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SOME TOPICS IN THE FUNCTION THEORY
OF COMPACT RIEMANN SURFACES

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Introduction

This is a leisurely survey of compact Riemann surfaces from the point of view of complex function theory rather than of algebraic geometry; so it focuses on functions and differential forms, uses the universal covering space freely, and assumes the standard properties of topological spaces. The deepest and most subtle parts of the subject really involve functions of several complex variables as well as functions of a single complex variable; so the tools of several complex variables, such as sheaf theory, are used fairly freely. The relevant background material that is assumed for reading this book, such as some standard properties of holomorphic functions of several variables, the theory of sheaves and sheaf cohomology, the topology of surfaces, the cohomology of groups, the structure of complex tori in several dimensions, are included as appendices, so that the book should be accessible to anyone who has a standard undergraduate background and is willing to go to the appendices as necessary. The book is a work in progress; included here are those chapters that are in fairly final form, and more will be added from time to time as time and enthusiasm permit. The eventual form of the book is hinted at by the table of contents, which lists other chapters currently being revised or at least being thought about. The material consists to a large extent of topics covered in graduate courses over a number of years at Princeton; some parts derive from earlier graduate courses and have been published previously in one form or another, but they have been fairly extensively rewritten and expanded for inclusion here. I must thank a great many graduate students who have suffered from these courses over the years; their comments, suggestions, corrections, and inspiration have been of fundamental importance, and I regret that I cannot acknowledge all of them by name here.
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Part I

Differentials and Integrals
Chapter 1

Divisors and Line Bundles

A Riemann surface \( M \) is a one-dimensional connected complex manifold\(^1\). The local properties of a meromorphic function at a point \( p \in M \) are described almost entirely by the order of the function at that point, denoted by \( \operatorname{ord}_p(f) \), which is a positive integer \( n \) if the function has a zero of order \( n \) at the point \( p \), is a negative integer \(-n\) if the function has a pole of order \( n \) at the point \( p \), and is zero otherwise. The global properties of a meromorphic function on a compact Riemann surface are determined almost completely when the order of the function is given at each point of the surface; and that is done most conveniently in terms of divisors. In general a divisor on a Riemann surface \( M \) is a mapping \( d : M \to \mathbb{Z} \) such that \( d(p) \neq 0 \) only at a discrete set of points \( p \in M \); that set of points is called the support of the divisor, and is denoted by \( |d| \). The set of all divisors on \( M \) clearly form an abelian group under pointwise addition of functions. The zero element of the group, called the zero divisor or the trivial divisor, is the divisor that is identically zero on \( M \). The group of divisors is partially ordered by setting \( d_1 \geq d_2 \) whenever \( d_1(p) \geq d_2(p) \) for all points \( p \in M \); in particular a divisor \( d \) is a positive divisor, traditionally also called an effective divisor, if \( d(p) \geq 0 \) for all points \( p \in M \). Any divisor \( d \) on \( M \) can be written uniquely as the difference \( d = d_+ - d_- \) of two effective divisors with disjoint supports, the divisors defined by

\[
(1.1) \quad d_+(p) = \max(d(p), 0), \quad d_-(p) = \max(-d(p), 0).
\]

On a compact Riemann surface \( M \) a divisor \( d \) is nonzero only at finitely many points of the surface; the degree of the divisor is the integer \( \deg d = \sum_{p \in M} d(p) \).

A customary and useful notation is to write a divisor in the form \( d = \sum_i \nu_i : p_i \) where \( \{p_i\} \) is a discrete set of points of \( M \) and \( d(p_i) = \nu_i \) while \( d(p) = 0 \) at all points other than those in the set \( \{p_i\} \); it is clear that for a divisor written this way \( \deg d = \sum_i \nu_i \) and \( |d| \subset \bigcup_i p_i \), where this inclusion is an equality of sets if \( \nu_i \neq 0 \) for all indices \( i \). The groups of divisors in the open subsets of

\(^1\)For the definitions and basic properties of complex manifolds and of holomorphic and meromorphic functions on complex manifolds see Appendix A.2.
a Riemann surface \( M \), with the obvious restriction homomorphisms, form a complete presheaf of abelian groups over \( M \); the associated sheaf is the sheaf of divisors on \( M \), denoted by \( \mathcal{D} \), and it is clear that \( \mathcal{D} \) is a fine sheaf over \( M \).

The group of divisors on any open subset \( U \subset M \) can be identified with the group \( \Gamma(U, \mathcal{D}) \) of sections of the sheaf \( \mathcal{D} \) over that subset.

To any meromorphic function \( f \) that is not identically zero on a Riemann surface \( M \) there can be associated its divisor \( d(f) \), defined by

\[
d(f)(p) = \text{ord}_p(f)
\]

for all points \( p \in M \). When this divisor is written as the difference

\[
d(f) = d^+(f) - d^-(f)
\]

of two effective divisors with disjoint supports, \( d^+(f) \) is the divisor of zeros of the function \( f \) and \( d^-(f) \) is the divisor of poles of the function; \( d^+(f) \) lists all zeros of the function \( f \) with their orders, while \( d^-(f) \) lists the poles of \( f \) with their orders. The mapping that associates to any nontrivial meromorphic function on \( M \) its divisor is a homomorphism

\[
d : \Gamma(M, \mathcal{M}^*) \longrightarrow \Gamma(M, \mathcal{D})
\]

from the multiplicative group of nontrivial meromorphic functions on \( M \) to the additive group of divisors on \( M \). The kernel of this homomorphism consists of those meromorphic functions having order zero at all points of \( M \), so is the multiplicative group \( \Gamma(M, \mathcal{O}^*) \) of nowhere vanishing holomorphic functions on \( M \); consequently (1.2) can be extended to the exact sequence

\[
0 \longrightarrow \Gamma(M, \mathcal{O}^*) \xrightarrow{\iota} \Gamma(M, \mathcal{M}^*) \xrightarrow{d} \Gamma(M, \mathcal{D})
\]

in which \( \iota \) is the inclusion mapping. The image of the homomorphism \( d \) consists of those divisors on \( M \) that are the divisors of global meromorphic functions on \( M \), customarily called principal divisors on \( M \). Two divisors \( \mathfrak{d}_1 \) and \( \mathfrak{d}_2 \) that differ by a principal divisor are called linearly equivalent divisors, and the linear equivalence of these two divisors is denoted by \( \mathfrak{d}_1 \sim \mathfrak{d}_2 \). It is clear that this is an equivalence relation in the usual sense, and that the principal divisors form one equivalence class. It is actually a nontrivial equivalence relation on compact Riemann surfaces; an initial necessary condition that divisors on a compact Riemann surface must satisfy in order to be principal divisors is the following.

**Theorem 1.1** If \( f \) is a nontrivial meromorphic function on a compact Riemann surface \( M \) then \( \deg \mathfrak{d}(f) = 0 \).

**Proof:** Select a finite triangulation\(^3\) of the compact Riemann surface \( M \) by 2-dimensional simplices \( \sigma_j \) such that the support of the divisor \( \mathfrak{d}(f) \) of the meromorphic function \( f \) is disjoint from the boundaries \( \partial \sigma_j \) of all of these simplices. By the residue theorem the degree of the divisor \( \mathfrak{d}(f) \) is given by

\[
\deg \mathfrak{d}(f) = \sum_j \frac{1}{2\pi i} \int_{\partial \sigma_j} d \log f(z) = \frac{1}{2\pi i} \int \sum_j \partial \sigma_j d \log f(z);
\]

and since \( \sum_j \partial \sigma_j = 0 \) it follows that \( \deg \mathfrak{d}(f) = 0 \), which suffices for the proof.

---

\(^2\)For the definition and basic properties of sheaves and presheaves see Appendix C.1.

\(^3\)For the topological properties of surfaces see Appendix D.
For any nontrivial meromorphic function $f$ on a compact Riemann surface $M$ it follows from the preceding theorem that $0 = \deg f = \deg d_+(f) - \deg d_-(f)$ and hence that $\deg d_+(f) = \deg d_-(f)$; this common value is called the degree of the meromorphic function $f$ and is denoted by $\deg f$. Clearly $\deg f \geq 0$ for any nontrivial meromorphic function on a compact Riemann surface, and $\deg f = 0$ if and only if $f$ is everywhere holomorphic and nowhere zero on $M$, hence is a nonzero complex constant as an immediate consequence of the maximum modulus theorem. Not all divisors of degree 0 on a compact Riemann surface $M$ are the divisors of meromorphic functions on $M$ though; but it is obvious that any germ of a divisor is the divisor of a germ of a nontrivial meromorphic function. Thus there is the exact sequence of sheaves

$$(1.4) \quad 0 \longrightarrow \mathcal{O}^* \xrightarrow{\iota} \mathcal{M}^* \xrightarrow{\delta} \mathcal{D} \longrightarrow 0$$

on any Riemann surface $M$, where $\iota$ is the natural inclusion homomorphism and $\delta$ is the homomorphism that associates to any germ of a nontrivial meromorphic function the germ of its divisor, the local form of the homomorphism $(1.2)$. The problem of determining which divisors are the divisors of meromorphic functions on $M$ then can be approached by using the exact cohomology sequence$^4$ associated to the exact sequence of sheaves $(1.4)$, which begins

$$(1.5) \quad 0 \longrightarrow \Gamma(M, \mathcal{O}^*) \xrightarrow{\iota} \Gamma(M, \mathcal{M}^*) \xrightarrow{\delta} \Gamma(M, \mathcal{D}) \xrightarrow{\delta} H^1(M, \mathcal{O}^*) \xrightarrow{\iota} H^1(M, \mathcal{M}^*) \xrightarrow{\delta} H^1(M, \mathcal{D})$$

where $\delta$ is the coboundary homomorphism. The first line of this exact sequence is just the exact sequence $(1.3)$; sheaf cohomology theory yields the coboundary homomorphism $\delta$ connecting the two lines and the homomorphisms between the higher dimensional cohomology groups. The exactness of the sequence $(1.5)$ at the group $\Gamma(M, \mathcal{D})$ is just the assertion that a divisor $\mathfrak{d} \in \Gamma(M, \mathcal{D})$ is a principal divisor if and only if $\delta(\mathfrak{d}) = 0 \in H^1(M, \mathcal{O}^*)$; and it follows that $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\delta(\mathfrak{d}_1) = \delta(\mathfrak{d}_2)$, so linear equivalence of divisors is just the equivalence relation defined by the group homomorphism $\delta$.

The preceding can be put into a more concrete form by tracing through the coboundary homomorphism in the exact cohomology sequence $(1.5)$. For any divisor $\mathfrak{d} \in \Gamma(M, \mathcal{D})$ on a Riemann surface $M$ there is a covering $\mathcal{U} = \{U_\alpha\}$ of $M$ by open subsets $U_\alpha \subset M$ such that the restriction of the divisor $\mathfrak{d}$ to each set $U_\alpha$ is the divisor of a nontrivial meromorphic function $f_\alpha$ in $U_\alpha$; the set of these functions can be viewed as a cochain $f \in C^0(\mathcal{U}, \mathcal{M})$, and the coboundary of this cochain is the 1-cocycle $\delta f = \lambda \in Z^1(\mathcal{U}, \mathcal{O}^*)$ that represents the image $\delta(\mathfrak{d}) \in H^1(M, \mathcal{O}^*)$. Explicitly as the multiplicative analogue of $(C.7)$

$$(1.6) \quad \lambda_{\alpha\beta} = \frac{f_\beta}{f_\alpha};$$

$^4$For the cohomology of sheaves see Appendix C.2.
the functions $\lambda_{\alpha\beta}$ are holomorphic and nowhere vanishing in the intersections $U_\alpha \cap U_\beta$, since $f_\alpha$ and $f_\beta$ have the same divisors there, and they clearly satisfy the skew-symmetry condition $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}^{-1}$ and the cocycle condition

$$\lambda_{\alpha\beta} \lambda_{\beta\gamma} \lambda_{\gamma\alpha} = 1 \quad \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$  

(1.7)  

The cocycle $\lambda$ depends on the choices of the local meromorphic functions $f_\alpha$; but any other meromorphic function $\tilde{f}_\alpha$ in $U_\alpha$ having the same divisor as $f_\alpha$ must be of the form $\tilde{f}_\alpha = f_\alpha h_\alpha$ for a nowhere vanishing holomorphic function $h_\alpha$ in the set $U_\alpha$, and the cocycle associated to the functions $\tilde{f}_\alpha$ is $\lambda_{\alpha\beta} = h_\alpha^{-1} \lambda_{\alpha\beta} h_\beta$ which is a cohomologous cocycle and represents the same cohomology class in $H^1(M, \mathcal{O}^*)$. The image of the coboundary homomorphism $\delta$ consists of those cohomology classes in $H^1(M, \mathcal{O}^*)$ that are trivial in $H^1(M, \mathcal{M}^*)$, hence that are represented for a sufficiently fine open covering $\mathcal{U} = \{U_\alpha\}$ of $M$ by cocycles $\lambda_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ that have the form (1.6) for some meromorphic functions $f_\alpha$ in the sets $U_\alpha$.

Elements of the cohomology group $H^1(M, \mathcal{O}^*)$ of a Riemann surface $M$ can be viewed as holomorphic line bundles\footnote{For the definition and basic properties of holomorphic line bundles see Appendix B, and for the description of holomorphic line bundles in terms of sheaf cohomology see Appendix C.2.} over $M$. To any divisor $d \in \Gamma(M, \mathcal{D})$ there thus can be associated the holomorphic line bundle $\delta(d) \in H^1(M, \mathcal{O}^*)$; but it is more convenient and more customary to associate to a divisor $d$ the holomorphic line bundle $\zeta_d = \delta(-d)$, called the line bundle of the divisor $d$. If $d$ is locally the divisor of meromorphic functions $f_\alpha$ then $-d$ is locally the divisor of the meromorphic functions $f_\alpha^{-1}$, so as in (1.6) the line bundle $\zeta_d$ is described by the coordinate transition matrices

$$\zeta_{d\alpha\beta} = \frac{f_{\alpha}^{-1}}{f_{\beta}} = \frac{f_{\alpha}}{f_{\beta}},$$  

(1.8)  

thus $f_\alpha = \zeta_{d\alpha\beta} f_\beta$ in each intersection $U_\alpha \cap U_\beta$, so the meromorphic functions $f_\alpha$ describe a meromorphic cross-section of the line bundle $\zeta_d$. In particular for the trivial divisor $d = 0$ the local functions $f_\alpha$ are holomorphic and nowhere vanishing, so the associated line bundle is the trivial line bundle $\zeta_0 = 1$. The vector space of meromorphic cross-sections of a holomorphic vector bundle $\lambda$ over the Riemann surface $M$ is denoted by $\Gamma(M, \mathcal{M}(\lambda))$, and the subspace of holomorphic cross-sections is denoted by $\Gamma(M, \mathcal{O}(\lambda))$. In general if $f \in \Gamma(M, \mathcal{M}(\lambda))$ is a meromorphic cross-section of a holomorphic line bundle $\lambda$ over a Riemann surface $M$ and is described by meromorphic functions $f_\alpha$ in coordinate neighborhoods $U_\alpha$ the order of the cross-section $f$ at a point $p \in M$ can be defined by $\text{ord}_p(f) = \text{ord}_p(f_\alpha)$ whenever $p \in U_\alpha$, for $\text{ord}_p(f_\alpha) = \text{ord}_p(\lambda_{\alpha\beta} f_\beta) = \text{ord}_p(f_\beta)$ if $p \in U_\alpha \cap U_\beta$ since $\lambda_{\alpha\beta}$ is holomorphic and nowhere vanishing there; the divisor $d(f)$ of the cross-section $f$ then is defined by $d(f)(p) = \text{ord}_p(f)$ for all points $p \in M$, just as for meromorphic functions on $M$. In these terms the relation between divisors and line bundles can be summarized as follows.
Theorem 1.2 The line bundles $\zeta_d$ of divisors $d$ on a Riemann surface $M$ are precisely the holomorphic line bundles over $M$ that have nontrivial meromorphic cross-sections. A holomorphic line bundle $\lambda$ over $M$ is the line bundle $\lambda = \zeta_{\mathcal{O}(f)}$ of the divisor of any meromorphic cross-section $f \in \Gamma(M, \mathcal{M}(\lambda))$. Two divisors $d_1$ and $d_2$ determine the same line bundle $\zeta_{d_1} = \zeta_{d_2}$ if and only if the divisors are linearly equivalent.

Proof: That the line bundle $\zeta_d$ of a divisor $d$ has nontrivial meromorphic cross-sections is an immediate consequence of definition (1.8). On the other hand if $\lambda$ is a holomorphic line bundle over $M$ and $f \in \Gamma(M, \mathcal{M}(\lambda))$ is a nontrivial meromorphic cross-section described by meromorphic functions $f_\alpha$ in coordinate neighborhoods $U_\alpha$ then these functions satisfy (1.8), and that is just the condition that $\lambda = \zeta_{\mathcal{O}(f)}$ is the line bundle of the divisor $\mathcal{O}(f)$. Finally since $-d_1 \sim -d_2$ if and only if $d_1 \sim d_2$ it follows from the exact sequence (1.5) that the divisors $d_1$ and $d_1$ are linearly equivalent if and only if $\zeta_{d_1} = \zeta_{d_2}$. That suffices to conclude the proof.

The line bundle of the divisor $\mathcal{O} = 1 \cdot p$ for a single point $p \in M$ is denoted by $\zeta_p$ and is called a point bundle over the Riemann surface $M$; the bundle $\zeta_p$ thus is characterized as that holomorphic line bundle over $M$ having a nontrivial holomorphic cross-section with a simple zero at the point $p \in M$ and no other zeros on $M$. For any divisor $d = \sum_i \nu_i \cdot p_i$ in $M$ it is obvious that $\zeta_d = \prod_i \zeta_{\nu_i p_i}$, so all line bundles of divisors on $M$ can be built up from point bundles over $M$. If $f_1, f_2 \in \Gamma(M, \mathcal{M}(\lambda))$ are two nontrivial meromorphic cross-sections of a holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ their quotient $f_1/f_2$ is a meromorphic function on $M$, so from Theorem 1.1 it follows that $0 = \deg \mathcal{O}(f_1/f_2) = \deg \mathcal{O}(f_1) - \deg \mathcal{O}(f_2)$; consequently the degrees of the divisors of all nontrivial meromorphic cross-sections of a holomorphic line bundle $\lambda$ over a compact Riemann surface are the same. This observation can be used to define the characteristic class or Chern class of a holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ by

\begin{equation}
(1.9) \quad c(\lambda) = \deg \mathcal{O}(f) \text{ for any } f \in \Gamma(M, \mathcal{M}(\lambda)), \ f \neq 0;
\end{equation}

in particular

\begin{equation}
(1.10) \quad c(\zeta_d) = \deg \mathcal{O} \text{ for any divisor } \mathcal{O}.
\end{equation}

The characteristic class is defined in this way though only for those holomorphic line bundles that have nontrivial meromorphic cross-sections. It will be demonstrated later in this chapter that all holomorphic line bundles over a compact Riemann surface do have nontrivial meromorphic cross-sections; thus the characteristic class actually is well defined for any holomorphic line bundle over $M$. However the characteristic class of a line bundle really is a purely topological invariant of the line bundle and can be described in various other ways, which provide definitions of the characteristic class for any holomorphic line bundle without recourse to the basic existence theorem. One alternative definition is
through a curvature integral expressed in terms of the differential operators $\partial$ and $\overline{\partial}$.

**Theorem 1.3** Let $\lambda$ be a holomorphic line bundle over a compact Riemann surface $M$ described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ in terms of a covering of $M$ by open sets $U_\alpha$. If $r_\alpha > 0$ are $C^\infty$ functions in the sets $U_\alpha$ such that $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in $U_\alpha \cap U_\beta$ then $\overline{\partial}\partial \log r_\alpha = \overline{\partial}\partial \log r_\beta$ in $U_\alpha \cap U_\beta$, so the local differential forms $\overline{\partial}\partial \log r_\alpha$ describe a global differential form of degree 2 on the surface $M$. The integral

$$\frac{1}{2\pi i} \int_M \overline{\partial}\partial \log r_\alpha$$

is independent of the choice of the functions $r_\alpha$; and if $\lambda$ has a nontrivial meromorphic cross-section the value of this integral is the characteristic class $c(\lambda)$ of the line bundle $\lambda$.

**Proof:** If $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in $U_\alpha \cap U_\beta$ then $\log r_\alpha = \log \lambda_{\alpha\beta} + \log \overline{\lambda_{\alpha\beta}} + \log r_\beta$ in that intersection; and since $\log \lambda_{\alpha\beta}$ is holomorphic $\partial \log \lambda_{\alpha\beta} = \partial \log \overline{\lambda_{\alpha\beta}} = 0$ and $\partial \log \lambda_{\alpha\beta} = d \log \lambda_{\alpha\beta}$, so

$$\partial \log r_\alpha = d \log \lambda_{\alpha\beta} + \partial \log r_\beta$$

and $\overline{\partial}\partial \log r_\alpha = \overline{\partial}\partial \log r_\beta$ as asserted. For any other $C^\infty$ functions $s_\alpha > 0$ in the sets $U_\alpha$ satisfying $s_\alpha = |\lambda_{\alpha\beta}|^2 s_\beta$ in $U_\alpha \cap U_\beta$ it is evident that $s_\alpha = h r_\alpha$ where $h > 0$ is a $C^\infty$ function defined on the entire surface $M$; then $\partial \log h$ is a $C^\infty$ differential form on the compact manifold $M$ so $\int_M \overline{\partial}\partial \log h = \int_M d(\partial \log h) = 0$ by Stokes’s Theorem and consequently

$$\int_M \overline{\partial}\partial \log s_\alpha = \int_M \left( \overline{\partial}\partial \log h + \overline{\partial}\partial \log r_\alpha \right) = \int_M \overline{\partial}\partial \log r_\alpha,$$

showing that the value of the integral (1.11) is independent of the choice of the functions $r_\alpha$. If $f \in \Gamma(M, M(\lambda))$ is a nontrivial meromorphic cross-section of the line bundle $\lambda$ and $\delta(f) = \sum_j \nu_j \cdot p_j$ then by definition $c(\lambda) = \deg \delta(f) = \sum_j \nu_j$. Choose coordinate neighborhoods $U_\alpha$ covering $M$ such that each point $p_j$ is contained in an open disc $D_j \subset U_\alpha$ for a coordinate neighborhood $U_\alpha$ and $\overline{D}_j \cap U_\beta = \emptyset$ whenever $\beta \neq \alpha_j$. Set $r_\alpha = |f_\alpha|^2$ if $\alpha = \alpha_j$ for any $j$; and let $r_\alpha_j$ be a modification of the function $|f_{\alpha_j}|^2$ within the disc $D_j$ so that $r_\alpha_j$ is a $C^\infty$ and strictly positive function in $U_\alpha$, as is clearly possible. The functions $r_\alpha$ so defined then are $C^\infty$ in the coordinate neighborhoods $U_\alpha$ and satisfy $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in any intersection $U_\alpha \cap U_\beta$; and $\overline{\partial}\partial \log r_\alpha = \overline{\partial}\partial (\log f_\alpha + \log f_{\alpha_j}) = 0$ in the complement of the union $\bigcup_j D_j$ since the functions $f_\alpha$ are holomorphic

---

6The differential operators $\partial$ and $\overline{\partial}$ are defined in Appendix A.1.
there. Then from the residue calculus and Stokes’s theorem it follows that
\[
\frac{1}{2\pi i} \int_M \partial \partial \log r_\alpha = \frac{1}{2\pi i} \sum_j \int_{D_j} \partial \partial \log r_{\alpha_j} = \frac{1}{2\pi i} \sum_j \int_{\partial D_j} d \partial \log r_{\alpha_j} = \sum_j \nu_j = c(\lambda)
\]
since \( \partial \log r_{\alpha_j} = d \log f_{\alpha_j} \) on \( \partial D_j \), and that suffices to conclude the proof.

It is easy to see that for any holomorphic line bundle \( \lambda \) over a compact Riemann surface \( M \) there exist functions \( r_\alpha \) satisfying the conditions of the preceding theorem for any finite open covering \( \{ U_\alpha \} \) of \( M \). Indeed choose open subsets \( V_\alpha \) such that \( V_\alpha \subset U_\alpha \) and that the sets \( V_\alpha \) also cover \( M \). For any open set \( U_\alpha \) of the initial covering there exists as usual a \( C^\infty \) function \( r_\alpha \alpha \) with support in \( U_\alpha \) such that \( r_\alpha \alpha (p) \geq 0 \) in \( U_\alpha \) and \( r_\alpha \alpha > 0 \) in \( V_\alpha \); and in terms of this function set \( r_{\alpha \beta} = \lambda_{\beta \alpha} r_\alpha \alpha \) for \( p \in U_\alpha \cap U_\beta \) and \( r_{\alpha \beta}(p) = 0 \) for \( p \in U_\beta \sim U_\alpha \), from which it is clear that \( r_{\beta \gamma}(p) = \lambda_{\gamma \beta} r_{\alpha \gamma} \) for \( p \in U_\beta \cap U_\gamma \). The sums \( r_\alpha = \sum_{\delta} r_{\alpha \delta} \) then satisfy the conditions of the theorem. The integral (1.11) thus is a well defined invariant associated to any line bundle \( \lambda \), and by the preceding theorem this invariant is equal to the characteristic class of \( \lambda \) if that bundle has a nontrivial meromorphic cross-section; this thus provides a definition of the characteristic class of an arbitrary holomorphic line bundle over a compact Riemann surface, which reduces to the preceding definition (1.9) for those holomorphic line bundles that have nontrivial meromorphic cross-sections.

**Corollary 1.4** If \( \lambda \) is a holomorphic line bundle over a compact Riemann surface \( M \) and \( c(\lambda) < 0 \) then the bundle \( \lambda \) has no nontrivial holomorphic cross-sections.

**Proof:** If \( \lambda \) has a nontrivial holomorphic cross-section \( f \) then \( c(\lambda) \) is defined by (1.9) so that \( c(\lambda) = \deg \delta(f) \geq 0 \), and that suffices for the proof.

**Corollary 1.5** If \( \lambda \) is a holomorphic line bundle over a compact Riemann surface \( M \) and \( c(\lambda) = 0 \) then
\[
\dim \Gamma(M, \mathcal{O}(\lambda)) = \begin{cases} 1 & \text{if } \lambda \text{ is analytically equivalent to the trivial bundle,} \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof:** If the line bundle \( \lambda \) has characteristic class \( c(\lambda) = 0 \) and has a nontrivial holomorphic cross-section \( f \) then \( c(\lambda) \) is defined by (1.9) so that \( 0 = c(\lambda) = \deg \delta(f) \), and consequently the cross-section \( f \) is holomorphic and nowhere vanishing on \( M \). Thus when the line bundle is defined by coordinate transition functions \( \lambda_{\alpha \beta} \) in intersections \( U_\alpha \cap U_\beta \) of coordinate neighborhoods on \( M \) the
cross-section $f$ is described by holomorphic and nowhere vanishing functions $f_\alpha$ in the sets $U_\alpha$ such that $\lambda_{\alpha\beta} = f_\alpha / f_\beta$ in $U_\alpha \cap U_\beta$; and that is just the condition that the line bundle $\lambda$ is analytically equivalent to the trivial line bundle. On the other hand any holomorphic cross-section of the trivial line bundle $\lambda = M \times \mathbb{C}$ over $M$ is just a holomorphic function on the compact Riemann surface $M$ so by the maximum modulus theorem is constant; thus $\dim \Gamma(M, \mathcal{O}(\lambda)) = 1$ for the trivial line bundle $\lambda$, and that suffices for the proof.

An alternative approach to the characteristic class of a holomorphic line bundle uses the cohomology groups of sheaves. Over an arbitrary Riemann surface $M$ there is the exact sequence of sheaves

$0 \to \mathbb{Z} \overset{\iota}{\to} \mathcal{C} \overset{e}{\to} \mathcal{C}^* \to 0$

in which $\iota$ is the natural inclusion mapping of the sheaf of locally constant integer-valued functions, the trivial sheaf with stalk $\mathbb{Z}$, into the sheaf $\mathcal{C}$ of germs of complex-valued continuous functions and $e$ is the homomorphism that associates to a germ $f \in \mathcal{C}$ the germ $e(f) = \exp 2\pi i f \in \mathcal{C}^*$ of a nowhere vanishing complex-valued continuous function. The associated exact cohomology sequence contains the segment

$H^1(M, \mathcal{C}) \overset{e}{\to} H^1(M, \mathcal{C}^*) \overset{\delta}{\to} H^2(M, \mathbb{Z}) \overset{\iota}{\to} H^2(M, \mathcal{C})$,

in which $H^1(M, \mathcal{C}^*)$ can be viewed as the set of topological line bundles over $M$. Since $\mathcal{C}$ is a fine sheaf $H^1(M, \mathcal{C}) = H^2(M, \mathcal{C}) = 0$, so this segment of an exact sequence reduces to the isomorphism

(1.14) $\delta : H^1(M, \mathcal{C}^*) \overset{\cong}{\longrightarrow} H^2(M, \mathbb{Z})$.

The image $\delta(\lambda) \in H^2(M, \mathbb{Z})$ of a topological line bundle $\lambda \in H^1(M, \mathcal{C}^*)$ under this isomorphism thus characterizes the topological equivalence class of $\lambda$ completely, and every cohomology class in $H^2(M, \mathbb{Z})$ is the image of some topological line bundle over $M$. There is a similar exact sequence of sheaves

(1.15) $0 \to \mathbb{Z} \overset{\iota}{\to} \mathcal{E} \overset{e}{\to} \mathcal{E}^* \to 0$

involving $C^\infty$ functions rather than merely continuous functions; and since $\mathcal{E}$ also is a fine sheaf the associated exact cohomology sequence leads in the same way to an isomorphism

(1.16) $\delta : H^1(M, \mathcal{E}^*) \overset{\cong}{\longrightarrow} H^2(M, \mathbb{Z})$

analogous to (1.14) but in which $H^1(M, \mathcal{E}^*)$ can be viewed as the set of $C^\infty$ line bundles over $M$. These isomorphisms can be described more concretely by tracing through the coboundary homomorphism in the exact sequence of sheaves, just as was done for the coboundary homomorphism in the exact sequence (1.5). If the line bundle $\lambda$ is described by coordinate transition functions $\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{C}^*)$ or $Z^1(\mathfrak{U}, \mathcal{E}^*)$ there are single-valued branches of the logarithms

(1.17) $f_{\alpha\beta} = \frac{1}{2\pi i} \log \lambda_{\alpha\beta}$
in the intersections $U_\alpha \cap U_\beta$ after passing to a suitable refinement of the covering if necessary. The image $\delta(\lambda) \in H^2(M, \mathbb{Z})$ is the cohomology class represented by the integral cocycle $n \in Z^1(\mathcal{U}, \mathbb{Z})$ for which

$$n_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}.$$  \hfill (1.18)

Since this is the same construction for either of the homomorphisms (1.14) or (1.16) it is evident that the image $\delta(\lambda) \in H^2(M, \mathbb{Z})$ of a $C^\infty$ line bundle $\lambda$ is the same cohomology class whether calculated through (1.14) or (1.16); consequently a topological line bundle over an arbitrary Riemann surface $M$ is topologically equivalent to a $C^\infty$ line bundle, and two $C^\infty$ line bundles are equivalent if and only if they are topologically equivalent. Of course a holomorphic line bundle $\lambda$ can be viewed either as a $C^\infty$ line bundle or as a topological line bundle, and either of these structures is described completely by the cohomology class $\delta(\lambda) \in H^2(M, \mathbb{Z})$.

Since $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ when $M$ is a compact connected two-dimensional orientable manifold the cohomology class $\delta \lambda \in H^2(M, \mathbb{Z})$ associated to a line bundle $\lambda$ over a compact Riemann surface $M$ can be identified with an integer. It is useful in the present discussion to view the integral cohomology group as a subgroup $H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{C}) \cong \mathbb{C}$ of the complex cohomology group, since the latter group can be handled analytically through the deRham isomorphism. There is of course a choice in the identification of a cohomology class in $H^2(M, \mathbb{C})$ with a complex number, since the identification $H^2(M, \mathbb{C}) \cong \mathbb{C}$ is determined only up to a linear mapping of $\mathbb{C}$; the choice made here is convenient for present purposes. A cohomology class $c \in H^2(M, \mathbb{C})$ is described by a cocycle $c_{\alpha\beta\gamma} \in Z^2(\mathcal{U}, \mathbb{C})$ in terms of a covering $\mathcal{U} = \{U_\alpha\}$ of the surface $M$. When viewed as a cocycle in $Z^2(\mathcal{U}, \mathcal{E})$ the cocycle $c_{\alpha\beta\gamma}$ is cohomologous to zero after a suitable refinement of the covering $\mathcal{U}$ since $\mathcal{E}$ is a fine sheaf; so

$$c_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}$$  \hfill (1.19)

for some $C^\infty$ functions $f_{\alpha\beta}$ in the intersections $U_\alpha \cap U_\beta$. Since $c_{\alpha\beta\gamma}$ are constants

$$0 = d c_{\alpha\beta\gamma} = d f_{\beta\gamma} - d f_{\alpha\gamma} + d f_{\alpha\beta},$$

and consequently the differential forms $d f_{\alpha\beta}$ form a cocycle in $Z^1(\mathcal{U}, \mathcal{E}^1)$. The sheaf $\mathcal{E}^1$ of $C^\infty$ differential forms of degree 1 also is a fine sheaf, so after a further refinement of the covering $\mathcal{U}$ if necessary.

$$d f_{\alpha\beta} = \phi_\beta - \phi_\alpha$$  \hfill (1.20)

for some $C^\infty$ differential 1-forms $\phi_\alpha$ in the neighborhoods $U_\alpha$. Then

$$0 = d d f_{\alpha\beta} = d \phi_\beta - d \phi_\alpha$$

so the differential 2-forms $d \phi_\alpha$ and $d \phi_\beta$ agree in the intersection $U_\alpha \cap U_\beta$; consequently these local differential forms describe a differential 2-form on the entire compact Riemann surface $M$. The deRham isomorphism

$$I : H^2(M, \mathbb{C}) \longrightarrow \mathbb{C}$$  \hfill (1.21)

\footnote{For details see the discussion in Appendix D.2.}
associates to the cohomology class $c \in H^2(M, \mathbb{C})$ represented by the cocycle $e_{\alpha\beta\gamma}$ the complex number

(1.22) \[ I(c) = \int_M d\phi_\alpha. \]

In terms of this isomorphism the characteristic class of a holomorphic line bundle can be described as follows.

**Theorem 1.6** If $\delta \lambda \in H^2(M, \mathbb{C})$ is the cohomology class associated to a holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ through the isomorphism (1.15) or (1.16) then $c(\lambda) = -I(\delta \lambda)$ is the characteristic class of that line bundle.

**Proof:** The cohomology class $\delta \lambda \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{C})$ is represented by the cocycle $n_{\alpha\beta\gamma}$ defined in (1.14) in terms of the functions $f_{\alpha\beta}$ of (1.17); so

\[ I(\delta \lambda) \in \mathbb{C} \]

is the complex number defined through equations (1.20) and (1.22) in terms of the functions $f_{\alpha\beta}$. It follows from (1.12) that

\[ df_{\alpha\beta} = \frac{1}{2\pi i} \partial \log r_\alpha - \frac{1}{2\pi i} \partial \log r_\beta \]

in terms of the functions $r_\alpha$ of Theorem 1.3, so in (1.20) it is possible to take $\phi_\alpha = -\frac{1}{2\pi i} \partial \log r_\alpha$; then $d\phi_\alpha = -\frac{1}{2\pi i} \partial \partial \log r_\alpha$ and consequently in (1.22)

\[ I(\delta \lambda) = \int_M d\phi_\alpha = -\frac{1}{2\pi i} \int_M \partial \partial \log r_\alpha = -c(\lambda) \]

by Theorem 1.3, which suffices for the proof.

That a negative sign appears in the formula for the characteristic class $c(\lambda)$ in the preceding theorem reflects the definition of the line bundle of a divisor $d$ as the line bundle $\zeta_d = \delta(-d)$ in order that $c(\zeta_d) = \deg d$. Further properties of holomorphic line bundles follow from a further examination of the differential operator $\partial\bar{\partial}$, beginning with a variant of the Cauchy Integral Formula that provides a partial inverse to that differential operator. In the statement of this result the *support* of a complex valued function in the plane is the closure of the set of points at which the function is nonzero.

**Theorem 1.7** If $g$ is a $C^\infty$ function with compact support in the complex plane $\mathbb{C}$ there is a $C^\infty$ function $f$ in $\mathbb{C}$ such that $\partial f/\partial \bar{\zeta} = g$.

**Proof:** When the complex variable $\zeta$ is expressed in polar coordinates as $\zeta = re^{i\theta}$ then

\[ \frac{d\zeta}{\zeta} \wedge \frac{d\zeta}{\zeta} = 2ie^{-i\theta} dr \wedge d\theta, \]

so this differential form remains bounded near the origin although not actually defined at the origin. Since $g$ is $C^\infty$ and has compact support the integral

(1.23) \[ f(z) = \frac{i}{2\pi} \int_{\mathbb{C}} g(z + \zeta) \frac{d\zeta \wedge d\bar{\zeta}}{\zeta} \]
is a $C^\infty$ function of the variable $z$ in the entire plane and can be differentiated by differentiating under the integral sign. For any fixed point $z \in \mathbb{C}$ let $D$ be a disc centered at the origin in the plane of the variable $\zeta$ and having sufficiently large radius that $g(z + \zeta)$ vanishes for all points $\zeta \notin D$; and let $D_\epsilon$ be another disc centered at the origin in the plane of the variable $\zeta$ contained in $D$ and having radius $\epsilon$. Then

$$\frac{\partial f(z)}{\partial \zeta} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{D \sim D_\epsilon} g(z + \zeta) \frac{d\zeta}{\zeta} = g(z)$$

by Stokes’s Theorem, since the differential form $g(z + \zeta) \zeta^{-1} d\zeta$ vanishes on $\partial D$. If the circle $D_\epsilon$ is parametrized by $\zeta = \epsilon e^{i\theta}$ then

$$\frac{\partial f(z)}{\partial \zeta} = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{\theta=0}^{2\pi} g(z + \epsilon e^{i\theta}) d\theta = g(z)$$

as desired, to conclude the proof.

Generally the function $f$ of the preceding theorem does not have compact support, although of course it must be holomorphic outside the support of $g$. For many purposes only the following local version of this theorem is of interest.

**Corollary 1.8** If $g$ is the germ of a $C^\infty$ function at a point in the complex plane there is a germ $f$ of a $C^\infty$ function at that point such that $\partial f/\partial \zeta = g$.

**Proof:** Any germ of a $C^\infty$ function can be represented by a $C^\infty$ function of compact support in the complex plane, so this assertion is an immediate consequence of the preceding theorem.

For any Riemann surface $M$ and any integers $0 \leq p, q \leq 1$ the vector spaces $\Gamma(U, \mathcal{E}^{(p,q)})$ of $C^\infty$ differential forms of type $(p,q)$ over open subsets $U \subset M$ clearly form a complete presheaf over $M$; the associated sheaf is denoted by $\mathcal{E}^{(p,q)}$, and the spaces $\Gamma(U, \mathcal{E}^{(p,q)})$ can be identified with the spaces of sections of this sheaf. The differential operator $\bar{\partial}$ is a homomorphism

$$\bar{\partial} : \Gamma(U, \mathcal{E}^{(p,0)}) \longrightarrow \Gamma(U, \mathcal{E}^{(p,1)})$$

between these presheaves of complex vector spaces so induces a sheaf homomorphism

$$(1.24) \quad \bar{\partial} : \mathcal{E}^{(p,0)} \longrightarrow \mathcal{E}^{(p,1)}$$

between the associated sheaves. If $p = 0$ so $\mathcal{E}^{(0,0)} = \mathcal{E}$ it follows from the Cauchy-Riemann equations that the kernel of the sheaf homomorphism (1.24) is the sheaf $\mathcal{O}$ of germs of holomorphic functions on $M$, and it follows from Corollary 1.8 that for any germ $\phi = g d\zeta \in \mathcal{E}^{(0,1)}$ there is a germ $f \in \mathcal{E}$ such
that \( \partial f/\partial \bar{z} = g \) and hence that \( \overline{\partial} f = \phi \); thus there is the exact sequence of sheaves

\[
0 \longrightarrow \mathcal{O} \overset{\iota}{\longrightarrow} \mathcal{E} \overset{\overline{\partial}}{\longrightarrow} \mathcal{E}^{(0,1)} \longrightarrow 0
\]

over \( M \), in which \( \iota \) is the natural inclusion homomorphism. If \( p = 1 \) the kernel of the sheaf homomorphism (1.24) is the sheaf of germs of differential forms \( \phi = f dz \) of type \((1,0)\) for which \( \partial f/\partial \bar{z} = 0 \), so for which the function \( f \) is holomorphic; these are called germs of holomorphic differential forms, often also called germs of holomorphic abelian differentials or of abelian differentials of the first kind, and the sheaf of such germs is denoted by \( \mathcal{O}^{(1,0)} \). It follows again from Corollary 1.8 that for any germ \( \phi = g dz \wedge d\bar{z} \in \mathcal{E}^{(1,1)} \) there is a germ \( f \in \mathcal{E} \) such that \( -\langle \partial f/\partial \bar{z} \rangle = g \) and hence \( \overline{\partial}(-f dz) = -g d\bar{z} \wedge dz = g dz \wedge d\bar{z} = \phi \); thus there is also the exact sequence of sheaves

\[
0 \longrightarrow \mathcal{O}^{(1,0)} \overset{\iota}{\longrightarrow} \mathcal{E}^{(1,0)} \overset{\overline{\partial}}{\longrightarrow} \mathcal{E}^{(1,1)} \longrightarrow 0
\]

over \( M \), in which \( \iota \) is the natural inclusion homomorphism.

More generally suppose that \( \lambda \) is a holomorphic line bundle over the Riemann \( M \) and that \( \lambda \) is described by a coordinate bundle \( \{ U_\alpha, \lambda_{\alpha\beta} \} \) in terms of a covering of \( M \) by open coordinate neighborhoods \( U_\alpha \). The vector spaces \( \Gamma(U, \mathcal{E}(\lambda)) \), \( \Gamma(U, \mathcal{O}(\lambda)) \) and \( \Gamma(U, \mathcal{M}(\lambda)) \) of \( \mathcal{C}^\infty \), holomorphic and meromorphic cross-sections of \( \lambda \) in the open subsets \( U \subset M \) clearly form complete presheaves of abelian groups over \( M \); the associated sheaves are denoted by \( \mathcal{E}(\lambda), \mathcal{O}(\lambda) \) and \( \mathcal{M}(\lambda) \) respectively, and the spaces \( \Gamma(U, \mathcal{E}(\lambda)), \Gamma(U, \mathcal{O}(\lambda)) \) and \( \Gamma(U, \mathcal{M}(\lambda)) \) of cross-sections of \( \lambda \) over \( U \) can be identified with the spaces of sections of the corresponding sheaves over \( U \). The vector spaces \( \Gamma(U, \mathcal{E}^{(p,q)}(\lambda)) \) consisting of \( \mathcal{C}^\infty \) differential forms \( \phi_\alpha \) of type \((p,q)\) in the intersections \( U \cap U_\alpha \) such that \( \phi_\alpha = \lambda_{\alpha\beta} \phi_\beta \) in the \( U \cap U_\alpha \cap U_\beta \) also form a complete presheaf of abelian groups over \( M \); the associated sheaf is denoted by \( \mathcal{E}^{(p,q)}(\lambda) \), and the vector space \( \Gamma(U, \mathcal{E}^{(p,q)}(\lambda)) \) can be identified with the space of sections of the sheaf \( \mathcal{E}^{(p,q)}(\lambda) \) over \( U \). Of course \( \mathcal{E}(\lambda) = \mathcal{E}^{(0,0)}(\lambda) \) so the sheaf \( \mathcal{E}(\lambda) \) also can be considered as a sheaf of germs of differential forms that are cross-sections of the line bundle \( \lambda \). If \( \phi = \{ \phi_\alpha \} \in \Gamma(U, \mathcal{E}^{(p,0)}(\lambda)) \) then \( \overline{\partial} \phi_\alpha = \lambda_{\alpha\beta} \overline{\partial} \phi_\beta \) in each intersection \( U_\alpha \cap U_\beta \) since the coordinate transition functions \( \lambda_{\alpha\beta} \) are holomorphic; thus the differential operator \( \overline{\partial} \) describes homomorphisms

\[
\overline{\partial} : \Gamma(U, \mathcal{E}^{(p,0)}(\lambda)) \longrightarrow \Gamma(U, \mathcal{E}^{(p,1)}(\lambda))
\]

of presheaves, which induce homomorphisms

\[
\overline{\partial} : \mathcal{E}^{(p,0)}(\lambda) \longrightarrow \mathcal{E}^{(p,1)}(\lambda)
\]

of the associated sheaves. The kernel of this homomorphism for \( p = 0 \) is the sheaf \( \mathcal{O}(\lambda) \) of germs of holomorphic cross-sections of the line bundle \( \lambda \) and for \( p = 1 \) is the sheaf \( \mathcal{E}^{(1,0)}(\lambda) \) of germs of holomorphic differential forms that are cross-sections of the line bundle \( \lambda \).
Corollary 1.9 If $\lambda$ is a holomorphic line bundle over an arbitrary Riemann surface $M$ there are exact sequences of sheaves of the form

\begin{align*}
0 & \longrightarrow \mathcal{O}^{(p,0)}(\lambda) \xrightarrow{\iota} \mathcal{E}^{(p,0)}(\lambda) \xrightarrow{\partial} \mathcal{E}^{(p,1)}(\lambda) \longrightarrow 0
\end{align*}

for $p = 0$ or $1$, in which $\iota$ is the inclusion mapping.

**Proof:** In a coordinate neighborhood $U_\alpha \subset M$ the exact sequence (1.28) reduces to the exact sequence (1.25) for $p = 0$ and to the exact sequence (1.26) for $p = 1$, and that suffices for the proof.

The same result clearly holds for differential forms that are cross-sections of a holomorphic vector bundle of any rank over a Riemann surface $M$, since such cross-sections are described by finite sets of exact sequences (1.25) or (1.26).

From the exact sequence of sheaves (1.28) there follows as usual an associated exact sequence of cohomology groups, leading to the following result.

**Theorem 1.10 (Theorem of Dolbeault)** If $\lambda$ is a holomorphic line bundle over an arbitrary Riemann surface $M$ then

\begin{align*}
H^1(M, \mathcal{O}(\lambda)) & \cong \frac{\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))}{\partial \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))}, \\
H^q(M, \mathcal{O}(\lambda)) & = 0 \text{ for } q \geq 2.
\end{align*}

**Proof:** The long exact cohomology sequence associated to the exact sequence of sheaves (1.28) for $p = 0$ includes the segment

\begin{align*}
\Gamma(M, \mathcal{E}(\lambda)) \xrightarrow{-\partial} \Gamma(M, \mathcal{E}^{(0,1)}(\lambda)) \xrightarrow{\delta} H^1(M, \mathcal{O}(\lambda)) \xrightarrow{\iota} H^1(M, \mathcal{E}(\lambda));
\end{align*}

and since $\mathcal{E}(\lambda)$ is a fine sheaf $H^1(M, \mathcal{E}(\lambda)) = 0$ so the homomorphism $\delta$ yields the first isomorphism of the theorem. The same exact cohomology sequence also includes the segments

\begin{align*}
H^{q-1}(M, \mathcal{E}^{(0,1)}(\lambda)) \xrightarrow{\delta} H^q(M, \mathcal{O}(\lambda)) \xrightarrow{\iota} H^q(M, \mathcal{E}(\lambda))
\end{align*}

for all $q \geq 1$; since $\mathcal{E}^{(0,1)}(\lambda)$ is a fine sheaf $H^{q-1}(M, \mathcal{E}^{(0,1)}(\lambda)) = H^q(M, \mathcal{E}(\lambda)) = 0$ for all $q \geq 2$, from which it follows that $H^q(M, \mathcal{O}(\lambda)) = 0$ for all $q \geq 2$. That suffices to conclude the proof.

The Theorem of Dolbeault also holds for holomorphic vector bundles as well as for holomorphic line bundles, with essentially the same proof. Further information about the first cohomology groups $H^1(M, \mathcal{O}(\lambda))$ can be obtained by strengthening Theorem 1.7 as follows.

**Theorem 1.11** If $g$ is a $C^\infty$ function in an open subset $U \subseteq \mathbb{C}$ there is a $C^\infty$ function $f$ in $U$ such that $\partial f/\partial \bar{z} = g$. 
Proof: It is sufficient to prove the theorem for a connected set $U$, so that will be assumed in the proof. Select a sequence of connected open subsets $U_n \subset U$ such that (i) $\overline{U}_n$ is compact, (ii) $\overline{U}_n \subset U_{n+1}$, (iii) $\bigcup_{n=1}^{\infty} U_n = U$, (iv) any function holomorphic in $U_{n-1}$ can be approximated uniformly in $\overline{U}_{n-2}$ by functions holomorphic in $U_n$. The existence of such a sequence of subsets is a standard result in complex analysis, rather like the Runge theorem. It will then be demonstrated by induction on $n$ that there is a sequence of functions $f_n$ such that (v) $f_n$ is a $C^\infty$ function in $U_n$, (vi) $\partial f_n/\partial \overline{z} = g$ in $U_n$, (vii) $f_n(z) - f_{n-1}(z)$ is holomorphic in $U_{n-1}$, (viii) $|f_n(z) - f_{n-1}(z)| < 2^{-n}$ for all $z \in \overline{U}_{n-2}$. For each index $n$ there is a $C^\infty$ function $h_n$ on the set $U$ such that $\partial h_n/\partial \overline{z} = g$ in $U_n$; for it is possible to modify the function $g$ outside $U_n$ so that it has compact support in $U$. If $n = 1$ take $f_1 = h_1$, and there is nothing further to show. If $n \geq 2$ suppose that the functions $f_1, \ldots, f_{n-1}$ have been determined so that they satisfy (v), (vi), (vii) and (viii). Both $h_n$ and $f_{n-1}$ are $C^\infty$ functions in $U_{n-1}$ and $\partial (h_n - f_{n-1})/\partial \overline{z} = g - g = 0$ in $U_{n-1}$ so $h_n - f_{n-1}$ actually is holomorphic in $U_{n-1}$. By (iv) there exists a holomorphic function $g_n$ in $U_n$ such that $|h_n - f_{n-1} - g_n| < 2^{-n}$ in $\overline{U}_{n-2}$. The functions $f_1, \ldots, f_{n-1}, f_n = h_n - g_n$ then also satisfy (v), (vi), (vii) and (viii), which completes the induction. The next step is to show that the sequence $f_n$ converges to a $C^\infty$ function $f$ in $U$ and that this limit has the desired properties.

If $k \geq n + 2$ it follows from (viii) that $|f_k(z) - f_{k-1}(z)| < 2^{-k}$ for all $z \in U_n$ and from (vii) that the functions $f_k(z) - f_{k-1}(z)$ are holomorphic in $U_n$; so the series $\sum_{k=n+2}^{\infty} (f_k(z) - f_{k-1}(z))$ is a uniformly convergent series of holomorphic functions in $U_n$. If $m \geq n + 2$ and $z \in U_n$

$$f_m(z) = f_{n+1}(z) + \sum_{k=n+2}^{m} (f_k(z) - f_{k-1}(z));$$

but then sequence $f_m(z)$ converges uniformly in $U_n$ to the function

$$f(z) = f_{n+1}(z) + \sum_{k=n+2}^{\infty} (f_k(z) - f_{k-1}(z))$$

that differs from $f_{n+1}(z)$ by a holomorphic function, and consequently $f(z)$ is a $C^\infty$ function in $U_n$ and $\partial f/\partial \overline{z} = \partial f_{n+1}/\partial \overline{z} = g$ in $U_n$. That is true for all sets $U_n$, and that suffices to conclude the proof.

**Corollary 1.12** If $U \subset \mathbb{C}$ is an open subset of the complex plane and $\lambda$ is a holomorphic line bundle over $U$ then $\lambda$ is analytically trivial and $H^p(U, \mathcal{O}(\lambda)) = 0$ for all $p > 0$.

**Proof:** Consider the exact sequence of sheaves

$$(1.29) \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O} \xrightarrow{c} \mathcal{O}^* \longrightarrow 0$$

---

over the subset $U \subset \mathbb{C}$ in which $\iota$ is the natural inclusion mapping of the sheaf $Z$ of locally constant integer-valued functions to the sheaf $O$ of germs of holomorphic functions and $e$ is the homomorphism that associates to a germ $f \in O$ the germ $e(f) = \exp 2\pi i f \in O^*$ of a nowhere vanishing holomorphic function; this is just the holomorphic version of the exact sheaf sequences (1.13) and (1.15). The associated exact cohomology sequence contains the segment

\[(1.30) \quad H^1(U, O) \xrightarrow{\iota^*} H^1(U, O^*) \xrightarrow{\delta} H^2(U, \mathbb{Z}).\]

It follows from the preceding theorem that for any differential form $\phi = gdz \in \Gamma(U, E(0, 1))$ there is a function $f \in \Gamma(U, E)$ for which $\partial f/\partial z = g$ and hence $\bar{\partial}f = gdz = \phi$; consequently $H^1(U, O) = 0$ by the Theorem of Dolbault, Theorem 1.10, for the special case of the trivial line bundle. Furthermore $H^2(U, \mathbb{Z}) = 0$ for an arbitrary open subset of the complex plane. It therefore follows from the exact sequence (1.30) that $H^1(U, O^*) = 0$, which is just the condition that any holomorphic line bundle over $U$ is analytically trivial. Thus if $\lambda$ is a holomorphic line bundle over $U$ then $\lambda \cong 1$, so by what has just been proved $H^1(U, O(\lambda)) = H^1(U, O) = 0$. Of course $H^p(U, O(\lambda)) = 0$ for all $p > 1$ by the Dolbeault Theorem again, and that suffices to conclude the proof.

An application of the preceding corollary yields a method for calculating the cohomology groups of Riemann surfaces with coefficients in the sheaf $O(\lambda)$ of germs of holomorphic cross-sections of a holomorphic line bundle.

**Theorem 1.13 (Theorem of Leray)** If $\mathcal{U}$ is a covering of the Riemann surface $M$ by open coordinate neighborhoods then for any holomorphic line bundle $\lambda$ over $M$ and any integer $p \geq 0$ the natural homomorphism

$$\iota^*_\mathcal{U} : H^p(\mathcal{U}, O(\lambda)) \longrightarrow H^p(M, O(\lambda))$$

is an isomorphism.

**Proof:** Since any intersection of coordinate neighborhoods in the covering $\mathcal{U}$ is again a coordinate neighborhood, so can be viewed as an open subset of $\mathbb{C}$, it follows from Corollary1.12 that $H^p(U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}, O(\lambda)) = 0$ for all $p > 0$. Thus the covering $\mathcal{U}$ is a Leray covering of the Riemann surface $M$ for the sheaf $O(\lambda)$, so by the general Theorem of Leray as discussed in Appendix C.2 the natural homomorphisms $\iota^*_\mathcal{U}$ are isomorphisms for all indices $p \geq 0$. That suffices for the proof.

The identification $H^1(M, O(\lambda)) \cong H^1(\mathcal{U}, O(\lambda))$ is very convenient for explicit calculations in these cohomology groups; it is worth examining this in more detail, since such calculations will be used repeatedly in the subsequent discussion. Suppose that the line bundle $\lambda$ is described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha}\}$ in terms of a covering of the Riemann surface $M$ by open coordinate neighborhoods $U_\alpha$. The local coordinates in the bundle $\lambda$ describe any point in $\lambda$ over the coordinate neighborhood $U_\alpha$ as a pair $(p, f_\alpha)$ where $p \in U_\alpha$ is the projection of the point to the base space $M$ and $f_\alpha \in \mathbb{C}$ is the fibre coordinate of...
the point. Thus if \( s \in \Gamma(M, \mathcal{O}(\lambda)) \) is a holomorphic cross-section of the bundle \( \lambda \) then for any point \( p \in U_\alpha \) the value \( s(p) \) of the cross-section \( s \) at the point \( p \) is described by the pair \( (p, f_\alpha(p)) \) in terms of the fibre coordinate \( f_\alpha(p) \in \mathbb{C} \); and the value \( f_\alpha(p) \) is a holomorphic function of the point \( p \in U_\alpha \), so the cross-section \( s \) over the neighborhood \( U_\alpha \) can be identified with the holomorphic function \( f_\alpha \) in the coordinate neighborhood \( U_\alpha \subset M \). A cochain \( s \in C^p(\mathcal{U}, \mathcal{O}(\lambda)) \) consists of sections \( s_{\alpha_0 \cdots \alpha_p} \in \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{O}(\lambda)) \) for each ordered set of \( p+1 \) open subsets \( U_{\alpha_0}, \ldots, U_{\alpha_p} \) of the covering \( \mathcal{U} \), where these sections are skew-symmetric in the indices \( \alpha_0, \ldots, \alpha_p \); and the section \( s_{\alpha_0 \cdots \alpha_p} \) can be identified with a holomorphic function \( f_{\alpha_0 \cdots \alpha_p} \) in the intersection \( U_{\alpha_p} \cap \cdots \cap U_{\alpha_0} \) in terms of the fibre coordinate over \( U_{\lambda_\alpha} \) when the intersection is viewed as a subset of the last coordinate neighborhood \( U_{\alpha_p} \). This identification will be used consistently in the subsequent discussion; thus cochains in \( C^p(\mathcal{U}, \mathcal{O}(\lambda)) \) will be identified without further comment as collections of holomorphic functions in the intersections \( U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \subset M \) in terms of the fibre coordinates in \( \lambda \) over \( U_{\alpha_p} \). Some care must be taken when using this identification though. For example the skew-symmetry of the sections \( s_{\alpha_0 \cdots \alpha_p} \) in the indices \( \alpha_0, \ldots, \alpha_p \) does not mean that the holomorphic functions \( f_{\alpha_0 \cdots \alpha_p} \) that represent these sections are skew symmetric in these indices. The section \( s_{\alpha_0 \cdots \alpha_p} \) is identified with a holomorphic function \( f_{\alpha_0 \cdots \alpha_p} \) in terms of the fibre coordinates of \( \lambda \) over \( U_{\alpha_p} \), while for any permutation \( \pi \in \mathbb{S}_{p+1} \) of the integers \( 0, 1, \ldots, p \) the section \( s_{\alpha_{\pi(0)} \cdots \alpha_{\pi(p)}} \) is identified with a holomorphic function \( f_{\alpha_{\pi(0)} \cdots \alpha_{\pi(p)}} \) in terms of the fibre coordinates of \( \lambda \) over the subset \( U_{\alpha_{\pi(p)}} \); when the identity \( s_{\alpha_0 \cdots \alpha_p} = (\text{sign } \pi) \cdot s_{\alpha_{\pi(0)} \cdots \alpha_{\pi(p)}} \) is expressed in terms of the fibre coordinate over the coordinate neighborhood \( U_{\alpha_p} \) it takes the form

\[
(1.31) \quad f_{\alpha_0 \cdots \alpha_p} = (\text{sign } \pi) \lambda_{\alpha_{\pi(0)} \alpha_{\pi(p)}} f_{\alpha_{\pi(0)} \cdots \alpha_{\pi(p)}}
\]

since the fibre coordinates over a point in \( U_{\alpha_p} \cap U_{\alpha_{\pi(p)}} \) are related by \( f_{\alpha_p} = \lambda_{\alpha_{\pi(0)} \alpha_{\pi(p)}} \cdot f_{\alpha_{\pi(0)} \alpha_{\pi(p)}} \). Thus a 1-cochain \( s \in C^1(\mathcal{U}, \mathcal{O}(\lambda)) \) consists of sections \( s_{\alpha_0 \alpha} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}(\lambda)) \) that are skew-symmetric in the indices \( \alpha, \beta \) and is identified with a collection of holomorphic functions \( f_{\alpha \beta} \in \Gamma(U_{\alpha} \cap U_{\beta}, \mathcal{O}) \) satisfying the skew-symmetry condition

\[
(1.32) \quad f_{\alpha \beta} = -\lambda_{\beta \alpha} f_{\beta \alpha} \quad \text{in } U_{\alpha} \cap U_{\beta}.
\]

A 0-cochain \( s \in C^0(\mathcal{U}, \mathcal{O}(\lambda)) \) consists just of sections \( s_{\alpha} \in \Gamma(U_\alpha, \mathcal{O}(\lambda)) \) and is identified with a collection of holomorphic functions \( f_{\alpha} \in \Gamma(U_\alpha, \mathcal{O}) \); but there is of course no skew-symmetry involved in this case.

The coboundary of a 0-cochain \( s \in C^0(\mathcal{U}, \mathcal{O}(\lambda)) \) is the 1-cochain \( (\delta s)_{\alpha} = s_{\beta} - s_{\alpha} \); and if the 0-cochain is identified with a collection of holomorphic functions \( f_{\alpha} \in \Gamma(U_\alpha, \mathcal{O}) \) and its coboundary \( \delta s \) is identified with a collection of holomorphic functions \( (\delta f)_{\alpha} \in \Gamma(U_\alpha \cap U_{\beta}, \mathcal{O}) \) then in terms of the fibre coordinates over the coordinate neighborhood \( U_{\beta} \)

\[
(1.33) \quad (\delta f)_{\alpha \beta} = f_{\beta} - \lambda_{\beta \alpha} f_{\alpha} \quad \text{in } U_{\alpha} \cap U_{\beta}.
\]
In this case \((\delta f)_{\alpha\beta} = -\lambda_{\beta\alpha}(f_{\alpha} - \lambda_{\alpha\beta}f_{\beta}) = -\lambda_{\beta\alpha}(f_{\beta})_{\beta\alpha}\), so the skew-symmetry condition (1.32) holds automatically. The 0-cochain \(s\) thus is a 0-cocycle if and only if the functions \(f_{\alpha}\) satisfy

\[
(1.34) \quad f_{\alpha} = \lambda_{\alpha\beta}f_{\beta} \quad \text{in} \quad U_{\alpha} \cap U_{\beta};
\]

that is just the condition that the functions \(f_{\alpha}\) are a cross-section of the line bundle \(\lambda\), yielding the usual identification \(Z^0(U, O(\lambda)) \cong \Gamma(M, O(\lambda))\). Correspondingly the coboundary of a 1-cochain \(s \in \mathcal{C}^1(U, O(\lambda))\) is the 2-cochain \((\delta s)_{\alpha\beta\gamma} = s_{\beta\gamma} - s_{\alpha\gamma} + s_{\alpha\beta}\); and if the 1-cochain \(s\) is identified with a collection of holomorphic functions \(f_{\alpha\beta} \in \Gamma(U_{\alpha} \cap U_{\beta}, O)\) and its coboundary \(\delta s\) is identified with a collection of holomorphic functions \((\delta f)_{\alpha\beta\gamma} \in \Gamma(U_{\alpha} \cap U_{\beta} \cap U_{\gamma}, O)\) then in terms of the fibre coordinates over the coordinate neighborhood \(U_{\gamma}\)

\[
(1.35) \quad (\delta f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + \lambda_{\gamma\beta}f_{\alpha\beta} \quad \text{in} \quad U_{\alpha} \cap U_{\beta} \cap U_{\gamma}.
\]

The 1-cochain \(s\) thus is a 1-cocycle if and only if the functions \(f_{\alpha\beta}\) satisfy

\[
(1.36) \quad f_{\alpha\gamma} = \lambda_{\gamma\beta}f_{\alpha\beta} + f_{\beta\gamma} \quad \text{in} \quad U_{\alpha} \cap U_{\beta} \cap U_{\gamma}.
\]

This description of the groups \(H^1(M, O(\lambda))\) can be used to demonstrate that they are finite-dimensional vector spaces when the Riemann surface \(M\) is compact, by introducing an appropriate topology on the space of cochains for a fixed coordinate covering \(U\). There are various ways of doing this; perhaps the simplest is to use a Hilbert space topology, even though it is not intrinsically defined. It is a standard result in complex analysis\(^9\) that the subspace

\[
\Gamma_2(U, O) = \left\{ f \in \Gamma(U, O) \mid \int_U |f(z)|^2 dx \wedge dy < \infty \right\}
\]

of square-integrable holomorphic functions on an open subset \(U \subset \mathbb{C}\) is a Hilbert space with the inner product

\[
(f, g) = \int_U f(z)\overline{g(z)} dx \wedge dy
\]

and the corresponding norm \(\|f\|^2 = (f, f)\). Furthermore if \(U\) and \(V\) are open sets and \(U \subset V\) then the restriction mapping

\[
\rho_{UV} : \Gamma_2(V, O) \rightarrow \Gamma_2(U, O)
\]

is a bounded linear mapping between these two Hilbert spaces; if \(\overline{U} \subset V\) is compact this restriction mapping is even a compact mapping by Vitali’s Theorem, so the image of any bounded subset of \(\Gamma_2(V, O)\) has compact closure in the space \(\Gamma_2(U, O)\).

Theorem 1.14 (Finite Dimensionality Theorem) If \( \lambda \) is a holomorphic line bundle over a compact Riemann surface \( M \) then the cohomology groups \( H^0(M, \mathcal{O}(\lambda)) \) and \( H^1(M, \mathcal{O}(\lambda)) \) are finite-dimensional complex vector spaces.

Proof: Since \( M \) is compact there is a finite covering \( \mathfrak{U} \) of \( M \) by open coordinate neighborhoods \( W_\alpha \). Choose open subsets \( U_\alpha \subset W_\alpha \) and \( V_\alpha \subset W_\alpha \) that form coverings \( \mathfrak{U} \) and \( \mathfrak{V} \) of \( M \) where \( \overline{U}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset W_\alpha \), so that \( \overline{U}_\alpha \) and \( \overline{V}_\alpha \) are compact. If \( \mu_1^* \) and \( \mu_2^* \) are the group homomorphisms (C.15) induced by the refining mappings corresponding to the inclusions \( V_\alpha \subset W_\alpha \) and \( U_\alpha \subset V_\alpha \) respectively and \( \iota_1^* \) is the natural homomorphism (C.16) from the cohomology group of the covering \( \mathfrak{U} \) to the cohomology of the space \( M \) then by the Theorem of Leray, Theorem (1.13), the homomorphisms \( \iota_1^*: \iota_2^* = \iota_1^* \circ \mu_2^* \) and \( \iota_2^* = \iota_1^* \circ \mu_2^* \) from the sequence

\[
(1.37) \quad H^p(\mathfrak{U}, \mathcal{O}(\lambda)) \xrightarrow{\mu_1^*} H^p(\mathfrak{V}, \mathcal{O}(\lambda)) \xrightarrow{\mu_2^*} H^p(\mathfrak{U}, \mathcal{O}(\lambda)) \xrightarrow{\iota_1^*} H^p(M, \mathcal{O}(\lambda))
\]

are isomorphisms; consequently the homomorphisms \( \mu_1^* \) and \( \mu_2^* \) also are isomorphisms. In the identification

\[
C^p(\mathfrak{U}, \mathcal{O}(\lambda)) \cong \bigoplus_\alpha \Gamma(V_{\alpha_0} \cap \cdots \cap V_{\alpha_p}, \mathcal{O})
\]

of \( p \)-cochains for the covering \( \mathfrak{U} \) with collections of holomorphic functions in the finitely many intersections of \( p + 1 \)-tuples of sets from the covering \( \mathfrak{U} \) let \( C^p_2(\mathfrak{V}, \mathcal{O}(\lambda)) \subset C^p(\mathfrak{V}, \mathcal{O}(\lambda)) \) be the subgroup consisting of cochains that are identified with square integrable holomorphic functions, so that

\[
C^p_2(\mathfrak{U}, \mathcal{O}(\lambda)) \cong \bigoplus_\alpha \Gamma_2(V_{\alpha_0} \cap \cdots \cap V_{\alpha_p}, \mathcal{O})
\]

this exhibits \( C^p_2(\mathfrak{U}, \mathcal{O}(\lambda)) \) as a finite direct sum of Hilbert spaces and hence as a Hilbert space itself. The coordinate transition functions \( \lambda_{\alpha\beta} \) of the line bundle \( \lambda \) are holomorphic in \( W_\alpha \cap W_\beta \) so are bounded in \( V_\alpha \cap V_\beta \); hence the coboundary mappings (1.33) take square integrable cochains into square integrable cochains and are bounded linear mappings \( \delta : C^p_2(\mathfrak{U}, \mathcal{O}(\lambda)) \rightarrow C^{p+1}_2(\mathfrak{U}, \mathcal{O}(\lambda)) \) between Hilbert spaces. The kernels \( Z^p_2(\mathfrak{U}, \mathcal{O}(\lambda)) \) of these mappings consequently are closed subspaces of a Hilbert space so also are Hilbert spaces. The same considerations of course also apply to the covering \( \mathfrak{U} \).

If \( p = 0 \) then \( H^0(M, \mathcal{O}(\lambda)) = \Gamma(M, \mathcal{O}(\lambda)) \). Since cross-sections can be described by holomorphic functions in the subsets \( W_\alpha \) and \( V_\alpha \subset W_\alpha \) it follows that all holomorphic cross-sections of \( \lambda \) lie in \( Z^0_2(\mathfrak{U}, \mathcal{O}(\lambda)) \) and consequently that \( Z^0_2(\mathfrak{V}, \mathcal{O}(\lambda)) = \Gamma(M, \mathcal{O}(\lambda)) = H^0(M, \mathcal{O}(\lambda)) \); the same of course holds for the covering \( \mathfrak{U} \). The isomorphism \( \mu_2^* \) in (1.37) thus can be viewed as an isomorphism

\[
\mu_2^* : Z^0_2(\mathfrak{V}, \mathcal{O}(\lambda)) \rightarrow Z^0_2(\mathfrak{U}, \mathcal{O}(\lambda))
\]

between Hilbert spaces. Since \( U_\alpha \subset \overline{U}_\alpha \subset \overline{V}_\alpha \) this is a compact mapping; thus a bounded open neighborhood of the origin in \( Z^0_2(\mathfrak{V}, \mathcal{O}(\lambda)) \) is mapped to an open
neighborhood of the origin in \( Z^0_2(\mathcal{M}, \mathcal{O}(\lambda)) \) having compact closure. A Hilbert space with an open neighborhood of the origin having compact closure is necessarily finite dimensional, so \( Z^0_2(\mathcal{M}, \mathcal{O}(\lambda)) \cong H^0(M, \mathcal{O}(\lambda)) \) is finite dimensional.

