B. Riemann Surfaces

§1. The topology of surfaces.

It is convenient to establish from the outset some conventions and notation for dealing with the topology of surfaces. Thorough discussions can be found in the books by H. Seifert and W. Threlfall (Lehrbuch der Topologie, Tuebner, 1934) and by L. V. Ahlfors and L. Sario (Riemann Surfaces, Princeton Univ. Press, 1960), for instance. If M is a compact connected two-dimensional topological manifold of genus g then it is just a sphere with g handles, and can be dissected into a contractible set by cutting it along $2g$ paths issuing from a common base point $p_0 \in M$ but otherwise disjoint, one pair of paths for each handle as in figure 2. The complement of this set of paths is a polygonal region Δ with $4g$ boundary arcs, in pairs corresponding to the two sides of each of the $2g$ dissecting paths; M can be recovered from Δ by pasting these pairs of boundary arcs together. The dissecting paths are oriented as shown in figure 2, and the corresponding orientations of the boundary arcs of Δ are as shown in figure 3.

If $\tilde{M}$ is the universal covering space of M, the choice of a point $z_0 \in \tilde{M}$ lying over $p_0 \in M$ establishes a canonical isomorphism between the covering translation group Γ of $\tilde{M}$ over M and the fundamental group $\pi_1(M,p_0)$, by associating to any $T \in \Gamma$ the homotopy class in $\pi_1(M,p_0)$ of the image in M of any path from z_0 to Tz_0 in $\tilde{M}$. The choice of another base point z_0 changes this isomorphism by an inner automorphism. With a slight but not really confusing abuse of notation, let α_j, β_j also denote the lifts of the loops α_j, β_j to paths in $\tilde{M}$ beginning at the base point z_0; the ends of these paths will then be the points $A_j z_0, B_j z_0$, where $A_j, B_j \in \Gamma$ correspond to the homotopy classes of the original loops α_j, β_j in $\pi_1(M,p_0)$. The lift of the polygonal region Δ to $\tilde{M}$ having α_1 as one boundary arc then has as remaining boundary arcs the others indicated in figure 4, as can readily be verified by tracing out the boundary path. In this figure and subsequently $C_j = [A_j, B_j] = A_j B_j A_j^{-1} B_j^{-1}$. Note that the oriented boundary of Δ is the path

(1) $$\partial\Delta = \sum_{j=1}^{g} [C_1 \cdots C_{j-1}\alpha_j + C_1 \cdots C_{j-1}A_j\beta_j - C_1 \cdots C_j B_j \alpha_j - C_1 \cdots C_j \beta_j] \; .$$

The elements $A_1, \ldots, A_g, B_1, \ldots, B_g$ generate the group Γ ; it is evident from figure 4 that they satisfy the relation

(2) $$C_1 \, C_2 \cdots C_g = 1 \; ,$$

and in fact all relations among these generators are consequences of this one relation. The subset $\Delta \subset \tilde{M}$ will be called the _fundamental_ _polygon_ associated to this marking. The sets $T\Delta$ as T ranges over the group Γ have disjoint interiors, and their union together with all translates of their boundary arcs cover the entire covering surface $\tilde{M}$.

Any choice of a base point $z_0 \in \tilde{M}$ and a set of dissecting paths $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ in this form will be called a _marking_ of the surface M . There are of course a great many such choices. However two markings will be considered as _equivalent_ if the base points coincide and they can be mapped to one another by a homeomorphism of M that preserves this base point and that is homotopic to the identity through homeomorphisms that also preserve this base point. The generators $A_1, \ldots, A_g, B_1, \ldots, B_g$ of Γ corresponding to equivalent markings are of course the same.

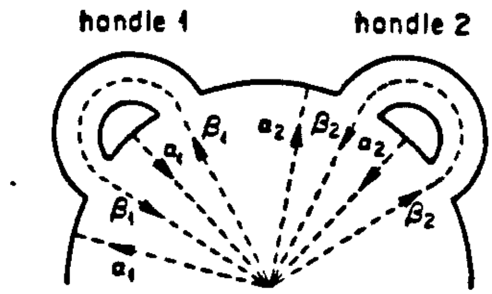


FIG. 2

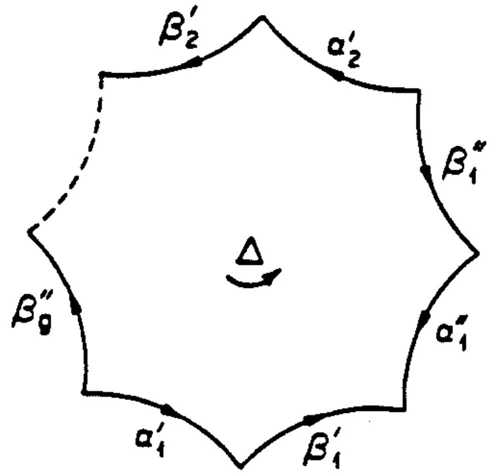


FIG. 3

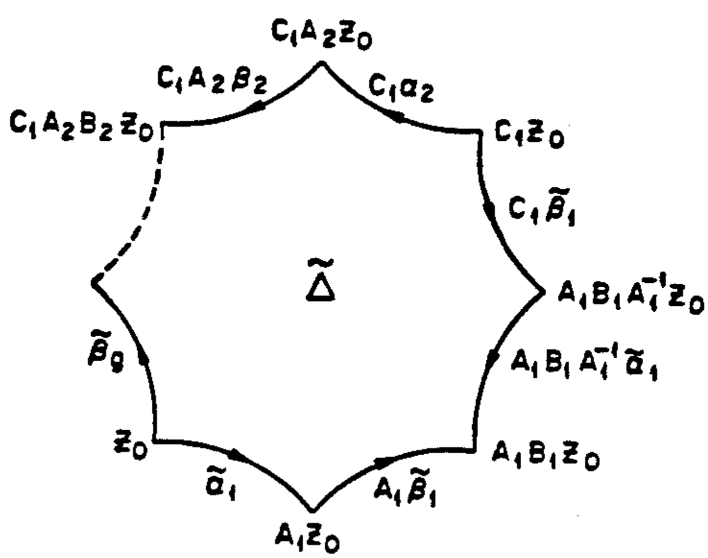


FIG. 4

A compact connected Riemann surface of genus g is a two-dimensional topological surface, and the surfaces considered here will normally all be marked surfaces. The terminology and notation as introduced here will be used quite freely with little further comment in the remainder of the discussion.

§2. Abelian differentials.

Let M be a compact connected Riemann surface of genus g. Thus M is a one-dimensional complex manifold, so has a coordinate covering $\{U_\alpha, z_\alpha\}$ by open subsets U_α in each of which there is a complex local coordinate z_α, and these local coordinates are holomorphic functions of one another in the intersections $U_\alpha \cap U_\beta$. A holomorphic or meromorphic function on M is a function that is holomorphic or meromorphic in each coordinate neighborhood, in terms of the given local coordinate there. A holomorphic or meromorphic differential on M is a differential form ω that in a coordinate neighborhood U_α can be written $\omega|U_\alpha = f_\alpha(z_\alpha)dz_\alpha$, where $f_\alpha(z_\alpha)$ is a holomorphic or meromorphic function of the local coordinate z_α; in an intersection $U_\alpha \cap U_\beta$ such a form has equivalent expressions $\omega|U_\alpha \cap U_\rho = f_\alpha(z_\alpha)dz_\alpha = f_\beta(z_\beta)dz_\beta$, so $f_\alpha(z_\alpha) = f_\beta(z_\beta) \cdot \dfrac{dz_\beta}{dz_\alpha}$. A basic part of the theory of Riemann surfaces consists of existence theorems, asserting that there always exist at least some such functions and differentials on any Riemann surface. These results will be assumed known, although what is assumed will be stated explicitly, but some of the consequences of these results will be discussed in detail, to establish the notation and terminology to be used here. These topics are discussed in the books by Ahlfors and Sario, by R. C. Gunning (Lectures on Riemann Surfaces, Princeton Univ. Press, 1966), and by H. M. Farkas and I. Kra (Riemann Surfaces, Springer-Verlag, 1980).

The first basic existence theorem is that there are nontrivial holomorphic differential forms on M whenever $g > 0$, and that the set of all such forms comprise a g-dimensional complex vector space. These forms are sometimes called the Abelian differentials or the diffentials of the first kind on M. Any such form ω is automatically a closed differential form, and can be viewed as a Γ-invariant differential form on the universal covering space $\tilde{M}$. Then since $\tilde{M}$ is simply connected the integral $w(z) = \int_{z_0}^{z} \omega$ is a well defined holomorphic function on $\tilde{M}$, and is uniquely determined by the conditions that $dw = \omega$ and that $w(z_0) = 0$; such functions will be called the <u>Abelian integrals</u> on M, and will always be normalized to vanish at the base point $z_0 \in \tilde{M}$. Note that $dw(Tz) - dw(z) = \omega(Tz) - \omega(z) = 0$ for any covering translation $T \in \Gamma$, so that $w(Tz) - w(z) = \omega(T)$ is a complex constant called the <u>period</u> of ω associated to the element $T \in \Gamma$. The mapping $\omega : \Gamma \to \mathbb{C}$ that associates to each element $T \in \Gamma$ the period $\omega(T)$ is clearly a group homomorphism. Note that this period is easily seen to be independent of the base point, since a change in base point merely has the effect of adding a constant to $w(z)$, and can be described alternatively as $\omega(T) = \int_{\gamma_T} \omega$ where γ_T is any closed loop on M representing the homotopy class of the element in $\pi_1(M,p_0)$ represented by the element $T \in \Gamma$.

To any meromorphic function f on M there is associated as customary its <u>divisor</u> $\mathfrak{D}(f) = \sum_p \nu_p(f) \cdot p$, a formal sum over the points $p \in M$, where the coefficient $\nu_p(f) \in \mathbb{Z}$ is the order of the function f at the point p ; at all but finitely many points it is the case that $\nu_p(f) = 0$, and such points can be left out of the formal sum as convenient, so that the divisor can be described alternatively as an element of the free Abelian group generated by the points of M. Similarly there can be associated a divisor $\mathfrak{D}(\omega)$ to any meromorphic differential form ω on M, merely setting $\mathfrak{D}(\omega)|U_\alpha = \mathfrak{D}(f_\alpha)$ when $\omega|U_\alpha = f_\alpha(z_\alpha)dz_\alpha$; this is a well defined divisor on M, since evidently $\mathfrak{D}(f_\alpha) = \mathfrak{D}(f_\beta)$ in $U_\alpha \cap U_\beta$. These divisors can be viewed as Γ-invariant divisors on $\tilde{M}$, and it is often convenient to do so. At any pole of a meromorphic function or differential form the principal part consists of the negative powers of the Laurent expansion of the given function, or of the coefficient of the given differential form, in terms of any local coordinate system at that point; the principal part depends of course on the choice of the local coordinate system. In the case of the principal part of a meromorphic differential form the residue is well defined, simply as the integral around the singularity of interest, and that piece of the principal part is independent of the choice of the local coordinate system. It should be noted that a meromorphic function really has no well defined residue, independent of the choice of local coordinate system.

The second basic existence theorem is that for any finite principal part on M with total residue (the sum of the residues at all poles) equal to zero, there exists a meromorphic differential having the prescribed principal part. The total residue of a differential ω is just the integral $\int_{\partial\Delta}\omega$, where it can be supposed that there are no poles on $\partial\Delta$, and that integral is equal to zero since ω is Γ-invariant and the integrals along corresponding boundary arcs clearly cancel; the residue condition is thus necessary. That it is sufficient is the main point of course. The simplest instance of this existence theorem is the assertion that for any point $p \in M$ there exists a meromorphic differential form having a double pole with residue zero at that point and no other singularities on M. Such a form is called a meromorphic Abelian differential of the second kind, or just a differential of the second kind on M. If ω_p is such a differential then ω_p can be viewed as a meromorphic Γ-invariant differential form on $\tilde{M}$; it is a closed differential form outside its singularities, and its integral around any closed curve avoiding these singularities is zero since $\tilde{M}$ is simply connected and the residue is zero at each singularity. The integral $w_p(z) = \int_{z_0}^{z} \omega_p$ is a well defined meromorphic function on $\tilde{M}$ and is uniquely determined by the conditions that $dw_p = \omega_p$ and that $w_p(z_0) = 0$, where this last normalization of course only makes sense when $z_0 \neq p$. This will be called an <u>Abelian integral of the second kind</u> on M, and can as here be normalized to vanish at the base point $z_0 \in M$ provided that z_0 does not represent the point p; but there are more convenient normalizations that avoid this complication, and that will be discussed later. Just as for ordinary abelian integrals $w_p(Tz) - w_p(z) = \omega_p(T)$ is a constant for any $T \in \Gamma$, and is called the period of w_p associated to the element T; these periods comprise the period homomorphism $\omega_p \in \mathrm{Hom}(\Gamma, \mathbb{C})$.

Another instance of the same existence theorem is the assertion that for any pair of distinct points p_+, p_- of M there exists a meromorphic differential form having simple poles at these points, with residue $+1$ at p_+ and residue -1 at p_-, and no other singularities on M. Such a form is called a <u>meromorphic Abelian differential of the third kind</u>, or just a <u>differential of the third kind</u> on M. If ω_{p_+,p_-} is such a differential then it can be viewed as a meromorphic Γ-invariant differential form on $\tilde{M}$; it is a closed differential form on the simply connected manifold $\tilde{M}$ outside its singularities, but does not have a single valued integral on $\tilde{M}$ because it has nonzero residues at these singularities. To remedy that, choose a simple path δ on M from p_- to p_+, an oriented arc with boundary $\partial\delta = p_+ - p_-$. Now ω_{p_+,p_-} can be viewed as a holmorphic differential form on $\tilde{M}$ outside the lifts to $\tilde{M}$ of this path δ, and the integral of this form around any closed path in the complement of the lifts of δ is zero since any such path must surround as many points covering p_+ as covering p_- . The integral $w_\delta(z) = \int_{z_0}^{z} \omega_{p_+,p_-}$ is a well defined holomorphic function on $\tilde{M}$ outside the lifts of δ , and is uniquely characterized by the conditions that $dw_\delta = \omega_{p_+,p_-}$ and that $w_\delta(z_0) = 0$. This will be called an <u>Abelian integral of the third kind</u> on Γ , and again can be normalized to vanish at the base point $z_0 \in \tilde{M}$ provided that z_0 does not represent a point on δ , although a more convenient normalization will be discussed later. Again $w_\delta(Tz) - w_\delta(z) = \omega_\delta(T)$ is a constant for any $T \in \Gamma$, and is called the <u>period</u> of w_δ associated to the element T ; these periods comprise the period homomorphism $\omega_\delta \in \mathrm{Hom}(\Gamma,\mathbb{C})$.

Now consider a marked Riemann surface M of genus $g > 0$, and let $\omega_1,\ldots,\omega_g$ be a basis for the Abelian differentials on M ; the corresponding Abelian integrals $w_1,\ldots,w_g$ are normalized to vanish at the base point $z_0 \in \tilde{M}$. The periods of these abelian integrals are completely determined by the values $\omega_i(A_j),\omega_i(B_j)$, and it is convenient to arrange these values as the entries in the associated $g \times 2g$ <u>period matrix</u> $\Lambda = (\Lambda',\Lambda'')$; here Λ',Λ'' are $g \times g$ matrix blocks with

$$\lambda'_{ij} = \omega_i(A_j) \ , \ \lambda''_{ij} = \omega_i(B_j) \ .$$

A basic property of this matrix is given by the following result due to Riemann.

<u>Theorem 1</u>. The period matrix $\Lambda = (\Lambda',\Lambda'')$ has the properties that

(i) $\Lambda' \, {}^t\Lambda''$ is a symmetric matrix,

(ii) $i(\Lambda' \, {}^t\bar{\Lambda}'' - \Lambda'' \, {}^t\bar{\Lambda}')$ is a positive definite Hermitian matrix.

Proof. View the differentials ω_i and their integrals w_i as holomorphic differential forms and functions on $\tilde{M}$. The product $w_i\omega_j$ is then a holomorphic differential form on $\tilde{M}$ for which $d(w_i\omega_j) = \omega_i \wedge \omega_j = 0$, so by Stokes's theorem

$$0 = \int_{\Delta} d(w_i\omega_j) = \int_{\partial\Delta} w_i\omega_j .$$

The boundary $\partial\Delta$ has the form given in (1.1), so that the integral over $\partial\Delta$ can be written as a sum of integrals over translates of the paths α_j, β_j on $\tilde{M}$; upon using the periods of the integrals to simplify this expression, noting in particular that the period associated to any commutator C_j is zero, it follows that

$$\begin{aligned}
0 &= \textstyle\sum_{k=1}^{g} \{\int_{\alpha_k} w_i(C_1\cdots C_{k-1}z)\omega_j(C_1\cdots C_{k-1}z) - w_i(C_1\cdots C_kB_kz)\omega_j(C_1\cdots C_kB_kz) \\
&\qquad + \int_{\beta_k} w_i(C_1\cdots C_{k-1}A_kz)\omega_j(C_1\cdots C_{k-1}A_kz) - w_i(C_1\cdots C_kz)\omega_j(C_1\cdots C_kz)\} \\
&= \textstyle\sum_{k=1}^{g} \{\int_{\alpha_k} w_i(z)\,\omega_j(z) - [w_i(z) + \omega_i(B_k)]\omega_j(z) \\
&\qquad + \int_{\beta_k} [w_i(z) + \omega_i(A_k)]\omega_j(z) - w_i(z)\omega_j(z)\} \\
&= \textstyle\sum_{k=1}^{g} \{-\omega_i(B_k)\int_{\alpha_k}\omega_j(z) + \omega_i(A_k)\int_{\beta_k}\omega_j(z)\} \\
&= \textstyle\sum_{k=1}^{g} \{\omega_i(A_k)\,\omega_j(B_k) - \omega_i(B_k)\omega_j(A_k)\} ,
\end{aligned}$$

thus proving the first assertion of the theorem.

Proof. Note first that for any period matrix $\Lambda = (\Lambda', \Lambda'')$ the $g \times g$ block Λ' is necessarily nonsingular. Indeed if Λ' were singular there would be some nonzero vector $c \in \mathbb{C}^g$ such that ${}^t c\Lambda' = 0$; but then

$$i\ {}^t c(\Lambda' {}^t\bar{\Lambda}'' - \Lambda'' {}^t\bar{\Lambda}')\bar{c} = i({}^t c\Lambda')({}^t\bar{\Lambda}''\bar{c}) - i({}^t c\Lambda'')^t({}^t\overline{c\Lambda'}) = 0 ,$$

in contradiction to part (ii) of the preceding theorem. Note next that changing the basis for the Abelian differentials by a linear transformation $C \in GL(g,\mathbb{C})$ has the effect of replacing the period matrix Λ by $C\Lambda$. It is evident from these two observations that there is a unique basis for the Abelian differentials for which the period matrix has the form $\Lambda = (I,\Omega)$. In this special case the conclusions of the previous theorem are (i) that Ω is a symmetric matrix and (ii) that $i(\bar{\Omega} - \Omega) = 2\ \mathrm{Im}\Omega$ is positive definite, thereby concluding the proof.

The Abelian differentials of this corollary are called the canonical Abelian differentials for the marked Riemann surface M ; they depend on the particular marking chosen, and are uniquely determined by it. They are characterized by the period conditions

$$\omega_i(A_j) = \delta^i_j \ ; \tag{1}$$

the remaining periods $\omega_{ij} = \omega_i(B_j)$ then form a matrix $\Omega = \{\omega_{ij}\} \in \mathfrak{H}_g$. Throughout the remainder of the discussion here the canonical Abelian differentials will always be the basis chosen for the space of Abelian differentials.

Proof. Note first that for any period matrix $\Lambda = (\Lambda', \Lambda'')$ the $g \times g$ block Λ' is necessarily nonsingular. Indeed if Λ' were singular there would be some nonzero vector $c \in \mathbb{C}^g$ such that ${}^tc\Lambda' = 0$; but then

$$i\,{}^tc(\Lambda'\,{}^t\bar{\Lambda}'' - \Lambda''\,{}^t\bar{\Lambda}')\bar{c} = i({}^tc\Lambda')({}^t\bar{\Lambda}''\bar{c}) - i({}^tc\Lambda'')^t({}^t\overline{c\Lambda'}) = 0 ,$$

in contradiction to part (ii) of the preceding theorem. Note next that changing the basis for the Abelian differentials by a linear transformation $C \in GL(g,\mathbb{C})$ has the effect of replacing the period matrix Λ by $C\Lambda$. It is evident from these two observations that there is a unique basis for the Abelian differentials for which the period matrix has the form $\Lambda = (I,\Omega)$. In this special case the conclusions of the previous theorem are (i) that Ω is a symmetric matrix and (ii) that $i(\bar{\Omega} - \Omega) = 2\,\mathrm{Im}\Omega$ is positive definite, thereby concluding the proof.

The Abelian differentials of this corollary are called the canonical Abelian differentials for the marked Riemann surface M ; they depend on the particular marking chosen, and are uniquely determined by it. They are characterized by the period conditions

(1) $$\omega_i(A_j) = \delta^i_j ;$$

the remaining periods $\omega_{ij} = \omega_i(B_j)$ then form a matrix $\Omega = \{\omega_{ij}\} \in \mathfrak{H}_g$. Throughout the remainder of the discussion here the canonical Abelian differentials will always be the basis chosen for the space of Abelian differentials.

The period matrix $\Lambda = (I,\Omega)$ of the canonical Abelian differentials on the marked Riemann surface M is of the form considered in the discussion of theta functions in part A, and conforming to the terminology used there the $g \times g$ matrix block Ω will also be called the <u>period matrix</u> of the marked surface M. The columns of $\Lambda = (I,\Omega)$ generate a lattice subgroup $\underline{L} = (I,\Omega)\mathbb{Z}^{2g} \subseteq \mathbb{C}^g$, and the associated complex torus $J = \mathbb{C}^g/\underline{L}$ will be called the <u>Jacobi variety</u> of the marked surface M; this torus has the natural marking corresponding to its representation in this form. There are very strong interconnections between the surface M and its Jacobi variety J, and the examination of these will be a prominent motif in the subsequent discussion. As a preamble to these considerations, let w denote the column vector composed of the g canonical Abelian integrals, defining a holomorphic mapping

$$(2) \qquad w : \tilde{M} \to \mathbb{C}^g .$$

For any element $T \in \Gamma$ it then follows that $w(Tz) = w(z) + \omega(T)$ for some vector $\omega(T) \in \underline{L}$; in particlar $\omega(A_j) = \delta_j$ and $\omega(B_j) = \Omega\delta_j$. The mapping $\omega : \Gamma \to \underline{L}$ defined by this period mapping is clearly a surjective group homomorphism, the canonical generators of Γ being mapped to the canonical generators of $\underline{L}$; this identifies $\underline{L}$ with the abelianization of Γ, so the kernel is precisely the commutator subgroup of Γ. The mapping (2) then induces a holomorphic mapping

$$(3) \qquad w : M = \tilde{M}/\Gamma \to J = \mathbb{C}^g/\underline{L} ,$$

often called the Abel-Jacobi mapping.

There are corresponding canonical meromorphic Abelian differentials on a marked Riemann surface M. For any point $p \in M$ the differential of the second kind ω_p is uniquely determined up to a constant factor and the addition of an Abelian differential. It is always possible to choose this additive Abelian differential in such a way that $\omega_p(A_j) = 0$. The resulting meromorphic differential will be called the canonical differential of the second kind with pole p on the marked Riemann surface M, and is uniquely determined up to a nonzero constant factor. Similarly for any distinct points p_+, p_- on M and simple path δ with $\partial\delta = p_+ - p_-$ the differential of the third kind ω_{p_+,p_-} is uniquely determined up to the addition of an Abelian differential, and the latter can always be chosen so that $\omega_\delta(A_j) = 0$. The resulting meromorphic differential will be called the canonical differential of the third kind corresponding to the arc δ on the marked Riemann surface M, and is uniquely determined by this arc δ; it will be denoted by ω_δ subsequently.

§3. The cross-ratio function.

