{jk} p_{jk} x_j x_k$ associated to the double differential, and conversely to each quadratic form in g variables there is associated a double differential (7). It is evident that the double differential $w'(z,a)$ has the additional property that $w'(z,z) = 0$ for all $z \in \tilde{M}$ precisely when the associated quadratic form belongs to the Petri space $\underline{P}_2$ of quadratic forms vanishing on the canonical curve; such double differentials will therefore be called <u>Petri double differentials</u>. To any basis $p_i(x) = \sum_{jk} p_{jk}^i x_j x_k$ for the Petri space $\underline{P}_2$ there thus corresponds a basis $w_i'(z,a) = \sum_{jk} p_{jk}^i w_j'(z) w_k'(a)$ for the Petri double differentials.

Now Theorem 10(i) asserts that the particular double differentials $w_t'(z,a)$ for points $t \in \underline{\underline{\Theta}}^1$ are Petri double differentials, so can be written

$$w_t'(z,a) = \sum_i \phi_i(t) w_i'(z,a) \tag{8}$$

in terms of any choice of basis $w_i'(z,a)$ for the space of Petri double differentials. The coefficients $\phi_i(t)$ are uniquely determined so must as usual be holomorphic functions of the variable $t \in \mathbb{C}^g$. When t is translated by a lattice vector $\lambda \in \underline{\underline{L}}$ then $\theta(t+\lambda) = \xi(\lambda,t)\theta(t)$ so that

$$\partial_{jk}\theta(t+\lambda) = \xi(\lambda,t)\partial_{jk}\theta(t) + \partial_j\theta(t)\partial_k\xi(\lambda,t)$$
$$+ \partial_k\theta(t)\partial_j\xi(\lambda,t) + \theta(t)\,\partial_{jk}\xi(\lambda,t) \; ;$$

in particular $\partial_{jk}\theta(t+\lambda) = \xi(\lambda,t)\,\partial_{jk}\theta(t)$ whenever $t \in \underline{\underline{\Theta}}^1$, and consequently

$$\phi_i(t+\lambda) = \xi(\lambda,t)\phi_i(t) \qquad \text{provided } t \in \underline{\underline{\Theta}}^1 .$$

Furthermore $\phi_i(t) = 0$ for all indices i and some point $t \in \underline{\underline{\Theta}}^1$ precisely when $w_t'(z,a)$ is identically zero in z and a, hence precisely when $\partial_{jk}\theta(t) = 0$ for all j, k so that $t \in \underline{\underline{\Theta}}^2$. The functions ϕ_i can be used to define a holomorphic mapping

$$\phi\colon \underline{\underline{\Theta}}^1 \sim \underline{\underline{\Theta}}^2 \longrightarrow \mathbb{P}^{d_2-1}$$

from the subset $\underline{\underline{\Theta}}^1 \sim \underline{\underline{\Theta}}^2 \subset J$ into the projective space of dimension d_2-1 where $d_2 = \dim \underline{\underline{P}}_2$. This can be viewed as a concrete description of the mapping that associates to the point $t \in \underline{\underline{\Theta}}^1$ the matrix $\partial_{jk}\theta(t)$, taking into account the special properties that this matrix has been shown to have; it will thus be called the <u>second-order Gauss mapping</u>, to follow the terminology used in section 3. Although the second-order Gauss mapping has really only been defined in the complement of the proper subvariety $\underline{\underline{\Theta}}^2$ of the analytic variety $\underline{\underline{\Theta}}^1$, just as for the ordinary Gauss mapping its graph can be extended to a well defined analytic variety over all of $\underline{\underline{\Theta}}^1$. The closure of the image of the Gauss mapping is consequently a well defined analytic and hence algebraic subvariety of $\mathbb{P}^{d_2-1}$.

The result of Theorem 10(iii) shows that the second-order Gauss mapping can be described just in terms of the standard function theory on Riemann surfaces, without any reference to theta functions. It is in many ways the analogue for the second-order Gauss mapping of the result of Corollary 1 to Theorem 4 for the first-order Gauss mapping, the descriptions of the Abelian differential $w_t'(z)$ not involving theta functions. The usefulness of this will become apparent in the subsequent discussion. The second-order Gauss mapping is a very interesting auxiliary tool that is still not terribly well understood. It is not hard to see directly from the Corollaries to Theorem 9 that the matrix $\partial_{jk}\theta(t)$ has rank ≤ 4 at all points $t \in \underline{\underline{\Theta}}^1$, but a more interesting proof will be given in the next section; something can be said as well about the rank of that matrix at points $t \in \underline{\underline{\Theta}}$, but that will not be needed here so will not be pursued further. The deepest general result known is the proof by M. Green ("Quadrics of rank four in the ideal of the canonical curve", Invent. Math. <u>75</u> (1984), 85-104) of an older conjecture that the image of the second-order Gauss mapping does not lie in any proper linear subspace of the space of Petri double differentials. The question of the extent to which the image of the second-order Gauss mapping lies in nontrivial quadric cones is more interesting in the present context, and will be discussed later.