If \( p = 1 \) the square integrable first cohomology group is defined by

\[
H^1_2(\mathcal{M}, \mathcal{O}(\lambda)) = \frac{Z^1_2(\mathcal{M}, \mathcal{O}(\lambda))}{\delta C^1_2(\mathcal{M}, \mathcal{O}(\lambda))};
\]

this is a well defined complex vector space but cannot be viewed as a Hilbert space since the subspace \( \delta C^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \subset Z^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \) has not been shown to be a closed linear subspace, although that will follow from the conclusion of the finite dimensionality theorem. The natural inclusion of square integrable cochains into all cochains induces a linear mapping

\[
\rho^*_\mathcal{M} : H^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \longrightarrow H^1(\mathcal{M}, \mathcal{O}(\lambda))
\]

between these two complex vector spaces, which will be shown to be an isomorphism. For this purpose first suppose that \( f_{\alpha\beta} \in Z^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \) is a cocycle that is cohomologous to zero in \( Z^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \); thus \( f_{\alpha\beta} \) is the coboundary of a cochain \( f_\alpha \in C^0(\mathcal{M}, \mathcal{O}(\lambda)) \) satisfying (1.33). Any point \( p \in \partial V_\alpha \) is contained in some set \( V_\beta \). The function \( f_{\alpha\beta} \) is square-integrable in \( V_\alpha \cap V_\beta \) by assumption, the function \( f_\beta \) is continuous in a full open neighborhood of the point \( p \) since \( p \in V_\beta \), and the function \( \lambda_{\alpha\beta} \) is holomorphic in \( W_\alpha \cap W_\beta \) and hence is bounded in a full open neighborhood of the point \( p \); consequently the function \( f_\alpha \) is also square integrable in an open neighborhood of the point \( p \) in \( V_\alpha \). The closure \( \overline{V}_\alpha \) is compact, so finitely many of these neighborhoods cover the boundary of that set, and it follows that \( f_\alpha \) is square integrable on the full set \( V_\alpha \); consequently \( f_\alpha \in C^0(\mathcal{M}, \mathcal{O}(\lambda)) \), which shows that \( \rho^*_\mathcal{M} \) is injective. Since \( \overline{V}_\alpha \subset W_\alpha \) and \( \overline{V}_\alpha \) is compact the isomorphism \( \mu^*_1 \) in (1.37) factors through square-integrable cohomology so can be written as the composition \( \mu^*_1 = \rho^*_\mathcal{M} \circ \nu^*_1 \) of the linear mappings \( \nu^*_1 \) and \( \rho^*_\mathcal{M} \) in the sequence

\[
H^1(\mathcal{M}, \mathcal{O}(\lambda)) \xrightarrow{\nu^*_1} H^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \xrightarrow{\rho^*_\mathcal{M}} H^1(\mathcal{M}, \mathcal{O}(\lambda));
\]

and since \( \rho^*_\mathcal{M} \) has just been shown to be an injection and the composition \( \rho^*_\mathcal{M} \circ \nu^*_1 = \mu^*_1 \) is an isomorphism it follows that \( \rho^*_\mathcal{M} \) is also an isomorphism as asserted.

The arguments just applied to the covering \( \mathcal{M} \) also can be applied to the covering \( \mathfrak{M} \); hence the linear mapping

\[
\rho^*_\mathfrak{M} : H^1_2(\mathfrak{M}, \mathcal{O}(\lambda)) \longrightarrow H^1(\mathfrak{M}, \mathcal{O}(\lambda))
\]

analogous to (1.38) is also an isomorphism. To conclude the proof of the theorem it then suffices to show that \( H^1_2(\mathfrak{M}, \mathcal{O}(\lambda)) \) is a finite dimensional complex vector space. In view of the isomorphisms (1.38) and (1.39) the isomorphism \( \mu^*_2 \) in (1.37) induces an isomorphism

\[
\mu^*_2 \mathfrak{M} : H^1_2(\mathcal{M}, \mathcal{O}(\lambda)) \longrightarrow H^1_2(\mathfrak{M}, \mathcal{O}(\lambda))
\]
between these two complex vector spaces. Moreover since $\overline{U}_\alpha \subset V_\alpha$ the homomorphism

$$\mu_{2Z}^* : Z_2^1(\mathfrak{M}, \mathcal{O}(\lambda)) \rightarrow Z_2^1(\mathfrak{U}, \mathcal{O}(\lambda))$$

(1.41)

corresponding to this inclusion is a compact linear mapping between these two Hilbert spaces. Consider then the Hilbert space

$$A = C^0_2(\mathfrak{U}, \mathcal{O}(\lambda)) \oplus Z_2^1(\mathfrak{M}, \mathcal{O}(\lambda))$$

and the bounded linear mapping

$$\delta, \mu_{2Z}^* : A \rightarrow Z_2^1(\mathfrak{U}, \mathcal{O}(\lambda))$$

defined by $\langle \delta, \mu_{2Z}^*(f,g) = \delta f + \mu_{2Z}^*(g) \rangle$. The mapping $\delta, \mu_{2Z}^*$ is surjective; for since (1.40) is an isomorphism of vector spaces any square-integrable cocycle of the covering $\mathfrak{U}$ must be cohomologous to a square-integrable cocycle coming from the covering $\mathfrak{M}$. The difference $\delta, \mu_{2Z}^* - (0, \mu_{2Z}^*)$ is just the coboundary mapping

$$\delta : C^0_2(\mathfrak{U}, \mathcal{O}(\lambda)) \rightarrow Z_2^1(\mathfrak{U}, \mathcal{O}(\lambda)),$$

so the theorem is a consequence of the following general lemma\(^\text{10}\).

**Lemma 1.15** If $X$ and $Y$ are Hilbert spaces and if both $\phi : X \rightarrow Y$ and $\psi : X \rightarrow Y$ are bounded linear mappings where $\phi$ is surjective and $\psi$ is compact then $Y/(\phi - \psi)(X)$ is finite dimensional.

**Proof:** Let $\phi^* : Y \rightarrow X$ and $\psi^* : Y \rightarrow X$ be the adjoint mappings to $\phi$ and $\psi$ respectively; then $\phi^*$ is an injective mapping with closed range and $\psi^*$ is a compact mapping. The first step is to show that the kernel $K$ of the mapping $\phi^* - \psi^*$ is a finite-dimensional subspace of $Y$. For this purpose suppose that $\{y_n\}$ is any bounded sequence of elements of $K$. Since $\psi^*$ is compact then after passing to a subsequence if necessary the sequence $\psi^*(y_n)$ converges; consequently the sequence $\phi^*(y_n) = \psi^*(y_n)$ also converges. Since $\phi^*$ is injective and has closed range it is a homeomorphism between $Y$ and its range; hence the sequence $\{y_n\}$ converges, which shows that $K$ is locally compact hence finite dimensional. The next step is to show that $\phi^* - \psi^*$ has closed range. Indeed after factoring out by $K$ it can be assumed that $\phi^* - \psi^*$ is injective. Consider a sequence of elements $y_n \in Y$ such that $(\phi^* - \psi^*)(y_n) \rightarrow x$. If $\{y_n\}$ has a bounded subsequence then as before it is possible to assume that $\psi^*(y_n)$ converges; but then $\phi^*(y_n) = (\phi^* - \psi^*)(y_n) + \psi^*(y_n)$ converges, so again $y_n$ converges to an element $y$ and $(\phi^* - \psi^*)(y) = x$. On the other hand if $\|y_n\| \rightarrow \infty$ the elements $y'_n = y_n/\|y_n\|$ have norm 1 and

$$f(\phi^* - \psi^*)(y'_n) = \frac{1}{\|y_n\|}(\phi^* - \psi^*)(y_n) \rightarrow 0;$$

\(^\text{10}\)For the properties of Hilbert space used in the proof of this lemma see for instance W. Rudin *Functional Analysis*, (McGraw-Hill, 1991).
again it can be assumed that $\psi^*(y_n)$ converges, hence that $\phi^*(y_n)$ and $y'_n$ converge, and if $y' = \lim y_n$ then $\|y'\| = 1$ and $(\phi^* - \psi^*)(y') = 0$, which contradicts the assumption that $\phi^* - \psi^*$ is one-to one so this case cannot occur. To conclude the proof of the lemma using the results of the preceding two steps note that since $\phi^* - \psi^*$ has closed range the same is true of $\phi - \psi$; so the quotient space $Y' = Y/(\phi - \psi)(X)$ is a Hilbert space. The surjective mapping $\phi$ induces a surjective mapping $\phi' : X \rightarrow Y'$, and the compact mapping $\psi$ induces a compact mapping $\psi' : X \rightarrow Y'$; and since $\phi' = \psi'$ the space $Y'$ is locally compact hence finite dimensional as desired, which suffices to conclude the proof.

The finite dimensionality theorem also holds for cohomology groups with coefficients in the sheaf of germs of holomorphic cross-sections of a complex vector bundle; for the essential part of the proof really uses only the compactness of the operation of restriction of holomorphic functions to compact subsets of their domain of definition, and that is true either for single functions or for vectors of functions with the supremum norm. Before examining these cohomology groups further it is useful first to consider yet another modification of the solution to the $\overline{\partial}$ equation in Theorem 1.7.

**Lemma 1.16** For any $\delta > 0$ there is a $C^\infty$ function $s$ in the complex plane with support in the disc $|z| \leq \delta/2$ such that for any bounded open subsets $U \subset V \subset \mathbb{C}$ for which the distance from $U$ to $\mathbb{C} \sim V$ is greater than $\delta$ and for any $C^\infty$ function $f$ in the complex plane with support contained in $U$ there is a $C^\infty$ function $g$ in the complex plane with support contained in $V$ for which

\begin{equation}
(1.42) \quad f(z) = \frac{\partial g(z)}{\partial z} + \frac{i}{2} \int_U f(\zeta) s(\zeta - z) d\zeta \wedge d\overline{\zeta}.
\end{equation}

**Proof:** Choose a $C^\infty$ function $r(z)$ in the complex plane such that

\[
r(z) = \begin{cases} 
1 & \text{if } |z| \leq \frac{\delta}{4} \\
0 & \text{if } |z| \geq \frac{\delta}{2}
\end{cases}
\]

and set

\[
s(z) = \begin{cases} 
0 & \text{if } |z| < \frac{\delta}{4} \\
-\frac{1}{\pi} \frac{\partial}{\partial z} \left( \frac{r(z)}{z} \right) & \text{if } 0 < |z| < \delta \\
0 & \text{if } |z| > \frac{\delta}{2}.
\end{cases}
\]

Clearly the function $s(z)$ is $C^\infty$ in the entire complex plane and its support is contained in the disc $|z| \leq \delta/2$. For any $C^\infty$ function $f$ in $\mathbb{C}$ with support in $U$ set

\begin{equation}
(1.43) \quad g(z) = \frac{i}{2\pi} \int_{\mathbb{C}} f(z + \zeta) r(\zeta) \frac{d\zeta \wedge d\overline{\zeta}}{\zeta},
\end{equation}
reminiscent of the integral (1.23) used in the proof of Theorem 1.7. The function $g$ thus defined is a $C^\infty$ function in the plane of the complex variable $\zeta$ and can be differentiated by differentiating under the integral sign. Furthermore if the distance from $z$ to $U$ exceeds $\delta/2$ then $g(z) = 0$ since $f(z + \zeta) = 0$ whenever $|\zeta| \leq \delta/2$ while $r(\zeta) = 0$ whenever $\zeta \geq \delta/2$; thus the support of the function $g$ is contained in $V$. For any fixed point $z \in U$ let $D$ be a disc centered at the origin in the plane of the complex variable $\zeta$ with radius sufficiently large that $f(z + \zeta) = 0$ whenever $\zeta \not\in D$, and let $D_\epsilon$ be another disc centered at the origin in the plane of the complex variable $\zeta$ with radius $\epsilon$. Then

$$\frac{\partial g(z)}{\partial \zeta} = \frac{i}{2\pi} \int_C \frac{\partial f(z + \zeta)}{\partial \zeta} \frac{r(\zeta) d\zeta \wedge d\zeta}{\zeta} = \frac{i}{2\pi} \int_C \frac{\partial f(z + \zeta)}{\partial \zeta} r(\zeta) \frac{d\zeta \wedge d\zeta}{\zeta}$$

$$= \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{D \sim D_\epsilon} \frac{\partial f(z + \zeta)}{\partial \zeta} \frac{r(\zeta) d\zeta \wedge d\zeta}{\zeta}$$

$$= \lim_{\epsilon \to 0} \frac{i}{2\pi} \left( \int_{D \sim D_\epsilon} d\left( f(z + \zeta) \frac{r(\zeta)}{\zeta} d\zeta \right) - \int_{D \sim D_\epsilon} f(z + \zeta) \frac{\partial}{\partial \zeta} \left( \frac{r(\zeta)}{\zeta} \right) d\zeta \wedge d\zeta \right)$$

$$= \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{\partial D_\epsilon} \frac{f(z + \zeta)}{\zeta} d\zeta - \frac{i}{2} \int_C f(z + \zeta) s(\zeta) d\zeta \wedge d\zeta,$$

where Stokes’s Theorem is used to replace the integral over $D \sim D_\epsilon$ with the integral over the boundary of this region and the integrand vanishes on the boundary of $D$ while $r(\zeta) = 1$ for $\zeta \in \partial D_\epsilon$ for sufficiently small $\epsilon$. The integral over $\partial D_\epsilon$ is just $f(z)$, as in the proof of Theorem 1.7, and the desired result then follows from a change of variable in the second integral.

The traditional approach to the further examination of the cohomology group $H^1(M, \mathcal{O}(\lambda))$ for a compact Riemann surface is through potential theory11. The discussion here however will follow an alternative approach introduced by J-P. Serre12 which proceeds by considering the space of linear functionals

$$(1.44) \quad T : H^1(M, \mathcal{O}(\lambda)) \to \mathbb{C}$$

on the finite dimensional complex vector space $H^1(M, \mathcal{O}(\lambda))$. By the Theorem of Dolbeault, Theorem 1.10,

$$(1.45) \quad H^1(M, \mathcal{O}(\lambda)) \cong \frac{\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))}{\partial \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))}$$

11The use of potential theory in Riemann surfaces goes back to Riemann’s inaugural dissertation in Göttingen in 1851, “Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Größe”, Collected Works, pp. 1 - 48, and was a crucial tool in the great classical work on Riemann surfaces by H. Weyl, Die Idee der Riemannschen Fläche (Teubner, 1923); [English translation The Concept of a Riemann Surface (Addison-Wesley, 1955)].

so the linear functionals on $H^1(M, O(\lambda))$ can be identified with the linear functionals on $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ that vanish on the subspace $\overline{\Gamma}(M, \mathcal{E}^{(0,0)}(\lambda))$. It suffices just to consider continuous linear functionals when the infinite dimensional complex vector space $\Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ has a suitable structure as a topological vector space. To introduce the appropriate topology suppose that $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ and is described by a coordinate bundle $\{V_\alpha, \lambda_\alpha\}$ in terms of a finite covering of $M$ by open coordinate neighborhoods $V_\alpha$; and let $z_\alpha = x_\alpha + iy_\alpha$ be the local coordinates in $V_\alpha$. Introduce on $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ for $q = 0$ or $1$ the norms

$$
(1.46) \quad ||\phi||_N = \sup_\alpha \sup_{\nu_1 + \nu_2 \leq N} \sup_{z_\alpha \in V_\alpha} \left| \frac{\partial^{\nu_1 + \nu_2} f_\alpha(x_\alpha, y_\alpha)}{\partial x_\alpha^{\nu_1} \partial y_\alpha^{\nu_2}} \right|
$$

for a cross-section $\phi \in \Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ where $\phi_\alpha = f_\alpha$ for $q = 0$ and $\phi_\alpha = f_\alpha dz_\alpha$ for $q = 1$. It is evident that these are norms in the customary sense that (i) $||\phi||_N \geq 0$ and this is an equality if and only if $\phi = 0$, (ii) $||\phi_1 + \phi_2||_N \leq ||\phi_1||_N + ||\phi_2||_N$, (iii) $||c \phi||_N = |c| \cdot ||\phi||_N$ for any complex constant $c$; and it is also evident that $||\phi||_N \leq ||\phi||_{N+1}$. Such a collection of norms determines on the vector space $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ the structure of a locally convex topological vector space by taking as a basis for the open neighborhoods of the origin the sets $V_{N,k}$ consisting of those cross-sections $\phi \in \Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ such that $||\phi||_N < 1/k$ for positive integers $N, k$; the norms (1.46) are continuous functions in this topology, and the topology can be defined equivalently by the translation invariant metric

$$
(1.47) \quad \rho(\phi, \psi) = \sup_N \frac{c_N \|\phi - \psi\|_N}{\Gamma + \|\phi - \psi\|_N}
$$

where $c_N$ are positive numbers such that $\lim_{N \to -\infty} c_N = 0$. It is evident from the definition (1.46) of these norms that any Cauchy sequence in $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ converges in this topology, so the topological vector space $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ is complete in this norm, hence is a Fréchet space. A linear functional

$$
(1.48) \quad T : \Gamma(M, \mathcal{E}^{(0,q)}(\lambda)) \to \mathbb{C}
$$

is continuous in this topology if and only if there are positive integers $M, N$ such that

$$
(1.49) \quad |T(\phi)| \leq M \|\phi\|_N
$$

for all $\phi \in \Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$.

**Lemma 1.17** The linear mapping

$$
\mathcal{S} : \Gamma(M, \mathcal{E}^{(0,0)}(\lambda)) \to \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))
$$

is a continuous linear mapping between these two Fréchet spaces, and its image is a closed linear subspace of $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$; consequently linear functionals on the quotient space $H^1(M, \mathcal{O}(\lambda))$ can be identified with continuous linear functionals on the Fréchet space $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ that vanish on the subspace $\partial\Gamma(M, \mathcal{E}^{(0,0)}(\lambda)) \subset \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$.

**Proof:** To simplify the notation for this proof let $A = \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ and $B = \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$. The linear mapping $\overline{\partial} : A \longrightarrow B$ is continuous as an immediate consequence of the obvious inequality $||\overline{\partial}\phi||_N \leq ||\phi||_{N+1}$. The kernel $K$ of this linear mapping is a closed subspace of $A$, so the quotient $A/K$ also is a Fréchet space and the induced mapping $\overline{\partial} : A/K \longrightarrow B$ is a continuous linear mapping with a trivial kernel. If $L \subset B$ is a finite dimensional subspace of $B$ complementary to $\overline{\partial}(A/K)$ the product $(A/K) \times L$ is a Fréchet space and the mapping $(\overline{\partial} + i) : (A/K) \times L \longrightarrow B$ defined by $(\overline{\partial} + i)(a,l) = \overline{\partial}(a) + l$ for $a \in A$, $l \in L$ is a surjective continuous linear mapping with a trivial kernel; hence by the open mapping theorem it is an isomorphism of Fréchet spaces. Since $(A/K) \times 0 \subset (A/K) \times L$ is a closed subspace it follows that its isomorphic image $(\overline{\partial} + i)((A/K) \times 0) = \overline{\partial}A$ is a closed subspace of $B$. In that case the quotient space $B/\overline{\partial}A$ with the topology it inherits from $B$ also is a Fréchet space; and any linear functional on the finite dimensional quotient space $B/\overline{\partial}A$ is necessarily continuous hence amounts to a continuous linear functional on $B$ that vanishes on $\overline{\partial}A$, which suffices to conclude the proof.

For an example of such a continuous linear functional suppose that $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ and is described by a coordinate bundle $\{V_\alpha, \lambda_{\alpha\beta}\}$ in terms of a finite covering $\mathcal{U}$ of $M$ by coordinate neighborhoods $V_\alpha$. A cross-section $\tau \in \Gamma(M, \mathcal{E}^{(1,0)}(\lambda^{-1}))$ is described by $C^\infty$ differential forms $\tau_\alpha$ of type $(1,0)$ in the coordinate neighborhoods $V_\alpha$ such that $\tau_\alpha = \lambda_{\alpha\beta}^{-1} \tau_\beta$ in intersections $V_\alpha \cap V_\beta$. If $\phi \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ is described by $C^\infty$ differential forms $\phi_\alpha$ of type $(0,1)$ in the coordinate neighborhoods $V_\alpha$ such that $\phi_\alpha = \lambda_{\alpha\beta}\phi_\beta$ in intersections $V_\alpha \cap V_\beta$ then $\tau_\alpha \wedge \phi_\alpha = \lambda_{\alpha\beta}^{-1} \tau_\beta \wedge \lambda_{\alpha\beta}\phi_\beta = \tau_\beta \wedge \phi_\beta$ in $V_\alpha \cap V_\beta$: so the product $\tau \wedge \phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ is a differential form of type $(1,1)$ defined on the entire compact Riemann surface $M$. The integral

\begin{equation}
T_\tau(\phi) = \int_M \tau_\alpha \wedge \phi_\alpha
\end{equation}

is clearly a continuous linear functional (1.48) for $q = 1$, and is a nontrivial linear functional so long as $\tau \neq 0$. If $g \in \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ is a cross-section described by $C^\infty$ functions $g_\alpha$ in the coordinate neighborhoods $V_\alpha$ such that $g_\alpha = \lambda_{\alpha\beta} g_\beta$ in $V_\alpha \cap V_\beta$ then $g_\alpha \tau_\alpha = \lambda_{\alpha\beta} g_\beta \cdot \lambda_{\alpha\beta}^{-1} \tau_\beta = g_\beta \tau_\beta$ in $V_\alpha \cap V_\beta$, so these local products describe a global differential form of type $(1,0)$ on $M$. By Stokes’s Theorem $\int_M d(g_\alpha \tau_\alpha) = 0$, hence $0 = \int_M d(g_\alpha \tau_\alpha) = \int_M \overline{\partial}(g_\alpha \tau_\alpha) = \int_M \overline{\partial}g_\alpha \wedge \tau_\alpha + \int_M g_\alpha \overline{\partial}\tau_\alpha$ and consequently

\begin{equation}
T_\tau(\overline{\partial}g_\alpha) = \int_M g_\alpha \cdot \overline{\partial}\tau_\alpha.
\end{equation}
Therefore $T_\tau(\partial g) = 0$ for all cross-sections $g \in \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ if and only if $\int_M g_\alpha \cdot \overline{\partial} \tau_\alpha = 0$ for all cross-sections $g$, so that $\overline{\partial} \tau_\alpha = 0$ and $\tau_\alpha$ are holomorphic differential forms; thus the linear functional $T_\tau$ vanishes on the linear subspace $\overline{\Gamma}(M, \mathcal{E}^{(0,0)}(\lambda))$ if and only if $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$, and in that case determines a linear functional (1.44). The deeper result is the converse assertion that all linear functionals (1.44) are of this form.

**Theorem 1.18 (Serre Duality Theorem)** If $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ the continuous linear functionals on the topological vector space $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ that vanish on the closed linear subspace $\overline{\Gamma}(M, \mathcal{E}^{(0,0)}(\lambda))$ are precisely the linear functionals $T_\tau$ for cross-sections $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$.

**Proof:** Suppose that $\lambda$ is defined by a coordinate bundle $\{W_\alpha, \lambda_{\alpha\beta}\}$ for a finite covering $\mathfrak{W}$ of the surface $M$ by bounded coordinate neighborhoods $W_\alpha \subset M$ with local coordinates $z_\alpha$. Choose open subsets $U_\alpha$ and $V_\alpha$ that form coverings $\mathfrak{U}$ and $\mathfrak{V}$ of $M$ with $\overline{U}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset W_\alpha$, so that $\overline{U}_\alpha$ and $\overline{V}_\alpha$ are compact; and choose a positive constant $\delta > 0$ such that the coordinate neighborhood $W_\alpha$ is viewed as a bounded open subset of the complex plane of the variable $z_\alpha$ both the distance from $U_\alpha$ to the complement of $V_\alpha$ and the distance from $V_\alpha$ to the complement of $W_\alpha$ are greater than $\delta$ for all $\alpha$. Introduce on the vector space $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ the topology defined by the norms (1.46) in terms of the covering of $M$ by the coordinate neighborhoods $V_\alpha$. For any index $\alpha$ let $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda)) \subset \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ be the subspace consisting of those cross-sections $\phi \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ with support contained in $V_\alpha$ and let $\Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ be the vector space consisting of ordinary $C^\infty$ differential forms of type $(0,1)$ with support in the coordinate neighborhood $V_\alpha$. To any differential form $\phi_\alpha = f_\alpha \, dz_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ there can be associated the cross-section $\iota_\alpha(\phi_\alpha) \in \Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$ for which $\phi_\beta = \lambda_{\beta\alpha} \phi_\alpha$ in any intersection $V_\alpha \cap V_\beta$ and $\phi_\beta = 0$ otherwise; all elements of $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$ arise in this way, so the mapping

$$\iota_\alpha : \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)}) \longrightarrow \Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$$

thus defined is an isomorphism between these two complex vector spaces. The vector space $\Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ can be made into a topological vector space by the norms

$$\|\phi_\alpha\|_{N, \alpha} = \sup_{\nu_1 + \nu_2 \leq N} \sup_{z_\alpha \in V_\alpha} \left| \frac{\partial^{\nu_1 + \nu_2} f_\alpha(x_\alpha, y_\alpha)}{\partial x_\alpha^{\nu_1} \partial y_\alpha^{\nu_2}} \right|$$

for any differential form $\phi_\alpha = f_\alpha \, dz_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$; and with this topology $\Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ also is a Fréchet space. For any $\phi_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ clearly

$$\|\iota_\alpha(\phi_\alpha)\|_n = \sup_{\beta} \sup_{\nu_1 + \nu_2 \leq N} \left| \frac{\partial^{\nu_1 + \nu_2} (\lambda_{\beta\alpha}(z_\beta) f_\alpha(z_\beta))}{\partial x_\beta^{\nu_1} \partial y_\beta^{\nu_2}} \right|$$

$$= \|\phi_\alpha\|_{N, \alpha} + \sup_{\beta \neq \alpha} \sup_{\nu_1 + \nu_2 \leq N} \sup_{z_\beta \in V_\beta} \left| \frac{\partial^{\nu_1 + \nu_2} (\lambda_{\beta\alpha}(z_\beta) f_\alpha(z_\beta))}{\partial x_\beta^{\nu_1} \partial y_\beta^{\nu_2}} \right|. $$
It follows from this identity first that \( \|\tau_\alpha(\phi_\alpha)\|_N \geq \|\phi_\alpha\|_{N,\alpha} \) and second that \( \|\tau_\alpha(\phi_\alpha)\|_N \leq C_N \|\phi_\alpha\|_{N,\alpha} \) for some constant \( C_N > 0 \), since the functions \( \lambda g_\beta(z_\beta) \) and all their partial derivatives are defined in \( W_\alpha \cap W_\beta \supset \mathcal{V}_\alpha \cap \mathcal{V}_\beta \) and hence are uniformly bounded in \( \mathcal{V}_\alpha \cap \mathcal{V}_\beta \) as are all the partial derivatives \( \partial^{k_1+k_2}x_\alpha / \partial^{k_1}x_\beta \partial^{k_2}y_\beta \) and \( \partial^{k_1+k_2}y_\alpha / \partial^{k_1}x_\beta \partial^{k_2}y_\beta \); therefore the linear mapping (1.52) is an isomorphism of Fréchet spaces. If the support of the differential form \( \phi_\alpha = f_\alpha d\zeta_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)}) \) is contained in the subset \( U_\alpha \subset V_\alpha \) and the coordinate neighborhood \( W_\alpha \) is viewed as an open subset of the complex plane of the variable \( z_\alpha \) it follows from Lemma 1.16 that \( f_\alpha = \partial g_\alpha / \partial \zeta_\alpha + h_\alpha \) for \( C^\infty \) functions \( g_\alpha \) and \( h_\alpha \) with supports contained in \( V_\alpha \). The function \( g_\alpha \) can be extended to a section \( \tau_\alpha(g_\alpha) \in \Gamma(M, \mathcal{E}(\lambda)) \) by the obvious analogue of the construction of the isomorphism (1.52); and when the differential forms \( \phi_\alpha = f_\alpha d\zeta_\alpha \) and \( \psi_\alpha = h_\alpha d\zeta_\alpha \) are extended by the isomorphism (1.52) then \( \phi_\alpha = \partial g_\alpha + \psi_\alpha \) so

\[
\tau_\alpha(\phi_\alpha) = \partial \tau_\alpha(g_\alpha) + \tau_\alpha(\psi_\alpha)
\]

since \( \tau_\alpha(\partial g_\alpha) = \partial \tau_\alpha(g_\alpha) \). If \( T : \Gamma(M, \mathcal{E}^{(0,1)}(\lambda)) \rightarrow \mathbb{C} \) is a continuous linear functional that vanishes on the subspace \( \partial \Gamma(M, \mathcal{E}(\lambda)) \) it then follows that (1.54)

\[
T(\tau_\alpha(\phi)) = T(\tau_\alpha(\psi)).
\]

As in (1.42) the function \( h_\alpha(z_\alpha) \) is given explicitly by the integral

\[
h_\alpha(z_\alpha) = i \int_{U_\alpha} f_\alpha(\zeta_\alpha) s(\zeta_\alpha - z_\alpha) d\zeta_\alpha \wedge d\bar{\zeta}_\alpha,
\]

which can be written as a limit of Riemann sums so that

\[
\psi_\alpha(z_\alpha) = h_\alpha(z_\alpha) d\zeta_\alpha = \lim \sum_j f_\alpha(\zeta_j) s(\zeta_j - z_\alpha) d\zeta_j \cdot \Delta_j
\]

for local elements of area \( \Delta_j \). For any fixed point \( \zeta_j \in U_\alpha \) the expression \( s(\zeta_j - z_\alpha) d\zeta_\alpha \) is a \( C^\infty \) differential form with support contained in \( V_\alpha \), since the support of the function \( s(z) \) is contained in a disc of radius \( \delta/2 \) about the origin; the extensions \( \tau_\alpha(s(\zeta_j - z_\alpha) d\zeta_\alpha) \) thus are well defined elements of \( \Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda)) \). Since \( s(z) \) is a \( C^\infty \) function it follows from the continuity of the functional \( T \) that the images

\[
T(\tau_\alpha(s(\zeta_j - z_\alpha) d\zeta_\alpha)) = \tau_\alpha(\zeta_j)
\]

are \( C^\infty \) functions of the variable \( \zeta_j \in U_\alpha \). The Riemann sums and their partial derivatives converge uniformly to the integral over \( U_\alpha \), so it follows further from the continuity of the functional \( T \) that

\[
T(\tau_\alpha(\psi_\alpha)) = T\left(\lim \sum_j f_\alpha(\zeta_j) s(\zeta_j - z_\alpha) d\zeta_j \cdot \Delta_j\right)
\]

\[
= \lim \sum_j f_\alpha(\zeta_j) T(s(\zeta_j - z_\alpha) d\zeta_\alpha) : \Delta_j
\]

\[
= \lim \sum_j f_\alpha(\zeta_j) \tau_\alpha(\zeta) \Delta_j = \frac{i}{2} \int_{U_\alpha} f_\alpha(\zeta) \tau_\alpha(\zeta) d\zeta \wedge d\bar{\zeta};
\]
where $\tau_\alpha = t_\alpha(\zeta)d\zeta$. If the support of $\phi \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ is contained in the intersection $U_\alpha \cap U_\beta$ then $\phi$ can be considered as being an element of either $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$ or $\Gamma_\beta(M, \mathcal{E}^{(0,1)}(\lambda))$, so

$$T(\iota_\alpha(\phi_\alpha)) = \frac{i}{2} \int_{U_\alpha} \tau_\alpha \wedge \phi_\alpha$$

and if that holds for any such form $\phi$ then necessarily $\tau_\alpha = \tau_\beta \lambda_{\beta\alpha}$ in $U_\alpha \cap U_\beta$ so that the differential forms $\tau_\alpha$ in the coordinate neighborhoods $U_\alpha$ are a cross-section $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$. If $\rho_\alpha$ is a $\mathcal{C}^\infty$ partition of unity subordinate to the covering $\{U_\alpha\}$ then any cross-section $\phi \in \Gamma(M, \mathcal{O}^{(0,1)}(\lambda))$ can be written as the sum $\phi = \sum_\alpha \rho_\alpha \phi$ of cross-sections $\rho_\alpha \phi \in \Gamma(M, \mathcal{O}^{(0,1)}(\lambda))$; if $\phi$ is described by differential forms $\phi_\alpha$ in the subsets $U_\alpha$ then $\rho_\alpha \phi$ is described by the differential forms $\rho_\alpha \phi_\alpha$ in the subsets $U_\beta$. In particular the product $\rho_\alpha \phi_\alpha$ can be viewed as a differential form $\rho_\alpha \phi_\alpha \in \Gamma_0(U_\alpha, \mathcal{E}^{(0,1)})$, and then $\iota_\alpha(\rho_\alpha \phi_\alpha) = \rho_\alpha \phi$ so that by (1.55)

$$T(\phi) = \sum_\alpha T(\rho_\alpha \phi) = \sum_\alpha \frac{i}{2} \int_{U_\alpha} \tau_\alpha \wedge \rho_\alpha \phi_\alpha = \frac{i}{2} \sum_\alpha \int_{U_\alpha} \rho_\alpha \cdot (\tau_\alpha \wedge \phi_\alpha).$$

However in an intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods $\tau_\alpha \wedge \phi_\alpha = \tau_\beta \wedge \phi_\beta$ as noted before, so these local differential forms describe a global differential form $\tau \wedge \phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ on the entire Riemann surface $M$ and consequently

$$T(\phi) = \frac{i}{2} \int_{M} \sum_\alpha \rho_\alpha \cdot (\tau \wedge \phi) = \int_{M} \left(\frac{i}{2}\tau\right) \wedge \phi = T_{i\tau/2}(\phi);$$

and since $T$ vanishes on the subspace $\mathfrak{H}\Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ it follows as a consequence of (1.51) as before that $i\tau/2 \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$, which suffices to conclude the proof.

**Corollary 1.19** If $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$

$$\dim H^1(M, \mathcal{O}(\lambda)) = \dim \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1})).$$
Proof: Since $H^1(M, \mathcal{O}(\lambda))$ is a finite dimensional complex vector space its dimension is equal to the dimension of its dual space, which in view of (1.45) and the preceding theorem is isomorphic to the vector space $\Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$, and that suffices for the proof.

Theorem 1.18 also holds for holomorphic vector bundles just as for holomorphic line bundles, with the proper interpretation. Thus for cross-sections $\phi = \{\phi_\alpha\} \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ and $\tau = \{\tau_\alpha\} \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^*))$ viewed as column vectors of differential forms, where the dual vector bundle $\lambda^*$ is defined by the dual coordinate transition functions $\lambda^*_{\alpha\beta} = t_{\lambda_{\alpha\beta}}^{-1}$, it follows that $t_{\tau_\alpha} \wedge \phi_\alpha = t_{\tau_\beta} \wedge \phi_\beta$ in $U_\alpha \cap U_\beta$ and consequently that

\[(1.57) \quad T_\tau(\phi) = \int_M t_{\tau_\alpha} \wedge \phi_\alpha\]

is a well defined linear functional on the vector space $\Gamma(M, \mathcal{O}^{(1,0)}(\lambda^*))$. The rest of the proof of the Serre Duality Theorem carries through unchanged, although in the final statement for vector bundles the dual vector bundle $\lambda^*$ replaces the inverse line bundle $\lambda^{-1}$. 
Chapter 2

The Riemann-Roch Theorem

The basic topological invariant of a holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ is its characteristic class $c(\lambda) \in \mathbb{Z}$, which characterizes the underlying topological line bundle completely. The basic analytic invariant is the dimension of the finite dimensional vector space $\Gamma(M, \mathcal{O}(\lambda))$, denoted by

\begin{equation}
\gamma(\lambda) = \dim \Gamma(M, \mathcal{O}(\lambda)).
\end{equation}

This section will begin the investigation of the relations between these two invariants. For the simplest Riemann surface, the Riemann sphere or equivalently the one-dimensional complex projective space $\mathbb{P}^1$, this relation is somewhat anomalous but can be described completely quite easily. The Riemann sphere $\mathbb{P}^1$ is no doubt quite familiar, since it arises naturally in a number of contexts in almost all discussions of basic complex analysis. It is constructed from two copies $U_0$ and $U_1$ of the complex plane, with the complex coordinates $z_0$ and $z_1$ respectively, by identifying nonzero values $z_0$ and $z_1$ whenever $z_0 = 1/z_1$; thus the two sets $U_0$ and $U_1$ form a coordinate covering of the resulting Riemann surface $\mathbb{P}^1$, and the intersection $U_0 \cap U_1$ consists of all points $z_0 \neq 0$ in the coordinate neighborhood $U_0$ and all points $z_1 \neq 0$ in the coordinate neighborhood $U_1$. The surface $\mathbb{P}^1$ is topologically a two-sphere, the one-point compactification of the complex plane $U_0$ that arises by the addition of the point $z_1 = 0$ to the complex plane $U_0$. A customary notation is to denote points in $U_0$ by the complex variable $z = z_0$ and to denote the point $z_1 = 0$ by $z = \infty$, viewed as the point added to compactify the plane $U_0$.

**Theorem 2.1** For divisors $\mathfrak{d}$ on the Riemann sphere $\mathbb{P}^1$

(i) $\mathfrak{d}$ is a principal divisor if and only if $\deg \mathfrak{d} = 0$;

(ii) $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\deg \mathfrak{d}_1 = \deg \mathfrak{d}_2$; and

(iii) if $\zeta_\mathfrak{d}$ is the line bundle of a divisor $\mathfrak{d}$ of degree $n = \deg \mathfrak{d} \geq 0$ then

\begin{equation}
\gamma(\zeta_\mathfrak{d}) = \deg \mathfrak{d} + 1 = c(\zeta_\mathfrak{d}) + 1
\end{equation}
and the cross-sections in $\Gamma(M, \mathcal{O}(\zeta))$ can be identified with polynomials of degree $n$ in the variable $z_0$.

**Proof:** (i) By definition the principal divisors on $\mathbb{P}^1$ are those divisors that are the divisors of meromorphic functions on $\mathbb{P}^1$. A rational function $f(z_0)$ of the complex variable $z_0$ can be viewed as a meromorphic function on the Riemann surface $\mathbb{P}^1$; $f(z_0)$ is of course a meromorphic function of the variable $z_0$ in the coordinate neighborhood $U_0$, and $f(1/z_1)$ is a meromorphic function of the variable $z_1$ in the coordinate neighborhood $U_1$. If

$$f(z_0) = \prod_i (z_0 - a_i)^{n_i} = \prod_i \left( \frac{1-a_1 z_1}{z_1} \right)^{n_i},$$

where $n_i$ are positive or negative integers then $\mathfrak{d}(f) = \sum_i n_i \cdot p_i - (\sum_i n_i) \cdot \infty$ where $p_i \in U_0$ are the points with coordinates $z_0 = a_i, z_1 = 1/a_i$, so $\mathfrak{d}(f)$ is a divisor of degree zero; any divisor on $\mathbb{P}^1$ of degree zero is of this form, so any divisor of degree zero on $\mathbb{P}^1$ is the divisor of a meromorphic function on $\mathbb{P}^1$. Conversely the divisor of an arbitrary meromorphic function on $\mathbb{P}^1$ is of degree zero by Theorem 1.1, and that suffices to demonstrate (i).

(ii) By definition $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\mathfrak{d}_1 - \mathfrak{d}_2$ is a principal divisor; so since $\deg(\mathfrak{d}_1 - \mathfrak{d}_2) = \deg \mathfrak{d}_1 - \deg \mathfrak{d}_2$ it is evident that (ii) follows from (i).

(iii) It follows from (ii) that if $\mathfrak{d}$ is a divisor of degree $n$ and $p \in \mathbb{P}^1$ is any point in the Riemann sphere then $\mathfrak{d} \sim n \cdot p$. In particular if the point $p$ has local coordinates $z_0 = a, z_1 = 1/a$ for a nonzero complex number $a \in \mathbb{C}$ then the function $f_0(z_0) = z_0 - a$ is holomorphic in $U_0$ with a simple zero at $p$ and at no other point of $U_0$ while the function $f_1(z_1) = 1 - a z_1$ is holomorphic in $U_1$ with a simple zero at $p$ and at no other point of $U_1$; then as in (1.8) the point bundle $\zeta_0$ is described by the coordinate transition functions

$$(2.3) \quad \zeta_{a,01} = \frac{f_0(z_0)}{f_1(z_1)} = \frac{z_0 - a}{1 - a z_1} = z_0$$

in $U_0 \cap U_1$, so the line bundle $\zeta_0$ of the divisor $\mathfrak{d}$ is described by the coordinate transition functions $\zeta_{a,01} = z_0^a$ in $U_0 \cap U_1$. A holomorphic cross-section $f \in \Gamma(\mathbb{P}^1, \mathcal{O}(\zeta))$ is described by an entire function $f_0(z_0)$ of the variable $z_0$ and an entire function $f_1(z_1)$ of the variable $z_1$ such that $f_0(z_0) = z_0^n f_1(1/z_0)$ for $z_0 \neq 0$. The function $f_1(1/z_0)$ is bounded for large values of $z_0$, so the entire function $f_0(z_0)$ is bounded by $C|z_0|^n$ for some constant $C > 0$ for large values of $z_0$; if $n \geq 0$ the function $f_0(z_0)$ consequently must be a polynomial of degree $n$ in the variable $z$. Conversely if $f_0(z_0)$ is a polynomial of degree $n$ in the variable $z_0$ then $f_1(z_1) = z_1^n f_0(1/z_1)$ is a polynomial of degree $n$ in the variable $z_1$, and the entire functions $f_0(z_0)$ and $f_1(z_1)$ describe a holomorphic cross-section of the bundle $\zeta_0$. Since the space of polynomials $f_0(z_0)$ of degree $n$ has dimension $n + 1$ it follows that $\gamma(\zeta_0) = n + 1$, and that suffices to conclude the proof.

**Theorem 2.2** Any holomorphic line bundle over the Riemann sphere $\mathbb{P}^1$ is the line bundle of a divisor on $\mathbb{P}^1$ so has nontrivial meromorphic cross-sections.
Proof: The Riemann sphere is covered by the two coordinate neighborhoods $U_0$ and $U_1$, so alternatively can be viewed as covered by the two discs

$$D_0 = \{ z_0 \in U_0 \mid |z_0| < 2 \} \quad \text{and} \quad D_1 = \{ z_1 \in U_1 \mid |z_1| < 2 \}$$

for which

$$D_0 \cap D_1 = \left\{ z_0 \in U_0 \mid \frac{1}{2} < |z_0| < 2 \right\}.$$ 

Since any holomorphic line bundle is analytically trivial over each disc $D_i$ by Corollary 1.12 it follows that any holomorphic line bundle $\lambda$ over $\mathbb{P}^1$ can be described by a coordinate line bundle $(D_0, \lambda_{0,0})$ in terms of this covering, where the coordinate transition function $\lambda_{01}$ in the intersection $D_0 \cap D_1$ is a holomorphic and nowhere vanishing function of the complex variable $z_0$ in the annulus $1/2 < |z_0| < 2$. A local branch of the holomorphic function $\log \lambda_{01}(z_0)$ near the point $z_0 = 1$ can be continued analytically once around the origin in this annulus, and upon this continuation its value will increase by $2\pi i n$ for some integer $n$; hence $f(z_0) = \log (z_0^n \lambda_{01}(z_0))$ is a single-valued holomorphic function in the annulus, and $\lambda_{01}(z_0) = z_0^n \exp f(z_0)$. By using the Cauchy integral formula the function $f(z_0)$ can be represented as usual as the difference $f(z_0) = f_0(z_0) - f_1(z_1)$ of a holomorphic function $f_0(z_0)$ in the disc $D_0$ and a holomorphic function in the exterior of the circle $|z_0| = 1/2$, where the latter function can be viewed equivalently as a holomorphic function $f_1(z_1)$ in the disc $D_1$. The exponentials $h_j(z_j) = \exp f_j(z_j)$ are holomorphic and nowhere vanishing functions in the discs $D_j$, and $\lambda_{01}(z_0) = z_0^n h_0(z_0)/h_1(z_1)$ in the intersection $D_1 \cap D_2 \subset \mathbb{P}^1$; hence the holomorphic line bundle $\lambda$ is analytically equivalent to the holomorphic line bundle defined by the coordinate transition function $z_0^n$, and it is evident from (2.3) that this is the line bundle of a divisor of degree $n$. That suffices to conclude the proof.

The two preceding theorems provide a complete characterization of holomorphic line bundles over the Riemann sphere $\mathbb{P}^1$ and a description of their properties.

Corollary 2.3 There is a unique holomorphic line bundle $\zeta$ of characteristic class $c(\zeta) = 1$ on $\mathbb{P}^1$, and $\zeta_p = \zeta$ for any point $p \in M$. For any integer $n$ the line bundle $\zeta^n$ is the unique holomorphic line bundle of characteristic class $c(\zeta^n) = n$ on $\mathbb{P}^1$, and $\gamma(\zeta^n) = \max(n + 1, 0)$.

Proof: The point bundle $\zeta_p$ for any point $p \in \mathbb{P}^1$ has characteristic class $c(\zeta_p) = 1$ by (1.10), so $c(\zeta^n_p) = n$ for any integer $n$. Any holomorphic line bundle $\lambda$ over $\mathbb{P}^1$ of characteristic class $c(\lambda) = n$ is the line bundle of a divisor of degree $n$ by Theorem 2.2, so $\lambda$ is linearly equivalent to the line bundle $\zeta^n_p$ by Theorem 2.1 (ii); and $\gamma(\lambda) = 0$ if $n < 0$ by Corollary 1.4 while $\gamma(\lambda) = n + 1$ if $n \geq 0$ by Theorem 2.1 (iii). That suffices for the proof.

Although there is not an equally simple description of all holomorphic line bundles over more general Riemann surfaces in terms of point bundles, nonetheless point bundles play a significant role in the study of holomorphic line bundles over arbitrary compact Riemann surfaces.
CHAPTER 2. RIEMANN-ROCH THEOREM

Theorem 2.4 The point bundles over an arbitrary compact Riemann surface $M$ can be characterized as those holomorphic line bundles $\lambda$ over $M$ such that

(2.4) \[ c(\lambda) = 1 \quad \text{and} \quad \gamma(\lambda) > 0; \]

and if $\zeta_p$ is a point bundle over $M$ then

(2.5) \[ \gamma(\zeta_p) = \begin{cases} 2 & \text{if } M = \mathbb{P}^1, \\ 1 & \text{if } M \neq \mathbb{P}^1. \end{cases} \]

Proof: If $\zeta_p$ is a point bundle over $M$ then $c(\zeta_p) = 1$ by (1.10); and since there is a nontrivial holomorphic cross-section of $\zeta_p$ necessarily $\gamma(\zeta_p) > 0$. Conversely if $\lambda$ is a holomorphic line bundle over $M$ for which $\gamma(\lambda) > 0$ then $\lambda = \zeta_d$ is the line bundle of some positive divisor $d$ on $M$; and if $c(\zeta_d) = 1$ then $\deg d = 1$ by (1.10) so $d = 1 \cdot p$ for some point $p \in M$. If $\gamma(\zeta_d) > 1$ for a point bundle $\zeta_d$ over $M$ choose two linearly independent holomorphic cross-sections $f_1, f_2 \in \Gamma(M, \mathcal{O}(\zeta_p))$ and let their divisors be $d(f_1) = 1 \cdot p_1$ and $d(f_2) = 1 \cdot p_2$. If $p_1 = p_2$ the quotient $f = f_1/f_2$ is a function that is holomorphic everywhere on the compact Riemann surface $M$, so by the maximum modulus theorem it must be a constant; but that contradicts the assumption that the two functions are linearly independent. Thus $p_1 \neq p_2$, and the quotient $f = f_1/f_2$ is a nonconstant meromorphic function on $M$ with the divisor $d(f) = 1 \cdot p_1 - 1 \cdot p_2$. This function can be viewed as a holomorphic mapping from the Riemann surface $M$ to the Riemann sphere $\mathbb{P}^1$ in the obvious manner: near any regular point the function $f$ takes finite values in the coordinate neighborhood $U_0 \subset \mathbb{P}^1$, while near its pole the function $1/f$ takes finite values in the coordinate neighborhood $U_1 \subset \mathbb{P}^1$. This mapping takes the single point $p_1$ to the origin in the coordinate neighborhood $U_0$, and the single point $p_2$ to the point $\infty$ in the coordinate neighborhood $U_0$, or equivalently to the origin in the coordinate neighborhood $U_1$. For any complex number $c$ the function $f(z_0) - c$ also has a single simple pole, so it must have a single simple zero; thus the function $f$ itself takes a single point of $M$ to the complex value $c$. Altogether the function $f$ describes a one-to-one holomorphic mapping from $M$ to $\mathbb{P}^1$, so $M$ is analytically equivalent to $\mathbb{P}^1$. Then $\gamma(\zeta_d) = 2$ by Corollary 2.3, and that suffices to conclude the proof of the theorem.

The preceding theorem provides a characterization of the Riemann sphere among all compact Riemann surfaces and also yields the following properties of point bundles over compact Riemann surfaces other than the Riemann sphere.

Corollary 2.5 If $M$ is a compact Riemann surface other than the Riemann sphere $\mathbb{P}^1$ the point bundles $\zeta_p$ for distinct points $p \in M$ are analytically inequivalent.

Proof: If $\zeta_p = \zeta_q$ for two distinct points $p, q \in M$ then this bundle has one cross-section with a simple zero at $p$ and no other points and another cross-section with a simple zero at $q$ and no other points; but then $\gamma(\zeta) > 1$ so by the preceding theorem $M$ is the Riemann sphere, which suffices for the proof.
Thus if $M$ is a compact Riemann surface other than the Riemann sphere $\mathbb{P}^1$ the mapping that associates to each point $p \in M$ the point bundle $\zeta_p$ is a one-to-one correspondence between points of $M$ and holomorphic line bundles $\lambda$ over $M$ with $c(\lambda) = 1$ and $\gamma(\lambda) > 0$. Point bundles often are used in the study of other holomorphic line bundles over compact Riemann surfaces through an application of the following observation.

**Lemma 2.6** If $\zeta_p$ is a point bundle and $\lambda$ is any other holomorphic line bundle on a compact Riemann surface $M$

\begin{equation}
\gamma(\lambda) \leq \gamma(\lambda \zeta_p) \leq \gamma(\lambda) + 1;
\end{equation}

and $\gamma(\lambda \zeta_p) = \gamma(\lambda)$ if and only if all holomorphic cross-sections of the bundle $\lambda \zeta_p$ vanish at the point $p$.

**Proof:** If $h \in \Gamma(M, \mathcal{O}(\zeta_p))$ is a nontrivial holomorphic cross-section of the point bundle $\zeta_p$ multiplication by $h$ clearly yields an injective homomorphism

$$xh : \Gamma(M, \mathcal{O}(\lambda)) \longrightarrow \Gamma(M, \mathcal{O}(\lambda \zeta_p)),$$

the image of which consists precisely of all holomorphic cross-sections of the bundle $\lambda \zeta_p$ that vanish at the point $p$ since $h(p) = 0$; in particular therefore $\gamma(\lambda) \leq \gamma(\lambda \zeta_p)$. If $\gamma(\lambda \zeta_p) = 0$ then $\gamma(\lambda) = 0$ and the asserted result holds trivially. Otherwise choose a basis $f_1, \ldots, f_n \in \Gamma(M, \mathcal{O}(\lambda \zeta_p))$ where $n = \gamma(\lambda \zeta_p)$. If $f_i(p) = 0$ for all of these cross-sections then the mapping $xh$ is surjective and $\gamma(\lambda) = \gamma(\lambda \zeta_p)$. If for instance $f_1(p) \neq 0$ then the mapping $xh$ is not surjective, so $\gamma(\lambda) \leq \gamma(\lambda \zeta_p) - 1$; the differences $g_i(z) = f_i(z) - (f_i(p)/f_1(p))f_1(z)$ for $2 \leq i \leq n$ are $n-1$ linearly independent holomorphic cross-sections of the bundle $\lambda \zeta_p$ that vanish at the point $p$ so are the images under the injective homomorphism $xh$ of $n-1$ linearly independent holomorphic cross-sections of $\lambda$, and consequently $\gamma(\lambda) \geq n-1 = \gamma(\lambda \zeta_p) - 1$. That suffices to conclude the proof.

An application of this auxiliary result yields a preliminary upper bound for the dimension $\gamma(\lambda)$ in terms of the characteristic class $c(\lambda)$ of a holomorphic line bundle.

**Theorem 2.7** If $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ and if $c(\lambda) > 0$ then $\gamma(\lambda) \leq c(\lambda) + 1$. If the equality holds for any holomorphic line bundle over $M$ then $M = \mathbb{P}^1$.

**Proof:** The result will be demonstrated by induction on the integer $n = c(\lambda)$. The case $n = 1$ follows immediately from Theorem 2.4. Assume therefore that the result has been demonstrated for all integers strictly less than $n$, and consider a holomorphic line bundle $\lambda$ for which $c(\lambda) = n > 1$. If $\gamma(\lambda) \geq n + 1$ then from the preceding lemma it follows that $\gamma(\lambda \zeta_p^{-1}) \geq \gamma(\lambda) - 1 \geq n$ for a point bundle $\zeta_p$; but $c(\lambda \zeta_p^{-1}) = n-1$ so from the induction hypothesis it follows that $M = \mathbb{P}^1$, and in that case $\gamma(\lambda) = n + 1$ by Corollary 2.3. Thus $\gamma(\lambda) \leq n + 1$ and equality holds only when $M = \mathbb{P}^1$, which concludes the induction step and the proof.
which divisor is the trivial divisor. The base divisor of cross-sections; the cross-sections have no common zeros precisely when this

\[(2.7)\]

\[d(f_1, f_2, \ldots)(a) = \inf ord_a(f_i).\]

The degree of this divisor is the number of common zeros of the collection of cross-sections; the cross-sections have no common zeros precisely when this divisor is the trivial divisor. The base divisor of a holomorphic line bundle \(\lambda\) for which \(\gamma(\lambda) > 0\) is the positive divisor \(b(\lambda)\) defined by

\[(2.8)\]

\[b(\lambda)(a) = \inf \left\{ \text{ord}_a(f) \left| f \in \Gamma(M, \mathcal{O}(\lambda)), \ f \neq 0 \right. \right\},\]

or equivalently it is the divisor of common zeros of the set of all nontrivial holomorphic cross-sections of the line bundle \(\lambda\); of course if \(\gamma(\lambda) = 0\) the only holomorphic cross-section of \(\lambda\) is that which vanishes identically, so the base divisor \(b(\lambda)\) is undefined. If \(b(\lambda) = \sum_i \nu_i \cdot a_i\) then all holomorphic cross-sections of the bundle \(\lambda\) vanish at the point \(a_i\) to order at least \(\nu_i\), and there are cross-sections that vanish at the point \(a_i\) to order exactly \(\nu_i\). The points that appear in the base divisor \(b(\lambda)\) with strictly positive coefficients are called the base points of the line bundle \(\lambda\); thus a point \(a \in M\) is a base point of a line bundle \(\lambda\) if and only if \(\gamma(\lambda) > 0\) and all holomorphic cross-sections of \(\lambda\) vanish at the point \(a\). If \(d \geq 0\) is a positive divisor and \(\gamma(\zeta_0) > 0\) clearly \(d = b(\zeta_0) + d'\) for another positive divisor \(d' \geq 0\), since there is a holomorphic cross-section of the line bundle \(\zeta_0\) that vanishes at the divisor \(d\); in particular if \(a\) is a base point of the line bundle \(\zeta_0\) of a divisor \(d \geq 0\) for which \(\gamma(\zeta_0) > 0\) then the point \(a\) must appear in the divisor \(d\). A holomorphic line bundle \(\lambda\) is base-point-free if \(\gamma(\lambda) > 0\) and \(b(\lambda) = 0\), or equivalently if \(\gamma(\lambda) > 0\) and for any point of \(M\) there exists a holomorphic cross-section of \(\lambda\) that is nonzero at that point. In particular the trivial line bundle \(\lambda = 1\) is base-point-free, since its holomorphic cross-sections are complex constants; indeed it is the only line bundle \(\lambda\) with \(c(\lambda) = 0\) that is base-point-free, since \(\gamma(\lambda) = 0\) for any other line bundle \(\lambda \neq 1\) with \(c(\lambda) = 0\). The set of base-point-free holomorphic line bundles on a compact Riemann surface \(M\) is denoted by \(\mathcal{B}(M)\), or simply by \(\mathcal{B}\) if it is either clear from context or irrelevant just which Riemann surface \(M\) is being considered.

**Theorem 2.8** If \(\lambda\) and \(\sigma\) are base-point-free holomorphic line bundles on a compact Riemann surface \(M\) their product \(\lambda \sigma\) also is base-point-free.

**Proof:** If \(\lambda, \sigma\) are base-point-free line bundles then for any point \(a \in M\) there are cross-sections \(f \in \Gamma(M, \mathcal{O}(\lambda))\) and \(g \in \Gamma(M, \mathcal{O}(\sigma))\) such that \(f(a) \neq 0\) and
$g(a) \neq 0$; and then the cross-section $fg \in \Gamma(M, \mathcal{O}(\lambda \sigma))$ has the property that $f(a)g(a) \neq 0$, so the product bundle also is base-point-free. That suffices for the proof.

The set $\mathcal{B}(M)$ of base-point-free holomorphic line bundles on a compact Riemann surface thus is closed under multiplication of line bundles, so sometimes it is called the \textit{semigroup of base-point-free holomorphic line bundles} on $M$. As a consequence the characteristic classes $c_i$ of the base-point-free holomorphic line bundles $\lambda \in \mathcal{B}(M)$ form a semigroup of nonnegative integers, called the \textit{Lüroth semigroup} of the surface $M$ and denoted by $\mathcal{L}(M)$. For some purposes it is convenient to use one or the other of the following two alternative characterizations of base-point-free holomorphic line bundles.

**Lemma 2.9** On a compact Riemann surface $M$ a holomorphic line bundle $\lambda$ is base-point-free if and only if it has two holomorphic cross-sections with no common zeros on $M$.

**Proof:** If $\lambda$ has two holomorphic cross-sections with no common zeros then of course it is base-point-free. Conversely suppose that $\lambda$ is base-point-free and let $f_i \in \Gamma(M, \mathcal{O}(\lambda))$ be a basis for the space of holomorphic cross-sections for $0 \leq i \leq n$. If $n = 0$ the cross-section $f_0$ must have no zeros, so it and the zero cross-section have no common zeros. If $n = 1$ the two cross-sections $f_0$ and $f_1$ must have no common zeros. If $n \geq 2$ consider the divisor $D(f_0) = \sum_j a_j \cdot a_j$ of the first of these cross-sections. A cross-section $f = \sum_{i=1}^n x_i f_i$ vanishes at the point $a_1$ if and only if the coefficients $x_i$ satisfy the linear equation $\sum_{i=1}^n x_i f_i(a_1) = 0$. Since $f_i(a_1) \neq 0$ for at least one index $i \geq 1$ and the equation involves at least two variables the set of solutions is a proper linear subspace of the space $\mathbb{C}^n = \{ (x_1, \ldots, x_n) \}$ of all the coefficients $x_i$; so its complement, the set of coefficients for which $f(a_1) \neq 0$, is a dense open subset of $\mathbb{C}^n$. The same is true for each of the points $a_j$, and since the intersection of finitely many dense open subsets of $\mathbb{C}^n$ is again a dense open subset there must exist coefficients $(x_1, \ldots, x_n)$ describing a holomorphic cross-section $f$ that does not vanish at any of the points $a_j$; this cross-section and the cross-section $f_0$ thus have no common zeros, and that suffices for the proof.

**Lemma 2.10** On a compact Riemann surface $M$ a holomorphic line bundle $\lambda$ is base-point-free if and only if $\gamma(\lambda \zeta_a^{-1}) = \gamma(\lambda) - 1$ for all points $a \in M$; equivalently a holomorphic line bundle $\lambda$ is not base-point-free if and only if there is a point $a \in M$ for which $\gamma(\lambda \zeta_a^{-1}) = \gamma(\lambda)$, and the point $a$ is then a base point of the bundle $\lambda$ if $\gamma(\lambda) > 0$.

**Proof:** Note first that if $\gamma(\lambda \zeta_a^{-1}) = \gamma(\lambda) - 1$ then $\gamma(\lambda) > 0$. Lemma 2.6 asserts that $\gamma(\lambda \zeta_a^{-1}) \leq \gamma(\lambda) \leq \gamma(\lambda \zeta_a^{-1}) + 1$ for any line bundle $\lambda$ and any point $a \in M$, and that $\gamma(\lambda \zeta_a^{-1}) = \gamma(\lambda)$ if and only if all holomorphic cross-sections of the bundle $\lambda$ vanish at the point $a$; the present lemma is an obvious consequence, and that suffices for the proof.
Although not all holomorphic line bundles are base-point-free, any line bundle \( \lambda \) for which \( \gamma(\lambda) > 0 \) can be described by its base divisor and an associated base-point-free holomorphic line bundle as follows.

**Theorem 2.11** On a compact Riemann surface \( M \) a holomorphic line bundle \( \lambda \) with \( \gamma(\lambda) > 0 \) is uniquely expressible as the product \( \lambda = \lambda_0 \zeta_\delta(\lambda) \) of a base-point-free line bundle \( \lambda_0 \) and the line bundle \( \zeta_\delta(\lambda) \) of the base divisor \( \delta(\lambda) \) of \( \lambda \), and \( \gamma(\zeta_\delta(\lambda)) = 1 \). For any nontrivial holomorphic cross-section \( h \in \Gamma(M, \mathcal{O}(\zeta_\delta(\lambda))) \) multiplication by \( h \) is an injective linear homomorphism of the form (2.9). If \( \gamma(\lambda) > 0 \) set \( \lambda_0 = \lambda \zeta_\delta^{-1}(\lambda) \) where \( \delta(\lambda) \) is the base divisor of \( \lambda \); the base divisor \( \delta(\lambda) \) of course is determined uniquely by the line bundle \( \lambda \), hence so are the line bundles \( \zeta_\delta(\lambda) \) and \( \lambda_0 \). Multiplication by any nontrivial holomorphic cross-section \( h \in \Gamma(M, \mathcal{O}(\zeta_\delta(\lambda))) \) is an injective linear homomorphism of the form (2.9). If \( \delta(h) = \delta(\lambda) \) then it is clear from the definition of the base divisor that for any cross-section \( f \in \Gamma(M, \mathcal{O}(\lambda)) \) the quotient \( f / h \) is everywhere holomorphic, hence is a cross-section \( f_0 = f / h \in \Gamma(M, \mathcal{O}(\lambda_0)) \) for which \( (\times h)(f_0) = f \); consequently the homomorphism \( \times h \) also is surjective, hence is an isomorphism. Therefore if \( f_i \in \Gamma(M, \mathcal{O}(\lambda_0)) \) is a basis for the space of holomorphic cross-sections of the line bundle \( \lambda_0 \) then \( h f_i \in \Gamma(M, \mathcal{O}(\lambda)) \) is a basis for the space of holomorphic cross-sections of the line bundle \( \lambda = \lambda_0 \zeta_\delta(\lambda) \); hence \( \delta(\lambda_0) \) is the divisor of common zeros of the cross-sections \( f_i \) while \( \delta(\lambda) \) is the divisor of common zeros of the cross-sections \( h f_i \) so \( \delta(\lambda) = \delta(\lambda) + \delta(\lambda_0) \) and consequently \( \delta(\lambda_0) = \emptyset \) so \( \lambda_0 \) is base-point-free. Finally if \( g \in \Gamma(M, \mathcal{O}(\zeta_\delta(\lambda))) \) is an arbitrary nontrivial holomorphic cross-section, not necessarily vanishing at the base divisor \( \delta(\lambda) \), nonetheless the corresponding homomorphism \( \times g \) also is an isomorphism, since it is injective and the two vector spaces of cross-sections have the same dimension; but then \( \delta(\lambda) = \delta(g) = \delta(h) \) so \( g \) is necessarily a constant multiple of \( h \), hence \( \gamma(\zeta_\delta(\lambda)) = 1 \). That suffices to conclude the proof.

This expression of a holomorphic line bundle \( \lambda \) for which \( \gamma(\lambda) > 0 \) as the product of a base-point-free line bundle \( \lambda_0 \) and the line bundle \( \zeta_\delta(\lambda) \) associated to the base divisor \( \delta(\lambda) \) is called the base decomposition of the line bundle \( \lambda \). As an illustrative example of the base decomposition of a line bundle, if \( \gamma(\lambda) = 1 \) then \( \lambda \) is the line bundle of a unique positive divisor \( \delta \) and \( \lambda \) has the base decomposition as the product \( \lambda = 1 \cdot \zeta_\delta \) of the base-point-free identity line bundle \( 1 \) and the line bundle \( \zeta_\delta \) of the base divisor \( \delta = \delta(\lambda) \). To each base-point-free line bundle \( \lambda_0 \) there can be associated the set of holomorphic line bundles with base decomposition \( \lambda = \lambda_0 \zeta_\delta \), parametrized by the appropriate positive divisors \( \delta \); alternatively to each positive divisor \( \delta \) for which \( \gamma(\zeta_\delta) = 1 \) there can be associated the set of holomorphic line bundles \( \lambda \) with the base decomposition \( \lambda = \lambda_0 \zeta_\delta \), parametrized by the appropriate set of base-point-free line bundles...
Theorem 2.12  (i) If $\lambda_0$ is a base-point-free holomorphic line bundle on a compact Riemann surface $M$ and $b \geq 0$ is a positive divisor on $M$ the product $\lambda = \lambda_0 \xi^b$ is the base decomposition of the line bundle $\lambda$ if and only if $\gamma(\lambda) = \gamma(\lambda_0)$.

(ii) If $\lambda_0$ is a base-point-free holomorphic line bundle on $M$ then for any point $a \in M$ either $\gamma(\lambda_0 \xi^a) = \gamma(\lambda_0)$, in which case $\lambda_0 \xi^a$ is the base decomposition of this product, or $\gamma(\lambda_0 \xi^a) = \gamma(\lambda_0) + 1$, in which case $\lambda_0 \xi^a$ is base-point-free.

(iii) If $\lambda = \lambda_0 \xi^b$ is the base decomposition of $\lambda$ for a base-point-free line bundle $\lambda_0$ over $M$ then for any point $a \in M$ either $\gamma(\lambda_0 \xi^a) = \gamma(\lambda)$, in which case $\lambda_0 \xi^{a+\lambda}$ is the base decomposition of the line bundle $\lambda_0 \xi^a$, or $\gamma(\lambda_0 \xi^a) = \gamma(\lambda) + 1$, in which case the divisor $b$ can be written uniquely as the sum $b = b' + b''$ of two positive divisors, one of which may be trivial, where $\lambda_0 = \lambda_0 \xi^{b'} + a$ is base-point-free and $\lambda_0 \xi^a = \lambda_0 \xi^{b''}$ is the base decomposition of the line bundle $\lambda_0 \xi^a$.

Proof: (i) If $\lambda = \lambda_0 \xi^b$ is the base decomposition of $\lambda$ then $\gamma(\lambda) = \gamma(\lambda_0)$ by the preceding theorem. Conversely if $\gamma(\lambda_0 \xi^b) = \gamma(\lambda_0)$ for a base-point-free line bundle $\lambda_0$ and a divisor $b \geq 0$, and if $h \in \Gamma(M, \mathcal{O}(\xi^b))$ is a nontrivial holomorphic cross-section for which $\delta(h) = b$, multiplication by $h$ is an injective homomorphism

$$\times h : \Gamma(M, \mathcal{O}(\lambda_0)) \longrightarrow \Gamma(M, \mathcal{O}(\lambda_0 \xi^b))$$

which also must be surjective since the two spaces of holomorphic cross-sections have the same dimension; consequently all holomorphic cross-sections of the line bundle $\lambda_0 \xi^b$ are multiples of $h$ so must vanish at the divisor $b$. If all of these cross-sections vanish at a divisor $b+a$ where $a \geq 0$ is a nontrivial positive divisor then all of the holomorphic cross-sections of the line bundle $\lambda_0$ also vanish at $a$, which is impossible since the line bundle $\lambda_0$ is base-point-free. Thus $b$ is the base divisor of the line bundle $\lambda_0 \xi^b$, and hence this is the base decomposition of the bundle $\lambda = \lambda_0 \xi^b$.

(ii) By part (i) if $\gamma(\lambda_0 \xi^a) = \gamma(\lambda_0)$ then $\lambda_0 \xi^a$ is the base decomposition of the product $\lambda_0 \xi^a$. If $\gamma(\lambda_0 \xi^a) > \gamma(\lambda_0)$ then $\gamma(\lambda_0 \xi^a) = \gamma(\lambda_0) + 1$ by Lemma 2.6; so if $f_i \in \Gamma(M, \mathcal{O}(\lambda_0))$ is a basis for the space of holomorphic cross-sections of the bundle $\lambda_0$ and $h \in \Gamma(M, \mathcal{O}(\xi^a))$ is a nontrivial holomorphic cross-section of the point bundle $\xi^a$ then a basis for the holomorphic cross-sections of the product bundle $\lambda_0 \xi^a$ consists of the products $hf_i$ together with one additional cross-section $g$. If these cross-sections have a common zero then it must be the point $a$ alone, for that is the only common zero of the products $hf_i$ since the cross-sections $f_i$ have no common zeros; but then $g$ also must vanish at $a$, and that is a contradiction since if all the cross-sections of the product $\lambda_0 \xi^a$ vanish at $a$ then $\gamma(\lambda_0 \xi^a) = \gamma(\lambda_0)$ by Lemma 2.6.