The canonical differential of the third kind ω_δ on a marked Riemann surface M is really an everywhere defined meromorphic differential on $\tilde{M}$, but its integral can only be defined as a single valued function in the complement of the lifts to $\tilde{M}$ of the arc δ. The problem is of course that the integral $w_\delta(z) = \int_{z_0}^{z} \omega_\delta$ has logarithmic branch points at the points of $\tilde{M}$ lying over p_+ and p_-, where $\partial\delta = p_+ - p_-$, so the value $w_\delta(z)$ changes by $\pm 2\pi i$ when continued analytically around any such branch point; however the function

$$(1) \qquad p_\delta(z) = \exp w_\delta(z)$$

remains single valued upon such analytic continuation. At any point of $\tilde{M}$ lying over p_+ the differential ω_δ has a simple pole with residue $+1$, so in a local coordinate z centered at that point has principal part $\frac{1}{z}\,dz$; its integral is $\log z$ + (holomorphic function), so $p_\delta(z)$ extends to a holomorphic function with a simple zero at that point. On the other hand at any point of $\tilde{M}$ lying over p_- the differential $\omega_\delta(z)$ has a simple pole with residue -1, so in a local coordinate z centered at that point has principal part $-\frac{1}{z}\,dz$; its integral is $-\log z$ + (holomorphic function), so $p_\delta(z)$ extends to a meromorphic function with a simple pole at that point. Thus $p_\delta(z)$ is a meromorphic function on $\tilde{M}$, with simple zeros at the points of $\tilde{M}$ lying over p_+, simple poles at the points of $\tilde{M}$ lying over p_-, and no other zeros or singularities. Moreover it follows from the periodicity properties of the integral $w_\delta(z)$ that

$$p_\delta(A_j z) = p_\delta(z) \ , \ p_\delta(B_j z) = p_\delta(z) \cdot \exp \omega_\delta(B_j) \tag{2}$$

for the generators A_j, B_j of Γ. Further properties of this auxiliary function follow from a further analysis of the periods of the integral $w_\delta(z)$.

Theorem 2. If δ is a simple path on the marked Riemann surface M disjoint from the paths α_j, β_j of the marking then

$$\omega_\delta(B_j) = 2\pi i \int_\delta \omega_j \ .$$

If δ', δ'' are two disjoint simple paths on M, disjoint from the paths α_j, β_j of the marking, then

$$\int_{\delta'} \omega_{\delta''} = \int_{\delta''} \omega_{\delta'} \ .$$

Proof. Let δ denote a lifting of the given path to the fundamental polygon $\Delta \subseteq \tilde{M}$, with $\partial\delta = z_+ - z_-$ for some points $z_+, z_- \in \Delta$. The product $w_j(z)\ \omega_\delta(z)$ of the Abelian integral $w_j(z)$ and the differential $\omega_\delta(z)$ is a meromorphic differential form in Δ; it has a simple pole at z_+ with residue $w_j(z_+)$ and a simple pole at z_- with residue $w_j(z_-)$, so by the residue theorem

$$\begin{aligned} 2\pi i[w_j(z_+) - w_j(z_-)] &= \int_{\partial\Delta} w_j(z)\ \omega_\delta(z) \\ &= \textstyle\sum_{k=1}^{g} \{\int_{\alpha_k} w_j(C_1\cdots C_{k-1}z)\omega_\delta(C_1\cdots C_{k-1}z) - w_j(C_1\cdots C_k B_k z)\omega_\delta(C_1\cdots C_k B_k z) \\ &\quad + \int_{\beta_k} w_j(C_1\cdots C_{k-1}A_k z)\omega_\delta(C_1\cdots C_{k-1}A_k(z) - w_j(C_1\cdots C_k z)\omega_\delta(C_1\cdots C_k z)\} \end{aligned}$$

$$= \sum_{k=1}^{g} \{-\omega_{jk} \int_{\alpha_k} \omega_\delta(z) + \delta_k^j \int_{\beta_k} \omega_\delta(z)\}$$

$$= \omega_\delta(B_j) ,$$

since $\omega_\delta(A_j) = 0$ by the normalization chosen. This yields the first assertion of the theorem, since

$$\int_\delta \omega_j = w_j(z_+) - w_j(z_-) .$$

Next let δ', δ'' denote liftings of the other given paths to Δ, with $\partial\delta' = z'_+ - z'_-$ and $\partial\delta'' = z''_+ - z''_-$, and choose disjoint simple closed curves γ', γ'' in Δ so that γ' encircles δ' and γ'' encircles δ''. The differential form $w_{\delta'}(z)\omega_{\delta''}(z)$ is holomorphic in Δ outside the interiors of these loops γ' and γ'', so that

$$\int_{\partial\Delta} w_{\delta'}(z)\ \omega_{\delta''}(z) = \int_{\gamma'} w_{\delta'}(z)\ \omega_{\delta''}(z) + \int_{\gamma''} w_{\delta'}(z)\omega_{\delta''}(z) .$$

Now the integral over $\partial\Delta$ can be calculated just as in the first part of the proof, and is readily seen to vanish since $\omega_{\delta'}(A_j) = \omega_{\delta''}(A_j) = 0$ by the chosen normalization. Inside γ'' the differential form $w_{\delta'}(z)\omega_{\delta''}(z)$ is a well defined meromorphic differential form, having a simple pole at z''_+ with residue $w_{\delta'}(z''_+)$, a simple pole at z''_- with residue $-w_{\delta'}(z''_-)$, and no other singularities; so by the residue theorem

$$\int_{\gamma''} w_{\delta'}(z)\omega_{\delta''}(z) = 2\pi i[w_{\delta'}(z''_+) - w_{\delta'}(z''_-)] = 2\pi i \int_{\delta'} \omega_{\delta''} .$$

Inside γ' however the function $w_{\delta'}(z)$ is not single valued; but both $w_{\delta'}(z)$ and $w_{\delta''}(z)$ are well defined near γ', so by Stokes's theorem

$$0 = \int_{\gamma'} d[w_{\delta'}(z)w_{\delta''}(z)] = \int_{\gamma'} w_{\delta'}(z)\omega_{\delta''}(z) + \int_{\gamma'} w_{\delta''}(z)\omega_{\delta'}(z) .$$

The differential form $w_{\delta''}(z)\omega_{\delta'}(z)$ is a well defined meromorphic differential form inside γ', so as above

$$\begin{aligned}\int_{\gamma'} w_{\delta'}(z)\omega_{\delta''}(z) &= -\int_{\gamma'} w_{\delta''}(z)\omega_{\delta'}(z)\\ &= -2\pi i[w_{\delta''}(z'_+) - w_{\delta''}(z'_-)] = -2\pi i \int_{\delta'} \omega_{\delta''} .\end{aligned}$$

The desired result follows readily from this.

Now consider any four distinct points z_1, z_2, a_1, a_2 in the fundamental polygon $\Delta \subseteq \tilde{M}$, and choose disjoint simple paths δ, γ inside Δ with $\partial\delta = z_1 - z_2$ and $\partial\gamma = a_1 - a_2$. The second identity of the preceding theorem shows that

$$w_\gamma(z_1) - w_\gamma(z_2) = w_\delta(a_1) - w_\delta(a_2) ,$$

and exponentiating this yields the identity

$$p_\gamma(z_1)/p_\gamma(z_2) = p_\delta(a_1)/p_\delta(a_2) .$$

It is thus possible to define a function $p(z_1,z_2,a_1,a_2)$ of any four distinct points of Δ by setting

(3) $$p(z_1,z_2,a_1,a_2) = p_\gamma(z_1)/p_\gamma(z_2) = p_\delta(a_1)/p_\delta(a_2) .$$

The first equality in (3) shows that this function extends to a meromorphic function of $(z_1,z_2) \in \tilde{M} \times \tilde{M}$ for any fixed $(a_1,a_2) \in \Delta \times \Delta$, $a_1 \neq a_2$, and the second shows similarly that it extends to a meromorphic function of $(a_1,a_2) \in \tilde{M} \times \tilde{M}$ for any fixed $(z_1,z_2) \in \Delta \times \Delta$, $z_1 \neq z_2$. A general theorem of W. Rothstein (Ein neuer Beweis des Hartogschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen. Math. Z. 53 (1950), 84-95.) then shows that this extended function is a meromorphic function of all four variables $(z_1,z_2,z_3,z_4) \in \Delta^4$, and the functional equations (2) extend this further to a meromorphic function on $\tilde{M}^4$. This is called the cross-ratio function of the marked Riemann surfaces M, and is

Theorem 3. The cross-ratio function for the marked Riemann surface M is a meromorphic function on the four dimensional complex manifold $\tilde{M}^4$ such that:

(i) $p(z_1,z_2,a_1,a_2)$ has simple zeros along the subvariety
$\cup_{T\in\Gamma}[(z_1 = Ta_1) \cup (z_2 = Ta_2)]$;

(ii) $p(z_1,z_2,a_1,a_2)$ has simple poles along the subvariety
$\cup_{T\in\Gamma}[(z_1 = Ta_2) \cup (z_2 = Ta_1)]$;

(iii) $p(z,z,a_1,a_2) = p(z_1,z_2,a,a) = 1$;

(iv) $p(A_j z_1,z_2,a_1,a_2) = p(z_1,z_2,a_1,a_2)$,
$p(B_j z_1,z_2,a_1,a_2) = p(z_1,z_2,a_1,a_2)\exp 2\pi i[w_j(a_1) - w_j(a_2)]$;

(v) $p(z_1,z_2,a_1,a_2) = p(a_1,a_2,z_1,z_2)$
$= p(z_2,z_1,a_1,a_2)^{-1} = p(z_1,z_2,a_2,a_1)^{-1}$.

These properties uniquely characterize the cross-ratio function.

Proof. Condition (v) follows immediately from the defining formula (3) whenever z_1, z_2, a_1, a_2 are distinct points in Δ, and must then hold throughout $\tilde{M}^4$ by the identity theorem for meromorphic functions. Condition (iv) follows from the first equality in (3), together with the observation (2) and the first part of Theorem 2. Combining (iv) and (v) shows that similar functional equations hold in all variables, so that it is only necessary to prove the remaining assertions within the fundamental polygon Δ. Whenever z_2, a_1, a_2 are distinct points of Δ it follows from the first equality in (3) that $p(z_1, z_2, a_1, a_2)$ as a function of z_1 has simple zeros at the points Ta_1, simple poles at the points Ta_2, and takes the value 1 at the point z_2; similar results hold for the other variables by (v), and that suffices to demonstrate the first three conditions as well. If $f(z_1, z_2, a_1, a_2)$ is any other function satisfying the same conditions then $p(z_1, z_2, a_1, a_2)/f(z_1, z_2, a_1, a_2) = h(z_1, z_2, a_1, a_2)$ is a holomorphic and nowhere vanishing function on $\tilde{M}^4$ as a consequence of (i) and (ii). As a function of z_1 alone it is Γ-invariant by (iv), hence must be a constant, so is really independent of z_1. It then follows from (v) that it is a constant in all variables, and (iii) shows that that constant is 1, thereby concluding the proof.

It is worth noting explicitly that if the path δ is deformed continuously holding the end points fixed then the period normalization fixing the canonical differential of the third kind ω_δ is unchanged; thus that differential really only depends on the homotopy class of the path δ. Of course the integral is only defined as a single valued function in the complement of a fixed path, but that is another matter. The homotopy class of δ is in turn fully determined by giving the end points of any lift of δ to $\tilde{M}$, so that ω_δ is fully specified merely by giving those points. This observation can be derived alternatively by noting from (1) and (3) that

$$\text{(4)} \qquad \omega_\delta(z) = \frac{d}{dz} \log p_\delta(z)dz = \frac{d}{dz} \log \frac{p_\delta(z)}{p_\delta(z_1)}\, dz$$
$$= \frac{d}{dz} \log p(z, z_1, a_1, a_2)dz$$

for any $z_1 \in \tilde{M}$, where $\partial\delta = a_1 - a_2$ for some lift of the path δ to $\tilde{M}$. It is often convenient to denote this differential just by $\omega_{a_1,a_2}(z)$.

The unique Riemann surface of genus $g = 0$ is the projective line $M = \mathbf{P}^1$, which is simply connected so that $\tilde{M} = M$, and in this case the cross-ratio function is the classical cross-ratio

$$p(z_1, z_2, a_1, a_2) = \frac{(z_1 - a_1)}{(z_1 - a_2)} \Big/ \frac{(z_2 - a_1)}{(z_2 - a_2)} .$$

The cross-ratio function in general has a corresponding analytic behavior, at least in the sense that it has a simple zero at a_1, a simple pole at a_2, and the value 1 at z_2, whence the terminology. That also suggests the possibility of factoring the cross-ratio function correspondingly in general; it remains only to find the analogue of the function $z - a$ for an arbitrary Riemann surface, and that will be taken up after some further necessary preliminaries are treated.

§4. Factors of automorphy.

Although the line bundles that arise naturally when dealing with meromorphic functions and forms on Riemann surfaces can be described in local terms, it is more natural in dealing with these bundles in connection with theta functions to describe them globally, by factors of automorphy as had arisen earlier in the discussion of the functional equations of theta functions. This does require some further knowledge of the function-theoretic properties of the universal covering space $\tilde{M}$. The simplest approach is perhaps just to assume the strongest result, the general theorem of uniformization. Whenever the genus g of M is strictly positive the universal covering space $\tilde{M}$ is a noncompact simply connected Riemann surface, and is biholomorphically equivalent either to the complex plane $\mathbb{C}$ in case $g = 1$ or to the unit disc $D \subset \mathbb{C}$ in case $g > 1$. In both cases the function-theoretic properties are quite well known and are treated in the books by Ahlfors and Sario and by Farkas and Kra already referred to.

If $\underset{\sim}{\mathfrak{d}}$ is any divisor on M then $\underset{\sim}{\mathfrak{d}}$ can be viewed as a Γ-invariant divisor on $\tilde{M}$. Whenever $g > 0$ so that $\tilde{M}$ is either the complex plane or the unit disc it then follows from Weierstrass's theorem that there exists a meromorphic function f on $\tilde{M}$ with that as its divisor. This function need not be invariant under Γ; but its divisor certainly is, so that $f(Tz)/f(z) = \mu(T,z)$ is a holomorphic and nowhere vanishing function of $z \in \tilde{M}$ for any fixed covering translation $T \in \Gamma$, and it is easy to see from this that

$$(1) \qquad \mu(ST,z) = \mu(S,Tz)\mu(T,z)$$

whenever $S,T \in \Gamma$. A nowhere vanishing function μ on $\Gamma \times \tilde{M}$ that satisfies (1) as a function of $T \in \Gamma$ and is holomorphic as a function of $z \in \tilde{M}$ is called a factor of automorphy for the action of the group Γ on $\tilde{M}$. For any such factor of automorphy μ, a meromorphic function f on $\tilde{M}$ such that $f(Tz) = \mu(T,z) \cdot f(z)$ for all $T \in \Gamma$ is called a meromorphic relatively automorphic function for μ. The holomorphic such functions will be called the holomorphic relatively automorphic functions for μ, or more commonly just the relatively automorphic functions for μ; the set of these latter functions form a complex vector space that will be denoted by $\Gamma(\mu)$, and the dimension of this space will be denoted by $\gamma(\mu)$. It is apparent that the product of any two factors of automorphy is again a factor of automorphy, as is their quotient; so the set of factors of automorphy for Γ form an abelian group, the identity of which is the factor of automorphy for which $\mu(T,z) = 1$. Two factors of automorphy μ,ν are called equivalent if there is a holomorphic and nowhere vanishing function h on $\tilde{M}$ such that $h(Tz) = h(z)\mu(T,z)/\nu(T,z)$ for all $T \in \Gamma$ and $z \in \tilde{M}$; this is evidently an equivalence relation in the usual sense.

Now to any divisor $\mathfrak{d}$ on M there has as above been associated a factor of automorphy μ, which admits a meromorphic relatively automorphic function f for which $\mathfrak{d}(f) = \mathfrak{d}$. This function f is clearly determined uniquely up to a holomorphic nowhere vanishing function on $\tilde{M}$, so that the factor of automorphy is determined uniquely up to equivalence, by the divisor $\mathfrak{d}$. On the other hand it is quite possible that different divisors determine equivalent factors of automorphy; two divisors $\mathfrak{d},\mathfrak{d}'$ will be called linearly equivalent if they do determine equivalent factors of automorphy, and that will be indicated by writing $\mathfrak{d} \sim \mathfrak{d}'$. Clearly $\mathfrak{d} \sim \mathfrak{d}'$ precisely when there exist meromorphic functions f,f' on M such that $\mathfrak{d}(f) = \mathfrak{d}$, $\mathfrak{d}(f') = \mathfrak{d}'$, and f, f' are meromorphic relatively automorphic functions for the same factor of automorphy; but this condition just means that the quotient $g = f/f'$ is a Γ-invariant meromorphic function on $\tilde{M}$, hence is really a meromorphic function on M , so that $\mathfrak{d} \sim \mathfrak{d}'$ precisely when there is a meromorphic function g on M such that $\mathfrak{d}(g) = \mathfrak{d} - \mathfrak{d}'$. This last condition is the classical notion of linear equivalence of divisors, and indicates the possibility of other relations between factors of automorphy and meromorphic functions on M . To pursue this line of thought further, consider a divisor $\mathfrak{d}$ on M and its associated factor of automorphy μ ; thus there is a meromorphic relatively automorphic function f for this factor of automorphy, such that $\mathfrak{d}(f) = \mathfrak{d}$. Then whenever $h \in \Gamma(\mu)$ the function h/f is a meromorphic function g on M such that $\mathfrak{d}(g) + \mathfrak{d} = \mathfrak{d}(gf) = \mathfrak{d}(h) \geqq 0$, where $\mathfrak{d} = \sum_p \nu_p \cdot p \geqq 0$ means that $\nu_p \geqq 0$ for all p ; conversely whenever g is a meromorphic function on M such that $\mathfrak{d}(g) + \mathfrak{d} \geqq 0$ then $\mathfrak{d}(gf) \geqq 0$ so that $h = gf \in \Gamma(\mu)$. Thus there is a natural isomorphism between the vector space $\Gamma(\mu)$ and the vector space of meromorphic functions g on M such that $\mathfrak{d}(g) + \mathfrak{d} \geqq 0$.

It can be observed just as in the earlier discussion of factors of automorphy for a complex torus that the basic functional equation (1) is just the condition that Γ acts as a group of holomorphic automorphisms of the product manifold $\tilde{M} \times E$ by defining $T(z,t) = (Tz, \mu(T,z) \cdot t)$; the quotient space $(\tilde{M} \times E)/\Gamma$ under this action maps naturally to the quotient space $\tilde{M}/\Gamma = M$, and describes a complex line bundle over M . The relatively automorphic functions are sections of this bundle, and equivalent factors of automorphy correspond to biholomophically equivalent line bundles.

The simplest nontrivial factors of automorphy for the action of Γ on $\tilde{M}$ are those that are independent of z, called the <u>flat factors of automorphy</u>; for these, condition (1) reduces to the assertion that $\mu(ST) = \mu(S)\mu(T)$, that is, that $\mu \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$. Any such representation is uniquely determined by its values on the canonical generators $A_1,\ldots,A_g,B_1,\ldots,B_g$ of Γ, and these values can be prescribed arbitrarily. It is convenient to introduce the two homomorphisms $\sigma_t,\rho_t \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ that for any vector $t = {}^t(t_1,\ldots,t_g) \in \mathbb{C}^g$ are defined by

$$(2) \qquad \begin{array}{ll} \sigma_t(A_j) = \exp 2\pi i t_j & \rho_t(A_j) = 1 \\ \sigma_t(B_j) = 1 & \rho_t(B_j) = \exp 2\pi i t_j \, . \end{array}$$

Then any $\phi \in \mathrm{Hom}(\Gamma,\mathbb{C})$ can be written as the product $\phi = \sigma_s\rho_t$ for some vectors $s,t \in \mathbb{C}^g$, and these vectors are uniquely determined up to elements of $\mathbb{Z}^g$. To see one instance in which these factors of automorphy arise, note that the functional equation (iv) in Theorem 3 can be restated in the form

$$(3) \qquad p(Tz_1,z_2,a_1,a_2) = \rho_{w(a_1-a_2)}(T)\cdot p(z_1,z_2,a_1,a_2) \, ,$$

so that the cross-ratio function $p(z_1,z_2,a_1,a_2)$ viewed as a function of the variable z_1 alone is a meromorphic relatively automorphic function for the factor of automorphic $\rho_{w(a_1-a_2)}$. As for the general properties of these factors of automorphy, note the following.

Theorem 4. Any flat factor of automorphy is equivalent to ρ_t for some $t \in \mathbb{C}^g$, and two factors of automorphy ρ_s, ρ_t are equivalent precisely when $s - t \in L$, the lattice subgroup generated by the period matrix of M.

Proof. The factor of automorphy $\phi = \sigma_s \rho_t$ is equivalent to the identity factor of automorphy precisely when there is a holomorphic nowhere vanishing function h on $\tilde{M}$ such that $h(Tz) = \phi(T)h(z)$ for all $T \in \Gamma$. Since $\tilde{M}$ is simply connected there is always a single valued branch of $w(z) = \frac{1}{2\pi i} \log h(z)$, and the functional equation for h is equivalent to the condition that

$$(4) \qquad \begin{aligned} w(A_j z) &= w(z) + s_j + m_j \\ w(B_j z) &= w(z) + t_j + n_j \end{aligned}$$

for some integers m_j, n_j. The differential $dw(z)$ is then invariant under Γ, so must be an Abelian differential and hence be expressible as a linear combination of the canonical Abelian differentials in the form

$$dw(z) = \sum_i c_i \omega_i(z)$$

for some constants c_i; equivalently

$$(5) \qquad w(z) = c + \sum_i c_i w_i(z)$$

for some constant $c = w(z_0)$. Since $w(z)$ has the periods given by (4) while $w_i(z)$ have the known periods as discussed in §2, it follows from (5) that

$$s_j + m_j = c_j$$
$$t_j + n_j = \sum_i c_i \omega_{ij} \ ,$$

so that $\phi = \sigma_s \rho_t$ is equivalent to the identity factor precisely when

$$t_j + n_j = \sum_i (s_i + m_i)\omega_{ij} = \sum_i \omega_{ji}(s_i + m_i)$$

for some integers n_j, m_j, hence when

$$t - \Omega s \in L \ .$$

The desired result follows readily from this observation.

As another preparatory interjection, the degree of a divisor $\mathfrak{z} = \sum_p \nu_p \cdot p$ on M is defined to be the integer $\deg \mathfrak{z} = \sum_p \nu_p$. It was already observed that the canonical Abelian integrals on M can be used as the components of a holomorphic mapping $w : \tilde{M} \to \mathbb{C}^g$ that induces a holomorphic mapping $w : M \to J$, where $J = \mathbb{C}^g / L$ is the Jacobi variety of M. Since J is an abelian group this Abel-Jacobi mapping naturally extends to a homomorphism from the free abelian group generated by the points of M to the group J, that is, from the group of divisors on M to J, merely setting $w(\mathfrak{z}) = w(\sum_p \nu_p \cdot p) = \sum_p \nu_p w(p)$. In these terms the analytical importance of flat factors of automorphy lies in the following result.

Theorem 5. The factor of automorphy associated to a divisor $\mathfrak{z}$ on M is equivalent to a flat factor of automorphy precisely when $\deg \mathfrak{z} = 0$.

If $\deg \mathfrak{d} = 0$ then $\rho_{w(\mathfrak{d})} \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ is a flat factor of automorphy associated to $\mathfrak{d}$.

<u>Proof</u>. If $\phi \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ is a factor of automorphy associated to some divisor $\mathfrak{d}$ on M then there is a meromorphic relatively automorphic function f for ϕ such that $\mathfrak{d}(f) = \mathfrak{d}$. It can be supposed that this divisor is disjoint from the paths $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ of the marking of M. Then $\deg \mathfrak{d}$ is just the total order of the function f in the canonical fundamental polygon $\Delta \subseteq \tilde{M}$, so by the residue theorem

$$\begin{aligned}2\pi i \deg \mathfrak{d} &= \int_{\partial\Delta} d \log f(z) \\ &= \textstyle\sum_{k=1}^{g} \{\int_{\alpha_k} d \log f(C_1\cdots C_{k-1}z) - d \log f(C_1\cdots C_k B_k z) \\ &\qquad + \int_{\beta_k} d \log f(C_1\cdots C_{k-1}A_k z) - d \log f(C_1\cdots C_k z)\} \\ &= 0\end{aligned}$$

since $d \log f(Tz) = d \log[\phi(T)f(z)] = d \log f(z)$ for all $T \in \Gamma$.