§7. Semicanonical functions of second order.

A natural and useful extension of the discussion in section 4 involves the consideration of the function $\theta(t+w(z_1+z_2-a_1-a_2))$. As a function of the variable z_1 alone for instance this is just

$$f_{r-t+w(a_1+a_2-z_2)} \in \Gamma(\rho_{r-t+w(a_1+a_2-z_2)}\zeta^g) = \Gamma(\rho_{-t}\zeta_{a_1}\zeta_{a_2}\zeta_{z_2}{}^{-1}\sigma),$$

where $f_t(z) \in \Gamma(\rho_t\zeta^g)$ is as in (1.1) and $\sigma = \rho_r\zeta^{g-1}$ is the semicanonical factor of automorphy associated to the Riemann point $r \in \mathbb{C}^g$. The related expression

$$(1) \qquad \theta(t+w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)\ q(a_1,a_2) \prod_{j,k=1}^{2} q(z_j,a_k)^{-1}$$

is a well defined meromorphic function of the variables $(z_1,z_2,a_1,a_2) \in \tilde{M}^4$ for any fixed point $t \in \mathbb{C}^g$. If z_2,a_1,a_2 represent distinct points of M this expression is a well defined meromorphic function of the single remaining variable $z_1 \in \tilde{M}$; as such it is a meromorphic relatively automorphic function for the factor of automorphy $\rho_{-t}\sigma$, a meromorphic semicanonical function in the terminology introduced in section 4, with singularities at most simple poles at the points Γa_1 and Γa_2. The expression (1) is symmetric in the variables z_1 and z_2, so has the corresponding properties as a function of the variable z_2; it is moreover invariant when the z's are interchanged with the a's and t is replaced by $-t$, since the theta function is an even function of its argument, so has analogous properties as a function of a_1 or of a_2. The expression (1) is a natural extension to pairs of variables of the semicanonical function $\theta(t+w(z-a))\ q(z,a)^{-1}$, and can be reduced to the latter function as follows.

Theorem 11. For any point $t \in \mathbb{C}^g$ and any points $z_1, z_2, a_1, a_2 \in \tilde{M}$

$$- \theta(t)\theta(t+w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)\ q(a_1,a_2) \prod_{j,k=1}^{2} q(z_j,a_k)^{-1}$$

$$= \det\begin{pmatrix} \theta(t+w(z_1-a_1))\ q(z_1,a_1)^{-1} & \theta(t+w(z_1-a_2))\ q(z_1,a_2)^{-1} \\ \theta(t+w(z_2-a_1))\ q(z_2,a_1)^{-1} & \theta(t+w(z_2-a_2))\ q(z_2,a_2)^{-1} \end{pmatrix} .$$

Proof. As already observed, for all points $(z_2,a_1,a_2) \in \tilde{M}^3$ outside the proper holomorphic subvariety $D = \{z_2 \in \Gamma a_1\} \cup \{z_2 \in \Gamma a_2\} \cup \{a_1 \in \Gamma a_2\}$ the expression (1) as a function of the variable $z_1 \in \tilde{M}$ is a well defined meromorphic relatively automorphic function for the factor of automorphy $\rho_{-t}\sigma$, and has as singularities at most simple poles at the points $\Gamma a_1 \cup \Gamma a_2$; the residue at the point a_1 is

$$\lim_{z_1 \to a_1} \theta(t+w(z_1+z_2-a_1-a_2))\ \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(z_2,a_1)\ q(z_2,a_2)}$$

$$= -\ \theta(t+w(z_2-a_2))\ q(z_2,a_2)^{-1},$$

and the residue at the point a_2 is

$$\lim_{z_1 \to a_2} \theta(t+w(z_1+z_2-a_1-a_2))\ \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_1)\ q(z_2,a_1)\ q(z_2,a_2)}$$

$$= +\ \theta(t+w(z_2-a_1))\ q(z_2,a_1)^{-1}.$$

On the other hand the function $\theta(t+w(z_1-a_1))\ q(z_1,a_1)^{-1}$ is also a meromorphic relatively automorphic function for the factor of automorphy $\rho_{-t}\sigma$, with singularities at most simple poles at the points Γa_1, and its residue at the point a_1 is clearly just $\theta(t)$; moreover $\theta(t+w(z_1-a_2))\ q(z_1,a_2)^{-1}$ is yet another meromorphic relatively automorphic function for the same factor of automorphy, with singularities at most simple poles at the points Γa_2, and its residue at the point a_2 is also just $\theta(t)$. Consequently so long as $\theta(t) \neq 0$ the expression (1) differs from

$$(2) \quad -\theta(t)^{-1}\theta(t+w(z_2-a_2))\ q(z_2,a_2)^{-1} \cdot \theta(t+w(z_1-a_1))\ q(z_1,a_1)^{-1}$$
$$+\ \theta(t)\ \theta(t+w(z_2-a_1))\ q(z_2,a_1)^{-1} \cdot \theta(t+w(z_1-a_2))\ q(z_1,a_2)^{-1}$$