(iii) If $\lambda = \lambda_0 \xi^b$ is the base decomposition of $\lambda$ then $\gamma(\lambda) = \gamma(\lambda_0)$ by the
preceding theorem. Again by part (i) if $\gamma(\lambda\zeta_a) = \gamma(\lambda) = \gamma(\lambda_0)$ then $\lambda\zeta_a = \lambda_0\zeta_{b+a}$ is the base decomposition of the product $\lambda\zeta_a$. If $\gamma(\lambda\zeta_a) > \gamma(\lambda)$ then $\gamma(\lambda\zeta_a) = \gamma(\lambda) + 1$ by Lemma 2.6; so if $f_i \in \Gamma(M, \mathcal{O}(\lambda_0))$ is a basis for the space of holomorphic cross-sections of the line bundle $\lambda_0$ and $h_b \in \Gamma(M, \mathcal{O}(\zeta_b))$ and $h_a \in \Gamma(M, \mathcal{O}(\zeta_a))$ are nontrivial holomorphic cross-sections then the products $f_i h_b h_a$, together with one other cross-section $g$ are a basis for $\Gamma(M, \mathcal{O}(\lambda\zeta_a))$. Not all of these cross-sections vanish at the point $a$, since otherwise it would follows from Lemma 2.6 as before that $\gamma(\lambda\zeta_a) = \gamma(\lambda)$. If all these cross-sections vanish at a point $b \neq a$ then since the cross-sections $f_i$ have no common zeros it must be the case that $h_b(b) = 0$; hence if $b''$ is the base divisor of the line bundle $\lambda\zeta_a$ then $b''$ must be part of the divisor $b$, so $b = b' + b''$ for another divisor $b' \geq 0$. Thus the base decomposition of the line bundle $\lambda\zeta_a$ has the form $\lambda\zeta_a = \lambda_0\zeta_{b''}$ for a base-point-free line bundle $\lambda_0$, and hence $\lambda_0 = \lambda_0\zeta_{b''+a}$. That suffices to conclude the proof.

To examine further relations between the dimension $\gamma(\lambda)$ of the space of holomorphic cross-sections of a general holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ and the characteristic class $c(\lambda)$ of that line bundle it is convenient to introduce the expression

$$\chi(\lambda) = \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)),$$

a finite integer as a consequence of the Finite Dimensionality Theorem, Theorem 1.14; this is called the **Euler characteristic** of the line bundle $\lambda$.

**Lemma 2.13** If $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ then

$$\chi(\lambda\zeta_p) - c(\lambda\zeta_p) = \chi(\lambda) - c(\lambda)$$

for any point bundle $\zeta_p$.

**Proof:** If $h \in \Gamma(M, \mathcal{O}(\zeta_p))$ is a nontrivial holomorphic cross-section multiplication by $h$ is an injective sheaf homomorphism $\times h : \mathcal{O}(\lambda) \rightarrow \mathcal{O}(\lambda\zeta_p)$; this leads to the exact exact sequence of sheaves

$$\begin{array}{cccc}
0 & \rightarrow & \mathcal{O}(\lambda) & \times h \\
& & \mathcal{O}(\lambda\zeta_p) & \rightarrow \mathcal{S} & \rightarrow 0,
\end{array}$$

in which $\mathcal{S}$ is the quotient sheaf. Since the cross-section $h$ has a simple zero at the point $p$ and is otherwise nonvanishing it follows that at any point $a \in M$ other than $p$ the homomorphism $\times h$ is an isomorphism, and consequently $\mathcal{S}_a = 0$. On the other hand if $z$ is a local coordinate centered at the point $p$ the elements in the stalk $\mathcal{O}_p(\lambda\zeta_p)$ can be identified with germs of holomorphic functions of the variable $z$ at the origin, and the functions that are in the image of the homomorphism $\times h$ are those functions that vanish at the point $p$. Therefore associating to the germ $f$ of a holomorphic function representing an element in the stalk $\mathcal{O}_p(\lambda\zeta_p)$ the value $f(p)$ is a mapping $\mathcal{O}_p(\lambda\zeta_p) \rightarrow \mathbb{C}$ with kernel the image of multiplication by $h$; and that in turn yields an identification $\mathcal{S}_p \cong \mathbb{C}$. 

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**CHAPTER 2. RIEMANN-ROCH THEOREM**
The precise identification depends of course on the choice of a local coordinate \( z \) in terms of which the bundle \( \lambda \) is trivialized, but is really not needed at all; more important are first that \( H^0(M, S) = \Gamma(M, S) \cong \mathbb{C} \) and second that the sheaf \( S \) is a fine sheaf, since it is nontrivial at just a single point of \( M \), so \( H^1(M, S) = 0 \). Therefore the exact cohomology sequence associated to the exact sequence of sheaves (2.12) begins

\[
0 \rightarrow H^0(M, \mathcal{O}(\lambda)) \xrightarrow{\times h} H^0(M, \mathcal{O}(\lambda \zeta_p^n)) \rightarrow \mathbb{C} \rightarrow \rightarrow H^1(M, \mathcal{O}(\lambda)) \xrightarrow{\times h} H^1(M, \mathcal{O}(\lambda \zeta_p^n)) \rightarrow 0.
\]

The alternating sum of the dimensions of the spaces in an exact sequence of vector spaces such as this is zero, as can be seen most simply by decomposing the exact sequence into a collection of short exact sequences; consequently

\[
0 = \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^0(M, \mathcal{O}(\lambda \zeta_p^n)) + 1
- \dim H^1(M, \mathcal{O}(\lambda)) + \dim H^1(M, \mathcal{O}(\lambda \zeta_p^n))
= \chi(\lambda) - \chi(\lambda \zeta_p) + 1.
\]

Since \( c(\lambda \zeta_p^n) = c(\lambda) + n \) this yields the desired result, thereby concluding the proof.

An immediate consequence of this lemma is the fundamental existence theorem for compact Riemann surfaces.

**Theorem 2.14 (Existence Theorem)** A holomorphic line bundle on a compact Riemann surface \( M \) has nontrivial meromorphic cross-sections; indeed for any choice of a base point \( p \in M \) any holomorphic line bundle \( \lambda \) over \( M \) has nontrivial meromorphic cross-sections with poles at most at the point \( p \in M \).

**Proof:** Iterating the preceding lemma yields the result that

\[
\chi(\lambda \zeta_p^n) - c(\lambda \zeta_p^n) = \chi(\lambda) - c(\lambda)
\]

for any integer \( n \), or explicitly

\[
\dim H^0(M, \mathcal{O}(\lambda \zeta_p^n)) - \dim H^1(M, \mathcal{O}(\lambda \zeta_p^n)) - c(\lambda) - n
= \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)) - c(\lambda)
\]

since \( c(\lambda \zeta_p^n) = c(\lambda) + n \); and hence

\[
\dim H^0(M, \mathcal{O}(\lambda \zeta_p^n)) = n + \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda))
+ \dim H^1(M, \mathcal{O}(\lambda \zeta_p^n))
\geq n + \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)).
\]

It is evident from this that \( \dim \Gamma(M, \mathcal{O}(\lambda \zeta_p^n)) = \dim H^0(M, \mathcal{O}(\lambda \zeta_p^n)) > 0 \) for sufficiently large \( n > 0 \), so there is a nontrivial holomorphic cross-section \( f \) of the
bundle $\lambda \zeta^n_p$ for some $n > 0$; and if $h \in \Gamma(M, \mathcal{O}(\zeta_p))$ is a nontrivial holomorphic section of the point bundle $\zeta_p$ the quotient $f/h^n$ is a nontrivial meromorphic cross-section of the bundle $\lambda$ with poles at most at the point $p \in M$, which concludes the proof.

**Corollary 2.15** Any holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ is the line bundle of a divisor on $M$, so there is the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}^*) \overset{i}{\longrightarrow} \Gamma(M, \mathcal{M}^*) \overset{\partial}{\longrightarrow} \Gamma(M, \mathcal{D}) \overset{\delta}{\longrightarrow} H^1(M, \mathcal{O}^*) \longrightarrow 0.$$  

**Proof:** By the Existence Theorem any line bundle $\lambda$ has a nontrivial meromorphic cross-section, and then $\lambda$ is the line bundle of the divisor of this cross-section. That means that the coboundary mapping in the exact cohomology sequence (1.5) is surjective, so that sequence reduces to the exact sequence of the present corollary, which suffices for the proof.

The exactness of the cohomology sequence of the preceding corollary was demonstrated by showing that the coboundary homomorphism in the exact sequence (1.5) is surjective; and since $\mathcal{D}$ is a fine sheaf $H^1(M, \mathcal{D}) = 0$, so it follows further from (1.5) that

$$(2.14) \quad H^1(M, \mathcal{M}^*) = 0 \quad \text{on any compact Riemann surface } M.$$  

A related result worth including here is the following.

**Theorem 2.16** If $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ then $H^1(M, \mathcal{M}(\lambda)) = 0$.

**Proof:** If $f_{\alpha\beta} \in Z^1(U, \mathcal{M}(\lambda))$ is a cocycle in a coordinate covering $\mathcal{U}$ of the Riemann surface $M$ choose a divisor $\partial$ on $M$ such that $\deg \partial > 2g - 2 - c(\lambda)$ and $\partial + \partial(f_{\alpha\beta}) \geq 0$ and let $h \in \Gamma(M, \mathcal{O}(\zeta_{\partial}))$ be a holomorphic cross-section with $\partial(h) = \partial$. Since $f_{\alpha\gamma} = \lambda_{\alpha\beta}f_{\alpha\beta} + f_{\beta\gamma}$ in $U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$ by (1.36) while $h_{\gamma} = \zeta_{\alpha,\gamma\beta}h_{\beta}$ in $U_{\beta} \cap U_{\gamma}$ it follows that $f_{\alpha\gamma}h_{\gamma} = \zeta_{\alpha,\gamma\beta}\lambda_{\gamma\beta}f_{\alpha\beta}h_{\beta} + f_{\beta\gamma}h_{\gamma}$, which is just the condition that the holomorphic functions $f_{\alpha\beta}h_{\beta}$ describe a cocycle in $Z^1(\mathcal{U}, \mathcal{O}(\zeta_{\partial})).$ By Corollary 1.19 to the Serre Duality Theorem $\dim H^1(M, \mathcal{O}(\zeta_{\partial}\lambda)) = \dim \Gamma(M, \mathcal{O}(\zeta_{\partial})\mathcal{O}(\zeta_{\partial}^{-1}\lambda)\mathcal{O}(\zeta_{\partial})^{-1})$ and $\dim \Gamma(M, \mathcal{O}(\zeta_{\partial})\mathcal{O}(\zeta_{\partial}^{-1}\lambda)\mathcal{O}(\zeta_{\partial})^{-1}) = \dim \Gamma(M, \mathcal{O}(\kappa_{\partial}\lambda^{-1})) = 0$ since $c(\kappa_{\partial}^{-1}\lambda^{-1}) = 2g - 2 - \deg \partial - c(\lambda) < 0$; consequently after passing to a refinement of the covering if necessary there will be holomorphic functions $g_{\alpha}$ in the sets $U_{\alpha}$ such that $f_{\alpha\beta}h_{\beta} = g_{\beta} - \lambda_{\beta\alpha}g_{\alpha}$ in the intersections $U_{\alpha} \cap U_{\beta}$. The quotients $f_{\alpha} = g_{\alpha}/h_{\alpha}$ are then meromorphic functions in the sets $U_{\alpha}$ such that $f_{\alpha\beta} = f_{\beta} - \lambda_{\beta\alpha}f_{\alpha}$ in the intersections $U_{\alpha} \cap U_{\beta}$, which is the condition that the meromorphic cocycle $f_{\alpha\beta}$ is cohomologous to zero, and that concludes the proof.

Yet another consequence of the Existence Theorem is an explicit formula for the Euler characteristic of any holomorphic line bundle over a compact Riemann surface. In this formula the *arithmetic genus* of a compact Riemann surface $M$ is defined to be the integer

$$(2.15) \quad g_{\alpha} = \dim H^1(M, \mathcal{O}) = \dim \Gamma(M, \mathcal{O}(1,0)).$$
where the two dimensions are equal by Corollary 1.19 to the Serre Duality Theorem for the case of the trivial line bundle $\lambda = 1$.

**Theorem 2.17 (Riemann-Roch Theorem)** If $\lambda$ is a holomorphic line bundle over a compact Riemann surface $M$ of arithmetic genus $g_a$

\[ \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)) = c(\lambda) + 1 - g_a. \quad (2.16) \]

**Proof:** Any holomorphic line bundle $\lambda$ over $M$ by the Existence Theorem, so $\lambda$ can be written as a product $\lambda = \prod_i \zeta_i^{n_i}$; it then follows by iterating Lemma 2.13 that $\chi(\lambda) - c(\lambda) = \chi(1) - c(1)$ where $1$ is the trivial holomorphic line bundle. The characteristic class of the trivial bundle is $c(1) = 0$; and since a holomorphic cross-section of the trivial line bundle is a holomorphic function on the compact Riemann surface $M$, so is a constant by the maximum modulus theorem, it follows that $\dim H^0(M, \mathcal{O}) = \dim \Gamma(M, \mathcal{O}) = 1$. The Euler class of the trivial bundle therefore is $\chi(1) = \dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O})) = 1 - g_a$, and that suffices for the proof.

There are various alternative forms of the Riemann-Roch Theorem that are frequently used. To establish some auxiliary results in preparation for the discussion of these alternative forms, a holomorphic differential form $\phi \in \Gamma(M, \mathcal{O}^{(1,0)})$ on an arbitrary Riemann surface $M$ can be written explicitly in a local coordinate neighborhood $U_\alpha$ with local coordinate $z_\alpha$ in the form $\phi = f_\alpha dz_\alpha$ for a holomorphic function $f_\alpha$ in $U_\alpha$. In the intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods $\phi = f_\alpha dz_\alpha = f_\beta dz_\beta$ and consequently

\[ f_\alpha = \frac{dz_\beta}{dz_\alpha} f_\beta = \left( \frac{dz_\alpha}{dz_\beta} \right)^{-1} f_\beta. \quad (2.17) \]

The derivatives

\[ \kappa_{\alpha\beta} = \frac{dz_\beta}{dz_\alpha} = \left( \frac{dz_\alpha}{dz_\beta} \right)^{-1} \quad (2.18) \]

are holomorphic and nowhere vanishing functions in the intersections $U_\alpha \cap U_\beta$ of pairs of sets, and it follows immediately from the chain rule for differentiation that $\kappa_{\alpha\beta} \cdot \kappa_{\beta\gamma} \cdot \kappa_{\gamma\alpha} = 1$ in any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$; thus $\{U_\alpha, \kappa_{\alpha\beta}\}$ is a holomorphic coordinate line bundle over $M$, describing a holomorphic line bundle. The same construction can be applied to the union of any two coordinate coverings of the surface, from which it is evident that the line bundle described by these coordinate line bundles is independent of the choice of a coordinate covering of the surface; that line bundle is called the canonical bundle of the Riemann surface $M$, and is denoted by $\kappa$. By (2.17) the coefficients $f_\alpha$ of a holomorphic differential form on $M$ are a holomorphic cross-section of the canonical line bundle $\kappa$, so there results the natural identification $\mathcal{O}^{(1,0)} \cong \mathcal{O}(\kappa)$ or more generally

\[ \mathcal{O}^{(1,0)}(\lambda) \cong \mathcal{O}(\kappa \lambda) \quad \text{for any line bundle } \lambda; \quad (2.19) \]
These observations can be used to examine the arithmetic genus of a compact Riemann surface through a holomorphic form of the deRham exact sequence of differential forms.

The germ of a holomorphic differential form on a Riemann surface is the germ $$\phi$$ of a differential form of type $$(1,0)$$ such that $$\partial \phi = 0$$, as on page 14. Clearly the exterior derivative of the germ of a holomorphic function is the germ of a holomorphic differential form. Conversely a germ $$\phi$$ of a holomorphic differential form on a Riemann surface is the germ of a closed differential 1-form since $$d\phi = \partial \phi = 0$$, so by the Poincaré lemma $$\phi$$ is the exterior derivative of the germ of a function $$f$$; but if $$\phi = df = \partial f + \bar{f}$$ then $$\bar{f} = 0$$ since $$\phi$$ is of type $$(1,0)$$, so $$f$$ is the germ of a holomorphic function. Thus there is the exact sequence of sheaves

$$0 \to \mathbb{C} \to O \xrightarrow{d} O^{(1,0)} \to 0$$

on an arbitrary Riemann surface, the holomorphic version of the deRham exact sequence of sheaves. From this exact sequence of sheaves there follows the exact cohomology sequence

$$0 \to \Gamma(M, \mathbb{C}) \xrightarrow{\iota} \Gamma(M, O) \xrightarrow{d} \Gamma(M, O^{(1,0)}) \xrightarrow{\delta} H^1(M, \mathbb{C}) \xrightarrow{\iota}$$

$$\xrightarrow{\iota} H^1(M, O) \xrightarrow{d} H^1(M, O^{(1,0)}) \xrightarrow{\delta} H^2(M, \mathbb{C}) \to 0$$

since $$H^2(M, O) = 0$$ by the Dolbeault Theorem, Theorem 1.10. If $$M$$ is compact every holomorphic function is constant, so $$\Gamma(M, \mathbb{C}) = \Gamma(M, O) = \mathbb{C}$$. Furthermore it follows from the Serre Duality Theorem in the form of Corollary 1.19 and the observation (2.19) that

$$\dim H^1(M, O^{(1,0)}) = \dim H^1(M, O(\kappa)) = \dim \Gamma(M, O^{(1,0)}(\kappa^{-1})) = \dim \Gamma(M, O) = 1$$

while $$\dim H^2(M, \mathbb{C}) = 1$$ as well. Thus for a compact Riemann surface this exact cohomology sequence reduces to the short exact sequence

$$0 \to \Gamma(M, O^{(1,0)}) \xrightarrow{\delta} H^1(M, \mathbb{C}) \xrightarrow{\iota} H^1(M, O) \to 0.$$  \hspace{1cm} (2.21)

In this sequence $$\dim H^1(M, \mathbb{C}) = 2g$$ where $$g$$ is the topological genus$^1$ of the compact surface $$M$$. On the other hand by the Serre Duality Theorem in the form of Corollary 1.19 again it also follows that $$g_a = \dim H^1(M, O) = \dim \Gamma(M, O^{(1,0)}),$$ so (2.21) implies that

$$g_a = \dim H^1(M, O) = \dim \Gamma(M, O^{(1,0)}) = \frac{1}{2} \dim H^1(M, \mathbb{C}) = g.$$  \hspace{1cm} (2.22)

Thus the arithmetic genus $$g_a$$ of a compact Riemann surface is equal to its topological genus $$g$$; this common value subsequently will be called simply the genus of the surface. The Riemann-Roch Theorem then can be restated in terms of the topological genus $$g$$ rather than the arithmetic genus $$g_a$$, and can be put into the following form in terms of the canonical bundle $$\kappa$$.

$^1$See the discussion of the topology of surfaces in Appendix D.
Theorem 2.18 (Canonical Bundle Theorem) The canonical bundle \( \kappa \) of a compact Riemann surface \( M \) of genus \( g \) is characterized by the conditions that

\[
(2.23) \quad c(\kappa) = 2g - 2 \quad \text{and} \quad \gamma(\kappa) = g.
\]

The Riemann-Roch Theorem on \( M \) can be rephrased in terms of the canonical bundle as the identity

\[
(2.24) \quad \gamma(\lambda) - \gamma(\kappa \lambda^{-1}) = c(\lambda) + 1 - g
\]

for any holomorphic line bundle \( \lambda \).

Proof: From the Serre Duality Theorem in the form of Corollary 1.19 and from (2.19) it follows that \( \dim H^1(M, \mathcal{O}(\lambda)) = \dim \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1})) = \gamma(\kappa \lambda^{-1}) \); and with these observations the Riemann-Roch Theorem, Theorem 2.17, takes the form (2.24), with the genus \( g \) in place of the arithmetic genus \( g_a \) in view of (2.22). Since \( \gamma(\kappa) = \dim \Gamma(M, \mathcal{O}^{(1,0)}) = g \) by (2.22) and since \( \gamma(\kappa \kappa^{-1}) = \gamma(1) = 1 \) it follows immediately from (2.24) for the special case in which \( \lambda = \kappa \) that \( c(\kappa) = 2g - 2 \). On the other hand if \( \lambda \) is a holomorphic line bundle with \( c(\lambda) = 2g - 2 \) and \( \gamma(\lambda) = g \) then it follows from (2.24) that \( g - \gamma(\kappa \lambda^{-1}) = 2g - 2 + 1 - g = g - 1 \), hence that \( \gamma(\kappa \lambda^{-1}) = 1 \); but since \( c(\kappa \lambda^{-1}) = 0 \) it follows from Corollary 1.5 that \( \kappa \lambda^{-1} \) is the identity bundle, hence that \( \lambda = \kappa \). That suffices to conclude the proof.

Corollary 2.19 There is a unique compact Riemann surface of genus \( g = 0 \), the Riemann sphere or one-dimensional complex projective space \( \mathbb{P}^1 \). The canonical bundle \( \kappa \) of \( \mathbb{P}^1 \) has characteristic class \( c(\kappa) = -2 \), so \( \mathbb{P}^1 \) has no nontrivial holomorphic differential forms.

Proof: If \( M \) is a compact Riemann surface of genus \( g = 0 \) then \( c(\kappa) = -2 \) by (2.23) of the preceding theorem, and it then follows from Corollary 1.4 that \( \gamma(\kappa) = 0 \) so \( M \) has no nontrivial holomorphic differential forms. For any point bundle \( \mathcal{O}_p \) over \( M \), it follows from the Riemann-Roch Theorem in the form (2.24) that \( \gamma(\mathcal{O}_p) = c(\mathcal{O}_p) + 1 \) since \( c(\kappa \mathcal{O}_p^{-1}) < 0 \), so \( M \) is the Riemann sphere as a consequence of Theorem 2.7. That suffices to conclude the proof.

Corollary 2.20 Compact Riemann surfaces of genus \( g = 1 \) can be characterized either as those compact Riemann surfaces for which the canonical bundle is topologically trivial or as those compact Riemann surfaces for which the canonical bundle is analytically trivial.

Proof: That compact Riemann surfaces of genus \( g = 1 \) are characterized by the condition that their canonical bundle \( \kappa \) is topologically trivial follows immediately from (2.23) of the preceding theorem, since the bundle \( \kappa \) is topologically trivial precisely when \( c(\kappa) = 0 \). Of course any analytically trivial bundle is topologically trivial; and conversely if the canonical bundle of a surface is topologically trivial then \( g = 1 \) and it follows from (2.23) that \( \gamma(\kappa) = 1 \) and hence by Corollary 1.5 that \( \kappa \) is analytically trivial. That suffices for the proof.
The Riemann-Roch Theorem in the form given in Theorem 2.18 can be rephrased in a more symmetric way in terms of an auxiliary expression that is useful in various other contexts as well. The Clifford index of a holomorphic line bundle $\lambda$ on a compact Riemann surface is defined by

\[(2.25) \quad C(\lambda) = c(\lambda) - 2(\gamma(\lambda) - 1),\]

so is another integral invariant associated to a holomorphic line bundle.

**Corollary 2.21 (Brill-Noether Formula)** The Clifford index of a holomorphic line bundle $\lambda$ over a compact Riemann surface satisfies the symmetry condition

\[(2.26) \quad C(\lambda) = C(\kappa\lambda^{-1}),\]

where $\kappa$ is the canonical bundle of the surface.

**Proof:** By the definition of the Clifford index and the Riemann-Roch Theorem in the form (2.24) it follows that

\[
C(\lambda) = c(\lambda) - 2(\gamma(\lambda) - 1) \\
= c(\lambda) - 2(\gamma(\kappa\lambda^{-1}) + c(\lambda) + 1 - g - 1) \\
= c(\kappa\lambda^{-1}) - 2(\gamma(\kappa\lambda^{-1}) - 1) = C(\kappa\lambda^{-1}),
\]

which concludes the proof.

There is yet another formulation of the Riemann-Roch Theorem that is quite commonly used, one that is expressed entirely in terms of meromorphic functions and differential forms and avoids any mention of line bundles. It is customarily expressed in terms of the complex vector spaces $L(\mathfrak{d})$ associated to divisors $\mathfrak{d}$ on the surface $M$, defined by

\[(2.27) \quad L(\mathfrak{d}) = \left\{ f \in \Gamma(M, \mathcal{M}) \middle| \mathfrak{d}(f) + \mathfrak{d} \geq 0 \right\}.
\]

Note that vector spaces $L(\mathfrak{d}_1)$ and $L(\mathfrak{d}_2)$ are isomorphic whenever the divisors $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are linearly equivalent; indeed if $\mathfrak{d}_1 \sim \mathfrak{d}_2$ there is a meromorphic function $g$ on $M$ with $\mathfrak{d}(g) = \mathfrak{d}_1 - \mathfrak{d}_2$ and multiplication by $g$ defines an isomorphism

\[(2.28) \quad \times g : L(\mathfrak{d}_1) \xrightarrow{\cong} L(\mathfrak{d}_2).
\]

Thus the dimension of the vector space $L(\mathfrak{d})$ depends only on the linear equivalence class of the divisor $\mathfrak{d}$. If $\mathfrak{d} = \sum_i \nu_i \cdot p_i$ for distinct points $p_i \in M$ then $L(\mathfrak{d})$ consists of those meromorphic functions $f$ on $M$ having a zero at $p_i$ of order at least $-\nu_i$ if $\nu_i \leq 0$ and having a pole at $p_i$ of order at most $\nu_i$ if $\nu_i \geq 0$. These vector spaces of meromorphic functions have played a major role in the study of function theory on compact Riemann surfaces from the earliest period. If $\zeta_{\mathfrak{d}}$ is the line bundle associated to the divisor $\mathfrak{d}$ and $h \in \Gamma(M, \mathcal{M}(\zeta_{\mathfrak{d}}))$ is a
meromorphic cross-section with $\mathfrak{d}(h) = \mathfrak{d}$ then multiplication by $h$ defines an isomorphism

\begin{equation}
\times h : L(\mathfrak{d}) \xrightarrow{\cong} \Gamma(M, \mathcal{O}(\zeta_\mathfrak{d})),
\end{equation}

so that

\begin{equation}
\dim L(\mathfrak{d}) = \gamma(\zeta_\mathfrak{d}).
\end{equation}

There is a corresponding definition for differential forms, expressed in terms of meromorphic differential forms. A germ of a meromorphic differential form is an expression of the form $f \, dz$ for the germ of a meromorphic function $f \in \mathcal{M}$; the sheaf of germs of meromorphic differential forms is denoted by $\mathcal{M}^{(1,0)}$, and the global sections in $\Gamma(M, \mathcal{M}^{(1,0)})$ are called meromorphic differential forms on $M$. These are not quite a subset of the space of $C^\infty$ differential forms of type $(1,0)$ on $M$, since the meromorphic differential forms are not differentiable at their singularities; but the space of holomorphic differential forms can be identified in the obvious way with a subspace of the space of meromorphic differential forms. Of course there is again the natural identification $\Gamma(M, \mathcal{M}^{(1,0)}) \cong \Gamma(M, \mathcal{M}(\kappa))$, so that meromorphic differential forms can be identified with meromorphic cross-sections of the canonical bundle in the same way that holomorphic differential forms can be identified with holomorphic cross-sections of the canonical bundle; and the notion of the divisor of a meromorphic differential form is consequently well defined. In these terms, let

\begin{equation}
L^{(1,0)}(\mathfrak{d}) = \left\{ f \, dz \in \Gamma(M, \mathcal{M}^{(1,0)}) \mid \mathfrak{d}(f) + \mathfrak{d} \geq 0 \right\}.
\end{equation}

Again if $\mathfrak{d} = \sum_i \nu_i \cdot p_i$ for distinct points $p_i \in M$ then $L^{(1,0)}(\mathfrak{d})$ consists of those meromorphic differential forms $f \, dz$ on $M$ having a zero at $p_i$ of order at least $-\nu_i$ if $\nu_i \leq 0$ and having a pole at $p_i$ of order at most $\nu_i$ if $\nu_i \geq 0$. If $h \in \Gamma(M, \mathcal{M}(\zeta_\mathfrak{d}))$ is a meromorphic cross-section with $\mathfrak{d}(h) = \mathfrak{d}$ then multiplication by $h$ defines an isomorphism

\begin{equation}
\times h : L^{(1,0)}(\mathfrak{d}) \xrightarrow{\cong} \Gamma(M, \mathcal{O}^{(1,0)}(\zeta_\mathfrak{d})) = \Gamma(M, \mathcal{O}(\kappa \zeta_\mathfrak{d})),
\end{equation}

and since $\dim \Gamma(M, \mathcal{O}^{(1,0)}(\zeta_\mathfrak{d})) = \dim \Gamma(M, \mathcal{O}(\kappa \zeta_\mathfrak{d}))$ it follows that

\begin{equation}
\dim L^{(1,0)}(\mathfrak{d}) = \gamma(\kappa \zeta_\mathfrak{d}).
\end{equation}

In these terms, the Riemann-Roch Theorem can be rephrased as follows.

**Corollary 2.22 (Riemann-Roch Theorem)** If $\mathfrak{d}$ is a divisor on a compact Riemann surface $M$ of genus $g$

\begin{equation}
\dim L(\mathfrak{d}) - \dim L^{(1,0)}(-\mathfrak{d}) = \deg \mathfrak{d} + 1 - g.
\end{equation}

**Proof:** This follows immediately from the Riemann-Roch Theorem in the form of equation (2.24) in Corollary 2.18, applied to the line bundle $\lambda = \zeta_\mathfrak{d}$, in view of the identifications (2.30) and (2.33), and that suffices for a proof.
A slight variant of this version of the Riemann-Roch Theorem replaces the meromorphic differential forms by their divisors. The divisors on $M$ that are associated to the canonical bundle are called \textit{canonical divisors} on $M$, and are customarily denoted by $\mathfrak{t}$; it is important to keep in mind that $\mathfrak{t}$ does not represent a single divisor, but rather any of a large class of linearly equivalent divisors. The divisor of any holomorphic differential form on $M$ is a positive canonical divisor, for instance, and the divisor of any meromorphic differential form on $M$ is a not necessarily positive canonical divisor. Alternatively, a canonical divisor is any divisor $\mathfrak{t}$ with the property that $\zeta_{\mathfrak{t}} = \kappa$. A divisor $\mathfrak{d}'$ is residual to a divisor $\mathfrak{d}$ if the sum of these divisors is a canonical divisor, that is, if $\mathfrak{d}' + \mathfrak{d} = \mathfrak{t}$; the divisor $\mathfrak{d}$ of course is then residual to the divisor $\mathfrak{d}'$, so that this is a dual relationship between divisors. In these terms, the Riemann-Roch Theorem can be rephrased yet again as follows.

**Corollary 2.23 (Riemann-Roch Theorem)** If $\mathfrak{d}'$ is the residual divisor to a divisor $\mathfrak{d}$ on a compact Riemann surface $M$ of genus $g$ then

$$\dim L(\mathfrak{d}) - \dim L(\mathfrak{d}') = \deg \mathfrak{d} + 1 - g. \tag{2.35}$$

**Proof:** This follows immediately from the preceding Corollary 2.22 upon noting that $\dim L(\mathfrak{d}') = \dim L(\mathfrak{t} - \mathfrak{d}) = \gamma(\mathcal{O}_{-\mathfrak{d}}) = \gamma(k\mathcal{O}_{-\mathfrak{d}}) = \dim L^{(1,0)}(-\mathfrak{d})$ as a consequence of (2.33), and that suffices for the proof.

The Riemann-Roch Theorem in the form given in Corollary 2.22 can be rewritten for the special case of nontrivial positive divisors in terms of a useful auxiliary expression involving holomorphic differential forms. On a compact Riemann surface $M$ of genus $g > 0$ choose a coordinate covering $\{ U_{\alpha} \}$ with local coordinates $\{ z_{\alpha} \}$ and a basis $\omega_i = f_{i\alpha}dz_{\alpha}$ for the space of holomorphic differential forms for $1 \leq i \leq g$; that there are $g$ differential forms in a basis follows from (2.22). If $\mathfrak{d} = \sum_{j=1}^n \nu_j \cdot p_j$ is a positive divisor of degree $r = \sum_{j=1}^n \nu_j > 0$ in which $p_j$ are distinct points and if $p_j \in U_{\alpha_j}$, the \textit{Brill-Noether matrix} $\Omega_{\alpha_1 \ldots \alpha_n}(\mathfrak{d})$ of this divisor in terms of the local coordinates $z_{\alpha_j}$ in $U_{\alpha_j}$ is the $g \times r$ matrix in which row $i$ for $1 \leq i \leq g$ is

$$\begin{align*}
  f_{i\alpha_1}(p_1), f'_{i\alpha_1}(p_1), \frac{1}{2}f''_{i\alpha_1}(p_1), & \ldots, \frac{1}{(\nu_1 - 1)!}f^{(\nu_1 - 1)}_{i\alpha_1}(p_1), \\
  f_{i\alpha_2}(p_2), f'_{i\alpha_2}(p_2), \frac{1}{2}f''_{i\alpha_2}(p_2), & \ldots, \frac{1}{(\nu_2 - 1)!}f^{(\nu_2 - 1)}_{i\alpha_2}(p_2), \\
  \cdots & \cdots \\
  f_{i\alpha_n}(p_n), f'_{i\alpha_n}(p_n), \frac{1}{2}f''_{i\alpha_n}(p_n), & \ldots, \frac{1}{(\nu_n - 1)!}f^{(\nu_n - 1)}_{i\alpha_n}(p_n),
\end{align*} \tag{2.36}$$

where the derivatives of the function $f_{i\alpha_j}$ at the point $p_j \in U_{\alpha_j}$ are with respect to the local coordinate $z_{\alpha_j}$. One extreme case is that in which $\mathfrak{d}$ is a positive divisor consisting of $r$ distinct points $p_j$, in which case

$$\text{row } i \text{ of } \Omega_{\alpha_1 \ldots \alpha_n}(p_1 + \cdots + p_r) = \left\{ f_{i\alpha_1}(p_1), \ldots, f_{i\alpha_r}(p_r) \right\}. \tag{2.37}$$

Another extreme case is that in which $\mathfrak{d}$ is a multiple of a single point $p \in U_{\alpha}$, in which case

$$\text{row } i \text{ of } \Omega_{\alpha}(r \cdot p) = \left\{ f_{i\alpha}(p), f'_{i\alpha}(p), \ldots, \frac{1}{(r - 1)!}f^{(r-1)}_{i\alpha}(p) \right\}. \tag{2.38}$$
If \( \{U_\alpha, \kappa_{\alpha\beta}\} \) is the holomorphic coordinate bundle describing the canonical bundle \( \kappa \) in terms of the chosen coordinates and \( p \in U_\alpha \cap U_\beta \) then upon differentiating (2.17) and noting that \( d/dz_{\alpha} = \kappa_{\alpha\beta} d/dz_{\beta} \) by the chain rule for differentiation it follows that the coefficient functions \( f_{\alpha\alpha}(p) \) and \( f_{i\beta}(p) \) and their derivatives at the point \( p \) with respect to the local coordinates \( z_{\alpha} \) and \( z_{\beta} \) respectively are related by

\[
\begin{align*}
f_{\alpha\alpha}(p) &= \kappa_{\alpha\beta}(p) f_{i\beta}(p) \\
f'_{\alpha\alpha}(p) &= \kappa_{\alpha\beta}(p)^2 f'_{i\beta}(p) + \kappa_{\alpha\beta}(p) \kappa'_{\alpha\beta}(p) f_{i\beta}(p) \\
f''_{\alpha\alpha}(p) &= \kappa_{\alpha\beta}(p)^3 f''_{i\beta}(p) + 3 \kappa_{\alpha\beta}(p)^2 \kappa'_{\alpha\beta}(p) f'_{i\beta}(p) \\
&\quad + (\kappa_{\alpha\beta}(p) \kappa'_{\alpha\beta}(p)^2 + \kappa_{\alpha\beta}(p)^2 \kappa''_{\alpha\beta}(p)) f_{i\beta}(p)
\end{align*}
\]

and so on,

where \( \kappa'_{\alpha\beta} \) denotes the derivative of the function \( \kappa_{\alpha\beta} \) with respect to the variable \( z_{\beta} \) and correspondingly for the higher derivatives. It is a straightforward matter to verify using (2.36) and (2.39) that

\[\tag{2.40} \Omega_{\alpha_1 \ldots \alpha_n}(d) = \Omega_{\beta_1 \ldots \beta_n}(d) \cdot K_{\alpha\beta} \]

for the nonsingular \( r \times r \) matrix

\[
K_{\alpha\beta} = \begin{pmatrix} \kappa_{\alpha\beta} & \kappa_{\alpha_1 \beta_1} \kappa'_{\alpha_1 \beta_1} & \kappa_{\alpha_1 \beta_1} \kappa'_{\alpha_1 \beta_1}^2 + \kappa_{\alpha_1 \beta_1}^2 \kappa''_{\alpha_1 \beta_1} & \cdots \\ 0 & \kappa_{\alpha_1 \beta_1}^2 & 3 \kappa_{\alpha_1 \beta_1} \kappa'_{\alpha_1 \beta_1} & \cdots \\ 0 & 0 & \kappa_{\alpha_1 \beta_1}^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]

for which

\[\tag{2.41} \det K_{\alpha\beta} = \kappa_{\alpha_1 \beta_1}^{\frac{1}{2} \nu_1(\nu_1+1)} \kappa_{\alpha_2 \beta_2}^{\frac{1}{2} \nu_2(\nu_2+1)} \cdots \kappa_{\alpha_n \beta_n}^{\frac{1}{2} \nu_n(\nu_n+1)}. \]

Since the matrix \( K_{\alpha\beta} \) is nonsingular the rank of the matrix \( \Omega_{\alpha_1 \ldots \alpha_n}(d) \) is independent of the choice of local coordinates; so when considering merely the rank of the Brill-Noether matrix the notation can be simplified by dropping the subscripts indicating the choice of local coordinates at the points of the divisor.

The rank of the matrix \( \Omega(d) \) also clearly is independent of the choice of a basis for the space of holomorphic differential forms on \( M \). The Riemann-Roch Theorem then takes the following form in terms of the Brill-Noether matrix \( \Omega(d) \).

**Theorem 2.24 (Riemann-Roch Theorem)** If \( d \) is a nontrivial positive divisor on a compact Riemann surface \( M \) of genus \( g > 0 \)

\[\tag{2.42} \gamma(z_d) = \dim L(d) = \deg d - \text{rank } \Omega(d) + 1, \]

where \( \Omega(d) \) is the Brill-Noether matrix of the divisor \( d \).
Proof: For a positive divisor $\mathcal{D}$ the vector space $L^{(1,0)}(-\mathcal{D})$ of meromorphic differential forms $\omega = f_{\alpha}dz_{\alpha}$ such that $\mathcal{D}(f_{\alpha}) - \mathcal{D} \geq 0$ consists of the holomorphic differential forms that vanish on the divisor $\mathcal{D}$. A holomorphic differential form $\omega = f_{\alpha}dz_{\alpha}$ can be written in terms of the basis $\omega_{i} = f_{i\alpha}dz_{\alpha}$ as the sum $\omega = \sum c_{i}\omega_{i}$ for some complex constants $c_{i}$, so $f_{\alpha} = \sum_{i=1}^{g} c_{i} f_{i\alpha}$. If $c = (c_{1}, \ldots, c_{g})$ is the row vector formed from these constants and the divisor $\mathcal{D}$ is nontrivial and has the Brill-Noether matrix $\Omega(\mathcal{D})$ then the entries in the row vector $c \cdot \Omega(\mathcal{D})$ are just the values of the function $f_{\alpha}(z_{\alpha})$ and of its derivatives at the points of the divisor $\mathcal{D}$, paralleling the entries in row $i$ of the matrix $\Omega(\mathcal{D})$. Consequently the holomorphic differential form $\omega = f dz$ vanishes on the divisor $\mathcal{D}$ precisely when $c \cdot \Omega(\mathcal{D}) = 0$, so

$$\dim L^{(1,0)}(-\mathcal{D}) = \dim \left\{ c \in \mathbb{C}^{g} \mid c \cdot \Omega(\mathcal{D}) = 0 \right\} = g - \text{rank } \Omega(\mathcal{D}).$$

Substituting this into the Riemann-Roch formula of Corollary 2.22 yields the desired result and thereby concludes the proof.

The Riemann-Roch Theorem has legions of applications, as will become apparent as the discussion proceeds; a few will be discussed in the remainder of this chapter.

**Theorem 2.25** Let $M$ be a compact Riemann surface of genus $g$.

(i) Any holomorphic line bundle $\lambda$ on $M$ with $c(\lambda) \geq 2g$ is base-point-free.

(ii) If $g > 0$ all holomorphic line bundles $\lambda$ on $M$ for which $c(\lambda) = 2g - 1$ are base-point-free except for those bundles of the form $\lambda = \kappa \omega_{a}$ for some point $a \in M$, and none of the latter bundles is base-point-free.

(iii) If $g \geq 0$ the canonical bundle $\kappa$ is base-point-free; all other holomorphic line bundles $\lambda$ on $M$ for which $c(\lambda) = 2g - 2$ also are base-point-free except for those bundles of the form $\lambda = \kappa \omega_{a} \omega_{b}^{-1}$ for two distinct points $a, b \in M$, and none of the latter bundles is base-point-free.

**Proof:** (i) If $\lambda$ is a holomorphic line bundle over $M$ for which $c(\lambda) \geq 2g$ then $c(\kappa \lambda^{-1}) < c(\kappa \lambda^{-1} \omega_{a}) < 0$ for any point $a \in M$, so $\gamma(\kappa \lambda^{-1}) = \gamma(\kappa \lambda^{-1} \omega_{a}) = 0$ by Corollary 1.4 and it follows from the Riemann-Roch Theorem in the form of equation (2.24) that $\gamma(\lambda) = c(\lambda) + 1 - g \geq g + 1 > 0$ and $\gamma(\lambda \omega_{a}^{-1}) = c(\lambda \omega_{a}^{-1}) + 1 - g = \gamma(\lambda) - 1$; so by Lemma 2.10 the bundle $\lambda$ is base-point-free.

(ii) If $c(\lambda) = 2g - 1$ then $c(\kappa \lambda^{-1}) < 0$ so $\gamma(\kappa \lambda^{-1}) = 0$ by Corollary 1.4 and it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = c(\lambda) + 1 - g = g > 0$. If $\lambda$ is not base-point-free then by Lemma 2.10 there is a point $a \in M$ such that $\gamma(\lambda \omega_{a}^{-1}) = \gamma(\lambda) = g$; and since $c(\lambda \omega_{a}^{-1}) = 2g - 2$ the Canonical Bundle Theorem, Theorem 2.18, shows that $\lambda \omega_{a}^{-1} = \kappa$. On the other hand if $\lambda = \kappa \omega_{a}$ for a point $a \in M$ then $\gamma(\lambda \omega_{a}^{-1}) = \gamma(\kappa) = g = \gamma(\lambda)$ so by Lemma 2.10 the bundle $\kappa \omega_{a}$ is not base-point-free.

(iii) Since $g > 0$ it follows from Theorem 2.4 that $\gamma(\omega_{a}) = 1$ for any point $a \in M$; then by the Riemann-Roch Theorem $\gamma(\kappa \omega_{a}^{-1}) = \gamma(\omega_{a}) + g - 2 = g - 1 = \gamma(\kappa) - 1$, since $\gamma(\kappa) = g$ by the Canonical Bundle Theorem, Theorem 2.18, so by Lemma 2.10 the bundle $\kappa$ is base-point-free. If $\lambda$ is a line bundle for which
The case $g = 1$ is slightly special, for $\gamma(\lambda) = g - 1 = 0$ as just demonstrated so $\lambda$ is not base-point-free; however $c(\lambda) = 0$ so $c(\lambda \zeta_b) = 1$ for any point $b \in M$, and from the Riemann-Roch Theorem it follows that $\gamma(\lambda \zeta_b) = \gamma(\kappa \lambda^{-1} \zeta_b^{-1}) + 1 = 1$, since $c(\kappa \lambda^{-1} \zeta_b^{-1}) < 0$, so $\lambda \zeta_b = \zeta_a$ for some point $a \in M$ by Theorem 2.4 and consequently $\lambda = \zeta_a \zeta_b^{-1}$. In the more general case if $g > 1$ and the bundle $\lambda$ is not base-point-free then by Lemma 2.10 there is a point $a \in M$ for which $\gamma(\lambda \zeta_a^{-1}) = \gamma(\lambda) = g - 1$. It then follows from the Riemann-Roch Theorem that $\gamma(\kappa \lambda^{-1} \zeta_a) = \gamma(\lambda \zeta_a^{-1}) + 2 - g = 1$, and since $c(\kappa \lambda^{-1} \zeta_a) = 1$ then $\kappa \lambda^{-1} \zeta_a = \zeta_b$ is a point bundle by Theorem 2.4; consequently $\lambda = \kappa \zeta_a \zeta_b^{-1}$, where $a \neq b$ since $\lambda \neq \kappa$. Conversely if $\lambda = \kappa \zeta_a \zeta_b^{-1}$ for points $a \neq b$ on the Riemann surface $M$ then $\lambda \neq \kappa$ so $\gamma(\lambda) = g - 1$, and $\gamma(\lambda \zeta_a^{-1}) = \gamma(\kappa \zeta_a^{-1}) = \gamma(\zeta_b) = g - 2 = g - 1$ as well; hence by Lemma 2.10 the bundle $\lambda$ is not base-point-free, which suffices to conclude the proof.

Since the canonical bundle of a compact Riemann surface of genus $g > 0$ is base-point-free by part (iii) of the preceding theorem, it follows that in part (ii) the product $\lambda = \kappa \zeta_a$ is the base decomposition of $\lambda$ for any point $a \in M$. For a Riemann surface of small genus the preceding theorem provides a characterization of the base-point-free holomorphic line bundles on that surface.

**Corollary 2.26** (i) On a compact Riemann surface of genus $g = 0$ all holomorphic line bundles $\lambda$ for which $c(\lambda) \geq 0$ are base-point-free; consequently the Lüroth semigroup of such a surface is

\[
\mathcal{L}(\mathbb{P}^1) = \{ n \in \mathbb{Z} \mid n \geq 0 \}.
\]

(ii) On a compact Riemann surface of genus $g = 1$ the base-point-free holomorphic line bundles are the identity bundle and all bundles $\lambda$ for which $c(\lambda) \geq 2$; consequently the Lüroth semigroup of such a surface is

\[
\mathcal{L}(M) = \{ n \in \mathbb{Z} \mid n = 0 \text{ or } n \geq 2 \} \quad \text{if } M \text{ is of genus } 1.
\]

**Proof:** (i) For a compact Riemann surface of genus $g = 0$ it follows from Theorem 2.25 (i) that all holomorphic line bundles $\lambda$ for which $c(\lambda) \geq 0$ are base-point-free.

(ii) For a compact Riemann surface of genus $g = 0$ it follows from Theorem 2.25 (ii) that all holomorphic line bundles $\lambda$ for which $c(\lambda) \geq 2$ are base-point-free. If $c(\lambda) = 1$ then it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = 1$; hence $\lambda$ is a point bundle $\lambda = \zeta_a$ by Theorem 2.4, and therefore $\lambda$ is not base-point-free. If $c(\lambda) = 0$ then either $\lambda$ is the identity bundle, which is base-point-free, or $\gamma(\lambda) = 0$ and $\lambda$ is not base-point-free. That suffices for the proof.
Theorem 2.27 Let \( \lambda \) be a holomorphic line bundle over a compact Riemann surface \( M \) of genus \( g > 0 \).

(i) If \( \gamma(\kappa\lambda^{-1}) = 0 \)

\[
(2.46) \quad \gamma(\lambda\zeta_b) = \gamma(\lambda) + \deg \mathfrak{d} \quad \text{for any positive divisor } \mathfrak{d}.
\]

(ii) If \( \kappa\lambda^{-1} \) is base-point-free

\[
(2.47) \quad \gamma(\lambda\zeta_a) = \gamma(\lambda) \quad \text{for any point } a \in M.
\]

(iii) If \( \gamma(\kappa\lambda^{-1}) > 0 \) and this line bundle has the base decomposition \( \kappa\lambda^{-1} = \lambda_0\zeta_b \) for a base-point-free line bundle \( \lambda_0 \) and a positive divisor \( b \) then

\[
(2.48) \quad \gamma(\lambda\zeta_b\zeta_a) = \gamma(\lambda) + \deg b,
\]

\[
(2.49) \quad \gamma(\lambda\zeta_b) = \gamma(\lambda) + \deg b \quad \text{for any point } a \in M,
\]

\[
(2.50) \quad \gamma(\lambda\zeta_a) = \gamma(\lambda) \quad \text{for any point } a \notin b.
\]

Proof: (i) It follows from the Riemann-Roch Theorem in the form of equation (2.24) that for any divisor \( \mathfrak{d} \) on \( M \)

\[
(2.51) \quad \gamma(\lambda\zeta_a) - \gamma(\lambda) = \gamma(\kappa\lambda^{-1}\zeta_a^{-1}) - \gamma(\kappa\lambda^{-1}) + \deg \mathfrak{d}.
\]

If \( \gamma(\kappa\lambda^{-1}) = 0 \) then \( \gamma(\kappa\lambda^{-1}\zeta_a^{-1}) = 0 \) for any positive divisor \( \mathfrak{d} \), since multiplication by a nontrivial holomorphic cross-section \( f \in \Gamma(M, \zeta_a) \) is an injective linear mapping

\[
\times f : \Gamma(M, \kappa\lambda^{-1}\zeta_a^{-1}) \longrightarrow \Gamma(M, \kappa\lambda^{-1});
\]

and in that case (2.46) is an immediate consequence of (2.51).

(ii) If \( \kappa\lambda^{-1} \) is base-point-free then \( \gamma(\kappa\lambda^{-1}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}) - 1 \) for any point \( a \in M \), and in that case (2.47) is an immediate consequence of (2.51) for the divisor \( \mathfrak{d} = 1 \cdot a. \)

(iii) If \( \kappa\lambda^{-1} = \lambda_0\zeta_b \) is the base decomposition of this line bundle then \( \gamma(\lambda_0\zeta_b) = \gamma(\lambda_0) \) by Theorem 2.12 (i), or since \( \lambda_0 = \kappa\lambda^{-1}\zeta_b^{-1} \) equivalently \( \gamma(\kappa\lambda^{-1}) = \gamma(\kappa\lambda^{-1}\zeta_b^{-1}) \); and in that case (2.48) is an immediate consequence of (2.51) for the divisor \( \mathfrak{d} = \zeta_b \). Since \( \lambda_0 \) is base-point-free \( \gamma(\lambda_0\zeta_a^{-1}) = \gamma(\lambda_0) - 1 \) for any point \( a \in M \) by Lemma 2.10, or equivalently \( \gamma(\kappa\lambda^{-1}\zeta_b^{-1}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}\zeta_b^{-1}) - 1 \), and it then follows from (2.51) for the line bundle \( \lambda\zeta_b \) in place of \( \lambda \) and the divisor \( \mathfrak{d} = 1 \cdot a \) that \( \gamma(\lambda\zeta_b\zeta_a) = \gamma(\lambda\zeta_b) \); this together with (2.48) yields (2.49). Finally since \( b \) is the common divisor of all the holomorphic cross-sections of the bundle \( \lambda_0\zeta_b \), not all of these cross-sections vanish at a point \( a \notin b \), and it then follows from Lemma 2.6 that \( \gamma(\lambda_0\zeta_b\zeta_a^{-1}) = \gamma(\lambda_0\zeta_b) - 1 \) or equivalently \( \gamma(\kappa\lambda^{-1}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}) - 1 \); and in that case (2.50) is an immediate consequence of (2.51) for the divisor \( \mathfrak{d} = 1 \cdot a. \) That suffices for the proof.

In view of these observations, it is useful to consider on any compact Riemann surface \( M \) of genus \( g > 0 \) the base divisor \( b(\kappa\lambda^{-1}) \) for a holomorphic line bundle
\( \lambda \) for which \( \gamma(\kappa \lambda^{-1}) > 0 \); this is called the dual base divisor of the line bundle \( \lambda \), and the points appearing in this divisor with positive coefficients are called the dual base points of that line bundle.

**Corollary 2.28** (i) If \( M \) is a compact Riemann surface of genus \( g > 0 \) and \( \lambda \) is a holomorphic line bundle over \( M \) with the dual base divisor \( b(\kappa \lambda^{-1}) = a = a_1 + \cdots + a_n \)

\[
(2.52) \quad \gamma(\lambda \zeta_b) = \begin{cases} 
\gamma(\lambda) + 1 & \text{if } b \in a, \\
\gamma(\lambda) & \text{if } b \notin a.
\end{cases}
\]

(ii) If in addition the bundle \( \lambda \) is base-point-free then whenever \( a = a' + a'' \) for some positive divisors \( a' \) and \( a'' \) the line bundle \( \lambda \zeta_{a'} \) is also base-point-free, and \( \gamma(\lambda \zeta_{a''} \zeta_b) = \gamma(\lambda \zeta_{a''}) \) for any point \( b \notin a \).

**Proof:** (i) If \( a = \emptyset \) the line bundle \( \kappa \lambda^{-1} \) is base-point-free and (2.52) is just (2.47) of the preceding theorem. Otherwise the line bundle \( \kappa \lambda^{-1} \) has the base decomposition \( \kappa \lambda^{-1} = \lambda_0 \zeta_a \) for a base-point-free holomorphic line bundle \( \lambda_0 \) and the positive divisor \( a = a_1 + \cdots + a_n \). It then follows from (2.50) in the preceding theorem that \( \gamma(\lambda \zeta_b) = \gamma(\lambda) \) for any point \( b \notin a \). On the other hand it follows from (2.48) in the preceding theorem that

\[
\gamma(\lambda \zeta_{a_1 + \cdots + a_n}) = \gamma(\lambda) + n,
\]

while by Lemma 2.6

\[
\gamma(\lambda \zeta_{a_1 + \cdots + a_{i-1}}) \leq \gamma(\lambda \zeta_{a_1 + \cdots + a_{i-1} + a_i}) \leq \gamma(\lambda \zeta_{a_1 + \cdots + a_{i-1}}) + 1
\]

for \( 1 \leq i \leq n \); it is then evident from the two preceding equations that

\[
(2.53) \quad \gamma(\lambda \zeta_{a_1 + \cdots + a_i}) = \gamma(\lambda \zeta_{a_1 + \cdots + a_{i-1}}) + 1
\]

for \( 1 \leq i \leq n \). In particular \( \gamma(\lambda \zeta_{a_1}) = \gamma(\lambda) + 1 \), hence \( \gamma(\lambda \zeta_b) = \gamma(\lambda) + 1 \) for any point \( b \in a \).

(ii) If \( \lambda \) is base-point-free then since \( \gamma(\lambda \zeta_{a_1}) = \gamma(\lambda) + 1 \) by (2.53) it follows from Theorem 2.12 (ii) that \( \lambda \zeta_{a_1} \) is base-point-free; then since \( \gamma(\lambda \zeta_{a_1 + a_2}) = \gamma(\lambda \zeta_{a_1}) + 1 \) by (2.53) it also follows from Theorem 2.12 (ii) that \( \lambda \zeta_{a_1 + a_2} \) is base-point-free; and by repeating this argument it follows that all the line bundles \( \zeta_{a_1 + \cdots + a_i} \) for \( 1 \leq i \leq n \) are base-point-free. For any point \( b \notin a \) it follows from (2.49) and (2.50) of Theorem 2.27 (iii) that \( \gamma(\lambda \zeta_b \zeta_a) = \gamma(\lambda \zeta_b) + n \); it is then possible to apply to the line bundle \( \lambda \zeta_b \) the argument leading to (2.53) in part (i) of the proof to show that

\[
\gamma(\lambda \zeta_b \zeta_{a_1 + \cdots + a_i}) = \gamma(\lambda \zeta_b \zeta_{a_1 + \cdots + a_{i-1}}) + 1
\]

for \( 1 \leq i \leq n \), and consequently that

\[
\gamma(\lambda \zeta_b \zeta_{a_1 + \cdots + a_i}) = \gamma(\lambda \zeta_b) + i
\]
for $1 \leq i \leq n$. On the other hand it follows from (2.53) that
\[ \gamma(\lambda a + \ldots + a_i) = \gamma(\lambda) + i, \]
and since $\gamma(\lambda \zeta_i) = \gamma(\lambda)$ by (2.50) it follows that
\[ \gamma(\lambda \zeta_b \zeta_{a_1 + \ldots + a_i}) = \gamma(\lambda \zeta_{a_1 + \ldots + a_i}), \]
which suffices to conclude the proof.

For most base-point-free holomorphic line bundles $\lambda$ on a compact Riemann surface $M$ of genus $g > 0$ the bundle $\kappa \lambda^{-1}$ is not base-point-free; indeed all bundles $\lambda$ for which $\gamma(\lambda) \geq 2g$ are base-point-free by Theorem 2.25, but $\gamma(\kappa \lambda^{-1}) = 0$ for all of these bundles. On the other hand there are base-point-free holomorphic line bundles $\lambda$ on $M$ such that $\kappa \lambda^{-1}$ is base-point-free; for instance the identity bundle $1$ is base-point-free, as observed on page 36, and the canonical bundle $\kappa = \kappa \cdot 1^{-1}$ is base-point-free by Theorem 2.25 (iii). A pair of holomorphic line bundles $(\lambda_1, \lambda_2)$ is called a dual pair of base-point-free holomorphic line bundles over $M$ if $\lambda_1 \lambda_2 = \kappa$ is the canonical bundle of $M$: the pair of holomorphic line bundles $(1, \kappa)$ thus is an example of a dual pair of base-point-free holomorphic line bundles. Of course it may be the case that $\lambda_1 = \lambda_2$ for a particular dual pair of base-point-free holomorphic line bundles, as for instance the pair $(1, 1)$ on a surface of genus $g = 1$ since in that case $\kappa = 1$ by Corollary 2.20. Although the line bundles appearing in dual pairs of base-point-free holomorphic line bundles are somewhat special base-point-free line bundles, nonetheless many base-point-free holomorphic line bundles on $M$ can be expressed in terms of dual pairs of base-point-free line bundles over $M$.

**Corollary 2.29** To any base-point-free holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ of genus $g > 0$ for which $\gamma(\kappa \lambda^{-1}) > 0$ there correspond a unique dual pair of base-point-free holomorphic line bundles $(\lambda_1, \lambda_2)$ over $M$ and a unique positive divisor $a$ such that $\lambda_1 = \lambda a$ and $\lambda_2 = \kappa \lambda^{-1} \zeta_a^{-1}$.

**Proof:** If $\lambda$ is a base-point-free holomorphic line bundle over $M$ for which $\gamma(\kappa \lambda^{-1}) > 0$ either the line bundle $\kappa \lambda^{-1}$ is base-point-free, in which case $(\lambda, \kappa \lambda^{-1})$ already is a dual pair of base-point-free holomorphic line bundles, or the line bundle $\kappa \lambda^{-1}$ has a base decomposition $\kappa \lambda^{-1} = \lambda_2 \zeta_a$ for a base-point-free holomorphic line bundle $\lambda_2$ and a positive divisor $a$, where the divisor $a$ is uniquely determined as the base divisor of the line bundle $\kappa \lambda^{-1}$ and hence the line bundle $\lambda_2$ also is uniquely determined. In the latter case it follows from Corollary 2.28 (ii) that the line bundle $\lambda_1 = \lambda \zeta_a$ is base-point-free; and since $\kappa \lambda^{-1} = \lambda_2 \zeta_a$ then $\kappa = \lambda \zeta_a \cdot \lambda_2 = \lambda_1 \cdot \lambda_2$, so $(\lambda_1, \lambda_2)$ is a dual pair of base-point-free line bundles over $M$. That suffices to conclude the proof.

**Corollary 2.30** If $M$ is a compact Riemann surface of genus $g > 0$ and $r$ is an integer in Lüroth semigroup $\mathcal{L}(M)$ of $M$ such that $0 \leq r \leq g - 2$ and that $r + 1$ is not in the Lüroth semigroup of $M$, then any holomorphic line bundle $\lambda$ such that $c(\lambda) = r$ is part of a dual pair $(\lambda, \lambda_0)$ of base-point-free holomorphic line bundles over $M$. 
Proof: The integers in the Lüroth semigroup of $M$ are by definition the characteristic classes of base-point-free holomorphic line bundles over $M$; so any $r \in \mathcal{L}(M)$ is the characteristic class of some base-point-free holomorphic line bundle $\lambda$ over $M$. If $0 \leq r \leq g$ then by the Riemann-Roch theorem in the form of Theorem 2.18 it follows that $\gamma(\kappa^{-1}) = \gamma(\lambda) + g - 1 - r \geq g - 1 - r > 0$, and consequently there is a base decomposition $\kappa\lambda^{-1} = \lambda_0\gamma$, for a base-point-free holomorphic line bundle $\lambda_0$ and a positive divisor $\mathfrak{a}$. If $\mathfrak{a} \neq \emptyset$ then for any point $a \in \mathfrak{a}$ it follows from Corollary 2.28 (ii) that the line bundle $\lambda\mathfrak{a}_a$ also is base-point-free, and therefore that $r + 1 \in \mathcal{L}(M)$. By assumption that is not the case, and therefore $\mathfrak{a} = \emptyset$ so $\kappa\lambda^{-1} = \lambda_0$ is base-point-free and consequently $(\lambda, \lambda_0)$ is a dual pair of base-point-free holomorphic line bundles over $M$, which concludes the proof.

The Riemann-Roch Theorem in the form of Theorem 2.24 yields an effective lower bound for the dimension $\gamma(\zeta_\mathfrak{d})$ of the space of holomorphic cross-sections of the line bundle $\zeta_\mathfrak{d}$ of a positive divisor $\mathfrak{d} \geq 0$.

**Theorem 2.31** If $\mathfrak{d} \geq 0$ is a positive divisor on a compact Riemann surface $M$ of genus $g > 0$ then

$$(2.54) \quad \gamma(\zeta_\mathfrak{d}) - 1 = \dim L(\mathfrak{d}) - 1 \geq \max(0, \deg \mathfrak{d} - g),$$

and this lower bound is attained for some positive divisors $\mathfrak{d}$ of any degree.

Proof: For the trivial divisor $\mathfrak{d} = 0$ of course $\deg \mathfrak{d} = 0$ and $\gamma(\zeta_\mathfrak{d}) = \gamma(1) = 1$, so the asserted inequality holds trivially as an equality. The Brill-Noether matrix $\Omega_\mathfrak{d}$ for a positive divisor $\mathfrak{d}$ of degree $r > 0$ on a compact Riemann surface $M$ of genus $g > 0$ is a $g \times r$ matrix, and consequently rank $\Omega_\mathfrak{d} \leq \min(r, g)$; hence from the Riemann-Roch Theorem in the form of equation (2.42) it follows that $\gamma(\zeta_\mathfrak{d}) = \deg \mathfrak{d} - \text{rank } \Omega(\mathfrak{d}) + 1 \geq r - \min(r, g) + 1 = r + \max(-r, -g) + 1 = \max(0, r - g) + 1$, which is (2.54). If $U \subset M$ is a coordinate neighborhood with a local coordinate $z$ then $r$ distinct points in $U$ can be described by their $r$ distinct coordinate values $z_j$; so if $f_i(z)dz$ is a basis for the holomorphic differentials on $M$ for $1 \leq i \leq g$ then as in (2.37) the Brill-Noether matrix for the divisor $\mathfrak{d} = z_1 + \cdots + z_r$ is the $g \times r$ matrix $\{f_i(z_j)\}$. Since the holomorphic functions $f_i(z)$ are linearly independent, rank $\{f_i(z_j)\} = \min(r, g)$ for general sets of $r$ distinct points of this coordinate neighborhood; this is a quite standard result, but for completeness it will be discussed briefly in the next lemma. It follows that in general $\gamma(\zeta_\mathfrak{d}) - 1 = r - \text{rank } \{f_i(z_j)\} = r - \min(r, g) = \max(0, r - g)$, which suffices to conclude the proof.

**Lemma 2.32** If $f_i(z)$ are $g$ linearly independent holomorphic functions in a connected open subset $U \subset \mathbb{C}$ then rank $\{f_i(z_j)\} = \min(r, g)$ for all points $z = (z_1, z_2, \ldots, z_r) \in U \setminus U^r$ outside a proper holomorphic subvariety of $U^r$.

Proof: For convenience of notation let $f(z) = \{f_i(z)\}$ be the vector valued function formed from the functions $f_i(z)$. First it will be demonstrated by
induction on $r$ that if $f_i(z)$ are $g$ linearly independent holomorphic functions in $U$ then

$$\text{rank } (f(z_1) \ f(z_2) \ \cdots \ f(z_r)) = \min(r, g)$$

for at least one point $(z_1, \ldots, z_r) \in U^r$. It is clearly enough just to show that for $r \leq g$. The result is trivially true for $r = 1$, so assume it holds for $r - 1$ and consider the matrix in (2.55) where $r \leq g$. By the inductive hypothesis there will be some points $z_1, \ldots, z_{r-1} \in U$ so that the vectors $f(z_1), \ldots, f(z_{r-1})$ are linearly independent. If the result for the case $r$ were not true then for any point $z \in U$ the vector $f(z)$ would be a linear combination $f(z) = \sum_{i=1}^{r-1} a_i(z)f(z_i)$ for some $a_i(z)$ depending of course on the point $z$; but since $r - 1 < g$ there are constants $b_j \in \mathbb{C}$, not all zero, such that $\sum_{j=1}^{g} b_j f_j(z_i) = 0$ for $1 \leq i \leq r - 1$, and then $\sum_{j=1}^{g} b_j f_j(z) = 0$ for all $z$, a contradiction since the functions $f_i(z)$ are assumed to be linearly independent. That shows that the set of points $(z_1, \ldots, z_r) \in U^r$ at which rank $\{f_i(z_j)\} < r$ for any $r \leq g$ is a proper subset of $U^r$. Since this set clearly is a holomorphic subvariety of $U^r$ it must be a proper holomorphic subvariety, which suffices for the proof.

The preceding lemma was of interest here for the Brill-Noether matrix for sets of distinct points $z_i$. Another extreme case for a comparable result is that finitely many holomorphic functions in a connected open subset $U \subset \mathbb{C}$ are linearly independent if and only if their Wronskian, the Brill-Noether matrix for a divisor $n \cdot z$, is nonsingular. The nontrivial direction of that assertion though is rather more difficult to show, indeed is not true just for $\mathcal{C}^\infty$ functions, as demonstrated by Böcher\(^2\).

A positive divisor $\mathfrak{d} \geq 0$ for which the difference $\gamma(\zeta_0) - 1$ exceeds the lower bound of the preceding theorem is called a special positive divisor\(^3\) while a divisor for which the difference $\gamma(\zeta_0) - 1$ attains that lower bound is called a general positive divisor; thus

$$\mathfrak{d} \geq 0 \text{ is a special positive divisor if } \gamma(\zeta_0) - 1 > \max(0, \deg \mathfrak{d} - g),$$

$$\mathfrak{d} \geq 0 \text{ is a general positive divisor if } \gamma(\zeta_0) - 1 = \max(0, \deg \mathfrak{d} - g),$$

and any positive divisor $\mathfrak{d} \geq 0$ is either special or general. By the preceding theorem there are general positive divisors of any degree $r \geq 0$ on a compact Riemann surface of genus $g > 0$; indeed in the proof of that theorem it was


\(^3\)There is some variety in the literature in what is meant by the term “special positive divisor”. Traditionally the index of speciality of a positive divisor $\mathfrak{d}$ on a compact Riemann surface of genus $g$ is defined to be the difference $g - \text{rank } \Omega(\mathfrak{d}) = \dim L^{1,0}(\mathfrak{d}) - \text{dim } \mathfrak{d}$, and special positive divisors are defined to be those positive divisors for which this index is positive, hence those positive divisors $\mathfrak{d}$ for which rank $\Omega(\mathfrak{d}) < g$. On the other hand the definition adopted here seems quite commonly used in informal discussions of properties of positive divisors, and reflects more closely the most interesting aspect of the discussion of these divisors. The two notions obviously agree for divisors of degree at least $g$. 
demonstrated that general positive divisors actually are general in a fairly natural sense, which will be made more precise in the discussion of subvarieties of special positive divisors on page 268. It is worth noting explicitly here some common special and general positive divisors.

**Corollary 2.33** On a compact Riemann surface $M$ of genus $g > 0$ the divisor $1 \cdot p$ for a point $p \in M$ is a general positive divisor. Equivalently not all holomorphic differential forms on $M$ vanish at any point $p \in M$.

**Proof:** For any point $p$ of a compact Riemann surface $M$ of genus $g > 0$ it follows from Theorem 2.4 that $\gamma(\zeta_p) - 1 = 0 = \max(0, 1 - g)$ so the divisor $1 \cdot p$ is a general positive divisor. It follows from Theorem 2.24 that rank $\Omega(p) = 2 - \gamma(\zeta_p) = 1$ for the Brill-Noether matrix $\Omega(1 \cdot p)$ of this divisor; and since as in (2.38) the Brill-Noether matrix for this divisor is the $g \times 1$ matrix
\[
\Omega(1 \cdot p) = \begin{pmatrix} f_{1\alpha}(p) \\ \vdots \\ f_{g\alpha}(p) \end{pmatrix}
\]
where $f_{i\alpha}dz_\alpha$ are the holomorphic differential forms on $M$ it follows that not all of these differential forms vanish at the point $p$, which suffices for the proof.

**Corollary 2.34** On a compact Riemann surface $M$ of genus $g > 0$ a positive divisor $d$ with $\deg d > 2g - 2$ is a general positive divisor. Equivalently rank $\Omega(d) = g$ for the $g \times \deg d$ Brill-Noether matrix $\Omega(d)$ of any positive divisor $d$ with $\deg d > 2g - 2$.

**Proof:** If $d$ is a positive divisor and $\deg d > 2g - 2$ then $c(\kappa\zeta^{-1}_d) < 0$ so $\gamma(\kappa\zeta^{-1}_d) = 0$ by Corollary 1.4, and it then follows from the Riemann-Roch Theorem in the form of equation (2.24) that $\gamma(\zeta_d) - 1 = \deg d - g = \max(0, \deg d - g)$ so $d$ is a general positive divisor. From Theorem 2.24 it then follows that rank $\Omega(d) = \deg d + 1 - \gamma(\zeta_d) = g$, and that suffices for the proof.