On the other hand if $\mathfrak{d} = \sum_p \nu_p \cdot p$ is a divisor with $\deg \mathfrak{d} = \sum_p \nu_p = 0$, for which it can again be assumed that $\mathfrak{d}$ is disjoint from the paths of the marking of M, then choose some points $a_j \in \Delta$ so that $\mathfrak{d}$ is represented by the divisor $\mathfrak{d} = \sum_j \nu_j \cdot a_j$ on $\tilde{M}$. Choose any points $a,b \in \Delta$ distinct from one another and from the points a_j, and introduce the meromorphic function f on $\tilde{M}$ defined in terms of the cross-ratio function by

$$f(z) = \Pi_j\, p(z,b,a_j,a)^{\nu_j}.$$

It follows from (i) and (ii) of Theorem 3 that f has order ν_j at the point a_j, order $-\sum_j \nu_j = 0$ at the point b, and order zero otherwise; thus $\mathfrak{d}(f) = \mathfrak{d}$ when viewed as a divisor on M, so that the factor of automorphy $\mu(T,z) = f(Tz)/f(z)$ represents the divisor $\mathfrak{d}$. However from (3) it is apparent that

$$\begin{aligned}\mu(T,z) &= \Pi_j \rho_{w(a_j - a)}(T)^{\nu_j} \\ &= \rho_{w(\sum \nu_j a_j - \sum \nu_j a)}(T) = \rho_{w(\mathfrak{d})}(T)\end{aligned}$$

as desired, to complete the proof.

This provides a useful explicit factor of automorphy describing any divisor of degree 0; to extend it to an equally useful explicit factor of automorphy describing any divisor, it is necessary to have some factor describing a divisor of degree 1. Provisionally then suppose that ζ is a factor of automorphy describing the divisor $1 \cdot p_0$, where $p_0 \in M$ is the base point of M; the problem of finding an explicit form for such a factor of automorphy will be taken up in the next section. In terms of this auxillary factor of automorphy, the following result holds.

Corollary. If ζ is a factor of automorphy associated to the divisor $1 \cdot p_0$ then

$$\zeta_{\mathfrak{d}} = \rho_{w(\mathfrak{d})} \cdot \zeta^{\deg \mathfrak{d}}$$

is a factor of automorphy associated to an arbitrary divisor $\mathfrak{d}$ on M.

Proof. If $\nu = \deg \mathfrak{d}$ then $\mathfrak{d} - \nu\cdot p_0$ has degree 0 so by the preceding theorem is represented by the factor of automorphy $\rho_{w(\mathfrak{d}-\nu\cdot z_0)} = \rho_{w(\mathfrak{d})}$; here explicit use is made of the fact that $z_0 \in \tilde{M}$ representing $p_0 \in M$ is the base point of M , since the Abelian integrals are normalized so that $w(z_0) = 0$. The divisor $\mathfrak{d} = (\mathfrak{d} - \nu\cdot p_0) + \nu\cdot p_0$ is then represented by the factor of automorphy $\rho_{w(\mathfrak{d})} \cdot \zeta^{\nu}$ as desired.

To complete the discussion of factors of automorphy to some extent, one further general existence theorem will be quoted here, the assertion that every factor of automorphy admits a nontrivial meromorphic relatively automorphic function. Any factor of automorphy is therefore equivalent to one associated to some divisor on M , hence to one of the form $\rho_t\zeta^{\nu}$ for some point $t \in \mathbb{C}^g$ and some integer $\nu \in \mathbb{Z}$, with ν the degree of the divisor. The integer ν will also be called the degree or Chern class of the factor of automorphy. Two factors $\rho_t\zeta^{\nu}$ and $\rho_s\zeta^{\mu}$ are equivalent precisely when $\nu = \mu$ and $t - s \in \mathcal{L}$. This shows that the space of equivalence classes of factors of automorphy, or what is the same thing the space of linear equivalence classes of divisors on M , is naturally represented by the nonconnected complex manifold $J \times \mathbb{Z}$.

§5. Theta factors of automorphy.

Consider now the holomorphic mapping $w : \tilde{M} \to \mathbb{C}^g$ defined by the canonical Abelian integrals on the marked Riemann surface M, and the associated period homorphism $\omega : \Gamma \to \underline{L}$ from the covering translation group Γ to the lattice subgroup $\underline{L} = (I,\Omega)\mathbb{Z}^{2g} \subseteq \mathbb{C}^g$ defined by the period matrix of M; here $w(Tz) = w(z) + \omega(T)$ for any $T \in \Gamma$. If $\mu(\lambda,w)$ is a factor of automorphy for the action of the lattice subgroup $\underline{L}$ on $\mathbb{C}^g$, it clearly induces a factor of automorphy $\mu(T,z)$ for the action of the covering translation group Γ on $\tilde{M}$ by setting $\mu(T,z) = \mu(\omega(T),w(z))$, and equivalent factors of automorphy for $\underline{L}$ are easily seen to induce equivalent factors of automorphy for Γ.

In particular the flat factors of automorphy σ_s, ρ_t for $\underline{L}$ as defined in A(3.3) induce the corresponding flat factors of automorphy σ_s, ρ_t for Γ as defined in (4.2); in this case it follows from A(5.2) and Theorem 4 that flat factors for $\underline{L}$ are equivalent precisely when the induced flat factors for Γ are equivalent, a stronger result than can be expected to hold in general. Next the theta factor of automorphy ξ for $\underline{L}$ as defined in A(3.2) induces a factor of automorphy for Γ, explicitly that for which

$$\xi(A_j,z) = 1 \ , \ \xi(B_j,z) = \exp - 2\pi i[w_j(z) + \frac{1}{2}\,\omega_{jj}] \ .$$

It is worth noting explicitly here that since any commutator $C \in [\Gamma,\Gamma]$ lies in the kernel of the period homorphism $\omega : \Gamma \to \underline{L}$ it must automatically be the case that $\xi(C,z) = 1$; in particular $\xi(C_j,z) = 1$ for the commutators $C_j = [A_j,B_j]$. Now the factor of automorphy ξ must be equivalent to one in the standard form $\rho_t\zeta^\nu$, so the first problem to be faced is that of determining the appropriate parameters $t \in \mathbb{C}^g$, $\nu \in \mathbb{Z}$. As a convenient notation introduce the Riemann point $r \in \mathbb{C}^g$, the point having coordinates

$$(2) \qquad r_j = \sum_{k=1}^{g} \int_{\alpha_k} w_j(z)\omega_k(z) + \sum_{k=1}^{g}\omega_{jk} - \frac{1}{2}\,\omega_{jj} \;;$$

this will be viewed alternatively either as a point $r \in \mathbb{C}^g$ or as a point $r \in J = \mathbb{C}^g/\underline{L}$, as convenient.

Theorem 6. For the induced theta factor of automorphy ξ on a marked Riemann surface of genus g

$$\xi \sim \rho_r\zeta^g \,.$$

Proof. There exists some meromorphic relatively automorphic function f for the factor of automorphy ξ, and if $\underline{\vartheta} = \underline{\vartheta}(f)$ then $\xi \sim \rho_{w(\underline{\vartheta})}\cdot\zeta^{\deg\underline{\vartheta}}$. It can be supposed that the divisor of f is disjoint from the paths of the marking of M, and then by the residue theorem

$$\begin{aligned} 2\pi i \deg\underline{\vartheta} &= \int_{\partial\Delta} d\log f(z) \\ &= \sum_{k=1}^{g}\{\int_{\alpha_k} d\log f(C_1\cdots C_{k-1}z) - d\log f(C_1\cdots C_kB_kz) \\ &\qquad + \int_{\beta_k} d\log f(C_1\cdots C_{k-1}A_kz) - d\log f(C_1\cdots C_kz)\} \\ &= \sum_{k=1}^{g}\{-\int_{\alpha_k} d\log\xi(B_k,z) + \int_{\beta_k} d\log\xi(A_k,z)\} \\ &= \sum_{k=1}^{g} 2\pi i\{\int_{\alpha_k}\omega_k(z)\} = 2\pi i\, g \,. \end{aligned}$$

On the other hand if $\mathfrak{d}$ is represented by the divisor $\sum_i \nu_i \cdot a_i$ where $a_i \in \Delta$ and if $w_j(z)$ is any Abelian integral then $w_j(z) d \log f(z)$ is a meromorphic differential form in Δ having simple poles at the points a_i with residues $\nu_i\, w_j(a_i)$ there, so by the residue theorem

$$\begin{aligned} 2\pi i\, w_j(\mathfrak{d}) &= 2\pi i \sum_i \nu_i \cdot w_j(a_i) = \int_{\partial\Delta} w_j(z)\, d \log f(z) \\ &= \sum_{k=1}^{g} \{ \int_{\alpha_k} w_j(z)\, d\log f(z) - [w_j(z) + \omega_{jk}][d \log f(z) - 2\pi i\, \omega_k(z)] \\ &\qquad + \int_{\beta_k} [w_j(z) + \delta_k^j]\, d \log f(z) - w_j(z)\, d \log f(z) \} \\ &= \sum_{k=1}^{g} \{ \int_{\alpha_k} - \omega_{jk}\, d \log f(z) + 2\pi i [w_j(z)\omega_k(z) + \omega_{jk}\omega_k(z)] \\ &\qquad + \int_{\beta_k} \delta_k^j\, d \log f(z) \} . \end{aligned}$$

Here

$$\int_{\alpha_k} d \log f(z) = \log f(A_k z_0) - \log f(z_0) = 2\pi i\, m_k$$

$$\int_{\beta_k} d \log f(z) = \log f(B_k z_0) - \log f(z_0) = 2\pi i [- \tfrac{1}{2}\, \omega_{kk} + n_k]$$

for some integers m_k, n_k, since $f(z)$ is relatively automorphic for the factor of automorphy ξ, and consequently

$$w_j(\mathfrak{d}) = - \tfrac{1}{2}\, \omega_{jj} + \sum_{k=1}^{g} \{ \int_{\alpha_k} w_j(z)\omega_k(z) + (1 - m_k)\omega_{jk} + n_k \delta_k^j \} ,$$

or equivalently

$$v(\mathfrak{z}) = r + \lambda$$

for the lattice vector $\lambda = (I,\Omega)\binom{n}{-m} \in \mathcal{L}$; that suffices for the proof, since $\rho_\lambda \sim 1$.

The products $w_j(z)\omega_k(z)$ are holomorphic differential forms on the universal covering space $\tilde{M}$ and their integrals $\int_{\alpha_\ell} w_j(z)\omega_k(z)$ and $\int_{\beta_\ell} w_j(z)\omega_k(z)$ are interesting further invariants called <u>quadratic period classes</u>; the Riemann point is a particular linear combination of these quadratic period classes, aside from the term $\frac{1}{2}\omega_{jj}$ and the lattice vector λ with components $\lambda_j = \sum_{k=1}^{g}\omega_{jk}$. The ordinary period classes are really determined merely by the homology classes of the paths of the marking, but the quadratic period actually depend on the homotopy classes; the precise extent to which that is the case was discussed by E. Jablow (Quadratic vector classes of Riemann surfaces, Duke J. Math., 1985), and a characterization of the Riemann constant as the only linear combination of quadratic period classes dependent just on the homology class of the marking can be found there as well.

In view of the preceding theorem, it is tempting to try $\rho_t\xi^{1/g}$ as the explicit form for the basic factor of automorphy ζ associated to the divisor $1 \cdot p_0$, for some parameter $t \in \mathbb{C}^g$; some care must be taken in choosing appropriate branches of the g-th root of course, but if it is possible to define a factor of automorphy this way then it is clear from the first part of the proof of Theorem 6 that it will represent some divisor of degree 1 , and by an appropriate choice of t that divisor can be made to be $1 \cdot p_0$. This can indeed be accomplished. As a convenient notation for this purpose introduce the point $s \in \mathbb{C}^g$ having coordinates

(3) $$s_j = r_j + \frac{1}{2}\omega_{jj} ,$$

where $r \in \mathbb{C}^g$ is the Riemann point.

Theorem 7. The factor of automorphy ζ for the covering translation group of a marked Riemann surface of genus g defined by

$$\zeta(A_j,z) = 1 , \ \zeta(B_j,z) = \exp -\frac{2\pi i}{g}[w_j(z) + s_j]$$

represents the divisor $1 \cdot p_0$.

Proof. It is first necessary to demonstrate that this really defines a factor of automorphy for the group Γ . Recall that Γ is generated by the elements $A_1,\ldots,A_g,B_1,\ldots,B_g$ subject only to the relation $C_1 \ldots C_g = 1$ and its consequences; thus Γ is isomorphic to the quotient of the free group on the letters $A_1,\ldots,A_g,B_1,\ldots,B_g$ by the normal subgroup generated by the word $C_1 \ldots C_g$. Now from the expressions for $\zeta(A_j,z)$ and $\zeta(B_j,z)$ and the functional equation $\zeta(ST,z) = \zeta(S,Tz)\zeta(T,z)$ there can be deduced a unique expression for $\zeta(T,z)$ for any word T in the letters $A_1,\ldots,A_g,B_1,\ldots,B_g$, and it is sufficient to show that $\zeta(T,z) = 1$ whenever $T = SC_1 \ldots C_g S^{-1}$ for any word S . For this purpose note that $1 = \zeta(TT^{-1},z) = \zeta(T,T^{-1}z)\zeta(T^{-1},z)$, so that

$$\zeta(T^{-1},z) = \zeta(T,T^{-1}z)^{-1} ,$$

and consequently

$$\begin{aligned}\zeta(C_j,z) &= \zeta(A_jB_jA_j^{-1}B_j^{-1},z)\\ &= \zeta(A_j,B_jA_j^{-1}B_j^{-1}z)\,\zeta(B_j,A_j^{-1}B_j^{-1}z)\,\zeta(A_j,A_j^{-1}B_j^{-1}z)^{-1}\zeta(B_j,B_j^{-1}z)^{-1}\\ &= \exp -\frac{2\pi i}{g}\,[w_j(A_j^{-1}B_j^{-1}z) - w_j(B_j^{-1}z)]\\ &= \exp -\frac{2\pi i}{g}\ ;\end{aligned}$$

therefore at least

$$\zeta(C_1\cdots C_g,z) = \zeta(C_1,C_2\cdots C_g z)\cdots\zeta(C_g,z) = (\exp -\frac{2\pi i}{g})^g = 1\ .$$

Moreover if $R = C_1 \ldots C_g$ then for any word S

$$\begin{aligned}\zeta(SRS^{-1},z) &= \zeta(S,RS^{-1}z)\ \zeta(R,S^{-1}z)\ \zeta(S,S^{-1}z)^{-1}\\ &= \zeta(S,S^{-1}z)\cdot 1\cdot \zeta(S,S^{-1}z)^{-1} = 1\ ,\end{aligned}$$

since R acts trivially on $\tilde{M}$.

Thus ζ is a well defined factor of automorphy, so there is some parameter $t \in \mathbb{C}^g$ such that $\rho_t\zeta$ represents the divisor $1 \cdot p_0$. There is a slight complication here though, in that the base point p_0 lies on the boundary of the fundamental polygon. Therefore choose a point $z_1 \in \Delta$ representing some other point $p_1 \in M$, and consider first the problem of determining the parameter $t_1 \in \mathbb{C}^g$ such that $\rho_{t_1}\zeta$ represents the divisor $1 \cdot p_1$, hence such that $\rho_{t_1}\zeta$ admits a relatively automorphic function f with $\underset{\sim}{\vartheta}(f) = 1 \cdot p_1$. As in the proof of the preceding theorem it follows from the residue theorem that

$$\begin{aligned}
2\pi i\, w_j(z_1) &= \int_{\partial\Delta} w_j(z)\, d\log f(z) \\
&= \sum_{k=1}^{g} \Big\{\int_{\alpha_k} w_j(z)\, d\log f(z) - [w_j(z) + \omega_{jk}][d\log f(z) - \frac{2\pi i}{g}\,\omega_k(z)] \\
&\qquad + \int_{\beta_k} [w_j(z) + \delta_k^j]\, d\log f(z) - w_j(z)\, d\log f(z)\Big\} \\
&= \sum_{k=1}^{g} \Big\{\int_{\alpha_k} - \omega_{jk}\, d\log f(z) + \frac{2\pi i}{g}\,[w_j(z)\,\omega_k(z) + \omega_{jk}\omega_k(z)] \\
&\qquad + \int_{\beta_k} \delta_k^j\, d\log f(z)\ .
\end{aligned}$$

Here

$$\begin{aligned}
\int_{\alpha_k} d\log f(z) &= \log f(A_k z_0) - \log f(z_0) = 2\pi i\, m_k\ , \\
\int_{\beta_k} d\log f(z) &= \log f(B_k\, z_0) - \log f(z_0) \\
&= 2\pi i[t_{1k} - \frac{1}{g}\, s_k + n_k]\ ,
\end{aligned}$$

for some integers m_k, n_k , since f is relatively automorphic for the factor of automorphy $\rho_{t_1}\zeta$, and consequently

$$\begin{aligned}
w_j(z_1) &= \sum_{k=1}^{g}\{-\omega_{jk}m_k + \frac{1}{g}\int_{\alpha_k} w_j(z)\omega_k(z) + \frac{1}{g}\,\omega_{jk} \\
&\qquad + \delta_k^j[t_{1k} - \frac{1}{g}\, s_k + n_k]\} \\
&= \frac{1}{g}\,[r_j + \frac{1}{2}\,\omega_{jj} - s_j] + t_{1j} + \sum_{k=1}^{g}(\delta_k^j n_k - \omega_{jk}m_k) \\
&= t_{1j} + \lambda_j
\end{aligned}$$

for the lattice vector $\lambda = (I,\Omega)\binom{n}{-m} \in \underset{\sim}{L}$. Thus

$$w_j(z_1) = t_1 + \lambda ,$$

so that $\rho_{t_1}\zeta \sim \rho_{w(z_1)}\zeta$ represents the divisor $1 \cdot p_1$; upon letting z_1 approach z_0 it follows by continuity that $\zeta = \rho_{w(z_0)}\zeta$ represents the divisor $1 \cdot p_0$ as desired, thus concluding the proof.

It should be noted that with this choice of an explicit form for the basic factor of automorphy ζ , the equivalence relation of Theorem 6 actually becomes the identity

$$\xi = \rho_r \zeta^g \tag{4}$$

where $r \in \mathbb{C}^g$ is the Riemann point. With the result of Theorem 7 and the observations made in the preceding section, very explicit standard forms have been obtained for any factor of automorphy for the covering translation group; and (4) indicates that these standard forms are quite convenient for the study of theta functions on Riemann surfaces.

§6. The prime function.

Choose any two distinct points a_1, a_2 of the marked Riemann surface M, and any holomorphic relatively automorphic functions $f_j \in \Gamma(\rho_{w(a_j)}\zeta)$ such that $\mathcal{J}(f_j) = 1 \cdot a_j$. In terms of these functions f_j and the cross-ratio function set

$$(1) \qquad q(z_1,z_2) = p(z_1,a_1,z_2,a_2)f_2(z_1)f_1(z_2) .$$

The result is a meromorphic function on $\tilde{M} \times \tilde{M}$, which will be called a prime function for the marked Riemann surface M; it has the following properties.

Theorem 8. A prime function $q(z_1,z_2)$ for the marked Riemann surface M is a holomorphic function on the two-dimensional complex manifold $\tilde{M}^2$ such that:

(i) $q(z_1,z_2)$ has simple zeros along the subvariety $\cup_{T\in\Gamma}(z_1 = Tz_2)$;

(ii) $q(Tz_1,z_2) = q(z_1,z_2)\rho_{w(z_2)}(T)\ \zeta(T,z_1)$ for all $T \in \Gamma$;

(iii) $q(z_1,z_2) = -q(z_2,z_1)$.

These properties characterize the prime function uniquely up to a nonzero constant factor.

Proof. For any fixed point $z_2 \in \tilde{M}$ the cross-ratio function $p(z_1,a_1,z_2,a_2)$ as a meromorphic function of $z_1 \in \tilde{M}$ has simple zeros at the points Tz_2 and simple poles at the points Ta_2 by Theorem 3, while $f_2(z_1)$ has simple zeros at the points Ta_2 by choice; the product $q(z_1,z_2)$ is consequently a holomorphic function of $z_1 \in \tilde{M}$ with simple zeros at the points Tz_2. The same argument with a reversal of the roles of the variables z_1 and z_2 shows that $q(z_1,z_2)$ is a holomorphic function of $z_2 \in \tilde{M}$ for each fixed $z_1 \in \tilde{M}$, and it then follows from the theorem of Hartogs that $q(z_1,z_2)$ is necessarily a holomorphic function of both variables. The zero locus is as asserted in (i). Furthermore, again for any fixed point $z_2 \in \tilde{M}$, it follows from Theorem 3 as noted in (4.2) that $p(z_1,a_1,z_2,a_2)$ as a function of z_1 is a meromorphic relatively automorphic function for the factor of automorphy $\rho_{w(z_2-a_2)}$, so since $f_2 \in \Gamma(\rho_{w(a_2)}\zeta)$ it is evident that $q(z_1,z_2) \in \Gamma(\rho_{w(z_2)}\zeta)$ as a function of z_1, as desired. The same argument shows of course that $q(z_1,z_2) \in \Gamma(\rho_{w(z_1)}\zeta)$ as a function of z_2. The quotient $q(z_1,z_2)/q(z_2,z_1)$, which is a holomorphic and nowhere vanishing function on $\tilde{M}^2$ from the first part of this proof, is consequently invariant under the action of Γ on each factor, so must be a constant. Thus $q(z_1,z_2) = cq(z_2,z_1)$ for some constant $c \in \mathbb{C}$, and since $q(z_1,z_2)$ has a simple zero at $z_1 = z_2$ it follows readily that $c = -1$ as desired. Finally if $q'(z_1,z_2)$ is any other function satisfying the three conditions of the theorem then it follows from (i) that the quotient $q'(z_1,z_2)/q(z_1,z_2)$ is a holomorphic and nowhere vanishing function on $\tilde{M}^2$, and it follows from (ii) and (iii) that it is invariant under the action of Γ in each variable, so it must be a constant.

The prime function, like the cross-ratio function, depends very much on the marking of the surface M; but unlike the cross-ratio function it is not really uniquely determined, but only determined up to a nonzero constant factor. At a later point in the discussion it will prove convenient to normalize the prime function further, but for the present it will be left undetermined to the extent admissible. The prime function was defined in terms of the cross-ratio function by (1); conversely the cross-ratio function can be defined in terms of the prime function by

$$p(z_1,z_2,a_1,a_2) = \frac{q(z_1,a_1)\ q(z_2,a_2)}{q(z_1,a_2)\ q(z_2,a_1)} . \tag{2}$$

Indeed it is quite evident that this expression has the appropriate zeros and poles, and satisfies the functional equations demanded in Theorem 3, so by the uniqueness part of that theorem must actually be the cross-ratio function. Note that changing the prime function by a constant factor does not change the cross-ratio function. In the special case $g = 0$ the prime function is of course just the linear polynomial $q(z_1,z_2) = z_1 - z_2$ up to a constant factor, but is in this case a meromorphic function, and (2) is then the standard defining equation for the cross-ratio in P^1. In the general case $g > 0$ the prime function is the replacement for the linear function in the cross-ratio as given by (2).

In terms of the prime function $q(z_1,z_2)$ set

$$\phi(z_1,z_2) = d_{z_1} q(z_1,z_2) = \frac{\partial q(z_1,z_2)}{\partial z_1}\, dz_1 \tag{3}$$

and

$$\phi(z) = \phi(z,z) . \tag{4}$$

Here $\phi(z_1,z_2)$ is a holomorphic differential form in $z_1 \in \tilde{M}$ and a holomorphic function in $z_2 \in \tilde{M}$, hence $\phi(z)$ is a holomorphic differential form on $\tilde{M}$; the latter will be called the <u>canonical form</u> for the marked Riemann surface M, and is determined uniquely by the prime function. Since $q(z_1,z_2)$ has a simple zero at $z_1 = z_2$ it must be the case that $\partial q(z_1,z_2)/\partial z_1|_{z_1=z_2} \neq 0$, so the differential form $\phi(z)$ is nowhere zero on $\tilde{M}$. The quotients

$$(5) \qquad \kappa(T,z) = \phi(z)/\phi(Tz)$$

then determine a factor of automorphy for the action of the covering translation group Γ on $\tilde{M}$, called the <u>canonical factor of automorphy</u>; this factor is not equivalent to the constant factor even though $\phi(z)$ is holomorphic and nowhere vanishing, since $\phi(z)$ is a differential form rather than a function. Any holomorphic differential form $\omega(z)$ on $\tilde{M}$ can be written uniquely as $\omega(z) = f(z)\phi(z)$ for the holomorphic function $f(z) = \omega(z)/\phi(z)$, and $\omega(z)$ is Γ-invariant hence is a differential form on M precisely when $f(Tz) = \kappa(T,z)\,f(z)$ for all $T \in \Gamma$. This sets up a natural one-to-one correspondence between the Abelian differentials on M and the relatively automorphic functions $\Gamma(\kappa)$. Of course this can be done, if not canonically so, for any holomorphic and nowhere vanishing differential form on $\tilde{M}$; the advantage of the canonical form is that the associated factor of automorphy is already in the standard form.