by an element of $\Gamma(\rho_{-t}\sigma)$, a holomorphic relatively automorphic function for the factor of automorphy $\rho_{-t}\sigma$. Note that $\gamma(\rho_{-t}\sigma) = \gamma(\rho_{r-t}\zeta^{g-1}) > 0$ precisely when $r-t \in W_{g-1} = r - \underline{\underline{\theta}}$, hence when $t \in \underline{\underline{\theta}}$, so that the expressions (1) and (2) must actually be equal whenever $t \notin \underline{\underline{\theta}}$. That thus establishes the desired identity for all points $(t,z_1,z_2,a_1,a_2) \in \mathbb{C}^g \times \tilde{M}^4$ outside the proper holomorphic subvariety $\underline{\underline{\theta}} \times \tilde{M} \times D$; and it then holds for all points of $\mathbb{C}^g \times \tilde{M}^4$ by continuity, to conclude the proof.

<u>Corollary 1</u>. For any point $t \in \mathbb{C}^g$ and any points $z,a \in \tilde{M}$

$$\theta(t+w(z-a))\ \theta(t-w(z-a))\ q(z,a)^{-2}$$
$$= -\ w_t'(z)w_t'(a) + \theta(t)w_t'(z,a) + \theta(t)^2 w_a'(z)$$

in terms of the differential forms $w_t'(z) = \sum_j \partial_j\theta(t)w_j'(z)$ and $w_t'(z,a) = \sum_{j,k} \partial_{jk}\theta(t)w_j'(z)w_k'(a)$ considered before.

<u>Proof</u>. Multiply the formula of the preceding theorem by $q(z_1,a_1)\ q(z_2,a_2)$, apply the differential operator $\partial^2/\partial z_1\partial z_2$, and take the limit in the result as z_1 tends to a_1 and z_2 tends to a_2. On the left-hand side this yields

$$-\theta(t) \lim_{z_i \to a_i} \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \theta(t+w(z_1+z_2-a_1-a_2)) \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} \right\}$$

$$= -\theta(t) \lim_{z_i \to a_i} \left\{ \sum_{jk} \partial_{jk}\theta(t+w(z_1+z_2-a_1-a_2))w_j'(z_1)w_k'(z_2) \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} \right.$$

$$+ \sum_j \partial_j\theta(t+w(z_1+z_2-a_1-a_2))w_j'(z_1) \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} (w_{z_2}(z_1) - w_{z_2}(a_1))$$

$$+ \sum_j \partial_j\theta(t+w(z_1+z_2-a_1-a_2))w_j'(z_2) \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} (w_{z_1}(z_2) - w_{z_1}(a_2))$$

$$\left. + \theta(t+w(z_1+z_2-a_1-a_2)) \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} [w_{z_2}'(z_1)+(w_{z_2}(z_1)-w_{z_2}(a_1))(w_{z_1}(z_2)-w_{z_1}(a_2))]\right\}$$

$$= -\theta(t) \left\{-w_t'(a_1,a_2) + 0 + 0 - \theta(t)w_{z_2}'(z_1)\right\}$$

and on the right-hand side it yields

$$\lim_{z_i \to a_i} \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \theta(t+w(z_1-a_1))\ \theta(t+w(z_2-a_2)) - \theta(t+w(z_1-a_2))\ \theta(t+w(z_2-a_1)) \frac{q(z_1,a_1)\ q(z_2,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} \right\}$$

$$= \lim_{z_i \to a_i} \left\{ \sum_{jk} \partial_j\theta(t+w(z_1-a_1))w_j'(z_1)\ \partial_k\theta(t+w(z_2-a_2))w_k'(z_2) - \theta(t+w(z_1-a_2))\ \theta(t+w(z_2-a_1)) \frac{\partial_1 q(z_1,a_1)\ \partial_1 q(z_2,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} + \text{terms with a factor } q(z_1,a_1) \text{ or } q(z_2,a_2) \right\}$$

$$= w_t'(a_1)\ w_t'(a_2) + \theta(t+w(a_1-a_2))\ \theta(t+w(a_2-a_1))\ q(a_1,a_2)^{-2} .$$

Comparing these two expressions yields the desired result upon the obvious change of notation.