**Corollary 2.35** On a compact Riemann surface $M$ of genus $g > 1$ a positive divisor of degree $2g - 2$ is a special positive divisor if and only if it is a canonical divisor; all positive divisors of degree $2g - 2$ other than canonical divisors are general positive divisors.

**Proof:** A positive divisor $d$ of degree $2g - 2$ is a special positive divisor if and only if $\gamma(\zeta_d) - 1 > \max(0, g - 2)$, hence if and only if $\gamma(\zeta_d) \geq \max(2, g)$; thus $\gamma(\zeta_d) \geq g$ when $g > 1$, hence $\zeta_d$ is the canonical bundle by the Canonical Bundle Theorem, Theorem 2.18, so $d$ is a positive canonical divisor and that suffices for the proof.

This last corollary is a special case of the more general observation that the special positive divisors on compact Riemann surfaces of genus $g > 1$ arise from positive canonical divisors. To make this more precise, a nontrivial positive
divisor $\mathfrak{d}$ is said to be *part of a positive canonical divisor* if its residual divisor $\mathfrak{d}'$ is also a positive divisor, that is, if there is a positive divisor $\mathfrak{d}'$ such that $\mathfrak{d} + \mathfrak{d}' = \mathfrak{f}$; in particular a positive canonical divisor itself is part of a positive canonical divisor.

**Corollary 2.36** A nontrivial special positive divisor on a compact Riemann surface of genus $g > 0$ is part of a positive canonical divisor; and conversely any positive divisor $\mathfrak{d}$ of $\deg \mathfrak{d} \geq g$ that is part of a positive canonical divisor is a special positive divisor.

**Proof:** Combining the Riemann-Roch Theorem in the form of equation (2.42) with the definition (2.56) shows that a positive divisor $\mathfrak{d}$ of degree $r > 0$ with the Brill-Noether matrix $\Omega(\mathfrak{d})$ is a special positive divisor if and only if $r - \rank \Omega(\mathfrak{d}) = \gamma(\zeta_0) - 1 > \max(0, r - g)$, hence if and only if $\rank \Omega(\mathfrak{d}) < \min(r, g)$. Thus if $\mathfrak{d}$ is a special positive divisor then $\rank \Omega(\mathfrak{d}) < g$, so if the Brill-Noether matrix is defined in terms of a basis $f_i(z)dz$ for the holomorphic differential forms on $M$ there is a nontrivial row vector $c \in \mathbb{C}^g$ such that $c \cdot \Omega(\mathfrak{d}) = 0$; then $\sum_i c_i f_i(z)dz$ is a nontrivial holomorphic differential form that vanishes at the divisor $\mathfrak{d}$, hence $\mathfrak{d}$ is part of the positive canonical divisor that is the divisor of this holomorphic differential form. Conversely if $\mathfrak{d}$ is part of a positive canonical divisor then there is a nontrivial holomorphic differential form that vanishes on $\mathfrak{d}$, so that $\rank \Omega(\mathfrak{d}) < g$; and if $r \geq g$ that is just the condition that $\mathfrak{d}$ is a special divisor. That suffices for the proof.

On a compact Riemann surface of genus $g > 0$ the line bundle of the trivial divisor $\mathfrak{d} = 0$ is the identity bundle $\zeta_0 = 1$, and since $\gamma(\zeta_0) - 1 = 0 = \max(0, -g)$ it follows that the trivial divisor is a general positive divisor. A divisor $\mathfrak{d}$ for which $\deg \mathfrak{d} = 1$ is a point bundle, so also is a general positive divisor by Corollary 2.33. On the other hand any positive divisor $\mathfrak{d}$ for which $\deg \mathfrak{d} > 2g - 2$ is a general divisor by Corollary 2.34. Consequently on a compact Riemann surface of genus $g > 0$

(2.57) \[ 2 \leq \deg \mathfrak{d} \leq 2g - 2 \] for any special positive divisor $\mathfrak{d} \geq 0$.

The upper bound $2g - 2$ for the degrees of the special positive divisors is effective for compact Riemann surfaces of genus $g > 1$ by Corollary 2.35. Thus the investigation of special positive divisors can be limited to an examination of special positive divisors with degrees limited to the values (2.57); this will be taken up again in the discussion of maximal sequences in Chapter 13.
Chapter 3

Holomorphic Differentials: Jacobi and Picard Varieties

Holomorphic differential forms on a compact Riemann surface often are called holomorphic abelian differentials or abelian differentials of the first kind. They are of interest only on surfaces of genus $g > 0$, since as noted earlier there are no nontrivial holomorphic differential forms on a compact Riemann surface of genus $g = 0$. When a compact Riemann surface $M$ of genus $g > 0$ is identified with the quotient $M = \tilde{M}/\Gamma$ of its universal covering space $\tilde{M}$ by the group $\Gamma$ of covering translations\(^1\) a holomorphic function on $M$ can be viewed alternatively as a $\Gamma$-invariant holomorphic function on $\tilde{M}$. It is convenient to be able to pass back and forth between these two perspectives quite freely, and often will be done without explicit comment; in particular no attempt will be made to use a notation that distinguishes between these perspectives. That should not cause any confusion, since in most cases the relevant interpretation will be apparent from the context and generally it does not matter anyway. A holomorphic abelian differential on $M$ can be viewed as a $\Gamma$-invariant holomorphic differential form on $\tilde{M}$ in the same way, and will be viewed as such without further comment whenever it is convenient to do so. On the other hand the integral

\[(3.1) \quad w(z,a) = \int_a^z \omega\]

of a holomorphic abelian differential $\omega$ on the Riemann surface $M$ is a well defined holomorphic function of the variables $(z,a) \in M \times M$ locally, since $\omega$ is a closed differential form, but is inevitably a multiple-valued function of these variables in the large; however the monodromy theorem ensures that this integral is a single-valued holomorphic function on the simply connected complex manifold $\tilde{M} \times \tilde{M}$, independent of the choice of the path of integration on $\tilde{M}$.

\(^1\)A survey of the topological properties of surfaces prerequisite to the discussion here can be found in Appendix D.
This function is called a *holomorphic abelian integral* on the Riemann surface $M$, although of course really it is defined as a holomorphic function on the universal covering space $\tilde{M}$ in both variables. A holomorphic abelian integral clearly satisfies the symmetry condition $w(z, a) = -w(a, z)$, and $w(z, z) = 0$ for all points $z \in \tilde{M}$. It is more convenient in many circumstances to view a holomorphic abelian integral as a function of the first variable only, and to allow it to be modified by an arbitrary additive constant; in such cases the function is denoted by $w(z)$ rather than $w(z, a)$, but also is called a holomorphic abelian integral. It must be kept in mind though that the abelian integral $w(z)$ is determined by the abelian differential $\omega$ only up to an arbitrary additive constant. For any choice of the abelian integral $w(z)$ the integral (3.1) is given by $w(z, a) = w(z) - w(a)$.

**Lemma 3.1** The holomorphic abelian integrals $w(z)$ on a compact Riemann surface $M$ of genus $g > 0$ can be characterized as those holomorphic functions on the universal covering space $\tilde{M}$ of the surface that satisfy

\[(3.2) \quad w(Tz) = w(z) + \omega(T) \quad \text{for all} \quad T \in \Gamma\]

for a group homomorphism $\omega \in \text{Hom}(\Gamma, \mathbb{C})$, where $\Gamma$ is the covering translation group of $M$.

**Proof:** If $w(z)$ is a holomorphic abelian integral on $M$ then $dw(Tz) = dw(z)$ for any covering translation $T \in \Gamma$, since $dw = \omega$ is invariant under $\Gamma$; therefore $w(Tz) = w(z) + \omega(T)$ for some complex constant $\omega(T)$. For any two covering translations $S, T \in \Gamma$

\[
w(STz) = w(z) + \omega(ST) = w(S \cdot Tz) = w(Tz) + \omega(S) = w(z) + \omega(T) + \omega(S);
\]

consequently $\omega(ST) = \omega(S) + \omega(T)$ so the mapping $T \mapsto \omega(T)$ is a group homomorphism $\omega \in \text{Hom}(\Gamma, \mathbb{C})$. Conversely if $w(z)$ is a holomorphic function on $\tilde{M}$ that satisfies (3.2) for some homomorphism $\omega \in \text{Hom}(\Gamma, \mathbb{C})$ then $\omega = dw$ is invariant under $\Gamma$ so is a holomorphic abelian differential on $M$, and the function $w(z)$ is the integral of $\omega$ and hence is a holomorphic abelian integral. That suffices for the proof.

Clearly the homomorphism $\omega$ in (3.2) is unchanged when the abelian integral $w(z)$ is replaced by $w(z) + c$ for a complex constant $c$, so it is determined uniquely by the abelian differential $\omega = dw$; it is called the *period class* of the holomorphic abelian differential $\omega$. The mapping that associates to a holomorphic abelian differential its period class is a homomorphism $\Gamma(M, \mathcal{O}^{1,0}) \rightarrow \text{Hom}(\Gamma, \mathbb{C})$ between these two additive abelian groups.

**Lemma 3.2** A holomorphic abelian differential on a compact Riemann surface of genus $g > 0$ is determined uniquely by its period class.
Proof: If the period class \( \omega \in \text{Hom}(\Gamma, \mathbb{C}) \) of a holomorphic abelian differential \( \omega \) is identically zero then in view of (3.2) any associated holomorphic abelian integral is a \( \Gamma \)-invariant holomorphic function \( \omega \) on \( \tilde{M} \), or equivalently is a holomorphic function on the compact Riemann surface \( M \); so by the maximum modulus theorem it must be constant and therefore \( \omega = dw = 0 \), which suffices for the proof.

A homomorphism \( \omega \in \text{Hom}(\Gamma, \mathbb{C}) \) is necessarily trivial on the commutator subgroup \( [\Gamma, \Gamma] \subset \Gamma \), so induces a homomorphism from the abelianized group \( \Gamma/[[\Gamma, \Gamma]] \cong H_1(M) \) to the complex numbers and therefore can be viewed as an element in the dual group \( \text{Hom}(H_1(M), \mathbb{C}) = H^1(M, \mathbb{C}) \); and conversely any cohomology class \( \omega \in H^1(M, \mathbb{C}) = \text{Hom}(H_1(M), \mathbb{C}) \) is induced by a unique homomorphism \( \omega \in \text{Hom}(\Gamma, \mathbb{C}) \). A homomorphism \( \omega \in \text{Hom}(\Gamma, \mathbb{C}) \) consequently is determined uniquely either by its values \( \omega(\tau_j) \) on a basis \( \tau_j \in H_1(M) \) for the homology of \( M \) or by its values \( \omega(T_j) \) on covering transformations \( T_j \in \Gamma \) generating the covering translation group \( \Gamma \) of \( M \). Under the canonical isomorphism \( \pi_{z_0} : \Gamma \rightarrow \pi_1(M, \pi(z_0)) \) between the covering translation group and the fundamental group of the surface determined by the choice of a base point \( z_0 \in \tilde{M} \), as discussed in Appendix D.1, an element \( T \in \Gamma \) is associated to the homotopy class of the image \( \pi(\tilde{T}) \subset M \) under the covering projection \( \pi : \tilde{M} \rightarrow M \) of any path \( \tilde{\tau} \subset \tilde{M} \) from the base point \( z_0 \in \tilde{M} \) to the point \( Tz_0 \in \tilde{M} \). If \( \omega \) is a holomorphic abelian differential on \( M \) and \( w(z) \) is its integral then \( \int_{\pi(\tilde{T})} \omega = \int_{\tilde{T}} \omega = w(Tz_0) - w(z_0) = \omega(T) \), so the period class represents the periods of the holomorphic differential form in the customary sense. The integral of course depends only on the homology class represented by the path \( \pi(\tilde{T}) \subset M \).

Holomorphic abelian differentials on a compact Riemann surface \( M \) of genus \( g > 0 \) can be identified with holomorphic cross-sections of the canonical bundle \( \kappa \) of \( M \), extending the local identification (2.19); consequently by the Canonical Bundle Theorem, Theorem 2.18, the set of holomorphic abelian differentials form a complex vector space of dimension \( g \). If \( \omega_i \in \Gamma(M, \mathcal{O}^{(1,0)}) \) for \( 1 \leq i \leq g \) is a basis for the holomorphic abelian differentials on \( M \) and \( \tau_j \in H_1(M) \) for \( 1 \leq j \leq 2g \) is a basis for the homology of \( M \) the values \( \omega_{ij} = \int_{\tau_j} \omega_i \) can be viewed as forming a \( g \times 2g \) complex matrix \( \Omega \); this is called the period matrix of the Riemann surface in terms of the bases \( \omega_i \) and \( \tau_j \). An arbitrary holomorphic abelian differential \( \omega \) on \( M \) can be expressed as the sum \( \omega = \sum_{i=1}^{g} c_i \omega_i \) for some complex constants \( c_i \), and an arbitrary homology class \( \tau \) on \( M \) can be expressed as the sum \( \tau = \sum_{j=1}^{2g} n_j \tau_j \) for some integers \( n_j \); consequently \( \omega(\tau) = \sum_{i=1}^{g} \sum_{j=1}^{2g} c_i \omega_i(n_j \tau_j) = \sum_{i=1}^{g} \sum_{j=1}^{2g} c_i \omega_{ij} n_j \), or in matrix notation \( \omega(\tau) = c^T \Omega n \) for the column vectors \( c = \{c_i\} \in \mathbb{C}^g \) and \( n = \{n_j\} \in \mathbb{Z}^{2g} \). Thus all the periods of the holomorphic abelian differentials on \( M \) can be expressed in this way in terms of the period matrix \( \Omega \) for any choice of bases \( \omega_i \) and \( \tau_j \).

**Theorem 3.3** The period matrix \( \Omega \) of a compact Riemann surface of genus...
$g > 0$ is a nonsingular period matrix.\(^2\)

**Proof:** By definition the period matrix $\Omega$ is a nonsingular period matrix if and only if its columns are linearly independent over the real numbers; and by Lemma F.1 that is equivalent to the condition that the associated full period matrix $\begin{pmatrix} \Omega & \Omega \end{pmatrix}$ is a nonsingular $2g \times 2g$ matrix in the usual sense. If $\Omega = A + iB$ for some $g \times 2g$ real matrices $A$ and $B$ then since

\[
\begin{pmatrix} \Omega & \Omega \end{pmatrix} = \begin{pmatrix} 1 & i1 \\ 1 & -i1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\]

where $I$ is the $g \times g$ identity matrix, the full period matrix is nonsingular if and only if the $2g \times 2g$ real matrix $C = \begin{pmatrix} A \\ B \end{pmatrix}$ is nonsingular. If the matrix $C$ is singular there is a nontrivial real row vector $y \in \mathbb{R}^{2g}$ such that $yC = 0$. If $y = (y_1, y_2)$ for some real row vectors $y_i \in \mathbb{R}^g$ then $y_1A + y_2B = 0$. The vector $c = y_1 - iy_2 \in \mathbb{C}^g$ consequently is a nontrivial complex row vector such that

\[
\Re(c\Omega) = \Re\left((y_1 - iy_2)(A + iB)\right) = y_1 + y_2 = 0,
\]

where $\Re(z)$ denotes the real part of the complex vector $z$. The periods of the nontrivial holomorphic abelian differential $\omega = \sum_i c_i \omega_i$ on a basis for the homology group $H_1(M)$ are the entries of the vector $c\Omega$ so are purely imaginary; consequently the period $\omega(T)$ is purely imaginary for any covering translation $T \in \Gamma$. If $w(z)$ is the integral of the nontrivial holomorphic abelian differential $\omega$ then $|\exp w(Tz)| = |\exp (w(z) + \omega(T))| = |\exp w(z)|$ for every covering translation $T \in \Gamma$ since $|\exp \omega(T)| = 1$; hence $|\exp w(z)|$ is a well defined continuous function on the compact Riemann surface $M$ so by the maximum modulus theorem the holomorphic function $\exp w(z)$ must be constant. The abelian integral $w(z)$ itself then is also constant and $\omega = dw = 0$, a contradiction since $\omega \neq 0$. That suffices to conclude the proof.

The essence of the proof of the preceding theorem is the observation that a nontrivial holomorphic abelian differential cannot have purely imaginary periods, or of course purely real periods either; that is one restriction on the possible period matrices of Riemann surfaces, but there are deeper restrictions that will be discussed later. The following simple consequences of the preceding theorem are useful in various circumstances.

**Corollary 3.4** If $\omega_i \in \Gamma(M, O^{1,0})$ for $1 \leq i \leq g$ is a basis for the holomorphic abelian differentials on a compact Riemann surface $M$ of genus $g > 0$ then the closed differential 1-forms $\omega_i$ and $\overline{\omega}_i$ for $1 \leq i \leq g$ are a basis for the deRham group $\mathcal{H}^1(M)$ of closed differential forms of degree 1 on $M$ modulo exact differential forms.

**Proof:** The period class of any closed differential form $\phi$ of degree 1 on $M$ is determined by the periods $\phi(\tau_j)$ on a basis $\tau_j \in H_1(M)$; and a collection

\(^2\)The definition and properties of nonsingular period matrices are discussed in Appendix F.1.
of $2g$ closed differential forms $\phi_i$ form a basis for the deRham group precisely when the $2g \times 2g$ complex matrix \( \{ \phi_i(\tau_j) \} \) is nonsingular. In particular for the differential forms $\phi_i = \omega_i$ for $1 \leq i \leq g$ and $\phi_i = \omega_{i-g}$ for $g+1 \leq i \leq 2g$ the period matrix is \( \{ \phi_i(\tau_j) \} = (\Omega) \); and since this is a nonsingular matrix as a consequence of the preceding theorem it follows that these differential forms are a basis for the deRham group, thereby concluding the proof.

**Corollary 3.5** An element $T \in \Gamma$ in the covering translation group of a compact Riemann surface $M$ of genus $g > 0$ is contained in the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ if and only if $\omega(T) = 0$ for the period classes $\omega$ of all holomorphic abelian differentials on $M$. 

**Proof:** It was already noted that the period class $\omega$ of a holomorphic abelian differential vanishes on any element $T \in [\Gamma, \Gamma]$. Conversely if the period classes of all holomorphic abelian differentials vanish on an element $T \in \Gamma$ then the preceding corollary shows that the period classes of all differential one-forms also vanish on $T$; that means that homotopy class represented by the covering translation $T$ determines the trivial homology class in the natural homomorphism $\Gamma \rightarrow H_1(M) = \Gamma/[\Gamma, \Gamma]$, which suffices for the proof.

**Theorem 3.6** The period matrices of a compact Riemann surface $M$ of genus $g > 0$ for all choices of bases for the holomorphic abelian differentials on $M$ and for the homology of $M$ are a full class of equivalent $^3$ period matrices.

**Proof:** Two bases $\tilde{\omega}_i$ and $\omega_i$ for the holomorphic abelian differentials on $M$ are related by $\tilde{\omega}_i = \sum_{k=1}^g a_{ik} \omega_k$ for an arbitrary nonsingular complex matrix $A = \{a_{ik}\} \in \text{Gl}(g, \mathbb{C})$, and two bases $\tilde{\tau}_j$ and $\tau_j$ for the homology of $M$ are related by $\tilde{\tau}_j = \sum_{l=1}^{2g} \tau_l q_{lj}$ for an arbitrary invertible integral matrix $Q = \{q_{lj}\} \in \text{Gl}(2g, \mathbb{Z})$. The associated period matrices $\Omega = \{\omega_i(\tau_j)\}$ and $\Omega = \{\omega_k(\tau_l)\}$ then are related by $\tilde{\omega}_i(\tilde{\tau}_j) = \sum_{k=1}^g \sum_{l=1}^{2g} a_{ik} \omega_k(\tau_l) q_{lj}$, or in matrix terms $\tilde{\Omega} = A \Omega Q$. That is just the condition (F.1) that these two period matrices are equivalent period matrices, which suffices for the proof.

A period matrix $\Omega$ for the Riemann surface $M$ of genus $g > 0$ describes a lattice subgroup $L(\Omega) = \Omega \mathbb{Z}^{2g} \subset \mathbb{C}^g$, which in turn describes a complex torus $J(\Omega) = \mathbb{C}^g/L(\Omega) = \mathbb{C}^g/\Omega \mathbb{Z}^{2g}$. Since the period matrices of the surface $M$ are equivalent period matrices, by the preceding theorem, it follows from Corollary F.9 that the complex tori described by these period matrices are biholomorphic complex manifolds; thus there is really a unique such complex manifold, called the Jacobi variety of the Riemann surface $M$ and denoted by $J(M)$. The Jacobi varieties of compact Riemann surfaces play a major role in almost any discussion of Riemann surfaces.

If $\omega_i$ is a basis for the holomorphic abelian differentials on a compact Riemann surface $M$ of genus $g > 0$ the associated integrals $w_i(z)$ can be taken

---

$^3$The equivalence of period matrices is defined and discussed in appendix F.1.
as the components of a column vector \( \tilde{w}(z) = \{w_i(z)\} \in \mathbb{C}^g \); the mapping that associates to any point \( z \in \tilde{M} \) the vector \( \tilde{w}(z) \) is a holomorphic mapping \( \tilde{w} : \tilde{M} \rightarrow \mathbb{C}^g \). For any covering translation \( T \in \Gamma \) the values \( \omega_i(T) \) of the period classes of these holomorphic abelian differentials can be viewed correspondingly as the components of a column vector \( \omega(T) = \{\omega_i(T)\} \in \mathbb{C}^g \); and the mapping that associates to a covering translation \( T \in \Gamma \) the vector \( \omega(T) \) is a group homomorphism \( \omega \in \text{Hom}(\Gamma, \mathbb{C}^g) \), which of course can be viewed alternatively as a group homomorphism \( \omega \in \text{Hom}(H_1(M), \mathbb{C}^g) \). The image \( \omega(\tau_j) \in \mathbb{C}^g \) of one of the homology classes \( \tau_j \) forming a basis for the homology \( H_1(M) \) of the surface \( M \) is just column \( j \) of the period matrix \( \Omega \) of the surface \( M \) for the bases \( \omega_i \) and \( \tau_j \), or equivalently is one of the generators of the lattice subgroup \( \mathcal{L}(\Omega) \); consequently the image subgroup \( \omega(\Gamma) = \omega(H_1(M)) \subset \mathbb{C}^g \) is precisely the lattice subgroup \( \mathcal{L}(\Omega) \subset \mathbb{C}^g \). Each of the integrals \( w_i(z) \) satisfies (3.2), so altogether

\[
(3.3) \quad \tilde{w}(Tz) = \tilde{w}(z) + \omega(T) \quad \text{for all } T \in \Gamma.
\]

This shows that points of \( \tilde{M} \) that are mapped to one another under the covering translation group \( \Gamma \), and hence have the same image under the covering projection \( \pi : \tilde{M} \rightarrow M \), have as their images under the covering projection \( \pi : \tilde{M} \rightarrow \mathbb{C}^g \) points of \( \mathbb{C}^g \) that are mapped to one another by translation by vectors in the lattice subgroup \( \mathcal{L}(\Omega) \), and hence have the same image under the covering projection \( \pi : \mathbb{C}^g \rightarrow J(\Omega) \); consequently the mapping \( \tilde{w} : \tilde{M} \rightarrow \mathbb{C}^g \) induces a holomorphic mapping \( w : M \rightarrow J(\Omega) \), and these mappings together with the covering projection mappings \( \pi \) form the commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{w}} & \mathbb{C}^g \\
\pi \downarrow & & \downarrow \pi \\
M = \tilde{M}/\Gamma & \xrightarrow{w} & J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega).
\end{array}
\]

The abelian integrals \( w_i(z) \) are determined only up to arbitrary complex constants, so the mapping \( w \) is determined only up to an arbitrary translation in the complex torus \( J(\Omega) \). The choice of another basis for the holomorphic abelian differentials has the effect of replacing the period matrix \( \Omega \) by \( A\Omega \) for a nonsingular matrix \( A \in \text{Gl}(g, \mathbb{C}) \) and correspondingly replacing the mapping \( w \) by \( Aw \); this can be viewed as the result of composing the initial holomorphic mapping \( w : M \rightarrow J(\Omega) \) with the composition of that mapping and the biholomorphic mapping \( A : J(\Omega) \rightarrow J(A\Omega) \) between these complex tori, so it really amounts to the same mapping from \( M \) to the Jacobi variety \( J(M) \) of the Riemann surface \( M \). The mapping \( w : M \rightarrow J(M) \) so defined is called the Abelian-Jacobi mapping of the Riemann surface into its Jacobi variety. Again though this mapping is determined only up to arbitrary translations in the complex torus \( J(M) \).

**Theorem 3.7** A holomorphic mapping \( f : M \rightarrow T \) from a compact Riemann surface \( M \) of genus \( g > 0 \) to a complex torus \( T \) can be factored uniquely as the
composition \( f = h \circ w \) of the Abel-Jacobi mapping \( w : M \to J(M) \) from \( M \) to its Jacobi variety \( J(M) \) and a holomorphic mapping \( h : J(M) \to T \) from the Jacobi variety to the complex torus \( T \), and this property characterizes the Abel-Jacobi mapping.

**Proof:** Suppose that the torus \( T \) is represented as the quotient \( T = \mathbb{C}^h/\mathcal{L} \) for a lattice subgroup \( \mathcal{L} \subset \mathbb{C}^h \). A holomorphic mapping \( f : M \to T \) lifts to a holomorphic mapping \( \tilde{f} : \tilde{M} \to \mathbb{C}^h \) from the universal covering space \( \tilde{M} \) of \( M \) to the universal covering space \( \mathbb{C}^h \) of the torus \( T \); and a mapping \( \tilde{f} : M \to \mathbb{C}^h \) induces a mapping \( f : M \to T \) between the quotient spaces if and only if for any covering translation \( T \in \Gamma \) and any point \( z \in \tilde{M} \) there is a lattice vector \( \lambda \in \mathcal{L} \) such that

\[
\tilde{f}(Tz) = \tilde{f}(z) + \lambda.
\]

Since the lattice subgroup \( \mathcal{L} \subset \mathbb{C}^h \) is discrete the lattice vector \( \lambda \) in (3.5) must be independent of the point \( z \in \tilde{M} \) so can be viewed as a function \( \lambda(T) \) of the covering translation \( T \in \Gamma \); and it is evident from (3.5) that \( \lambda(ST) = \lambda(S) + \lambda(T) \) for any two covering translations \( S, T \in \Gamma \), so the function \( \lambda(T) \) is a group homomorphism \( \lambda \in \text{Hom}(\Gamma, \mathbb{C}) \). By Lemma 3.1 the component functions \( f_i(z) \) of the mapping \( \tilde{f} \) must be holomorphic integrals on \( M \); so if \( w_j(z) \) is a basis for the holomorphic abelian integrals on \( M \) then \( f_i(z) = \sum_{j=1}^g a_{ij} w_j(z) + a_i \) for some uniquely determined complex constants \( a_{ij}, a_i \), or in matrix notation

\[
\tilde{f}(z) = A\tilde{w}(z) + a
\]

for the matrix \( A = \{a_{ij}\} \in \mathbb{C}^{h \times g} \), the vector \( a = \{a_i\} \in \mathbb{C}^h \) and the mapping \( \tilde{w} : \tilde{M} \to \mathbb{C}^g \) described by the component functions \( w_j(z) \). It follows from (3.2) and (3.5) that \( \lambda(T) = \tilde{f}(Tz) - \tilde{f}(z) = A(w(Tz) - w(z)) = A\omega(T) \) for any covering translation \( T \in \Gamma \); the vectors \( \omega(T) \) generate the lattice subgroup \( \mathcal{L}(\Omega) \) described by the period matrix \( \Omega \) of the surface \( M \) in terms of the chosen basis for the holomorphic abelian integrals while \( \lambda(T) \in \mathcal{L} \), so \( A\mathcal{L}(\Omega) \subset \mathcal{L} \) and as in Theorem F.6 the affine mapping \( \tilde{h}(t) = At + a \) from \( \mathbb{C}^g \) to \( \mathbb{C}^h \) induces a holomorphic mapping \( h : J(M) \to T \) from the Jacobi variety \( J(M) = \mathbb{C}^g/\mathcal{L}(\Omega) \) of \( M \) to the complex torus \( T = \mathbb{C}^h/\mathcal{L} \). The holomorphic mapping \( \tilde{w} \) induces the Abel-Jacobi mapping \( w : M \to J(\Omega) \) as in the commutative diagram (3.4), so since \( \tilde{f} = h \circ \tilde{w} \) by (3.6) it follows that \( f = h \circ w \). To show that this property characterizes the Abel-Jacobi mapping suppose that \( w_0 : M \to J_0(M) \) is a holomorphic mapping from \( M \) to another complex torus \( J_0(M) \) such that any holomorphic mapping \( f : M \to T \) from \( M \) to a complex torus \( T \) can be factored uniquely as the composition \( f = h_0 \circ w_0 \) of the mapping \( w_0 : M \to J_0 \) and a holomorphic mapping \( h_0 : J_0 \to T \). Then in particular for this mapping \( w_0 \) and for the Abel-Jacobi mapping \( w \) there are unique holomorphic mappings \( h_0 : J_0(M) \to J(M) \) and \( h : J(M) \to J_0(M) \) such that \( w = h_0 \circ w_0 \) and \( w_0 = h \circ w \). The uniqueness implies that \( h \circ h_0 \) and \( h_0 \circ h \) are both identity mappings and consequently that \( h : J(M) \to J_0(M) \) is a biholomorphic mapping, thus
identifying the complex torus $J_0(M)$ with the Jacobi variety $J(M)$. That suffices for the proof.

Paralleling the exact sequence of sheaves (2.20) over $M$ is the exact sequence of sheaves

$$
0 \longrightarrow \mathbb{C}^* \overset{\iota}{\longrightarrow} \mathcal{O}^* \overset{\text{dl}}{\longrightarrow} \mathcal{O}^{(1,0)} \longrightarrow 0
$$

in which $\iota$ is the natural inclusion homomorphism and $\text{dl}(f) = df/f = d \log f$ for any germ $f$ of a nowhere vanishing holomorphic function. The exactness of (3.7) follows immediately from the exactness of (2.20), since $df/f = \text{dl}(f) = 0$ precisely when $f$ is constant and a germ $\phi \in \mathcal{O}^{(1,0)}$ can be written $\phi = dg$ for some germ $g \in \mathcal{O}$ so $\phi = \text{dl}(f)$ where $f = \exp g$.

**Theorem 3.8** On a compact Riemann surface $M$ of genus $g > 0$ there is the exact sequence

$$
0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \overset{\delta}{\longrightarrow} H^1(M, \mathbb{C}^*) \overset{\iota}{\longrightarrow} H^1(M, \mathcal{O}^*) \overset{\text{dl}}{\longrightarrow} H^1(M, \mathcal{O}^{(1,0)}) \longrightarrow 0,
$$

where $\iota$ is the homomorphism induced by the natural inclusion $\mathbb{C}^* \subset \mathcal{O}^*$ and $c(\lambda) \in \mathbb{Z}$ is the characteristic class of a holomorphic line bundle $\lambda \in H^1(M, \mathcal{O}^*)$.

**Proof:** The exact cohomology sequence arising from the exact sequence of sheaves (3.7) begins

$$
0 \longrightarrow \Gamma(M, \mathbb{C}^*) \overset{\iota}{\longrightarrow} \Gamma(M, \mathcal{O}^*) \overset{\text{dl}}{\longrightarrow} \Gamma(M, \mathcal{O}^{(1,0)}) \overset{\delta}{\longrightarrow} \delta \longrightarrow H^1(M, \mathbb{C}^*) \overset{\iota}{\longrightarrow} H^1(M, \mathcal{O}^*) \overset{\text{dl}}{\longrightarrow} H^1(M, \mathcal{O}^{(1,0)}) \longrightarrow \cdots
$$

Since $M$ is compact every holomorphic function on $M$ is constant, by the maximum modulus theorem, so the homomorphism $\iota : \Gamma(M, \mathbb{C}^*) \longrightarrow \Gamma(M, \mathcal{O}^*)$ is an isomorphism and consequently the homomorphism $\text{dl} : \Gamma(M, \mathcal{O}^*) \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)})$ is the zero homomorphism; thus (3.9) reduces to the exact sequence

$$
0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \overset{\delta}{\longrightarrow} H^1(M, \mathbb{C}^*) \overset{\iota}{\longrightarrow} H^1(M, \mathcal{O}^*) \overset{\text{dl}}{\longrightarrow} H^1(M, \mathcal{O}^{(1,0)}) \longrightarrow \cdots
$$

In terms of an open coordinate covering $\mathcal{U} = \{U_\alpha\}$ of $M$, a cohomology class $\lambda \in H^1(M, \mathcal{O}^*)$ can be represented by a cocycle $\lambda_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and its image $\text{dl}(\lambda) \in H^1(M, \mathcal{O}^{(1,0)})$ then is represented by the cocycle $d \log \lambda_{\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{O}^{(1,0)})$. By the Theorem of Dolbeault, Theorem 1.10, for the special case that the line bundle $\lambda$ is the canonical bundle $\lambda = \kappa$ so that $\mathcal{O}^{(1,0)} \cong \mathcal{O}(\kappa)$, there is the isomorphism

$$
H^1(M, \mathcal{O}^{(1,0)}) \cong \frac{\Gamma(M, \mathcal{E}^{(1,1)})}{\partial \Gamma(M, \mathcal{E}^{(1,0)})}.
$$

Explicitly this isomorphism is induced by the mapping that associates to a cross-section $\phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ the cohomology class of the cocycle $\phi_{\alpha\beta} = \phi_\beta - \phi_\alpha \in$
as noted following the proof of Theorem 1.3, there are $C^\infty$ functions $r_\alpha > 0$ in the coordinate neighborhoods $U_\alpha$ such that $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in the intersections $U_\alpha \cap U_\beta$, hence such that $d \log \lambda_{\alpha\beta} = \partial \log r_\alpha - \partial \log r_\beta$; therefore for the cocycle $\phi_{\alpha\beta} = d \log \lambda_{\alpha\beta}$ it is possible to take $\phi_\alpha = -\partial \log r_\alpha$, so the cohomology class $d\lambda(\lambda) \in H^1(M, \mathcal{O}^{(1,0)})$ can be represented in the isomorphism (3.11) by the differential form $\lambda = \bar{\partial} \phi_\alpha = -\bar{\partial} \partial \log r_\alpha$. By the Serre Duality Theorem, Theorem 1.18, again for the special case of the line bundle $\lambda = \kappa$, the dual space to the vector space (3.11) consists of the linear functionals (1.50) associated to the holomorphic cross-sections $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\kappa^{-1})) = \Gamma(M, \mathcal{O})$; since $M$ is compact these cross-sections are merely complex constants, so the vector space (3.11) can be identified with the complex numbers $\mathbb{C}$ by associating to the global differential form $\phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ the integral $\int_M c \phi$ for any choice of a normalizing complex constant $c \in \mathbb{C}$. For present purposes take $c = -1/(2\pi i)$, so the differential form $\phi = -\bar{\partial} \partial \log r_\alpha$ corresponds to the complex constant

$$\frac{1}{2\pi i} \int_M \bar{\partial} \partial \log r_\alpha,$$

which by Theorem 1.3 is just the characteristic class $c(\lambda)$ of the line bundle $\lambda$. Thus the homomorphism $d\lambda : H^1(M, \mathcal{O}^*) \to H^1(M, \mathcal{O}^{(1,0)})$ in (3.10) can be identified with the mapping that associates to a line bundle $\lambda \in H^1(M, \mathcal{O}^*)$ its characteristic class $c(\lambda) \in \mathbb{Z}$, and that suffices to conclude the proof.

**Corollary 3.9** A holomorphic line bundle $\lambda$ over a compact Riemann surface of genus $g > 0$ is holomorphically equivalent to a flat line bundle if and only if $c(\lambda) = 0$.

**Proof:** This is an immediate consequence of the exact sequence (3.8) of the preceding theorem, and is included explicitly here only as a convenience for later reference.

For later purposes it is useful to keep in mind the explicit form of the coboundary homomorphism $\delta$ in the exact sequence (3.8). If $\mathfrak{U}$ is a covering of $M$ by contractible open coordinate neighborhoods $U_\alpha$ such that any nonempty intersection $U_\alpha \cap U_\beta$ is connected, a holomorphic abelian differential $\omega \in \Gamma(M, \mathcal{O}^{(1,0)})$ can be written locally as the exterior derivative $\omega(z) = d w_\beta(z)$ of a holomorphic function $w_\beta(z)$ in each coordinate neighborhood $U_\alpha$; the image $\delta \omega \in H^1(M, \mathcal{C}^*)$ is the cohomology class of the cocycle

$$\lambda_{\alpha\beta} = \exp 2\pi i (w_\beta(z) - w_\alpha(z)) \quad \text{for} \quad z \in U_\alpha \cap U_\beta,$$

where $w_\beta(z) - w_\alpha(z)$ is a complex constant in the connected set $U_\alpha \cap U_\beta$ since $dw_\alpha(z) = dw_\beta(z) = \omega(z)$ there. The cohomology group $H^1(M, \mathcal{O}^*)$ can be identified with the group of holomorphic line bundles over the Riemann surface $M$, and the cohomology group $H^1(M, \mathcal{C}^*)$ can be identified correspondingly with the group of flat line bundles over $M$; the natural inclusion $H^1(M, \mathcal{C}^*) \to H^1(M, \mathcal{O}^*)$ is the mapping that associates to a flat line bundle the holomorphic line bundle it represents. Of course this inclusion is not injective, since distinct flat line bundles may well be holomorphically equivalent; nor
is it surjective, since not all holomorphic line bundles can be described by flat coordinate bundles. The situation really is described by the exact sequence (3.8) in Theorem 3.8. As in (1.14) the condition that \( c(\lambda) = 0 \) is equivalent to the condition that the line bundle \( \lambda \) is topologically trivial; thus the holomorphic line bundles that can be represented by flat line bundles are precisely those that are topologically trivial. The subgroup of topologically trivial holomorphic line bundles on a compact Riemann surface \( M \) is called the Picard group of \( M \), and is denoted by \( P(M) \); and in these terms the preceding theorem can be rephrased as follows.

**Corollary 3.10** On a compact Riemann surface \( M \) of genus \( g > 0 \) there is the exact sequence

\[
0 \longrightarrow \Gamma(M, \mathcal{O}(1,0)) \overset{\delta}{\longrightarrow} H^1(M, \mathbb{C}^*) \overset{p}{\longrightarrow} P(M) \longrightarrow 0
\]

where \( P(M) \) is the Picard group of the surface \( M \), \( p \) is the homomorphism that associates to a flat line bundle the holomorphic line bundle it represents, and \( \delta \) is the coboundary homomorphism defined by (3.12).

**Proof:** This also follows immediately from the exact sequence (3.8) of the preceding theorem, since the kernel of the mapping \( c \) in the exact sequence (3.8) is precisely the Picard group \( P(M) \) of the surface \( M \); so no further proof is necessary.

There is a usefully explicit alternative description of the group of line bundles on a compact Riemann surface \( M \) of genus \( g > 0 \). A continuous factor of automorphy for the action of the covering translation group \( \Gamma \) on the universal covering space \( \tilde{M} \) of \( M \) is a mapping \( \lambda : \Gamma \times \tilde{M} \longrightarrow \mathbb{C}^* \) that is continuous on \( \tilde{M} \) and satisfies

\[
\lambda(ST, z) = \lambda(S, Tz) \lambda(T, z) \quad \text{for all } S, T \in \Gamma, z \in \tilde{M}.
\]

The factor of automorphy is **holomorphic** if the functions \( \lambda(T, z) \) are holomorphic functions of the variable \( z \in \tilde{M} \), and is **flat** if the functions \( \lambda(T, z) \) are constant in the variable \( z \in \tilde{M} \); clearly a flat factor of automorphy is just a group homomorphism \( \lambda \in \text{Hom}(\Gamma, \mathbb{C}^*) \), and any flat factor of automorphy is also a holomorphic factor of automorphy. The set of factors of automorphy form an abelian group under multiplication \( \lambda_1(T, z) \cdot \lambda_2(T, z) \) of the functions \( \lambda(T, z) \); the holomorphic factors of automorphy are a subgroup of the group of all factors of automorphy, and the flat factors of automorphy are a subgroup of the group of holomorphic factors of automorphy. Two factors of automorphy \( \lambda_1(T, z) \) and \( \lambda_2(T, z) \) are **equivalent** if there is a continuous mapping \( f : \tilde{M} \longrightarrow \mathbb{C}^* \) such that

\[
\lambda_1(T, z) = f(Tz) \lambda_2(T, z) f(z)^{-1} \quad \text{for all } T \in \Gamma, z \in \tilde{M};
\]

in that case \( \lambda_2(T, z) = f(Tz)^{-1} \lambda_1(T, z) f(z) \), so this relation is symmetric. Two holomorphic factors of automorphy are **holomorphically equivalent** if they are
equivalent and the function $f(z)$ in (3.15) is holomorphic. Analogously two flat factors of automorphy could be considered flatly equivalent if they are equivalent and the function $f(z)$ in (3.15) is constant; but that is the case only when $\lambda_1(T) = \lambda_2(T)$, so flat equivalence just amounts to coincidence and is not worth introducing as a separate notion. It is clear that equivalence of factors of automorphy and holomorphic equivalence of holomorphic factors of automorphy are equivalence relations in the usual sense. A \textit{relatively automorphic function} for a factor of automorphy $\lambda(T, z)$ is a continuous function $f(z)$ on the universal covering space $\tilde{M}$ such that

\begin{equation}
(3.16) \hspace{1cm} f(Tz) = \lambda(T, z)f(z) \hspace{0.5cm} \text{for all } T \in \Gamma, \ z \in \tilde{M}.
\end{equation}

A \textit{holomorphic relatively automorphic function} for a holomorphic factor of automorphy is a relatively automorphic function that is holomorphic in the variable $z \in M$. Analogously a flat relatively automorphic function for a flat factor of automorphy could be defined as a relatively automorphic function that is constant in the variable $z \in \tilde{M}$; but clearly there is such a function only when the flat factor of automorphy is the trivial factor $\lambda(T) = 1$ for all $T \in \Gamma$, so this also is not a useful auxiliary notion. A comparison of equations (3.15) and (3.16) shows that two factors of automorphy $\lambda_1(T, z)$ and $\lambda_2(T, z)$ are equivalent if and only if there is a nowhere vanishing relatively automorphic function for the factor of automorphy $\lambda_1(T, z) \cdot \lambda_2(T, z)^{-1}$, or of course equivalently for the factor of automorphy $\lambda_2(T, z) \cdot \lambda_1(T, z)^{-1}$; and if these factors of automorphy are holomorphic they are holomorphically equivalent if and only if there is a holomorphic nowhere vanishing relatively automorphic function for the factor of automorphy $\lambda_1(T, z) \cdot \lambda_2(T, z)^{-1}$, or of course equivalently for the factor of automorphy $\lambda_2(T, z) \cdot \lambda_1(T, z)^{-1}$.

If $\lambda(T, z)$ is a factor of automorphy for the action of the covering translation group $\Gamma$ of the Riemann surface $M$ then to each covering translation $T \in \Gamma$ there can be associated the mapping $T_\lambda: \tilde{M} \times \mathbb{C} \rightarrow \tilde{M} \times \mathbb{C}$ defined by

$$T_\lambda(z, t) = (Tz, \lambda(T, z)t) \in \tilde{M} \times \mathbb{C} \hspace{0.5cm} \text{for} \hspace{0.5cm} (z, t) \in \tilde{M} \times \mathbb{C}.$$ 

It follows from the defining condition (3.14) for a factor of automorphy that for any two covering translations $S, T \in \Gamma$ the associated mappings satisfy

$$(ST)_\lambda(z, t) = (STz, \lambda(ST, z)t) = (S \cdot Tz, \lambda(STz) \cdot \lambda(T, z)) = S_\lambda(Tz, \lambda(T, z)t) = S_\lambda(T_\lambda(z, t));$$

so this exhibits the covering translation group as a group of continuous mappings of the space $M \times \mathbb{C}$ to itself, or of holomorphic mappings if the factor of automorphy is holomorphic. The group action on the product $\tilde{M} \times \mathbb{C}$ commutes with the action of the covering translation group on the universal covering space $\tilde{M}$ itself under the natural projection $\pi: \tilde{M} \times \mathbb{C} \rightarrow \tilde{M}$, yielding the commutative
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diagram

\[
\begin{array}{c}
\tilde{M} \times \mathbb{C} \xrightarrow{T} \tilde{M} \times \mathbb{C} \\
\tilde{M} \xrightarrow{T} \tilde{M};
\end{array}
\]

(3.17)

it follows that the projection \( \pi: \tilde{M} \times \mathbb{C} \rightarrow \tilde{M} \) induces a mapping \( \pi_\lambda: \lambda \rightarrow M \) between the quotient spaces \( \lambda = (\tilde{M} \times \mathbb{C})/\Gamma \) and \( M = \tilde{M}/\Gamma \).

**Theorem 3.11** If \( \lambda(T, z) \) is a factor of automorphy for the action of the covering translation group \( \Gamma \) of a compact Riemann surface \( M \) of genus \( g > 0 \) the quotient space \( \lambda = (\tilde{M} \times \mathbb{C})/\Gamma \) has the natural structure of a line bundle over \( M \) with the projection mapping \( \pi_\lambda: \lambda \rightarrow M \). Relatively automorphic functions for the factor of automorphy \( \lambda(T, z) \) correspond to cross-sections of \( \lambda \); and the line bundles corresponding to two factors of automorphy are equivalent if and only if the factors of automorphy are equivalent. A holomorphic factor of automorphy \( \lambda(T, z) \) describes in this way a holomorphic line bundle \( \lambda \), and holomorphic relatively automorphic functions for \( \lambda(T, z) \) correspond to holomorphic cross-sections of \( \lambda \); the line bundles described by two holomorphic factors of automorphy are holomorphically equivalent if and only if the factors of automorphy are holomorphically equivalent. A flat factor of automorphy \( \lambda(T, z) \) describes in this way a flat line bundle \( \lambda \) over \( M \).

**Proof:** The quotient space \( \lambda \) can be described most conveniently by using a coordinate covering \( \tilde{U} \) of the Riemann surface \( M \) by connected and simply connected coordinate neighborhoods \( U_\alpha \) such that the intersections \( U_\alpha \cap U_\beta \) of pairs of these coordinate neighborhoods also are connected; there exist such coordinate coverings on any Riemann surface. The inverse image of a coordinate neighborhood \( U_\alpha \subset M \) under the covering projection \( \pi: \tilde{M} \rightarrow M \) is a disjoint collection of open coordinate neighborhoods \( \tilde{U}_\alpha \subset \tilde{M} \), each of which is biholomorphic to \( U_\alpha \) under the covering projection \( \pi: \tilde{U}_\alpha \rightarrow U_\alpha \); the sets \( \tilde{U}_\alpha \subset \tilde{M} \) form a coordinate covering \( \tilde{U} \) of the universal covering space \( \tilde{M} \). The images \( U_\alpha \cap \tilde{U}_\alpha \subset M \) can be taken as another coordinate covering \( \pi(\tilde{U}) \) of the Riemann surface \( M \) itself, although it must be kept in mind that the point sets \( U_\alpha \) for all indices \( i \) actually coincide with the point set \( U_\alpha \) although they are considered as being different sets of the covering. If \( U_\alpha \cap U_\beta \neq \emptyset \) then since this intersection is connected by assumption it follows that for any two components \( \tilde{U}_\alpha \) and \( \tilde{U}_\beta \); there is a uniquely determined covering translation \( T_{\alpha \beta} \in \Gamma \) such that

\[
\tilde{U}_\alpha \cap \pi^{-1}(U_\alpha \cap U_\beta) = T_{\alpha \beta} \left( \tilde{U}_\beta \cap \pi^{-1}(U_\alpha \cap U_\beta) \right).
\]

Of course the point sets \( U_\alpha \cap U_\beta \) for any indices \( i, j \) coincide with the point set \( U_\alpha \cap U_\beta \); but \( \tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset \) in \( \tilde{M} \) if and only if \( T_{\alpha \beta} = I \). Since
$T \tilde{U}_\alpha \cap \tilde{U}_\alpha = \emptyset$ for any covering translation $T \in \Gamma$ other than the identity it follows that

$$T_\lambda(\tilde{U}_\alpha \times \mathbb{C}) \cap (\tilde{U}_\alpha \times \mathbb{C}) = \emptyset \quad \text{if} \quad T \neq I;$$

therefore the set $\tilde{U}_\alpha \times \mathbb{C}$ can be identified with a subset of the quotient space $(\tilde{M} \times \mathbb{C})/\Gamma = \lambda$, and a local coordinate $z_\alpha$ in $\tilde{U}_\alpha$ and the variable $t_\alpha \in \mathbb{C}$ can be used as local coordinates $(z_\alpha, t_\alpha)$ in this subset of the quotient space $\lambda$. In terms of these coordinates the mapping $\pi_\lambda : \lambda \longrightarrow \tilde{M}$ is just the natural projection $\tilde{U}_\alpha \times \mathbb{C} \longrightarrow \tilde{M}$. Points $(z_\alpha, t_\alpha) \in \tilde{U}_\alpha \times \mathbb{C}$ and $(z_\beta, t_\beta) \in \tilde{U}_\beta \times \mathbb{C}$ represent the same point in the quotient space $\lambda$ if and only if

$$(3.18) \quad z_\alpha = T_{\alpha, \beta} z_\beta \quad \text{and} \quad t_\alpha = \lambda(T_{\alpha, \beta}, z_\beta) t_\beta;$$

this is a linear relation between the fibre coordinates that is continuous in the local coordinates on $\tilde{M}$, so determines on $\lambda$ the structure of a complex line bundle over $M$. Of course if the factor of automorphy is holomorphic the line bundle is holomorphic, and if the factor of automorphy is flat the line bundle is flat. If $f(z)$ is a relatively automorphic function for the factor of automorphy $\lambda(T, z)$ then the restrictions of this function to the coordinate neighborhoods $\tilde{U}_\alpha \subset \tilde{M}$ satisfy $f(z_\alpha) = \lambda(T_{\alpha, \beta}, z_\beta) f(z_\beta)$ so they describe a cross-section of the line bundle $\lambda$. Conversely a cross-section of $\lambda$ is described by functions $f_{\alpha i}(z)$ in the coordinate neighborhoods $\tilde{U}_\alpha$, such that

$$(3.19) \quad f_{\alpha i}(z_\alpha) = \lambda(T_{\alpha, \beta}, z_\beta) f_{\beta j}(z_\beta)$$

whenever $z_\alpha \in \tilde{U}_\alpha$, $z_\beta \in \tilde{U}_\beta$, $\pi(z_\alpha) = \pi(z_\beta)$. Since $\lambda(T_{\alpha, \beta}, z) = 1$ whenever $\tilde{U}_\alpha \cap \tilde{U}_\beta \neq \emptyset$ it follows that $f_{\alpha i}(z) = f_{\beta j}(z)$ for any point $z \in \tilde{U}_\alpha \cap \tilde{U}_\beta$, so the local functions $f_{\alpha i}(z_\alpha)$ combine to form a single global function on the entire covering space $\tilde{M}$; and it follows from (3.19) that this function is a relatively automorphic function for the factor of automorphy $\lambda$. Two factors of automorphy $\lambda_1(T, z)$ and $\lambda_2(T, z)$ are equivalent if and only if the factor of automorphy $\lambda_1(T, z) \lambda_2(T, z)^{-1}$ has a nowhere vanishing relatively automorphic function, and two line bundles $\lambda_1$ and $\lambda_2$ are equivalent if and only if the line bundle $\lambda_1 \lambda_2^{-1}$ has a nowhere vanishing cross-section; since relatively automorphic functions correspond to cross-sections of the line bundle they describe it follows that two factors of automorphy are equivalent if and only if the line bundles they describe are equivalent. The same result of course holds for holomorphic line bundles, and that suffices to conclude the proof.

The preceding theorem shows that a factor of automorphy for the covering translation group of a compact Riemann surface $M$ describes a line bundle over $M$, and that two factors of automorphy are equivalent if and only if they describe equivalent line bundles over $M$; thus the mapping that associates to a factor of automorphy the line bundle it describes is an injective mapping from the multiplicative group of equivalence classes of factors of automorphy into the group $H^1(M, \mathcal{O}^*)$ of line bundles over $M$. The corresponding statement holds for holomorphic factors of automorphy and flat factors of automorphy, so that
the appropriate equivalence classes of these special factors of automorphy are mapped injectively into the groups $H^1(M, \mathbb{C}^\ast)$ and $H^1(M, \mathbb{C}^\ast)$ respectively. In all three cases the mapping will be shown to be surjective as well, so line bundles over $M$ can be identified with equivalence classes of factors of automorphy for the covering translation group of $M$ and correspondingly for holomorphic and flat line bundles. This really is of interest here just for the special case of flat factors of automorphy; the proof is simplest for that case, an almost immediate consequence of the observation that any flat line bundle over a simply connected surface such as the universal covering space $\tilde{M}$ is a trivial flat bundle, and will be given in the next corollary. The corresponding result for the general case will be demonstrated in Chapter 6 by constructing an explicit holomorphic factor of automorphy with any prescribed characteristic class.

**Corollary 3.12** The mapping that associates to a flat factor of automorphy in $\text{Hom}(\Gamma, \mathbb{C}^\ast)$ the flat line bundle that it describes is an isomorphism $\phi : \text{Hom}(\Gamma, \mathbb{C}^\ast) \cong H^1(M, \mathbb{C}^\ast)$.

**Proof:** If a flat factor of automorphy describes a trivial flat line bundle that line bundle has a flat nowhere vanishing cross-section; this cross-section corresponds to a nowhere vanishing flat relatively automorphic function for the flat factor of automorphy, which as noted means that the flat factor of automorphy is trivial. Thus the homomorphism $\phi : \text{Hom}(\Gamma, \mathbb{C}^\ast) \rightarrow H^1(M, \mathbb{C}^\ast)$ is injective, and to conclude the proof it is only necessary to show that it is also surjective. For this purpose let $\mathcal{U} = \{U_\alpha\}$ be a coordinate covering of $M$ with the properties as in the proof of the preceding theorem. A flat line bundle $\lambda$ over $M$ is described by constant coordinate transition functions in each nonempty intersection $U_\alpha \cap U_\beta$; it is convenience to denote these coordinate transition functions by $\lambda(U_\alpha, U_\beta)$ for this proof. The same line bundle $\lambda$ can be described in terms of the covering $\pi(\mathcal{U})$ by the coordinate transition functions $\lambda(U_\alpha \cap U_\beta) = \lambda(U_\alpha, U_\beta)$ in each nonempty intersection $U_\alpha \cap U_\beta$. The bundle $\lambda$ induces a flat line bundle $\tilde{\lambda}$ over the universal covering space $\tilde{M}$ described in terms of the covering $\tilde{\mathcal{U}}$ by the coordinate transition functions $\tilde{\lambda}(\tilde{U}_{\alpha_1} \cap \tilde{U}_{\beta_j}) = \lambda(U_\alpha, U_\beta)$ in each nonempty intersection $\tilde{U}_{\alpha_1} \cap \tilde{U}_{\beta_j}$; it is evident from its definition that this coordinate bundle is invariant under the covering translation group $\Gamma$. Since the coordinate transition functions are constants a constant cross-section of the bundle $\tilde{\lambda}$ over a coordinate neighborhood $\tilde{U}_{\alpha_1} \subset \tilde{M}$ can be extended to a constant cross-section over any coordinate neighborhood $\tilde{U}_{\beta_j} \subset \tilde{M}$ that meets $\tilde{U}_{\alpha_1}$; this cross-section can be extended further in the same way, and since $\tilde{M}$ is simply connected the usual monodromy argument shows that there results a well defined cross-section of the induced line bundle $\lambda$ over $\tilde{M}$. This cross-section is described by complex constants $\phi(\tilde{U}_{\alpha_1})$ in the coordinate neighborhoods $\tilde{U}_{\alpha_1} \subset \tilde{M}$ such that

$$\phi(\tilde{U}_{\alpha_1}) = \tilde{\lambda}(\tilde{U}_{\alpha_1} \cap \tilde{U}_{\beta_j}) \phi(\tilde{U}_{\beta_j}) \quad \text{if} \quad \tilde{U}_{\alpha_1} \cap \tilde{U}_{\beta_j} \neq \emptyset.$$  

(3.20)

Set $\phi(U_{\alpha_1}) = \phi(\tilde{U}_{\alpha_1})$ and note that the coordinate transition functions

$$\sigma(U_{\alpha_1}, U_{\beta_j}) = \phi(U_{\alpha_1})^{-1} \lambda(U_{\alpha_1}, U_{\beta_j}) \phi(U_{\beta_j}) \quad \text{for} \quad U_{\alpha_1} \cap U_{\beta_j} \neq \emptyset$$  

(3.21)
describe a flat coordinate bundle $\sigma$ over $M$ in terms of the covering $\pi(\tilde{U})$ and that this bundle is flatly equivalent to the initial coordinate bundle $\lambda$. For any covering translation $T \in \Gamma$ and any coordinate neighborhood $\tilde{U}_{\alpha_i} \subset \tilde{M}$ set $\lambda(\tilde{U}_{\alpha_i}, T) = \sigma(TU_{\alpha_i}, U_{\alpha_i})$, which is well defined since $TU_{\alpha_i} \cap U_{\alpha_i} \neq \emptyset$. If $\tilde{U}_{\alpha_i} \cap \tilde{U}_{\beta_j} \neq \emptyset$ then of course $T\tilde{U}_{\beta_j} \cap \tilde{TU}_{\alpha_i} \neq \emptyset$ as well and it is evident upon comparing (3.20) and (3.21) that $\sigma(U_{\alpha_i}, U_{\beta_j}) = \sigma(TU_{\beta_j}, TU_{\alpha_i}) = 1$; so from the compatibility conditions for the coordinate transition functions $\sigma(U_{\alpha_i}, U_{\beta_j})$ it follows that whenever $U_{\alpha_i} \cap U_{\beta_j} \neq \emptyset$

$$\lambda(\tilde{U}_{\alpha_i}, T) = \sigma(TU_{\alpha_i}, U_{\alpha_i}) = \sigma(TU_{\beta_j}, TU_{\alpha_i})\sigma(TU_{\alpha_i}, U_{\alpha_i})\sigma(U_{\alpha_i}, U_{\beta_j}) = \sigma(TU_{\beta_j}, U_{\beta_j}) = \lambda(\tilde{U}_{\beta_j}, T).$$

Thus the constants $\lambda(\tilde{U}_{\alpha_i}, T)$ are independent of the coordinate neighborhood $\tilde{U}_{\alpha_i} \subset \tilde{M}$ so can be labeled just $\lambda(T)$. Then $\lambda(ST) = \lambda(\tilde{U}_{\alpha_i}, ST) = \sigma(STU_{\alpha_i}, U_{\alpha_i}) = \sigma(\sigma(STU_{\alpha_i}, TU_{\alpha_i})\sigma(TU_{\alpha_i}, U_{\alpha_i})) = \lambda(T\tilde{U}_{\alpha_i}, S)\lambda(U_{\alpha_i}, T) = \lambda(S)\lambda(T)$ for any two covering translations $S, T \in \Gamma$, so the constants $\lambda(T)$ describe a homomorphism $\lambda \in \text{Hom}(\Gamma, \mathbb{C}^*)$. As in the proof of the preceding theorem this factor of automorphy describes a flat line bundle over $M$ for which the coordinate transition functions in an intersection $U_{\alpha_i} \cap U_{\beta_j}$ are the constants $\lambda(T_{\alpha_i,\beta_j}) = \lambda(\tilde{U}_{\beta_j}, T_{\alpha_i,\beta_j}) = \sigma(T_{\alpha_i,\beta_j}, U_{\beta_j}) = \sigma(U_{\alpha_i}, U_{\beta_j})$ which are the coordinate transition functions of the flat line bundle $\sigma$ flatly equivalent to $\lambda$. That suffices to conclude the proof.

**Theorem 3.13** On a compact Riemann surface $M$ of genus $g > 0$ with universal covering space $\tilde{M}$ and covering transformation group $\Gamma$ there is the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta_0} \text{Hom}(\Gamma, \mathbb{C}^*) \xrightarrow{p_0} P(M) \longrightarrow 0,$$

where $P(M)$ is the Picard group of the surface, $p_0$ is the homomorphism that associates to a flat factor of automorphy in $\text{Hom}(\Gamma, \mathbb{C}^*)$ the holomorphic line bundle it describes, and $\delta_0$ is the homomorphism that associates to a holomorphic abelian differential $\omega \in \Gamma(M, \mathcal{O}^{(1,0)})$ the flat factor of automorphy

$$\lambda(T) = \exp -2\pi i \omega(T)$$

where $\omega(T)$ is the period class of the differential $\omega$.

**Proof:** The isomorphism of Corollary 3.12 can be combined with the exact sequence (3.13) of Corollary 3.10 to yield the commutative diagram of exact sequences

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^1(M, \mathbb{C}^*) \xrightarrow{p} P(M) \longrightarrow 0$$

(3.24)
in which \( \delta_0 = \phi^{-1} \cdot \delta \) and \( p_0 = p \cdot \phi \); and it merely remains to describe the homomorphisms \( \delta_0 \) and \( p_0 \) more explicitly. Clearly \( p_0 \) associates to a flat factor of automorphy the holomorphic line bundle represented by the flat line bundle described by that factor of automorphy, or more briefly the holomorphic line bundle represented by the flat line bundle \( M_\delta \) in which \( \delta \).

**CHAPTER 3. HOLOMORPHIC DIFFERENTIALS**

The period class can be associated the flat factor of automorphy \( \tilde{\tau} \) of automorphy. In terms of a basis \( \tau_j \in H_1(M) \) for the homology group of the surface \( M \) associate to any complex vector \( t = \{t_j\} \in \mathbb{C}^g \) the homomorphism \( \rho_t \in \text{Hom}(\Gamma, \mathbb{C}^*) = \text{Hom}(H_1(M), \mathbb{C}^*) \) for which

\[
(3.26) \quad \rho_t(\tau_j) = \exp 2\pi i t_j.
\]
Any homomorphism in $\text{Hom}(\Gamma, \mathbb{C}^*)$ is of this form for some vector $t \in \mathbb{C}^{2g}$, two vectors $t_1, t_2 \in \mathbb{C}^{2g}$ determine the same homomorphism if and only if they differ by an integral vector, and the mapping $\rho$ that associates to a vector $t \in \mathbb{C}^{2g}$ the homomorphism $\rho_t \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a homomorphism from the additive group $\mathbb{C}^{2g}$ to the multiplicative group $\text{Hom}(\Gamma, \mathbb{C}^*)$ of flat factors of automorphy, yielding the exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{C}^{2g} \xrightarrow{\rho} \text{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow 0$$

called the canonical parametrization of flat factors of automorphy for the surface $M$ associated to the basis $\tau_j$ for the homology group $H_1(M)$. For another basis $\tilde{\tau}_j \in H_1(M)$ there is a corresponding exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{C}^{2g} \xrightarrow{\tilde{\rho}} \text{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow 0.$$

The two homomorphisms $\rho$ and $\tilde{\rho}$ are surjective, so for any vector $t \in \mathbb{C}^{2g}$ there will be at least one vector $\tilde{t} \in \mathbb{C}^{2g}$ such that $\tilde{\rho}(\tilde{t}) = \rho(t) = \rho_t$. If $\tilde{\tau}_j = \sum_{k=1}^{2g} q_{jk} \tau_k$ for a nonsingular matrix $Q = \{q_{jk}\} \in \text{Gl}(2g, \mathbb{Z})$ it follows from (3.26) that

$$\exp 2\pi i \tilde{t}_j = \tilde{\rho}(\tilde{\tau}_j) = \rho_t(\tilde{\tau}_j) = \rho_t \left( \sum_{k=1}^{2g} q_{jk} \tau_k \right) = \prod_{k=1}^{2g} \rho_t(\tau_k)^{q_{jk}} = \exp 2\pi i \sum_{k=1}^{2g} q_{jk} t_k,$$

and consequently $\tilde{t}_j = n_j + \sum_{k=1}^{2g} q_{jk} t_k$ for some integers $n_j \in \mathbb{Z}$, or in matrix terms $\tilde{t} = n + Qt$ for the vector $n = \{n_j\} \in \mathbb{Z}^{2g}$. Since $\tilde{\rho}_n = 1$ for any integral vector $n \in \mathbb{Z}^{2g}$ by (3.27), the two canonical parametrizations of flat factors of automorphy are related by $\tilde{\rho}_n = \rho_t$; and since linear transformation defined by the matrix $Q \in \text{Gl}(2g, \mathbb{Z})$ maps the subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ to itself, there is the commutative diagram of exact sequences

$$0 \longrightarrow \mathbb{Z}^{2g} \xrightarrow{i} \mathbb{C}^{2g} \xrightarrow{\rho} \text{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathbb{Z}^{2g} \xrightarrow{i} \mathbb{C}^{2g} \xrightarrow{\tilde{\rho}} \text{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow 0,$$

showing the effect of a change of basis for the homology group $H_1(M)$ on the exact sequence (3.27). It follows that the complex Lie group structure on the group $\text{Hom}(\Gamma, \mathbb{C}^*)$ arising from the identification $\text{Hom}(\Gamma, \mathbb{C}^*) \cong \mathbb{C}^{2g}/\mathbb{Z}^{2g}$ in the exact sequence (3.27) is independent of the choice of a basis for the homology group $H_1(M)$; thus the group $\text{Hom}(\Gamma, \mathbb{C}^*)$ has a uniquely determined structure as a noncompact complex manifold.

When the parametrization (3.27) of the group $\text{Hom}(\Gamma, \mathbb{C}^*)$ of flat line bundles is composed with the surjective homomorphism $\rho_0$ in the exact sequence (3.22)
describing the Picard group \( P(M) \) of the Riemann surface \( M \), there results a surjective homomorphism
\[
P = p_0 \cdot \rho : \mathbb{C}^{2g} \longrightarrow P(M)
\]
called the \textit{canonical parametrization} of the Picard group associated to the basis for the homology group \( H_1(M) \); this parametrization can be described more explicitly as follows.

**Theorem 3.14** If \( M \) is a compact Riemann surface of genus \( g > 0 \) then for any bases \( \omega_i \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_j \in H_1(M) \) there is the exact sequence
\[
0 \longrightarrow \mathbb{Z}^{2g} + \Omega \mathbb{C}^g \overset{\iota}{\longrightarrow} \mathbb{C}^{2g} \overset{P}{\longrightarrow} P(M) \longrightarrow 0,
\]
where \( \iota \) is the natural inclusion homomorphism, \( \Omega \) is the period matrix of \( M \) and \( P \) is the canonical parametrization of the Picard group \( P(M) \) in terms of these bases.

**Proof:** In addition to the canonical parametrization (3.27) of flat factors of automorphy introduce the parametrization
\[
\sigma : \mathbb{C}^g \xrightarrow{\cong} \Gamma(M, \mathcal{O}^{(1,0)})
\]
that associates to any vector \( s = (s_1, \ldots, s_g) \in \mathbb{C}^g \) the holomorphic abelian differential \( \sigma(s) = \sum_{k=1}^{g} s_k \omega_k \) in terms of the basis \( \{\omega_k\} \); and introduce as well the linear mapping \( \Omega : \mathbb{C}^g \longrightarrow \mathbb{C}^{2g} \) defined by the negative of the transpose of the period matrix \( \Omega \) of the surface \( M \) in terms of the bases \( \{\omega_k\} \) and \( \{\tau_j\} \). These homomorphisms can be combined in the following diagram
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}^{2g} & + & \Omega \mathbb{C}^g & \overset{\iota}{\longrightarrow} & \mathbb{C}^{2g} & \overset{P}{\longrightarrow} & P(M) & \longrightarrow & 0 \\
& & \uparrow & & \iota & & \sigma & \cong & \rho & & \delta_0 & & \delta_0 \cdot \sigma
\end{array}
\]
(3.34)

The first row is the exact sequence (3.22) of Theorem 3.13 and the long column is the exact sequence (3.27) in the canonical parametrization of flat factors of automorphy. That this is a commutative diagram will be demonstrated by showing that \( \rho \cdot (-\Omega) = \delta_0 \cdot \sigma \). If \( s \in \mathbb{C}^g \) the image \( \omega = \sigma(s) \) is the holomorphic abelian differential \( \omega = \sum_{k=1}^{g} s_k \omega_k \), and by (3.23) the image \( \delta_0 \cdot \sigma(s) \) is the
homomorphism $\lambda_1 \in \text{Hom}(\Gamma, \mathbb{C}^*) = \text{Hom}(H_1, \mathbb{C}^*)$ for which

$$\lambda_1(\tau_j) = \exp -2\pi i \omega(T_j) = \exp -2\pi i \sum_{k=1}^{g} s_k \omega_k(T_j)$$

$$= \exp -2\pi i \sum_{k=1}^{g} s_k \omega_k;$$

by (3.26) the image $\lambda_2 = \rho \cdot (-\Omega(s)) \in \text{Hom}(\Gamma, \mathbb{C}^*) = \text{Hom}(H_1, \mathbb{C}^*)$ is the homomorphism for which

$$\lambda_2(\tau_j) = \rho_{-\Omega, \omega}(\tau_j) = \exp -2\pi i \sum_{k=1}^{g} s_k \omega_k.$$

Comparing these two equations shows that $\rho \cdot (-\Omega) = \delta_0 \cdot \sigma$, hence that the diagram (3.34) is commutative. By definition the canonical parametrization $P : \mathbb{C}^{2g} \longrightarrow P(M)$ of the Picard group is the composition $P = p_0 \cdot \rho$. For the exactness of the sequence (3.32), it is clear from the diagram (3.34) that $(p_0 \cdot \rho)(n - \Omega s) = (p_0 \cdot \rho \cdot \iota)(n) + (p_0 \cdot \delta_0 \cdot \sigma)(s) = 0$ for any $n \in \mathbb{Z}^{2g}$ and $s \in \mathbb{C}^g$; conversely if $t \in \mathbb{C}^{2g}$ and $(p_0 \cdot \rho)(t) = 0$ then $\rho(t) = (\delta_0 \cdot \sigma)(s) = (\rho \cdot (-\Omega))(s)$ for some $s \in \mathbb{C}^g$, hence $\rho(t + \Omega s) = 0$ so $t + \Omega s = \iota(n)$ for some $n \in \mathbb{Z}^{2g}$ and therefore $t \in \mathbb{Z}^{2g} - \Omega \mathbb{C}^g = \mathbb{Z}^{2g} + \Omega \mathbb{C}^g$. That suffices to conclude the proof.