<u>Theorem 9</u>. The canonical factor of automorphy on the marked Riemann surface M of genus g is

$$\kappa(T,z) = \rho_k(T)\ \zeta(T,z)^{2g-2} ,$$

where

$$k_j = 2r_j$$

in terms of the Riemann point $r \in \Gamma^g$.

Proof. It follows from Theorem 8 (ii) and (3) that

$$\begin{aligned}\phi(Tz_1,z_2) &= dq(Tz_1,z_2)\\ &= \frac{\partial}{\partial z_1}\,[\rho_{w(z_2)}(T)\ \zeta(T,z_1)\ q(z_1,z_2)]\ dz_1\\ &= \rho_{w(z_2)}(T)\ \zeta(T,z_1)\ [\phi(z_1,z_2) + q(z_1,z_2)\ d\log \zeta(T,z_1)]\end{aligned}$$

and

$$\begin{aligned}\phi(z_1,Tz_2) &= \frac{\partial}{\partial z_1}\ q(z_1,Tz_2)\ dz_1\\ &= \frac{\partial}{\partial z_1}\,[\rho_{w(z_1)}(T)\ \zeta(T,z_2)\ q(z_1,z_2)]\ dz_1\\ &= \rho_{w(z_1)}(T)\ \zeta(T,z_2)\ [\phi(z_1,z_2) + q(z_1,z_2)\ d\log \rho_{w(z_1)}(T)]\ ,\end{aligned}$$

and consequently

$$\begin{aligned}\phi(Tz_1,Tz_2) &= \rho_{w(Tz_2)}(T)\ \zeta(T,z_1)[\phi(z_1,Tz_2) + q(z_1,Tz_2)\ d\log \zeta(T,z_1)]\\ &= \rho_{w(Tz_2)}(T)\zeta(T,z_1)\{\rho_{w(z_1)}(T)\zeta(T,z_2)[\phi(z_1,z_2) + q(z_1,z_2)\ d\log \rho_{w(z_1)}(T)]\\ &\qquad + q(z_1,Tz_2)\ d\log \zeta(T,z_1)\}\ ;\end{aligned}$$

then setting $z_1 = z_2 = z$ and recalling that $q(z,z) = q(z,Tz) = 0$ yield the result that

$$\phi(Tz) = \rho_{2w(z)+\omega(T)}(T)\ \zeta(T,z)^2\phi(z) .$$

Thus the canonical factor of automorphy is

$$\kappa(T,z) = \rho_{-2w(z)-\omega(T)}(T)\ \zeta(T,z)^{-2} .$$

Using the definition of the representation ρ_t and Theorem 7 yields the more explicit form

$$\begin{aligned}\kappa(A_j,z) &= 1 ,\\ \kappa(B_j,z) &= \exp 2\pi i[-2w_j(z) - \omega_{jj}] \cdot \exp 2\,\frac{2\pi i}{g}\,[w_j(z) + s_j]\\ &= \exp -\frac{2\pi i}{g}\,[(2g-2)(w_j(z) + s_j) - 2g(s_j - \tfrac{1}{2}\,\omega_{jj})]\\ &= \rho_k(B_j) \cdot \zeta(B_j,z)^{2g-2} ,\end{aligned}$$

where $k_j = 2(s_j - \frac{1}{2}\,\omega_{jj}) = 2r_j$ as desired, thereby concluding the proof.

To interpret this construction in an alternative manner, note that if $\phi(z)$ is the canonical form on $\tilde{M}$ then

$$u(z) = \int_{z_0}^{z} \phi \tag{6}$$

is a well defined holomorphic function on $\tilde{M}$ such that $du(z) = \phi(z)$ is nowhere vanishing; thus this function defines a locally biholomorphic mapping $u : \tilde{M} \to \mathbb{C}$, so can be viewed as exhibiting $\tilde{M}$ as spread out as a locally unbranched holomorphic covering over an open subset $D \subseteq \mathbb{C}$. It should be noted that it is not asserted that $u : \tilde{M} \to D$ is a covering space in the usual sense; that may be the case, but is not known to be so.

It is merely asserted that $\tilde{M}$ is exhibited by u as being an analytic configuration over D in the sense of Weierstrass, or a Riemann domain over D in the sense used in the theory of analytic functions of several complex variables. The function u can be used as a local coordinate system in a neighborhood of any point of $\tilde{M}$, and the resulting special system of coordinates will be called the <u>canonical coordinates</u> on $\tilde{M}$. This coordinatization is uniquely determined by the marking up to a constant factor, and will be uniquely determined if the prime function is.

One great advantage of having a coordinatization defined by a global holomorphic function on $\tilde{M}$ is that it is then possible to define differentiation globally: if $f(z)$ is a holomorphic function on $\tilde{M}$ then its derivative $f'(z) = df(z)/dz$ with respect to the canonical coordinates is a well defined holomorphic function on $\tilde{M}$, and so are the second derivative $f''(z) = d^2f(z)/dz^2$ and all higher derivatives. This will be used freely in the sequel, and whenever a derivative of a function on $\tilde{M}$ is indicated it is always to be interpreted as the derivative with respect to the canonical coordinates on $\tilde{M}$. Note from (6) that in terms of the canonical coordinates $\phi = du$; thus the function $f(z)$, when a holomorphic differential form $\omega(z)$ on $\tilde{M}$ is written as $\omega(z) = f(z)\ \phi(z) = f(u)du$, is just the coefficient of that form as expressed in canonical coordinates, and in particular the identification of Abelian differentials on M with relatively automorphic functions in $\Gamma(\kappa)$ merely amounts to considering the coefficient when the differential is expressed in canonical coordinates. When the canonical Abelian differentials are expressed in canonical coordinates as $\omega_j(z) = v_j'(z)dz$, where $v_j'(z)$ is the derivative of the Abelian integral $v_j(z)$ with respect to the canonical coordinates, then the functions $w_j'(z) \in \Gamma(\kappa)$ are a basis for the space of relatively automorphic functions for the factor of automorphy $\kappa(T,z) = \rho_k(T)\ \zeta(T,z)^{2g-2}$.

For any point $a \in \tilde{M}$ and any transformation $T \in \Gamma$ there is an open neighborhood U of a such that U and TU are both mapped homeomorphically to open subsets of $\mathbb{C}$ by the canonical coordinate function u ; there is consequently a well defined biholomorphic mapping $\hat{T} : u(U) \to u(TU)$ such that $\hat{T}(u(z)) = u(T(z))$ whenever $z \in U$. The mapping $\hat{T}$ can be continued analytically along any path in $\tilde{M}$ beginning at a ; this does not necessarily mean that it can be continued analytically along any path in D beginning at u(a) , but it does lead to a definition of $\hat{T}$ as a holomorphic mapping $\hat{T} : D \to D$ although possibly multiple valued and possibly with interior singularities. Since $du(z) = \phi(z)$ it does follow upon differentiating the identity $\hat{T}(u(z)) = u(T(z))$ that $\hat{T}'(u(z))\phi(z) = \phi(Tz)$, hence by Theorem 9 that

$$\hat{T}'(u(z)) = \kappa(T,z)^{-1} = \rho_{-k}(T)\ \zeta(T,z)^{-2g+2} . \tag{7}$$

In particular for the transformation $T = A_j$ it follows that $\hat{A}_j'(u(z)) = 1$, and therefore that $\hat{A}_j(u(z)) = u(z) + a_j$ for some constant a_j ; this relation is preserved under analytic continuation, so that D must be left invariant under the mapping $\hat{A}_j : u \to u + a_j$. It may be the case that $a_j = 0$, so this need not be very significant. On the other hand for the commutator $C_j = [A_j, B_j]$ it was noted that $\zeta(C_j, z) = \exp 2\pi i/g$, and consequently $\hat{C}_j'(u(z)) = \exp(2\pi i/g)(2-2g) = \exp 4\pi i/g$ and $\hat{C}_j(u) = (\exp 4\pi i/g)u + c_j$ for some constant c_j . There are thus at least some nontrivial holomorphic mappings of D to itself. Very little is known about the geometric properties of this mapping u, such as whether it is ever one-to-one or even a covering mapping. The only case in which it is easy to calculate it is that in which $g = 1$, and there it is the covering mapping $u(z) = \exp -2\pi iz$ when $\tilde{M}$ is identified with the complex plane.

Another advantage of the canonical coordinates that follows from (7) is that differentiation with respect to these coordinates preserves functional equations expressed in terms of the standard factors of automorphy. For example if $f \in \Gamma(\rho_t \zeta^\nu)$, so that f is a holomorphic function on $\tilde{M}$ satisfying $f(Tz) = \rho_t(T)\ \zeta(T,z)^\nu f(z)$, then

$$df(Tz) = \rho_t(T)\zeta(T,z)^\nu[df(z) + \nu f(z)\ d\log\zeta(T,z)] ,$$

so it follows from (7) that for differentiation with respect to the canonical coordinates

$$(8) \qquad f'(Tz) = \rho_{k+t}(T)\ \zeta(T,z)^{2g-2+\nu}[f'(z) + \nu\ f(z)\ \frac{d}{dz}\log\zeta(T,z)] .$$

On the other hand for the prime function when expressed in terms of canonical local coordinates it follows from (3) and (4) that

$$\frac{\partial q(z,a)}{\partial z}\Big|_{z=a} = 1 ,$$

so since $q(a,a) = 0$ there is a local Taylor expansion in these coordinates of the form

$$q(z,a) = (z - a) + (z - a)^2\ q_2(z,a)$$

where $q_2(z,a)$ is holomorphic near the diagonal. Now $q(z,a) = -q(z,a)$ by Theorem 8, and therefore $q_2(z,a) = -q_2(a,z)$ as well so that necessarily $q_2(z,a) = (z - a)q_3(z,a)$; thus the Taylor expansion actually has the form

$$(9) \qquad q(z,a) = (z - a) + (z - a)^3\, q_3(z,a)$$

in canonical coordinates, where $q_3(z,a)$ is holomorphic near the diagonal and $q_3(z,a) = q_3(a,z)$. The coordinate functions z and a are really globally defined on $\tilde{M}$, so that $q_3(z,a)$ necessarily extends to a meromorphic function on $\tilde{M} \times \tilde{M}$; the possibility of singularities occurring arises from the possibility that $z - a = 0$ for some points $z \neq a$, since the canonical coordinates may not be one-to-one.

When viewed as points of the Jacobi variety J , the canonical point k and the Riemann point r are related by $k = 2r$, since in the formula for their relationship given in Theorem 9 the term $\sum_{j=1}^{g} \omega_{ij}$ represents a point in the lattice subgroup $\underline{L}$. The factor of automorphy $\rho_r\zeta^{g-1}$ is thus a square root of the canonical bundle $\kappa = \rho_k\zeta^{2g-2}$, in the sense that $(\rho_r\zeta^{g-1})^2 = \kappa$. There are of course several distinct square roots of the canonical bundle; indeed there are clearly 2^{2g} such up to equivalence, given by $\rho_{r+\epsilon}\zeta^{g-1}$ where ϵ runs over the 2^{2g} distinct points of order two in the group J, represented by the 2^{2g} half periods modulo $\underline{L}$. The factor of automorphy $\sigma = \rho_r\zeta^{g-1}$ is an intrinsically distinguished one of these square roots, and will be called the <u>semicanonical factor of automorphy</u>, avoiding the temptations to call it the canonical semicanonical factor of automorphy. (The letter σ will now have two distinct standard uses; that should lead to no confusion, since the representation $\sigma \in$

$\mathrm{Hom}(\Gamma, E^*)$ will seldom be needed in the sequel.) The theta factor of automorphy is $\xi = \rho_\tau \zeta^g = \sigma\zeta$, so that the prime function $q(z,a)$ can be viewed as a relatively automorphic function for the factor of automorphy $\rho_{w(a)}\xi\sigma^{-1}$ as a function of z, and correspondingly as a function of a. A function $f \in \Gamma(\sigma)$ is sometimes considered as describing a half-order differential $f(z)\sqrt{dz}$, in the sense that this expression is Γ-invariant when properly interpreted and hence represents an analytic entity of one sort or another on the Riemann surface M. Analogously the prime function $q(z,a)$ can be considered as describing an analytic entity $E(z,a) = q(z,a)/\sqrt{dz}\sqrt{da}$, which transforms by the factor of automorphy $\rho_{w(a)}\xi$ in the variable z and correspondingly in the variable a. The factor of automorphy ξ can be considered more primitive than its g-th root ζ; at least it arose naturally earlier in the subject. The entity $E(z,a)$ is essentially Klein's prime form; it was used by Klein in his treatment of Riemann surfaces and theta functions, and is also used by Fay in place of the prime function $q(z,a)$ that will be adopted here. For this point see the discussion in Fay's book (John D. Fay, Theta Functions on Riemann Surfaces, Springer-Verlag, Lectures Notes in Math. vol. 352, 1973).

§7. Canonial meromorphic differentials.

The canonical meromorphic Abelian differentials on M can be expressed quite simply and conveniently in terms of the prime function on M, as in the following result. It may be recalled that the canonical differential of the second kind has so far only been determined up to a constant factor; the expression of this differential in terms of the prime function suggests a unique determination for this differential, one that is particularly natural in terms of the canonical coordinates on $\tilde{M}$ and that will be used consistently henceforth.

Theorem 10. For any point $a \in \tilde{M}$, the canonical differential of the second kind with a double pole at a can be written

$$w_a'(z) = \frac{\partial^2}{\partial z\, \partial a} \log q(z,a),$$

and in terms of the canonical local coordinate on $\tilde{M}$ near a has a Laurent expansion with principal part

$$w_a'(z) = \frac{1}{(z-a)^2} + \cdots .$$

For any points $a_1, a_2 \in \tilde{M}$ representing distinct points of $\tilde{M}$ and any path δ in $\tilde{M}$ such that $\partial\delta = a_1 - a_2$ and δ projects to a simple path on M, the canonical differential of the third kind associated to δ can be written

$$w_\delta'(z) = \frac{\partial}{\partial z} \log \frac{q(z,a_1)}{q(z,a_2)} .$$

Proof. Consider first the function

$$w_a(z) = \frac{\partial}{\partial a} \log q(z,a),$$

a well defined meromorphic function on $\tilde{M} \times \tilde{M}$ as a function of the two variables z,a , with simple poles along the subvarieties $z = Ta$ for $T \in \Gamma$. As a function of z alone it satisfies the functional equation

$$(1)\qquad \begin{aligned} v_a(Tz) &= \frac{\partial}{\partial a} \log \left[\rho_{v(a)}(T)\ \zeta(T,z)\ q(z,a)\right] \\ &= v_a(z) + \frac{\partial}{\partial a} \log\ \rho_{v(a)}(T); \end{aligned}$$

so in particular

$$(2)\qquad v_a(A_j z) = v_a(z)\ ,\ v_a(B_j z) = v_a(z) + 2\pi i\ w'_j(a),$$

where differentiation is as usual in terms of the canonical coordinates on $\tilde{M}$. The derivative of this function of z is clearly a Γ-invariant meromorphic differential $w'_a(z)$ on $\tilde{M}$, hence is a meromorphic differential on M, and $v_a(z)$ is its integral. This differential evidently has the form and Laurent expansion as in the statement of the theorem, so is a meromorphic differential of the second kind with a double pole at a; moreover since its integral is invariant under the transformations A_j by (2) it has trivial periods under the transformations A_j, so must be the canonical differential of the second kind as asserted. Next by combining formulas (3.4) and (6.2) it follows that

$$\begin{aligned} w'_\delta(z) &= \frac{\partial}{\partial z} \log\ p(z,z_1,a_1,a_2) \\ &= \frac{\partial}{\partial z} \log \frac{q(z,a_1)\ q(z_1,a_2)}{q(z,a_2)\ q(z_1,a_1)} = \frac{\partial}{\partial z} \log \frac{q(z,a_1)}{q(z,a_2)} \end{aligned}$$

as desired, thereby concluding the proof.

Corollary. The canonical differential of the second kind $w'_a(z)$ is a meromorphic function of the two variables $(z,a) \in \tilde{M} \times \tilde{M}$, and satisfies the symmetry condition

$$w'_a(z) = w'_z(a).$$

The canonical differential of the third kind $w'_{\delta}(z)$ is a meromorphic function of the three variables $(z,a_1,a_2) \in \tilde{M} \times \tilde{M} \times \tilde{M}$, where $\partial\delta = a_1 - a_2$, and satisfies the conditions that

$$\frac{\partial}{\partial a_1} w'_{\delta}(z) = w'_{a_1}(z), \quad \frac{\partial}{\partial a_2} w'_{\delta}(z) = - w'_{a_2}(z),$$

where as always differentiation is with respect to the canonical coordinates on $\tilde{M}$.

This assertion is such an immediate consequence of the formulas of the preceding theorem that it needs no further formal proof, but is well worth stating explicitly for emphasis. Since the canonical differential of the third kind $w'_{\delta}(z)$ is determined fully by the boundary points of δ it is really more convenient to denote it by $w'_{a_1,a_2}(z)$ where $\partial\delta = a_1 - a_2$; the integral does depend quite definitely on the entire path δ though, since it can only be defined as a single-valued function in the complement of $\Gamma\,\delta \subset \tilde{M}$.

The normalization of the canonical differential of the second kind that has hereby been adopted is that for which the principal part has the indicated form in terms of canonical local coordinates on $\tilde{M}$; it is thereby uniquely determined by the marking on M. It is perhaps worth observing explicitly that, although the prime function is only determined up to a constant factor, this indeterminacy does not carry through to the canonical differentials because of the form in which those differentials are expressed in terms of the prime function. It should also be noted that the result of the preceding theorem has the obvious extension to more general canonical meromorphic differentials; for instance $\partial^3 \log q(z,a)/\partial z\, \partial a^2$ is the natural canonical differential of the second kind with a triple pole at the point a, and so on.

With the canonical differentials now well established, it is worth discussing their periods in somewhat more detail. Note that any element $T \in \Gamma$ can be written uniquely in the form

$$T = A_1^{\alpha_1} \cdots A_g^{\alpha_g} B_1^{\beta_1} \cdots B_g^{\beta_g} C \tag{3}$$

for some integers α_j, β_j and some element C in the commutator subgroup, $C \in [\Gamma,\Gamma] \subset \Gamma$. The integers α_j, β_j of course depend on the element T, so should really be written $\alpha_j(T)$, $\beta_j(T)$; as functions of T it is evident that they are actually group homomorphisms $\alpha_j, \beta_j \in \mathrm{Hom}\,(\Gamma, \mathbb{Z})$, and are uniquely determined by the conditions that

$$\begin{aligned} &\alpha_j(A_k) = \delta_k^j, \quad \alpha_j(B_k) = 0, \\ &\beta_j(A_k) = 0, \quad \beta_j(B_k) = \delta_k^j. \end{aligned} \tag{4}$$

For the period class of the canonical differential of the second kind it is clear from these general observations that

$$\omega_a(T) = \sum_{j=1}^{g} [\alpha_j(T)\ \omega_a(A_j) + \beta_j(T)\ \omega_a(B_j)],$$

and hence from (1) and (2) that

$$\omega_a(T) = \frac{\partial}{\partial a} \log \rho_{v(a)}(T) = 2\pi i \sum_{j=1}^{g} \beta_j(T)\ w_j'\ (a) \tag{5}$$

in terms of the canonical coordinates. For the period class of the canonical differential of the third kind it is evident from Theorem 2 that

$$\omega_\delta(T) = 2\pi i \sum_{j=1}^{g} \beta_j(T) \int_\delta \omega_j. \tag{6}$$

The basic normalization here is reflected in the fact that the periods only involve the homomorphisms β_j and not the homomorphisms α_j at all.

The preceding discussion can be carried further, to yield equally convenient normalizations for the canonical meromorphic Abelian integrals on M; the resulting normal forms will be used consistently in the sequel. These forms have the advantage of not involving the values of the integrals at any particular points on $\tilde{M}$, thereby avoiding the problems that arise when these points are singular points.

Theorem 11. The canonical Abelian integral of the second kind on the marked Riemann surface M can be taken in the form

$$w_a(z) = \frac{\partial}{\partial a} \log q(z,a),$$

where the differentiation is taken in the canonical local coordinates on $\tilde{M}$; it is then characterized as that meromorphic function of the variables $(z,a) \in \tilde{M} \times \tilde{M}$, with simple poles along the subvarieties $z = Ta$ for $T \in \Gamma$ but no other singularities, such that

$$(7) \quad w_a(Tz) = w_a(z) + 2\pi i \sum_{j=1}^{g} \beta_j(T)\, w'_j(a) \qquad \text{for all } T \in \Gamma,$$

$$(8) \quad w_{Tz}(a)\, \kappa(T,z)^{-1} = w_z(a) - \frac{2\pi i}{g} \sum_{j=1}^{g} \beta_j(T)\, w'_j(z) \qquad \text{for all } T \in \Gamma,$$

and in terms of the marking on M and the associated fundamental polygon $\Delta \subset \tilde{M}$,

$$(9) \quad \int_{\alpha_j} w_z(a)\, dz = 0 \qquad \text{for} \qquad a \in \Delta \text{ and } j = 1,\ldots,g.$$

Proof. The derivative $w'_a(z)$ is precisely the canonical Abelian differential of the second kind of the preceding theorem, so that $w_a(z)$ can be taken as the associated canonical Abelian integral, and (7) is merely a restatement of the period formula (5). Perhaps the easiest way to demonstrate the next desired result is first to note that since the canonical factor of automorphy can be written variously as

$$\kappa(T,z) = \rho_k(T)\ \zeta(T,z)^{2g-2} = \rho_{-2v(z)-\omega(T)}(T)\ \zeta(T,z)^{-2}$$

it follows immediately that

(10) $$\zeta(T,z)^{-2g} = \rho_{2v(z)\,+\,\omega(T)+k}(T) = \rho_{\omega(T)+k}(T)\ \rho_{v(z)}(T)^2;$$

thus

$$-2g\,\frac{\partial}{\partial z}\log\,\zeta\,(T,z) = 2\,\frac{\partial}{\partial z}\log\,\rho_{v(z)}(T),$$

so by (5)

(11) $$\frac{\partial}{\partial z}\log\,\zeta(T,z) = -\frac{2\pi i}{g}\sum_{j=1}^{g}\beta_j(T)\ w'_j(z).$$

Consequently

$$w_{Tz}(a)\ T'(z) = \frac{\partial}{\partial z}\log\,q(a,Tz)$$

$$= \frac{\partial}{\partial z}\log\,[q(a,z)\ \zeta(T,z)\ \rho_{v(a)}(T)]$$

$$= w_z(a) - \frac{2\pi i}{g}\sum_{j=1}^{g}\beta_j(T)\ w'_j(z),$$

thereby establishing (8) since $\kappa(T,z)^{-1} = T'(z)$ in canonical local coordinates on $\tilde{M}$. This shows incidentally that $w_z(a)$ is more naturally viewed as a differential form in the variable z, since then the preceding functional equation takes the form

(12) $$w_{Tz}(a)\ dTz = w_z(a)\ dz - \frac{2\pi i}{g}\sum_{j=1}^{g}\beta_j(T)\ \omega_j(z).$$

If $a \in A$ the differential form $w_z(a)\,dz$ is well defined along the paths α_j, β_j of the marking, so its integral is also well defined. Note in particular that

$$\int_{\alpha_j} w_z(a)\, dz = \int_{\alpha_j} \frac{\partial}{\partial z} \log q(a,z)\, dz$$

$$= \log q(a, A_j z_o) - \log q(a, z_o)$$

$$= 2\pi i\, n_j$$

for some integer n_j, since $q(a,A_j z_o) = q(a,z_o)$; here of course $\log q(a,z)$ refers to any chosen branch of the logarithm along the path α_j. Similarly note that

$$\int_{\beta_j} w_z(a)\, dz = \int_{\beta_j} \frac{\partial}{\partial z} \log q(a,z)\, dz$$

$$= \log q(a, B_j z_o) - \log q(a, z_o)$$

$$= 2\pi i\, [w_j(a) - \frac{1}{g} s_j + m_j]$$

for some integer m_j, since

$$q(a, B_j z_o) = q(a, z_o)\ \rho_{w(a)}(B_j)\ \zeta(B_j, z_o)$$

$$= q(a, z_o) \exp 2\pi i\, [w_j(a) - \frac{1}{g} s_j]$$

where the factor of automorphy ζ has the explicit form as in Theorem 7. On the other hand for the canonical Abelian integral $w_k(z)$ it follows from the residue theorem that

$$\int_{\partial\Delta} w_k(z)\, w_z(a)\, dz = 2\pi i\, w_k(a)$$

when $a \in \Delta$, since $w_z(a)\, dz$ has a simple pole with residue 1 at the point a; but this integral can also be calculated by using the known general form of the canonical fundamental polygon and the period relation (12), with the result that

$$\int_{\partial\Delta} v_k(z)\, v_z(a)\, dz$$

$$= \sum_{j=1}^{g} \int_{C_1\cdots C_{j-1}\alpha_j - C_1\cdots C_j B_j \alpha_j + C_1\cdots C_{j-1}A_j\beta_j - C_1\cdots C_j\beta_j} v_k(z)\, v_z(a)\, dz$$

$$= \sum_j \int_{\alpha_j} \{v_k(z) v_z(a) dz - [v_k(z) + \omega_{kj}]\, [v_z(a)\, dz - \frac{2\pi i}{g}\, v'_j(z)\, dz]\}$$

$$+ \sum_j \int_{\beta_j} \{[v_k(z) + \delta_k^j]\, v_z(a)\, dz - v_k(z)\, v_z(a)\, dz\}$$

$$= \frac{2\pi i}{g} \sum_j \int_{\alpha_j} v_k(z)\, v'_j(z)\, dz - \sum_j \omega_{kj} \int_{\alpha_j} v_z(a)\, dz + \frac{2\pi i}{g} \sum_j \omega_{kj} \int_{\alpha_j} v'_j(z)\, dz$$

$$+ \int_{\beta_k} v_z(a)\, dz$$

$$= \frac{2\pi i}{g}\, (s_k - \sum_j \omega_{kj}) - 2\pi i \sum_j \omega_{kj} n_j + \frac{2\pi i}{g} \sum_j \omega_{kj}$$

$$+ 2\pi i\, (v_k(a) - \frac{1}{g}\, s_k + m_k)$$

$$= 2\pi i\, (v_k(a) + m_k - \textstyle\sum_j \omega_{kj}\, n_j),$$

where the Riemann point r_j is as in (5.2) and $s_j = r_j + \omega_{jj}$. Comparing these two expressions for the integral over $\partial\Delta$ shows that $m_k - \sum_j \omega_{kj} n_j = 0$ for all indices k, and therefore $m_k = n_k = 0$ for all indices k; in particular (9) holds as desired.