For points $t \in \underline{\underline{\Theta}}$ this result reduces to the formula given in Corollary 1 to Theorem 7; it can therefore be viewed as an extension of that earlier result to a statement that is valid for more general values of the parameter t than just points in the theta locus. In this interpretation the differential forms $w_t'(z)$ and $w_t'(z,a)$ are defined for all values $t \in \mathbb{C}^g$ by the formulas $w_t'(z) = \sum_j \partial_j\theta(t)\ w_j'(z)$ and $w_t'(z,a) = \sum_{jk} \partial_{jk}\theta(t)\ w_j'(z)\ w_k'(a)$; both were considered earlier primarily for special values of the parameter t. In an alternative interpretation, the expresson $\theta(t+w(z-a))\ \theta(t-w(z-a))\ q(z,a)^{-2}$ +

$+ w_t'(z)\, w_t'(a)$ is holomorphic in t, and by Corollary 1 to Theorem 7 it vanishes at the locus $\theta(t) = 0$; this expression is therefore divisible by $\theta(t)$, and the formula of the present corollary exhibits the quotient and indeed provides explicitly the second order ~~theorem~~ terms in an expansion in powers of $\theta(t)$.

For a point $t \in \underline{\theta}$ the left-hand side of the formula of Theorem 11 vanishes, so the theorem itself reduces to the assertion that the determinant on the right-hand side vanishes identically in the variables z_j, a_k; this is just the result (4.4) obtained earlier in the discussion of semicanonical functions. However something nontrivial can be derived from the theorem by applying the differential operator $\partial/\partial t_j$ throughout, since $\partial_j \theta(t)$ may not vanish. If $t \in \underline{\underline{\theta}}^1$ then all these new formulas are also trivial, but the situation can be salvaged by applying the differential operator $\partial^2/\partial t_j \partial t_k$. The results in this case are sufficiently interesting to merit a bit more discussion here.

<u>Corollary 2</u>. For any point $t \in \underline{\underline{\theta}}^1 \subseteq \mathbb{C}^g$, any points $z_1, z_2, a_1, a_2 \in \tilde{M}$, and any indices $1 \leq j,\ k \leq g$,

$$- \partial_{jk}\theta(t)\ \theta(t+w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)\ q(a_1,a_2) \prod_{\mu,\nu=1}^{2} q(z_\mu,a_\nu)^{-1}$$

$$= f_j(t;z_1,a_1)\ f_k(t;z_2,a_2) + f_k(t;z_1,a_1)\ f_j(t;z_2,a_2)$$

$$- f_j(t;z_1,a_2)\ f_k(t;z_2,a_1) - f_k(t;z_1,a_2)\ f_j(t;z_2,a_1)$$

where $f_j(t;z,a) = f_j(-t;a,z) = q(z,a)^{-1}\partial_j\theta(t+w(z-a))$ are holomorphic functions of $z,a \in \tilde{M}$ that represent elements of $\Gamma(\rho_{r-t}\zeta^{g-1})$ as functions of z and elements of $\Gamma(\rho_{r+t}\zeta^{g-1})$ as functions of a.

<u>Proof</u>. Consider again the formula of Theorem 11 for arbitrary points $t \in \mathbb{C}^g$, and to simplify the notation set $\alpha_{\mu\nu} = q(z_\mu, a_\nu)\ \theta(t+w(z_\mu - a_\nu))$. Then apply the differential operator $\partial^2/\partial t_j \partial t_k$ to that formula, and consider the result for a point $t \in \underline{\underline{\theta}}^1$. Since $\theta(t) = \partial_j\theta(t) = \partial_k\theta(t) = 0$ when $t \in \underline{\underline{\theta}}^1$ the only nontrivial term that remains on the left-hand side is that in which the differentiation is all applied to the factor $\theta(t)$, yielding $\partial_{jk}\theta(t)$. On the other hand the right-hand side is the determinant $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$. Now from Riemann's theorem, Theorem 2, it follows that $\alpha_{\mu\nu} = 0$ for all values of z_μ, a_ν whenever $t \in \underline{\underline{\theta}}^1$, so the only nontrivial terms that remain on the right-hand side are those in which each factor is differentiated at least once, namely the terms $\partial_j\alpha_{11}\ \partial_k\alpha_{22} + \partial_k\alpha_{11}\ \partial_j\alpha_{22} - \partial_j\alpha_{12}\ \partial_k\alpha_{21} - \partial_k\alpha_{12}\ \partial_j\alpha_{21}$. The result altogether is the formula of the corollary, since $\partial_j\alpha_{11} = f_j(t;z_1,a_1)$ and so on. The function $f_j(t;z,a)$ is clearly a holomorphic in the variables z and a, since $\partial_j\theta(t) = 0$ for $t \in \underline{\underline{\theta}}^1$, and it follows quite readily from (5.3) that as a function of z it transforms as a relatively automorphic function for the factor of automorphy $\rho_{r-t}\zeta^{g-1}$, since $\theta(t+w(z-a)) = 0$ for all z,a. Finally since $\theta(t)$ is an even function of t necessarily $\partial_j\theta(t+w(z-a)) =$ $-\ \partial_j\theta(-t+w(a-z))$, and that suffices to conclude the proof.