The exact sequence (3.32) depends on the bases $\omega_i \in \Gamma(M, O(1,0))$ for the holomorphic abelian differentials on $M$ and $\tau_j \in H_1(M)$ for the homology of $M$. There is a corresponding exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} + \hat{\Omega} \mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{\hat{P}} P(M) \longrightarrow 0$$

for the period matrix $\hat{\Omega}$ and the canonical parametrization $\hat{P}$ of the Picard group associated to other bases $\hat{\omega}_i \in \Gamma(M, O(1,0))$ and $\hat{\tau}_j \in H_1(M)$. If $\hat{\omega}_i = \sum_{k=1}^{g} a_{ik} \omega_k$ and $\hat{\tau}_j = \sum_{l=1}^{2g} q_{jl} \tau_l$ for nonsingular matrices $A = \{a_{ik}\} \in \text{Gl}(g, \mathbb{C})$ and $Q = \{q_{jl}\} \in \text{Gl}(2g, \mathbb{Z})$ the two period matrices are related by

$$(3.35) \quad \hat{\Omega} = \{\hat{\omega}_i(\hat{\tau}_j)\} = \{\sum_{k=1}^{g} \sum_{l=1}^{2g} a_{ik} \omega_k(q_{jl}\tau_l)\} = \{\sum_{k=1}^{g} \sum_{l=1}^{2g} a_{ik} \omega_k q_{jl}\}$$

$$= A \Omega Q.$$

The canonical parametrizations $\rho$ and $\hat{\rho}$ of flat factors of automorphy are related as in (3.30), so that $\hat{\rho}_{Q\tau} = \rho_t$ and consequently $\hat{P}(Q\tau_t) = p_0(\hat{\rho}_{Q\tau}) = p_0(\rho_t) = P(t)$ for all $t \in \mathbb{C}^{2g}$; and the kernel of the homomorphism $\hat{P}$ is the subgroup $\mathbb{Z}^{2g} + \hat{\Omega} \mathbb{C}^g = \mathbb{Z}^{2g} + Q \Omega A \mathbb{C}^g = Q(\mathbb{Z}^{2g} + \Omega \mathbb{C}^g)$ in view of (3.35). Thus there is the commutative diagram of exact sequences

$$0 \xrightarrow{} \mathbb{Z}^{2g} + \Omega \mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{\hat{P}} P(M) \xrightarrow{} 0$$

$$Q \xrightarrow{\approx} Q \xrightarrow{\approx} Q$$

$$(3.36) \quad 0 \longrightarrow \mathbb{Z}^{2g} + \Omega \mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{\hat{P}} P(M) \longrightarrow 0$$
in which the vertical homomorphisms are isomorphisms; and this shows that the canonical parametrization \( P \) of the Picard group \( P(M) \) transforms canonically under a change in the basis for the homology group \( H_1(M) \) of the surface \( M \).

**Corollary 3.15** The Picard group \( P(M) \) of a compact Riemann surface \( M \) of genus \( g > 0 \) has a uniquely defined structure as the complex torus described by the inverse period matrix\(^4\) to a period matrix of \( M \).

**Proof:** For any bases \( \omega_i \in \Gamma(M, \mathcal{O}^{1,0}) \) for the holomorphic abelian differentials on \( M \) and \( \tau_j \in H_1(M) \) for the homology of \( M \) the exact sequence (3.32) of Theorem 3.14 yields the isomorphism

\[
P(M) \cong \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + \Omega \mathbb{C}^g};
\]

and by Corollary F.15 in Appendix F.1 this quotient group is the complex torus \( J(\Pi) \) described by the inverse period matrix \( \Pi \) to the period matrix \( \Omega \). Since the period matrices of a compact Riemann surface \( M \) for any choices of bases for the holomorphic abelian differentials and the homology of \( M \) are equivalent period matrices by Theorem 3.6, their inverse period matrices are also equivalent period matrices as noted on page 401; hence the structure of \( P(M) \) as a complex torus is intrinsically defined, and is independent of the choices of these bases. That suffices for the proof.

It should be kept in mind that the Picard group \( P(M) \) has a natural group structure, so when viewed as a complex torus it is not just a complex manifold but is an abelian compact Lie group with a specified identity element, the trivial complex line bundle represented by the origin \( 0 \in \mathbb{C}^{2g} \) in the exact sequence (3.32). When only the structure of a complex manifold is relevant, the Picard group often is called the *Picard variety* of the Riemann surface \( M \), but still is denoted by \( P(M) \). The Picard variety \( P(M) \) like the Jacobi variety \( J(M) \) of a compact Riemann surface is a complex torus of dimension \( g \) canonically associated to the Riemann surface \( M \). The two complex tori \( J(\Omega) \) and \( P(M) \) clearly are closely related, indeed are defined by period matrices that are inverse to one another; their relationship will be discussed further in Theorem 3.23. Before turning to that, though, it is useful to consider a formulation of the preceding corollary that provides a convenient and more explicit description of the line bundles forming the Picard group \( P(M) \).

**Corollary 3.16** If \( M \) is a compact Riemann surface of genus \( g > 0 \) then for any choice of bases \( \omega_i \in \Gamma(M, \mathcal{O}^{1,0}) \) for the holomorphic abelian differentials on \( M \) and \( \tau_j \in H_1(M) \) for the homology of \( M \) there is the exact sequence

\[
0 \rightarrow \frac{\Pi \mathbb{Z}^{2g}}{\Pi \mathbb{C}^g} \rightarrow \Pi \mathbb{C}^g \rightarrow P(D) \rightarrow P(M) \rightarrow 0
\]

\(^4\)The inverse period matrix is defined in Appendix F.1, and its basic properties are discussed there.
where $\Omega$ is the period matrix of $M$ in terms of these bases, $\Pi$ is the inverse period matrix to $\Omega$, $\iota$ is the natural inclusion homomorphism and $P_0$ is the restriction of the canonical parametrization $P : \mathbb{C}^g \to P(M)$ of the Picard group to the linear subspace $\mathfrak{T}C^g \subset \mathbb{C}^{2g}$. Consequently any line bundle in $P(M)$ is holomorphically equivalent to the flat line bundle $\rho_{\Omega}t$ for some vector $t \in \mathbb{C}^g$; and flat line bundles $\rho_{\Omega}t_1$ and $\rho_{\Omega}t_2$ are holomorphically equivalent if and only if $t_2 - t_1 \in \Pi \mathbb{Z}^{2g}$.

**Proof:** Since by Theorem 3.14 the canonical parametrization of the Picard group is a homomorphism $P : \mathbb{C}^{2g} \to P(M)$ with the subgroup $\mathfrak{T}C^g \subset \mathbb{C}^{2g}$ in its kernel, it follows from the direct sum decomposition

\[(3.39) \quad \mathbb{C}^{2g} = \mathfrak{T}C^g \oplus \mathfrak{T}C^g \]

of (F.9) in Appendix F.1 that the restriction of the homomorphism $P$ to the subgroup $\mathfrak{T}C^g \subset \mathbb{C}^{2g}$ is a surjective group homomorphism

\[(3.40) \quad P_0 : \mathfrak{T}C^g \to P(M);\]

and it follows further from Theorem 3.14 that the kernel of the restriction $P_0$ is the intersection $(\mathbb{Z}^{2g} + \mathfrak{T}C^g) \cap \mathfrak{T}C^g$. By using the natural projection operators (F.11) for the direct sum decomposition (3.39) as given in Appendix F.1, any $n \in \mathbb{Z}^{2g}$ can be decomposed as $n = \mathfrak{T}\Pi n + \mathfrak{T}\Pi n$, and consequently

\[(\mathbb{Z}^{2g} + \mathfrak{T}C^g) \cap \mathfrak{T}C^g = \mathfrak{T}\Pi \mathbb{Z}^{2g}.\]

That suffices to demonstrate the exactness of the sequence (3.38); the remaining statement of the corollary as an immediate consequence of this exactness, and that concludes the proof.

A different explicit description of the Picard group $P(M)$ arises by restricting the exact sequence (3.32) of Theorem 3.14 to the real linear subspace $\mathbb{R}^{2g} \subset \mathbb{C}^{2g}$ rather than to the complex linear subspace $\mathfrak{T}C^g \subset \mathbb{C}^{2g}$. It is apparent from the definition (3.26) that $|\rho_t(\tau_j)| = 1$ for all the vectors $\tau_j \in H_1(M)$ precisely when $t \in \mathbb{R}^{2g}$; so under the canonical parametrization (3.27) of flat factors of automorphy, real vectors $t \in \mathbb{R}^{2g}$ parametrize precisely those flat factors of automorphy for which $|\rho_t(T)| = 1$ for all $T \in \Gamma$, called **unitary flat factors of automorphy**. The flat line bundles represented by these factors of automorphy under the isomorphism of Corollary 3.12 are those that can be described by flat coordinate bundles $f_{\alpha\beta}$ for some coordinate covering $\mathcal{U} = \{U_\alpha\}$ of $M$ such that $|f_{\alpha\beta}| = 1$; they are called **unitary flat line bundles**. Since any holomorphic function $f_{\alpha\beta}$ for which $|f_{\alpha\beta}| = 1$ is necessarily constant, unitary flat line bundles can be characterized alternatively as those holomorphic line bundles that can be represented in terms of some coordinate covering $\mathcal{U} = \{U_\alpha\}$ of the surface $M$ by holomorphic coordinate line bundles $\rho_{\alpha\beta}$ such that $|\rho_{\alpha\beta}| = 1$ for all intersections $U_\alpha \cap U_\beta$. 


Theorem 3.18 In terms of the natural complex structures on the groups of flat line bundles and of flat factors of automorphy over a compact Riemann surface \( M \) of genus \( g > 0 \), the mappings
\[
p : H^1(M, \mathbb{C}^*) \longrightarrow P(M), \quad p_0 : \text{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow P(M)
\]
are holomorphic mappings exhibiting the complex manifolds $H^1(M, \mathbb{C}^*)$ and $\text{Hom}(\Gamma, \mathbb{C}^*)$ as holomorphic fibre bundles over the complex torus $P(M)$ with fibre the complex vector space $\mathbb{C}^g$ and group the lattice subgroup $\Pi \mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ acting on the vector space $\mathbb{C}^g$ by translation, where $\Pi$ is the inverse period matrix of $\mathbb{C}^{2g}$.

**Proof:** Choose bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, and let $\Omega$ be the period matrix of the surface $M$ in terms of these bases and $\Pi$ be the inverse period matrix to a period matrix $\tilde{\Pi}$ of $\mathbb{C}^{2g}$. The group $\text{Hom}(\Gamma, \mathbb{C}^*)$ has the complex structure arising from its identification with the quotient of $\mathbb{C}^{2g}$ by the action of the subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ by translation, as in the canonical parametrization (3.28) of flat factors of automorphy for the surface $M$. The group $P(M)$ has the structure of the complex torus arising as the quotient of the complex manifold $t \Omega \mathbb{C}^g$ by the lattice subgroup $t \Omega \Pi \mathbb{Z}^{2g} \subset t \Omega \mathbb{C}^g$ by translation, as in the exact sequence (3.38) of Corollary 3.16. The holomorphic mapping $P_0$ in the latter exact sequence can be viewed alternatively as the result of factoring the canonical parametrization $P = p_0 \cdot \rho : \mathbb{C}^{2g} \rightarrow P(M)$ of the Picard group through the natural projection

\[
\tilde{\Pi} : \mathbb{C}^{2g} \rightarrow \tilde{\Pi} \mathbb{C}^g
\]

in the direct sum decomposition (3.39), yielding the commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}^{2g} & \xrightarrow{\rho} & \text{Hom}(\Gamma, \mathbb{C}^*) \\
\mid \downarrow \tilde{\Pi} \mid & & \downarrow p_0 \\
\tilde{\Pi} \mathbb{C}^{2g} & \xrightarrow{P_0} & P(M).
\end{array}
\]

When $n \in \mathbb{Z}^{2g}$ is decomposed as $n = \tilde{\Pi} n + \tilde{\Pi} \Pi n$ in the direct sum decomposition (3.39), the action of $n$ on $\mathbb{C}^{2g}$ by translation decomposes correspondingly as the mapping of $\mathbb{C}^{2g} = \tilde{\Pi} \mathbb{C}^g + \tilde{\Pi} \mathbb{C}^{2g}$ to itself given by

\[
(\tilde{\Pi} s, \tilde{\Pi} t) \rightarrow (\tilde{\Pi}(s + \Pi n), \tilde{\Pi}(t + \Pi n));
\]

hence the action of $n$ on $\mathbb{C}^{2g}$ by translation commutes with the action of $\tilde{\Pi} \Pi n$ on the complex manifold $\tilde{\Pi} \mathbb{C}^g$ by translation, in the commutative diagram (3.44), so the mapping (3.43) induces the holomorphic mapping

\[
p_0 : \text{Hom}(\Gamma, \mathbb{C}^*) = \frac{\tilde{\Pi} \mathbb{C}^g + \tilde{\Pi} \mathbb{C}^{2g}}{\mathbb{Z}^{2g}} \rightarrow \frac{\tilde{\Pi} \mathbb{C}^g}{\tilde{\Pi} \Pi \mathbb{Z}^{2g}} = P(M)
\]

between the quotient spaces. If $U_\alpha \subset \tilde{\Pi} \mathbb{C}^g$ is an open subset that is disjoint from any of its translates under the action of the lattice subgroup $\tilde{\Pi} \Pi \mathbb{Z}^{2g}$ then the subset

\[
\tilde{\Pi} \mathbb{C}^g \oplus U_\alpha \subset \tilde{\Pi} \mathbb{C}^g \oplus \tilde{\Pi} \mathbb{C}^{2g}
\]

is an open subset of $\mathbb{C}^{2g}$ that is disjoint from any of its translates under the action (3.45) of the group $\mathbb{Z}^{2g}$, so this subset can be identified with the inverse
image $p_0^{-1}(U_\alpha) \subset \text{Hom}(\Gamma, \mathbb{C}^*)$; thus if the points of the open subset $U_\alpha$ are parametrized as $\Pi t_\alpha \in U_\alpha$ for local coordinates $t_\alpha \in \mathbb{C}^g$ and if the points in the linear subspace $\Omega C^g$ are parametrized as $\Omega s_\alpha \in \Omega C^g$ for local coordinates $s_\alpha \in \mathbb{C}^g$ then the pairs $(s_\alpha, t_\alpha) \in \mathbb{C}^g \times \mathbb{C}^g$ can be taken as local coordinates in $p_0^{-1}(U_\alpha) = \Omega C^g \oplus U_\alpha$. In terms of these local coordinates the holomorphic mapping (3.46) is just the restriction of the natural projection $\mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}^g$ to the second factor, so it exhibits the mapping $p_0$ as a local fibration with fibre $\mathbb{C}^g$. Over an intersection $U_\alpha \cap U_\beta \subset \Omega C^g$ it is possible to take the same fibre coordinates $s_\alpha = s_\beta$ in the fibres $\Omega C^g$; thus when viewed as a fibre bundle over the entire complex manifold $\Omega C^g$ rather than over the quotient $P(M)$, the fibration (3.46) is a trivial fibre bundle. However for another local coordinate system in $P(M)$ corresponding to the translate

\begin{equation}
U_\beta = U_\alpha + \Pi n_{\beta \alpha} \quad \text{for} \quad n_{\beta \alpha} \in \mathbb{Z}^{2g}
\end{equation}

it follows from (3.45) that the fibre coordinates are related by

\begin{equation}
s_\beta = s_\alpha + \Pi n_{\beta \alpha};
\end{equation}

of course this also holds over the intersection $U_\alpha \cap U_\beta$, for which $n_{\beta \alpha} = 0$. Altogether then the holomorphic mapping (3.46) exhibits the complex manifold $\text{Hom}(\Gamma, \mathbb{C}^*)$ as a holomorphic fibre bundle over $P(M)$ with fibres the complex vector space $\mathbb{C}^g$ and coordinate transition functions in the fibres given by (3.49). The complex manifold $H^1(M, \mathbb{C}^*)$ is biholomorphic to the complex manifold $\text{Hom}(\Gamma, \mathbb{C}^*)$ through the isomorphism $\phi$ of Corollary 3.12, and the two mappings $\phi$ and $p_0$ are related as in the commutative diagram (3.24) in the proof of Theorem 3.13; through that diagram the results just derived for the mapping $p_0$ extend immediately to the corresponding results for the mapping $\phi$, and that suffices to conclude the proof.

The fibre bundles (3.42) are described quite explicitly by the coordinate transition functions (3.49) in terms of a covering of the manifold $P(M)$ by coordinate neighborhoods represented by subsets $U_\alpha \subset \Omega C^g$. These bundles also can be described globally, paralleling the use of factors of automorphy in describing line bundles over a Riemann surface; and the global description is useful in examining these fibre bundles a bit more closely.

**Corollary 3.19** Let $M$ be a compact Riemann surface of genus $g > 0$ and $\Pi$ be the inverse period matrix to a period matrix of $M$.

(i) Cross-sections of the fibre bundles (3.42) can be identified with mappings $\tilde{f} : \mathbb{C}^g \to \mathbb{C}^g$ from the universal covering space of the manifold $P(M)$ to the fibres $\mathbb{C}^g$ such that

\begin{equation}
\tilde{f}(t + \Pi n) = \tilde{f}(t) + \Pi n \quad \text{for all} \quad n \in \mathbb{Z}^{2g};
\end{equation}

and these mappings in turn are in one-to-one correspondence with mappings

\[ f : J(\Pi) \to J(\Pi) \]
between the complex tori described by the period matrices $\Pi$ and $\bar{\Pi}$.

(ii) The fibre bundles (3.42) are topologically trivial.

(iii) Holomorphic cross-sections of the fibre bundles (3.42) are in one-to-one correspondence with with triples $(A, Q, a_0)$ where $(A, Q)$ is a Hurwitz relation from the period matrix $\Pi$ to the period matrix $\bar{\Pi}$ and $a_0 \in J(\Pi)$.

(iv) The fibre bundles (3.42) are holomorphically trivial if and only if the complex tori $J(\Pi)$ and $J(\Pi)$ are isogenous.

**Proof:**

(i) With the notation as in the proof of the preceding theorem, cross-sections of the bundle (3.46) are described by mappings $f_\alpha : U_\alpha \rightarrow C^g$ that agree in intersections $U_\alpha \cap U_\beta$ and that satisfy

\[
(3.51) \quad f_\alpha(\Omega t + \Omega n) = f_\alpha(\Omega t) + \Omega n
\]

for all points $\Omega t \in U_\alpha$ and all $n \in \mathbb{Z}^{2g}$. Since these local mappings agree in intersections $U_\alpha \cap U_\beta$ they can be combined to yield a mapping $\tilde{f} : \Omega C^g \rightarrow C^g$; and as a consequence of (3.51) this global mapping satisfies $\tilde{f}(\Omega t + \Omega n) = \tilde{f}(\Omega t) + \Omega n$, which is just (3.50) in terms of the parametrization of the linear subspace $\Omega C^g$ by parameter values $t \in C^g$. Conversely any mapping $f$ satisfying (3.50) when viewed as a mapping defined on the parametrized subspace $\Omega C^g$ restricts to coordinate neighborhoods $U_\alpha \subset \Omega C^g$ to yield local mappings satisfying (3.51), and these local mappings describe a cross-section of the fibre bundle. The condition that a mapping $\tilde{f} : C^g \rightarrow C^g$ satisfies (3.50) is equivalent to the condition that this mapping commutes with the natural projections $p_i$ to the quotient groups in the commutative diagram

\[
\begin{array}{ccc}
C^g & \xrightarrow{\tilde{f}} & C^g \\
p_1 | & & | p_2 \\
\Omega C^g / \Omega \mathbb{Z}^{2g} & \xrightarrow{\tilde{f}} & C^g / \Omega \mathbb{Z}^{2g}
\end{array}
\]

and consequently that the mapping $\tilde{f}$ induces a mapping $f$ between the quotient groups.

(ii) There is a real linear topological homeomorphism $\phi : J(\Pi) \rightarrow J(\bar{\Pi})$, since any two complex tori of the same dimension are homeomorphic. When the component functions of this homeomorphism $\phi$ are identified with real linear mappings $f_i : C^g \rightarrow C^g$ satisfying (3.50) the linear functions $f_i$ are linearly independent, so any continuous cross-section $\tilde{f}$ can be written uniquely as a linear combination $\tilde{f} = \sum_{i=1}^{g} c_i f_i$ for some constants $c_i$; consequently the fibre bundles (3.42) are topologically trivial.

(iii) As in (i) holomorphic cross-sections of the fibre bundle (3.42) are in one-to-one correspondence with holomorphic mappings $f : J(\Pi) \rightarrow J(\bar{\Pi})$; and by Theorem F.9 in Appendix F.1 these holomorphic mappings are in one-to-one correspondence with with triples $(A, Q, a_0)$ where $(A, Q)$ is a Hurwitz relation from the period matrix $\Pi$ to the period matrix $\bar{\Pi}$ and $a_0 \in J(\Pi)$.

(iv) Since by (iii) holomorphic cross-sections of the fibre bundle (3.42) are constant complex linear functions, the fibre bundle (3.42) is holomorphically trivial.
if and only if there are \( g \) holomorphic cross-sections \( f_i \) that are linearly independent complex linear functions; when viewed as the coordinate functions of a mapping \( f: \mathcal{J}(\Pi) \to J(\Pi) \) as in (i), the condition that these coordinate functions are linearly independent complex linear functions is equivalent to the condition that the holomorphic mapping \( f: \mathcal{J}(\Pi) \to J(\Pi) \) is locally biholomorphic, hence that this mapping is an isogeny between the two complex tori. That suffices to conclude the proof.

The Picard and Jacobi varieties of a compact Riemann surface \( M \) of genus \( g > 0 \) have additional properties arising from the multiplicative structure in the cohomology group \( H^1(M) \). If \( \tau_j \in H_1(M) \) is a basis for the homology of \( M \) and \( \phi_j \in \Omega^1(M) \) is a dual basis for the first deRham group of the surface, so that \( \phi_j \) are closed differential forms of degree 1 on \( M \) such that \( \int_{\tau_j} \phi_j = \delta^k_j \) for \( 1 \leq k \leq 2g \) in terms of the Kronecker delta, the intersection matrix of the surface \( M \) in terms of the basis \( \{ \tau_j \} \) is the skew-symmetric integral matrix \( P = \{ p_{jk} \} \) where \( p_{jk} = \int_{\tau_j} \phi_j \wedge \phi_k \). If \( \omega_i \in \Gamma(M, \mathcal{O}(1,0)) \) is a basis for the holomorphic abelian differentials then \( \omega_i \sim \sum_{j=1}^{2g} \omega_{ij} \phi_j \) in terms of the basis \( \phi_j \), where \( \sim \) denotes cohomologous differential forms and \( \{ \omega_{ij} \} = \Omega \) is the period matrix of the surface in terms of these bases.

**Theorem 3.20 (Riemann Matrix Theorem)** If \( M \) is a compact Riemann surface of genus \( g > 0 \), and if \( \Omega \) is the period matrix and \( P \) is the intersection matrix of that surface in terms of bases \( \omega_i \in \Gamma(M, \mathcal{O}(1,0)) \) and \( \tau_j \in H_1(M) \), then

(i) \( \Omega P \bar{\Omega} = 0 \) and

(ii) \( i \Omega P \bar{\Omega} \) is a positive definite Hermitian matrix.

**Proof:** First \( \omega_i \wedge \omega_j = 0 \) for any two holomorphic abelian differentials \( \omega_i, \omega_j \), since the exterior product is a differential form of type \((2,0)\) so must vanish identically; consequently

\[
0 = \int_M \omega_i \wedge \omega_j = \sum_{kl} \omega_{ik} \omega_{jl} \int_M \phi_k \wedge \phi_l = \sum_{kl} \omega_{ik} \omega_{jl} p_{kl},
\]

which in matrix terms is (i). Next in a coordinate neighborhood \( U_\alpha \) with local coordinate \( z_\alpha = x_\alpha + iy_\alpha \) a holomorphic abelian differential \( \omega \) can be written \( \omega = f_\alpha \, dz_\alpha \) for some holomorphic function \( f_\alpha \); consequently

\[
i \omega \wedge \overline{\omega} = i |f_\alpha|^2 dz_\alpha \wedge d\overline{z_\alpha} = 2 |f_\alpha|^2 dx_\alpha \wedge dy_\alpha.
\]

Since \( dx_\alpha \wedge dy_\alpha \) is the local element of area in the canonical orientation of the Riemann surface \( M \) the integral of this differential form is non-negative, indeed is strictly positive so long as \( \omega \neq 0 \). Therefore if \( \omega = \sum_i c_i \omega_i \) for some complex constants \( c_i \) and if \( h_{ij} = i \int_{\tau_j} \omega_i \wedge \overline{\omega_j} \) then

\[
0 \leq i \int_M \omega \wedge \overline{\omega} = i \sum_{ij} c_i \overline{c_j} \int_M \omega_i \wedge \overline{\omega_j} = \sum_{ij} c_i \overline{c_j} h_{ij},
\]
and equality occurs only when \( c_i = 0 \) for all \( i \); this is just the condition that the matrix \( H = \{ h_{ij} \} \), which is readily seen to be Hermitian, is positive definite. Furthermore

\[
h_{ij} = i \int_M \sum_{kl} w_{ik} \phi_k \wedge \bar{\omega}_{jl} \phi_l = i \sum_{kl} \omega_{ik} \bar{\omega}_{jl} p_{kl},
\]

or in matrix terms \( H = i \Omega P' \bar{\Omega} \); consequently (ii) follows, and that concludes the proof.

Traditionally condition (i) of the preceding theorem is called Riemann’s equality and condition (ii) is called Riemann’s inequality. These two conditions taken together amount to the condition that \( \Omega \) is a Riemann matrix\(^5\) with principal matrix \( P \). Since \( \det P = 1 \) by Corollary D.2 in Appendix D.2 the entries of the matrix \( P \) are relatively prime, so \( P \) is a primitive principal matrix for the Riemann matrix \( \Omega \); the pair \((\Omega, P)\) thus describes a polarized Riemann matrix called the polarized period matrix of the surface \( M \) in terms of these bases. It was demonstrated in Theorem 3.6 that the period matrices of a Riemann surface for various choices of bases are equivalent period matrices; the corresponding result holds for the polarized period matrices as well.

**Corollary 3.21** The polarized period matrices \((\Omega, P)\) of a Riemann surface \( M \) of genus \( g > 0 \) for arbitrary choices of bases for the holomorphic abelian differentials and the homology of \( M \) are a full equivalence class of polarized Riemann matrices.

**Proof:** Two bases \( \omega_i \) and \( \tilde{\omega}_i \) for the holomorphic abelian differentials on the Riemann surface \( M \) are related by \( \tilde{\omega}_i = \sum_{k=1}^g a_{ik} \omega_k \) for an arbitrary nonsingular complex matrix \( A = \{ a_{ik} \} \in \text{GL}(g, \mathbb{C}) \), and two bases \( \tau_j \) and \( \tilde{\tau}_j \) for the homology of the surface \( M \) are related by \( \tilde{\tau}_j = \sum_{l=1}^{2g} \tau_{lj} q_{lj} \) for an arbitrary invertible integral matrix \( Q = \{ q_{lj} \} \in \text{GL}(2g, \mathbb{Z}) \). The two period matrices for these two bases then are related by \( \tilde{\Omega} = A \Omega Q \), as in Theorem 3.6. If \( \phi_j \) and \( \tilde{\phi}_j \) are bases for the deRham group of \( M \) dual to the bases \( \tau_j \) and \( \tilde{\tau}_j \) for the homology of \( M \) then \( \tilde{\phi}_m = \sum_{n=1}^{2g} \phi_n q_{nm} \) for an invertible integral matrix \( Q = \{ q_{lj} \} \in \text{GL}(2g, \mathbb{Z}) \); and \( \delta_{mn} = \int_{\tilde{\tau}_j} \tilde{\phi}_m = \sum_{l=1}^{2g} q_{lj} \int_{\tau_j} \phi_l \bar{\phi}_n = \sum_{l,m=1}^{2g} \eta_{lm} \delta_{mn} = \sum_{l=1}^{2g} \eta_{lj} q_{lm} \) so \( \tilde{Q} Q = I \) and consequently \( \tilde{Q} = Q^{-1} \).

Therefore the intersection matrices \( P \) and \( \tilde{P} \) for the two homology bases are related by \( \tilde{P}_{jk} = \int_M \tilde{\phi}_j \wedge \tilde{\phi}_k = \sum_{l,m=1}^{2g} \tilde{q}_{lj} \tilde{q}_{mk} \int_M \phi_l \wedge \phi_m = \sum_{l,m=1}^{2g} \tilde{q}_{lj} \tilde{q}_{mk} P_{lm} \), or in matrix terms \( P = \tilde{Q} P \tilde{Q} = Q^{-1} P Q^{-1} \). Altogether then the two polarized Riemann matrices \((\Omega, P)\) and \((\tilde{\Omega}, \tilde{P})\) are related by \( \tilde{\Omega} = A \Omega Q \) and \( \tilde{P} = Q^{-1} P Q^{-1} \) for arbitrary matrices \( A \in \text{GL}(g, \mathbb{C}) \) and \( Q \in \text{GL}(2g, \mathbb{Z}) \); and by (F.33) in Appendix F.3 that is precisely the description of a complete equivalence class of polarized Riemann matrices, so that suffices for the proof.

\(^5\)The definition of a Riemann matrix and of various related notions, and a survey of some of the basic properties of Riemann matrices, are given in Appendix F.3.
It follows from Theorem D.1 in Appendix D.2 and its corollary that the intersection matrices for a compact Riemann surface of genus \( g > 0 \) are precisely the \( 2g \times 2g \) integral matrices \( QJQ^{-1} \) where \( Q \in \text{Gl}(2g, \mathbb{Z}) \) is an arbitrary invertible integral matrix and \( J \) is the basic \( 2g \times 2g \) integral skew-symmetric matrix

\[
J = \begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix};
\]

thus the polarized period matrix of a compact Riemann surface of genus \( g > 0 \) is always equivalent to a polarized Riemann matrix of the form \((\Omega, J)\), a principally polarized Riemann matrix. By Lemma F.22 in Appendix F.3 any principally polarized Riemann matrix is in turn equivalent to a normalized principally polarized Riemann matrix, one of the form \(((I \ Z), J)\) where \( I \) is the identity matrix of rank \( g \) and \( Z \in \mathfrak{H}_g \) is a matrix in the Siegel upper half-space of rank \( g \), although not to a unique normalized principally polarized Riemann matrix; but by Lemma F.23 in Appendix F.3 two normalized principally polarized Riemann matrices \(((I \ Z), J)\) and \(((I \ \tilde{Z}), J)\) are equivalent if and only if

\[
(3.53) \quad \tilde{Z} = (A + ZC)^{-1}(B + ZD)
\]

for a symplectic modular matrix

\[
Q = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}).
\]

Thus if \( A_g = \mathfrak{H}_g / \text{Sp}(2g, \mathbb{Z}) \) is the quotient of the complex manifold \( \mathfrak{H}_g \) by the action (3.53) of the symplectic modular group on the Siegel upper half-space, the moduli space of equivalence classes of principally polarized Riemann matrices, there is associated to any compact Riemann surface \( M \) of genus \( g > 0 \) a unique point \( Z(M) \in A_g \), the point in the quotient space \( A_g \) represented by the matrix \( Z \) for any normalized principally polarized period matrix for the surface \( M \). The image point is called the \textit{Riemann modulus} of the Riemann surface \( M \). That of course raises the questions (i) whether the mapping that associates to a compact Riemann surface \( M \) of genus \( g > 0 \) its Riemann modulus \( Z(M) \in A_g \) is an injective mapping, so that the Riemann moduli describe the biholomorphic equivalence classes of Riemann surfaces; and (ii) how to describe or characterize the image in \( A_g \) of the set of Riemann moduli for all compact Riemann surfaces of genus \( g > 0 \). These are basic questions in the further study of compact Riemann surfaces, to be considered in somewhat more detail later. For the present, though, the normalized principally polarized period matrices of a Riemann surface will be used just to determine canonical bases for the spaces of holomorphic abelian differentials.

If \( M \) is a marked Riemann surface\(^6\) of genus \( g > 0 \) the marking determines a basis for the homology \( H_1(M) \) consisting of homology classes \( \tau_j = \alpha_j \), \( \tau_{g+j} = \beta_j \) for \( 1 \leq j \leq g \). In terms of this basis the intersection matrix of the surface

\(^6\)The notion and properties of marked surfaces are discussed in Appendix D.2.
$M$ is the basic skew-symmetric matrix $J$, by Theorem D.1 of Appendix D.2, so the period matrix $\Omega$ of the surface is a principally polarized Riemann matrix; and this matrix can be reduced to a normalized principally polarized Riemann matrix $\Omega = (I\ Z)$, where $Z \in \mathcal{H}_g$ represents the Riemann modulus of the surface $M$.

**Theorem 3.22** If $M$ is a marked Riemann surface of genus $g > 0$, with the marking representing homology classes $\alpha_j, \beta_j \in H_1(M)$, there is a uniquely determined basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on $M$ such that

\begin{equation}
\int_{\alpha_j} \omega_i = \delta_i^j \text{ for } 1 \leq i \leq g;
\end{equation}

the remaining periods are the entries

\begin{equation}
\int_{\beta_j} \omega_i = z_{ij}
\end{equation}

in a matrix $Z \in \mathcal{H}_g$ in the Siegel upper half-space of rank $g$ that represents the Riemann modulus for the Riemann surface $M$.

**Proof:** If $\tilde{\omega}_j$ is a basis for the holomorphic abelian differentials on $M$ and $\tau_j \in H_1(M)$ is the basis for the homology of $M$ arising from the marking of $M$ then the period matrix of the surface in terms of these bases is a principally polarized Riemann matrix $\Omega$. By Lemma F.22 there is a uniquely determined nonsingular matrix $A = \{a_{ij}\} \in \text{Gl}(g, \mathbb{C})$ such that $A^{-1}\Omega$ is a normalized principally polarized Riemann matrix, for which $A^{-1}\Omega = (I\ Z)$ where $Z \in \mathcal{H}_g$. Then $(I\ Z)$ is the period matrix of the surface $M$ in terms of the basis $\omega_i = \sum_{j=1}^{g} a_{ij} \tilde{\omega}_j$ for the holomorphic abelian differentials on $M$ in terms of the given basis $\tau_j \in H_1(M)$, so these abelian differentials have the periods (3.54) and (3.55) and that suffices for the proof.

The basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on a marked Riemann surface satisfying the conditions of the preceding theorem is called the canonical basis for the holomorphic abelian differentials on that marked surface. Such bases are particularly simple for many purposes and are quite commonly used in the study of compact Riemann surfaces; but they do depend on the choice of a marking, so are not intrinsic to the surface itself.

Another important consequence of the Riemann Matrix Theorem is that the Jacobi and Picard varieties of a compact Riemann surface are biholomorphic complex manifolds. Indeed by Theorem F.20 in Appendix F.3, if $(\Omega, P)$ is a polarized Riemann matrix for which $\det P = 1$ and $\Pi$ is the inverse period matrix to $\Omega$ then $(\Pi, \ tP^{-1})$ is a polarized Riemann matrix equivalent to $(\Omega, P)$, where the equivalence is exhibited by the Hurwitz relation $(\Omega P^{-1} \Pi, P)$ from the period matrix $\Pi$ to the period matrix $\Omega$; and it follows from this that the complex tori $J(\Omega)$ and $J(\Pi)$ are biholomorphic complex manifolds. The explicit form of this biholomorphic mapping is often quite useful in the examination of Riemann surfaces.
Theorem 3.23 If $M$ is a compact Riemann surface of genus $g > 0$ and $(\Omega, P)$ is the polarized period matrix of that surface in terms of bases for the holomorphic abelian differentials and the homology of the surface then the linear mapping $\Omega P : \mathbb{C}^{2g} \rightarrow \mathbb{C}^g$ defined by the $g \times 2g$ complex matrix $\Omega P$ defines a biholomorphic mapping

\begin{equation}
(\Omega P)^* : \mathbb{C}^{2g} / \mathbb{Z}^{2g} + \Omega \mathbb{C}^g \rightarrow \mathbb{C}^g / \Omega \mathbb{Z}^{2g}
\end{equation}

from the Picard variety

$$P(M) = \mathbb{C}^{2g} / \mathbb{Z}^{2g} + \Omega \mathbb{C}^g$$

of the surface $M$ onto its Jacobi variety

$$J(M) = \mathbb{C}^g / \Omega \mathbb{Z}^{2g}.$$ 

Proof: Since $\Omega P \Omega = 0$ by Riemann’s equality, the linear subspace $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ is in the kernel of the linear mapping described by the matrix $\Omega P$; and since rank $\Omega P = g$ the subspace $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ is precisely the kernel of this linear mapping. Further $\det P = 1$ by Corollary D.2 in Appendix D.2, so $P \in \text{Gl}(2g, \mathbb{Z})$ and hence $P \mathbb{Z}^{2g} = \mathbb{Z}^{2g}$; consequently $\Omega P \mathbb{Z}^{2g} = \Omega \mathbb{Z}^{2g}$, which suffices to conclude the proof.
Chapter 4

Meromorphic Differentials of the Second Kind and Double Differentials

Meromorphic differential forms on a compact Riemann surface $M$ often are called **meromorphic abelian differentials** on $M$. The sheaf $\mathcal{M}^{(1,0)}$ of germs of meromorphic abelian differentials was defined on page 47; and paralleling the identification (2.19) on page 43 there is the natural identification $\mathcal{M}^{(1,0)}(\lambda) \cong \mathcal{M}(\kappa \lambda)$ of the sheaf of germs of meromorphic abelian differentials that are cross-sections of a holomorphic line bundle $\lambda$ with the sheaf of germs of meromorphic cross-sections of the product $\kappa \lambda$ of the line bundle $\lambda$ with the canonical bundle $\kappa$ of the Riemann surface $M$. The divisors of meromorphic abelian differentials are the not necessarily positive canonical divisors $\kappa$; as noted in the discussion of divisors of holomorphic abelian differentials, $\kappa$ does not denote a single divisor but rather a linear equivalence class of divisors on $M$.

If $a \in U_\alpha$ is a pole of the meromorphic abelian differential $\mu$ in the coordinate neighborhood $U_\alpha \subset M$ and if $\gamma$ is a simple closed path in $U_\alpha$ that encircles the point $a$ once in the positive orientation and contains no other singularities of the differential $\mu$ on the path itself or in its interior the integral

$$\text{res}_a(\mu) = \frac{1}{2\pi i} \int_\gamma \mu$$

is called the **residue** of the abelian differential $\mu$ at the point $a$; since the differential $\mu$ is a $\mathcal{C}^\infty$ closed differential form except at its poles it is clear from Stokes’s Theorem that the value of this integral is independent of the choice of the path $\gamma$, subject to the stated conditions. If $z_\alpha$ is a local coordinate centered at the point $a \in M$ and the differential is written $\mu = f_\alpha dz_\alpha$ then the residue $\text{res}_a(\mu)$ is the coefficient of $z_\alpha^{-1}$ in the Laurent expansion of the function $f_\alpha$ in terms of this coordinate; that coefficient consequently is independent of the choice of the local coordinate. The corresponding coefficient in the Laurent expansion of
a meromorphic function, on the other hand, depends on the choice of the local coordinate; a direct calculation for the case of a simple pole is quite convincing. Thus the residue of a meromorphic function really is not well defined independently of the choice of a local coordinate system, while that of a meromorphic abelian differential is; that should be kept in mind to avoid possible confusion.

**Theorem 4.1** The sum of the residues of a meromorphic abelian differential at all its poles on a compact Riemann surface is zero.

**Proof:** Let \( a_i \) be the finitely many poles of a meromorphic abelian differential \( \mu \) on the surface \( M \). For each pole \( a_i \) let \( \gamma_i \) be a path encircling that pole once in the positive orientation and containing no other poles of the differential \( \mu \) on the path itself or in its interior, let \( \Delta_i \) be the interior of the path \( \gamma_i \), and assume further that these paths are chosen so that the closed sets \( \overline{\Delta}_i = \overline{\Delta}_i \cup \gamma_i \) are disjoint. Note that the path \( \bigcup_i \gamma_i \) can be viewed either as the boundary of the set \( \bigcup_i \Delta_i \) or as the negative of the boundary of the complementary set \( M \sim \bigcup_i \Delta_i \), that is, as the boundary of the latter set with the reversed orientation. The differential form \( \mu \) is a closed differential form except at the points \( a_i \), hence in particular \( \mu \) is a closed differential form on the set \( M \sim \bigcup_i \Delta_i \) so \( d\mu = 0 \) there; consequently

\[
2\pi i \sum_i \text{res}_{a_i}(\mu) = \int_{\bigcup_i \gamma_i} \mu = -\int_{\partial(M \sim \bigcup_i \Delta_i)} \mu = -\int_{M \sim \bigcup_i \Delta_i} d\mu = 0
\]

by Stokes’s Theorem, and that suffices to complete the proof.

A meromorphic function on a compact Riemann surface can be described uniquely up to an additive constant by specifying its singularities; and a meromorphic abelian differential can be described uniquely up to an additive holomorphic abelian differential by specifying its singularities. To make this more precise, the sheaf of germs of holomorphic functions \( \mathcal{O} \) is a subsheaf of the sheaf of germs of meromorphic functions \( \mathcal{M} \), so there is a well defined quotient sheaf \( \mathcal{P} = \mathcal{M}/\mathcal{O} \) called the sheaf of principal parts on \( M \). Similarly the sheaf of germs of holomorphic differential forms \( \mathcal{O}(1,0) \) is a subsheaf of the sheaf of germs of meromorphic differential forms \( \mathcal{M}(1,0) \), so there is a well defined quotient sheaf \( \mathcal{P}(1,0) = \mathcal{M}(1,0)/\mathcal{O}(1,0) \) called the sheaf of differential principal parts on \( M \).

There are thus the two exact sequences of sheaves

\[
0 \rightarrow \mathcal{O} \xrightarrow{\iota} \mathcal{M} \xrightarrow{p} \mathcal{P} \rightarrow 0, \tag{4.2}
\]

\[
0 \rightarrow \mathcal{O}(1,0) \xrightarrow{\iota} \mathcal{M}(1,0) \xrightarrow{p} \mathcal{P}(1,0) \rightarrow 0, \tag{4.3}
\]

in each of which \( \iota \) is the natural inclusion homomorphism and \( p \) is the homomorphism that associates to the germ of a meromorphic function or differential at a point \( a \in M \) its principal part at the point \( a \), the element in the quotient sheaf that it represents.
The sheaves $\mathcal{P}$ and $\mathcal{P}^{(1,0)}$ and the sheaf homomorphisms $p$ in these exact sequences can be described more concretely in quite familiar terms. A germ of a meromorphic function $f \in \mathcal{M}_a$ at any point $a \in M$ has a Laurent expansion in any local coordinate $z_\alpha$ centered at the point $a$, and the negative terms in that Laurent expansion determine the given germ uniquely up to the germ of a holomorphic function. Thus the image $p_a = p(f) \in \mathcal{P}_a$, the principal part of the germ $f$ at the point $a$, is described completely by the negative terms in the Laurent expansion of $f$ in terms of the local coordinate $z_\alpha$, and of course any Laurent series with finitely many negative terms describes the germ of some meromorphic function; hence $\mathcal{P}_a$ can be identified with the set of finite negative Laurent expansions in a local coordinate centered at the point $a$. The negative terms in the Laurent expansion in one local coordinate centered at the point $a$ completely determine the negative terms in the Laurent expansion in any other local coordinate centered at that point; but the explicit formula for the relation between these two Laurent expansions depends on the particular local coordinates. The germ of a meromorphic abelian differential $\mu$ at a point $a \in M$ can be expressed in terms of any local coordinate $z_\alpha$ centered at that point as $\mu = f_\alpha \cdot dz_\alpha$ where $f_\alpha \in \mathcal{M}_a$, and the principal part of the differential $\mu$ is determined completely by the principal part of the coefficient function $f_\alpha$. Thus $\mathcal{P}_a^{(1,0)}$ can be identified with $\mathcal{P}_a \cdot dz_\alpha$ in terms of any local coordinate $z_\alpha$ centered at the point $a$. Again the negative terms in the Laurent expansion in one local coordinate centered at the point $a$ completely determine the negative terms in the Laurent expansion in any other local coordinate centered at that point; but the explicit formula for the relation between these two Laurent expansions depends on the particular local coordinates. The one exceptional case is that of a meromorphic abelian differential having a simple pole with residue $r$; in that case the single negative term in the Laurent expansion for any local coordinate $z_\alpha$ centered at that point is always $(r/z_\alpha) \cdot dz_\alpha$. That this is the only case in which the coefficients of the negative terms in the Laurent expansion are independent of the choice of local coordinate is quite evident upon considering just simple changes of the local coordinate of the form $\tilde{z} = cz$. A section $p \in \Gamma(M, \mathcal{P})$, called a principal part on $M$, is described by listing principal parts of meromorphic functions at a discrete set of points of the Riemann surface $M$; correspondingly a section $p \in \Gamma(M, \mathcal{P}^{(1,0)})$, called a differential principal part on $M$, is described by listing the principal parts of meromorphic abelian differentials at a discrete set of points of the Riemann surface $M$. If a principal part or differential principal part $p$ consists of Laurent expansions with poles of order $\nu_i$ at distinct points $a_i \in M$ the divisor of that principal part is the divisor $d(p) = \sum_i \nu_i \cdot a_i$; it is locally just the polar divisor $d_+(f_\alpha)$ for any local meromorphic function or differential form $f_\alpha$ that has the principal part $p$.

**Theorem 4.2**  On a compact Riemann surface $M$ there are isomorphisms

$$\frac{\Gamma(M, \mathcal{P})}{p(\Gamma(M, \mathcal{M}))} \cong H^1(M, \mathcal{O}) \quad \text{and} \quad \frac{\Gamma(M, \mathcal{P}^{(1,0)})}{p(\Gamma(M, \mathcal{M}^{(1,0)}))} \cong H^1(M, \mathcal{O}^{(1,0)}),$$

where $p$ is the linear mapping that associates to a meromorphic function or
differential form on \( M \) its principal part.

**Proof:** The exact cohomology sequence associated to the exact sequence of sheaves (4.2) contains the segment

\[
\Gamma(M, \mathcal{M}) \xrightarrow{p} \Gamma(M, \mathcal{P}) \xrightarrow{\delta_1} H^1(M, \mathcal{O}) \rightarrow H^1(M, \mathcal{M});
\]

and since \( H^1(M, \mathcal{M}) = 0 \) by Corollary 2.16 that yields the first isomorphism. Similarly the exact cohomology sequence associated to the exact sequence of sheaves (4.3) contains the segment

\[
\Gamma(M, \mathcal{M}^{(1,0)}) \xrightarrow{p} \Gamma(M, \mathcal{P}^{(1,0)}) \xrightarrow{\delta_2} H^1(M, \mathcal{O}^{(1,0)}) \rightarrow H^1(M, \mathcal{M}^{(1,0)});
\]

and since \( H^1(M, \mathcal{M}^{(1,0)}) = H^1(M, \mathcal{M}(\kappa)) = 0 \) by Corollary 2.16 that yields the second isomorphism and concludes the proof.

This provides an interesting and useful alternative interpretation of the cohomology groups \( H^1(M, \mathcal{O}) \) and \( H^1(M, \mathcal{O}^{(1,0)}) \) as measures of the obstruction to a principal part or differential principal part being the principal part of a global meromorphic function or differential form. The cohomology groups can be eliminated from the statement of the preceding theorem by using the Serre Duality Theorem.

**Theorem 4.3** On a compact Riemann surface \( M \) there are isomorphisms

\[
\delta_1 : \frac{\Gamma(M, \mathcal{P})}{p(\Gamma(M, \mathcal{M}))} \cong \Gamma(M, \mathcal{O}^{(1,0)})^* \quad \text{and} \quad \delta_2 : \frac{\Gamma(M, \mathcal{P}^{(1,0)})}{p(\Gamma(M, \mathcal{M}^{(1,0)}))} \cong \mathbb{C},
\]

where for any principal part \( p \in \Gamma(M, \mathcal{P}) \) the image \( \delta_1(p) \) is the element in the dual space to \( \Gamma(M, \mathcal{O}^{(1,0)}) \) that takes the value

\[
\delta_1(p)(\omega) = \sum_{a \in M} \text{res}_a(p\omega)
\]

on a holomorphic abelian differential \( \omega \in \Gamma(M, \mathcal{O}^{(1,0)}) \), and for any differential principal part \( p \in \Gamma(M, \mathcal{P}^{(1,0)}) \)

\[
\delta_2(p) = \sum_{a \in M} \text{res}_a(p).
\]

**Proof:** The first isomorphism in Theorem 4.2 arises from the exact cohomology sequence associated to the exact sequence of sheaves (4.2). Explicitly for a principal part \( p \) on the Riemann surface \( M \) choose a covering \( \mathcal{U} \) of the Riemann surface \( M \) by open sets \( U_\alpha \) such that each pole of the principal part \( p \) is contained within a single coordinate neighborhood \( U_\alpha \), and such that there are meromorphic functions \( f_\alpha \) in the sets \( U_\alpha \) with the principal parts \( p(f_\alpha) = p|_{U_\alpha} \); the differences \( f_{\alpha\beta} = f_\beta - f_\alpha \) are holomorphic in the intersections \( U_\alpha \cap U_\beta \) and form a cocycle in \( Z^1(\mathcal{U}, \mathcal{O}) \) representing the image of the principal part \( p \) in the cohomology group \( H^1(M, \mathcal{O}) \). The functions \( f_\alpha \) can be modified so that they become
within the sets $U_\alpha$ without changing their values in any intersections $U_\alpha \cap U_\beta$; indeed merely multiply each function $f_\alpha$ by a $C^\infty$ function that is identically 0 near the pole in $U_\alpha$ and is identically 1 in all other subsets $U_\beta$ of the covering $\mathcal{U}$. The modified functions $\tilde{f}_\alpha$ then form a $C^\infty$ cochain with coboundary $f_{\alpha\beta}$, and the global differential form $\bar{\partial}\tilde{f}_\alpha \in \Gamma(M, \mathcal{E}^{(0,1)})$ represents the cohomology class of the cocycle $f_{\alpha\beta}$ under the isomorphism $H^1(M, \mathcal{O}) \cong \Gamma(M, \mathcal{E}^{(0,1)})/\bar{\partial}\Gamma(M, \mathcal{E})$ of the Theorem of Dolbeault, Theorem 1.10. By the Serre Duality Theorem, Theorem 1.18, the dual space to the quotient vector space $\Gamma(M, \mathcal{E}^{(0,1)})/\bar{\partial}\Gamma(M, \mathcal{E})$ is the vector space $\Gamma(M, \mathcal{O}^{(1,0)})$, where a cross-section $\tau \in \Gamma(M, \mathcal{O}^{(1,0)})$ associates to the cross-section $\omega \in \bar{\partial}\tilde{f}_\alpha$ the value $T_\omega(\bar{\partial}\tilde{f}_\alpha) = \int_M \omega \wedge \bar{\partial}\tilde{f}_\alpha$; of course this can be normalized for convenience by multiplying by an arbitrary complex constant. The integrand in this formula is identically zero except near the poles $p_i$ where the meromorphic functions $f_\alpha$ have been modified, since otherwise $\tilde{f}_\alpha = f_\alpha$ is holomorphic. Therefore if $\Delta_i$ are disjoint open neighborhoods of the distinct poles $a_i$ of the principal part $p$ within which the functions $f_\alpha$ have been modified then

$$T_\omega(\bar{\partial}\tilde{f}_\alpha) = \frac{1}{2\pi i} \sum_i \int_{\Delta_i} \bar{\partial}\tilde{f}_\alpha \wedge \omega = \frac{1}{2\pi i} \sum_i \int_{\Delta_i} d(\tilde{f}_\alpha \omega) = \frac{1}{2\pi i} \sum_i \int_{\partial\Delta_i} \tilde{f}_\alpha \omega$$

by Stokes’s Theorem, since the functions $\tilde{f}_\alpha$ and $f_\alpha$ coincide on the boundaries of the sets $\Delta_i$. That yields the first isomorphism. The same argument carries through for a differential principal part $p$, simply replacing the functions $f_\alpha$ by differential forms in applying the Serre Duality Theorem; the integration does not involve integrating against a holomorphic abelian differential but just against a constant. That yields the second isomorphism and concludes the proof of the theorem.

If $p_a \in \mathcal{P}_a$ is a principal part at the point $a$ and $f$ is a meromorphic function in an open neighborhood of $a$ with that principal part, and if $\omega$ is a holomorphic differential form in an open neighborhood of the point $a$, then the product $f \omega$ is a meromorphic abelian differential near the point $a$ that is determined by the principal part $p_a$ only up to the product of the holomorphic abelian differential $\omega$ and an arbitrary holomorphic function near that point; but the residue $\text{res}_a(f \omega)$ appearing in the statement of the preceding theorem is independent of the choice of this additive term, so is well defined. The preceding theorem provides criteria for determining which principal parts or differential principal parts on a compact Riemann surface are the principal parts of meromorphic functions or meromorphic abelian differentials on the surface.

**Corollary 4.4** On any compact Riemann surface $M$

(i) there exists a meromorphic function with the principal part $p \in \Gamma(M, \mathcal{P})$ if
and only if
\[ \sum_{a \in M} \text{res}_{a}(p \omega) = 0 \]
for all holomorphic abelian differentials \( \omega \in \Gamma(M, \mathcal{O}^{(1,0)}) \):

(ii) there exists a meromorphic abelian differential with the differential principal part \( p \in \Gamma(M, \mathcal{P}^{(1,0)}) \) if and only if
\[ \sum_{a \in M} \text{res}_{a}(p) = 0. \]

**Proof:** The first isomorphism of the preceding theorem shows that a principal part \( p \in \Gamma(M, \mathcal{P}) \) is the principal part of a meromorphic function, that is, is contained in the subgroup \( p(\Gamma(M, \mathcal{M})) \), if and only if \( \delta_{1}(p) \) is the trivial linear functional on the space of holomorphic abelian differentials, hence if and only if \( \sum_{a \in M} \text{res}_{a}(p \omega) = 0 \) for all holomorphic abelian differentials \( \omega \). Correspondingly the second isomorphism of the preceding theorem shows that a differential principal part \( p \in \Gamma(M, \mathcal{P}^{(1,0)}) \) is the principal part of a meromorphic differential, that is, is contained in the subgroup \( p(\Gamma(M, \mathcal{M}^{(1,0)})) \), if and only if \( 0 = \delta_{2}(p) = \sum_{a \in M} \text{res}_{a}(p) \). That suffices for the proof.

Part (ii) of the preceding corollary complements Theorem 4.1 by showing that the necessary condition given in Theorem 4.1 that a differential principal part on a compact Riemann surface \( M \) be the principal part of a meromorphic abelian differential also is sufficient. The meromorphic abelian differentials that have nontrivial poles but have residue zero at each pole are called *abelian differentials of the second kind*; and the meromorphic abelian differentials having at least one pole with a nonzero residue are called *abelian differentials of the third kind*. Correspondingly a *differential principal part of the second kind* is a differential principal part with zero residues at all its poles, while any other differential principal part is a *differential principal part of the third kind*. By the preceding corollary any differential principal part of the second kind is the principal part of an abelian differential of the second kind; and that differential is determined by its principal part uniquely up to the addition of an arbitrary abelian differential of the first kind. Just as in the case of holomorphic abelian differentials, meromorphic abelian differentials on a compact Riemann surface \( M \) of genus \( g > 0 \) can be viewed as \( \Gamma \)-invariant meromorphic differential forms on the universal covering space \( \tilde{M} \) of the surface \( M \), where \( \Gamma \) is the group of covering translations acting on \( \tilde{M} \). If \( \mu \) is an abelian differential of the second kind on \( M \) viewed as a meromorphic differential form on \( \tilde{M} \) then \( \int_{\gamma} \mu = 0 \) for any closed path \( \gamma \subset \tilde{M} \) that avoids the singularities of \( \mu \). Indeed since \( \tilde{M} \) is simply connected the path \( \gamma \) is the boundary \( \gamma = \partial \Delta \) of a domain \( \Delta \subset \tilde{M} \); and if \( a_{i} \) are the poles of \( \mu \) in \( \Delta \) and for each pole \( \gamma_{i} \) is a closed path in \( \Delta \) that encircles \( a_{i} \) once in the positive direction, such that the paths \( \gamma_{i} \) are disjoint and have disjoint interiors, then since \( \mu \) is a closed differential form in the complement of the points \( a_{i} \) in \( \Delta \) it follows from Stokes’s Theorem and the definition of the
residue that $\int_\gamma \mu = \sum_i \int_\gamma \mu = \sum_i \text{res } \omega_i(\mu) = 0$. Therefore the integral

\begin{equation}
\text{(4.4)}
\quad u(z,a) = \int_a^z \mu
\end{equation}

taken along any path in $\tilde{M}$ that avoids the poles of $\mu$ is a well defined meromorphic function of the variables $z,a \in \tilde{M}$ that is independent of the path of integration; such a function is called a \textit{meromorphic abelian integral} on the Riemann surface $M$, although of course really it is a meromorphic function on the universal covering space $\tilde{M}$ in both variables. A meromorphic abelian integral clearly satisfies the symmetry condition $u(z,a) = -u(a,z)$, and $u(z,z) = 0$ for all points $z \in \tilde{M}$. It is more convenient in many circumstances to view a meromorphic abelian integral as a function of the first variable only, and to allow it to be modified by an arbitrary additive constant; in such cases the function is denoted by $u(z)$ rather than $u(z,a)$, but also is called a meromorphic abelian integral, just as in the case of holomorphic abelian integrals. It must be kept in mind though that a meromorphic abelian integral $u(z)$ is determined by the meromorphic abelian differential $\nu$ only up to an arbitrary additive constant. For any choice of the meromorphic abelian integral $u(z)$ the integral (4.4) is given by $u(z,a) = u(z) - u(a)$ at its regular points.

\textbf{Lemma 4.5} The meromorphic abelian integrals $u(z)$ on a compact Riemann surface $M$ of genus $g > 0$ can be characterized as those meromorphic functions on the universal covering space $\tilde{M}$ of the surface $M$ that satisfy

\begin{equation}
\text{(4.5)}
\quad u(Tz) = u(z) + \mu(T) \quad \text{for all } T \in \Gamma
\end{equation}

for a group homomorphism $\mu \in \text{Hom}(\Gamma, \mathbb{C})$, where $\Gamma$ is the covering translation group of $M$.

\textbf{Proof:} As in the proof of the corresponding result, Lemma 3.1, for holomorphic abelian integrals, a meromorphic abelian integral $u(z)$ on a compact Riemann surface $M$ of genus $g > 0$ clearly satisfies (4.5) since $du(Tz) = du(z)$. The differential $\mu = du$ of any meromorphic function $u(z)$ on $\tilde{M}$ that satisfies (4.5) is invariant under $\Gamma$ and has zero residue at each pole so is a meromorphic abelian differential of the second kind; and the function $u(z)$ is an integral of $\mu$ so is a meromorphic abelian integral on $M$. That suffices for the proof.

Clearly the group homomorphism $\mu$ in (4.5) is unchanged when the abelian integral $u(z)$ is replaced by $u(z) + c$ for a complex constant $c$, so it is determined uniquely by the meromorphic abelian differential $\mu$; it is called the \textit{period class} of the abelian differential of the second kind $\mu = du$. The period class can be viewed either as an element of the group $H^1(\Gamma, \mathbb{C}) \cong \text{Hom}(\Gamma, \mathbb{C})$ or as an element of the group $\text{Hom}(H_1(M, \mathbb{C})) = H^1(M, \mathbb{C})$; in the latter case it associates to the homology class of a path $\tau \subset M$ that avoids the singularities of the abelian differential $\mu$ the period $\mu(\tau) = \int_\tau \mu$ in the usual sense.
**Lemma 4.6** A meromorphic abelian differential of the second kind on a compact Riemann surface $M$ of genus $g > 0$ is determined by its period class uniquely up to the derivative of a meromorphic function on $M$.

**Proof:** If the period class of a meromorphic abelian differential of the second kind is identically zero its integral is a meromorphic function on the Riemann surface $M$ itself and conversely, and that suffices for the proof.

**Theorem 4.7** On a compact Riemann surface $M$ of genus $g > 0$ let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{jk}\}$ be the intersection matrix of $M$ in terms of bases $\omega_i \in \Gamma(M, O^{(1,0)})$ and $\tau_j \in H_1(M)$.

(i) The periods $\mu(\tau_j)$ of an abelian differential $\mu$ of the second kind on $M$ satisfy

$$2g \sum_{j,k=1}^{g} \omega_{ij} p_{jk} \mu(\tau_k) = 2\pi i \sum_{a \in M} \text{res}_a(w_i \mu)$$

for $1 \leq i \leq g$, where $w_i(z) = \int_{z}^{\infty} \omega_i$ is the integral of the differential form $\omega$.

(ii) The periods $\mu'(\tau_j)$ and $\mu''(\tau_j)$ of any two abelian differentials $\mu', \mu''$ of the second kind on $M$ satisfy

$$2g \sum_{j,k=1}^{g} \mu'(\tau_j) p_{jk} \mu''(\tau_k) = 2\pi i \sum_{a \in M} \text{res}_a(u' \mu'')$$

where $u'(z) = \int_{a}^{z} \mu'$ is the integral of the meromorphic differential form $\mu'$.

**Proof:** (i) Let $u(z) = \int_{a}^{z} \mu$ be an integral of the differential of the second kind $\mu$, where $a$ is any point other than a pole of $\mu$; the periods of $\mu$ thus are given by $\mu(T) = u(Tz) - u(z)$ for all $T \in \Gamma$. Choose disjoint contractible open neighborhoods $\Delta_j$ of the poles $a_j$ of $\mu$ in $M$, and for each of these neighborhoods choose a connected component $\tilde{\Delta}_j$ of the inverse image $\pi^{-1}(\Delta_j) \subset \tilde{M}$ where $\pi: \tilde{M} \to M$ is the covering projection. The set $\tilde{\Delta}_j$ is thus homeomorphic to $\Delta_j$ under the covering projection $\pi$; and the complete inverse image $\pi^{-1}(\Delta_j) \subset \tilde{M}$ is the union of the disjoint open sets $T \tilde{\Delta}_j$ for all $T \in \Gamma$. Choose a $C^\infty$ real-valued function $r$ on $M$ that is identically one on an open neighborhood of $M \sim \cup_j \Delta_j$ and is identically zero in an open neighborhood of each pole $a_j$; this function also can be viewed as a $\Gamma$-invariant function on $\tilde{M}$. In terms of this auxiliary function introduce the smoothed integral

$$\tilde{u}(z) = \begin{cases} 
  u(z) & \text{for } z \in \tilde{M} \sim \cup_j \Gamma \tilde{\Delta}_j \\
  r(z)u(z) & \text{for } z \in \cup_j \tilde{\Delta}_j \\
  \tilde{u}(T^{-1}z) + \mu(T) & \text{for } z \in \cup_j T \tilde{\Delta}_j, \ T \neq I.
\end{cases}$$
Thus $\tilde{u}(z)$ is a $C^\infty$ function on $\tilde{M}$, $\tilde{u}(Tz) = \tilde{u}(z) + \mu(T)$ for any covering translation $T \in \Gamma$, and $\tilde{u}(z) = u(z)$ whenever $z \not\in \bigcup_j \Delta_j$. The differential form $\tilde{\mu} = d\tilde{u}$ is a $C^\infty$ closed $\Gamma$-invariant differential form on $\tilde{M}$, or equivalently is a $C^\infty$ closed differential form on $M$, that is holomorphic outside the set $\bigcup_j \Delta_j$ and that has the same periods as the meromorphic abelian differential $\mu$. Let $\phi_j$ be a basis for the first deRham group of $M$ dual to the chosen basis $\tau_j \in H_1(M)$, so that $\phi_j$ are $C^\infty$ closed differential forms of degree 1 on $M$ with the periods $\phi_j(\tau_k) = \delta_{jk}$. The $C^\infty$ differential form $\tilde{\mu}$ and the holomorphic abelian differentials $\omega_j$ can be expressed in terms of this basis by

$$\tilde{\mu} \sim \sum_{k=1}^{2g} \mu(\tau_k)\phi_k \quad \text{and} \quad \omega_j \sim \sum_{j=1}^{2g} \omega_{ij}\phi_j,$$

where $\sim$ denotes cohomologous differential forms, those that differ by exact differential forms. Then

$$\int_M \omega_i \wedge \tilde{\mu} = \sum_{j=1}^{2g} \int_M \omega_{ij}\phi_j \wedge \mu(\tau_k)\phi_k = \sum_{j,k=1}^{2g} \omega_{ij}p_{jk}\mu(\tau_k)$$

where $p_{jk} = \int_M \phi_j \wedge \phi_k$ are the entries of the intersection matrix of the surface $M$ in terms of these bases. On the other hand the differential form $\tilde{\mu}$ is holomorphic outside the set $\bigcup_j \Delta_j$, so that $\omega_i \wedge \tilde{\mu} = 0$ there, and consequently by Stokes's Theorem

$$\int_M \omega_i \wedge \tilde{\mu} = \int_{\bigcup_j \Delta_j} \omega_i \wedge \tilde{\mu} = \int_{\bigcup_j \Delta_j} d(w_i\tilde{\mu})$$

$$= \sum_j \int_{\partial \Delta_j} w_i \tilde{\mu} = \sum_j \int_{\partial \Delta_j} w_i \mu = 2\pi i \sum_j \text{res}_{a_j}(w_i \mu)$$

since $\mu = \tilde{\mu}$ on the boundary of the sets $\Delta_j$. Combining these two observations yields the formula of part (i), since $\text{res}_{a_j}(w_i \mu) = \text{res}_{a_j}(w_i p)$ for any meromorphic abelian differential $\mu$ with the differential principal part $p$.

(ii) Next for any two abelian differentials $\mu', \mu''$ of the second kind let $\{a_j\}$ be the union of the sets of poles of these two differentials, and choose disjoint contractible open neighborhoods $\Delta_j$ of these points and a $C^\infty$ function $r$ as in the preceding part of the argument; and in these terms introduce the smoothed versions $\tilde{u}'(z)$ and $\tilde{u}''(z)$ of the integrals $u'(z) = \int_a^z \mu'$ and $u''(z) = \int_a^z \mu''$, also as in the preceding part of the argument. Then for the $C^\infty$ closed differential forms $\tilde{\mu}' = d\tilde{u}'$ and $\tilde{\mu}'' = d\tilde{u}''$ it follows that

$$\int_M \tilde{\mu}' \wedge \tilde{\mu}'' = \sum_{j,k=1}^{2g} \int_M \mu'(\tau_j)\phi_j \wedge \mu''(\tau_k)\phi_k = \sum_{j,k=1}^{2g} \mu'(\tau_j)p_{jk}\mu''(\tau_k).$$

Again $\tilde{\mu}' \wedge \tilde{\mu}'' = 0$ outside the sets $\Delta_j$, since both $\tilde{\mu}'$ and $\tilde{\mu}''$ are holomorphic
differential forms there, and consequently by Stokes’s theorem
\[ \int_{\tilde{\mathcal{M}}} \tilde{\mu}' \wedge \tilde{\mu}'' = \sum_{j} \int_{\partial \Delta_j} \tilde{u}' \tilde{\mu}'' = \sum_{j} \int_{\partial \Delta_j} u' \mu'' = 2\pi i \sum_{j} \text{res}_{a_j}(u' \mu'') \]
since \( \tilde{u}' = u' \) and \( \tilde{\mu}'' = \mu'' \) on the boundary of the sets \( \Delta_j \). Combining these

The products \( w_i \mu \) and \( u' \mu'' \) in the preceding theorem are meromorphic dif-
ferential forms on \( \tilde{\mathcal{M}} \), so their residues are well defined at least on \( \tilde{\mathcal{M}} \); and

since \( w_i(Tz)\mu(Tz) = w_i(z)\mu(z) + \omega_i(T)\mu(z) \) and \( u'(Tz)\mu''(Tz) = u'(z)\mu''(z) \) + \( \mu'(T)\mu''(z) \), where the differentials \( \mu \) and \( \mu'' \) have zero residue, the residues are

the same at any two points of \( \tilde{\mathcal{M}} \) that are transforms of one another by covering translations, so these residues actually are well defined on the Riemann

surface \( \mathcal{M} \) itself. The expression (4.7) is not symmetric in the differentials \( \mu' \) and \( \mu'' \); indeed since the intersection matrix \( P \) is skew-symmetric the left-hand side changes sign when these two differentials are reversed, so the right-hand side also must change signs and thus

\[ \text{res}_{a}(u' \mu'') = -\text{res}_{a}(u'' \mu') . \]

Alternatively this is a simple consequence of Stokes’s theorem, since

\[ \text{res}_{a}(u' \mu'') + \text{res}_{a}(u'' \mu') = \frac{1}{2\pi i} \int_{\gamma} (u' \mu'' + u'' \mu') = \frac{1}{2\pi i} \int_{\gamma} d(u' u'') = 0 \]

for any simple closed path \( \gamma \) encircling only the singularity \( a \). An abelian differential of the second kind is determined by its principal part only up to the

addition of an arbitrary abelian differential of the first kind; but it is possible to normalize the abelian differentials of the second kind in terms of their period

classes so that there corresponds to each differential principal part of the second kind a uniquely determined normalized abelian differential of the second kind.

**Theorem 4.8** (i) For any differential principal part of the second kind \( p \) on a compact Riemann surface \( M \) of genus \( g > 0 \) there are a unique meromorphic abelian differential of the second kind \( \mu_p \) and a unique holomorphic abelian differential \( \omega_p \) such that \( \mu_p \) has the differential principal part \( p \) and has the same period class as the complex conjugate differential \( \overline{\omega}_p \).