To conclude the proof it is only necessary to show that the function $v_a(z)$ is completely characterized by the properties listed in the theorem.

For this purpose note that the difference between $w_a(z)$ and any other function with the same properties is a meromorphic function $f(a,z)$ on $\tilde{M} \times \tilde{M}$ with at most simple poles along the subvarieties $z = Ta$ for $T \in \Gamma$ as sole singularities, and as a consequence of (7) and (8) this function satisfies

$$(7') \qquad f(a,Tz) = f(a,z) \qquad \text{for all } T \in \Gamma ,$$

$$(8') \qquad f(Ta,z)\ \kappa(T,a)^{-1} = f(a,z) \qquad \text{for all } T \in \Gamma.$$

Condition (8') implies that for each fixed point $z \in \tilde{M}$ the function $f(a,z)$ as a function of $a \in \tilde{M}$ is just a meromorphic differential form on $\tilde{M}$; since this form has at most a single simple pole, and must also have residue zero, it must actually be holomorphic. Then (7') implies that for each fixed point $a \in \tilde{M}$ the function $f(a,z)$ as a function of $z \in \tilde{M}$ is a Γ-invariant holomorphic function so must be a constant. Thus $f(a,z) = f(a)$ is an Abelian differential in a and is constant in z, hence is really independent of z; but by (9) this differential form must have zero periods along all the paths α_j, and it must consequently vanish identically. Thus $f(a,z)$ is identically zero, so the uniqueness is established as desired and the proof thereby concluded.

<u>Corollary</u>. Let $a_1,a_2 \in \tilde{M}$ represent distinct points of M, and let δ be a path in $\tilde{M}$ such that $\partial\delta = a_1-a_2$ and δ projects to a simple path in M. The associated canonical Abelian differential of the third kind can be written

$$w'_\delta(z) = w'_{a_1,a_2}(z) = w_z(a_1) - w_z(a_2);$$

it is possible to take as the corresponding normalized Abelian integral of the third kind the holomorphic function on $\tilde{M} \sim \Gamma\delta$ given by

$$w_\delta(z) = \int_{x\in\delta} w_x(z)\, dx .$$

<u>Proof</u>. As for the first assertion, it follows from Theorem 10 that $w'_\delta(z) = \partial \log q(z,a_1)/\partial z - \partial \log q(z,a_2)/\partial z$, and from Theorem 11 that $\partial \log q(z,a)/\partial z = w_z(a)$. As for the second assertion, the function $w_\delta(z)$ given by the integral above is clearly a well defined holomorphic function in the set $\tilde{M} \sim \cup_{T\in\Gamma} T\delta$; by using the symmetry property of the Corollary to Theorem 10 it follows that

$$w'_\delta(z) = \int_{x\in\delta} w'_x(z)\, dx = \int_{x=a_2}^{a_1} w'_z(x)dx$$

$$= w_z(a_1) - w_z(a_2),$$

hence the derivative $w'_\delta(z)$ is the canonical differential of the third kind so $w_\delta(z)$ can be taken as the canonical integral of the third kind as desired.

The behavior of the differential $w'_\delta(z) = w'_{a_1,a_2}(z)$ as a function of the points $a_i \in \tilde{M}$ can be deduced immediately from the preceding theorem and its corollary; further detailed discussion is really not necessary. The normalization of the integral of the third kind given in this corollary will be used consistently henceforth here; it amounts to a simple prescription for a well defined branch of the function $\log [q(z,a_1) / q(z,a_2)]$, since it can be written equivalently as

$$w_\delta(z) = \int_{x\in\delta} \frac{\partial}{\partial x} \log q(z,x)\, dx.$$

It is worth noting that with this normalization, whenever the path δ is decomposed into a sum $\delta = \delta_1 + \delta_2$ by splitting it at some interior point $a \in \delta$, so that $\partial\delta_1 = a_1 - a$ and $\partial\delta_2 = a - a_2$, then $w_\delta(z) = w_{\delta_1}(z) + w_{\delta_2}(z)$.

Of course the integral formula for $w_\delta(z)$ yields a well defined holomorphic function in $\tilde{M} \sim \cup_{T\epsilon\Gamma} T\delta$ for any path $\delta \subset \tilde{M}$, whether simple or not; if the path is not simple the complement $\tilde{M} \sim \cup_{T\epsilon\Gamma} T\delta$ will have a number of components, in each of which the function $w_\delta(z)$ is well defined. For instance the integral $w_{\alpha_1}(z)$ is a well defined holomorphic function in each connected component of $\tilde{M} \sim \cup_{T\epsilon\Gamma} T\alpha_1$; actually by (17) this function is identically zero in that connected component containing A, and it follows quite easily from (15) that $w_{\alpha_1}(Tz) = w_{\alpha_1}(z) + 2\pi i\ \beta_1(T)$ so that $\frac{1}{2\pi i} w_{\alpha_1}(z)$ is an integer in each connected component.

To complement the preceding considerations, it might be worth digressing briefly here to discuss an alternative normalization of the Abelian integral of the second kind. The meromorphic Abelian differentials and integrals are determined by their singularities only up to the addition of an arbitrary holomorphic Abelian differential or integral. They have been normalized here in such a manner that they have zero periods on the elements $A_1,\ldots,A_g \epsilon \Gamma$. An alternative normalization is that for which the periods on the elements $A_1,\ldots,A_g$, $B_1,\ldots,B_g$, hence on all elements of Γ, are purely imaginary. The advantages of this alternative normalization are that it is intrinsic to the Riemann surface itself, in the sense that it is independent of the marking, and that the real parts of the integrals are well defined functions on the initial Riemann surface M, since they are Γ-invariant functions on $\tilde{M}$. The disadvantage is that these differentials and integrals are not holomorphic functions of the parameters describing the locations of the singularities; it is for this last reason that the alternative normalization is not used here. At any rate, the alternatively normalized Abelian integral of the second kind is the function $G(z,a) = w_a(z) + \sum_j ic_j\, w_j(z)$ for some constants $c_j \in \mathbb{R}$, and its period for any element $T \in \Gamma$ is then $G(T,a) = \omega_a(T) + \sum_j ic_j \omega_j(T)$; in particular

$$G(A_k,a) = ic_k, \quad G(B_k,a) = 2\pi i\, w'_k(a) + \sum_j ic_j\, \omega_{jk},$$

or in vector notation

$$\{G(A_k,a) : 1 \le k \le g\} = i\, \vec{c}, \quad \{G(B_k,a) : 1 \le k \le g\} = 2\pi i \vec{w}'(a) + i\, \Omega\, \vec{c}$$

where $\vec{c} = \{c_k : 1 \le k \le g\}$ and $\Omega = {}^t\Omega = \{\omega_{jk} : 1 \le j,\ k \le g\}$ is the period matrix. Recall that $\Omega = X + i\, Y$ where X, Y are $g \times g$ real symmetric matrices and Y is positive definite. These periods are purely imaginary precisely when $\vec{c} \in \mathbb{R}^g$ and

$$0 = \text{Re}\, \left[2\pi i\, \vec{w}'(a) + i\, \Omega\, \vec{c}\right] = -2\pi\, \text{Im}\, \vec{w}'(a) - Y\, \vec{c};$$

the vector $\vec{c}$ is thus uniquely determined and given by $\vec{c} = -2\pi\, Y^{-1}\, \text{Im}\, \vec{w}'(a)$, so that the alternatively normalized Abelian integral of the second kind is

$$G(z,a) = w_a(z) - 2\pi i\, {}^t\text{Im}\, \vec{w}'(a)\, Y^{-1}\, \vec{w}(z)$$

$$= w_a(z) + 2\pi\, {}^t\text{Re}\, \vec{w}'(a)\, Y^{-1}\, \vec{w}(z),$$

and is uniquely determined. Its real part is the <u>Green's function</u> for the Riemann surface M,

(13) $$g(z,a) = \text{Re}\, w_a(z) + 2\pi\, {}^t\text{Re}\, \vec{w}'(a)\, Y^{-1}\, \text{Re}\, \vec{w}(z),$$

a well defined harmonic function of z on the Riemann surface M with a singularity at the point $z=a$ having an expansion in canonical local coordinates centered at $a \in \vec{M}$ of the form

$$g(z,a) = -\text{Re}\, \frac{1}{z-a} + (\text{harmonic function}).$$

The Green's function is also harmonic in the parameter $a \in \vec{M}$, but is not Γ-invariant.

§8. Quadratic period functions.

The investigation of theta factors of automorphy for a Riemann surface M led naturally to a particular combination of quadratic period classes, the Riemann point $r \in \mathbb{C}^g$ defined in (5.2). The general quadratic period classes $\int_{\alpha_\ell} w_j(z)\ \omega_k(z)$ and $\int_{\beta} w_j(z)\ \omega_k(z)$ are interesting further invariants associated to the Riemann surface M; but aside from the Riemann point itself they will not be considered much further here since at present they seem most useful in investigating the effects of changes of markings, and that is another topic altogether. However there are analogous quadratic period classes involving meromorphic Abelian differentials, and they will play a role in the subsequent discussion so will be examined in some detail.

Introduce then the integrals

$$(1) \qquad \tilde{\phi}^j_k(z) = \int_{x \in \alpha_j} w_z(x)\ \omega_k(x)$$

for $1 \leq j,k \leq g$. It is clear that $\tilde{\phi}^j_k(z)$ is a well defined holomorphic function of the variable z on the open subset $\tilde{M} \sim \Gamma\, \alpha_j = \tilde{M} \sim \bigcup_{T \in \Gamma} T\alpha_j \subset \tilde{M}$, in terms of canonical local coordinates on $\tilde{M}$. It is sometimes more convenient to consider instead the associated differential forms

$$(2) \qquad \underline{\tilde{\phi}}^j_k(z) = \tilde{\phi}^j_k(z)\ dz,$$

which are well defined holomorphic differential forms on the open subset $\tilde{M} \sim \Gamma\, \alpha_j \subset \tilde{M}$ and are independent of the choice of coordinates on $\tilde{M}$. In terms of these differential forms, the transformation formula for the

Abelian integral of the second kind given in Theorem 11 implies that

$$(3)\quad \underline{\underline{\tilde{\phi}}}{}^j_k(Tz) = \int_{x\in a_j} \Big[w_z(x) - \frac{2\pi i}{g} \sum_{\ell=1}^{g} \beta_\ell(T)\, w'_\ell(z)\Big]\, \omega_k(x)\, dx$$

$$= \underline{\underline{\tilde{\phi}}}{}^j_k(z) - \frac{2\pi i}{g}\, \delta^j_k \sum_{\ell=1}^{g} \beta_\ell(T)\, \omega_\ell(z)$$

for all $T \in \Gamma$. Now the paths Γa_j decompose the surface $\tilde{M}$ into a countable union of separate connected components, each of which is the universal covering space of the connected noncompact surface $M - a_j$, and $\tilde{\phi}^j_k$ and $\underline{\underline{\tilde{\phi}}}{}^j_k$ are really defined separately on these various components. However it is a familiar result in complex analysis that functions defined by such integrals, which are of a form much like that of the Cauchy integral formula, can be continued analytically across the path of integration, but that the continuation will differ from the originally given function by a function determined from the integrand. That result can be used to derive the following.

<u>Theorem 12</u>. The restriction of the differential form $\underline{\underline{\tilde{\phi}}}{}^j_k$ to that connected component $\tilde{M}_o$ of $\tilde{M} - \Gamma a_j$ containing the fundamental polygon Δ can be continued analytically from $\tilde{M}_o$ to all of $\tilde{M}$. The continuation is a holomorphic differential form $\underline{\underline{\phi}}{}^j_k$ on $\tilde{M}$ that satisfies

$$\underline{\underline{\phi}}{}^j_k(Tz) = \underline{\underline{\phi}}{}^j_k(z) + 2\pi i\ \beta_j(T)\ \omega_k(z) - \frac{2\pi i}{g}\ \delta^j_k \sum_{\ell=1}^{g} \beta_\ell(T)\ \omega_\ell(z)$$

for all $T \in \Gamma$ and

$$\int_{a_\ell} \underline{\underline{\phi}}{}^j_k(z) = 2\pi i\ \delta^j_\ell\ \delta^k_\ell$$

for $1 \le \ell \le g$, and it is uniquely determined by these two properties.

<u>Proof</u> The first step is to show that the differential form $\underset{=}{\tilde{\phi}}{}_k^j(z) = \tilde{\phi}_k^j(z)\,dz$ can be continued analytically across all interior points of the path α_j, and that if ${}^+\underset{=}{\tilde{\phi}}{}_k^j(z)$ denotes the analytic continuation of $\underset{=}{\tilde{\phi}}{}_k^j(z)$ across α_j from the right-hand side then ${}^+\underset{=}{\tilde{\phi}}{}_k^j(z) - \underset{=}{\tilde{\phi}}{}_k^j(z) = 2\pi i\, \omega_k(z)$ in a neighborhood of α_j on its right-hand side. Indeed consider a point $z_0 \in \tilde{M}$ near α_j and on the right-hand side of α_j; choose a segment δ_1 of the path α_j near z_0, and a path δ_2 joining the end points of δ_1 but lying entirely on the left-hand side of α_j otherwise, as sketched in Figure 5. Then for all points $z \in \tilde{M}$ near z_0

$$\tilde{\phi}_k^j(z) = \int_{\alpha_j} w_z(x)\, w'_k(x)\, dx$$
$$= \int_{\delta_1 - \delta_2} w_z(x)\, dx + \int_{\alpha_j - \delta_1 + \delta_2} w_z(x)\, w'_k(x)\, dx.$$

The first integral on this last line is zero, since for z near enough to z_0 to lie outside the closed path $\delta_1 - \delta_2$ the function $w_z(x)\, w'_k(x)$ is holomorphic in x inside $\delta_1 - \delta_2$. The second integral does not really involve

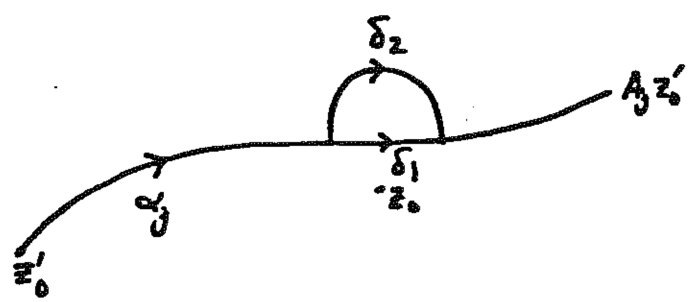


Figure 5

any integration along the segment δ_1, represents a holomorphic function of z that admits an analytic continuation across δ_1 and up to the path δ_2. It is clear from this that $\tilde{\phi}^j_k(z)$ can be continued analytically across δ_1, and that within the closed path $\delta_1 - \delta_2$ the analytic continuation is given by

$$^{+}\tilde{\phi}^j_k(z) = \int_{\alpha_j - \delta_1 + \delta_2} v_z(x)\, w'_k(x)\, dx;$$

therefore

$$^{+}\tilde{\phi}^j_k(z) - \tilde{\phi}^j_k(z) = \int_{\delta_2 - \delta_1} v_z(x)\, w'_k(x)\, dx$$

$$= -2\pi i \ \text{residue}_{x=z}\ v_z(x)\, w'_k(x)$$

$$= 2\pi i\, w'_k(z)$$

for all points z within the closed path $\delta_1 - \delta_2$, since in canonical local coordinates near z

$$v_z(x) = -\frac{1}{x-z} + (\text{holomorphic function of } x).$$

That establishes the asserted result, which can then be extended to hold in precisely the same way for the full paths $\Gamma\alpha_j$. Indeed note from the transformation formula of Theorem 11 that $v_{A_j z}(x)\, dA_j z = v_z(x)\, dz$, and consequently if $\alpha_j = \alpha'_j + \alpha''_j$ where α'_j is the portion of α_j from z_o to some interior point $z_1 \in \alpha_j$ and α''_j is the complementary portion then

$$\underset{=}{\tilde{\phi}^j_k}(z) = \int_{x \in \alpha'_j \cup \alpha''_j} v_z(x)\, dz \cdot w'_k(x)\, dx$$

$$= \int_{x \in A_j \alpha'_j \cup \alpha''_j} v_z(x)\, dz \cdot w'_k(x)\, dx;$$

this last integral is of the same form as that considered before, but has $A_j z_o$ as an interior point, so the same argument as before shows that $\underset{=}{\tilde{\phi}^j_k}(z)$ extends across the path $A_j \alpha'_j \cup \alpha''_j$ near $A_j z_o$. Next from (3) it follows that $\underset{=}{\tilde{\phi}^j_k}(Tz) - \underset{=}{\tilde{\phi}^j_k}(z)$ is a holomorphic differential form on all of $\tilde{M}$, so the properties already established for the analytic continuation through $\alpha_j \cup A_j \alpha'_j$ hold in precisely the same way through the image $T(\alpha_j \cup A_j \alpha'_j)$, and consequently hold for the full paths $\Gamma\alpha_j$. It is worth noting in this connection that the difference ${}^{+}\tilde{\underline{\underline{\phi}}}{}^j_k(z) - \tilde{\underline{\underline{\phi}}}{}^j_k(z) = -2\pi i\ \omega_k(z)$ is Γ-invariant.

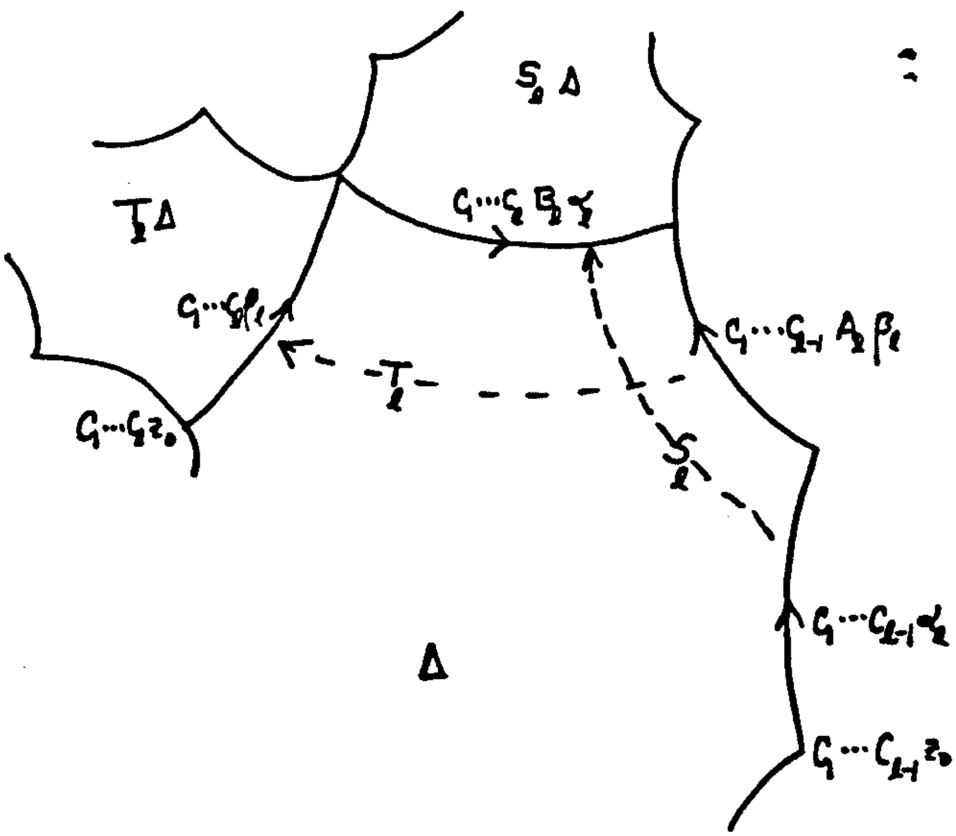
This shows that the differential form $\tilde{\underline{\underline{\phi}}}{}^j_k$ can be continued from any connected component of $\tilde{M} \sim \Gamma\alpha_j$ across any boundary path in $\Gamma\alpha_j$ to an adjacent connected component of $\tilde{M} \sim \Gamma\alpha_j$, and moreover that the continuation will differ from the initially given differential form in this new component by a global Abelian differential on $\tilde{M}$; the continuation is therefore holomorphic throughout this new component, and can similarly be continued across its boundary paths. Thus $\tilde{\underline{\underline{\phi}}}{}^j_k$ can be extended from the connected component $\tilde{M}_o$ of $\tilde{M} \sim \Gamma\alpha_j$ to become a holomorphic differential form $\underline{\underline{\phi}}{}^j_k$ on all of $\tilde{M}$.