For a point $t \in \underline{\underline{\theta}}^1 \sim \underline{\underline{\theta}}^2$ not all of the terms $\partial_{jk}\theta(t)$ vanish, so it follows from the preceding corollary that the functions $f_j(t;z,a)$ are not all trivial. Indeed for the special case $j = k$ the formula of the corollary can be rewritten

$$-\partial_{jj}\theta(t)\ \theta(t+w(z_1+z_2-a_1-a_2))\ q(z_1,z_2)\ q(a_1,a_2)\prod_{\mu,\nu=1}^{2} q(z_\mu,a_\nu)^{-1}$$

(3)

$$= 2\det\begin{pmatrix} f_j(z_1,a_1) & f_j(z_1,a_2)\\ f_j(z_2,a_1) & f_j(z_2,a_2)\end{pmatrix}$$

upon setting $f_j(z,a) = f_j(t;z,a)$ for simplicity, since the point t is here viewed as fixed, and it is evident from this <u>that</u> $f_j(z,a)$ <u>must be a nontrivial function on</u> $\tilde{M}\times\tilde{M}$ <u>whenever</u> $\partial_{jj}\ \theta(t)\neq 0$. Furthermore note that for fixed points a_1,a_2 the function $\theta(t+w(z_1+z_2-a_1-a_2))$ vanishes identically in z_1,z_2 precisely when $t-w(a_1+a_2)+W_2\subseteq \underline{\underline{\theta}} = W_{g-1}-r$, hence by B(9.12) precisely when

$$r+t-w(a_1+a_2)\in W_{g-1}\ominus W_2 = W_{g-3}$$

provided that $g\geq 3$; this cannot happen for all points a_1,a_2, for by B(9.12) again that would mean that $r+t\in W_{g-3}\ominus(-W_2) = W^2_{g-1}$ hence that $t\in -r+W^2_{g-1} = \underline{\underline{\theta}}^2$. If $\theta(t+w(z_1+z_2-a_1-a_2))$ does not vanish identically in z_1,z_2 and $q(a_1,a_2)\partial_{jj}\theta(t)\neq 0$ then the determinant in (3) does not vanish identically in z_1,z_2 either; that means that the two functions $f_j(z,a_1)$, $f_j(z,a_2)$ are linearly independent. <u>Thus</u> $f_j(z,a_1)$ $f_j(z,a_2)$ <u>are linearly independent elements of</u> $\Gamma(\rho_{r-t}\zeta^{g-1})$ <u>as functions of</u> z <u>precisely when</u> $\partial_{jj}\theta(t)\neq 0$ <u>and</u> a_1,a_2 <u>represent distinct points of M for which</u> $r+t-w(a_1+a_2)\notin W_{g-3}$; there are always some points a_1,a_2 for which this last condition is satisfied. It should be remarked that $\gamma(\rho_{r-t}\zeta^{g-1}) = 2$ whenever $t\in\underline{\underline{\theta}}^1\sim\underline{\underline{\theta}}^2$, so $f_j(z,a_1)$ and $f_j(z,a_2)$ are a basis for $\Gamma(\rho_{r-t}\zeta^{g-1})$ whenever they are linearly independent functions of z. On the other hand if $\partial_{jj}\theta(t) = 0$ then it follows readily from (3) that $f_j(z,a) = g_j(z)\ h_j(a)$ for some functions $g_j\in\Gamma(\rho_{r-t}\zeta^{g-1})$ and $h_j\in\Gamma(\rho_{r+t}\zeta^{g-1})$.

and if $\partial_{kk}\theta(t) = 0$ as well there arise in this way the two functions $g_j, g_k \in \Gamma(\rho_{r-t}\zeta^{g-1})$ and $h_j, h_k \in \Gamma(\rho_{r+t}\zeta^{g-1})$. In this case the formula of Corollary 3 can be written as the identity

$$(4)\quad \begin{aligned} &-\partial_{jk}\theta(t)\ \theta\big(t+w(z_1+z_2-a_1-a_2)\big)\ q(z_1,z_2)\ q(a_1,a_2)\ \prod_{\mu\nu} q(z_\mu,a_\nu)^{-1} \\ &= \det\begin{pmatrix} g_j(z_1) & g_j(z_2) \\ g_k(z_1) & g_k(z_2)\end{pmatrix} \det\begin{pmatrix} h_j(a_1) & h_j(a_2) \\ h_k(a_1) & h_k(a_2)\end{pmatrix} ,\end{aligned}$$

and it is clear from this that <u>whenever</u> $\partial_{jj}\theta(t) = \partial_{kk}\theta(t) = 0$ <u>but</u> $\partial_{jk}\theta(t) \neq 0$ <u>then</u> g_j, g_k <u>form a basis for</u> $\Gamma(\rho_{r-t}\zeta^{g-1})$. Since not all the derivatives $\partial_{jk}\theta(t)$ vanish there is always some basis for $\Gamma(\rho_{r-t}\zeta^{g-1})$ obtained in one of the two ways just indicated from the functions $f_j(z,a) = q(z,a)^{-1}\ \partial_j\theta(t+w(z-a))$.