(ii) The holomorphic abelian differential \( \omega_p \) is characterized by the conditions that

\[ \int_{M} \omega \wedge \overline{\omega}_p = 2\pi i \sum_{a \in M} \text{res}_{a}(w_p) \]

for all holomorphic abelian differentials \( \omega \), where \( w(z) = \int_{a}^{z} \omega \) is the integral of the holomorphic differential form \( \omega \).
(iii) If \( p' \) and \( p'' \) are two differential principal parts of the second kind on \( M \) then the associated abelian differentials of the second kind \( \mu_{p'} \) and \( \mu_{p''} \) satisfy

\[
\sum_{a \in M} \text{res}_a(u_{p'} \mu_{p''}) = 0,
\]

where \( u_{p'}(z) = \int_a^z \mu_{p'} \) is the integral of the meromorphic differential form \( \mu_{p'} \).

**Proof:** (i) Let \( \omega_i \in \Gamma(M, \mathcal{O}^{(1,0)}) \) be a basis for the space of holomorphic abelian differentials on the surface \( M \) and \( \tau_j \in H_1(M) \) be a basis for the homology of the surface \( M \), and in terms of these bases let \( \Omega = \{ \omega_{ij} \} \) be the period matrix and \( P = \{ p_{ij} \} \) be the intersection matrix of \( M \). As in (F.9) in Appendix F.1 there is the direct sum decomposition \( \mathbb{C}^{2g} = \mathfrak{H} \mathbb{C}^g \oplus \mathfrak{H} \mathbb{C}^g \), in which the subspace \( \mathfrak{H} \mathbb{C}^g \subset \mathbb{C}^{2g} \) consists of the period vectors \( \{ \omega(\tau_j) \} \) of the holomorphic abelian differentials \( \omega \) on the basis \( \tau_j \), and the subspace \( \mathfrak{H} \mathbb{C}^g \subset \mathbb{C}^{2g} \) consists of the period vectors \( \overline{\omega}(\tau_j) \) of the complex conjugates \( \overline{\omega} \) of the holomorphic abelian differentials on the basis \( \tau_j \). If \( \mu \) is an abelian differential of the second kind with differential principal part \( p \) then for any abelian differential of the first kind \( \omega \) the sum \( \mu + \omega \) is an abelian differential of the second kind with the same principal part \( p \), and all the abelian differentials of the second kind with the differential principal part \( p \) arise in this way. There is a unique abelian differential of the first kind \( \omega \) such that the period vector \( \{ \mu_p(\tau_j) \} = \{ \mu(\tau_j) + \omega(\tau_j) \} \) of the sum \( \mu_p = \mu + \omega \) is contained in the linear subspace \( \mathfrak{H} \mathbb{C}^g \subset \mathbb{C}^{2g} \), hence such that \( \mu_p(\tau_j) = \overline{\mu}_p(\tau_j) \) for a uniquely determined holomorphic abelian differential \( \omega_p \), thus demonstrating (i).

(ii) If \( \phi_j \) are closed real differential forms of a basis for the first deRham group of \( M \) dual to the basis \( \tau_j \), then from the homologies \( \omega_i = \sum_{j=1}^g \omega_i(\tau_j) \phi_j = \sum_{j=1}^g \omega_{ij} \phi_j \) and \( \omega_p = \sum_{j=1}^g \omega_p(\tau_j) \phi_k \) it follows that

\[
\int_M \omega_i \wedge \overline{\omega}_p = \int_M \sum_{j,k=1}^g \omega_{ij} \overline{\omega}_p(\tau_k) \phi_j \wedge \phi_k = \sum_{j,k=1}^g \omega_{ij} p_{jk} \overline{\omega}_p(\tau_k);
\]

and since \( \overline{\omega}_p(\tau_k) = \mu_p(\tau_k) \) as in (i) it follows from (4.6) in Theorem 4.7 that

\[
\sum_{j,k=1}^g \omega_{ij} p_{jk} \overline{\omega}_p(\tau_k) = \sum_{j,k=1}^g \omega_{ij} p_{jk} \mu_p(\tau_k) = 2\pi i \sum_{a \in M} \text{res}_a(w_i p)
\]

where \( w_i(z) = \int_a^z \omega_i \) is the integral of the holomorphic differential form \( \omega_i \). Combining the two preceding equations shows that (ii) holds for the basis \( \omega_i \), and consequently it holds for all holomorphic abelian differentials \( \omega \).

(iii) Finally if \( p' \) and \( p'' \) are two differential principal parts of the second kind to which correspond the meromorphic differential forms \( \mu_{p'} \) and \( \mu_{p''} \), as in (i), the associated holomorphic abelian differentials \( \omega_{p'} \) and \( \omega_{p''} \) satisfy \( \omega_{p'} \wedge \omega_{p''} = 0 \), since the product is a differential form of type \( (2,0) \) on the Riemann surface \( M \); so from the homologies \( \omega_{p'} = \sum_{j=1}^g \omega_{p'}(\tau_k) \phi_k \) and \( \omega_{p''} = \sum_{j=1}^g \omega_{p''}(\tau_k) \phi_k \)
it follows that
\[
0 = \int_M \omega' \wedge \omega'' = \int_M \sum_{j,k=1}^g \omega_p'(\tau_j) \phi_j \wedge \omega_p''(\tau_k) \phi_k = \sum_{j,k=1}^g \omega_p'(\tau_j) p_{jk} \omega_p''(\tau_k).
\]

Since \(\omega_p'(\tau_k) = \mu_p'(\tau_k)\) and \(\omega_p''(\tau_k) = \mu_p''(\tau_k)\) it follows from (4.7) in Theorem 4.7 that
\[
\sum_{j,k=1}^g \omega_p'(\tau_j) p_{jk} \omega_p''(\tau_k) = 2 \pi i \sum_{a \in \mathcal{M}} \text{res}_a(u_1 \mu_2).
\]

Combining the two preceding equations shows that (iii) holds and thereby concludes the proof.

The meromorphic abelian differential \(\mu_p\) of part (i) of the preceding theorem is called the normalized abelian differential of the second kind with the differential principal part \(p\), and the holomorphic abelian differential \(\omega_p\) is called its associated holomorphic abelian differential; both are determined uniquely by the differential principal part \(p\). Since linear combinations of the normalized meromorphic abelian differentials of the second kind have period classes and principal parts that are the corresponding linear combinations it is evident that
\[
(4.12) \quad \mu_{c_1 p_1 + c_2 p_2} = c_1 \mu_{p_1} + c_2 \mu_{p_2}
\]
for any differential principal parts \(p_1\) and \(p_2\) of the second kind and any complex constants \(c_1, c_2\).

**Corollary 4.9** On a compact Riemann surface \(M\) of genus \(g > 0\) let \(\Omega = \{\omega_{ij}\}\) be the period matrix and \(P = \{p_{ij}\}\) be the intersection matrix of \(M\) in terms of bases \(\omega_i \in \Gamma(M, \mathcal{O}^{1,0})\) and \(\tau_j \in H_1(M)\). For any differential principal part \(p\) of the second kind on \(M\) the periods of the normalized abelian differential of the second kind \(\mu_p\) are
\[
(4.13) \quad \mu_p(T) = -2 \pi \sum_{m,n=1}^g \sum_{a \in \mathcal{M}} g_{mn} \text{res}_a(w_{m,p}) \omega_n(T)
\]
for any covering translation \(T \in \Gamma\), where \(G = \{g_{mn}\} = i H^{-1}\) for the positive definite Hermitian matrix \(H = i \Omega P \Omega^T\).

**Proof:** The holomorphic abelian differential \(\omega_p\) associated to the normalized abelian differential of the second kind \(\mu_p\) can be written as a sum \(\omega_p = \sum_{i=1}^g c_i \omega_i\) for some complex constants \(c_i\), so its periods are \(\omega_p(\tau_k) = \sum_{i=1}^g c_i \omega_i(\tau_k) = \sum_{i=1}^g c_i \omega_{ik}\). Substituting this into (4.11) yields the identity
\[
(4.14) \quad 2 \pi i \sum_{a \in \mathcal{M}} \text{res}_a(w_{m,p}) = \sum_{j,k,l=1}^g \omega_{mj} p_{jk} \omega_{lk} = -i \sum_{l=1}^g h_{ml} \omega_l
\]
where $h_{nm}$ are the entries in the $g \times g$ complex matrix $H = i \Omega P \overline{\Omega}$. This matrix is positive definite Hermitian by Riemann’s inequality, Theorem 3.20 (ii), so $G = \{g_{mn}\}$ exists; and if $G = \{g_{mn}\}$ then upon multiplying (4.14) by $g_{mn}$ and summing over $m$ it follows that

$$
\tau_n = -2\pi \sum_{m=1}^{g} \sum_{a \in M} g_{mn} \text{Res}_a(w_m, p),
$$

hence that

$$
\mu_p(\tau_j) = \omega_p(\tau_j) = \sum_{n=1}^{g} \tau_n \omega_{nj} = -2\pi \sum_{m,n=1}^{g} g_{mn} \text{Res}_a(w_m, p) \omega_n(z),
$$

If the covering translation $T \in \Gamma$ corresponds to a homology class $\tau \in H_1(M)$ and $\tau \sim \sum_{j=1}^{g} n_j \tau_j$ for some integers $n_j$ then $\mu_p(T) = \mu_p(\tau) = \sum_{j=1}^{g} n_j \mu_p(\tau_j)$ and correspondingly $\omega_n(T) = \omega_n(\tau) = \sum_{j=1}^{g} n_j \omega_n(\tau_j) = \sum_{j=1}^{g} n_j \omega_{nj}$; multiplying both sides of the preceding equation by $n_j$ and summing over $j = 1, \ldots, g$ yields (4.14) and thereby concludes the proof.

Since $\omega_p(T) = \mu_p(T)$ it follows immediately from (4.13) that the associated holomorphic abelian differential is given by

$$
\omega_p(z) = -2\pi \sum_{m,n=1}^{g} \sum_{a \in M} g_{mn} \text{Res}_a(w_m, p) \omega_n(z).
$$

It is apparent from this that although the normalized abelian differentials of the second kind are complex linear functions of their differential principal parts as in (4.12), the associated holomorphic abelian differentials are complex conjugate functions of their differential principal parts; of course that also is clear from the definitions of these differentials, since the periods of the normalized abelian differentials and of their associated holomorphic abelian differentials are complex conjugates of one another. The explicit formulas (4.13) and (4.15) depend on the choice of bases $\omega_i$ for the holomorphic abelian differentials and $\tau_j$ for the homology of the surface $M$, but it is clear that $\mu_p(T)$ and $\omega_p(z)$ are independent of these choices. It may be comforting to verify that directly, and for that purpose as well as for later use it is convenient to rephrase (4.13) in matrix notation. If $\tilde{w} : \tilde{M} \rightarrow \mathbb{C}^g$ is the mapping defined by the column vector \{w_j(z)\} of holomorphic abelian integrals on $\tilde{M}$ and $\omega \in \text{Hom}(\Gamma, \mathbb{C}^g)$ is the group homomorphism defined by the column vector \{\omega_j(T)\} of period classes of the holomorphic abelian differentials, as in (3.3) and (3.4), then (4.13) can be written equivalently as

$$
\mu_p(T) = -2\pi \sum_{a \in M} \text{Res}_a(\tilde{w}^T \cdot G \cdot \omega(T)).
$$

Of course (4.15) can be rephrased correspondingly. A change of basis for the holomorphic abelian differentials on $M$ has the effect of replacing the vector $\tilde{w}$
by $A\omega$ and the vector $\varpi(T)$ by $A\varpi(T)$, and the form matrix $G$ by $A^{-1}GA^{-1}$ as in equation (F.41) in Appendix F.4; and this change clearly leaves the expression $\text{res}_a(T \varpi p) \cdot G \cdot \varpi(T)$ unchanged.

**Corollary 4.10** On a compact Riemann surface $M$ of genus $g > 0$ there is a meromorphic function with the principal part $p$ if and only if the normalized abelian differential of the second kind $\mu dp$ with the differential principal part $dp$ has a trivial period class.

**Proof:** If there is a meromorphic function $f$ with principal part $p$ then $df$ is a meromorphic abelian differential of the second kind with the principal part $p$ and with a trivial period class; since its period class is the same as that of the trivial holomorphic abelian differential $\omega = 0$ it follows from Theorem 4.8 (i) that $df$ is a normalized abelian differential of the second kind. Conversely if there is a normalized abelian differential of the second kind with principal part $dp$ and with trivial period class it is the differential of a meromorphic function with the period class $p$. That suffices for the proof.

**Theorem 4.11** Let $a_j \in M$ be $n$ distinct points of a compact Riemann surface $M$ of genus $g > 0$, let $d = \sum_{j=1}^{n} \nu_j \cdot a_j$ be a positive divisor with $\deg d = r \leq g$, and let $p_{jk}$ be principal parts of the form $p_{jk} = z_j^{-k}$ in terms of local coordinates $z_j$ centered at the points $a_j$. The period classes of the $r$ normalized abelian differentials of the second kind $\mu_{jk} = \mu dp_{jk}$ with the differential principal parts $dp_{jk}$ for $1 \leq j \leq n$, $1 \leq k \leq \nu_j$, are linearly dependent if and only if $d$ is a special positive divisor.

**Proof:** The period classes of the normalized abelian differentials of the second kind $\mu_{jk}$ are linearly dependent if and only if there are constants $c_{jk}$ not all of which are zero such that $\sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk} \mu_{jk}(T) = 0$ for all covering translations $T \in \Gamma$; and that is just the condition that the nontrivial normalized abelian differential of the second kind $\mu = \sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk} \mu_{jk}$ with the differential principal part $dp = \sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk} dp_{jk}$ has trivial period classes, which by Corollary 4.10 is equivalent to the condition there is a meromorphic function on $M$ with the principal part $p = \sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk} p_{jk}$. By Corollary 4.4 there is a meromorphic function on $M$ with principal part $p$ if and only if

$$0 = \sum_{a \in M} \text{res}_a(p \omega_i) = \sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk} \text{res}_a(p_{jk} \omega_i)$$

(4.17)

where $\omega_i$ is a basis for the holomorphic abelian differentials on $M$. If $\omega_i = f_i(z_j)dz_j$ is the expression of the differential $\omega_i$ in terms of the local coordinate $z_j$ centered at the point $a_j$ then

$$\text{res}_{a_j}(p_{jk} \omega_i) = \text{res}_{z_j=0}(z_j^{-k} f_i(z_j)) = \frac{1}{(k-1)!} f_i^{(k-1)}(a_j);$$
Corollary 4.12 If \( z \) is a local coordinate at a point \( a \in M \) of a compact Riemann surface \( M \) of genus \( g > 0 \), the period classes of the \( r \) normalized abelian differentials of the second kind \( \mu_k = \mu_{\text{dp}_k} \) with the differential principal parts \( \text{dp}_k = z^{-k-1}dz \) for \( 1 \leq k \leq r \leq g \) are linearly dependent if and only if \( r \cdot a \) is a special positive divisor.

Proof: This is just the special case of the preceding theorem in which \( n = 1 \), \( \nu_1 = r \), so nothing further is needed to conclude the proof.

Corollary 4.13 Let \( a_i \in M \) be \( r \leq g \) distinct points of a compact Riemann surface \( M \) of genus \( g > 0 \) and \( \text{dp}_i \) be the principal parts consisting of a simple pole of residue 1 at the point \( a_i \in M \) and no other singularities on \( M \). The period classes of the \( r \) normalized abelian differentials of the second kind \( \mu_i = \mu_{\text{dp}_i} \) with the differential principal parts \( \text{dp}_i \) for \( 1 \leq i \leq r \leq g \) are linearly dependent if and only if \( 1 \cdot a_1 + \cdots + 1 \cdot a_r \) is a special positive divisor.

Proof: This is just the special case of the preceding theorem in which \( \nu_j = 1 \) for \( 1 \leq j \leq r \), so nothing further is needed to conclude the proof.

As noted in the proof of Theorem 4.8 (i), if \( \Omega \) is the period matrix of a compact Riemann surface \( M \) of genus \( g > 0 \) in terms of bases \( \omega_j \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_j \in H_1(M) \) the vectors consisting of the period classes of the holomorphic abelian differentials on the basis \( \tau_j \) span the linear subspace \( \mathfrak{H}C^g \subset \mathbb{C}^{2g} \); and by definition the vectors consisting of the period classes of the normalzed abelian differentials of the second kind on the basis \( \tau_j \) lie in the complementary linear subspace \( \mathfrak{H}C^g \subset \mathbb{C}^{2g} \) in the direct sum decomposition \( \mathbb{C}^{2g} = \mathfrak{H}C^g \oplus \mathfrak{H}C^g \). Consequently when the period classes of all of these abelian differentials are viewed as elements of the complex vector space \( \text{Hom}(\Gamma, \mathbb{C}) \) of dimension \( 2g \), the period classes of the differentials of the first kind and the period classes of the normalized differentials of the second kind lie in complementary linear subspaces of dimension \( g \). The two preceding corollaries then can be interpreted alternatively as follows in terms of the vector spaces \( L^{(1,0)}(\mathfrak{d}) \) of meromorphic differential forms as defined in (2.31).

Corollary 4.14 If \( a \) is a point of a compact Riemann surface \( M \) of genus \( g > 0 \) and \( r \geq 0 \) then \( \dim L^{(1,0)}((r+1) \cdot a) = g + r \) and the differential forms in \( L^{(1,0)}((r+1) \cdot a) \) are linear combinations of abelian differentials of the first
and second kinds. If $0 \leq r \leq g$ the period classes of these differential forms span a linear subspace of $\text{Hom}(\Gamma, \mathbb{C})$ of dimension $g + r$ if and only if the divisor $r \cdot a$ is a general positive divisor.

**Proof:** By definition the vector space $L^{(1,0)}((r + 1) \cdot a)$ consists of those meromorphic differential forms on $M$ that have a pole of order at most $r + 1$ at the point $a$; and from the Riemann-Roch Theorem in the form of Theorem 2.22 it follows that $\dim L^{(1,0)}((r + 1) \cdot a) = g + r$, for by (2.33) $\dim L(- (r + 1) \cdot a) = \gamma(\zeta_a^{-(r+1)}) = 0$ since $c(\zeta_a^{-(r+1)}) < 0$. Since the total residue of any differential form in $L^{(1,0)}((r + 1) \cdot a)$ vanishes by Theorem 4.4 (ii), all of these differential forms must be of the first or second kind. A basis for the normalized abelian differentials of the second kind in $L$ consists of $r \cdot a$ of genus $g$, and the space $\text{Hom}(\Gamma, \mathbb{C})$ of all period classes of these differentials are linearly independent if and only if the divisor $r \cdot a$ is a general positive divisor, so since these period classes and those of the abelian differentials of the first kind lie in complementary linear subspaces of the space $\text{Hom}(\Gamma, \mathbb{C})$ of all period classes that suffices for the proof.

**Corollary 4.15** If $a_1, \ldots, a_r$ are $r$ distinct points of a compact Riemann surface $M$ of genus $g > 0$ and $\mathfrak{d} = 1 \cdot a_1 + \cdots + 1 \cdot a_r$ then $\dim L^{(1,0)}(2 \cdot \mathfrak{d}) = g + 2r$ and the subspace of $L^{(1,0)}(2 \cdot \mathfrak{d})$ spanned by abelian differentials of the first and second kinds has dimension $g + r$. If $1 \leq r \leq g$ the period classes of the abelian differentials of the first and second kinds in $L^{(1,0)}(2 \cdot \mathfrak{d})$ span a linear subspace of $\text{Hom}(\Gamma, \mathbb{C})$ of dimension $g + r$ if and only if the divisor $\mathfrak{d}$ is a general positive divisor.

**Proof:** As in the proof of the preceding corollary, it follows from the Riemann-Roch Theorem in the form of Theorem 2.22 that $\dim L^{(1,0)}(2 \cdot \mathfrak{d}) = g + 2r$. A basis for the normalized abelian differentials of the second kind in $L^{(1,0)}(2 \cdot \mathfrak{d})$ consists of $r$ differential forms with differential principal parts $z_j^{-2}dz_j$ for local coordinates $z_j$ centered at the points $a_j$ in $M$, for $1 \leq j \leq r$. By Theorem 4.11 the period classes of these differentials are linearly independent if and only if the divisor $\mathfrak{d}$ is a general positive divisor; so since these period classes and those of the abelian differentials of the first kind lie in complementary linear subspaces of the space $\text{Hom}(\Gamma, \mathbb{C})$ of all period classes that suffices for the proof.

The most general abelian differential of the second kind with the differential principal part $p$ is of the form $\mu_p + \omega$ for an arbitrary holomorphic abelian differential $\omega$. Some particular choices of the differential $\omega$ yield useful alternative normalizations of the abelian differentials of the second kind, which also are determined uniquely by their differential principal parts. If $M$ is a marked Riemann surface of genus $g > 0$, with the marking described by generators $A_j, B_j \in \Gamma$ of the covering translation group $\Gamma$ of the surface for $1 \leq j \leq g$, the associated canonical basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials is characterized by the period conditions $\omega_i(A_j) = \delta_{ij}$ for $1 \leq i, j \leq g$.

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1The definition and properties of markings of a surface are discussed in Appendix D.1.
as in Theorem 3.22. To any differential principal part of the second kind \( p \) associate the meromorphic abelian differential

\[
\hat{\mu}_p = \mu_p - \sum_{j=1}^{g} \mu_p(A_j) \omega_j,
\]

called the canonical abelian differential of the second kind on the marked surface \( M \) with the differential principal part \( p \).

**Theorem 4.16** Let \( M \) be a marked Riemann surface of genus \( g > 0 \), with the marking described by generators \( A_j, B_j \in \Gamma \) of the covering translation group \( \Gamma \) of the surface for \( 1 \leq j \leq g \), and let \( p \) be a differential principal part of the second kind on \( M \).

(i) The canonical abelian differential \( \hat{\mu}_p \) is characterized by the conditions that it has the differential principal part \( p \) and the periods

\[
\hat{\mu}_p(A_j) = 0 \quad \text{for} \quad 1 \leq j \leq g.
\]

(ii) The remaining periods of the differential \( \hat{\mu}_p \) are determined by (i) and

\[
\hat{\mu}_p(B_j) = 2\pi i \sum_{a \in M} \text{res}_a(w_j p) \quad \text{for} \quad 1 \leq j \leq g,
\]

where \( w_j(z) = \int_a^z \omega_j \) is the integral of the abelian differential \( \omega_j \).

**Proof:** (i) The defining equation (4.18) for the differential \( \hat{\mu}_p \) amounts to the conditions that \( \hat{\mu}_p \) has the same differential principal part as the normalized abelian differential of the second kind \( \mu_p \) and that

\[
\hat{\mu}_p(A_j) = \mu_p(A_j) - \sum_{k=1}^{g} \mu_p(A_k) \omega_k(A_j) = 0
\]

since \( \omega_k(A_j) = \delta_{jk}^i \).

(ii) The intersection matrix of the surface \( M \) in terms of the basis for the homology \( H_1(M) \) described by the marking is the basic skew-symmetric matrix \( J \) by Theorem D.1 in Appendix D.2; and by Theorem 3.22 the period matrix of \( M \) in terms of this basis for the homology \( H_1(M) \) and the canonical basis for the holomorphic abelian differentials on \( M \) has the form \( \Omega = (I \ Z) \) for a matrix \( Z \) in the Siegel upper half-space \( \mathfrak{H}_g \), a \( g \times g \) complex symmetric matrix with a positive definite imaginary part \( Y = \Re(Z) \). It then follows from Corollary 4.9 that

\[
\mu_p(A_j) = -2\pi \sum_{m=1}^{g} \sum_{a \in M} g_{mj} \text{res}_a(w_mp)
\]

\[
\mu_p(B_j) = -2\pi \sum_{m,n=1}^{g} \sum_{a \in M} g_{mn} \Re_{nj} \text{res}_a(w_mp)
\]
where

\[
H = i \begin{pmatrix} 1 & Z \\ -Z & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ Z \end{pmatrix} = i(Z - Z) = 2Y
\]

and \(G = H^{-1} = \frac{1}{2} Y^{-1}\). Since \(Z - Z = 2iY\) and \(GY = \frac{1}{2}\) it follows from (4.18) that

\[
\tilde{\mu}_p(B_j) = \mu_p(B_j) - \sum_{k=1}^g \mu_p(A_k) \omega_k(B_j) = \mu_p(B_j) - \sum_{k=1}^g \mu_p(A_k) z_{kj}
\]

\[
= -2\pi \sum_{m,k=1}^g \sum_{a \in M} g_{mk} z_{kj} \text{res}_a(w_m) + 2\pi \sum_{m,k=1}^g \sum_{a \in M} g_{mk} z_{kj} \text{res}_a(w_m)
\]

\[
= 2\pi \sum_{m,k=1}^g \sum_{a \in M} g_{mk} 2i z_{kj} \text{res}_a(w_m) = 2\pi i \sum_{a \in M} \text{res}_a(w_{kj})
\]

which suffices to conclude the proof.

The advantage of this normalization is the simplicity of the periods in (4.19) and (4.20), a convenience in some calculations; the disadvantage is that this normalization is not intrinsically determined, but depends on the marking of the surface. Another normalization, although one that will play almost no role in the subsequent discussion here, leads to harmonic rather than complex analytic functions. To any differential principal part of the second kind \(p\) on a compact Riemann surface \(M\) of genus \(g > 0\) associate the meromorphic abelian differential

\[
\tilde{\mu}_p = \mu_p - \omega_p.
\]

This differential has the differential principal part \(p\) and the periods \(\tilde{\mu}_p(T) = \mu_p(T) - \omega_p(T) = \omega_p(T)\) where \(\omega_p(T) = -2i \Im(\omega_p(T))\) where \(\Im(z)\) denotes the imaginary part of the complex number \(z\); it is readily seen to be determined uniquely by the conditions that it has the differential principal part \(p\) and purely imaginary periods, so it is called the real-normalized abelian differential of the second kind with the differential principal part \(p\). The disadvantage of this normalization is that the differential \(\tilde{\mu}_p\) is not a complex linear function of the differential principal part, since the associated abelian differential \(\omega_p\) is a conjugate linear function of the differential principal part \(p\); consequently it cannot be expected to depend analytically on other parameters such as the location of the poles. The advantage of this normalization on the other hand is that the period class of the differential \(\tilde{\mu}_p\) is purely imaginary, so the real part \(g_p(z) = \Re(\tilde{u}_p(z))\) of the integral \(\tilde{u}_p(z)\) of the closed differential form \(\tilde{\mu}_p\) is a well defined harmonic function on the Riemann surface \(M\) with the singularities of the real part of the meromorphic function \(\tilde{u}_p(z)\). This function, called the Green’s function of the differential principal part \(p\), is determined uniquely up to a real additive constant by the specified principal part and determines the differential \(\tilde{\mu}_p\) since it is easy to see that \(\tilde{\mu}_p = 2 \partial g_p\). Indeed if \(\tilde{u}_p = g_p + i h_p\) where \(g_p = \Re(\tilde{u}_p)\) and \(h_p = \Im(\tilde{u}_p)\)
then by the Cauchy-Riemann equations \( 0 = \partial u_p = \partial g_p + i \partial h_p \) so \( \partial h_p = i \partial g_p \) or equivalently \( \partial h_p = -i \partial g_p \) and therefore \( \mu_p = d\bar{u}_p = \partial \bar{u}_p = \partial g_p + i \partial h_p = 20\partial g_p \).

The simplest abelian differentials of the second kind are those having a single double pole with residue zero; and when viewed also as functions of the pole they will be shown to be meromorphic double differentials. A **meromorphic double differential** \( \mu(z, \zeta) \) on a compact Riemann surface \( M \) is an expression that is a well defined meromorphic differential form in each variable separately. More explicitly, if \( \{U_\alpha, z_\alpha\} \) and \( \{V_\beta, \zeta_\beta\} \) are two coordinate coverings of \( M \) then in each product \( U_\alpha \times V_\beta \) a meromorphic double differential \( \mu(z, \zeta) \) has the form

\[
\mu(z, \zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha \wedge d\zeta_\beta
\]

where \( f_{\alpha\beta}(z_\alpha, \zeta_\beta) \) is a meromorphic function of the variables \( (z_\alpha, \zeta_\beta) \in U_\alpha \times V_\beta \), and in an intersection \( (U_\alpha \times V_\beta) \cap (U_\gamma \times V_\delta) \)

\[
f_{\alpha\beta}(z_\alpha, \zeta_\beta) = \kappa_{\alpha\gamma}(z) \kappa_{\beta\delta}(\zeta) f_{\gamma\delta}(z_\gamma, \zeta_\delta)
\]

where \( \kappa_{\alpha\gamma}(z) = (dz_\alpha/dz_\gamma)^{-1} \) and \( \kappa_{\beta\delta}(\zeta) = (d\zeta_\beta/d\zeta_\delta)^{-1} \) are cocycles describing the canonical bundle \( \kappa \) over \( M \) in the two coordinate coverings; moreover it is required that (4.22) is a well defined meromorphic differential form in each variable separately, so that for each fixed point \( z_\alpha \in U_\alpha \) the expression \( f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\beta \) is a well defined meromorphic differential form in the variable \( \zeta_\beta \in V_\beta \) and correspondingly when the two variables are reversed. A consequence is that for a meromorphic double differential (4.22) the polar locus of the meromorphic function \( f_{\alpha\beta}(z_\alpha, \zeta_\beta) \) of two complex variables does not contain a product \( a_\alpha \times V_\beta \subset U_\alpha \times V_\beta \) or \( U_\alpha \times b_\beta \subset U_\alpha \times V_\beta \) for any points \( a_\alpha \in U_\alpha \), \( b_\beta \in V_\beta \); this restriction on the polar loci of meromorphic double differentials is significant and should be kept in mind throughout the subsequent discussion. In particular if \( \omega(z) = f(z) dz_\alpha \) is a holomorphic abelian differential on \( M \) and \( \mu(z) = g(z) dz_\alpha \) a meromorphic abelian differential on \( M \) with nontrivial singularities then the product \( \omega(z) \mu(z) = f(z)g(z) dz_\alpha d\zeta_\beta \) is not a meromorphic double differential with the definition adopted here. A double differential can be viewed alternatively as a meromorphic differential form

\[
\mu^*(z, \zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta) \, dz_\alpha \wedge d\zeta_\beta
\]

on the complex manifold \( M \times M \) expressed in terms of local product coordinate neighborhoods \( U_\alpha \times U_\beta \), when \( \{U_\alpha, z_\alpha\} \) and \( \{V_\beta, \zeta_\beta\} \) are viewed as coordinate coverings of the two separate factors and formally \( f_{\beta\alpha}(z_\alpha, \zeta_\beta) = -f_{\alpha\beta}(z_\alpha, \zeta_\beta) \), since (4.23) is just the formula for way in which a differential form \( \mu^*(z, \zeta) \) on \( M \times M \) transforms under a change of coordinates \( (z_\alpha, \zeta_\beta) = (f(z_\gamma), g(\zeta_\delta)) \) where \( f, g \) are holomorphic functions of a single complex variable; again though the singularities of the coordinates are restricted by the condition that the polar locus of the meromorphic function \( f_{\alpha\beta}(z_\alpha, \zeta_\beta) \) does not contain a product \( a_\alpha \times V_\beta \subset U_\alpha \times V_\beta \) or \( U_\alpha \times b_\beta \subset U_\alpha \times V_\beta \) for any points \( a_\alpha \in U_\alpha \), \( b_\beta \in V_\beta \). However the real interest here lies in the double differential \( \mu(z, \zeta) \) as an entity defined on the Riemann surface \( M \) itself, so this alternative viewpoint generally will not be
used. A meromorphic double differential \( \mu(z, \zeta) \) is symmetric if \( \mu(z, \zeta) = \mu(\zeta, z) \), so that if the point \( z \in U_\alpha \) has the local coordinate \( z_\alpha \) and the point \( \zeta \in V_\beta \) has the local coordinate \( \zeta_\beta \) then \( f_{\alpha \beta}(z_\alpha, \zeta_\beta) = f_{\beta \alpha}(\zeta_\beta, z_\alpha) \) when \( \mu(z, \zeta) \) is written explicitly as in (4.22). In particular if points \( z', z'' \in U_\alpha \) have local coordinates \( z'_\alpha, z''_\alpha \) then \( f_{\alpha \alpha}(z'_\alpha, z''_\alpha) = f_{\alpha \alpha}(z''_\alpha, z'_\alpha) \). Similarly the double differential \( \mu(z, \zeta) \) is skew symmetric if \( \mu(z, \zeta) = -\mu(\zeta, z) \), with the corresponding interpretation. Of course any meromorphic double differential \( \mu(z, \zeta) \) can be written uniquely as the sum \( \mu(z, \zeta) = \frac{1}{2} (\mu(z, \zeta) + \mu(\zeta, z)) + \frac{1}{2} (\mu(z, \zeta) - \mu(\zeta, z)) \) of a symmetric and a skew-symmetric double differential.

The singularities of a meromorphic double differential can be described formally in the same way as the singularities of a meromorphic abelian differential. The sheaf of germs of holomorphic double differentials on \( M \times M \) is a subsheaf of the sheaf of germs of meromorphic double differentials, and the quotient sheaf is the sheaf of double differential principal parts on \( M \times M \); the image of a meromorphic double differential in the quotient sheaf at a point of \( M \times M \) is its principal part at that point. However the quotient sheaf cannot be described as simply as in the case of ordinary meromorphic abelian differentials, since the singularities of meromorphic functions of several variables lie on holomorphic subvarieties of codimension one rather than just on isolated points. There is at least one important case in which a very simple intrinsic description of the singularities is possible, though, that in which the singular locus of the meromorphic double differential \( \mu(z, \zeta) \) is the diagonal subvariety \( D = \{ (z, z) | z \in M \} \subset M \times M \) and the differential has a double pole with zero residue along the diagonal. More explicitly, for a coordinate neighborhood \( U_\alpha \subset M \) in which the local coordinate is denoted by either \( z_\alpha \) or \( \zeta_\alpha \), the restriction of \( \mu(z, \zeta) \) to the product neighborhood \( U_\alpha \times U_\alpha \subset M \times M \) has the form

\[
\mu(z, \zeta) = \left( \frac{1}{(z_\alpha - \zeta_\alpha)^2} + f_\alpha(z_\alpha, \zeta_\alpha) \right) dz_\alpha d\zeta_\alpha
\]

where \( f_\alpha(z_\alpha, \zeta_\alpha) \) is holomorphic in \( U_\alpha \times U_\alpha \). Under a change of the local coordinate \( z_\alpha = h_\alpha(t_\alpha), \zeta_\alpha = h_\alpha(\tau_\alpha) \) in the coordinate neighborhood \( U_\alpha \)

\[
\mu(t, \tau) = \left( \frac{1}{(h_\alpha(t_\alpha) - h_\alpha(\tau_\alpha))^2} + f_\alpha(h_\alpha(t_\alpha), h_\alpha(\tau_\alpha)) \right) h'_\alpha(t_\alpha) h'_\alpha(\tau_\alpha) dt_\alpha d\tau_\alpha
\]

\[
= \left( \frac{f_{1\alpha}(t_\alpha, \tau_\alpha)}{(t_\alpha - \tau_\alpha)^2} + f_{2\alpha}(t_\alpha, \tau_\alpha) \right) dt_\alpha d\tau_\alpha
\]

where

\[
f_{1\alpha}(t_\alpha, \tau_\alpha) = \left( \frac{t_\alpha - \tau_\alpha}{h_\alpha(t_\alpha) - h_\alpha(\tau_\alpha)} \right)^2 h'_\alpha(t_\alpha) h'_\alpha(\tau_\alpha)
\]

and

\[
f_{2\alpha}(t_\alpha, \tau_\alpha) = f_\alpha(h_\alpha(t_\alpha), h_\alpha(\tau_\alpha)) h'_\alpha(t_\alpha) h'_\alpha(\tau_\alpha)
\]
are holomorphic functions in $U_\alpha \times U_\alpha$. Since $\lim_{\tau_\alpha \rightarrow t_\alpha} f_{1\alpha}(t_\alpha, \tau_\alpha) = 1$ it follows that $f_{1\alpha}(t_\alpha, \tau_\alpha) = 1 + (t_\alpha - \tau_\alpha) f_{3\alpha}(t_\alpha, \tau_\alpha)$ for some holomorphic function $f_{3\alpha}(t_\alpha, \tau_\alpha)$ in $U_\alpha \times U_\alpha$. From its definition it is clear that the function $f_{1\alpha}(t_\alpha, \tau_\alpha)$ is symmetric in the variables $t_\alpha, \tau_\alpha$; the function $f_{3\alpha}(t_\alpha, \tau_\alpha)$ hence must be skew-symmetric in these two variables, and consequently $f_{3\alpha}(t_\alpha, \tau_\alpha) = (t_\alpha - \tau_\alpha) f_{4\alpha}(t_\alpha, \tau_\alpha)$ for some holomorphic function $f_{4\alpha}(t_\alpha, \tau_\alpha)$ in $U_\alpha \times U_\alpha$. Thus in terms of the new local coordinates

$$
(4.26) \quad \mu(t, \tau) = \left( \frac{1}{(t_\alpha - \tau_\alpha)^2} + g_{\alpha}(t_\alpha, \tau_\alpha) \right) dt_\alpha d\tau_\alpha
$$

for the holomorphic function $g_{\alpha}(t_\alpha, \tau_\alpha) = f_{4\alpha}(t_\alpha, \tau_\alpha) + f_{2\alpha}(t_\alpha, \tau_\alpha)$. A comparison of (4.25) and (4.26) shows that the principal part of this meromorphic double differential has the same form for any local coordinate system on $M$, so its singularities can be specified completely merely by saying that it has the differential principal part $(z_\alpha - \zeta_\alpha)^{-2} d\zeta_\alpha$ along the diagonal of the manifold $M \times M$.

A meromorphic double differential of the second kind is a meromorphic double differential that is a differential of the second kind in each variable separately. For example, if there is a meromorphic double differential with principal part $(z_\alpha - \zeta_\alpha)^{-2} d\zeta_\alpha$ along the diagonal of the manifold $M \times M$ and no other singularities then it is a meromorphic double differential of the second kind. When a meromorphic double differential of the second kind is written explicitly as in (4.22), the expression $f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ is a well defined meromorphic abelian differential of the second kind in the variable $z_\alpha$ for each fixed point $\zeta_\beta \in V_\beta$, and it has well defined periods $g_{\beta}(\tau; \zeta_\beta) = \int_\tau f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ on any homology class $\tau \in H_1(M)$. If $\tau$ is represented by a closed path that avoids the poles of the differential $f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ then that path also avoids the poles of the differential $f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ for all points $\zeta_\beta$ sufficiently near $\zeta_\beta'$; hence $f_{\alpha\beta}(z_\alpha, \zeta_\beta)$ is continuous in the variable $z_\alpha \in \tau$ and holomorphic in the variable $\zeta_\beta$ for all points in an open neighborhood of the point $\zeta_\beta'$, so the period $g_{\beta}(\tau; \zeta_\beta) = \int_\tau f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ is a holomorphic function of the variable $\zeta_\beta$. This is one point at which the restriction on the polar loci of meromorphic double differentials is crucial. Since $f_{\alpha\beta}(z_\alpha, \zeta_\beta) = \kappa_{\beta\gamma}(\zeta_\beta) f_{\alpha\gamma}(z_\alpha, \zeta_\gamma)$ for $\zeta \in V_\beta \cap V_\gamma$, it follows that $g_{\beta}(\tau; \zeta_\beta) = \kappa_{\beta\gamma}(\zeta_\beta) g_{\gamma}^\prime(\tau; \zeta_\gamma)$ for $\zeta \in V_\beta \cap V_\gamma$ and consequently that

$$
(4.27) \quad \mu'(\tau; \zeta) = g_{\beta}^\prime(\tau; \zeta_\beta) d\zeta_\beta = \left( \int_{z_\alpha \in \tau} f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha \right) d\zeta_\beta = \int_{z \in \tau} \mu(z, \zeta)
$$

is a holomorphic abelian differential on $M$; it is called the first period class of the double differential $\mu(z, \zeta)$. The second period class $\mu''(\tau; z)$ is defined correspondingly by reversing the roles of the two variables, so

$$
(4.28) \quad \mu''(\tau; z) = \int_{\zeta \in \tau} \mu(z, \zeta).
$$
If the double differential $\mu(z, \zeta)$ is symmetric then clearly $\mu'(\tau; z) = \mu''(\tau; z)$; this common value is denoted by $\mu(\tau; z)$ and is called merely the period class of the symmetric double differential $\mu(z, \zeta)$. The first period class is a linear function of the homology class $\tau$ since clearly

\begin{equation}
\mu'(n'\tau' + n''\tau''; z) = n'\mu'(\tau'; z) + n''\mu'(\tau''; z)
\end{equation}

for any integers $n', n'' \in \mathbb{Z}$ and any homology classes $\tau', \tau'' \in H_1(M)$; hence the first period class can be viewed as a homomorphism

\begin{equation}
\mu' \in \text{Hom}(H_1(M), \Gamma(M, \mathcal{O}^{(1,0)})) = \text{Hom}(\Gamma, \Gamma(M, \mathcal{O}^{(1,0)})).
\end{equation}

The corresponding result of course holds for the second period class. Each of these period classes can be integrated again to yield the double period classes that associate to any homology classes $\tau', \tau'' \in H_1(M)$ the values

\begin{equation}
\begin{cases}
\mu'(\tau', \tau'') &= \int_{\zeta \in \tau''} \mu'(\tau'; \zeta) = \int_{\zeta \in \tau''} \int_{\tau \in \tau'} \mu(z, \zeta), \\
\mu''(\tau', \tau'') &= \int_{\tau \in \tau''} \mu''(\tau'; z) = \int_{\tau \in \tau''} \int_{\zeta \in \tau'} \mu(z, \zeta).
\end{cases}
\end{equation}

If the double differential $\mu(z, \zeta)$ is symmetric the two double period classes coincide; the common period class is denoted by $\mu(\tau', \tau'')$ and is called the double period class of the symmetric double differential. Since the double differential is meromorphic the order of the iterated integrals is significant; the difference

\begin{equation}
\int_{\zeta \in \tau''} \int_{\tau \in \tau'} \mu(z, \zeta) - \int_{\tau \in \tau'} \int_{\zeta \in \tau''} \mu(z, \zeta) = \mu'(\tau', \tau'') - \mu''(\tau'', \tau')
\end{equation}

does not necessarily vanish, even for a symmetric double differential $\mu(z, \zeta)$.

In terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on $M$ and $\tau_j \in H_1(M)$ for the homology of $M$, the first period class can be written

\begin{equation}
\mu'(\tau_j; \zeta) = \sum_{k=1}^{g} \lambda_{kj}^{1} \omega_k(\zeta) \quad \text{for} \quad 1 \leq j \leq 2g
\end{equation}

for some complex constants $\lambda_{kj}^{1}$; the coefficients in this expansion form a $g \times 2g$ matrix $\Lambda' = \{\lambda_{kj}^{1}\}$ called the first period matrix of the double differential $\mu(z, \zeta)$ in terms of these bases. The indexing convention is chosen so that this is a $g \times 2g$ matrix, hence has the same shape as the other matrices that have been called period matrices. The second period class can be written correspondingly

\begin{equation}
\mu''(\tau_j; z) = \sum_{k=1}^{g} \lambda_{kj}^{2} \omega_k(z) \quad \text{for} \quad 1 \leq j \leq 2g,
\end{equation}

where the coefficients form the second period matrix $\Lambda'' = \{\lambda_{kj}^{2}\}$ of the double differential $\mu(z, \zeta)$ in terms of these bases. The values of the double period
classes on the basis $\tau_i$ can be viewed as entries in the double period matrices, the $2g \times 2g$ matrices

$$N' = \{\mu' (\tau_i, \tau_j)\} \quad \text{and} \quad N'' = \{\mu'' (\tau_i, \tau_j)\};$$

and since

$$\left\{ \begin{array}{l}
\mu' (\tau_i, \tau_j) = \int_{\zeta \in \tau_j} \sum_{k=1}^{g} \lambda'_{ki} \omega_k (\zeta) = \sum_{k=1}^{g} \lambda'_{ki} \omega_{kj} \\
\mu'' (\tau_i, \tau_j) = \int_{z \in \tau_j} \sum_{k=1}^{g} \lambda''_{ki} \omega_k (z) = \sum_{k=1}^{g} \lambda''_{ki} \omega_{kj}
\end{array} \right.$$  

the double period matrices can be expressed in terms of the first and second period matrices as

$$N' = t' \Lambda' \Omega, \quad N'' = t'' \Lambda'' \Omega.$$  

If the double differential is symmetric $\Lambda' = \Lambda''$, which common value is denoted by $\Lambda$ and is called simply the period matrix of the double differential, and $N' = N''$, which common value is denoted by $N$ and is called the double period matrix of the symmetric double differential; thus for symmetric double differentials the double period matrix is

$$N = t' \Lambda \Omega.$$  

In terms of other bases $\hat{\omega}_i = \sum_{k=1}^{g} a_{ik} \omega_k$ and $\hat{\tau}_j = \sum_{l=1}^{2g} \tau_l q_{lj}$, where $A = \{a_{ij}\} \in \text{Gl}(g, \mathbb{C})$ and $Q = \{q_{lj}\} \in \text{Gl}(2g, \mathbb{Z})$ are arbitrary invertible matrices, the first period class is described by a matrix $\hat{\Lambda}' = \{\hat{\lambda}'_{ij}\}$; and by the linearity property (4.29)

$$\mu' (\hat{\tau}_j; \zeta) = \sum_{i=1}^{g} \hat{\lambda}'_{ij} \hat{\omega}_i (\zeta) = \sum_{i,k=1}^{g} \hat{\lambda}'_{ij} a_{ik} \omega_k (\zeta)$$

$$= \mu' \left( \sum_{l=1}^{2g} \tau_l q_{lj} ; \zeta \right) = \sum_{l=1}^{2g} q_{lj} \mu' (\tau_l ; \zeta) = \sum_{k=1}^{g} \sum_{l=1}^{2g} q_{lj} \lambda'_{ki} \omega_k (\zeta)$$

so that $t' \Lambda \hat{\Lambda}' = \Lambda' Q$ and hence

$$\hat{\Lambda}' = t' \Lambda^{-1} \Lambda' Q.$$  

Thus the first period matrices of the double differential $\mu (z, \zeta)$ in terms of any bases for the space of holomorphic abelian differentials and for the homology of the surface are equivalent period matrices. The second period matrix is described in the corresponding way, so the analogous formula holds for a change of bases for the second period matrix and consequently the second period matrices for any bases also are equivalent period matrices. Since $\hat{\Omega} = A \Omega Q$ it follows from (4.37) that the associated double period matrices are related by

$$N' = t' \hat{\Lambda}' \hat{\Omega} = t' Q N' Q$$

$^2$The equivalence of period matrices is defined in equation (F.1) in Appendix F.1.
and similarly for $\tilde{N}''$. Of course for a symmetric double differential correspondingly
\begin{equation}
\tilde{\Lambda} = 'A^{-1} \Lambda Q \quad \text{and} \quad \tilde{N} = 'QNQ
\end{equation}
under a change of basis for the holomorphic abelian differentials on $M$ described by the matrix $A$ and a change of basis for the homology of $M$ described by the matrix $Q$.

With these general observations about meromorphic double differentials out of the way, the discussion can turn to the special meromorphic double differentials arising as the simplest normalized meromorphic differentials of the second kind on a compact Riemann surface.

**Theorem 4.17** (i) On a compact Riemann surface $M$ of genus $g > 0$ there is a unique symmetric meromorphic double differential of the second kind $\mu_M(z, \zeta)$ with the differential principal part $(z_\alpha - \zeta_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal of the product manifold $M \times M$, such that $\mu_M(z, \zeta)$ is the normalized abelian differential of the second kind with that principal part in each variable separately.

(ii) If $\Omega$ is the period matrix of the Riemann surface $M$ and $P$ is the intersection matrix of that surface in terms of bases $\omega_i \in \Gamma(M, O^{(1,0)})$ and $\tau_j \in H_1(M)$ then the period matrix $\Lambda$ of the symmetric double differential $\mu_M(z, \zeta)$ in terms of these bases is
\begin{equation}
\Lambda = -2\pi G \Omega
\end{equation}
where $G = 'H^{-1}$ for the positive definite Hermitian matrix $H = i \Omega P \Omega^*$; thus the period matrix $\Lambda$ is equivalent to the complex conjugate $\overline{\Omega}$ of the period matrix $\Omega$ of the Riemann surface.

(iii) The double period matrix of the symmetric double differential $\mu_M(z, \zeta)$ in terms of these bases is
\begin{equation}
N = -2\pi \overline{\Omega} 'G \Omega
\end{equation}
so that
\begin{equation}
N = 'N = 2\pi i P^{-1}.
\end{equation}

**Proof:** (i) Select a coordinate covering of the surface $M$ by simply connected coordinate neighborhoods $U_\gamma \subset M$ with local coordinates $\zeta_\gamma$; these neighborhoods and their coordinates will be held fixed throughout the proof. Select a point in one of these coordinate neighborhoods $U_\alpha$ at which the local coordinate $\zeta_\alpha$ takes the particular value $z_\alpha$ and consider the normalized abelian differential of the second kind $\mu_\alpha$ with the differential principal part
\[ p = \frac{d\zeta_\alpha}{(\zeta_\alpha - z_\alpha)^2} \]
at that point and with no other singularities on $M$. This differential principal part is described fully by specifying the coordinate neighborhood $U_\alpha$ and the
value \( z_\alpha \) of the local coordinate \( \zeta_\alpha \) at the singularity; hence the differential \( \mu_\beta \) can be denoted unambiguously by \( \mu_{\alpha, z_\alpha} \). In any coordinate neighborhood \( U_\gamma \) of the covering this differential can be written \( \mu_{\alpha, z_\alpha} = f_{\alpha \gamma}(z_\alpha, \zeta_\gamma) d\zeta_\gamma \), where \( f_{\alpha \gamma}(z_\alpha, \zeta_\gamma) \) is a meromorphic function of the variable \( \zeta_\gamma \in U_\gamma \) with a pole at the point \( z_\alpha \) if that point also lies in the coordinate neighborhood \( U_\alpha \) but with no other poles. In particular in the coordinate neighborhood \( U_\alpha \) itself the function \( f_{\alpha \alpha}(z_\alpha, \zeta_\alpha) \) has a Laurent expansion at the point \( z_\alpha \) beginning

\[
    f_{\alpha \alpha}(z_\alpha, \zeta_\alpha) = \frac{1}{(\zeta_\alpha - z_\alpha)^2} + \cdots.
\]

The associated integral \( u_{\alpha, z_\alpha}(\zeta) = \int_{\zeta_\alpha}^{\zeta} \mu_{\alpha, z_\alpha} \) is a meromorphic function of the variable \( \zeta \in \tilde{M} \) with simple poles at the points of \( \tilde{M} \) having image \( z_\alpha \) under the covering projection \( \pi : \tilde{M} \rightarrow M \); indeed in any connected component of the inverse image \( \pi^{-1}(U_\alpha) \subset \tilde{M} \) and in terms of the local coordinate induced by \( \zeta_\alpha \) under the covering projection \( \pi \), the integral \( u_{\alpha, z_\alpha}(\zeta_\alpha) \) has a Laurent expansion at the point \( z_\alpha \) in terms of the local coordinate \( \zeta_\alpha \) beginning

\[
    u_{\alpha, z_\alpha}(\zeta_\alpha) = -\frac{1}{\zeta_\alpha - z_\alpha} + \cdots.
\]

There is another normalized abelian differential of the second kind \( \mu_{\beta, z_\beta} = f_{\beta \gamma}(z_\beta, \zeta_\gamma) d\zeta_\gamma \) with a principal part of the corresponding form at a point of the coordinate neighborhood \( U_\beta \) at which the local coordinate \( \zeta_\beta \) takes the value \( z_\beta \). It then it follows from equation (4.10) in Theorem 4.8 (iii) that

\[
0 = \sum_{\alpha \in M} \text{res}_\alpha(u_{\alpha, z_\alpha} \mu_{\beta, z_\beta})
\]

\[
= \text{res}_{z_\alpha} \left( -\frac{1}{\zeta_\alpha - z_\alpha} f_{\beta \alpha}(z_\beta, \zeta_\alpha) d\zeta_\alpha \right)
  + \text{res}_{z_\beta}(u_{\alpha, z_\alpha}(\zeta_\beta) \frac{d\zeta_\beta}{(\zeta_\beta - z_\beta)^2})
\]

\[
= -f_{\beta \alpha}(z_\beta, z_\alpha) + \frac{d}{d\zeta_\beta} u_{\alpha, z_\alpha}(\zeta_\beta) \bigg|_{\zeta_\beta = z_\beta} = -f_{\beta \alpha}(z_\beta, z_\alpha) + f_{\alpha \beta}(z_\alpha, z_\beta).
\]

As a consequence of this symmetry the functions \( f_{\alpha \beta}(z_\alpha, \zeta_\beta) \) also are meromorphic functions of the variable \( z_\alpha \in U_\alpha \), so by Rothstein's Theorem\(^3\) these functions are meromorphic functions in the product coordinate neighborhoods \( U_\alpha \times V_\beta \subset M \times M \); and consequently \( \mu(z, \zeta) = f_{\alpha \beta}(z_\alpha, \zeta_\beta) d\zeta_\alpha d\zeta_\beta \) is a symmetric meromorphic double differential of the second kind with the principal part \((\zeta_\alpha - z_\alpha)^{-2} d\zeta_\alpha d\zeta_\alpha\) along the diagonal and no other singularities. Furthermore by construction \( f_{\alpha \beta}(z_\alpha, \zeta_\beta) d\zeta_\beta \) is the normalized abelian differential of the second kind in the variable \( \zeta \) with this principal part at the point \( \zeta_\alpha = z_\alpha \); and by symmetry the same is true in the other variable as well. These properties determine this double differential uniquely.

\(^3\)Rothstein's Theorem is that a function of \( n \) complex variables that is meromorphic in each variable separately is a meromorphic function of all \( n \) variables; the theorem is an extension of Hartog's Theorem from holomorphic function to meromorphic functions, and is discussed on page 395 in Appendix A.1.
(ii) For a fixed point \( z_\alpha \in U_\alpha \) it follows from (i) that the differential form \( \mu_M(z_\alpha, \zeta) \) in the variable \( \zeta \) is the normalized abelian differential of the second kind with the differential principal part \((\zeta_\alpha - z_\alpha)^{-2}d\zeta_\alpha\); therefore by (4.13) in Corollary 4.9 its period on the homology class \( \tau_j \) is

\[
\mu_M(\tau_j; z_\alpha) = -2\pi \sum_{m,n=1}^g g_{mn} \overline{w}_n(z_\alpha) \frac{d\zeta_\alpha}{(\zeta_\alpha - z_\alpha)^2} dz_\alpha = -2\pi \sum_{m,n=1}^g g_{mn} \overline{w}_n w_m' \omega_m(z_\alpha);
\]

and since \( \mu_M(\tau_j, z_\alpha) = \sum_{m=1}^g \lambda_{mj} \omega_m(z_\alpha) \) by (4.33), for the special case of a symmetric double differential for which \( \Lambda' = \Lambda \), it follows that

\[
\lambda_{mj} = -2\pi \sum_{n=1}^g g_{mn} \overline{w}_n,
\]

which in matrix terms is (4.42).

(iii) The double period matrix is expressed in terms of the period matrix \( \Lambda \) as in (4.38), so from (4.42) it follows that

\[
N = '\Lambda \Omega = -2\pi \overline{\Omega}^1 G \Omega
\]

and consequently

\[
N - 'N = -2\pi \left( \overline{\Omega}^1 G \Omega - '\Omega G \overline{\Omega} \right) = -2\pi \left( \overline{\Omega} H^{-1} \Omega - '\Omega H^{-1} \overline{\Omega} \right) = 2\pi i P^{-1}
\]

by the inverse of the Riemann equality in the form of equation (F.39) in Appendix F.4. That concludes the proof.

The double differential \( \mu_M(z, \zeta) \) of the preceding theorem is the normalized double differential of the second kind on the Riemann surface \( M \); it is the analogue for Riemann surfaces of genus \( g > 0 \) of the familiar meromorphic double differential \((z - \zeta)^{-2}dzd\zeta\) on the Riemann sphere \( \mathbb{P}^1 \). In view of (4.44) the double period matrix of the second kind is not symmetric, so the difference in (4.32) is nonzero.

**Corollary 4.18** If \( \tau_j \in H_1(M) \) is a basis for the homology of a compact Riemann surface \( M \) of genus \( g > 0 \) for \( 1 \leq j \leq 2g \) then the 2g periods \( \mu(\tau_j; z) = \int_{\zeta \in \tau_j} \mu_M(z, \zeta) \) of the normalized double differential of the second kind \( \mu_M(z, \zeta) \) span the \( g \)-dimensional space of holomorphic abelian differentials on \( M \).

**Proof:** For bases \( \omega_i \in \Gamma(M, O^{(1,0)}) \) and \( \tau_j \in H_1(M) \), in terms of which the period matrix of the surface is the matrix \( \Omega \), the period matrix of the symmetric double differential \( \mu_M(z, \zeta) \) is \( \Lambda = -2\pi G \Omega \) by (4.42), where \( G \) is nonsingular and \( \text{rank } \Omega = g \); hence \( \text{rank } \Lambda = g \). The periods of the double differential are the differential forms \( \mu(\tau_j; z) = \sum_{k=1}^g \lambda_{kj} \omega_k(z) \) as in (4.34) for the case
of a symmetric double differential; and since rank Λ = g then among these 2g periods there are g linearly independent holomorphic abelian differentials. That suffices for the proof.

If \( \mu_M(z, \zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha d\zeta_\beta \) is the normalized double differential of the second kind on a compact Riemann surface \( M \) of genus \( g > 0 \) then in terms of the local coordinate \( \zeta_\beta \) in a coordinate neighborhood \( U_\beta \subset M \) set

\[
(4.45) \quad \mu^{(\nu)}_M(z; \zeta_\beta) = \left( \frac{\partial^\nu f_{\alpha\beta}(z_\alpha, \zeta_\beta)}{\partial^\nu \zeta_\beta} \right) dz_\alpha
\]

for any integer \( \nu \geq 0 \). It is evident from (4.23) that this is a well defined meromorphic differential form in the variable \( z \) for any fixed value of the local coordinate \( \zeta_\beta \); in particular for \( \nu = 0 \) it is just the abelian differential of the second kind on \( M \) that arises as the restriction of the double differential \( \mu(z, \zeta) \). For \( \nu > 1 \) though this differential form depends not just on the particular point represented by the coordinate \( \zeta_\beta \) but also on the choice of the local coordinate system.

**Corollary 4.19** The differential form \( \mu^{(\nu)}(z; \zeta_\beta) \) in the variable \( z \) on a compact Riemann surface \( M \) of genus \( g > 0 \) is the normalized abelian differential of the second kind with the differential principal part

\[
(4.46) \quad \frac{(\nu + 1)!}{(z_\beta - \zeta_\beta)^{\nu+2}} d\zeta_\beta.
\]

If \( \Omega \) is the period matrix and \( P \) is the intersection matrix of \( M \), in terms of bases \( \omega_i \in \Gamma(M, \mathcal{O}(1,0)) \) and \( \tau_j \in H_1(M) \), the periods of the differential form \( \mu^{(\nu)}(z; \zeta_\beta) \) are

\[
(4.47) \quad \mu^{(\nu)}_M(T; \zeta_\beta) = -2\pi i \sum_{m,n=1}^g g_{mn} w^{(\nu+1)}_m(\zeta_\beta) \omega_n(T)
\]

for any covering translation \( T \in \Gamma \), where \( G = iH^{-1} \) for the positive definite Hermitian matrix \( H = i\Omega P \Omega^T \).

**Proof:** By Theorem 4.17 (i) the differential form \( \mu^{(0)}_M(z; \zeta_\beta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha \) in the variable \( z \) for any fixed value \( \zeta_\beta \in U_\beta \) is the normalized abelian differential of the second kind with the differential principal part \( p_{\zeta_\beta} = (z_\beta - \zeta_\beta)^{-2} d\zeta_\beta \), and by Corollary 4.9 its periods are

\[
(4.48) \quad \mu_M(T; \zeta_\beta) = -2\pi i \sum_{m,n=1}^g g_{mn} \text{res}_{\zeta_\beta}(w_m p_{\zeta_\beta}) \omega_n(T)
\]

\[
= -2\pi i \sum_{m,n=1}^g g_{mn} w_m'(\zeta_\beta) \omega_n(T)
\]
for any covering translation $T \in \Gamma$, where $w'(\zeta_\beta) = dw(\zeta_\beta)/d\zeta_\beta$; thus $\mu_M^{(0)}(z; \zeta_\beta)$ has the desired properties for the case $\nu = 0$. The derivative $\mu_M^{(\nu)}(z; \zeta_\beta)$ then has the differential principal part

$$\frac{\partial^\nu}{\partial \zeta_\beta^\nu} \frac{1}{(z_\beta - \zeta_\beta)^2} = \frac{(\nu + 1)!}{(z_\beta - \zeta_\beta)^{\nu+2}};$$

and since the periods for a covering translation $T \in \Gamma$ are just the integrals from a point $a \in M$ to the image $Ta$ along any path in $M$ that avoids the singularities of the differential form it follows that

$$\mu_M^{(\nu)}(T; \zeta_\beta) = \int_a^{Ta} \frac{\partial^\nu}{\partial \zeta_\beta^\nu} f_{\alpha\beta}(z_\alpha, \zeta_\beta) \frac{d\zeta_\beta}{\partial \zeta_\beta} dz_\alpha = \frac{\partial^\nu}{\partial \zeta_\beta^\nu} \int_a^{Ta} f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha$$

$$= -2\pi \frac{\partial^\nu}{\partial \zeta_\beta^\nu} \sum_{m,n=1}^g g_{mn} w_m'(\zeta_\beta) \omega_n(T)$$

$$= -2\pi \sum_{m,n=1}^g g_{mn} w_m'(\zeta_\beta) \omega_n(T).$$

These periods are the complex conjugates of the periods of the holomorphic abelian differential $-2\pi \sum_{m,n=1}^g g_{mn} w_m'(\zeta_\beta) \omega_n(z)$, so by Theorem 4.8 (i) the differential form $\mu^{(n)}(z; \zeta_\beta)$ is a normalized abelian differential of the second kind. That suffices to conclude the proof.

For any choice of local coordinates $\zeta_\beta$ at its poles, any differential principal part of the second kind can be written as a unique linear combination of the differential principal parts (4.46); and since a normalized differential form of the second kind is determined uniquely by its principal part it follows from the preceding corollary that any normalized abelian differential of the second kind can be written uniquely as a linear combination of the normalized abelian differentials $\mu^{(n)}(z; \zeta_\beta)$. Thus the normalized abelian differentials of the second kind on a compact Riemann surface of genus $g > 0$ are holomorphic functions of their differential principal parts, in the sense that they are holomorphic functions of the local coordinates $\zeta_\beta$ describing the locations of the poles and of the coefficients in the Laurent expansion of the differential principal parts at these poles. Double differentials with the same principal part as the normalized double differential $\mu_M(z, \zeta)$ differ from $\mu_M(z, \zeta)$ by a double differential that is everywhere holomorphic. If $\mu(z, \zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha d\zeta_\beta$ is a holomorphic double differential and $\omega_i \in \Gamma(M, O^{1,0})$ is a basis for the space of holomorphic abelian differentials on $M$ then since $f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha$ is a holomorphic abelian differential in the variable $z$ for any fixed point $\zeta_\beta$ it can be written as the sum $f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha = \sum_{i=1}^g e_{i\beta}(\zeta_\beta) \omega_i(z)$ for some coefficients $e_{i\beta}(\zeta_\beta)$ depending on the point $\zeta_\beta$. If $\tau_j \in H_1(M)$ is a basis for the homology of the surface $M$ and $\omega_{ij} = \int_{\tau_j} \omega_i$ is the period matrix of the surface in terms of these bases then

$$\int_{\tau_j} \mu(z, \zeta) = \left( \int_{\tau_j} f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha \right) d\zeta_\beta = \sum_{j=1}^{2g} \omega_{ij} e_{i\beta}(\zeta_\beta) d\zeta_\beta;$$
the periods of the double differential $\mu(z, \zeta)$ are holomorphic abelian differentials, and since the period matrix $\Omega = \{\omega_{ij}\}$ has rank $g$ it follows that the expressions $e_{i\beta}(\zeta_\beta)d\zeta_\beta$ are holomorphic abelian differentials hence can be written $e_{i\beta}(\zeta_\beta)d\zeta_\beta = \sum_{i,j=1}^{g} e_{ij} \omega_i(z)\omega_j(\zeta)$ for some complex constants $e_{ij}$, and consequently a holomorphic double differential must be of the form

$$f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha d\zeta_\beta = \sum_{i,j=1}^{g} e_{ij} \omega_i(z)\omega_j(\zeta)$$

for some complex constants $e_{ij}$. Thus any meromorphic double differential on $M$ with the same differential principal part as $\mu_M(z, \zeta)$ must be of the form

$$(4.49) \quad \mu_{M,E}(z, \zeta) = \mu_M(z, \zeta) + \sum_{k,l=1}^{g} e_{kl} \omega_k(z)\omega_l(\zeta)$$

for $g \times g$ complex matrix $E = \{e_{ij}\}$. Such a double differential is called the basic double differential of the second kind on the compact Riemann surface $M$ described by the matrix $E$; the symmetric basic double differentials are those described by symmetric matrices $E$.

**Corollary 4.20** If $\Omega = \{\omega_{ij}\}$ is the period matrix of a compact Riemann surface $M$ of genus $g > 0$ in terms of bases $\omega_i$ for the holomorphic abelian differentials on $M$ and $\tau_j$ for the first homology of $M$, the period matrices of the basic double differential $\mu_{M,E}(z, \zeta)$ are

$$(4.50) \quad \Lambda'_E = \Lambda + \Omega E, \quad \Lambda''_E = \Lambda + E \Omega$$

where $\Lambda$ is the period matrix of the normalized double differential of the second kind $\mu_M(z, \zeta)$; correspondingly the double period matrices of the basic double differential $\mu_{M,E}(z, \zeta)$ are

$$(4.51) \quad N'_E = N + \Omega E \Omega, \quad N''_E = N + \Omega' E \Omega$$

where $N$ is the double period matrix of the normalized double differential of the second kind $\mu_M(z, \zeta)$, so

$$(4.52) \quad N'_E - N''_E = N - \Omega N = 2\pi i P^{-1},$$

which is independent of the matrix $E$.

**Proof:** From (4.49) it follows that the first period class of the basic double differential $\mu_{M,E}(z, \zeta)$ is

$$\mu'_{M,E}(\tau_j; \zeta) = \int_{\tau_j} \mu_{M,E}(z, \zeta)$$

$$= \mu'_M(\tau_j; \zeta) + \int_{\tau_j} \sum_{k,l=1}^{g} e_{kl} \omega_k(z)\omega_l(\zeta)$$

$$= \sum_{l=1}^{g} \lambda'_l(\zeta) + \sum_{k,l=1}^{g} e_{kl} \omega_k(\zeta)\omega_l(\zeta),$$
so its first period matrix is $\Lambda'_E = \Lambda' + tE\Omega$; and the second period class correspondingly is

$$\mu''_{M,E}(\tau_j; z) = \sum_{k=1}^{g} \lambda_{kj}'\omega_k(z) + \sum_{k,l=1}^{g} e_{kl}\omega_k(z)\omega_{lj}$$

so its second period matrix is $\Lambda''_E = \Lambda'' + E\Omega$. Since the double differential $\mu_M(z, \zeta)$ is symmetric $\Lambda' = \Lambda'' = \Lambda$, then by (4.37) the double period matrices are $N'_E = \Lambda'_E\Omega = \Lambda + t\Omega\Omega$ and $N''_E = \Lambda''_E\Omega = \Lambda + E\Omega\Omega$. Then by (4.52) then follows from (4.44), which suffices to conclude the proof.

That $N'_E - tN''_E = N - tN = 2\pi iP^{-1}$ reflects the fact that this difference is really determined by the topology of the singular locus of the double differential, rather than by the particular choice of the matrix $E$, although that point will not be pursued in the discussion here. On a marked Riemann surface one of the basic double differentials represents the canonical abelian differentials of the second type with a single double pole.

Theorem 4.21

(i) On a marked Riemann surface $M$ of genus $g > 0$, with the marking described by generators $A_j, B_j \in \Gamma$ of the covering translation group of $M$, there is a unique symmetric meromorphic double differential of the second kind $\hat{\mu}_M(z, \zeta)$ that has the principal part $(z_\alpha - \zeta_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal of the product manifold $M \times M$ and that is the canonical abelian differential of the second kind with that principal part in each variable separately.

(ii) The double differential $\hat{\mu}_M(z, \zeta)$ is the basic symmetric double differential of the second kind $\hat{\mu}_{M,E}(z, \zeta)$ described by the matrix $E = \pi Y^{-1}$, where the period matrix of $M$ in terms of the generators $A_j, B_j \in \Gamma$ describing the marking and of the associated canonical holomorphic abelian differentials $\omega_i$ is $\Omega = (I \ Z)$ for a matrix $Z = X + iY \in \mathbb{H}_g$.

(iii) The period matrix of the double differential $\hat{\mu}_M(z, \zeta)$ is $\hat{\Lambda} = (0 \ 2\pi i I)$ and its period class is determined by

$$\hat{\mu}_M(A_j; z) = 0, \quad \hat{\mu}_M(B_j; z) = 2\pi i \omega_j(z),$$

where $\omega_j \in \Gamma(M, \mathcal{O}(1, 0))$ are the canonical holomorphic abelian differentials on the marked Riemann surface $M$.

(iv) The double period matrix of the double differential $\hat{\mu}_M(z, \zeta)$ is

$$\hat{N} = 2\pi i \begin{pmatrix} 0 & 0 \\ 1 & Z \end{pmatrix},$$

so that $\hat{N} - t\hat{N} = -2\pi i J$ where $J$ is the basic skew-symmetric matrix.

Proof: The period matrix of the marked Riemann surface $M$ in terms of the generators $A_j, B_j \in \Gamma$ describing the marking and of the associated canonical holomorphic abelian differentials $\omega_i$ has the form $\Omega = (I \ Z)$ as in Theorem 3.22, where $Z = X + iY \in \mathbb{H}_g$, the Siegel upper half-space of rank $g$; and
the intersection matrix is the basic skew-symmetric matrix \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \).