To derive the transformational properties of the differential form $\underline{\underline{\phi}}{}^j_k$, consider the portion of the boundary of the fundamental polygon Δ indicated in Figure 6. The transformation $S_\ell = C_1 \cdots C_\ell B_\ell (C_1 \cdots C_{\ell-1})^{-1} = (C_1 \cdots C_{\ell-1} A_\ell)\ B_\ell (C_1 \cdots C_{\ell-1} A_\ell)^{-1} \in \Gamma$ takes the side $C_1 \cdots C_\ell \alpha_\ell$ to $C_1 \cdots C_\ell B_\ell \alpha_\ell$ and the transformation $T_\ell = C_1 \cdots C_\ell (C_1 \cdots C_{\ell-1} A_\ell)^{-1} = (C_1 \cdots C_{\ell-1} A_\ell B_\ell) A_\ell^{-1} (C_1 \cdots C_{\ell-1} A_\ell B_\ell)^{-1} \in \Gamma$ takes the side $C_1 \cdots C_{\ell-1} A_\ell \beta_\ell$ to $C_1 \cdots C_\ell \beta_\ell$; moreover $S_\ell \Delta$ is the translate of Δ that is adjacent to Δ along the side $C_1 \cdots C_\ell B_\ell \alpha_\ell$, while $T_\ell \Delta$ is the translate that is adjacent to Δ along the side $C_1 \cdots C_\ell B_\ell$. The 2g elements $S_1, \ldots, S_g$, $T_1, \ldots, T_g$ are another set of generators for Γ. Now if $\ell \neq j$ then $\Delta \cup S_\ell \Delta \cup T_\ell \Delta \subseteq \tilde{M}_o$, so that $\underline{\underline{\phi}}{}^j_k(z) = \tilde{\underline{\underline{\phi}}}{}^j_k(z)$ whenever

$z \in \Delta \cup S_\ell\Delta \cup T_\ell\Delta$; consequently $\underline{\underline{\varphi}}_k^j$ satisfies (3) for $T = S_\ell$ and $T = T_\ell$, and that is just the desired result since $\beta_j(S_\ell) = \beta_j(B_\ell) = 0$ and $\beta_j(T_\ell) = \beta_j(A_\ell^{-1}) = 0$ for $\ell \neq j$. If $\ell = j$ then at least $\Delta \cup T_j\Delta \subseteq M_0$ so that $\underline{\underline{\varphi}}_k^j$ satisfies (3) for $T = T_j$, and that is again the desired result since $\beta_j(T_j) = \beta_j(A_j^{-1}) = 0$. However Δ and $S_j\Delta$ are adjacent along the side $C_1 \cdots C_j B_j \alpha_j \subseteq \Gamma \alpha_j$ so lie in separate connected components of $\tilde{M} \sim \Gamma \alpha_j$; $\Delta \subseteq M_0$ so that $\underline{\underline{\varphi}}_k^j(z) = \tilde{\underline{\underline{\varphi}}}_k^j(z)$ whenever $z \in \Delta$, but from the results established in the first part of the proof it must be the case that

$$\underline{\underline{\varphi}}_k^j(z) = \tilde{\underline{\underline{\varphi}}}_k^j(z) + 2\pi i\ \omega_k(z) \quad \text{whenever } z \in S_j\Delta.$$

Figure 6



Then whenever $z \in \Delta$ so that $S_j z \in S_j \Delta$

$$\underline{\underline{\phi}}_k^j(S_j z) = \underline{\underline{\tilde{\phi}}}_k^j(S_j z) + 2\pi i\ \omega_k(S_j z)$$
$$= \underline{\underline{\tilde{\phi}}}_k^j(z) - \frac{2\pi i}{g}\ \delta_k^j \sum_{\ell=1}^{g} \beta_\ell(S_j)\ \omega_\ell(z) + 2\pi i\ \omega_k(S_j z)$$
$$= \underline{\underline{\phi}}_k^j(z) - \frac{2\pi i}{g}\ \delta_k^j \sum_{\ell=1}^{g} \beta_\ell(S_j)\ \omega_\ell(z) + 2\pi i\ \omega_k(S_j z).$$

The differential $\omega_k(z)$ is Γ-invariant, so that $\omega_k(S_j z) = \omega_k(z)$; and since S_j is conjugate to B_j as observed above then $\beta_j(S_j) = 1$ and this last formula can be rewritten

$$\underline{\underline{\phi}}_k^j(S_j z) = \underline{\underline{\phi}}_k^j(z) - \frac{2\pi i}{g}\ \delta_k^j \sum_{\ell=1}^{g} \beta_\ell(S_j)\ \omega_\ell(z) + 2\pi i\ \beta_j(S_j)\ \omega_k(z),$$

which is the desired result for $S_j = T$. Thus the desired transformational equation holds for all the elements $S_1, \ldots, S_g, T_1, \ldots, T_g$, and since they generate Γ it must consequently hold for all $T \in \Gamma$.

To determine the periods of these differential forms, for each ℓ in the range $1 \leq \ell \leq g$ choose a point $b_\ell \in C_1 \cdots C_{\ell-1} A_\ell \beta_\ell$ and a path $\tilde{a}_\ell$ from $T_\ell b_\ell$ to b_ℓ lying in Δ except for its end points; let σ_ℓ be that portion of the path $C_1 \cdots C_{\ell-1} A_\ell \beta_\ell$ from b_ℓ to $C_1 \cdots C_\ell B_\ell A_\ell z_0$, and let Δ_ℓ be that portion of the canonical fundamental polygon Δ with boundary

$$\partial \Delta_\ell = \tilde{\alpha}_\ell + \sigma_\ell - C_1 \cdots C_\ell B_\ell \alpha_\ell - T_\ell \sigma_\ell$$

as sketched in Figure 7. Assuming that b_ℓ is an interior point of the arc $C_1 \cdots C_{\ell-1} A_\ell \beta_\ell$ implies that the path $\tilde{\alpha}_\ell$ is disjoint from $\Gamma\ \sigma_j$ for all j; thus the path $\tilde{\alpha}_\ell \subset \tilde{M}_0 \subset \tilde{M}$ avoids all the points at which there are any complications in determining the values of the differential forms $\underline{\underline{\phi}}_k^j$, and consequently the periods along $\tilde{\alpha}_\ell$ at least can readily be calculated.

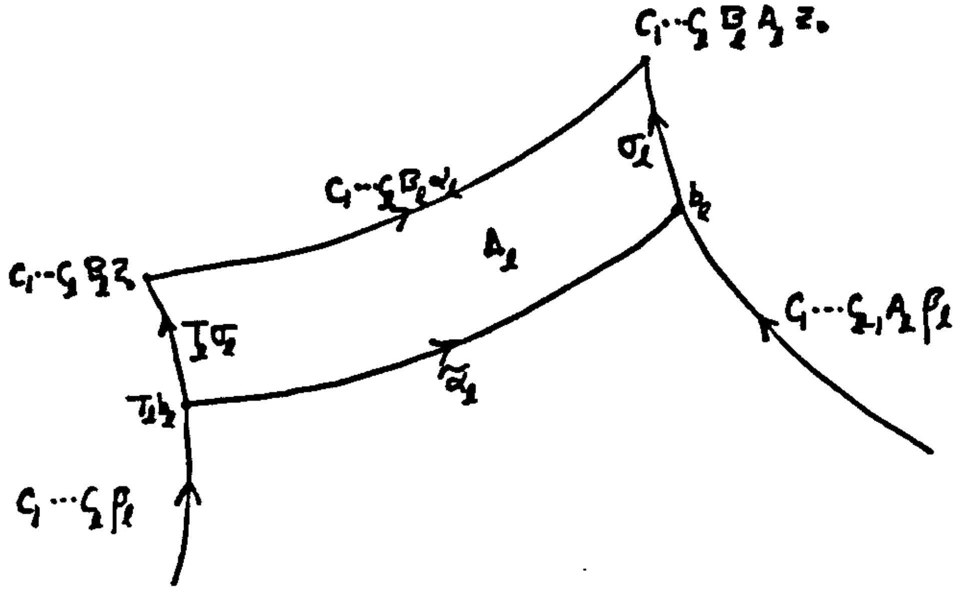


Figure 7

Indeed

$$(4)\quad \int_{\tilde{a}_\ell} \underline{\underline{\phi}}^{j}_{k}(z) = \int_{\tilde{a}_\ell} \tilde{\underline{\underline{\phi}}}^{j}_{k}(z) = \int_{z\in\tilde{a}_\ell} \Big(\int_{x\in a_j} v_z(x)\ v'_k(x)\ dx \Big)\ dz$$

$$= \int_{x\in a_j} \Big(\int_{z\in\tilde{a}_\ell} v_z(x)\ dz \Big)\ v'_k(x)\ dx\ ,$$

where the interchange of the order of integration is possible since the integral is continuous in the variables $(z,x) \in \tilde{a}_\ell \times a_j$. Now fix a point $x \in a_j$, let $y = C_1 \cdots C_j B_j x \in C_1 \cdots C_j B_j a_j \subset \partial\Delta$, and choose a sequence of points $y_\nu \in \Delta$ converging to $y \in \partial\Delta$; it can of course be supposed that

$y_\nu \in \Delta_\ell$ for all ν if $\ell \neq j$ and that $y_\nu \in \Delta_\ell$ for all ν if $\ell = j$. From the transformational formula (15) of Theorem 11 it follows that

$$\int_{z\in\tilde{\alpha}_\ell} w_z(y)dz = \int_{z\in\tilde{\alpha}_\ell} w_z(C_1 \cdots C_j B_j x)\, dz$$

$$= \int_{z\in\tilde{\alpha}_\ell} [w_z(x)\, dz + 2\pi i\, w'_j(z)\, dz]$$

$$= \int_{z\in\tilde{\alpha}_\ell} w_z(x)\, dz + 2\pi i\, \delta^j_k,$$

for $\int_{\tilde{\alpha}_\ell} w'_j(z)\, dz = \int_{\alpha_\ell} w'_j(z)\, dz = \delta^j_\ell$ since $\tilde{\alpha}_\ell$ is homologous to α_ℓ on M. On the other hand since the points y and y_ν do not lie on $\tilde{\alpha}_\ell$ the functions $w_z(y_\nu)$ converge to $w_z(y)$ uniformly in $z \in \tilde{\alpha}_\ell$, so that

$$\int_{z\in\tilde{\alpha}_\ell} w_z(y)\, dz = \lim_\nu \int_{z\in\tilde{\alpha}_\ell} w_z(y_\nu)\, dz.$$

Next since $w_z(y_\nu)$ has a simple pole at y_ν with residue 1 and $y_\nu \in \Delta_\ell$ precisely when $\ell = j$

$$\int_{\partial\Delta_\ell} w_z(y)\, dz = 2\pi i \operatorname*{residue}_{z\in\Delta_\ell} w_z(y_\nu) = 2\pi i\, \delta^j_\ell,$$

while from the transformational formula (16) and the integral formula (17) of Theorem 11

$$\int_{\partial\Delta_\ell} w_z(y_\nu)\, dz = \int_{\tilde{\alpha}_\ell + \sigma_\ell - C_1 \cdots C_\ell B_\ell \alpha_\ell - T_\ell \sigma_\ell} w_z(y_\nu)\, dz$$

$$= \int_{\tilde{\alpha}_\ell} w_z(y_\nu)\, dz + \int_{\sigma_\ell} w_z(y_\nu)\, dz$$

$$- \int_{\alpha_\ell} [w_z(y_\nu) - \frac{2\pi i}{g} w'_\ell(z)]\, dz - \int_{\sigma_\ell} w_z(y_\nu)\, dz$$

$$= \int_{\tilde{\alpha}_\ell} w_z(y_\nu)\, dz + \frac{2\pi i}{g},$$

since T_ℓ is conjugate to A_ℓ^{-1} ; therefore

$$\int_{\tilde{a}_\ell} w_z(y_\nu)\, dz = 2\pi i\, (\delta^j_\ell - \frac{1}{g}),$$

so that

$$\int_{\tilde{a}_\ell} w_z(x)\, dz = -\frac{2\pi i}{g},$$

and consequently (4) becomes

$$\int_{\tilde{a}_\ell} \underline{\underline{\phi}}^j_k(z) = -\frac{2\pi i}{g} \int_{a_j} w'_k(x)\, dx = -\frac{2\pi i}{g}\, \phi^j_k.$$

The desired result follows readily upon an application of the transformational formula for $\underline{\underline{\phi}}^j_k$ from the first part of the proof; for since $\underline{\underline{\phi}}^j_k$ is everywhere holomorphic

$$-\frac{2\pi i}{g}\, \phi^j_k = \int_{\tilde{a}_\ell} \underline{\underline{\phi}}^j_k(z) = \int_{T_\ell\sigma_\ell + C_1 \cdots C_\ell B_\ell a_\ell - \sigma_\ell} \underline{\underline{\phi}}^j_k(z)$$

$$= \int_{\sigma_\ell} \underline{\underline{\phi}}^j_k(z) + \int_{a_\ell} [\underline{\underline{\phi}}^j_k(z) + 2\pi i\, \phi^j_k \omega_k(z) - \frac{2\pi i}{g}\, \phi^j_k \omega_\ell(z)] - \int_{\sigma_\ell} \underline{\underline{\phi}}^j_k(z)$$

$$= \int_{a_\ell} \underline{\underline{\phi}}^j_k(z) + 2\pi i\, \delta^j_\ell\, \delta^k_\ell - \frac{2\pi i}{g}\, \phi^j_k.$$

It remains merely to show that the two properties that have now been established determine the differential form $\underline{\underline{\phi}}^j_k(z)$ uniquely. The difference between any two differential forms with these properties is an Abelian differential on M as a consequence of the first property, and has zero periods along all the paths a_ℓ as a consequence of the second property so must vanish identically; and that concludes the proof.

It should be noted that the differential forms $\underset{=}{\tilde{\phi}}{}^j_k$ can be continued analytically from any connected component of $\tilde{M} - \Gamma\, a_j$ to become holomorphic differential forms on all of $\tilde{M}$, and that any two such continuations differ by an Abelian differential on M; the differential form $\underset{=}{\phi}{}^j_k$ was just a convenient canonical choice of one such consideration. All continuations satisfy the same functional equation under the action of Γ, but have different periods along the paths a_ℓ. The canonical choice $\underset{=}{\phi}{}^j_k(z) = \phi^j_k(z)dz$ will be called a quadratic period form, and the coefficient $\phi^j_k(z)$ a quadratic period function. They will be of interest here primarily as canonical solutions of the functional equation of the preceding theorem. That equation is particularly simple for $j \neq k$, and a solution of the corresponding simple functional equation for $j = k$ can be concocted by combining the differential form $\underset{=}{\phi}{}^j_j$ and the canonical Abelian integral of the second kind; in terms of the functions rather than the differential forms the result is as follows.

Corollary. For any fixed point $a \in \tilde{M}$ the function

$$\phi^j_k(z;a) = \phi^j_k(z) - \delta^j_k\, v_z(a)$$

satisfies

$$\phi^j_k(Tz;a)\ \kappa(T,z)^{-1} = \phi^j_k(z;a) + 2\pi i\ \beta_j(T)\ w'_k(z)$$

for all $T \in \Gamma$.

Proof. This is an immediate consequence of the formulas of Theorems 11 and 12.

A few remarks about the significance of this functional equation should perhaps be inserted here. In terms of any factor of automorphy $\mu(T,z)$ the group Γ can be viewed as acting as a group of linear transformations on the vector space V of all holomorphic functions on $\tilde{M}$ by setting

$$f^T(z) = \mu(T,z)^{-1} f(Tz)$$

for any $f \in V$ and $T \in \Gamma$; the mapping $f \to f^T$ from V to V is clearly linear, and from the functional equation for a factor of automorphy it follows that

$$f^{ST}(z) = \mu(ST,z)^{-1} f(STz)$$
$$= \mu(T,z)^{-1} \mu(S,Tz)^{-1} f(STz) = \mu(T,z)^{-1} f^S(Tz)$$

hence that $f^{(ST)} = (f^S)^T$ for all $S, T \in \Gamma$. This action, and therefore the structure of V as a Γ-module, depend on the choice of the factor of automorphy, which will be indicated when necessary by writing V_μ. Now to any such Γ-module there are associated cohomology groups $H^p(\Gamma,V_\mu)$ for all integers $p \geq 0$. Formally let $C^p(\Gamma,V_\mu)$ be the vector space of all mappings $f : \Gamma^{p+1} \to V_\mu$ such that

$$f(T_0T, T_1T, \ldots, T_pT) = f(T_0,T_1,\ldots, T_p)^T$$

for all $T,T_0, \ldots, T_p \in \Gamma$, and let $\delta : C^p(\Gamma,V_\mu) \to C^{p+1}(\Gamma, V_\mu)$ be the linear mapping that sends an element $f \in C^p(\Gamma,V_\mu)$ to the mapping $\delta f : \Gamma^{p+2} \to V$ defined by

$$(\delta f)(T_0,T_1, \ldots, T_{p+1}) = \sum_{i=0}^{p+1} (-1)^i f(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{p+1});$$

it is clear that $\delta f \in C^{p+1}(\Gamma,V_\mu)$, and a straightforward calculation to verify that $\delta^2 = 0$. If $Z^p(\Gamma,V_\mu) \subset C^p(\Gamma,V_\mu)$ is the kernel of the mapping $\delta : C^p(\Gamma,V_\mu) \to C^{p+1}(\Gamma,V_\mu)$ and $B^p(\Gamma,V_\mu) = \delta C^{p-1}(\Gamma,V_\mu)$ then $B^p(\Gamma,V_\mu) \subseteq Z^p(\Gamma,V_\mu)$ and by definition

$$H^p(\Gamma,V_\mu) = Z^p(\Gamma,V_\mu) / B^p(\Gamma,V_\mu) ;$$

for $p = 0$ this is modified by setting $B^0(\Gamma,V_\mu) = 0$. In particular if $f \in C^0(\Gamma,V_\mu)$ then $f(I) = f_0(z) \in V_\mu$ is a holomorphic function on $\tilde{M}$, and

$f(T) = f(I)^T = \mu(T,z)^{-1} f_o(T_z)$ for any element $T \in \Gamma$; moreover $f \in Z^0(\Gamma,V_\mu)$ precisely when

$$0 = (\delta f)\ (I,T) = f(T) - f(I)$$
$$= \mu(T,z)^{-1} f_o(Tz) - f_o(z)$$

for all Γ, hence when $f_o(z)$ is relatively automorphic for μ. Thus $H^0(\Gamma,V_\mu) = \Gamma(\mu)$ is just the space of holomorphic relatively automorphic functions for the factor of automorphy $\mu(T,z)$. Next if $f \in C^1(\Gamma,V_\mu)$ then $f(I,T) = f_o(T,z)$ is a holomorphic function on $\tilde{M}$ for each $T \in \Gamma$, and $f(T_0,T_1) = f(I,T_1T_0^{-1})^{T_0} = \kappa(T_1,z)^{-1} f_o(T_1T_0^{-1},T_0z)$ for any elements $T_0,T_1 \in \Gamma$. In this case $f \in Z^1(\Gamma,V_0)$ precisely when

$$0 = (\delta f)(I,T_1,T_2) = f(T_1,T_2) - f(I,T_2) + f(I,T_1)$$
$$= \mu(T_1,z)^{-1} f_o(T_2T_1^{-1},\ T_1z) - f_o(T_2,z) + f(T_1,z),$$

or equivalently, upon setting $T_1 = T$ and $T_2 = ST$, when

$$f_o(ST,z) = \mu(T,z)^{-1} f_o(S,Tz) + f(T,z)$$

for all $S,\ T \in \Gamma$; and $f \in B^1(\Gamma,V_\mu)$ precisely when there is some element $g \in C^0(B,V_\mu)$ described by a holomorphic function $g(I) = g_0(z) \in V_\mu$ such that

$$f_o(T,z) = f(I,T) = (\delta g)\ (I,T) = g(T) - g(I)$$
$$= \mu(T,z^{-1})\ g_o(Tz) - g_o(z)$$

for all $T \in \Gamma$.

The functional equation of the Corollary to Theorem 12, and that of Theorem 12 itself when written out in terms of the coefficients $\phi_k^j(z)$ of the differential forms ϕ_k^j, are of just this form. Thus the functions

$$f_k^j(T,z) = 2\pi i\ \beta_j(T)\ w'_k(z)$$

can be viewed as cocycles $f_k^j \in Z^1(\Gamma,V_\kappa)$ representing some cohomology classes $[f_k^j] \in H^1(\Gamma,V_\kappa)$, for the Γ-module V_κ defined by the canonical factor of automorphy, and Theorem 12 implies that

$$[f_k^j] = \frac{1}{g}\,\delta_k^j \sum_{\ell=1}^{g} [f_\ell^\ell] \quad \text{in} \quad H^1(\Gamma,V_\kappa);$$

indeed

$$f_k^j - \frac{1}{g}\,\delta_k^j \sum_{\ell=1}^{g} f_\ell^\ell = \delta(\phi_k^j)$$

for the cochain $\phi_k^j \in C^0(\Gamma,V_\kappa)$ described by the holomorphic functions $\phi_k^j(z)$. Thus $[f_k^j] = 0$ whenever $j \neq k$, while $[f_\ell^\ell]$ is independent of ℓ or equivalently $[f_1^1] = [f_2^2] = \dots = [f_g^g]$. It is actually the case that $[f_1^1] \neq 0$; for if $[f_1^1] = 0$ there would exist some holomorphic function $g(z)$ on $\tilde{M}$ such that $g(Tz)\ \kappa(T,z)^{-1} - g(z) = 2\pi i\ \beta_1(T)w'_1(z)$, and then $g(z) - \phi_1^1(z) + w_z(a)$ would be a holomorphic differential form on M with a single nontrivial simple pole at the point a, an impossiblity. The cocyles $f_k^j(T,z)$ are really of a rather special form. Note that for the trivial factor of automorphy $\mu(T,z) = 1$ the constant functions are a well defined Γ-submodule $\mathbb{C}_1 \subset V_1$, the subscript here referring to the Γ-module structure defined by the trivial factor of automorphy; moreover an element $f_o \in C^1(\Gamma,\mathbb{C}_1)$ is a cocycle precisely when it is a homomorphism $f_o \in \mathrm{Hom}\,(\Gamma,\mathbb{C})$, while the only coboundaries are the trivial homomorphisms, so that $H^1(\Gamma,\mathbb{C}_1) = Z^1(\Gamma,\mathbb{C}_1) = \mathrm{Hom}\,(\Gamma,\mathbb{C})$. The cocycles $f_k^j \in Z^1(\Gamma,V_\kappa)$ are of the form of the product of the element $\beta_j \in \mathrm{Hom}(\Gamma,\mathbb{Z}) = H\,(\Gamma,\mathbb{Z})$ and the relatively automorphic function $w'_k(z) \in H^0(\Gamma,V_\kappa)$; and in general multiplication is easily seen to yield well defined homomorphism

$$H^1(\Gamma,\mathbb{Z}_1) \times H^0(\Gamma,V_\kappa) \to H^1(\Gamma,V_\kappa),$$

a natural product construction.

So far this is all quite formal and rather trivial, but it does reflect some very interesting and nontrivial structure; pursuing this in detail would be too much of a digression here, but a brief survey should be included to justify as much of a digression as already been intruded into this discussion. The point is that these formal cohomology groups $H^p(\Gamma, V_\mu)$ are isomorphic to the analytic cohomology groups $H^p(M,\mu)$ of the holomorphic line bundle μ associated to the factor of automorphy $\mu(T,z)$; all are finite-dimensional complex vector spaces, but only $H^0(M,\mu)$ and $H^1(M,\mu)$ are nontrivial and they are related to one another through the Serre duality theorem, which asserts that $H^1(M,\mu)$ is canonically isomorphic to the dual of $H^0(M,\kappa\mu^{-1})$. In particular then $H^1(\Gamma,V_\kappa) \cong H^1(M,\kappa) \cong H^0(M,1) = \mathbb{C}$; the cohomology class $[\phi^\ell_\ell] \in H^1(\Gamma,V_\kappa)$ thus generates this one-dimensional vector space. In view of the Serre duality theorem, the one-dimensional cohomology groups can generally be replaced by zero-dimensional cohomology groups, so are usually only considered incidentally as they arise in passing in standard arguments involving exact cohomology sequences. However there are cases in which they do arise naturally and merit independent consideration; the functional equations of Theorem 12 are one form in which they will arise in subsequent discussion here.

To turn now in another direction, there is more to be said about these functions $\phi^j_k(z)$ as quadratic period integrals rather than just as solutions of some interesting functional equations. First of all, the original definition (1) involves the integration of the product of one differential and one integral, and there are two choices of which is which. The other choice does not lead to much that is really very different though, since

$$\int_{x\epsilon\alpha_j} v_k(x)\ \omega_z(x) = \int_{\alpha_j} \frac{d}{dx}\left[v_z(x)\ v_k(x)\ dx - \int_{\alpha_j} v_z(x)\ v'_k(x)\ dx\right.$$

$$= v_z(A_j z_o)\ v_k(A_j z_o) - v_z(z_o)\ v_k(z_o) - \tilde{\phi}_k^j(z)$$

provided that $z \epsilon \Gamma \alpha_j$; this can be rewritten by using the known periods of the Abelian differentials of the first and second kinds as

$$(5)\quad \int_{x\epsilon\alpha_j} v_k(x)\ \omega_z(x) = \delta_k^j\ v_z(z_o) - \tilde{\phi}_k^j(z)$$

whenever $z \epsilon \Gamma\alpha_j$. In this connection it should be recalled that the normalization adopted for the canonical Abelian integral of the second kind involved its expression as a derivative of a prime function rather than its vanishing at the base point $z_o \epsilon \tilde{M}$; thus $v_z(z_o)$ is not necessarily zero.