As an alternative approach, for any vector $c = (c_1,\dots,c_g) \in \mathbb{C}^g$ consider the function $f_c(z,a) = \sum_j c_j f_j(z,a)$, which is again an element of $\Gamma(\rho_{r-t}\zeta^{g-1})$ as a function of z and an element of $\Gamma(\rho_{r+t}\zeta^{g-1})$ as a function of a. In these terms it is easy to see that Corollary 2 is equivalent to the assertion that

$$(5)\quad \begin{aligned} &-\sum_{jk} \partial_{jk}\theta(t)c_jc_k\ \theta\big(t+w(z_1+z_2-a_1-a_2)\big)\ q(z_1,z_2)\ q(a_1,a_2) \prod_{\mu,\nu} q(z_\mu,a_\nu)^{-1} \\ &= \det\begin{pmatrix} f_c(z_1,a_1) & f_c(z_1,a_2) \\ f_c(z_2,a_1) & f_c(z_2,a_2)\end{pmatrix}\end{aligned}$$

for all $c \in \mathbb{C}^g$. It is clear from this formula that $f_c(z,a) = g(z)\ h(a)$ <u>for some functions</u> $g \in \Gamma(\rho_{r-t}\zeta^{g-1})$ <u>and</u> $h \in \Gamma(\rho_{r+t}\zeta^{g-1})$ <u>precisely when</u> $\sum_{jk}\partial_{jk}\theta(t)c_jc_k = 0$. Somewhat more interesting and not quite so obvious is the assertion <u>that</u> $f_c(z,a) = 0$ <u>for all points</u> $(z,a) \in \tilde{M}\times\tilde{M}$ <u>precisely when</u> $\sum_k\partial_{jk}\theta(t)c_k = 0$ <u>for all indices</u> $1 \leq j \leq g$. Indeed if $0 = q(z,a)\ f_c(z,a) = \sum_j c_j\ \partial_j\theta(t+w(z-a))$ identically in z and a then upon differentiating this identity with respect to z and setting a = z it follows that $0 = \sum_{jk} c_j \partial_{jk}\theta(t) w_k'(z)$ for all z, hence that $\sum_j c_j \partial_{jk}\theta(t) = 0$ for all k since the differentials $w_k'(z)$ are linearly independent. On the other hand if $\sum_k \partial_{jk}\theta(t) c_k = 0$ for all indices j then $f_c(z,a) = g(z)\, h(a)$ by the preceding observation; now

$$g(a)\ h(a) = f_c(a,a) = \lim_{z\to a} \sum_j c_j \frac{\partial_j\theta(t+s(z-a))}{q(z,a)} = \sum_{jk} c_j \partial_{jk}\theta(t)\ w_k'(a) = 0$$

for all points $a \in \tilde{M}$, so that either g vanishes identically or h does and in either case $f_c(z,a) = g(z)\ h(a)$ vanishes identically. As a consequence of this the rank of the matrix $\partial_{jk}\theta(t)$ is equal to the number of linearly independent functions from among the $f_j(z,a)$ viewed as functions on $\tilde{M}\times\tilde{M}$, since this is just g minus the dimension of the space of all vectors $c \in \mathbb{C}^g$ satisfying the two conditions just shown to be equivalent. If g_1, g_2 are a basis of $\Gamma(\rho_{r-t}\zeta^{g-1})$ and h_1, h_2 are a basis of $\Gamma(\rho_{r+t}\zeta^{g-1})$ then any function $f_j(z,a)$ can be written $f_j(z,a) = \sum_{k\ell} c_j^{k\ell}\ g_k(z)\ h_\ell(a)$ from some uniquely determined constants $c^{k\ell}{}_j$; thus there are at most 4 linearly independent functions from among the $f_j(z,a)$, and hence rank $\partial_{jk}\theta(t) \leq 4$. It has already been noted that if $t \in \underline{\Theta} \sim \underline{\underline{\Theta}}^1$ then for any fixed a the functions $f_j(z,a)$ span $\Gamma(\rho_{r-t}\zeta^{g-1})$, so it is evident that rank $\partial_{jk}\theta(t) \geq 2$. Actually since the Petri functions vanish on the canonical curve, a quadric of rank ≤ 2 really amounts to a linear condition, and the canonical curve is not contained in any proper linear subspace, it follows readily that rank $\partial_{jk}\theta(t) \geq 3$. These observations can be summarized as follows.

Corollary 3. For any fixed point $t \in \underline{\underline{\Theta}}^1 \sim \underline{\underline{\Theta}}^2$ the rank of the matrix $\partial_{jk}\theta(t)$, which is either 3 or 4, is equal to the number of linearly independent functions on $\tilde{M} \times \tilde{M}$ from among the g functions $q(z,a)^{-1}\partial_j\theta(t+w(z-a))$ for $1 \leq j \leq g$; as functions of z alone the latter span the two-dimensional space $\Gamma(\rho_{r-t}\zeta^{g-1})$ for any fixed $a \in \tilde{M}$.

Another consequence of the second corollary in a slightly different direction is also worth noting.