Consequently the matrices \( G \) and \( H \) in Theorem 4.17 (ii) are \( H = i \Omega J \Omega = i(\overline{Z} - Z) = 2Y \) and \( G = \frac{1}{2}Y^{-1} \), so the period matrix \( \Lambda \) of the normalized double differential of the second kind \( \mu_M(z, \zeta) \) is \( \Lambda = -2\pi G\Omega = -\pi Y^{-1}\Omega \).

By (4.50) the symmetric basic double differential \( \mu_{M,E}(z, \zeta) \) described by a symmetric matrix \( E \) has the period matrix

\[
\Lambda_E = \Lambda + E\Omega = -\pi Y^{-1}\Omega + E\Omega = (\Lambda'_{E} \quad \Lambda''_{E})
\]

for the \( g \times g \) matrix blocks

\[
\Lambda'_E = E - \pi Y^{-1} \quad \text{and} \quad \Lambda''_E = (E - \pi Y^{-1})X + i(E + \pi Y^{-1})Y.
\]

There is a unique symmetric matrix \( E \) for which \( \Lambda'_E \) vanishes, the matrix \( E = \pi Y^{-1} \); and for this choice of the matrix \( E \) clearly \( \Lambda''_E = 2\pi iI \), so the period matrix is \( \Lambda_E = (0 \quad 2\pi iI) \) for the \( g \times g \) identity matrix \( I \) and by (4.33) the periods of the double differential \( \mu_{M,E}(z, \zeta) \) are as in (4.53). It follows that \( \mu_{M,E}(z, \zeta) \) is the canonical abelian differential of the second kind with the given principal part in each variable separately. Finally by (4.38) the double period matrix of this double differential is \( \tilde{N} = \Lambda\Omega = \frac{i}{4}(0 \quad 2\pi iI)(I \quad Z) \), which is (4.54). That suffices to conclude the proof.

The double differential \( \hat{\mu}_M(z, \zeta) \) is called the canonical double differential of the second kind on the marked Riemann surface \( M \). The canonical double differentials of the second kind for all the markings of a Riemann surface thus are just basic double differentials for various matrices \( E \) so are expressible directly in terms of the normalized double differential and the holomorphic abelian differentials as in (4.49). The basic double differentials in general can be described somewhat more intrinsically as follows.

**Theorem 4.22** If \( M \) is a compact Riemann surface of genus \( g > 1 \) then the only meromorphic double differentials of the second kind with singularities along the diagonal subvariety of \( M \times M \) and nowhere else are constant multiples of the basic double differentials of \( M \).

**Proof:** If \( \mu(z, \zeta) \) is a meromorphic double differential of the second kind with singularities only along the diagonal subvariety of \( M \times M \) then it can be written \( \mu(z, \zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha d\zeta_\beta \) in terms of a covering of the surface \( M \) by coordinate neighborhoods \( U_\alpha \), in which the local coordinate is denoted by either \( z_\alpha \) or \( \zeta_\alpha \); the coefficients \( f_{\alpha\beta}(z_\alpha, \zeta_\beta) \) are meromorphic functions in the coordinate neighborhood \( U_\alpha \times U_\beta \) with singularities only along the diagonal, and in a product neighborhood \( U_\alpha \times U_\alpha \) containing the diagonal these coefficients have the form

\[
f_{\alpha\alpha}(z_\alpha, \zeta_\alpha) = \frac{h_\alpha(z_\alpha, \zeta_\alpha)}{(z_\alpha - \zeta_\alpha)^n} dz_\alpha d\zeta_\alpha
\]
for some integer \( n \geq 2 \) where \( h_\alpha(z_\alpha, \zeta_\alpha) \) is holomorphic in \( U_\alpha \times U_\alpha \) and \( h_\alpha(z_\alpha, z_\alpha) \) is not identically zero. In an intersection \((U_\alpha \times U_\alpha) \cap (U_\beta \times U_\beta)\)

\[
h_\alpha(z_\alpha, \zeta_\alpha) = \left(\frac{z_\alpha - \zeta_\alpha}{z_\beta - \zeta_\beta}\right)^n \left(\frac{dz_\alpha}{dz_\beta}\right)^{-1} \left(\frac{\partial z_\alpha}{\partial z_\beta}\right)^{-1} h_\beta(z_\beta, \zeta_\beta);
\]

and upon taking the limit in this identity as \( \zeta_\beta \) approaches \( z_\beta \) and hence \( \zeta_\alpha \) approaches \( z_\alpha \) it follows that

\[
h_\alpha(z_\alpha, z_\alpha) = \left(\frac{dz_\alpha}{dz_\beta}\right)^{n-2} h_\beta(z_\beta, z_\beta).
\]

Thus the functions \( h_\alpha(z_\alpha, z_\alpha) \) describe a nontrivial holomorphic cross-section of the line bundle \( \kappa^{2-n} \), where \( \kappa \) is the canonical bundle of \( M \), and the characteristic class of this bundle is \( c(\kappa^{2-n}) = (2-n)c(\kappa) = -(n-2)(2g-2) \). If \( n > 2 \) then \( c(\kappa^{2-n}) < 0 \) since \( g > 1 \) and consequently \( h_\alpha(z_\alpha, z_\alpha) = 0 \), which implies that the meromorphic functions \( f_{\alpha \beta}(z_\alpha, \zeta_\beta) \) have poles of order strictly less than \( n \) along the diagonal, in contradiction to the assumption that the singularities are of order \( n \). If \( n = 2 \) then \( \kappa^{2-n} = 1 \) is the identity line bundle so the functions \( h_\alpha(z_\alpha, z_\alpha) \) must be constants, and consequently the differential is a multiple of a basic double differential on \( M \). That suffices to conclude the proof.

For a compact Riemann surface \( M \) of genus \( g = 1 \) the canonical bundle is trivial and the proof of the preceding theorem merely shows that \( h_\alpha(z_\alpha, z_\alpha) \) is a constant. To discuss this case further, suppose that \( M \) is in fact the one-dimensional complex torus \( M = \mathbb{C}/\mathcal{L} \); as will be demonstrated later, every compact Riemann surface of genus \( g = 1 \) actually is a one-dimensional complex torus. Since the canonical bundle of \( M \) is trivial, a meromorphic double differential with singularities only along the diagonal is of the form \( \mu(z, \zeta) = f(z, \zeta)dzd\zeta \) for a meromorphic function \( f(z, \zeta) \) on \( M \times M \) with singularities only along the diagonal; this function can be viewed as a meromorphic function on \( \mathbb{C}^2 \) that is invariant under the lattice subgroup \( \mathcal{L} \) in each variable and that has poles only at points \((z, \zeta) \in \mathbb{C}^2 \) for which \( z - \zeta \in \mathcal{L} \). The function \( g(z, \zeta) = f(z + \zeta, z - \zeta) \) then is a meromorphic function on \( \mathbb{C}^2 \), is invariant under the lattice \( \mathcal{L} \) in each variable, and has poles only at points \((z, \zeta) \) for which \( (z + \zeta) - (z - \zeta) = 2\zeta \in \mathcal{L} \); and \( f(z, \zeta) = g((z + \zeta)/2, (z - \zeta)/2) \). For any fixed point \( \zeta \in \mathbb{C} \) for which \( 2\zeta \notin \mathcal{L} \) the function \( g(z, \zeta) \) is holomorphic in the variable \( z \) in the entire complex plane and is invariant under the lattice \( \mathcal{L} \), so must be constant in \( z \). Consequently \( f(z, \zeta) = g((z - \zeta)/2) \) for a meromorphic function \( g(\zeta) \) in \( \mathbb{C} \) with poles at points \( \zeta \in \frac{1}{2}\mathcal{L} \); and for \( f(z, \zeta) \) to be invariant under the lattice \( \mathcal{L} \) in each variable separately the function \( g(\zeta) \) must be invariant under the lattice \( \frac{1}{2}\mathcal{L} \). Any such function \( g(\zeta) \) thus yields a meromorphic double differential on the compact Riemann surface \( M \) of genus \( g = 1 \); and this double differential is of the second kind whenever the function \( g(\zeta) \) has zero residue at each pole, and is a symmetric double differential whenever the function \( g(\zeta) \) is even. Thus there are symmetric meromorphic double differentials of the second kind with higher order singularities along the diagonal in this case.
There are meromorphic double differentials of the second kind on Riemann surfaces of genus \( g > 1 \) other than the basic double differentials of the second kind; but their singularities lie along holomorphic subvarieties \( V \subset M \times M \) other than the diagonal subvariety. The discussion of such double differentials will be postponed until after the discussion of the existence and properties of such subvarieties of the product manifold in Part II.
Any differential principal part \( p \) on a compact Riemann surface \( M \) can be written as the sum \( p = p_1 + p_2 \) of a differential principal part \( p_1 \) consisting of simple poles at a finite number of points of \( M \) and a differential principal part \( p_2 \) of the second kind on \( M \). By Corollary 4.4 (ii) the differential principal part \( p \) is the principal part of a meromorphic differential form on \( M \) if and only if the sum of the residues at the poles of \( p_1 \) is zero; and in that case \( p_1 \) can be written as a sum of differential principal parts, each of which consists of simple poles at two distinct points of \( M \) with residues that are negatives of one another.

That suggests that the examination of abelian differentials of the third kind on a compact Riemann surface \( M \) can begin by examining a differential principal part \( p(a_+, a_-) \) consisting of a simple pole with residue +1 at the point \( a_+ \), a simple pole with residue −1 at the point \( a_- \), and no other singularities on \( M \).

It follows from Corollary 4.4 (ii) that there is a meromorphic abelian differential \( \nu \) on \( M \) with the differential principal part \( p(a_+, a_-) \); and of course the other meromorphic abelian differentials on \( M \) with this differential principal part differ from \( \nu \) by holomorphic abelian differentials. To define the period class of the differential \( \nu \), choose an oriented simple path \( \delta \subset M \) from the point \( a_- \) to the point \( a_+ \) and a point \( z_- \in \tilde{M} \) such that \( \pi(z_-) = a_- \), where \( \pi : \tilde{M} \longrightarrow M \) is the covering projection from the universal covering space \( \tilde{M} \) to \( M \) with covering translation group \( \Gamma \). There is a unique path \( \tilde{\delta} \in \tilde{M} \) beginning at the point \( z_- \) such that \( \pi(\tilde{\delta}) = \delta \), and this path ends at a point \( z_+ \in \tilde{M} \) for which \( \pi(z_+) = a_+ \).

The inverse image of the path \( \delta \) under the covering projection \( \pi \) is the collection of paths \( \pi^{-1}(\delta) = \Gamma \tilde{\delta} = \bigcup_{T \in \Gamma} T \tilde{\delta} \), where the paths \( T \tilde{\delta} \subset \tilde{M} \) for distinct covering translations \( T \in \Gamma \) are disjoint. The complement \( M \sim \delta \subset M \) is a connected set, since the path \( \delta \) does not intersect itself; and \( \tilde{M} \sim \Gamma \tilde{\delta} \subset \tilde{M} \) is a connected
covering space over \( M \sim \delta \). When \( \nu \) is viewed as a \( \Gamma \)-invariant differential form on \( \tilde{M} \) its integral around any closed path \( \gamma \subset \tilde{M} \sim \Gamma \delta \) is zero. Indeed since \( \tilde{M} \) is simply connected \( \gamma \) is the boundary \( \gamma = \partial \Delta \) of a domain \( \Delta \subset \tilde{M} \), and for any component \( T \delta \subset \Delta \) it is possible to choose a closed path \( \gamma_T \subset \Delta \) that encircles \( T \delta \) once in such a manner that the paths \( \gamma_T \) are disjoint and have disjoint interiors; since \( \nu \) is a closed differential form in the complement of the paths \( T \delta \) it follows from Stokes’s Theorem that \( \int_{\gamma} \nu = \sum_T \int_{\gamma_T} \nu = 0 \), for the integral over \( \gamma_T \) is the sum of the residues of the differential form \( \nu \) at the two poles \( T z_+ \) and \( T z_- \) enclosed by that path hence is zero. Therefore the integral

\[
(5.1) \quad v^\delta(z, a) = \int_a^z \nu
\]

along any path that avoids the sets \( T \delta \) is a well defined holomorphic function of the variables \( z, a \in \tilde{M} \sim \Gamma \delta \) independent of the path of integration; this function is called the integral of the abelian differential of the third kind \( \nu \) with respect to the path \( \delta \), although of course it is really a holomorphic function on the open subset \( \hat{M} \sim \Gamma \delta \subset \hat{M} \) in both variables. This integral clearly satisfies the symmetry condition \( v^\delta(z, a) = -v^\delta(a, z) \), and \( v^\delta(z, z) = 0 \). It is more convenient in many circumstances to view this integral as a function of the first variable only, and to allow it to be modified by an arbitrary additive constant; in that case the simpler notation \( v^\delta(z) \) will be used, with the same cautions as in the cases of abelian integrals of the first and second kinds. For any covering translation \( T \in \Gamma \) the difference

\[
(5.2) \quad v^\delta(Tz) - v^\delta(z) = \nu^\delta(T)
\]

is a constant since \( dv^\delta(z) = \nu(z) \) is invariant under \( T \) and \( \tilde{M} \sim \Gamma \delta \) is connected; the mapping \( v^\delta : T \longrightarrow v^\delta(T) \) is a group homomorphism \( v^\delta \in \text{Hom}(\Gamma, \mathbb{C}) = \text{Hom}(H_1(M), \mathbb{C}) = H^1(\Gamma, \mathbb{C}) \) called the period class of the abelian differential \( \nu \) with respect to the path \( \delta \). The period class clearly is unchanged by adding an arbitrary constant to the function \( v^\delta(z) \), so it depends only on the abelian differential \( \nu \).

**Theorem 5.1** On a compact Riemann surface \( M \) of genus \( g > 0 \) let \( \Omega = \{\omega_{ij}\} \) be the period matrix and \( P = \{p_{jk}\} \) be the intersection matrix of the surface in terms of bases \( \omega_i \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_j \in H_1(M) \).

(i) If \( \delta \subset M \) is a simple path from \( a_- \) to \( a_+ \) the periods \( v^\delta(\tau_j) \) with respect to the path \( \delta \) of an abelian differential of the third kind \( \nu \) with the differential principal part \( p(a_+, a_-) \) satisfy

\[
(5.3) \quad \sum_{j,k=1}^{2g} \omega_{ij} p_{jk} v^\delta(\tau_k) = 2\pi i \int_{\Omega} \omega_i
\]

for \( 1 \leq i \leq g \).

(ii) If \( \delta', \delta'' \subset M \) are two disjoint paths from \( a_- \) to \( a_+ \) the periods \( v^{\delta',\delta''}(\tau_j) \)
with respect to the path $\delta'$ of an abelian differential of the third kind $\nu'$ with the differential principal part $p(a'_+ , a'_-)$ and the periods $\nu'^{\mu'}(\tau_k)$ with respect to the path $\delta''$ of an abelian differential of the third kind $\nu''$ with the differential principal part $p(a''_+, a''_-)$ satisfy

$$
\sum_{j,k=1}^{2g} \nu'^{\delta'}(\tau_j) p_{jk} \nu'^{\delta''}(\tau_k) = 2\pi i \left( \int_{\delta'} \nu' - \int_{\delta''} \nu'' \right).
$$

(iii) If $\delta \subset M$ is a simple path from $a_-$ to $a_+$, the periods $\nu^\delta(\tau_k)$ with respect to the path $\delta$ of an abelian differential $\nu$ of the third kind with the differential principal part $p(a_+, a_-)$ and the periods $\mu(\tau_j)$ of a meromorphic abelian differential $\mu$ of the second kind with poles at points $a_i \notin \delta$ satisfy

$$
\sum_{j,k=1}^{2g} \mu(\tau_j) p_{jk} \nu^\delta(\tau_k) = 2\pi i \int_\delta \mu - 2\pi i \sum_{a_i} \text{res}_{a_i}(\nu^\delta \mu)
$$

where $\nu^\delta(z) = \int_a^z \nu$ is an integral of the differential form $\nu$ on $\tilde{M} \sim \Gamma \tilde{\delta}$.

**Proof:** (i) Let $\nu^\delta(z) = \int_a^z \nu$ be the integral of the abelian differential of the third kind $\nu$ on $\tilde{M} \sim \Gamma \tilde{\delta}$; the periods of $\nu$ thus are given by $\nu^\delta(T) = \nu^\delta(Tz) - \nu^\delta(z)$. Choose a contractible open neighborhood $\Delta \subset M$ of the path $\delta \subset M$ and let $\Delta$ be that component of the inverse image $\pi^{-1}(\Delta) \in \tilde{M}$ for which $\tilde{\delta} \subset \tilde{\Delta}$, where $\pi : \tilde{M} \rightarrow M$ is the covering projection. The complete inverse image $\pi^{-1}(\Delta) \subset \tilde{M}$ consists of disjoint open sets $T\tilde{\Delta}$ for all $T \in \Gamma$, and the set $T\tilde{\Delta}$ contains the component $T\tilde{\delta}$ of the path $\pi^{-1}(\tilde{\delta})$. Choose a $C^\infty$ real-valued function $r$ on $M$ that is identically one on an open neighborhood of $M \sim \Delta$ and is identically zero in an open neighborhood of the path $\delta$ in $\Delta$; this function also will be viewed as a $\Gamma$-invariant function on $\tilde{M}$. In terms of this auxiliary function introduce the smoothed integral

$$
\nu^\delta(z) = \begin{cases} 
\nu^\delta(z) & \text{for } z \in \tilde{M} \sim \Gamma \tilde{\Delta} \\
r(z)\nu^\delta(z) & \text{for } z \in \tilde{\Delta} \\
\nu^\delta(T^{-1}z) + \nu^\delta(T) & \text{for } z \in T\tilde{\Delta}, \ T \neq I.
\end{cases}
$$

Thus $\nu^\delta(z)$ is a $C^\infty$ function on $\tilde{M}$, $\nu^\delta(Tz) = \nu^\delta(z) + \nu^\delta(T)$ for any covering translation $T \in \Gamma$, and $\nu^\delta(z) = \nu^\delta(z)$ whenever $z \notin \Gamma \tilde{\Delta}$. The differential form $\nu^\delta = d \nu^\delta$ then is a $C^\infty$ closed $\Gamma$-invariant differential form on $\tilde{M}$, or equivalently is a $C^\infty$ closed differential form on $M$, that is holomorphic outside the set $\Delta$ and that has the periods $\nu^\delta(T) = \nu^\delta(T)$ for all covering transformations $T \in \Gamma$. If $\phi_k$ is a basis for the first deRham group of $M$ that is dual to the chosen basis for the homology of $M$ then $\nu^\delta \sim \sum_{k=1}^{2g} \nu^\delta(\tau_k) \phi_k(z)$, where $\sim$ denotes cohomologous differential forms; and the abelian differentials of the first kind
can be written correspondingly as \( \omega_i \sim \sum_{j=1}^{2g} \omega_{ij} \phi_j \). Then
\[
\int_M \omega_i \wedge \tilde{\nu}^\delta = \sum_{j,k=1}^{2g} \int_M \omega_{ij} \phi_j \wedge \nu^\delta(\tau_k) \phi_k = \sum_{j,k=1}^{2g} \omega_{ij} p_{jk} \nu^\delta(\tau_k)
\]
where \( p_{jk} = \int_M \phi_j \wedge \phi_k \) are the entries of the intersection matrix \( P \) of the surface \( M \) in terms of these bases. On the other hand the differential form \( \tilde{\nu}^\delta(z) \) is holomorphic outside \( \Delta \), so that \( \omega_i \wedge \tilde{\nu}^\delta = 0 \) there, and consequently by Stokes’s Theorem and the residue theorem it follows that
\[
\int_M \omega_i \wedge \tilde{\nu}^\delta = \int_{\partial \Delta} w_i \tilde{\nu}^\delta = \int_{\partial \Delta} w_i \nu^\delta = 2\pi i \left( w_i(a_+) - w_i(a_-) \right) = 2\pi i \int_M \omega_i
\]
since \( \nu^\delta = \tilde{\nu}^\delta \) on the boundary of the disc \( \Delta \). Combining these two equations yields (5.3).

(ii) For meromorphic abelian differentials of the third kind \( \nu' \) with the differential principal part \( p(a'_+, a'_-) \) and \( \nu'' \) with the differential principal part \( p(a''_+, a''_-) \), and for disjoint paths \( \delta' \) from \( a'_- \) to \( a'_+ \) and \( \delta'' \) from \( a''_+ \) to \( a''_- \), introduce the smoothed functions \( \nu^{\delta'} \) and \( \nu''^{\delta''} \) in disjoint open neighborhoods \( \Delta' \) and \( \Delta'' \) of the paths \( \delta' \) and \( \delta'' \) as in the proof of part (i). Then for the \( C^\infty \) differential forms \( \nu^{\delta'} = d \tilde{\nu}^{\delta'} \) and \( \nu^{\delta''} = d \tilde{\nu}^{\delta''} \) on \( M \) it follows that
\[
\int_M \nu^{\delta'} \wedge \nu^{\delta''} = \sum_{i,j=1}^{2g} \int_M \nu^{\delta'(\tau_i)} \phi_i \wedge \nu^{\delta''(\tau_j)} \phi_j = \sum_{i,j=1}^{2g} \nu^{\delta'(\tau_i)} p_{ij} \nu^{\delta''(\tau_j)}.
\]
Again since both \( \tilde{\nu}^{\delta'} \) and \( \tilde{\nu}^{\delta''} \) are holomorphic outside \( \Delta' \cup \Delta'' \) it follows that \( \nu^{\delta'} \wedge \nu^{\delta''} = 0 \) outside \( \Delta' \cup \Delta'' \), and consequently from Stokes’s Theorem and the residue theorem it follows that
\[
\int_M \nu^{\delta'} \wedge \nu^{\delta''} = \int_{\Delta' \cup \Delta''} \nu^{\delta'} \wedge \nu^{\delta''}
\]
\[
= \int_{\Delta''} d(\nu^{\delta'} \nu^{\delta''}) - \int_{\Delta'} d(\nu^{\delta''} \nu^{\delta'})
\]
\[
= \int_{\partial \Delta''} \nu^{\delta''} \nu' - \int_{\partial \Delta'} \nu^{\delta'} \nu''
\]
\[
= 2\pi i \left( \nu^{\delta'}(a''_+) - \nu^{\delta'}(a''_-) \right) - 2\pi i \left( \nu^{\delta''}(a'_+) - \nu^{\delta''}(a'_-) \right)
\]
\[
= 2\pi i \left( \int_{\tilde{\nu}'} - \int_{\tilde{\nu}''} \right).
\]
Combining these two equations yields (5.4).
Let $f$ be a meromorphic differential of the third kind with poles at points $a_i$ and $\Delta$ be an open neighborhood of the path $\delta$ such that all of these open sets have disjoint closures. Introduce the smoothed integrals $\tilde{v}_\delta(z)$ as in the preceding part of the proof of the present theorem and $\tilde{u}(z)$ as in the proof of Theorem 4.7, where $\tilde{u}(z)$ is modified within the open sets $T\Delta_i$ covering $\Delta_i$ and $\tilde{v}_\delta$ is modified within the open sets $T\Delta$ covering $\Delta$. Since $\tilde{\mu} \sim \sum_{j=1}^{2g} \mu(\tau_j) \phi_j$ and $\tilde{\nu}_\delta \sim \sum_{k=1}^{2g} \nu^\delta(\tau_k) \phi_k$ it follows that

$$\int_M \tilde{\mu} \wedge \tilde{\nu}_\delta = \sum_{j,k=1}^{2g} \int_M \mu(\tau_j) \phi_j \wedge \nu^\delta(\tau_k) \phi_k = \sum_{i,j=1}^{2g} \mu(\tau_j) \nu^\delta(\tau_k).$$

Since $\tilde{\mu} \wedge \tilde{\nu}_\delta = 0$ outside the set $(\cup_i \Delta_i) \cup \Delta$ it further follows from Stokes’s Theorem and the residue theorem that

$$\int_M \tilde{\mu} \wedge \tilde{\nu}_\delta = \int_{\Delta \cup (\cup_i \Delta_i)} \tilde{\mu} \wedge \tilde{\nu}_\delta$$

$$= \int_{\Delta} d(\tilde{\mu} \tilde{\nu}_\delta) - \sum_i \int_{\Delta_i} d(\tilde{\mu} \tilde{\nu}_\delta)$$

$$= \int_{\partial \delta} \tilde{\nu}_\delta \tilde{\nu} - \sum_i \int_{\partial \Delta_i} \mu \tilde{\nu}_\delta$$

$$= 2\pi i \int_\delta \mu - 2\pi i \sum_i \text{res}_{\delta_i}(\nu^\delta \mu).$$

Combining these two equations yields (5.5) and thereby concludes the proof.

Although the integral and the periods of an abelian differential of the third kind $\nu$ with the differential principal part $p(a_+, a_-)$ depend on the choice of a simple path $\delta$ from the point $a_-$ to the point $a_+$, this dependence is rather limited.

**Lemma 5.2** If $\nu$ is a meromorphic abelian differential of the third kind on a compact Riemann surface $M$ of genus $g > 0$ and has the differential principal part $p(a_+, a_-)$, and if $\delta'$ and $\delta''$ are any two simple paths on $M$ from the point $a_-$ to the point $a_+$, then for any points $z, a \in M \sim \pi^{-1}(\delta' \cup \delta'')$ the integrals $\nu^\delta(z, a)$ and $\nu^{\delta''}(z, a)$ of $\nu$ with respect to these two paths satisfy

$$\nu^\delta(z, a) - \nu^{\delta''}(z, a) = 2\pi i n_{\delta', \delta''}(z, a)$$

where $n_{\delta', \delta''}(z, a) \in \mathbb{Z}$;

and the period classes of the differential $\nu$ for these two paths satisfy

$$\nu^\delta(T) - \nu^{\delta''}(T) = 2\pi i n_{\delta', \delta''}(T)$$

where $n_{\delta', \delta''}(T) \in \mathbb{Z}$.

**Proof:** For any points $z, a \in \tilde{M} \sim \pi^{-1}(\delta' \cup \delta'') \subset \tilde{M}$ the complex number $\nu^\delta(z, a) - \nu^{\delta''}(z, a)$ is the difference between the integrals of the meromorphic
abelian differential $\nu$ along two paths from the point $a$ to the point $z$ in the complement $\tilde{M} \sim \pi^{-1}(a_+ \cup a_-)$; thus it is the integral of $\nu$ along a closed path in $\tilde{M} \sim \pi^{-1}(a_+ \cup a_-)$ and consequently it is equal to the sum of the residues of $\nu$ at the poles enclosed by that path. Since the residue of $\nu$ at each pole is $\pm 1$ the integral is $2\pi i$ times an integer. Furthermore for any covering translation $T \in \Gamma$

$$\nu^\delta(T) - \nu^{\delta''}(T) = (v^\delta(Tz, a) - v^\delta(z, a)) - (v^{\delta''}(Tz, a) - v^{\delta''}(z, a))$$

$$= 2\pi i(n_{\delta', \delta''}(Tz, a) - n_{\delta', \delta''}(z, a))$$

so this too is an integer, and that suffices for the proof.

**Theorem 5.3** Let $\nu$ be a meromorphic abelian differential of the third kind on a compact Riemann surface $M$ of genus $g > 0$, with the differential principal part $p(a_+, a_-)$.

(i) For any choice of a simple path $\delta$ on $M$ from $a_-$ to $a_+$ the holomorphic function

$$q_\nu(z, a) = \exp v^\delta(z, a)$$

in the variables $z, a \in \tilde{M} \sim \partial$ extends to a meromorphic function $q_\nu(z, a)$ of the variables $z, a \in \tilde{M}$ that is independent of the choice of the path $\delta$. The extended function is multiplicatively skew-symmetric, in the sense that $q_\nu(z, a) = q_\nu(a, z)^{-1}$; and as a function of the variable $z \in \tilde{M}$ for a fixed point $a \in \tilde{M}$ it has simple zeros at the points $\pi^{-1}(a_+)$, simple poles at the points $\pi^{-1}(a_-)$ and no other zeros or poles on $\tilde{M}$.

(ii) For any choice of a simple path $\delta$ on $M$ from $a_-$ to $a_+$ and for any covering translation $T \in \Gamma$ the exponential

$$e_\nu(T) = \exp \nu^\delta(T)$$

is independent of the choice of the path $\delta$; and the mapping $T \mapsto e_\nu(T)$ is a group homomorphism $e_\nu \in \text{Hom}(\Gamma, \mathbb{C}^*)$.

(iii) The function $q_\nu(z, a)$ as a function of the variable $z \in \tilde{M}$ for a fixed point $a \in \tilde{M}$ is a meromorphic relatively automorphic function for the flat factor of automorphy defined by the homomorphism $e_\nu$.

**Proof:** (i) It is evident from the definition (5.1) that the function $q_\nu(z, a) = \exp v^\delta(z, a)$ can be extended holomorphically across the interior points of the paths $\Gamma \partial$ in both variables; and as a consequence of the preceding lemma the extension is a single valued nowhere vanishing holomorphic function of the variables $z, a \in \tilde{M}$ in the complement of the points of $M$ covering $a_+ \cup a_-$. It follows immediately from (5.1) that $v^\delta(z, a) = -v^\delta(a, z)$ and hence that $q_\nu(z, a)$ is multiplicatively skew-symmetric. Since the differential $\nu$ has a simple pole at $a_+$ with residue $+1$ its integral $v^\delta(z, a)$ as a function of the variable $z$ has a
logarithmic singularity at any point $z_+ \in \widetilde{M}$ for which $\pi(z_+) = a_+$; in an open neighborhood of such a point $z_+$ the integral can be written

$$\nu^\delta(z, a) = \log(z - z_+) + h(z)$$

for a holomorphic function $h(z)$, and consequently $q_{\nu}(z, a) = (z - z_+) \exp h(z)$ so this function has a simple zero at $z_+$ but is holomorphic and nonvanishing in this neighborhood otherwise. Correspondingly the differential $\nu$ has a simple pole at $z_-$ with residue $-1$, so its integral $\nu^\delta(z, a)$ has a logarithmic singularity at any point $z_- \in \widetilde{M}$ for which $\pi(z_-) = a_-$; in an open neighborhood of such a point $z_-$ the integral can be written

$$\nu^\delta(z, a) = -\log(z - z_-) + h(z)$$

for a holomorphic function $h(z)$, and consequently $q_{\nu}(z, a) = (z - z_-)^{-1} \exp h(z)$ so this function has a simple pole at $z_-$. Altogether then $q_{\nu}(z, a)$ is a well-defined meromorphic function of the variable $z \in \widetilde{M}$; and from the multiplicative skew-symmetry property already demonstrated it follows that $q_{\nu}(z, a)$ also is meromorphic in the variable $a \in \widetilde{M}$, so by Rothstein’s Theorem$^1$ it is a meromorphic function of the two variables $(z, a) \in \widetilde{M} \times \widetilde{M}$.

(ii) The exponential $e_{\nu}(T) = \exp \nu^\delta(T)$ of the additive group homomorphism $\nu^\delta \in \text{Hom}(\Gamma, \mathbb{C})$ is a multiplicative group homomorphism $e_{\nu} \in \text{Hom}(\Gamma, \mathbb{C}^*)$; and it follows from the preceding lemma that this homomorphism is independent of the choice of the path $\delta$ from $a_-$ to $a_+$.

(iii) Finally $q_{\nu}(Tz, a)/q_{\nu}(z, a) = \exp \left( \nu^\delta(Tz, a) - \nu^\delta(z, a) \right) = \exp \nu^\delta(T) = e_{\nu}(T)$ for any covering translation $T \in \Gamma$; that is just the condition that $q_{\nu}(z, a)$ as a function of the variable $z \in \widetilde{M}$ is a relatively automorphic function for the factor of automorphy $e_{\nu} \in \text{Hom}(\Gamma, \mathbb{C}^*)$, and that suffices to conclude the proof.

An abelian differential of the third kind with the differential principal part $p(a_+, a_-)$ is determined only up to the addition of arbitrary holomorphic abelian differentials. It is possible to normalize the abelian differentials of the third kind in terms of their period classes so that there is a unique abelian differential with that differential principal part; but the normalization depends on the choice of the path $\delta$ from $a_-$ to $a_+$ since it involves the periods $\nu^\delta(T)$ and not just their exponentials $e_{\nu}(T) = \exp \nu^\delta(T)$.

**Theorem 5.4**

(i) For any simple path $\delta$ from a point $a_-$ to a point $a_+$ on a compact Riemann surface $M$ of genus $g > 0$ there are a unique meromorphic abelian differential $\nu_\delta$ and a unique holomorphic abelian differential $\omega_\delta$ such that $\nu_\delta$ has the differential principal part $p(a_+, a_-)$ and that its period class $\nu^\delta_\delta$ with respect to the path $\delta$ is equal to the period class of the complex conjugate differential $\overline{\omega_\delta}$.

(ii) The holomorphic abelian differential $\omega_\delta$ is characterized by the condition that

$$\int_M \omega \wedge \overline{\omega_\delta} = 2\pi i \int_\delta \omega$$

$^1$For Rothstein’s Theorem see Appendix A.1.
for all holomorphic abelian differentials $\omega$.

(iii) If $\delta'$ is a simple path from a point $a_-'$ to a point $a_+$ on $M$ and $\delta''$ is a simple path from a point $a_-$ to a point $a_+$ on $M$, where the paths $\delta'$ and $\delta''$ are disjoint, then the meromorphic abelian differentials $\nu_{\delta'}$ and $\nu_{\delta''}$ satisfy

\[ \int_{\delta'} \nu_{\delta'} = \int_{\delta''} \nu_{\delta''}. \]

(iv) The abelian differential $\nu_\delta$ and the normalized abelian differential of the second kind $\mu_\delta$ with poles at points $a_i \notin \delta$ satisfy

\[ \int_\delta \mu_\delta = \sum_{a_i} \text{res}_{a_i} (v^\delta_3 p), \]

where $v^\delta_3(z) = \int_z^a \nu_\delta$ is the integral of the meromorphic abelian differential $\nu_\delta$ on $\bar{M} \sim \Gamma\delta$.

Proof: (i) Let $\omega_j \in \Gamma(M, \mathcal{O}^{(1,0)})$ be a basis for the space of holomorphic abelian differentials on the surface $M$ and $\tau_j \in H_1(M)$ be a basis for the homology of the surface $M$, and in terms of these bases let $\Omega = \{\omega_j\}$ be the period matrix and $P = \{p_{ij}\}$ be the intersection matrix of $M$. As in (F.9) in Appendix F.1 there is the direct sum decomposition $\mathbb{C}^{2g} = \mathfrak{H} \mathbb{C}^g \oplus \mathfrak{H}^* \mathbb{C}^g$, in which the subspace $\mathfrak{H} \mathbb{C}^g \subset \mathbb{C}^{2g}$ consists of the period vectors $\{\omega(\tau_j)\}$ of the holomorphic abelian differentials $\omega$ on the basis $\tau_j$ and the subspace $\mathfrak{H}^* \mathbb{C}^g \subset \mathbb{C}^{2g}$ consists of the period vectors $\{\overline{\omega(\tau_j)}\}$ of the complex conjugates $\overline{\omega}$ of the holomorphic abelian differentials $\omega$ on the basis $\tau_j$. If $\{\nu^\delta_j(\tau_j)\} \in \mathbb{C}^{2g}$ is the period vector of an abelian differential $\nu$ with the differential principal part $p(a_+, a_-)$ there is a unique holomorphic abelian differential $\omega$ such that the period vector of the sum $\nu_\delta = \nu + \omega$ is contained in the linear subspace $\mathfrak{H} \mathbb{C}^g \subset \mathbb{C}^{2g}$, hence such that period class $\nu^\delta_3 \in \text{Hom}(\Gamma, \mathbb{C})$ of the differential $\nu_\delta$ is the same as the period class of the complex conjugate of some holomorphic abelian differential $\overline{\omega}$. 

(ii) If $\phi_j$ are closed real differential forms of a basis for the first deRham group of $M$ dual to the basis $\tau_j$, then from the homologies $\omega_i \sim \sum_{j=1}^g \omega_{ij} \phi_j$ and $\omega_\delta \sim \sum_{j=1}^g \omega_{3j} (\tau_k) \phi_k$ it follows that

\[ \int_M \omega_i \wedge \overline{\omega_\delta} = \int_M \sum_{j,k=1}^g \omega_{ij} \phi_j \wedge \overline{\omega_3(\tau_k)} \phi_k = \sum_{j,k=1}^g \omega_{ij} \rho_{jk} \overline{\omega_3(\tau_k)}; \]

and since $\overline{\omega(\tau_k)} = v^\delta_3(\tau_k)$ as in (i) it follows from (5.3) in Theorem 5.1 that

\[ \sum_{j,k=1}^g \omega_{ij} \rho_{jk} \overline{\omega_3(\tau_k)} = \sum_{j,k=1}^g \omega_{ij} \rho_{jk} v^\delta_3(\tau_k) = 2\pi i \int_\delta \omega_i. \]

Combining the two preceding equations shows that (5.10) holds for the basis $\omega_i$, and consequently it holds for all holomorphic abelian differentials $\omega$. 

(iii) If \( \nu_{0'} \) is the meromorphic abelian differential of the third kind with the differential principal part \( p(a'_+, a'_-) \) and \( \nu_{0''} \) is the meromorphic abelian differential of the third kind with the differential principal part \( p(a''_+, a''_-) \) as in (i), then for disjoint paths \( \delta' \) from \( a'_- \) to \( a'_+ \) and \( \delta'' \) from \( a''_- \) to \( a''_+ \), the associated holomorphic differentials \( \omega_{\delta'} \) and \( \omega_{\delta''} \) satisfy \( \omega_{\delta'} \wedge \omega_{\delta''} = 0 \), since the product is a differential form of type (2, 0) on the Riemann surface \( M \). From the homologies

\[
0 = \int_M \omega_{\delta'} \wedge \omega_{\delta''} = \int_M \sum_{j,k=1}^g \omega_{\delta'}(\tau_j)\phi_j \wedge \omega_{\delta''}(\tau_k)\phi_k = \sum_{j,k=1}^g \omega_{\delta'}(\tau_j)p_{jk}\omega_{\delta''}(\tau_k).
\]

Since \( \omega_{\delta_k}(\tau_j) = \nu^j_\delta(\tau_j) \) by (i) it follows from (5.4) in Theorem 5.1 that

\[
\sum_{j,k=1}^g \omega_{\delta'}(\tau_j)p_{jk}\nu^j_{\delta''}(\tau_k) = \sum_{j,k=1}^g \nu^j_{\delta'}(\tau_j)p_{jk}\nu^j_{\delta''}(\tau_k) = 2\pi i \left( \int_{\delta'} \nu^j_{\delta'} - \int_{\delta''} \nu^j_{\delta''} \right).
\]

Combining these two equations shows that (5.11) holds.

(iv) The holomorphic abelian differentials \( \omega_p \) and \( \omega_\delta \) with period classes conjugate to the period classes of the meromorphic abelian differentials \( \mu_p \) and \( \nu_\delta \) satisfy \( \omega_p \wedge \omega_\delta = 0 \), since the product is a differential form of type (2, 0) on the Riemann surface \( M \); so from the homologies \( \omega_p \sim \sum_{j=1}^g \mu_p(\tau_j)\phi_j \) and \( \omega_\delta \sim \sum_{k=1}^g \nu^j_\delta(\tau_k)\phi_k \) it follows that

\[
0 = \int_M \omega_\delta \wedge \omega_p = \int_M \sum_{j,k=1}^g \mu_p(\tau_j)\phi_j \wedge \nu^j_\delta(\tau_k)\phi_k = \sum_{j,k=1}^g \mu_p(\tau_j)p_{jk}\nu^j_\delta(\tau_k).
\]

Since \( \omega_p(\tau_j) = \nu_p(\tau_j) \) and \( \omega_\delta(\tau_k) = \nu^j_\delta(\tau_k) \) it follows from (5.5) in Theorem 5.1 that

\[
\sum_{j,k=1}^g \mu_p(\tau_j)p_{jk}\nu^j_\delta(\tau_k) = 2\pi i \int_\delta \mu - 2\pi i \sum_{a_i} \text{res}_{a_i}(\nu^j_\delta \mu).
\]

Combining these two equations yields (5.12), and that suffices to conclude the proof.

The meromorphic abelian differential \( \nu_\delta \) of the preceding theorem with the differential principal part \( p(a_+, a_-) \) is called the \textit{normalized abelian differential of the third kind} with respect to the path \( \delta \), and the holomorphic abelian differential \( \omega_\delta \) is called the \textit{associated holomorphic abelian differential}; both are determined uniquely by the path \( \delta \) from the pole \( a_- \) to the pole \( a_+ \), and their period classes can be written explicitly in terms of that path.

**Corollary 5.5** On a compact Riemann surface \( M \) of genus \( g > 0 \) let \( \Omega = \{ \omega_{ij} \} \) be the period matrix and \( P = \{ p_{ij} \} \) be the intersection matrix in terms of bases \( \omega_i \in \Gamma(M, \mathcal{O}^{1,0}) \) and \( \tau_j \in H_1(M) \). For a differential principal part \( p(a_+, a_-) \)
and any simple path \( \delta \) from \( a_- \) to \( a_+ \) the periods of the normalized abelian differential of the third kind \( \nu_\delta \) with respect to the path \( \delta \) are

\[
(5.14) \quad \nu_\delta^j(T) = -2\pi i \sum_{m,n=1}^g g_{mn} \omega_n(T) \left( \int_\delta \omega_m \right)
\]

for any covering translation \( T \in \Gamma \), where \( G = \{ g_{ij} \} = iH^{-1} \) for the positive definite Hermitian matrix \( H = i\Omega P \bar{\Omega} \).

**Proof:** The associated holomorphic abelian differential \( \omega_\delta \) can be written as the sum

\[
\omega_\delta = \sum_{j=1}^g c_j \omega_j
\]

for some complex constants \( c_j \), so its periods are \( \omega_\delta(\tau_k) = \sum_{j=1}^g c_j \omega_j(\tau_k) \). Substituting this into (5.13) yields the identity

\[
2\pi i \int_\delta \omega_m = \sum_{j,k=1}^g \omega_{mj} p_{jk} \omega_\delta(\tau_k) = \sum_{j,k,l=1}^g \omega_{mj} p_{jk} \bar{\omega}_{lk} \bar{\tau}_l = -i \sum_{l=1}^g h_{ml} \bar{\tau}_l
\]

where \( h_{ml} \) are the entries in the \( g \times g \) matrix \( H = i\Omega P \bar{\Omega} \). The matrix \( H \) is positive definite Hermitian by Riemann’s inequality, Theorem 3.20 (ii), so \( G = iH^{-1} \) exists; and if \( G = \{ g_{mn} \} \) then upon multiplying the preceding equation by \( g_{mn} \) and summing over \( m \) it follows that

\[
c_n = -2\pi \sum_{m=1}^g g_{mn} \int_\delta \omega_m
\]

hence that

\[
(5.15) \quad \nu_\delta^j(\tau_j) = \bar{\omega}_{n \tau_j} = \sum_{n=1}^g c_n \omega_{nj} = -2\pi \sum_{m,n=1}^g g_{mn} \omega_{nj} \int_\delta \omega_m.
\]

If the covering translation \( T \in \Gamma \) corresponds to a homology class \( \tau \in H_1(M) \) and \( \tau \sim \sum_{j=1}^g n_j \tau_j \) for some integers \( n_j \) then \( \nu_\delta^j(T) = \nu_\delta^j(\tau) = \sum_{j=1}^g n_j \int_\delta \omega_\delta^j(\tau_j) \) and \( \bar{\omega}_n(T) = \bar{\omega}_n(\tau) = \sum_{j=1}^g n_j \omega_n(\tau_j) = \sum_{j=1}^g n_j \omega_{nj} \); multiplying both sides of (5.15) by \( n_j \) and summing over \( j = 1, \ldots, g \) yields (5.14) and thereby concludes the proof.

Although the explicit formula (5.14) depends on the choice of bases \( \omega_i \) for the holomorphic abelian differentials on \( M \) and \( \tau_j \) for the homology of \( M \), it is clear that the value \( \nu_\delta^j(T) \) is independent of these choices. It may be comforting, just as in the case of the corresponding result for meromorphic abelian differentials of the second kind discussed on page 101, to prove that directly; and for that purpose it is convenient to rewrite (5.14). Choose a point \( z_- \in M \) such that \( \pi(z_-) = a_- \), where \( \pi: \bar{M} \rightarrow M \) is the covering projection. A path \( \delta \) from \( a_- \) to \( a_+ \) in \( M \) has a unique lifting to a path in \( \bar{\delta} \subset \bar{M} \) beginning at the point \( z_- \), and the lifting is a simple path that ends at a point \( z_+ \in \bar{M} \) for which

\[
\pi(z_+) = a_+.
\]

If \( w_m(z, a) = \int_a^z \omega_m \) are the integrals of the differential forms \( \omega_m \) then

\[
\int_\delta \omega_m = \int_\delta dw_m = w_m(z_+, z_-);
\]

and

\[
\nu_\delta^j(T) = -2\pi i \sum_{m,n=1}^g g_{mn} \omega_n(T) \left( \int_\delta \omega_m \right)
\]
hence (5.14) can be written

\[ \nu_\delta(T) = -2\pi \sum_{m,n=1}^{g} w_{m}(z_+,z_-) g_{mn} \omega_n(T) \]  

for any covering translation \( T \in \Gamma \). To rewrite this formula in matrix terms, introduce the column vector \( \tilde{w}(z,a) = \{w_j(z,a)\} \) consisting of the integrals \( w_j(z,a) = \int_a^z \omega_j \) of the homomorphic abelian differentials \( \omega_j \) and the homomorphism \( \omega \in \text{Hom}(\Gamma, \mathbb{C}^g) \) where \( \omega(T) = \{\omega_j(T)\} \in \mathbb{C}^g \) is the column vector of period classes of the homomorphic abelian differentials. In these terms (5.16) takes the form

\[ \nu_\delta(T) = -2\pi \tilde{w}(z_+,z_-) G \omega(T) \]  

for all \( T \in \Gamma \). A change of basis for the homomorphic abelian differentials on \( M \) has the effect of replacing the vector \( \tilde{w}(z,a) \) by \( A \tilde{w}(z,a) \), the vector \( \omega(T) \) by \( A \omega(T) \), and the form matrix \( G \) by \( t^{-1} G A^{-1} G^{-1} \) as in equation (F.41) in Appendix F.4; this change clearly leaves (5.17) unchanged.

Since the associated homomorphic abelian differential \( \omega_\delta \) has periods that are the complex conjugate of the periods \( \nu_\delta(T) \) of the normalized abelian differential of the third kind it follows from (5.16) that

\[ \omega_\delta(z) = -2\pi \sum_{m,n=1}^{g} w_{m}(z_+,z_-) g_{mn} \omega_n(z); \]

thus \( \omega_\delta \) is determined just by the points \( z_+ \) and \( z_- \). Equivalently the differential \( \omega_\delta \) depends only on the homotopy type of the path \( \delta \); for any two homotopic paths in \( M \) from \( a_- \) to \( a_+ \) when lifted to paths in \( \tilde{M} \) beginning at the point \( z_- \) have the same end point, and since \( \tilde{M} \) is simply connected conversely any two paths in \( \tilde{M} \) from \( z_- \) to \( z_+ \) are homotopic so their projections under the covering projection \( \pi : \tilde{M} \rightarrow M \) are homotopic paths from \( a_- \) to \( a_+ \) in \( M \). To make the dependence on the points \( z_+ \) and \( z_- \) quite explicit, set

\[ \omega_\delta(z) = \omega_{z_+,z_-}(z). \]

As for the dependence on the choice of the initial point \( z_- \) for which \( \pi(z_-) = a_- \), it is evident from (5.18) that

\[ \omega_{Tz_+,Tz_-}(z) = \omega_{z_+,z_-}(z) \]  

for any \( T \in \Gamma \).

Since the normalized abelian differential of the third kind \( \nu_\delta(z) \) is determined uniquely by its principal part \( p(a_+,a_-) \) and its associated homomorphic abelian differential \( \omega_\delta(z) \), by Theorem 5.4 (i), it too is determined uniquely by the points \( z_+ \) and \( z_- \) and consequently can be denoted unambiguously by

\[ \nu_\delta(z) = \nu_{z_+,z_-}(z); \]
and of course (5.20) holds for this abelian differential as well. The integral
\[(5.22) \quad v_δ^{z_+, z_-}(z, a) = \int_a^z \nu_{z_+, z_-} \]
is defined as a holomorphic function on the complement \(\widetilde{M} \sim \Gamma \delta\), so in that sense still depends on the choice of the path \(\delta\); but by Theorem 5.3 its exponential is a meromorphic function
\[(5.23) \quad q(z, a; z_+, z_-) = q_{z_+, z_-}(z, a) = \exp v_δ^{z_+, z_-}(z, a) = \exp \int_a^z \nu_{z_+, z_-}\]
of the variables \(z, a \in \widetilde{M}\), which is called the cross-ratio function of the Riemann surface \(M\) or sometimes the intrinsic cross-ratio function of the Riemann surface \(M\) to be more specific. The group homomorphism or flat factor of automorphy associated to the abelian differential \(\nu_δ(z) = \nu_{z_+, z_-}(z)\) as in Theorem 5.3 (ii) is denoted correspondingly by \(e_δ^{z_+, z_-}(T)\). If \(\Omega\) is the period matrix and \(P\) is the intersection matrix of \(M\) in terms of bases \(\omega_i \in \Gamma(M, \mathcal{O}^{(1, 0)})\) and \(\tau_j \in H_1(M)\) then by (5.16)
\[(5.24) \quad e_{z_+, z_-}(T) = \exp -2\pi \sum_{m,n=1}^g w_m(z_+, z_-) g_{mn} \omega_n(T) \]
for any covering translation \(T \in \Gamma\). This flat factor of automorphy can be described alternatively as
\[(5.25) \quad e_{z_+, z_-}(T) = \rho_t(z_+, z_-)(T)\]
in terms of the canonical parametrization of flat factors of automorphy (3.26) associated to the basis \(\tau_j \in H_1(M)\), where \(t(z_+, z_-) = \{t_j(z_+, z_-)\} \in \mathbb{C}^{2g}\) is the complex vector for which \(e_{z_+, z_-}(\tau_j) = \exp 2\pi it_j(z_+, z_-)\); in view of (5.24) this is the vector with components
\[(5.26) \quad t_j(z_+, z_-) = i \sum_{m,n=1}^g w_m(z_+, z_-) g_{mn} \overline{w_{nj}}\]
for \(1 \leq j \leq 2g\), or in matrix terms, when all vectors are viewed as column vectors, is the vector
\[(5.27) \quad t(z_+, z_-) = i \overline{\Omega}^T G \overline{\hat{w}}(z_+, z_-).\]
Again although the preceding explicit formulas involve the choice of bases for the holomorphic abelian differentials on \(M\) and the homology of \(M\), the factor of automorphy \(e_{z_+, z_-}(T) = \rho_t(z_+, z_-)(T)\) is independent of these choices. The basic properties of the cross-ratio function can be summarized as follows.
Theorem 5.6 If $M$ is a compact Riemann surface $M$ of genus $g > 0$ with the universal covering space $\tilde{M}$ and the covering translation group $\Gamma$, the cross-ratio function of $M$ is a meromorphic function $q(z_1, z_2; z_3, z_4)$ on the complex manifold $\tilde{M} \times \tilde{M} \times \tilde{M} \times \tilde{M}$ that is characterized uniquely by the following properties:

(i) The function $q(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$, simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, but is otherwise holomorphic and nowhere vanishing; and

(ii) The function $q(z_1, z_2; z_3, z_4)$ has the symmetries

\[
q(z_1, z_2; z_3, z_4) = q(z_3, z_4; z_1, z_2) = q(z_2, z_1; z_3, z_4) = q(z_1, z_2; z_3, z_4)^{-1} = q(z_1, z_2; z_4, z_3)^{-1}.
\]

(iii) The function $q(z_1, z_2; z_3, z_4)$ as a function of the variable $z_1 \in \tilde{M}$ for any fixed points $z_2, z_3, z_4 \in \tilde{M}$ is a meromorphic relatively automorphic function for the canonically parametrized factor of automorphy $\rho_{t(z_3, z_4)}(T)$ described by the vector $t(z_3, z_4) \in \mathbb{C}^{2g}$ that for any bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ has the explicit form

\[
t(z_3, z_4) = i \frac{1}{\Omega} G \bar{w}(z_3, z_4)
\]

in which $\Omega$ is the period matrix, $P$ is the intersection matrix, and $G = \Omega^{-1}$ for the positive definite symmetric matrix $H = i \Omega P \frac{1}{\Omega}$ in terms of these bases.

Proof: Whenever $z_3, z_4$ are points of $\tilde{M}$ that are not equivalent under $\Gamma$ it follows from Theorem 5.3 that the cross-ratio function $q(z_1, z_2; z_3, z_4) = q(z_3, z_4; z_1, z_2)$ is a well defined meromorphic function of the variable $z_1 \in \tilde{M}$ that is relatively automorphic for the factor of automorphy $e_{z_3, z_4} = \rho_{t(z_3, z_4)}$ and that has simple zeros at the points $\Gamma z_3$, simple poles at the points $\Gamma z_4$, and no other zeros or poles on $\tilde{M}$. It is clear from the definition (5.23) that $q(z_1, z_2; z_3, z_4) = q(z_2, z_1; z_3, z_4)^{-1}$, from which the corresponding analyticity properties as a function of the variable $z_2 \in \tilde{M}$ follow immediately; and it is also clear from the definition that $q(z_1, z_1; z_3, z_4) = 1$. For two normalized abelian differentials $\nu_{z_1, z_2}$ and $\nu_{z_3, z_4}$ of the third kind, and for disjoint paths $\delta_1$ from $\pi(z_2)$ to $\pi(z_1)$ and $\delta_2$ from $\pi(z_4)$ to $\pi(z_3)$, it follows from Theorem 5.4 (iii) that

\[
\int_{\delta_1} \nu_{z_3, z_4} = \int_{\delta_2} \nu_{z_1, z_2};
\]

consequently

\[
q(z_1, z_2; z_3, z_4) = \exp \int_{\delta_1} \nu_{z_3, z_4} = \exp \int_{\delta_2} \nu_{z_1, z_2} = \exp \int_{\delta_3} \nu_{z_1, z_2} = \exp \int_{\delta_4} \nu_{z_2, z_4} = q(z_3, z_4; z_1, z_2).
\]

This symmetry, together with that already demonstrated, implies all the symmetries of (ii). From these symmetries and the already established analyticity properties of $q(z_1, z_2; z_3, z_4)$ as a function of the variables $z_1$ and $z_2$ it follows that $q(z_1, z_2; z_3, z_4)$ also is a meromorphic function of the variable $z_3$ and the variable $z_4$, hence by Rothstein’s Theorem\(^2\) it is a meromorphic function on

\(^2\)For Rothstein’s Theorem see Appendix A.1.
\[ \tilde{M} \times \tilde{M} \times \tilde{M} \times \tilde{M} \]; and altogether it has the zeros and poles as in (i). Finally
the quotient of any two functions satisfying the conditions of the theorem is necessarily a holomorphic and nowhere vanishing function on \( \tilde{M} \times \tilde{M} \times \tilde{M} \times \tilde{M} \) that is invariant under the group of covering translations in each factor, hence
is a constant; and from the normalization condition of (i) it is evident that this
constant is 1. That suffices to conclude the proof.

The preceding theorem also holds for surfaces of genus \( g = 0 \) to the extent
possible. On the Riemann sphere \( \mathbb{P}^1 \) the classical cross-ratio function is

\[ q(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \]  

(5.30)

This clearly is a meromorphic function on \( \mathbb{P}^1 \) in all four variables, with the
singularities as in Theorem 5.6 (i); and equally clearly it satisfies the symmetry
conditions of Theorem 5.6 (ii). Theorem 5.6 (iii) is not applicable, since the
covering translation group is trivial in this case; but the remaining properties
obviously characterize the cross-ratio function uniquely. The terminology of
course is suggested by this special case; but the role that the cross-ratio function
plays for general Riemann surfaces differs in many ways from its role in classical
projective geometry.

**Corollary 5.7** Let \( M \) be a compact Riemann surface of genus \( g > 0 \) with the
universal covering projection \( \pi : \widetilde{M} \longrightarrow M \); let \( \Omega \) be the period matrix and
\( P \) be the intersection matrix of \( M \) in terms of bases \( \omega_i \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and
\( \tau_j \in H_1(M) \), and set \( G = \text{det}^{-1} H \) for the positive definite Hermitian matrix
\( H = i \Omega P \Omega \). For any points \( z_+, z_- \in \tilde{M} \) the canonically parametrized flat
line bundle \( \rho_m(z_+, z_-) \) described by the vector \( t(z_+, z_-) = i \Omega \tilde{G} \tilde{w}(z_+, z_-) \), where
\( \tilde{w}(z, a) \in \mathbb{C}^g \) is the vector with entries \( w_i(z, a) = \int_a^z \omega_i \), is holomorphically
equivalent to the holomorphic line bundle \( \zeta_{\pi(z_+)} \zeta_{\pi(z_-)}^{-1} \) over \( M \).

**Proof:** For any fixed points \( a, z_+, z_- \in \tilde{M} \) it follows from the preceding theo-
rem that the cross-ratio function \( q(z, a; z_+, z_-) \) is a meromorphic relatively au-
tomorphic function for the canonically parametrized flat factor of automorphy
\( \rho_m(z_+, z_-) \) and has the divisor \( d = 1 \cdot a_+ - 1 \cdot a_- \) where \( a_+ = \pi(z_+) \), \( a_- = \pi(z_-) \);
consequently the flat line bundle \( \rho_m(z_+, z_-) \) is holomorphically equivalent to the
holomorphic line bundle \( \zeta_{a_+} \zeta_{a_-}^{-1} \), and that suffices for the proof.

It follows immediately from the definition (5.23) of the cross-ratio function
that the normalized abelian differential of the third kind with the principal part
\( p(z_+, z_-) \) can be written explicitly in terms of the cross-ratio function as

\[ \nu_{z_+, z_-} = \frac{\partial}{\partial z} \log q(z, a; z_+, z_-)dz \]

(5.31)

for any point \( a \in \tilde{M} \); and consequently its integral can be written

\[ \nu^\delta_{z_+, z_-}(z, a) = \int_a^z \nu_{z_+, z_-} = \log q(z, a; z_+, z_-) \]

(5.32)
for that branch of the logarithm for which $v_{+,-}^\delta(a,a) = \log q(a,a;z_+,z_-) = \log 1 = 0$. The normalized double differential of the second kind on a compact Riemann surface of genus $g > 0$ also can be written explicitly in terms of the cross-ratio function of the surface as follows.

**Theorem 5.8** The normalized double differential of the second kind $\mu_M(z,\zeta)$ on compact Riemann surface $M$ of genus $g > 0$ can be expressed in terms of the cross-ratio function $q(z_1,z_2;z_3,z_4)$ of that surface as

$$
\mu_M(z,\zeta) = \frac{\partial^2}{\partial z_\alpha \partial \zeta_\beta} \log q(z_\alpha,a;\zeta_\beta,b) \, dz_\alpha d\zeta_\beta 
$$

(5.33)

for any points $a,b \in \tilde{M}$.

**Proof:** When the normalized double differential of the second kind is written $\mu_M(z,\zeta) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha d\zeta_\beta$ in terms of a covering of the surface $M$ by coordinate neighborhoods $U_\alpha$, in which the local coordinates are denoted by either $z_\alpha$ or $\zeta_\alpha$, then $\mu_p(z) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha$ is the normalized meromorphic abelian differential of the second kind with the principal part $p = (z_\beta - \zeta_\beta)^{-2}dz_\beta$ at the fixed point $\zeta_\alpha \in U_\beta$. From Theorem 5.4 (iv) it then follows that for any simple path $\delta$ from $z_-$ to $z_+$ on $\tilde{M}$ and for the integral (5.32)

$$
\int_{z_-}^{z_+} f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha = \int_\delta \mu_p = \text{res}_{z_\beta = \zeta_\beta} \left( v_{+,-}^\delta (a,p) \right) 
$$

(5.33)

$$
= \text{res}_{z_\beta = \zeta_\beta} \left( \log q(z_\beta,a;z_+,z_-) \frac{dz_\beta}{(z_\beta - \zeta_\beta)^2} \right) 
$$

$$
= \frac{\partial}{\partial \zeta_\beta} \log q(z_\beta,a;z_+,z_-). 
$$

Differentiating the preceding equation with respect to the variable $z_+$ at the point $z_+ = z_\alpha \in U_\alpha$ shows that

$$
f_{\alpha\beta}(z_\alpha,\zeta_\beta) = \frac{\partial^2}{\partial z_\alpha \partial \zeta_\beta} \log q(\zeta_\beta,a;z_\alpha,z_-); 
$$

this is equivalent to the desired result and that suffices to conclude the proof.

In addition to the intrinsic cross-ratio function it is convenient to introduce, in parallel to the discussion of double differentials in the preceding chapter, the basic cross-ratio function associated to an arbitrary symmetric matrix $E = \{e_{kl}\}$

$$
q_E(z_1,z_2;z_3,z_4) = q(z_1,z_2;z_3,z_4) \exp \sum_{k,l=1}^{g} e_{kl} w_k(z_1,z_2) w_l(z_3,z_4) 
$$

(5.34)

where $w_i(z_1,z_2) = \int_{z_1}^{z_2} \omega_i$ are the integrals of the holomorphic abelian differentials $\omega_i$. These functions can be characterized as follows.
Theorem 5.9 If $M$ is a compact Riemann surface $M$ of genus $g > 0$ with the universal covering space $\tilde{M}$ and the covering translation group $\Gamma$, the basic cross-ratio functions of $M$ are meromorphic functions $\tilde{q}(z_1, z_2; z_3, z_4)$ on the complex manifold $\tilde{M} \times \tilde{M} \times \tilde{M} \times \tilde{M}$ that are characterized uniquely by the following properties:

(i) The function $\tilde{q}(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$ and simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, but is otherwise holomorphic and nowhere vanishing on $M^4$; and $\tilde{q}(z_1, z_2; z_3, z_4) = 1$ if $z_1 = z_2$ or $z_3 = z_4$.

(ii) The function $\tilde{q}(z_1, z_2; z_3, z_4)$ has the symmetries

$$
\tilde{q}(z_1, z_2; z_3, z_4) = \tilde{q}(z_3, z_4; z_1, z_2) = \tilde{q}(z_2, z_1; z_4, z_3) = \tilde{q}(z_2, z_1; z_3, z_4) = \tilde{q}(z_2, z_1; z_3, z_4)^{-1} = \tilde{q}(z_1, z_2; z_4, z_3)^{-1}.
$$

(iii) The function $\tilde{q}(z_1, z_2; z_3, z_4)$ as a function of the variable $z_1 \in \tilde{M}$ for any fixed points $z_2, z_3, z_4 \in \tilde{M}$ is a relatively automorphic function for a factor of automorphy described under the canonical parametrization of flat factors of automorphy in terms of any basis $\tau_j \in H_1(M)$ by $\rho_s(z_3, z_4)$ for some vector $s(z_3, z_4)$ depending only on the parameters $z_3, z_4$.

Proof: It is clear from the definition (5.34) of the basic cross-ratio function $q_E(z_1, z_2; z_3, z_4)$, in which $E$ is required to be a symmetric matrix, and from the characterization of the intrinsic cross-ratio function in Theorem 5.6, that a basic cross-ratio function satisfies (i) and (ii), and that for any covering translation $T \in \Gamma$

$$
q_E(Tz_1, z_2; z_3, z_4) = q(Tz_1, z_2; z_3, z_4) \cdot \exp \sum_{k,l=1}^{g} e_{kl}w_k(Tz_1, z_2)w_l(z_3, z_4)
$$

$$
= \rho_{t(z_3, z_4)}(T) q(z_1, z_2; z_3, z_4) \cdot \exp \sum_{k,l=1}^{g} e_{kl}(w_k(z_1, z_2) + \omega_k(T))w_l(z_3, z_4)
$$

$$
= \rho_{t(z_3, z_4)}(T) \left( \exp \sum_{k,l=1}^{g} e_{kl}\omega_k(T)w_l(z_3, z_4) \right) q_E(z_1, z_2; z_3, z_4);
$$

in particular for $T = T_j$

$$
q_E(T_jz_1, z_2; z_3, z_4) = \exp \left( 2\pi i t_j(z_3, z_4) + \sum_{k,l=1}^{g} e_{kl}\omega_k(T_j)w_l(z_3, z_4) \right) q_E(z_1, z_2; z_3, z_4)
$$

$$
= \rho_s(z_3, z_4)(T_j) q_E(z_1, z_2; z_3, z_4)
$$

where

$$
s_j(z_3, z_4) = t_j(z_3, z_4) + \frac{1}{2\pi i} \sum_{k,l=1}^{g} e_{kl}\omega_k(T_j)w_l(z_3, z_4),
$$

(5.36)
so a basic cross-ratio function also satisfies (iii).

Conversely suppose that \( \tilde{q}(z_1, z_2; z_3, z_4) \) is an arbitrary meromorphic function on \( \tilde{M} \times \tilde{M} \times M \times M \) that satisfies (i), (ii), and (iii). It follows from (i) and Theorem 5.6 (i) that

\[
\tilde{q}(z_1, z_2; z_3, z_4) = q(z_1, z_2; z_3, z_4) \exp h(z_1, z_2; z_3, z_4)
\]

for a holomorphic function \( h(z_1, z_2; z_3, z_4) \) on the simply connected complex manifold \( \tilde{M} \times \tilde{M} \times M \times M \), since the two cross-ratio functions have the same zeros and poles. In addition it follows that \( \exp h(z_1, z_2; z_3, z_4) = 1 \), and consequently \( h(z_1, z_2; z_3, z_4) = 2\pi i n \) for some integer \( n \); so after replacing the function \( h(z_1, z_2; z_3, z_4) \) by \( h(z_1, z_2; z_3, z_4) - 2\pi in \) it can be assumed that \( h(z_1, z_2; z_3, z_4) = 0 \). The functions \( \tilde{q}(z_1, z_2; z_3, z_4) \) and \( q(z_1, z_2; z_3, z_4) \) both satisfy the symmetries of part (ii), so \( \exp h(z_1, z_2; z_3, z_4) \) does as well. Therefore \( h(z_1, z_2; z_3, z_4) = h(z_3, z_4; z_1, z_2) = 2\pi in_1 \) for some integer \( n_1 \); and setting \( z_1 = z_2 \) and \( z_3 = z_4 \) shows that actually \( n_1 = 0 \). Furthermore \( h(z_1, z_2; z_3, z_4) + h(z_2, z_1; z_3, z_4) = 2\pi in_2 \) for another integer \( n_2 \); and setting \( z_1 = z_2 \) shows that \( n_2 = 0 \) as well. It follows from these observations that the function \( h(z_1, z_2; z_3, z_4) \) satisfies

\[
h(z_1, z_2; z_3, z_4) = h(z_3, z_4; z_1, z_2) = h(z_2, z_1; z_4, z_3) = -h(z_2, z_1; z_3, z_4) = -h(z_1, z_2; z_4, z_3).
\]

For any covering translation \( T \in \Gamma \) it follows from (iii) that

\[
h(Tz_1, z_2; z_3, z_4) = h(z_1, z_2; z_3, z_4) + g(T; z_3, z_4)
\]

for some holomorphic function \( g(T; z_3, z_4) \) of the variables \( z_3, z_4 \in \tilde{M} \); thus \( h(z_1, z_2; z_3, z_4) \) as a function of the variable \( z_1 \) alone is a holomorphic abelian integral, and since it vanishes when \( z_1 = z_2 \) it can be written in terms of a basis \( w_k(z, z_2) \) for the holomorphic abelian integrals as

\[
h(z_1, z_2; z_3, z_4) = \sum_{k=1}^{g} e_k(z_2, z_3, z_4)w_k(z_1, z_2)
\]

for some uniquely determined functions \( e_k(z_2, z_3, z_4) \), which consequently must be holomorphic functions of the variables \( z_2, z_3, z_4 \). In that case

\[
g(T; z_3, z_4) = h(Tz_1, z_2; z_3, z_4) - h(z_1, z_2; z_3, z_4)
\]

\[
= \sum_{k=1}^{g} e_k(z_2, z_3, z_4)\omega_k(T),
\]

and since this is the case for all \( T \in \Gamma \) it follows that the coefficients \( e_k(z_2, z_3, z_4) \) must be independent of the variable \( z_2 \). Upon interchanging the two pairs of variables it follows from the symmetry of the function \( h(z_1, z_2; z_3, z_4) \) that

\[
\sum_{k=1}^{g} e_k(z_3, z_4)w_k(z_1, z_2) = \sum_{k=1}^{g} e_k(z_1, z_2)w_k(z_3, z_4)
\]
and consequently that $e_k(z_3, z_4)$ is a holomorphic abelian integral in each variable as well; and since this integral vanishes when $z_3 = z_4$ it follows that $e_k(z_3, z_4) = \sum_{l=1}^{p} e_{kl} w_l(z_3, z_4)$, which shows that the function $\tilde{q}(z_1, z_2; z_4, z_3)$ is a basic cross-ratio function as defined in (5.34) and thereby concludes the proof.

**Corollary 5.10** Let $\Omega$ be the period matrix and $P$ be the intersection matrix of a compact Riemann surface $M$ of genus $g > 0$ in terms of any arbitrary bases $\omega_i \in \Gamma(M, O^{(1,0)})$ and $\tau_j \in H_1(M)$; and let $\tilde{w}(z, a) = \{w_i(z, a)\}$ be the column vector with entries the integrals $w_i(z, a) = \int_{a}^{z} \omega_i$. The basic cross-ratio function $q_E(z_1, z_2; z_3, z_4)$ for a symmetric matrix $E$ is a relatively automorphic function for the flat factor of automorphy described under the canonical parametrization of flat factors of automorphy in terms of these bases by $\rho_s(z_3, z_4)$ for the vector (5.37)

$$s(z_3, z_4) = i t^t \left( \Omega^t G - \frac{1}{2\pi} \Omega^t E \right) \tilde{w}(z_3, z_4)$$

where $G = \Omega^{-1}$ for the positive definite symmetric matrix $H = \Omega P \Omega$.