Next, the original definition (1) involves the integration only over the paths α_j, but there are of course analogous integrals over the paths β_j; however these latter integrals can be expressed in terms of the integrals (1) and other standard invariants. For instance whenever $z \epsilon \Delta$ then by the residue theorem

$$-2\pi i\ v_j(z)\ v'_k(z) = \int_{x\epsilon\partial\Delta} v_z(x)\ v_j(x)\ \omega_k(x)$$

$$= \sum_{\ell=1}^{g} \int_{x\epsilon C_1\cdots C_{\ell-1}\alpha_\ell + C_1\cdots C_{\ell-1}A_\ell\beta_\ell - C_z\cdots C_\ell B_\ell\alpha_\ell - C_1\cdots C_\ell\beta_\ell} v_z(x) v_j(x)\omega_k(x)$$

$$= \sum_{\ell=1}^{g} \int_{\alpha_\ell} v_z(x) v_j(x)\ \omega_k(x) - [v_z(x) + 2\pi i\ v'_\ell(z)]\ [v_j(x) + \omega_{j\ell}]\ \omega_k(x)$$

$$+ \sum_{\ell=1}^{g} \int_{\beta_\ell} v_z(x)\ [v_j(x) + \delta_\ell^j]\ \omega_k(x) - v_z(x)\ v_j(x)\ \omega_k(x)$$

$$= \sum_{\ell=1}^{g} \{\omega_{j\ell}\int_{\alpha_\ell} v_z(x)\omega_k(x) + 2\pi i\ v'_\ell(z)\ [\delta_\ell^k\ \omega_{j\ell} + \int_{\alpha_\ell} v_j(x)\ \omega_k(x)]\}$$

$$+ \int_{\beta_j} v_z(x)\ \omega_k(x)$$

hence

$$(6)\quad \int_{\beta_j} w_z(x)\,\omega_k(x) = \sum_{\ell=1}^{g} \left[\omega_{j\ell}\phi_k^{\ell}(z) + 2\pi i\, w'_{\ell}(z)\int_{\alpha_\ell} w_j(z)\,\omega_k(x)\right] + 2\pi i\,\left[\omega_{jk} - w_j(z)\right] w'_k(z)$$

whenever $z \in \Delta$, where all derivatives are as usual with respect to the canonical local coordinates on $\tilde{M}$. This expresses the quadratic period integrals over the paths β_j in terms of those over the paths α_j, together with the ordinary quadratic period classes and other standard invariants.

§9. Varieties of special positive divisors.

It is worth pausing to indicate here how the two basic general theorems about divisors on compact Riemann surfaces fit into the present discussion. First to any divisor $\underline{\mathfrak{d}}$ on M there is associated as in the Corollary to Theorem 5 the factor of automorphy $\rho_{w(\underline{\mathfrak{d}})} \cdot \zeta^{\deg \underline{\mathfrak{d}}}$, while as in the discussion in §4 two divisors are linearly equivalent precisely when their associated factors of automorphy are equivalent. Therefore two divisors $\underline{\mathfrak{d}}_1$ and $\underline{\mathfrak{d}}_2$ are linearly equivalent precisely when $\deg \underline{\mathfrak{d}}_1 = \deg \underline{\mathfrak{d}}_2$ and $w(\underline{\mathfrak{d}}_1) = w(\underline{\mathfrak{d}}_2)$ in J; this is Abel's theorem.

Next consider a factor of automorphy which admits some nontrivial holomorphic relatively automorphic functions, hence one of the form $\zeta_{\underline{\mathfrak{d}}} = \rho_{w(\underline{\mathfrak{d}})} \cdot \zeta^{\deg \underline{\mathfrak{d}}}$ for a positive divisor $\underline{\mathfrak{d}}$. As in the discussion in §4, the vector space $\Gamma(\zeta_{\underline{\mathfrak{d}}})$ is isomorphic to the space of meromorphic functions g on M such that $\underline{\mathfrak{d}}(g) + \underline{\mathfrak{d}} \geq 0$. If $\underline{\mathfrak{d}} = \sum \nu_i \cdot a_i$ for distinct points $a_i \in M$, the condition that $\underline{\mathfrak{d}}(g) + \underline{\mathfrak{d}} \geq 0$ means that g has as singularities at most poles of order ν_i at the points a_i. To any such function g there can be associated its principal parts at these points, the expression $\sum_{j=1}^{\nu_i} c_{ij}(z - a_i)^{-j}$; the mapping from g to the coefficients $\{c_{ij}\}$ is a linear mapping to a vector space of dimension $\sum_i \nu_i = \deg \underline{\mathfrak{d}}$, and the kernel consists of the everywhere holomorphic functions hence of the complex constants. This thus shows that $\gamma(\zeta_{\underline{\mathfrak{d}}}) = \dim \Gamma(\zeta_{\underline{\mathfrak{d}}}) \leq \deg \underline{\mathfrak{d}} + 1$, so in particular the spaces $\Gamma(\zeta_{\underline{\mathfrak{d}}})$ are finite-dimensional. A more precise estimate is the Riemann-Roch theorem, which asserts that on a compact Riemann surface of genus g

$$\gamma(\zeta_{\underline{\mathfrak{d}}}) = \gamma(\kappa\zeta_{\underline{\mathfrak{d}}}^{-1}) + \deg \underline{\mathfrak{d}} + 1 - g$$

where κ is the canonical factor of automorphy. This follows quite simply from the preceding discussion. First consider a divisor $\mathfrak{d} = a_1 + \ldots + a_r$ in which a_i are distinct points of M. To any meromorphic function g on M for which $\mathfrak{d}(g) + \mathfrak{d} \geq 0$ associate the differential of the second kind dg, which will have double poles at the points a_i and zero periods since its integral g is Γ-invariant. Any such differential arises from some function g in this way, and the kernel of the resulting linear mapping from functions to differentials is the one-dimensional space of constant functions; thus $\gamma(\zeta_{\mathfrak{d}}) - 1$ is just the dimension of the space of these particular differentials. Now any such differential can be written uniquely in the form

$$dg(z) = \sum_{i=1}^{g} x_i \omega_i(z) + \sum_{j=1}^{r} y_j \omega_{a_j}(z)$$

in terms of the canonical differentials of the first and second kinds, for some constants x_i, $y_j \in E$. In view of the known periods for the canonical Abelian differentials $\omega_i(z)$ and the period formulas (6.10), the condition that $dg(z)$ have zero period for the transformation A_i is that $x_i = 0$ and the condition that it have zero period for the transformation B_i is that

$$\sum_{j=1}^{r} y_j w_i'(a_j) = 0 ;$$

therefore in terms of the $g \times r$ matrix $W = \{w_i'(a_j) : 1 \leq i \leq g, 1 \leq j \leq r\}$

(2) $\quad \gamma(\zeta_{\mathfrak{d}}) - 1 = \dim\{y \in E^r : \sum_{j=1}^r y_j v_i'(a_j) = 0 \quad \text{for} \quad 1 \le i \le g\}$
$\qquad\qquad = r - \text{rank } W .$

On the other hand if $f \in \Gamma(\zeta_{\mathfrak{d}})$ has divisor $\mathfrak{d}(f) = \mathfrak{d}$ then multiplying a function in $\Gamma(\kappa\zeta_{\mathfrak{d}}^{-1})$ by f establishes an isomorphism between $\Gamma(\kappa\zeta_{\mathfrak{d}}^{-1})$ and the space of functions in $\Gamma(\kappa)$ that vanish on $\mathfrak{d}$. Any function in $\Gamma(\kappa)$ can be written uniquely in the form

$$h = \sum_{i=1}^g t_i v_i'(z)$$

in terms of the canonical derivatives $v_i'(z) \in \Gamma(\kappa)$, and therefore

(3) $\quad \gamma(\kappa\zeta_{\mathfrak{d}}^{-1}) = \dim\{t \in E^g : \sum_{i=1}^g v_i'(a_j) = 0 \quad \text{for} \quad 1 \le j \le r\}$
$\qquad\qquad = g - \text{rank } W .$

Combining (2) and (3) leads immediately to the identity $\gamma(\zeta_{\mathfrak{d}}) - 1 - r = -\text{rank } W = \gamma(\kappa\zeta_{\mathfrak{d}}^{-1}) - g$, which is the Riemann-Roch theorem. If there are some coincidences among the points a_i, a simple modification of this argument will give the proof. For example if $a_1 = a_2$ but the points are otherwise distinct then in examining $\gamma(\zeta_{\mathfrak{d}})$ it is necessary to consider differentials of the second kind with triple poles at a_1, so in place of $\omega_{a_1}(z)$ and $\omega_{a_2}(z)$ it is necessary to use $\omega_{a_1}(z)$ and $\partial\omega_{a_2}(z)/\partial a_1$; thus in the matrix W in place of $v_i'(a_1)$ and $v_i'(a_2)$ it is necessary to use $v_i'(a_1)$ and $v_i''(a_1)$. The same modification of the matrix W is needed in the analysis of $\gamma(\kappa\zeta_{\mathfrak{d}}^{-1})$, which involves functions in $\Gamma(\kappa)$ vanishing to the second order at a_1, so the proof proceeds in just the same way as before.

It may be helpful for those not terribly familiar with these theorems to see a typical and useful application of the Riemann-Roch theorem, in showing that the Abelian differentials on a compact Riemann surface of genus $g > 0$ have no common zeros. If all the functions in $\Gamma(\kappa)$ vanish at some point a then dividing these functions by $q(z,a)$ yields elements of $\Gamma(\kappa\zeta_a^{-1})$, so that $\gamma(\kappa\zeta_a^{-1}) = \gamma(\kappa) = g$. It then follows from the Riemann-Roch theorem that $\gamma(\zeta_a) = \gamma(\kappa\zeta_a^{-1}) + 1 + 1 - g = 2$, so that there are at least two linearly independent functions $f_1, f_2 \in \Gamma(\zeta_a)$. If $\mathfrak{J}(f_i) = a_i$ then $a_1 \neq a_2$, since otherwise f_1/f_2 would be a holomorphic Γ-invariant function hence a constant, contradicting the condition that f_1, f_2 are linearly independent. Therefore f_1/f_2 is a nontrivial Γ-invariant meromorphic function, so a meromorphic function on M with divisor $\mathfrak{J}(f) = a_1 - a_2$. This function takes every complex value precisely once on M outside a_2, since for any $c \in \mathbb{C}$ the function $f(z) - c$ has a simple pole at a_2 and must therefore have a simple zero at some other single point of M. Thus f can be viewed as a one-to-one holomorphic mapping $f : M \to \mathbb{P}^1$, which means M is really just the Riemann sphere $\mathbb{P}^1$ and contradicts the assumption that $g > 0$.

The canonical Abelian integrals on the marked Riemann surface are the coordinate functions of the holomorphic mapping $w : \tilde{M} \to \mathbb{C}^g$ that induces the Abel-Jacobi mapping $w : M \to J$ to the Jacobi variety of M, as discussed earlier. This mapping has the following properties.

Theorem 10. The Abel-Jacobi mapping $w : M \to J$ for a marked Riemann surface of genus $g > 0$ is a nonsingular biholomorphic mapping between M and its image $w(M) \subseteq J$.

Proof. If $w(z_1) = w(z_2)$ for distinct points z_1, z_2 in M then by Abel's theorem there would exist a meromorphic function f on M with $\underline{\mathcal{J}}(f) = z_1 - z_2$. But a function f with a simple pole on M takes every complex value c just once, since the function $f - c$ must have a single zero on M, hence establishes a biholomophic mapping $f : M \to \mathbb{P}^1$ in contradiction to the assumption that $g > 0$. The differential of the Abel-Jacobi map is given by the vector of canonical Abelian differentials $\omega(z) = dw(z)$, and these have no common zeros as already noted so the mapping $w : M \to J$ is nonsingular. That suffices to conclude the proof.

The image of the Abel-Jacobi mapping will be denoted by $w(M) = W_1 \subseteq J$; thus W_1 can be viewed as an embedding of the Riemann surface M in its Jacobi variety. It is worth noting as another consequence of the preceding theorem that the mapping $w : \tilde{M} \to \mathbb{C}^g$ is also a nonsingular holomorphic mapping, and in this case $w(z_1) = w(z_2)$ precisely when $z_1 = Tz_2$ for some $T \in \Gamma$ with period $\omega(T) = 0$, hence precisely when T lies in the commutator subgroup $[\Gamma, \Gamma] \subseteq \Gamma$ since the period homomorphism $\omega : \Gamma \to \underline{L}$ really amounts to the mapping of the group Γ to its abelianization. If $\hat{M} = w(\tilde{M}) \subseteq \mathbb{C}^g$ then $\hat{M}$ is an irreducible (or equivalently connected) complex submanifold of $\mathbb{C}^g$ and the universal covering mapping factors in the form $\tilde{M} \to \hat{M} \to M$; these coverings are Galois coverings, corresponding to the tower of subgroups $1 \subseteq [\Gamma, \Gamma] \subseteq \Gamma$, so $\hat{M} \to M$ is just the maximal Abelian covering of M with covering translation group $\underline{L} = \Gamma/[\Gamma, \Gamma]$.

Straightforward extensions of the Abel-Jacobi mapping lead to a collection of interesting and important further subvarieties of the Jacobi variety. For any integer $r \geq 0$ introduce the natural holomorphic mapping $w : M^r \to J$ defined by $w(z_1, \ldots, z_r) = w(z_1) + \ldots + w(z_r)$; the image is a holomorphic subvariety $w(M^r) = W_r \subseteq J$. (It is a general result in complex analysis that the image of a holomorphic variety under a proper holomorphic mapping is a holomorphic variety; in particular the image of a compact complex manifold under a holomorphic mapping is a holomorphic variety.) Note that the image does not really depend on the order of the points $z_1, \ldots, z_r$, but merely on the divisor $z_1 + \ldots + z_r$; thus W_r can be viewed as the image of the set of all positive divisors of degree r , under the natural extension of the Abel-Jacobi mapping to a homomorphism from the group of divisors on M to the group J . These subvarieties are consequently called the <u>subvarieties</u> <u>of</u> <u>positive</u> <u>divisors</u> in J . The condition that the mapping $w : M^r \to J$ be independent of the order of the factors is equivalent to the condition that it factors through the quotient space $M^{(r)} = M^r/\mathfrak{S}_r$ of the natural action of the symmetric group $\mathfrak{S}_r$ as permutations of the factors M . The quotient space $M^{(r)}$ can be viewed as the variety of divisors of degree r on M , with the natural analytic structure that it inherts. In many ways the mapping $\hat{w} : M^{(r)} \to J$ is the more natural one to consider than $w : M^r \to J$. The more detailed analysis of this topic is too much of a digression to pursue further here though. Note that W_r can also be characterized by

$$(4) \qquad W_r = \{t \in J : \gamma(\rho_t \zeta^r) > 0\} ;$$

indeed whenever the factor of automorphy $\rho_t \zeta^r$ admits a nontrivial holomorphic relatively automorphic function $f \in \Gamma(\rho_t \zeta^r)$ then $\mathfrak{D}(f) = z_1 + \ldots + z_r$ and the factors $\rho_t \zeta^r$ and $\rho_{w(\mathfrak{D})} \zeta^r$ are equivalent, so that t and $w(\mathfrak{D}) = w(z_1) + \ldots + w(z_r)$ represent the same point in J .

As the image of the connected hence irreducible complex manifold M^r, each subvariety W_r is an irreducible subvariety of the Jacobi variety. The Jacobian matrix of the mapping $w : M^r \to J$ at a point $(z_1,\ldots,z_r) \in M^r$ is the $g \times r$ complex matrix $\{w_i'(z_j) : 1 \leq i \leq g , 1 \leq j \leq r\}$, where the derivatives are taken in any coordinate system on M^r. If $r \leq g$ then there are points $(z_1,\ldots,z_r)$ at which this matrix is of rank r, since the g Abelian differentials $w_i'(z)\,dz = \omega_i(z)$ are linearly independent; at these points W_r is actually an r-dimensional manifold, so that the subvariety W_r is necessarily r-dimensional. It is convenient to set $W_0 = 0$, the image of the origin hence the zero element of the group $J = \mathbb{C}^g/\mathcal{L}$, and to set $W_r = \emptyset$ whenever $r < 0$; there is then the sequence of subvarieties

$$0 = W_0 \subset W_1 \subset W_2 \subset\ldots\subset W_{g-1} \subset W_g = W_{g+1} =\ldots= J .$$

The group structure of J can be used in the natural manner to define the sum $X + Y$ of any two subsets $X,Y \subseteq J$ by setting $X + Y = \{x + y : x \in X , y \in Y\}$. Since $W_r = \{w(z_1) +\ldots+ w(z_r) : z_i \in M\} = \{t_1 +\ldots+ t_r : t_i \in W_1\}$ it is evident that $W_r = W_1 + W_1 +\ldots+ W_1$, the sum of r copies of the image curve W_1; thus all these subvarieties W_r are determined quite simply by W_1 alone. It is clear from this in turn that

$$\text{(5)} \qquad W_r + W_s = W_{r+s}$$

for any indices $r,s \geq 0$. A rather deeper use of the group structure comes from introducing another operation $X \ominus Y$ on any two subsets

$X, Y \subseteq J$, defined following H. H. Martens by

$$X \ominus Y = \{t \in J : t + Y \subseteq X\} ;$$

thus $Y + (X \ominus Y) \subseteq X$, and $X \ominus Y$ is the largest subset of J for which this holds. In these terms

(6) $$W_r \ominus W_s = W_{r-s} \quad \text{whenever} \quad r < g .$$

To demonstrate this assertion note first that both sides are empty if $s > r$. If $s \leq r$ then $W_{r-s} \subseteq W_r \ominus W_s$ since $W_{r-s} + W_s \subseteq W_r$ by (5). On the other hand if $t \in W_r \ominus W_s$ then $t + w(z_1 + \dots + z_s) \in W_r$ for any points $z_j \in M$, so using the characterization (4) yields

$$0 < \gamma(\rho_{t+w(z_1+\dots+z_s)}\zeta^r) = \gamma(\rho_t \zeta^{r-s} \cdot \zeta_{z_1} \cdots \zeta_{z_s})$$

for any points $z_j \in M$, where $\zeta_{z_j} = \rho_{w(z_j)}\zeta$ is the factor of automorphy associated to the divisor z_j. This implies by using the Riemann-Roch theorem that

$$\gamma(\kappa\rho_{-t}\zeta^{g-r}\zeta_{z_1}^{-1} \cdots \zeta_{z_s}^{-1}) > g - 1 - r \geq 0$$

for any points $z_j \in M$, the last inequality following from the assumption that $r < g$; multiplying these relatively automorphic functions by $f \in \Gamma(\zeta_{z_1} \cdots \zeta_{z_s})$ where $\mathfrak{d}(f) = z_1 + \dots + z_s$ shows that there are strictly more than $g - 1 - r$ linearly independent functions in $\Gamma(\kappa\rho_{-t}\zeta^{g-r})$

vanishing at $z_1+\dots+z_s$ for any points $z_j \in M$, and it is clear from this that there must therefore be strictly more than $g-1-r+s$ linearly independent functions in $\Gamma(\kappa\rho_{-t}\zeta^{s-r})$. Thus $\gamma(\kappa\rho_{-t}\zeta^{s-r}) > g-1-r+s$, so from the Riemann-Roch theorem again

$$\gamma(\rho_t\zeta^{r-s}) > 0$$

and hence $t \in W_{r-s}$ as desired. This sort of argument, incidentially, is quite common in the subject, with one modification or another.

There is a further collection of interesting subvarieties of J, obtained by modifying the characterization (4) of the subvarieties $W_r \subseteq J$ by setting

$$W_r^\nu = \{t \in J : \gamma(\rho_t\zeta^r) > \nu\} \tag{7}$$

for any integers $\nu, r \geq 0$. For the special case $\nu = 0$ this is just the characterization (4), so that $W_r^0 = W_r$, and in general

$$W_r = W_r^0 \supseteq W_r^1 \supseteq W_r^2 \supseteq \dots .$$

If $r > 2g-2$ the Riemann-Roch theroem shows that $\gamma(\rho_t\zeta^r) = r+1-g$ for all points $t \in J$, so that

$$W_r^\nu = \begin{cases} J & \text{if } r > 2g-2,\ \nu \leq r-g \\ \emptyset & \text{if } r > 2g-2,\ \nu > r-g \end{cases};$$

the formal definition (7) is sometimes extended by setting $W_r^\nu = \emptyset$

whenever $\nu \geqq 0$, $r < 0$. If $t \in W_r = W_r^0$ then $t = w(\mathfrak{d})$ for some divisor $\mathfrak{d}$ of degree r , and $\gamma(\rho_t \zeta^r) = \gamma(\kappa\zeta_{\mathfrak{d}}^{-1}) + r + 1 - g$; if $\mathfrak{d} = z_1 + \dots + z_r$ consists of r distinct points where $r \leqq g$ then in general there will be $g - r$ linearly independent functions in $\Gamma(\kappa)$ vanishing at the divisor $\mathfrak{d}$, so that $\gamma(\kappa\zeta_{\mathfrak{d}}^{-1}) = g - r$ and $\gamma(\rho_t \zeta^r) = 1$. Thus the condition that $t = w(\mathfrak{d}) \in W_r^\nu$ for $\nu > 0$ and $r \leqq g$ means that the divisor is somewhat special, in that $\gamma(\zeta_{\mathfrak{d}})$ is larger than usual; for this reason the subsets W_r^ν are called the <u>subvarieties of special positive divisors</u>. It might be noted though that W_r^ν has not yet been shown to be a holomorphic subvariety of W_r .

The investigation of these subsets $W_r^\nu \subseteq J$ is an extensive subject in its own right, and will not much be discussed here. However a few rather simple observations about the set-theoretic interrelations between these subvarieties, extending what has just been shown for the simpler subvarieties $W_r = W_r^0$, will be included for use in the sequel. First, the Riemann-Roch theorem can be restated geometrically as the identity

$$W_r^\nu = k - W_{2g-2-r}^{\nu-r-1+g} , \tag{8}$$

where of course $k - X = \{k - x : x \in X\}$. Indeed $t \in W_r^\nu$ precisely when

$$\nu < \gamma(\rho_t \zeta^r) = \gamma(\rho_{k-t} \zeta^{2g-2-r}) + r + 1 - g ,$$

hence precisely when $k - t \in W_{2g-2-r}^{\nu-r-1+g}$ as desired. Next, there are the rather evident inclusions

(9) $\quad W_r^\nu + W_1 \subseteq W_{r+1}^\nu,\ W_r^\nu - W_1 \subseteq W_{r-1}^{\nu-1}$ if $\nu - 1 \geq 0$.

Indeed suppose that $t \in W_r^\nu$ and $a \in M$. Multiplication by the prime function $q(z,a)$ defines an injective linear mapping $\Gamma(\rho_t\zeta^r) \to \Gamma(\rho_t\zeta^r\cdot\zeta_a)$, so that $\gamma(\rho_{t+w(a)}\zeta^{r+1}) = \gamma(\rho_t\zeta^r\cdot\zeta_a) \geq \gamma(\rho_t\zeta^r) > \nu$ and consequently $t + w(a) \in W_{r+1}^\nu$. On the other hand $X = \{f \in \Gamma(\rho_t\zeta^r) : f(a) = 0\}$ is a linear subspace of $\Gamma(\rho_t\zeta^r)$ of codimension at most one, and division by the prime function $q(z,a)$ defines an injective linear mapping $X \to \Gamma(\rho_t\zeta^r\zeta_a^{-1})$; therefore $\gamma(\rho_{t-w(a)}\zeta^{r-1}) = \gamma(\rho_t\zeta_r\zeta_a^{-1}) \geq \dim X \geq \gamma(\rho_t\zeta^r) - 1 > \nu - 1$, so if $\nu - 1 \geq 0$ then $t - w(a) \in W_{r-1}^{\nu-1}$ as desired while if $\nu = 0$ this inequality conveys no information about the point t. By iteration it follows immediately that

(10) $\quad W_r^\nu + W_s \subseteq W_{r+s}^\nu,\ W_r^\nu - W_s \subseteq W_{r-s}^{\nu-s}$ if $\nu - s \geq 0$,

and these two inclusions can be combined as

(11) $\quad W_r^\nu + (W_s - W_s) \subseteq W_r^{\nu-s}$ if $\nu - s \geq 0$,

which is sometimes even more useful.