Corollary 4. For any point $t \in \underline{\underline{\Theta}}^1$ and any points $a_1,a_2,b_1,b_2 \in \tilde{M}$

$$w_t'(a_1,a_2)\ w_t'(b_1,b_2) - w_t'(a_1,b_1)\ w_t'(a_2,b_2) - w_t'(a_1,b_2)\ w_t'(a_2,b_1)$$
$$= q(a_1,a_2)^{-1}\left\{ w'_{t+w(a_1-a_2)}(b_1)\ \ w'_{t-w(a_1-a_2)}(b_2) + w'_{t+w(a_1-a_2)}(b_2)\ \ w'_{t-w(a_1-a_2)}(b_1)\right\}.$$

Proof. When multiplied by $q(z_1,a_1)\ q(z_2,a_2)$ the formula of Corollary 2 takes the form

$$\partial_{jk}\theta(t)\ \theta(t+w(z_1+z_2-a_1-a_2))\ \frac{q(z_1,z_2)\ q(a_1,a_2)}{q(z_1,a_2)\ q(a_1z_2)}$$
$$= \partial_j\theta(t+w(z_1-a_1))\ \partial_k\theta(t+w(z_2-a_2)) + \partial_k\theta(t+w(z_1-a_1))\ \partial_j\theta(t+w(z_2-a_2))$$
$$- \frac{q(z_1,a_1)\ q(z_2,a_2)}{q(z_1,a_2)\ q(z_2,a_1)}\left\{ \partial_j\theta(t+w(z_1-a_2))\ \partial_k\theta(t+w(z_2-a_1)) + \partial_k\theta(t+w(z_1-a_2))\ \partial_j\theta(t+w(z_2-a_1))\right\}.$$

Apply the differential operator $\partial^2/\partial z_1\partial z_2$ to this formula and take the limit as $z_1 \to a_1$ and $z_2 \to a_2$. Since $\theta(t) = \partial_j\theta(t) = 0$ for $t \in \Theta^1$ the only nontrivial terms on the left-hand side are those for which only the second theta factor is differentiated; the factors $q(z_1,a_1)$ and $q(z_2,a_2)$ must be differentiated to obtain something nontrivial on the right-hand side. The result is easily seen to be

$$\begin{aligned} &\partial_{jk}\theta(t) \sum_{\ell m} \partial_{\ell m}\,\theta(t)\; w'_\ell(a_1)\; w'_m(a_2) \\ &= \sum_{\ell m} \Big\{ \partial_{j\ell}\theta(t)\; w'_\ell(a_1)\; \partial_{km}\theta(t)\; w'_m(a_2) + \partial_{k\ell}\theta(t) w'_\ell(a_1)\; \partial_{jm}\theta(t)\; w'_m(a_2) \Big\} \\ &\quad - q(a_1,a_2)^{-2} \Big\{ \partial_j\theta\big(t+w(a_1-a_2)\big)\; \partial_k\theta\big(t-w(a_1-a_2)\big) \\ &\qquad\qquad + \partial_k\theta\big(t+w(a_1-a_2)\big)\; \partial_j\theta\big(t-w(a_1-a_2)\big) \Big\}. \end{aligned}$$

Multiply this by $w'_j(b_1)\; w'_k(b_2)$ and summing over j and k yields the desired result.

It may be recalled from Theorem 10 that $w'_t(z,z) = 0$ whenever $t \in \theta^1$, so in the limit as b_1 and b_2 tend to z the formula of the preceding corollary becomes

$$(6) \qquad w'_t(z,a_1)\; w'_t(z,a_2) = -q(a_1,a_2)^{-2}\; w'_{t+w(a_1-a_2)}(z)\; w'_{t-w(a_1-a_2)}(z),$$

an interesting result in itself. To see its significance note that if $t \in \underline{\underline{\theta}}^1 \sim \underline{\underline{\theta}}^2$ then for general points $a_1,a_2 \in M$ there are uniquely determined divisors of degree g-2 on M defined by the conditions

$$\begin{aligned} r-t-w(a_1) &= w(x_1 + \dots + x_{g-2}), & r+t-w(a_1) &= w(x'_1 + \dots + x'_{g-2}), \\ r-t-w(a_2) &= w(y_1 + \dots + y_{g-2}), & r+t-w(a_2) &= w(y'_1 + \dots + y'_{g-2}); \end{aligned}$$

these can be combined in four ways to yield canonical divisors, by adding two terms in which t cancels. It is easy to see from Corollary 1 to Theorem 4 that

$$\begin{aligned} \mathfrak{d}(w'_{t+w(a_1-a_2)}) &= a_1 + a_2 + x_1 + \dots + x_{g-2} + y'_1 + \dots + y'_{g-2}, \\ \mathfrak{d}(w'_{t-w(a_1-a_2)}) &= a_1 + a_2 + x'_1 + \dots + x'_{g-2} + y_1 + \dots + y_{g-2}, \end{aligned}$$

keeping in mind the observation that $w'_s(z) = -w'_{-s}(z)$. On the other hand from Theorem 10 it follows that

$$\mathcal{D}(w_t'(z,a_1)) = 2a_1 + x_1 + \dots + x_{g-2} + x_1' + \dots + x_{g-2}',$$

$$\mathcal{D}(w_t'(z,a_2)) = 2a_2 + y_1 + \dots + y_{g-2} + y_1' + \dots + y_{g-2}',$$

since in this case $w_t'(z,a) = w_{-t}'(z,a)$. The two sides of (6) thus correspond to two different ways of grouping the $4g-4$ points involved here into sums of two canonical divisors.