Here (10) and (11) are generally strict inclusions rather than equalitites as in (5), but are the best possible inclusions of this sort in that sense that

$$W_{r-s}^\nu = W_r^\nu \ominus W_s \quad \text{if } r - \nu \leq g - 1,$$

(12)

$$W_{r+s}^{\nu+s} = W_r^\nu \ominus (-W_s).$$

Indeed it is clear from (10) that $W_{s-r}^\nu \subseteq W_r^\nu \ominus W_s$ and $W_{r+s}^{\nu+s} \subseteq W_r^\nu \ominus (-W_s)$. To begin with the easier case of the reverse inclusions, if $t - W_s \subseteq W_r^\nu$ then

$$\gamma(\rho_t\zeta^{r+s}\zeta_{a_1}^{-1}\cdots\zeta_{a_s}^{-1}) = \gamma(\rho_{t-w(a_1+\cdots+a_s)}\zeta^r) > \nu$$

for any points $a_i \in M$; the subspace of $\Gamma(\rho_t\zeta^{r+s})$ consisting of those functions vanishing at $a_1 + \dots + a_s$ thus has dimension $> \nu \geq 0$, and since the points a_i are arbitrary it must be the case that $\gamma(\rho_t\zeta^{r+s}) > \nu + s$ so that $t \in W^{\nu+s}_{r+s}$. On the other hand if $t + W_s \subseteq W^{\nu}_r$ then

$$\gamma(\rho_t\zeta^{r-s}\zeta_{a_1}\cdots\zeta_{a_s}) = \gamma(\rho_{t+w(a_1+\dots+a_s)}\zeta^r) > \nu$$

for any points $a_i \in M$, so by the Riemann-Roch theorem $\gamma(\kappa\rho_{-t}\zeta^{s-r}\zeta_{a_1}\cdots\zeta^{-1}_{a_s}) > \nu + g - r - 1$; the subspace of $\Gamma(\kappa\rho_{-t}\zeta^{s-r})$ consisting of those functions vanishing at $a_1+\dots+a_s$ thus has dimension $> \nu + g - r - 1 \geq 0$ and since the points a_i are arbitrary it must be the case that $\gamma(\kappa\rho_{-t}\zeta^{s-r}) > \nu + g - r - 1 + s$ so by the Riemann-Roch theorem again $\gamma(\rho_t\zeta^{r-s}) > \nu$ and $t \in W^{\nu}_{r-s}$. The assertions (12) can be combined into

(13) $\qquad W^{\nu+s}_r = W^{\nu}_r \ominus (W_s - W_s)$ if $r - \nu \leq g - 1$.

Indeed $W^{\nu+s}_r \subseteq W^{\nu}_r \ominus (W_s - W_s)$ by (10), while if $t \in W^{\nu}_r \ominus (W_s - W_s)$ then $(t + W^{\nu}_s) - W_s \subseteq W_r$ so that $t + W_s \subseteq W^{\nu}_r \ominus (-W_s) = W^{\nu+s}_{r+s}$ and $t \in W^{\nu+s}_r$.

Incidentally note that (13) for the case $\nu = 0$ can be rewritten

$$\begin{aligned} W^s_r &= W_r \ominus (W_s - W_s) \\ &= \{t \in J : t + w_1 - w_2 \in W_r \text{ whenever } w_j \in W_s\} \\ &= \cap_{w_1, w_2 \in W_s} (W_r - w_1 + w_2) \end{aligned}$$

whenever $r \leq g - 1$; thus W^s_r is an intersection of holomorphic subvarieties so is itself a holomorphic subvariety of J whenever $r \leq g - 1$, and the Riemann-Roch theorem in the geometric form (8) shows that W^s_r is a holomorphic subvariety for higher values of r as well.

This leaves open the questions whether (10) and (11) are the best possible inclusions in the other variables as well, that is to say, whether the inclusions

$$W_s \subseteq W_r^\nu \ominus W_{r+s}^\nu, \; -W_s \subseteq W_{r-s}^{\nu-s} \ominus W_r^\nu, \; W_s - W_s \subseteq W_r^{\nu-s} \ominus W_r^\nu$$

are actually equalities in some cases. This is a considerably harder question, about which fairly little seems to be known in much generality. (Such problems are discussed in the papers by B. van Geemen and G. van der Geer, Kummer varieties and the moduli spaces of Abelian varieties, Amer. J. Math. 108 (1986), 615-642, and by G. E. Welters, The surface C - C on Jacobi varieties and second order theta functions, Acta Math. 157 (1986), 1-22)

§10. The canonical curve.

The Abel-Jacobi mapping, defined in terms of the canonical Abelian integrals, provides a canonical representation of any compact Riemann surface as a nonsingular holomorphic subvariety of its Jacobi variety, as discussed in the preceding section. The canonical Abelian differentials can be used to obtain another canonical representation of most compact Riemann surfaces, as follows. If $w'(z)$ denotes the vector having as components the g canonical Abelian differentials viewed as relatively automorphic functions $w'_j(z) \in \Gamma(\kappa)$ then associating to any point $z \in \tilde{M}$ the value $w'(z) \in \mathbb{C}^g$ yields a well defined holomorphic mapping $w' : \tilde{M} \to \mathbb{C}^g$. It was demonstrated in the preceding section that the functions $w'_j(z)$ have no common zeros; the point $w'(z) \in \mathbb{C}^g$ therefore represents a point $[w'(z)] \in \mathbb{P}^{g-1}$, and that yields a well defined holomorphic mapping $[w'] : \tilde{M} \to \mathbb{P}^{g-1}$. For any covering transformation $T \in \mathcal{Z}$ the vectors $w'(Tz)$ and $w'(z)$ differ only by a scalar factor $\kappa(T,z)$ so represent the same point in projective space; therefore the last mappng induces a holomorphic mapping $[w'] : M \to \mathbb{P}^{g-1}$. This mapping is called the <u>canonical mapping</u>, and its image is called the <u>canonical curve</u> associated to the Riemann surface M. This construction and hence the canonical curve depend on the marking, but otherwise are as canonical as the terminclogy suggests. As the image of a compact complex manifold under a holomorphic mapping, the canonical curve is a holomorphic subvariety of $\mathbb{P}^{g-1}$ by Remmert's proper mapping theorem; then by another standard result, Chow's theorem, the canonical curve is an algebraic subvariety of $\mathbb{P}^{g-1}$.

For the case $g = 1$ this whole construction is of course rather trivial mapping the Riemann surface M to the point $\mathbb{P}^0$. For the cases $g > 1$ the result of the construction depends to some extent on the nature of the Riemann surface M. Under a very standard terminology, a Riemann surface M of genus $g > 1$ is said to be hyperelliptic if $\gamma(\rho_t\zeta^2) > 1$ for some factor of automorphy $\rho_t\zeta^2$, or equivalently if there is some point $t \in W_2^1 \subseteq J$; thus if M is a nonhyperelliptic surface of genus $g > 1$ then $\gamma(\rho_t\zeta^2) \leq 1$ for all points $t \in \mathbb{C}^g$, or equivalently $W_2^1 = \emptyset$. The behavior of the canonical mapping differs radically according to whether the Riemann surface is hyperelliptic or not.

Theorem 14. For a nonhyperelliptic Riemann surface M of genus $g > 1$ the canonical mapping $[w']: M \to \mathbb{P}^{g-1}$ is a nonsingular biholomorphic mapping between M and its image, the canonical curve $[w'(M)] \subseteq \mathbb{P}^{g-1}$. The canonical curve is a nonsingular algebraic curve of degree $2g-2$ in $\mathbb{P}^{g-1}$.

Proof. To see that the canonical mapping is one-to-one, suppose to the contrary that $[w'(z_1)] = [w'(z_2)]$ for distinct points z_1 and z_2. There are $g-1$ linearly independent Abelian differentials vanishing at z_1, each of which can be written as ${}^tc \cdot w'(z)$ for some vector $c \in \mathbb{C}^g$; for each such differential ${}^tc \cdot w'(z_1) = 0$ by assumption, and since the vectors $w'(z_1)$ and $w'(z_2)$ are linearly dependent necessarily ${}^tc \cdot w'(z_2) = 0$ also, so each vanishes at the point z_2 as well. The set of differentials vanishing at z_1 and z_2 can be identified as usual with $\Gamma(\kappa\zeta_{z_1}^{-1}\zeta_{z_2}^{-1})$, and from an application of the Riemann-Roch theorem $\gamma(\zeta_{z_1}\zeta_{z_2}) = \gamma(\kappa\zeta_{z_1}^{-1}\zeta_{z_2}^{-1}) + 3-g \geq g-1 + 3-g > 1$, which means that M is hyperelliptic in contradiction to the hypothesis.

To see that the canonical mapping is nonsingular, suppose to the contrary that it is singular at some point $z_1 \in M$. If say $w'_g(z_1) \neq 0$ then the g-1 quotients $f_j(z) = w'_j(z)/w'_g(z)$ for $1 \leq j \leq g-1$ can be taken as the coordinate functions describing the canonical mapping near z_1, in a suitable inhomogeneous coordinate system in $\mathbb{P}^{g-1}$ in a neighborhood of the image point $w'(z_1)$. The condition that this mapping be singular at z_1 is just that

$$0 = f'_j(z_1) = \frac{w'_g(z_1)\, w''_j(z_1) - w'_j(z_1)\, w''_g(z_1)}{w'_g(z_1)^2} \quad \text{for } 1 \leq j \leq g-1,$$

which evidently amounts to the condition that the vectors $w'(z_1)$ and $w''(z_1)$ are linearly dependent. Then arguing as in the preceding paragraph shows that any g-1 linearly independent Abelian differentials vanishing at z_1 necessarily vanish to the second order at z_1, since their derivatives vanish there as well; thus $\gamma(\zeta_{z_1}{}^2) = \gamma(\kappa\zeta_{z_1}{}^{-2}) > 1$, which again contradicts the hypothesis that M is nonhyperelliptic.

The canonical mapping $[w'] : M \to \mathbb{P}^{g-1}$ is thus a one-to-one nonsingular mapping, so its image is a nonsingular one-dimensional subvariety and the mapping is a biholomorphic one between M and its image. If L is a hyperplane in $\mathbb{P}^{g-1}$ defined as the zero locus of some linear function $\ell(w)$ then the inverse image under the canonical mapping of the intersection $L \cap [w'(M)]$ is the subvariety $\{z \in M : \ell(w'(z)) = 0\}$; the function $\ell(w'(z))$ is a linear combination of Abelian differentials, hence is itself an Abelian differential, so vanishes at 2g-2 points, counting multiplicities, and hence $L \cap [w'(M)]$ consists of 2g-2 points, counting multiplicities, and the canonical curve $[w'(M)]$ is an algebraic curve of degree 2g-2 as asserted. That concludes the proof.

Next suppose that M is a hyperelliptic Riemann surface of genus $g > 1$. There is thus some factor of automorphy $\rho_e\zeta^2$ such that $\gamma(\rho_e\zeta^2) > 1$. If $\gamma(\rho_e\zeta^2) > 2$ then there are at least three linearly independent relatively automorphic functions for this factor of automorphy, so there are clearly at least two linearly independent such functions that vanish at the base point z_0; dividing them by the function in $\Gamma(\zeta)$ vanishing at z_0 yields at least two linearly independent functions in $\Gamma(\rho_e\zeta)$, but since $c(\rho_e\zeta) = 1$ and by hypothesis M has genus $g > 1$, it is as noted earlier impossible that $\gamma(\rho_e\zeta) > 1$. It is thus the case that $\gamma(\rho_e\zeta^2) = 2$; so choose two functions $f_1, f_2 \in \Gamma(\rho_e\zeta^2)$ that are a basis for this space of relatively automorphic functions. Note that f_1 and f_2 have no common zeros, since otherwise the quotient f_1/f_2 would be a meromorphic function with at most a simple pole as its singularity; but that is impossible, since either f_1/f_2 is holomorphic hence constant, contradicting the assumption that f_1 and f_2 are linearly independent, or f_1/f_2 describes a biholomorphic mapping from M to $\mathbb{P}^1$, contradicting the assumption that M has genus $g > 1$. Thus f_1/f_2 is a meromorphic function with either two distinct simple poles or one double pole, and describes a holomorphic mapping $\pi: M \to \mathbb{P}^1$ that exhibits M as a two-sheeted branched covering of $\mathbb{P}^1$. It is quite easy to see conversely, that any Riemann surface that can be exhibited as a two-sheeted branched covering of $\mathbb{P}^1$ is either hyperelliptic or of genus $g \leq 1$.

To simplify the notation set $\tau = \rho_e\zeta^2$; thus τ is a factor of automorphy with $c(\tau) = \gamma(\tau) = 2$, and $f_1, f_2 \in \Gamma(\tau)$ are a basis for this space of functions. For any integer $\nu \geq 1$ the $\nu + 1$ functions $f_1^\nu, f_1^{\nu-1}f_2, \ldots, f_1f_2^{\nu-1}, f_2^\nu$ belong to the space $\Gamma(\tau^\nu)$ and are easily seen to be linearly independent; indeed any linear relation among these functions can be viewed as a homogenous polynomial P of degree ν in two variables such that $P(f_1,f_2) = 0$, but such a polynomial can be written as a product of linear factors and there can be no nontrivial such relation since the functions f_1, f_2 are linearly independent. This shows that $\gamma(\tau^\nu) \geq \nu + 1$ whenever $\nu \geq 1$. For the special case $\nu = g-1$ the factor of automorphy $\kappa\tau^{1-g}$ has $\gamma(\kappa\tau^{1-g}) = \gamma(\tau^{g-1}) + 1-g \geq 1$; that means that $\kappa\tau^{1-g}$ is the identity factor of automorphy so that $\kappa = \tau^{g-1} = \rho_{(g-1)e}\zeta^{2g-2}$ The point $e \in J$ thus has the property that $(g-1)e = \kappa \in J$, the canonical point; furthermore the g functions $f_1^{g-1}, f_1^{g-2}f_2, \ldots, f_1 f_2^{g-2}, f_2^{g-1}$ can be taken as a basis for the space $\Gamma(\tau^{g-1}) = \Gamma(\kappa)$ of Abelian differentials. The canonical mapping in this case is the composition of the two-sheeted branched covering $\pi : M \to \mathbb{P}^1$ taking a point $z \in M$ to the point $[f_1(z), f_2(z)] \in \mathbb{P}^1$ and the mapping $\phi : \mathbb{P}^1 \to \mathbb{P}^g$ taking a point $[x_1, x_2] \in \mathbb{P}^1$ to the point $[x_1^{g-1}, x_1^{g-2}x_2, \ldots, x_1 x_2^{g-2}, x_2^{g-1}] \in \mathbb{P}^{g-1}$. This latter mapping is a well known biholomorphic imbedding of the Riemann sphere $\mathbb{P}^1$ in $\mathbb{P}^{g-1}$, where the image is the rational normal curve of degree g-1; in inhomogenous coordinates it has the more familiar form $t \to (t, t^2, \ldots, t^{g-1})$. Thus for a hyperelliptic Riemann surface M the canonical mapping is just a concrete realization of the two-sheeted branched covering $\pi : M \to \mathbb{P}^1$, where $\mathbb{P}^1$ is represented as the rational normal curve in $\mathbb{P}^{g-1}$. The canonical curve is then an algebraic curve of degree g-1 in $\mathbb{P}^{g-1}$.

Since $\pi : M \to \mathbb{P}^1$ is just the mapping defined by the meromorphic function f_2/f_1, it is clear that the inverse image $\pi^{-1}(x) = \{z \in M : f_2(z)/f_1(z) = c\}$ consists of the two points $z_1, z_2 \in M$ such that $(f_2 - cf_1) = z_1 + z_2$; thus $\rho_e\zeta^2 = \zeta_{z_1}\zeta_{z_2}$ so that $w(z_1 + z_2) = e$. The mapping π is intrinsically and hence uniquely determined by the canonical mapping, as just described, so it is quite evident that the point e is uniquely determined, or equivalently that on a hyperelliptic Riemann surface there is a unique factor of automorphy $\tau = \rho_e\zeta^2$ for which $c(\tau) = \gamma(\tau) = 2$. The point e is sometimes called the hyperelliptic point of the Jacobi variety, and can be described alternatively as $e = W_2^1$.

For any Riemann surface M of genus $g > 1$ the associated canonical curve is a well defined nonsingular algebraic curve in $\mathbb{P}^{g-1}$, hence is the set of zeros of an ideal of homogeneous polynomials in g variables. This ideal will be called the Petri ideal, since it was analyzed extensively by K. Petri. The polynomials P in this ideal are those for which $P(w'_1(z),\ldots,w'_g(z)) \equiv 0$ in $z \in \tilde{M}$, hence describe the algebraic relations among the canonical Abelian differentials on M. In the case of a hyperelliptic Riemann surface the Petri ideal is equivalent to the ideal of the rational normal curve, the ideal of all homogeneous polynomials P such that $P(1,t,\ldots,t^{g-1}) \equiv 0$ in $t \in \mathbb{C}$, a reasonably straightforward algebraic set. The space of homogeneous polynomials of degree n in g variables has dimension $\binom{g+n-1}{n}$, while the space of polynomials in one variable t that can be written as $P(1,t,\ldots,t^{g-1})$ where P is a homogenous polynomial of degree n is spanned by the terms $1,t,\ldots,t^{(g-1)n}$ hence has dimension $gn-n+1$; the terms of degree n in the Petri ideal are the space $\mathcal{P}_n$ of homogeneous polynomials P of degree n such that $P(1,t_1,\ldots,t^{g-1}) = 0$, hence

(1) $\dim \mathcal{P}_n = \binom{g+n-1}{n} - gn+n-1$ if M is hyperelliptic.

In particular $\dim \mathcal{P}_2 = \binom{g+1}{2} - 2g+1 = \binom{g-1}{2}$ and $\dim \mathcal{P}_3 = \binom{g+2}{3} - 3g + 2 = \frac{1}{6}(g+6)(g-2)(g-1)$. In the case of a nonhyperelliptic Riemann surface M on the other hand, if P is a homogeneous polynomial of degree n then $P(w_1'(z),\ldots,w_g'(z)) \in \Gamma(\kappa^n)$; by the Riemann-Roch theorem $\dim \Gamma(\kappa^n) = (2n-1)(g-1)$

whenever $n > 1$, and by a well known theorem of M. Noether the elements $P(w_1'(z),\dots,w_g'(z))$ span the full space $\Gamma(\kappa^n)$, so that

(2) $\dim \underset{\sim}{P}_n = \binom{g+n-1}{n} - (2n-1)(g-1)$ if M is nonhyperelliptic.

In particular $\dim \underset{\sim}{P}_2 = \binom{g+1}{2} - 3(g-1) = \binom{g-2}{2}$ and $\dim \underset{\sim}{P}_3 = \binom{g+2}{3} - 5(g-1) = \frac{1}{6}(g-3)(g^2+6g-10)$.

For a curve of genus 2 the canonical mapping $M \to \mathbb{P}^1$ can only be a branched covering mapping, with the canonical curve being $\mathbb{P}^1$ itself; thus M is necessarily hyperelliptic. For a curve of genus 3 the canonical mapping takes M to a plane algebraic curve $X \subseteq \mathbb{P}^2$; if M is hyperelliptic X is a rational curve of degree 2, while otherwise X is an algebraic curve of degree 4. In the former case it follows from (1) that $\dim P_2 = 1$; there is up to a constant factor a unique quadric equation $p_2(w_1'(z), w_2'(z), w_3'(z)) = 0$ among the Abelian differentials, and this quadric equation is the defining equation for the canonical curve and a generator of the full Petri ideal $\underset{\sim}{P}$. In the latter case it follows from (2) that $\dim \underset{\sim}{P}_2 = \dim \underset{\sim}{P}_3 = 0$ and $\dim \underset{\sim}{P}_4 = 1$; there is up to a constant factor a unique quartic equation $p_4(w_1'(z), w_2'(z), w_3'(z)) = 0$ among the Abelian differentials but no equation of lower degree, and this quartic equation is the defining equation for the canonical curve and a generator of the full Petri ideal. For a curve of genus 4 the canonical mapping takes M to an algebraic curve $X \subseteq \mathbb{P}^3$; if M is hyperelliptic X is a rational curve of degree 3, while otherwise X is an algebraic curve of degree 6. In the former case it follows from (1) that $\dim \underset{\sim}{P}_2 = 3$, so there are three linearly independent quadric equations among the Abelian differentials; the canonical curve can be described geometrically by two quadric equations, but it takes three to generate the full Petri ideal. In the latter case it follows from (2) that $\dim \underset{\sim}{P}_2 = 1$ and $\dim \underset{\sim}{P}_3 = 5$;

there is up to a constant factor a unique quadric equation $p_2(w_1'(z),\dots,w_4'(z)) = 0$ among the Abelian differentials, but there are five linearly independent cubic equations. In terms of homogeneous coordinates x_1,x_2,x_3,x_4 in $\mathbb{P}^4$ there is thus essentially one quadric equation $p_2(x_1,\dots,x_4)$ vanishing on the canonical curve $X \subseteq \mathbb{P}^4$; then $x_1p_2(x_1,\dots,x_4),\dots, x_4p_2(x_1,\dots,x_4)$ are four lineraly independent cubic equations vanishing on X, and there must exist another cubic equation $p_3\ (x_1,\dots,x_4)$ vanishing on X as well, determined uniquely only up to a constant factor and an arbitrary linear combination of the first four cubic equations. Since X is of degree 6 the two polynomials p_2,p_3 vanishing there must actually be the defining equations of X, the product of their degrees also being 6; these polynomials serve to generate the full Petri ideal.

There is one finer point about nonhyperelliptic Riemann surfaces of genus g=4 that will arise in the discussion later so should be mentioned here. The quadric polynomial $p_2 \in \underset{\sim}{P_2}$ is described by a 4 x 4 matrix, the rank of which is another evident invariant attached to the Riemann surface; this invariant will for short just be called the rank of the Riemann surface. It is always possible to choose a basis for the Abelian differentials on that Riemann surface, although generally not the canonical basis for these differentials, so that the quadric relation p_2 between the differentials is in a standard form. For rank 1 that relation can be taken in the form $w'_1(z)^2 = 0$; but such a relation is impossible, so this case cannot occur. For rank 2 that relation can be taken in either the form $w'_1(z)^2 + w'_2(z)^2 = 0$ or the form $w_1'(z)\ w_2'(z) = 0$, so again is impossible since the product of two nonzero functions cannot vanish identically. For rank 3 that relation can be taken in either the form $w'_1(z)^2 + w'_2(z)^2 + w'_3(z)^2 = 0$ or the form $w'_1(z) \cdot w'_2(z) = w'_3(z)^2$, and for rank 4 that relation can be taken in either the form $w'_1(z)^2 + w'_2(z)^2 + w'_3(z)^2 + w'_4(z)^2 = 0$ or the form $w'_1(z)\, w'_2(z) = w'_3(z)^2$. Both of these cases do arise, so that there are really the two separate classes of nonhyperelliptic Riemann surfaces of genus 4, those of rank 3 and those of rank 4. More can be said about this classification, but will not be needed here; for completeness it might just be mentioned that the general case is that of rank 4, in which the space of special positive divisors W_3^1 consists precisely of two points, while the surfaces of rank 3 are those special surfaces for which these two points happen to coincide so that W_3^1 consists of just a single point. The analysis leading to the connection between the rank and W_3^1 is not a difficult one, resting merely on the form of the quadric equations between the Abelian differentials.

For Riemann surfaces of genus $g > 4$ the analysis of the Petri ideal is rather more complicated, involving more than just the dimensions of the spaces P_n; it is enough for present purposes merely to survey rather quickly some of the results of this analysis that will be needed here, leaving details and proofs to other sources. For a hyperelliptic surface $\dim \mathcal{P}_2 = \binom{g-1}{2}$ by (1), and from standard properties of the rational normal curve, the elements of any basis for P_2 generate the ideal of the canonical curve; all relations among the Abelian differentials are consequences of the quadric relations. On the other hand for a nonhyperelliptic curve $\dim \mathcal{P}_2 = \binom{g-1}{2}$ and in general it is again the case that the elements of any basis for P_2 generate the ideal of the canonical curve, and all relations among the Abelian differentials are consequences of the quadric relations; this is a result of Petri, although the weaker assertion that the canonical curve is described geometrically as an intersection of quadrics is due to Enriques and Babbage. However there are exceptional nonhyperelliptic curves for which the ideal of the canonical curve is generated by bases for P_2 and P_3; these are curves for which the products of elements of P_2 by arbitrary linear forms do not span all of P_3, not on dimensional grounds but rather because of linear dependencies among products of element of P_2 by linear functions. The exceptional curves fall into two classes, the first consisting of these nonhyperelliptic curves of genus $g > 1$ for which $W_3^1 \neq \emptyset$, and the second consisting of nonsingular plane curves of degree five. If $W_3^1 \neq \emptyset$ but $W_2^1 = \emptyset$ then the surface can be represented as a three-sheeted branched covering of $\mathbb{P}^1$ but not as a covering with fewer sheets; such surfaces are usually called trigonal surfaces. These topics are discussed very nicely in the books by H. M. Farkas and I. Kra, (Riemann Surfaces, Springer-Verlag 1980), and by E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris, (Geometry of Algebraic Curves I, Springer-Verlag, 1985).