A related result in a slightly different direction, the analogue of Corollary 1 to Theorem 7 for semicanonical functions of second order, is as follows.

<u>Theorem 12</u>. For any point $t \in \underline{\underline{\theta}}^1 \subseteq \mathbb{C}^g$ and any points $z_1, z_2, a_1, a_2 \in \tilde{M}$

$$\theta(t+w(z_1+z_2-a_1-a_2))\,\theta(t-w(z_1+z_2-a_1-a_2))\,q(z_1,z_2)^2\,q(a_1,a_2)^2 \prod_{j,k=1}^{2} q(z_j,a_k)^{-2}$$

$$= w_t'(z_1,z_2)\, w_t'(a_1,a_2)$$

where $w_t'(z,a) = \sum_{j,k} \partial_{jk}\theta(t)\, w_j'(z)\, w_k'(a)$.

<u>Proof</u>. If $t \in \underline{\underline{\theta}}^2$ then it is evident from Theorem 2 that both sides of the asserted equality vanish identically, so the desired result holds trivially; for the remainder of the proof it can therefore be supposed that $t \in \underline{\underline{\theta}}^1 \sim \underline{\underline{\theta}}^2$, hence that $r \pm t \in W_{g-1}^1 \sim W_{g-1}^2$. The functions $\theta(t \pm w(z_1+z_2-a_1-a_2))$ do not then vanish identically in the variables z_1, z_2, a_1, a_2, so for general values of z_2, a_1, a_2 they are nontrivial holomorphic functions of the variable z_1, but these functions of z_1 have zeros at the points a_1 and a_2. The expression (1) is consequently a nontrivial holomorphic relatively automorphic function for the factor of automorphy $\rho_{-t}\sigma$ when viewed as a function of z_1; it vanishes at z_2 because of the factor $q(z_1,z_2)$, hence has the divisor $z_2 + x_1 + \dots + x_{g-2}$ on M where

(7) $\quad r-t-w(z_2) = w(x_1 + \dots + x_{g-2})$.

The point z_2 can be taken so that $r-t-w(z_2) \notin W^1_{g-2}$, so the divisor $x_1 +\dots+ x_{g-2}$ on M is uniquely determined by (7). The analogous expression but with t replaced by $-t$ is an element of $\Gamma(\rho_t\sigma)$ vanishing at the divisor $z_2 + y_1 +\dots+ y_{g-2}$ on M, where z_2 can also be taken so that $r+1 - w(z_2) \notin W^1_{g-2}$ as well, so that the divisor $y_1 +\dots+ y_{g-2}$ is determined uniquely by the analogue of (7). Now the product of these two expressions is an element of $\Gamma(\rho_{-t}\sigma \cdot \rho_t\sigma) = \Gamma(\sigma^2) = \Gamma(\kappa)$, hence an Abelian differential on M, and vanishes at the divisor $2z_2 + x_1 +\dots+ x_{g-2}$ on M; it then follows from Theorem 10 that this product, which is just the left-hand side of the formula of the present theorem, is a constant multiple of the differential form $w_t'(z_1,z_2)$ in z_1, where the constant depends of course on z_2,a_1,a_2. From this and the obvious symmetries it is quite evident that the left-hand side of the formula of the present theorem is actually equal to $c\; w_t'(z_1,z_2)\; w_t'(a_1,a_2)$ where c is independent of the variables z_1,z_2,a_1,a_2; thus

$$\frac{\theta(t+w(z_1+z_2-a_1-a_2))}{q(z_1,a_1)\;q(z_2,a_2)} \cdot \frac{\theta(t-w(z_1+z_2-a_1-a_2))}{q(z_1,a_1)\;q(z_2,a_2)} \cdot \frac{q(z_1,z_2)^2\;q(a_1,a_2)^2}{q(z_1,a_2)^2\;q(a_1,z_2)^2}$$
$$= c\; w_t'(z_1,z_2)\; w_t'(a_1,a_2).$$

To evaluate the constant c here simply take the limit as $z_1 \to a_1$ and $z_2 \to a_2$; the first term on the left-hand side evidently tends to

$$\sum_{jk} \partial_{jk}\theta(t)\; w_j'(a_1)\; w_k'(a_2) = w_t'(a_1,a_2)$$

as does the second term, so that $c = 1$ and the desired result is thereby demonstrated.