**Proof:** The intrinsic cross-ratio function is a relatively automorphic function for the flat factor of automorphy $\rho_t(z_3, z_4)$ for the vector $t(z_3, z_4)$ given by (5.27), and the basic cross-ratio function associated to a symmetric matrix $E = \Omega^t E$ is a relatively automorphic function for the flat factor of automorphy $\rho_s(z_3, z_4)$ for the vector $s(z_3, z_4)$ given by (5.36); so

$$s(z_3, z_4) = t(z_3, z_4) + \frac{1}{2\pi i} \Omega E \tilde{w}(z_3, z_4)$$

$$= i t^t G \tilde{w}(z_3, z_4) + \frac{1}{2\pi i} \Omega E \tilde{w}(z_3, z_4),$$

which reduces to (5.37), and that suffices to conclude the proof.

The characterization of basic cross-ratio functions in Theorem 5.9 is more appealing than the characterization of intrinsic cross-ratio functions in Theorem 5.6 in that the factor of automorphy is stated in a rather more general form; the explicit description of the factor of automorphy in Corollary 5.10 shows that the intrinsic cross-ratio function is characterized among the basic cross-ratio functions by having a factor of automorphy described by the complex conjugate period matrix alone. The basic double differential of the second kind on the Riemann surface $M$ described by a symmetric matrix $E$ can be expressed in terms of the basic cross-ratio function associated to that matrix, in an extension of Theorem 5.8; indeed it is evident from the definitions (4.49)
and (5.34) that
\[
\frac{\partial^2}{\partial z \partial \zeta} \log \frac{q}{E}(z, a; \zeta, b) dzd\zeta = \frac{\partial^2}{\partial z \partial \zeta} \log \left( q(z, a; \zeta, b) \cdot \exp \sum_{k,l=1}^{g} e_{kl}w_k(z, a)w_l(\zeta, b) \right)
\]
(5.38)
\[
= \mu_M(z, \zeta) + \sum_{k,l=1}^{g} e_{kl}\omega_k(z)\omega_l(\zeta)
\]
\[
= \mu_{M,E}(z, \zeta)
\]
for any symmetric matrix $E$. In particular the canonical double differential of the second kind $\hat{\mu}(z, \zeta)$ on a marked Riemann surface of genus $g > 0$, the basic double differential with the properties given in Theorem 4.21, can be expressed in terms of the corresponding basic cross-ratio function, called the \textit{canonical cross-ratio function} on the marked Riemann surface; this function is denoted by $\hat{q}(z_1, z_2; z_3, z_4)$, so
\[
\frac{\partial^2}{\partial z \partial \zeta} \log \hat{q}(z, a; \zeta, b) dzd\zeta = \hat{\mu}_M(z, \zeta).
\]
(5.39)

The canonical cross-ratio function can be characterized as follows.

\textbf{Theorem 5.11} On a marked compact Riemann surface $M$ of genus $g > 0$, with the marking described by generators $A_j, B_j \in \Gamma$, the canonical cross-ratio function $\hat{q}(z_1, z_2; z_3, z_4)$ is characterized by the following properties.

(i) The function $\hat{q}(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$ and simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, but is otherwise holomorphic and nowhere vanishing on $\tilde{M}^4$; and

\[
\hat{q}(z_1, z_2; z_3, z_4) = 1 \text{ if } z_1 = z_2 \text{ or } z_3 = z_4.
\]

(ii) The function $\hat{q}(z_1, z_2; z_3, z_4)$ has the symmetries

\[
\hat{q}(z_1, z_2; z_3, z_4) = \hat{q}(z_3, z_4; z_1, z_2) = \hat{q}(z_2, z_1; z_3, z_4) = \hat{q}(z_2, z_1; z_4, z_3) = \hat{q}(z_2, z_1; z_3, z_4)^{-1} = \hat{q}(z_1, z_2; z_4, z_3)^{-1}.
\]

(iii) For any fixed points $z_2, z_3, z_4$

\[
\hat{q}(A_j z_1, z_2; z_3, z_4) = \hat{q}(z_1, z_2; z_3, z_4),
\]
\[
\hat{q}(B_j z_1, z_2; z_3, z_4) = \hat{q}(z_1, z_2; z_3, z_4) \exp 2\pi i w_j(z_3, z_4)
\]

where $w_j(z_3, z_4) = \int_{z_4}^{z_3} \omega_j$ are the integrals of the canonical abelian differentials $\omega_j$ on the marked surface.

\textbf{Proof:} The intersection matrix of the surface $M$ in terms of the basis for $H_1(M)$ described by the generators $A_j, B_j \in \Gamma$ of the marking of $M$ is the basic
skew-symmetric matrix $J$, and by Theorem 4.21 the period matrix $\Omega$ of $M$ in terms of this basis and of the canonical holomorphic abelian differentials $\omega_i$ on the marked surface $M$ is the matrix $\Omega = (1 \quad Z)$ where $Z = X + iY \in S_g$, the Siegel upper half-space of rank $g$. By Theorem 4.21 the canonical double differential of the second kind on $M$ is the basic double differential described by the matrix $E = \pi Y^{-1}$. The basic cross-ratio function associated to this matrix $E$ is a relatively automorphic function for the canonically parametrized factor of automorphy described by the vector $s(z_3, z_4)$ given explicitly by (5.37). Since $G = \frac{1}{2} Y^{-1} = \frac{1}{2\pi} E$, as in the proof of Theorem 4.21, it follows that

$$i\left(\Omega G - \frac{1}{2\pi} \Omega E\right) = \frac{1}{2\pi} \left(\begin{array}{cc} 1 & \pi i w_1(z_3, z_4) \\ i & -\pi i w_1(z_3, z_4) \end{array}\right)$$

consequently $s(z_3, z_4) = i\left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array}\right)\hat{w}(z_3, z_4)$ so that

$$\rho_s(z_3, z_4)(A_j) = 1, \quad \rho_s(z_3, z_4)(B_j) = \exp 2\pi i w_j(z_3, z_4)$$

and therefore the basic cross-ratio function associated to this matrix $E$ satisfies (iii). Since this cross-ratio function is characterized uniquely by (i), (ii) and (iii), that suffices to conclude the proof.

The real-normalized cross-ratio function is quite useful for some purposes, even though it is not a holomorphic function in all variables; it is defined by

$$q(z_1, z_2; a, b) = q(z_1, z_2; a, b) \exp 2\pi \sum_{k, l=1}^g g_{kl}w_k(z_1, z_2)w_l(a, b)$$

in terms of bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{1,0})$ for the holomorphic abelian differentials and $\tau_j \in H_1(M)$ for the homology of a compact Riemann surface $M$ of genus $g > 0$, where $\Omega$ is the period matrix and $P$ is the intersection matrix of $M$, and $G = H^{-1}$ for the positive definite symmetric matrix $H = i\Omega P \Omega$. Although the definition (5.42) is expressed in terms of these bases it follows just as in the demonstration of the invariance of (5.17) that the function $\bar{q}(z_1, z_2; a, b)$ itself is intrinsically defined on $M$, independent of the choice of bases.

**Theorem 5.12** On a compact Riemann surface $M$ of genus $g > 0$ the real-normalized cross-ratio function $\bar{q}(z_1, z_2; a, b)$ is a meromorphic function of the variables $(a, b) \in \bar{M}$ with the following properties:

(i) The function $\bar{q}(z_1, z_2; a, b)$ has simple zeros along the subvarieties $z_1 = Ta$ and $z_2 = Tb$ and simple poles along the subvarieties $z_1 = Tb$ and $z_2 = Ta$ for all $T \in \Gamma$, but no other zeros or poles in the variables $(z_1, z_2)$, for any fixed point $(a, b)$; and it takes the value $1$ if $z_1 = z_2$ or $a = b$.

(ii) The function $\bar{q}(z_1, z_2; a, b)$ has the symmetries

$$\bar{q}(z_1, z_2; a, b) = \bar{q}(z_2, z_1; a, b)^{-1} = \bar{q}(z_1, z_2; b, a)^{-1}.$$
(iii) The function $\bar{q}(z_1, z_2; a, b)$ as a function of the variable $z_1 \in \tilde{M}$ for any fixed points $z_2, a, b \in \tilde{M}$ is a meromorphic relatively automorphic function for the canonically parametrized factor of automorphy $\rho_{r(a,b)}(T)$ described by the real vector $r(a, b) \in \mathbb{R}^{2g}$ that for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ is given by

\begin{equation}
(5.43) \quad r(a, b) = 2\Im \left( \Omega G \bar{w}(a, b) \right) = \frac{1}{2\pi i} \left( \Omega G \bar{w}(a, b) - \bar{\Omega} G \bar{w}(a, b) \right)
\end{equation}

in which $\Omega$ is the period matrix, $P$ is the intersection matrix, $G = \mathcal{H}^{-1}$ for the positive definite symmetric matrix $H = i \Omega P \bar{\Omega}$ in terms of these bases, and $\Im(z) = y$ is the imaginary part of the complex number $z = x + iy$; hence

\begin{equation}
(5.44) \quad |\rho_{r(a,b)}(T)| = 1 \quad \text{for all} \quad T \in \Gamma.
\end{equation}

Proof: Properties (i) and (ii) are immediate consequences of the corresponding properties of the intrinsic cross-ratio function in Theorem 5.6 and of the definition (5.42) of the real-normalized cross-ratio function, since the exponential is a nowhere vanishing function that is holomorphic in the variables $z_1, z_2 \in \tilde{M}$ and conjugate meromorphic in the variables $a, b \in \tilde{M}$, and $w_k(z_1, z_2) = -w_k(z_2, z_1)$. For the generator $T_j \in \Gamma$ corresponding to the basis element $\tau_j \in H_1(M)$ it follows from the definition (5.42) and Theorem 5.6 (iii) that

$$
\bar{q}(T_jz_1, z_2; a, b) = q(T_jz_1, z_2; a, b) \cdot \exp 2\pi \sum_{k,l=1}^{g} g_{kl} w_k(T_jz_1, z_2) \bar{w}_l(a, b)
$$

$$
= \rho_{r(a,b)}(T_j) q(z_1, z_2; a, b) \cdot \exp 2\pi \sum_{k,l=1}^{g} g_{kl} \left( w_k(z_1, z_2) + \omega_{kj} w_l(a, b) \right)
$$

$$
= q(z_1, z_2; a, b) \cdot \exp 2\pi \sum_{k,l=1}^{g} \left( -g_{kl} \bar{w}_k w_l(a, b) + g_{kl} \omega_{kj} \bar{w}_l(a, b) \right)
$$

$$
= q(z_1, z_2; a, b) \exp 2\pi i r_j(a, b) = \rho_{r(a,b)}(T_j) \bar{q}(z_1, z_2; a, b),
$$

and therefore $q(Tz_1, z_2; a, b) = \rho_{r(a,b)}(T) \bar{q}(z_1, z_2; a, b)$ for all $T \in \Gamma$. Since $r(a, b)$ is a real vector $|\rho_{r(a,b)}(T_j)| = 1$ for each generator $T_j \in \Gamma$ and therefore $|\rho_{r(a,b)}(T)| = 1$ for all $T \in \Gamma$; and that suffices for the proof.
Chapter 6

Abelian Factors of Automorphy

A holomorphic factor of automorphy for the action of the covering translation group $\Gamma$ on the universal covering space $\tilde{M}$ of a compact Riemann surface $M$ of genus $g > 0$ represents a holomorphic line bundle over the surface $M$, as in Theorem 3.11. Topologically trivial holomorphic line bundles can be represented by flat factors of automorphy, as in Theorem 3.13; and it will be demonstrated in this chapter that all holomorphic line bundles can be represented by some factors of automorphy, by exhibiting explicit factors of automorphy that represent holomorphic line bundles of any characteristic class. These factors of automorphy are modeled on those that arise in connection with some of the classical elliptic functions\(^1\). The sigma function of Weierstrass for a one-dimensional complex torus $M = \mathbb{C}/\mathcal{L}$ is a holomorphic function $\sigma(z)$ on the complex plane $\mathbb{C}$ that transforms under translations $T \in \mathcal{L}$ by functional equations of the form $\sigma(Tz) = \sigma(z) \cdot \exp \sigma_T(z)$ for some linear functions $\sigma_T(z)$; hence $\sigma(z)$ is a holomorphic relatively automorphic function for the factor of automorphy $\exp \sigma_T(z)$. The Jacobian theta function $\vartheta(z)$ for the complex torus $M$ is another example of a holomorphic relatively automorphic function for a factor of automorphy given by exponentials of linear functions. Linear functions on a complex torus $M$ are holomorphic abelian integrals on $M$; so the analogue of these classical factors of automorphy for a compact Riemann surface $M$ of genus $g > 1$ is a factor of automorphy of the form

$$
(6.1) \quad \zeta(T, z) = \exp 2\pi i \sigma(T, z)
$$

where for each covering translation $T \in \Gamma$ the function $\sigma(T, z)$ is a holomorphic abelian integral on the universal covering space $\tilde{M}$ of the surface $M$ so can be

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\(^1\) For the properties of the classical elliptic functions see for instance Whittaker and Watson, *Modern Analysis*, (Cambridge, 1902).
CHAPTER 6. ABELIAN FACTORS OF AUTOMORPHY

written

$$\sigma(T, z) = \sigma_0(T) + \sum_{i=1}^{g} \sigma_i(T) w_i(z, a)$$

for some mappings $\sigma_i : \Gamma \to \mathbb{C}$ for $0 \leq i \leq g$, where $w_i(z, a) = \int_{\gamma_i} \omega_i$ in terms of a basis $\omega_i(z) \in \Gamma(M, O^{1,0}(\gamma_i))$ for the holomorphic abelian differentials on $M$ and a base point $a \in \tilde{M}$.

Theorem 6.1 The functions $\zeta(T, z)$ are a factor of automorphy for the action of the covering translation group $\Gamma$ on the universal covering space $\tilde{M}$ of a compact Riemann surface $M$ of genus $g > 0$ if and only if

(i) the mappings $\sigma_i$ are group homomorphisms for $1 \leq i \leq g$, and

(ii) the mapping $\sigma_0$ satisfies the condition that $w(S, T) \in \mathbb{Z}$ for all $S, T \in \Gamma$, where

$$w(S, T) = \sigma_0(S) + \sigma_0(T) - \sigma_0(ST) + \sum_{i=1}^{g} \sigma_i(S) \omega_i(T)$$

and $\omega_i \in \text{Hom}(\Gamma, \mathbb{C})$ are the period classes of the holomorphic abelian differentials $\omega_i(z) = dw_i(z, a)$.

Proof: The functions $\zeta(T, z)$ are a factor of automorphy if and only if

$$\zeta(S, Tz) \zeta(T, z) \zeta(ST, z)^{-1} = 1$$

for all $S, T \in \Gamma$; and when $\zeta(T, z)$ has the explicit form (6.1) that is equivalent to the condition that the functions

$$\tilde{w}(S, T; z) = \sigma(S, Tz) + \sigma(T, z) - \sigma(ST, z)$$

take only integer values for all $S, T \in \Gamma$. If $\sigma(T, z)$ is defined by (6.2) then since $w_i(Tz, a) = w_i(z, a) + \omega_i(T)$ for all $T \in \Gamma$

$$\tilde{w}(S, T; z) = \sigma_0(S) + \sigma_0(T) - \sigma_0(ST)$$

$$+ \sum_{i=1}^{g} \left( \sigma_i(S) w_i(Tz, a) + \sigma_i(T) w_i(z, a) - \sigma_i(ST) w_i(z, a) \right)$$

$$= \sigma_0(S) + \sigma_0(T) - \sigma_0(ST) + \sum_{i=1}^{g} \sigma_i(S) \omega_i(T)$$

$$+ \sum_{i=1}^{g} \left( \sigma_i(S) + \sigma_i(T) - \sigma_i(ST) \right) w_i(z, a)$$

$$= w(S, T) + \sum_{i=1}^{g} \left( \sigma_i(S) + \sigma_i(T) - \sigma_i(ST) \right) w_i(z, a).$$
If the functions $\zeta(T, z)$ are a factor of automorphy then $\tilde{w}(S, T; z)$ is an integer for all $S, T \in \Gamma$, in particular is constant, and consequently

$$0 = d\tilde{w}(S, T; z) = \sum_{i=1}^{g} \left( \sigma_i(S) + \sigma_i(T) - \sigma_i(ST) \right) \omega_i(z).$$

Since the abelian differentials are linearly independent it follows that $\sigma_i(S) + \sigma_i(T) - \sigma_i(ST) = 0$ for all $S, T \in \Gamma$ and $1 \leq i \leq g$, so these mappings are group homomorphisms; and in addition $w(S, T) = \tilde{w}(S, T; z)$, so it is an integer for all $S, T \in \Gamma$. Conversely if the mappings $\sigma_i$ are group homomorphisms for $1 \leq i \leq g$ and if $w(S, T) \in \mathbb{Z}$ for all $S, T \in \Gamma$ then $\tilde{w}(S, T; z) = w(S, T) \in \mathbb{Z}$ so $\zeta(T, z)$ is a factor of automorphy. That suffices to conclude the proof.

These factors of automorphy are called the \textit{abelian factors of automorphy} described by the holomorphic abelian integrals $\sigma(T, z)$. Changing the values $\sigma_0(T)$ by arbitrary integers does not change the condition that (6.3) takes only integer values, nor does it change the factor of automorphy $\zeta(T, z)$; so there is no loss of generality in normalizing the mapping $\sigma_0$ so that

$$0 \leq \Re(\sigma_0(T)) < 1 \quad \text{for all} \quad T \in \Gamma,$$

where $\Re(\sigma_0(T))$ denotes the real part of the complex number $\sigma_0(T)$. For $S = I$, the identity element, (6.3) reduces to $w(I, T) = \sigma_0(I)$, so the condition that $w(I, T) \in \mathbb{Z}$ together with the normalization (6.5) imply that

$$\sigma_0(I) = 0 \quad \text{for the identity} \quad I \in \Gamma;$$

and it then follows from (6.3) that

$$w(I, T) = w(T, I) = 0 \quad \text{for all} \quad T \in \Gamma.$$

The conditions of Theorem 6.1 can be expressed quite conveniently in terms of the machinery of the cohomology of groups\footnote{A brief survey of the cohomology of groups can be found in Appendix E; detailed references to particular results mentioned in Appendix E will be included as these results are needed in the subsequent discussion.}. Mappings from the group $\Gamma$ to the complex numbers taking the identity $I \in \Gamma$ to $0 \in \mathbb{C}$ can be viewed as inhomogeneous 1-cochains of the group $\Gamma$ with coefficients in $\mathbb{C}$, as defined in (E.9); all group homomorphisms take the identity $I \in \Gamma$ to $0 \in \mathbb{C}$, as does the mapping $\sigma_0$ by (6.6), so the mappings $\sigma_i$ for $0 \leq i \leq g$ can be viewed as 1-cochains $\sigma_i \in C^1(\Gamma, \mathbb{C})$. Moreover the group homomorphisms $\sigma_i$ for $1 \leq i \leq g$ are 1-cocycles $\sigma_i \in Z^1(\Gamma, \mathbb{C})$, and the period classes $\omega_i$ of the holomorphic abelian differentials on $M$ also are 1-cocycles $\omega_i \in Z^1(\Gamma, \mathbb{C}) \cong H^1(\Gamma, \mathbb{C})$, when $\Gamma$ is viewed as acting trivially on $\mathbb{C}$, as in (E.17); and $Z^1(\Gamma, \mathbb{C}) \cong H^1(\Gamma, \mathbb{C})$ by (E.22). Mappings from $\Gamma \times \Gamma$ to the complex numbers that take $(I, T)$ and $(T, I)$ to $0 \in \mathbb{C}$ for all $T \in \Gamma$ are 2-cochains of the group $\Gamma$ with coefficients in $\mathbb{C}$, so in view of (6.7) the expression $w(S, T)$ can be viewed as a 2-cochain.
$w \in C^2(\Gamma, \mathbb{C})$. Since $w(S, T) = \tilde{w}(S, T; z)$ is given by (6.4), and the factor of automorphy has the form (6.1), it follows from (E.61) and the accompanying discussion of that equation in Appendix E.3 that the 2-cochain $w \in C^2(\Gamma, \mathbb{Z})$ is actually a 2-cocycle $w \in Z^2(\Gamma, \mathbb{Z})$ and represents the characteristic class of the factor of automorphy $\zeta(T, z)$. The coboundary of the 1-cochain $\sigma_0 \in C^1(\Gamma, \mathbb{C})$ is the 2-cocycle $\delta\sigma_0 \in Z^2(\Gamma, \mathbb{C})$ defined by

$$\delta\sigma_0(S, T) = \sigma_0(S) + \sigma_0(T) - \sigma_0(ST)$$

by (E.24). The 2-cochain $\sigma_i \cup \omega_i \in Z^2(\Gamma, \mathbb{C})$ for $1 \leq i \leq g$ defined in terms of the 1-cocycles $\sigma_i, \omega_i \in Z^1(\Gamma, \mathbb{C})$ by

$$\sigma_i(S, T) = \sigma_i(S) \cdot \omega_i(T) \quad \text{for} \quad 1 \leq i \leq g$$

is a 2-cocycle $\sigma_i \cup \omega_i \in Z^2(\Gamma, \mathbb{C})$, and the cohomology class it represents is the cup product $\sigma_i \cup \omega_i \in H^2(\Gamma, \mathbb{C})$ of the cohomology classes $\sigma_i, \omega_i \in H^1(\Gamma, \mathbb{C})$ represented by these two cocycles, as in (E.7) in Appendix E.3. In cohomological terms (6.3) can be written

$$w(S, T) = \delta\sigma_0(S, T) + \sum_{i=1}^{g} (\sigma_i \cup \omega_i)(S, T),$$

which expresses the condition that the cocycles $w(S, T)$ and $\sum_{i=1}^{g} (\sigma_i \cup \omega_i)(S, T)$ differ by the coboundary of the 1-cochain $\sigma_0(T)$ so represent the same cohomology class $w = \sum_{i=1}^{g} (\sigma_i \cup \omega_i) \in H^2(\Gamma, \mathbb{C})$.

**Corollary 6.2** If $\omega_i \in \text{Hom}(\Gamma, \mathbb{C})$ for $1 \leq i \leq g$ are the period classes of a basis $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on a compact Riemann surface $M$ of genus $g > 0$, and if $\sigma_i \in \text{Hom}(\Gamma, \mathbb{C})$ for $1 \leq i \leq g$ are homomorphisms of the covering translation group $\Gamma$ of $M$, there exists a cochain $\sigma_0 \in C^1(\Gamma, \mathbb{C})$ such that the holomorphic abelian integrals (6.2) describe an abelian factor of automorphy if and only if the cohomology class

$$\sum_{i=1}^{g} (\sigma_i \cup \omega_i) \in H^2(\Gamma, \mathbb{C})$$

is an integral cohomology class, a cohomology class contained in the subgroup $H^2(\Gamma, \mathbb{Z}) \subset H^2(\Gamma, \mathbb{C})$.

**Proof:** That the cohomology class $\sum_{i=1}^{g} (\sigma_i \cup \omega_i) \in H^2(\Gamma, \mathbb{C})$ lies in the subgroup $H^2(\Gamma, \mathbb{Z}) \subset H^2(\Gamma, \mathbb{C})$ is just the condition that there is a 1-cochain $\sigma_0$ such that the cocycle $\delta\sigma_0(S, T) + \sum_{i=1}^{g} (\sigma_i \cup \omega_i)(S, T)$ takes integral values; and by Theorem 6.1 as interpreted in (6.10) that is just the condition that the holomorphic abelian integrals (6.2) describe an abelian factor of automorphy. That suffices for the proof.

The 1-cocycle $\omega_i \in Z^1(\Gamma, \mathbb{C})$ is the period class of the holomorphic closed 1-form $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ on the Riemann surface $M$ under the period isomorphism $p_1$ of (E.39), as defined explicitly in (E.38); and correspondingly the
1-cocycles \( \sigma_i \in Z^1(\Gamma, \mathbb{C}) \) for \( 1 \leq i \leq g \) can be described as the period classes of some \( \mathcal{C}^\infty \) closed differential forms \( \sigma_i(z) \in \Gamma(M, \mathcal{E}^i_\mathcal{C}) \) on \( M \). In these terms the preceding corollary can be rephrased and extended as follows.

**Theorem 6.3** If \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{1,0}) \) for \( 1 \leq i \leq g \) is a basis for the holomorphic abelian differentials on a compact Riemann surface \( M \) of genus \( g > 0 \), with period classes \( \omega_i \in \text{Hom}(\Gamma, \mathbb{C}) \), and if \( \sigma_i(z) \in \Gamma(M, \mathcal{E}^i_\mathcal{C}) \) for \( 1 \leq i \leq g \) are \( \mathcal{C}^\infty \) closed differential forms on \( M \), with period classes \( \sigma_i \in \text{Hom}(\Gamma, \mathbb{C}) \), there exists a cochain \( \sigma_0 \in C^1(\Gamma, \mathbb{C}) \) such that the holomorphic abelian integrals (6.2) describe an abelian factor of automorphy if and only if

\[
\sum_{i=1}^g \int_M \sigma_i(z) \wedge \omega_i(z) = n \in \mathbb{Z}.
\]

The characteristic class of the holomorphic line bundle \( \zeta \) described by this factor of automorphy is \( c(\zeta) = n \).

**Proof:** The period class \( p_2(\phi_i(z)) \in H^2(\Gamma, \mathbb{C}) \) of the closed differential 2-form

\[
\phi_i(z) = \sigma_i(z) \wedge \omega_i(z)
\]

under the period isomorphism (E.48) is the cup product \( p_2(\phi_i(z)) = p_1(\sigma_i(z)) \cup p_1(\omega_i(z)) \) by Theorem E.7; so if \( \phi(z) = \sum_{i=1}^g \phi_i(z) \) then

\[
p_2(\phi(z)) = \sum_{i=1}^g p_1(\sigma_i(z)) \cup p_1(\omega_i(z)).
\]

In these terms Corollary 6.2 asserts that there is a cochain \( \sigma_0 \in C^1(\Gamma, \mathbb{C}) \) such that the abelian integrals (6.2) describe an abelian factor of automorphy if and only if \( p_2(\phi(z)) \in H^2(\Gamma, \mathbb{Z}) \subset H^2(\Gamma, \mathbb{C}) \). The image of the cohomology class \( p_2(\phi(z)) \in H^2(\Gamma, \mathbb{C}) \) under the isomorphism \( p \cdot p_2^{-1} \) of (E.51) is the complex number \( (p \cdot p_2^{-1})(p_2(\phi(z))) = p(\phi(z)) = \int_M \phi(z) \); and if that number is an integer it follows from Theorem E.6 that this cohomology class can be represented by an integral cocyle \( v(S, T) \in Z^2(\Gamma, \mathbb{Z}) \) and consequently that \( p_2(\phi(z)) \in H^2(\Gamma, \mathbb{Z}) \subset H^2(\Gamma, \mathbb{C}) \). Conversely if \( p_2(\phi(z)) \in H^2(\Gamma, \mathbb{Z}) \) and if this cohomology class is represented by an integral cocycle \( v(S, T) \in Z^2(\Gamma, \mathbb{Z}) \) then when the covering translation group \( \Gamma \) is given the presentation \( \Gamma = F/K \) as the quotient of a free group \( F \) on \( 2g \) generators modulo the normal subgroup \( K \subset F \) generated by a single commutator \( C \in [F, F] \), by the choice of a marking of the surface \( M \), it follows from Corollary E.5 that

\[
\int_M \phi(z) = (p \cdot p_2^{-1})(p_2(\phi(z))) = -p^*(p_2(\phi(z)))(C);
\]

and by Theorem E.4 this constant can be expressed in terms of the values \( v(S, T) \) as in (E.52), so it is an integer. That demonstrates the first assertion of the theorem. An abelian factor of automorphy has the form \( \zeta(T, z) = \exp 2\pi i \sigma(T, z) \), so by (E.61) the characteristic class of this factor of automorphy is the cohomology class represented by the 2-cocycle

\[
\sigma(S, Tz) + \sigma(T, z) - \sigma(ST, z) = w(S, T) \in Z^2(\Gamma, \mathbb{Z});
\]

and by (6.10) this cohomology class is just

\[
\sum_{i=1}^g \sigma_i \cup \omega_i = \sum_{i=1}^g p_1(\sigma_i(z)) \cup p_1(\omega_i(z)) = p_2(\phi(z)).
\]

It then follows from Theorem E.8 that the characteristic class of the holomorphic line bundle described by that factor of automorphy is \( (p \cdot p_2^{-1})(p_2(\phi(z))) = \ldots \).
\[ \int_M \phi(z) = n, \] which demonstrates the second assertion of the theorem and thereby concludes the proof.

The differential forms \( \sigma_i(z) \) can be taken to be linear combinations of the holomorphic abelian differentials \( \omega_i(z) \) and their complex conjugates as a consequence of Corollary 3.4, since only their periods really play a role; so it can be supposed that

\[
\sigma_i(z) = \sum_{j=1}^{g} \left( a_{ij} \omega_j(z) + b_{ij} \omega_j(z) \right) \quad \text{for} \quad 1 \leq i \leq g
\]

for some \( g \times g \) complex matrices \( A = \{a_{ij}\}, B = \{b_{ij}\} \), and consequently that

\[
\sigma_i(T) = \sum_{j=1}^{g} \left( a_{ij} \omega_j(T) + b_{ij} \omega_j(T) \right) \quad \text{for} \quad 1 \leq i \leq g
\]

in terms of the period classes \( \omega_j \in \text{Hom}(\Gamma, \mathbb{C}) \). Then (6.2) takes the form

\[
\sigma(T, z) = \sigma_0(T) + \sum_{i,j=1}^{g} \left( a_{ij} \omega_j(T) + b_{ij} \omega_j(T) \right) w_i(z, a),
\]

or in matrix notation

\[
\sigma(T, z) = \sigma_0(T) + \tilde{\omega}(z, a) \left( A \omega(T) + B \omega(T) \right)
\]

where \( \tilde{\omega}(z, a) = \{w_i(z, a)\} \) is the vector of abelian integrals and \( \omega(T) = \{\omega_i(T)\} \) is the vector of the periods of these integrals, both viewed as column vectors in \( \mathbb{C}^g \) as on page 64.

**Theorem 6.4** Let \( M \) be a compact Riemann surface of genus \( g > 0 \) and \( \Omega \) and \( P \) be the period matrix and intersection matrix of \( M \) in terms of some bases \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_j \in \text{H}_1(M) \). For any matrices \( A, B \in \mathbb{C}^{g \times g} \) there is a mapping \( \sigma_0 : \Gamma \rightarrow \mathbb{C} \) such that the abelian integrals (6.16) describe an abelian factor of automorphy if and only if \( i \text{tr}(\overline{AH}) \in \mathbb{Z} \), where \( H = i \Omega P \overline{\Omega} \) and \( \text{tr}(X) \) denotes the trace of the matrix \( X \); and the characteristic class of the holomorphic line bundle \( \zeta \) described by this factor of automorphy is \( c(\zeta) = i \text{tr}(\overline{AH}) \).

**Proof:** Since the homomorphisms \( \sigma_i \in \text{Hom}(\Gamma, \mathbb{C}) \) described by the matrices \( A, B \) are the period classes of the differential forms (6.13) it follows from Theorem 6.3 that these homomorphisms describe an abelian factor of automorphy for some cochain \( \sigma_0 \in C^1(\Gamma, \mathbb{C}) \) if and only if

\[
\sum_{i=1}^{g} \int_M \sigma_i(z) \wedge \omega_i(z) = n \in \mathbb{Z},
\]
and that the integer \( n \) is the characteristic class of the holomorphic line bundle described by this factor of automorphy. When the holomorphic abelian differentials are written as the sums \( \omega_i(z) = \sum_{j=1}^{g} \omega_{ij} \phi_j(z) \), where \( \phi_j(z) \in \Gamma(M, \mathcal{E}_k^1) \) are real closed \( C^\infty \) differential forms dual to the homology basis \( \tau_j \in H_1(M) \), the intersection matrix \( P \) of the surface \( M \) has the entries

\[
P_{ij} = \int_M \phi_i(z) \wedge \omega_j(z);
\]

and since \( \omega_j(z) \wedge \omega_i(z) = 0 \)

\[
\sum_{i=1}^{g} \int_M \sigma_i(z) \wedge \omega_i(z) = \sum_{i,j=1}^{g} \int_M \left( a_{ij} \omega_j(z) + b_{ij} \omega_j(z) \right) \wedge \omega_i(z)
\]

\[
= \sum_{i,j=1}^{g} \sum_{k,l=1}^{2g} \int_M \left( a_{ij} \omega_{jk} \phi_k(z) \right) \wedge \omega_{il} \phi_l(z)
\]

\[
= \sum_{i,j,k,l=1}^{g} a_{ij} \omega_{jk} \rho_{kl} \omega_{il} = \text{tr}(\Omega P^t \Omega) = i \text{tr}(AH),
\]

which suffices for the proof.

It is more convenient to write an abelian factor of automorphy as the product

\[
\zeta_{A,B,a}(T, z) = \lambda_0(T) \xi_{A,B,a}(T, z)
\]

of an auxiliary mapping

\[
\lambda_0 : \Gamma \longrightarrow \mathbb{C}^*
\]

and a root factor

\[
\xi_{A,B,a}(T, z) = \exp 2\pi i \tilde{w}(z, a) \left( A \omega(T) + B \omega(T) \right)
\]

for any matrices \( A, B \in \mathbb{C}^{g \times g} \) and any point \( a \in \tilde{M} \) on the universal covering surface of a compact Riemann surface \( M \) of genus \( g > 0 \), where \( \tilde{w}(z, a) = \{ w_i(z, a) \} \) is the vector of abelian integrals on \( M \) in terms of a basis \( \omega_i \in \Gamma(M, \mathcal{O}^{1,0}) \) for the holomorphic abelian differentials and \( \omega(T) = \{ \omega_i(T) \} \) is the corresponding vector of period classes of these differentials.

**Corollary 6.5** Let \( M \) be a compact Riemann surface of genus \( g > 0 \) and \( \Omega \) and \( P \) be the period matrix and intersection matrix of \( M \) in terms of bases \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{1,0}) \) and \( \tau_j \in H_1(M) \). Then \( \xi_{A,B,a}(T, z) \) is a root factor for some abelian factors of automorphy if and only if \( i \text{tr}(AH) \in \mathbb{Z} \) for the positive definite Hermitian matrix \( H = i \Omega P^t \Omega \), while \( B \) can be quite arbitrary. If \( \lambda_0 \) is an auxiliary mapping for this root factor, all other auxiliary mappings are the products \( \rho \lambda_0 \) for arbitrary flat factors of automorphy \( \rho \in \text{Hom}(\Gamma, \mathbb{C}^*) \). The holomorphic line bundles described by these factors of automorphy all have characteristic class \( n = i \text{tr}(AH) \); and any holomorphic line bundle of characteristic class \( n \) on the surface \( M \) can be described by an abelian factor of automorphy of this form.
Proof: This is for the most part just a restatement of the result of the preceding theorem in terms of the root factor \( \xi_{A,B,a}(T, z) \) as defined in (6.19). If \( \zeta_{A,B,a}(T, z) = \lambda_0(T)\xi_{A,B,a}(T, z) \) is an abelian factor of automorphy then \( \zeta_{A,B,a}(T, z) = \lambda_0(T)\xi_{A,B,a}(T, z) \) is another factor of automorphy if and only if the quotient

\[
\frac{\zeta_{A,B,a}'(T, z)}{\zeta_{A,B,a}(T, z)} = \frac{\lambda_0'(T)}{\lambda_0(T)} = \rho(T)
\]

is also a factor of automorphy; and that factor of automorphy must be a representation \( \rho \in \text{Hom}(\Gamma, \mathbb{C}^*) \), since it is independent of the variable \( z \in \mathbb{M} \). Any other holomorphic line bundle of the same characteristic class can be represented by the product of the factor of automorphy \( \zeta_{A,B,a}(T, z) \) and a flat factor of automorphy \( \rho \in \text{Hom}(\Gamma, \mathbb{C}^*) \), since by Corollary 3.9 any topologically trivial holomorphic line bundle can be represented by a flat factor of automorphy. That suffices for the proof.

It is clear from the definition (6.19) that the root factors \( \xi_{A,B,a}(T, z) \) satisfy

\[
\xi_{A_1+A_2,B_1+B_2,a}(T, z) = \xi_{A_1,B_1,a}(T, z) \cdot \xi_{A_2,B_2,a}(T, z)
\]

for any matrices \( A_1, A_2, B_1, B_2 \in \mathbb{C}^{g \times g} \); and since \( \tilde{w}(z, a) = \tilde{w}(z, b) + \tilde{w}(b, a) \) it is also clear that

\[
\xi_{A,B,a}(T, z) = \xi_{A,B,a}(T, b)\xi_{A,B,b}(T, z)
\]

for any points \( a, b, z \in \mathbb{M} \). Furthermore since \( \omega \in \text{Hom}(\Gamma, \mathbb{C}^g) \) depends only on the homology class represented by a covering translation \( T \in \Gamma \) the root factor also can be viewed as a function \( \xi_{A,B,a}(\tau, z) \) of homology classes \( \tau \in H_1(M) \), and

\[
\xi_{A,B,a}(\sigma + \tau, z) = \xi_{A,B,a}(\sigma, z) \cdot \xi_{A,B,a}(\tau, z)
\]

for any homology classes \( \sigma, \tau \in H_1(M) \); thus for any fixed points \( z, a \in \mathbb{M} \) the root factor \( \xi_{A,B,a}(\tau, z) \) when viewed as a function of the variable \( \tau \in H_1(M) \) is a homomorphism from the homology group \( H_1(M) \) into the multiplicative group \( \mathbb{C}^* \). For many purposes it is sufficient just to know that there are factors of automorphy representing all holomorphic line bundles; however for some purposes it is necessary or at least useful to know the explicit forms of these factors of automorphy. There are several standard forms of root factors on any compact Riemann surface of genus \( g > 0 \).

Theorem 6.6 Let \( M \) be a compact Riemann surface of genus \( g > 0 \), let \( P \) be the period matrix and \( \Omega \) be the intersection matrix of \( M \) in terms of bases \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_j \in H_1(M) \), and let \( H = i\Omega \mathcal{P}^{-1} \) and \( G = \text{exp}^{-1} = H^{-1} \). The \( g \times g \) complex matrices \( A = \frac{\mu}{g}G \) and \( B = 0 \) determine the root factor

\[
\xi_{n,a}(T, z) = \xi_{\frac{n}{g}G\omega,0,a}(T, z) = \exp \frac{2\pi i}{g} \tilde{w}(z, a)G\omega(T)
\]
for some abelian factors of automorphy. The holomorphic line bundles described by these factors of automorphy all have characteristic class $n$; and any holomorphic line bundle of characteristic class $n$ can be described by a factor of automorphy of this form. The root factor $\xi_{n,n}(T,z)$ is independent of the choice of the bases $\omega_i(z) \in \Gamma(M,\mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$.

**Proof:** The matrix $H = i\Omega P\overline{\Omega}$ is a positive definite Hermitian matrix by the Riemann Matrix Theorem, Theorem 3.20, so its inverse transpose $G = H^{-1}$ is also a positive definite Hermitian matrix. If $A = \frac{ni}{g}G = \frac{ni}{g}H^{-1}$ then $i\text{tr}(A\overline{H}) = n \in \mathbb{Z}$, so by Corollary 6.5 there are abelian factors of automorphy with the root factor (6.23), and any holomorphic line bundle of characteristic class $n$ can be represented by one of these abelian factors of automorphy. Other bases $\omega'_k(z) \in \Gamma(M,\mathcal{O}^{(1,0)})$ and $\tau'_l \in H_1(M)$ for the holomorphic abelian differentials on $M$ and the homology of $M$ are related to the bases $\omega_i(z) \in \Gamma(M,\mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ by

\[
\omega_i(z) = \sum_{k=1}^{2g} c_{ik}\omega'_k(z) \quad \text{and} \quad \tau_j = \sum_{l=1}^{2g} q_{lj}\tau'_l
\]

for some matrices $C = \{c_{ik}\} \in \text{Gl}(g,\mathbb{C})$ and $Q = \{q_{lj}\} \in \text{Gl}(2g,\mathbb{Z})$. Then if $\phi_i \in \Gamma(M,\mathcal{E}^*_i)$ are the $C^\infty$ real differential forms on $M$ dual to the homology base $\tau_j$, so that $\int_{\tau_j} \phi_i = \delta^*_j$, and if $\phi'_m$ are the differential forms dual to the homology base $\tau'_m$, these differential forms are related by $\phi_i = \sum_{m=1}^{2g} q_{mi}^*\phi'_m$ for some matrix $Q^* = \{q_{mj}^*\}$ and

\[
\delta^*_j = \int_{\tau_j} \phi_i = \sum_{m=1}^{2g} q_{mi}^*\phi'_m
\]

so I = $'QQ^*$ or equivalently $Q^* = 'Q^{-1}$. The intersection matrices $P$ and $P'$ for the two homology bases are related by

\[
p_{ij} = \int_M \phi_i \wedge \phi_j = \sum_{l,m=1}^{2g} q_{il}\phi'_l \wedge q_{mj}\phi'_m = \sum_{l,m=1}^{2g} q_{il}q_{mj}^*p'_{lm}
\]

and the period matrices $\Omega$ and $\Omega'$ are related by

\[
\omega_{ij} = \int_{\tau_j} \omega_i(z) = \sum_{k=1}^{2g} c_{ik}\omega'_k(z) = \sum_{l=1}^{2g} q_{lj}c_{ik} \int_{\tau'_l} \omega'_k = \sum_{l=1}^{2g} q_{lj}c_{ik}\omega'_{kl}
\]

thus

\[
P = 'Q^*P'Q^* = Q^{-1}P'Q^{-1} \quad \text{and} \quad \Omega = C\Omega'Q.
\]
It follows that
\[ H = i \Omega' P' \Omega' = i C\Omega' Q \cdot Q^{-1} P' \cdot Q^{-1} \cdot \overline{Q} \overline{\Omega} \overline{C} = i C\Omega' P' \Omega' \overline{C} = CH' \overline{C} \]
where \( H' = i \Omega' P' \Omega' \) is the matrix corresponding to \( H \) for the bases \( \omega_k' \) and \( \tau_l' \); hence the associated matrices \( G \) and \( G' \) are related by
\[ (6.26) \quad G = t H^{-1} = t C^{-1} (H')^{-1} \overline{C} = t C^{-1} \overline{G} \overline{C}. \]

Since \( \tilde{w}(z, a) = C \tilde{w}'(z, a) \) and \( \omega(T) = C \omega'(T) \) it follows finally that
\[ (6.27) \quad t \tilde{w}(z, a) G \omega(T) = t \tilde{w}'(z, a) t C^{-1} \overline{G} \overline{C} \cdot \overline{C} \omega'(T) = t \tilde{w}'(z, a) G' \omega'(T), \]
so the formula (6.23) for the root factor is the same in terms of any bases for the holomorphic abelian differentials on \( M \) and of the homology of \( M \) and that serves to conclude the proof.

The root factor (6.23) is called the intrinsic root factor of characteristic class \( n \) on the Riemann surface \( M \), since it is independent of the choice of bases for the holomorphic abelian differentials and for the homology of \( M \) so is really intrinsically defined on the Riemann surface \( M \). For a marked Riemann surface \( M \) the intrinsic root factor can be expressed somewhat more directly in terms of the period matrix of the surface. A marking is described by generators \( A_j, B_j \in \Gamma \) of the covering translation group \( \Gamma \) of the surface for \( 1 \leq j \leq g \), and the homology classes associated to these generators are the natural homology basis for \( M \). The period matrix of \( M \) in terms of the associated canonical basis for the holomorphic abelian differentials has the form \( \Omega = (1 \quad Z) \) where \( Z = X + i Y \) is a matrix in the Siegel upper half-space \( \mathcal{H}_g \) of rank \( g \), as in Theorem 3.22; the matrix \( Z \) is symmetric, and its imaginary part \( Y \) is a positive definite real symmetric matrix.

**Corollary 6.7** If \( M \) is a marked Riemann surface of genus \( g > 0 \) and if \( \Omega = (1 \quad Z) \) for \( Z = X + i Y \in \mathcal{H}_g \) is the period matrix of \( M \) in terms of the homology basis associated to the marking and the associated canonical basis \( \omega_i(z) \) for the holomorphic abelian differentials, the intrinsic root factor of characteristic class \( n \) can be written
\[ (6.28) \quad \xi_n, a(T, z) = \exp \frac{\pi n}{g} t \tilde{w}(z, a) Y^{-1} \omega(T). \]

**Proof:** The intersection matrix of the surface \( M \) in terms of the homology basis associated to the marking is the basic skew-symmetric matrix \( P = J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), and the period matrix in terms of the associated canonical basis \( \omega_i \in \Gamma(M, \mathcal{O}^{1,0}) \) of holomorphic abelian differentials has the form \( \Omega = (1 \quad Z) \); hence
\[ (6.29) \quad H = i \Omega P \Omega = i (1 \quad Z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ Z \end{pmatrix} = 2Y \quad \text{so} \quad G = \frac{1}{2} Y^{-1}. \]

\(^3\)The definition and properties of markings of a surface are discussed in Appendix D.1.
In terms of these bases the intrinsic root factor (6.23) is determined by the matrices \( A = \frac{n}{g} G = \frac{n}{g} Y^{-1} \) and \( B = 0 \), so has the form (6.28), and that suffices for the proof.

For some purposes it is more convenient, and in many ways it is more traditional, to use a slightly different but closely related root factor on marked Riemann surfaces.

**Theorem 6.8** Let \( M \) be a marked compact Riemann surface of genus \( g > 0 \) with the period matrix \( \Omega = (I \ Z) \) in terms of the homology basis associated to the marking and the associated canonical basis \( \omega_i(z) \) for the holomorphic abelian differentials, where \( Z = X + iY \in \mathcal{H}_g \). The \( g \times g \) complex matrices \( A = B = \frac{n}{g^2} Y^{-1} \) determine the root factor

\[
\hat{\xi}_{n,a}(T, z) = \xi_{\frac{n}{g^2} Y^{-1}, \frac{n}{g^2} Y^{-1}, a}(T, z) = \exp -\frac{2\pi i n}{g} \tilde{\omega}(z, a)Y^{-1}\Im(\omega(T))
\]

for some abelian factors of automorphy, where \( \Im(\omega(T)) \) denotes the imaginary part of the period vector \( \omega(T) \). The holomorphic line bundles described by these abelian factors of automorphy all have characteristic class \( n \), and any holomorphic line bundle of characteristic class \( n \) can be described by an abelian factor of automorphy of this form. The root factor \( \hat{\xi}_{n,a}(T, z) \) has the property that

\[
\hat{\xi}_{n,a}(A_j, z) = 1 \quad \text{for } 1 \leq j \leq 2g
\]

where \( A_j, B_j \in \Gamma \) are the generators of the marking of \( M \).

**Proof:** The intrinsic root factor is determined by the matrices \( A = \frac{n}{g^2} Y^{-1} \) and \( B = 0 \), and the associated abelian factors of automorphy describe holomorphic line bundles of characteristic class \( n \), as in Corollary 6.7; so \( \text{tr}(A^T) = n \), and it then follows from Corollary 6.6 that the matrices \( A = B = \frac{n}{g^2} Y^{-1} \) also determine a root factor, that the associated abelian factors of automorphy all describe holomorphic line bundles of characteristic class \( n \), and that any holomorphic line bundle of characteristic class \( n \) can be described by an abelian factor of automorphy of this form. Explicitly

\[
\xi_{\frac{n}{g^2} Y^{-1}, \frac{n}{g^2} Y^{-1}, a}(T, z) = \exp 2\pi i \tilde{\omega}(z, a)\left( -\frac{n}{2g} Y^{-1}\omega(T) + \frac{n}{2g} Y^{-1}\omega(T) \right)
\]

\[
= \exp -\frac{\pi n}{g} \tilde{\omega}(z, a)Y^{-1}\left( -\omega(T) + \omega(T) \right);
\]

and since \( -\omega(T) + \omega(T) = 2i \Im(\omega(T)) \) this reduces to (6.30), and that suffices for the proof.

The root factor (6.30) is called the **canonical root factor** of characteristic class \( n \) on the marked Riemann surface \( M \), since it follows the pattern for the definition of the canonical meromorphic abelian differentials in Chapters 4 and 5. This root factor though is not intrinsic to the Riemann surface, as can be verified by reviewing the last part of the proof of Theorem 6.6. However the intrinsic and canonical root factors of characteristic class \( n \) are closely related as a consequence of the following general observation.
CHAPTER 6. ABELIAN FACTORS OF AUTOMORPHY

Theorem 6.9 If $B$ is a symmetric $g \times g$ matrix and $\lambda_B : \Gamma \to \mathbb{C}^*$ is the mapping defined by

$$\lambda_B(T) = \exp -\pi i \omega(T) B \omega(T)$$

for any $T \in \Gamma$, two abelian factors of automorphy of the form $\lambda_0(T) \xi_{A,B,a}(T,z)$ and $\lambda_0(T) \lambda_B(T) \xi_{A,0,a}(T,z)$ are holomorphically equivalent.

Proof: The function

$$h_B(z) = \exp -\pi i \tilde{w}(z,a) B \tilde{w}(z,a)$$

is holomorphic and nowhere vanishing on the universal covering space $\tilde{M}$. Since the matrix $B$ is symmetric it follows that

$$h_B(Tz) = \exp -\pi i \tilde{w}(Tz,a) B \tilde{w}(Tz,a)$$
$$= \exp -\pi i \left( \tilde{w}(z,a) + \omega(T) \right) B \left( \tilde{w}(z,a) + \omega(T) \right)$$
$$= \exp -\pi i \left( \tilde{w}(z,a) B \tilde{w}(z,a) + 2 \tilde{w}(z,a) B \omega(T) + \omega(T) B \omega(T) \right)$$
$$= h_B(z) \cdot \xi_{0,-B,a}(T,z) \cdot \lambda_B(T);$$

consequently using (6.20)

$$h_B(Tz) \lambda_0(T) \xi_{A,B,a}(T,z) h_B(z)^{-1} = \xi_{0,-B,a}(T,z) \lambda_B(T) \cdot \lambda_0(T) \xi_{A,B,a}(T,z)$$
$$= \lambda_0(T) \lambda_B(T) \cdot \xi_{A,0,a}(T,z),$$

which suffices to conclude the proof.

Corollary 6.10 If $M$ is a marked Riemann surface of genus $g > 0$ and if $\Omega = (I \ Z)$ for $Z = X + i Y \in \mathbb{H}_g$ is the period matrix of $M$ in terms of the homology basis associated to the marking and the associated canonical basis $\omega_i(z)$ for the holomorphic abelian differentials, an abelian factor of automorphy $\lambda_0(T) \xi_{n,a}(T,z)$ defined by the canonical root factor is holomorphically equivalent to the abelian factor of automorphy $\lambda_0(T) \lambda_{\frac{n}{2g} Y^{-1}}(T) \xi_{n,a}(T,z)$ defined by the intrinsic root factor.

Proof: The canonical root factor is determined by the matrices $A = B = \frac{n}{2g} Y^{-1}$, as in Corollary 6.8, while the intrinsic root factor is determined by the matrices $A = \frac{n}{2g} Y^{-1}$ and $B = 0$ as in Theorem 6.6; so the asserted result is an immediate consequence of the preceding theorem, and that suffices for the proof.

There are yet other root factors, but they are even less intrinsic than the canonical abelian factor of automorphy and appear to be of less use. The simplest of them is the minimal root factor of characteristic class $n$ associated to an abelian differential $\omega$ on the Riemann surface $M$; it is minimal in the sense that it involves only the integral and the period class of the single holomorphic abelian differential $\omega$, and it has the following explicit form.
Theorem 6.11 If $M$ is a compact Riemann surface $M$ of genus $g > 0$ and $\omega(z) = dw(z,a)$ is a holomorphic abelian differential having the period class $\omega \in \text{Hom}(\Gamma, \mathbb{C})$ there is a positive constant $h > 0$ such that

$$(6.34) \quad \xi_{n,a,\omega}(T,z) = \exp \frac{2\pi n}{h}w(z,a)\omega(T).$$

is the root factor for some abelian factors of automorphy. The holomorphic line bundles described by these factors of automorphy all have characteristic class $n$; and any holomorphic line bundle of characteristic class $n$ can be described by an abelian factor of automorphy of this form.

Proof: Choose a basis $\omega_i(z)$ for the holomorphic abelian differentials on $M$ such that $\omega_1(z) = \omega(z)$, and choose any basis for the homology of $M$. If $P$ is the intersection matrix and $\Omega$ is the period matrix of $M$ in terms of these bases then $H = \{h_{ij}\} = i\Omega P \Omega^t$ is positive definite Hermitian matrix, so the entry $h_{11}$ is a strictly positive real number. If $A$ is the $g \times g$ complex matrix with the single nonzero entry $a_{11} = nih^{-1}$ then the matrix $AH$ has the first row

$$n_i, n_i h_{12}, n_i h_{13}, \ldots, n_i h_{1g},$$

while all entries on the remaining rows are zero; consequently $i \text{tr}(AH) = n$, so by Corollary 6.5 this matrix $A$ and the matrix $B = 0$ determine the root factor

$$\xi_{A,B,a}(T,z) = \exp \frac{2\pi n}{h}w_1(z,a)\omega_1(T)$$

for some abelian factors of automorphy. The holomorphic line bundles described by these abelian factors of automorphy all have characteristic class $n$, and as in Corollary 6.5 any holomorphic line bundle with characteristic class $n$ can be represented by an abelian factor of automorphy of this form. That suffices for the proof.

The more detailed examination of the auxiliary mappings associated to any of these root factors is in some ways a bit more complicated but in many ways a bit more interesting than the examination of the root factors themselves. The condition that (6.17) is a factor of automorphy is just that

$$\lambda_0(ST)\xi_{A,B,a}(ST,z) = \lambda_0,\lambda_0(A,B,a)(S,Tz) \cdot \lambda_0,\lambda_0(A,B,a)(T,z),$$

for all $S, T \in \Gamma$, or equivalently in view of (6.22) and (6.19) that

$$\lambda_0(ST) = \lambda_0(S)\lambda_0(T) \frac{\xi_{A,B,a}(S,Tz)\xi_{A,B,a}(T,z)}{\xi_{A,B,a}(ST,z)},$$

for all $S, T \in \Gamma$.
for all $S, T \in \Gamma$; and since $\hat{w}(Tz, a) - \hat{w}(z, a) = \omega(T)$ this can be written

$$
\lambda_0(ST) = \lambda_0(S)\lambda_0(T) \exp \phi_{A,B}(S, T)
$$

in terms of the homomorphism $\phi_{A,B} \in \text{Hom}(\Gamma \times \Gamma, \mathbb{C})$ defined by

$$
\phi_{A,B}(S, T) = 2\pi i \omega(T) \left( A\omega(S) + B\omega(S) \right).
$$

The existence of a mapping $\lambda_0 : \Gamma \rightarrow \mathbb{C}^*$ satisfying (6.35) is a cohomological condition, which of course is equivalent to the cohomological condition for the existence of the mapping $\sigma_0$ given in Theorem 6.4. If $T_1, \ldots, T_{2g} \in \Gamma$ is a set of generators of the covering translation group representing a basis $\tau_1, \ldots, \tau_{2g} \in H_1(M)$, it follows from (6.35) that an auxiliary mapping for the root factor $\xi_{A,B,a}(T, z)$ is determined fully by its values on the generators $T_j$ for $1 \leq j \leq 2g$; and since an auxiliary mapping can be modified by multiplying by an arbitrary homomorphism in $\text{Hom}(\Gamma, \mathbb{C}^*)$, there is a uniquely determined auxiliary mapping $\lambda_{0,\{T_j\}}$ normalized by requiring that

$$
\lambda_{0,\{T_j\}}(T_j) = 1 \quad \text{for } 1 \leq j \leq 2g.
$$

A covering translation $T \in \Gamma$ represents a cohomology class $\tau = \sum_{j=1}^{2g} n_j \tau_j \in H_1(M)$, so the quotient $C = (T_1^{n_1} \cdots T_{2g}^{n_{2g}})^{-1} T \in \Gamma$ represents the trivial cohomology class in $H_1(M) \cong \Gamma/\Gamma, \Gamma]$ and consequently lies in the commutator subgroup $[\Gamma, \Gamma]$; thus any covering translation $T \in \Gamma$ can be written uniquely as the product $T = T_1^{n_1} \cdots T_{2g}^{n_{2g}} C$ for some integers $n_j \in \mathbb{Z}$ and an element $C \in [\Gamma, \Gamma]$.

**Theorem 6.12** The normalized auxiliary mapping $\lambda_{0,\{T_j\}}$ for the root factor $\xi_{A,B,a}(T, z)$ in terms of generators $T_j \in \Gamma$ is given by

$$
\lambda_{0,\{T_j\}}(T_1^{n_1} \cdots T_{2g}^{n_{2g}} C) =
$$

$$
= \lambda_{0,\{T_j\}}(C) \cdot \exp \left( \sum_{j=1}^{2g} \frac{n_j}{2} \phi_{A,B}(T_j, T_j) + \sum_{1 \leq j < k \leq 2g} n_j n_k \phi_{A,B}(T_j, T_k) \right)
$$

in terms of the homomorphism $\phi_{A,B} \in \text{Hom}(\Gamma \times \Gamma, \mathbb{C})$ for any $C \in [\Gamma, \Gamma]$.

**Proof:** It will be demonstrated first by induction on $n$ that if $\lambda_{0,\{T_j\}}(S) = 1$ then

$$
\lambda_{0,\{T_j\}}(S^n) = \exp \left( \frac{n}{2} \phi_{A,B}(S, S) \right) \quad \text{for } n \geq 1.
$$

Since $\binom{n}{2} = 0$ for $n = 1$ this clearly holds for the initial case $n = 1$. If (6.39) holds for $n = 1$ and for some $n \geq 1$ then since $\phi_{A,B}(S, S^n) = n\phi_{A,B}(S, S)$ it
follows from (6.35) for \( T = S^n \) that
\[
\lambda_{0,\{T_j\}}(S^{n+1}) = \lambda_{0,\{T_j\}}(S \cdot S^n) = \lambda_{0,\{T_j\}}(S) \cdot \lambda_{0,\{T_j\}}(S^n) \cdot \exp \phi_{A,B}(S, S^n)
\]
\[
= 1 \cdot \exp \left( \frac{n}{2} \phi_{A,B}(S, S) \right) \cdot \exp \left( n \phi_{A,B}(S, S) \right)
\]
\[
= \exp \left( \frac{n}{2} \phi_{A,B}(S, S) + n \phi_{A,B}(S, S) \right);
\]
and \( \binom{n}{2} + n = \binom{n+1}{2} \) so that establishes (6.39) for \( n + 1 \) and hence also demonstrates the inductive step. Next it will be demonstrated by induction on the integer \( l \) in the range \( 1 \leq l \leq 2g \) that
\[
\lambda_{0,\{T_j\}}(T_1^{n_1} \cdot \ldots \cdot T_l^{n_l}) = \exp \left( \sum_{j=1}^{l} \frac{n_j}{2} \phi_{A,B}(T_j, T_j) + \sum_{1 \leq j < k \leq l} n_j n_k \phi_{A,B}(T_j, T_k) \right)
\]
This holds for \( l = 1 \) by (6.39). If it is assumed to hold for \( l = 1 \) and for some \( l < 2g \) then since \( \phi_{A,B}(T_1^{n_1} \cdot \ldots \cdot T_l^{n_l} \cdot T_{l+1}^{n_{l+1}}) = \sum_{j=1}^{l} n_j n_{l+1} \phi_{A,B}(T_j, T_{l+1}) \) it follows from (6.35) and (6.39) that
\[
\lambda_{0,\{T_j\}}(T_1^{n_1} \cdot \ldots \cdot T_l^{n_l} \cdot T_{l+1}^{n_{l+1}}) = \lambda_{0,\{T_j\}}(T_1^{n_1} \cdot \ldots \cdot T_l^{n_l}) \cdot \lambda_{0,\{T_j\}}(T_{l+1}^{n_{l+1}}) \cdot \exp \phi_{A,B}(T_1^{n_1} \cdot \ldots \cdot T_l^{n_l}, T_{l+1}^{n_{l+1}})
\]
\[
= \exp \left( \sum_{j=1}^{l} \frac{n_j}{2} \phi_{A,B}(T_j, T_j) + \sum_{1 \leq j < k \leq l} n_j n_k \phi_{A,B}(T_j, T_k) \right)
\]
\[
+ \left( \frac{n_{l+1}}{2} \right) \phi_{A,B}(T_{l+1}, T_{l+1}) + \sum_{j=1}^{l} n_j n_{l+1} \phi_{A,B}(T_j, T_{l+1}^{n_{l+1}}),
\]
and that is equivalent to (6.40) for \( l + 1 \) so establishes the inductive step. In particular (6.40) holds for \( l = 2g \) and since \( \phi_{A,B}(T, C) = 1 \) for any \( T \in \Gamma \) and any \( C \in [\Gamma, \Gamma] \) it follows from (6.35) that \( \lambda_{0,\{T_j\}}(T C) = \lambda_{0,\{T_j\}}(T) \lambda_{0,\{T_j\}}(C) \) for any \( T \in \Gamma \) and \( C \in [\Gamma, \Gamma] \), so (6.38) follows immediately from (6.40) for \( l = 2g \), and that suffices for the proof.

What is not determined in the preceding theorem is the value \( \zeta_{0,\{T_j\}}(C, z) \) for a commutator \( C \in [\Gamma, \Gamma] \). If \( S, T \in [\Gamma, \Gamma] \) then \( \omega(S) = \omega(T) = 0 \) so \( \phi_{A,B}(S, T) = 1 \), and it then follows from (6.35) that the restriction of the auxiliary mapping \( \lambda_0 \) to the commutator subgroup is a homomorphism
\[
\lambda_0|_{[\Gamma, \Gamma]} \in \text{Hom}([\Gamma, \Gamma], \mathbb{C}^*);
\]
and since \( \xi_{A,B,a}(C, z) = 1 \) whenever \( C \in [\Gamma, \Gamma] \) it is clear that
\[
\zeta_{A,B,a}(C, z) = \lambda_0(C) \quad \text{for all } C \in [\Gamma, \Gamma].
\]
Theorem 6.13  Any auxiliary mapping $\lambda_0$ for the root factor $\xi_{A,B,a}(T,z)$ satisfies

\[ \lambda_0([S,T]) = \exp \left( \phi_{A,B}(S,T) - \phi_{A,B}(T,S) \right) \]  

in terms of the homomorphism $\phi_{A,B} \in \text{Hom}(\Gamma \times \Gamma, C)$ for any $S$, $T \in \Gamma$, and

\[ \lambda_0(TCT^{-1}) = \lambda_0(C) \]

for any $C \in [\Gamma, \Gamma]$; hence the homomorphism $\lambda_0|_{[\Gamma, \Gamma]} \in \text{Hom}([\Gamma, \Gamma], \mathbb{C}^*)$ is determined fully by the matrices $A$ and $B$ alone.

Proof: It follows from (6.19) that

\[ \xi_{A,B,a}(S,Tz) = \exp 2\pi i \overline{\phi(S)} \left( \omega(S) + B\omega(S) \right) \]

hence that

\[ \xi_{A,B,a}(S,Tz) = \xi_{A,B,a}(S,z) \exp \phi_{A,B}(S,T). \]

Since $\zeta(T^{-1},z) = \zeta(T,T^{-1}z)^{-1}$ for any factor of automorphy

\[ \zeta([S,T],z) = \zeta(ST \cdot (TS)^{-1},z) = \zeta(ST, S^{-1}T^{-1}z) \cdot \zeta(TS, S^{-1}T^{-1}z)^{-1} \]

\[ = \frac{\zeta(S, TS^{-1}T^{-1}z) \cdot \zeta(T, S^{-1}T^{-1}z)}{\zeta(T, T^{-1}z) \cdot \zeta(S, S^{-1}T^{-1}z)}. \]

In particular for the factor of automorphy (6.17)

\[ \lambda_0([S,T]) = \xi_{A,B,a}([S,T],z) \]

\[ = \frac{\lambda_0(S)\xi_{A,B,a}(S, TS^{-1}T^{-1}z) \cdot \lambda_0(T)\xi_{A,B,a}(T, S^{-1}T^{-1}z)}{\lambda_0(T)\xi_{A,B,a}(T, T^{-1}z) \cdot \lambda_0(S)\xi_{A,B,a}(S, S^{-1}T^{-1}z)} \]

\[ = \frac{\xi_{A,B,a}(S,z) \exp \phi_{A,B}(S, TS^{-1}T^{-1}z) \cdot \xi_{A,B,a}(T,z) \exp \phi_{A,B}(T, S^{-1}T^{-1})}{\xi_{A,B,a}(T,z) \exp \phi_{A,B}(T, T^{-1}) \cdot \xi_{A,B,a}(S,z) \exp \phi_{A,B}(S, S^{-1}T^{-1})} \]

\[ = \exp \left( \phi_{A,B}(S,T) - \phi_{A,B}(T,S) \right), \]

and that establishes the identity (6.43). Then for any $C \in [\Gamma, \Gamma]$ by (6.42)

\[ \lambda_0(TCT^{-1}) = \xi_{A,B,a}(TCT^{-1},z) \]

\[ = \xi_{A,B,a}(T, CT^{-1}z) \cdot \xi_{A,B,a}(C, T^{-1}z) \]

\[ = \frac{\lambda_0(T)\xi_{A,B,a}(T, CT^{-1}z) \cdot \lambda_0(C)}{\lambda_0(T)\xi_{A,B,a}(T, T^{-1}z)} \]

\[ = \lambda_0(C), \]
which establishes the identity (6.44). The subgroup \([\Gamma, \Gamma] \subset \Gamma\) is the normal subgroup generated by commutators, so it is evident from (6.43) and (6.44) that the homomorphism \(\lambda_0|_{[\Gamma, \Gamma]}\) is determined fully by the matrices \(A\) and \(B\), and that suffices to conclude the proof.

The simplest abelian factors of automorphy are those for which the auxiliary mapping is the identity homomorphism on the commutator subgroup, hence for which

\[(6.46)\]  
\[\zeta(C, z) = \lambda_0(C) = 1 \quad \text{for all } C \in [\Gamma, \Gamma].\]

Since any two auxiliary mappings for a given root factor differ by the product with a representation \(\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)\), and every such representation is the identity homomorphism on the commutator subgroup, it is evident that the condition (6.46) depends solely on the root factor; the root factors for which all auxiliary mappings satisfy (6.46) are called hyperabelian root factors, and the associated abelian factors of automorphy are called hyperabelian factors of automorphy.

Lemma 6.14 If \(\zeta(T, z)\) is a hyperabelian factor of automorphy for a compact Riemann surface \(M\) of genus \(g > 0\) then for any covering translations \(S, T \in \Gamma\) and \(C \in [\Gamma, \Gamma]\)

\[(6.47)\]  
\[\zeta(CT, z) = \zeta(TC, z) = \zeta(T, Cz) = \zeta(T, z)\]

and

\[(6.48)\]  
\[\zeta(ST, z) = \zeta(TS, z);\]

consequently the factor of automorphy \(\zeta(T, z)\) depends only on the homology class \(\tau \in H_1(M)\) represented by a covering translation, so it can be viewed as a function \(\zeta(\tau, z)\) of \(\tau \in H_1(M)\) and \(z \in \tilde{M}\).

Proof: If \(\zeta(T, z)\) is a hyperabelian factor of automorphy then for any covering translations \(T \in \Gamma\) and \(C \in [\Gamma, \Gamma]\) it follows from (6.46) that

\[\zeta(CT, z) = \zeta(C, Tz)\zeta(T, z) = \zeta(T, z)\]

and

\[\zeta(TC, z) = \zeta(TCT^{-1}, T, z) = \zeta(TCT^{-1}, Tz)\zeta(T, z) = \zeta(T, z)\]

since \(TCT^{-1} \subset [\Gamma, \Gamma] \subset \Gamma\) also, and then \(\zeta(T, Cz) = \zeta(TC, z)\zeta(C, z)^{-1} = \zeta(T, z)\). Furthermore \(\zeta(ST, z) = \zeta([S, T] \cdot TS, z) = \zeta([S, T], TSz)\zeta(TS, z) = \zeta(TS, z)\) for any \(S, T \in \Gamma\). That suffices for the proof.
Theorem 6.15 Let $M$ be a compact Riemann surface of genus $g > 0$ and let $Ω$ and $P$ be the period matrix and intersection matrix of $M$ in terms of bases $ω_i(z) ∈ Γ(M, O^{1,0})$ and $τ_j ∈ H_1(M)$. The function $ξ_{A,B,a}(T, z)$ is a hyperabelian root factor if and only if the matrices $A, B ∈ C^{g×g}$ satisfy

\[(6.49) \quad ΩATA - ΩΩΩ = N ∈ Z^{2g×2g} \quad \text{and} \quad B = tB;\]

and the holomorphic line bundles described by the abelian factors of automorphy with this root factor all have characteristic class $\frac{1}{2}\text{tr}(NP)$.

**Proof:** The intersection matrix $P$ is integral and skew-symmetric, so for the matrix $H = ΩPΩ$:

\[i \text{tr}(AΩH) = \text{tr}(AΩPΩ) = \text{tr}(PΩAΩ) = \text{tr}′(PΩAΩ)\]

\[= -\text{tr}(ΩΩPNP) = -\text{tr}(ΩΩΩΩN) \cdot P\]

\[= -\text{tr}(ΩΩΩΩP) + \text{tr}(NP) = -i \text{tr}(AΩH) + \text{tr}(NP)\]

where $N$ is the matrix (6.49), and consequently

\[(6.50) \quad i \text{tr}(AΩH) = \frac{1}{2} \text{tr}(NP).\]

By Corollary 6.5 the function $ξ_{A,B,a}(T, z)$ is a root factor for some abelian factors of automorphy if and only if the matrices $A, B ∈ C^{g×g}$ satisfy $i \text{tr}(AΩH) ∈ Z$, and the holomorphic line bundles described by these abelian factors of automorphy all have characteristic class $\frac{1}{2}\text{tr}(NP)$. It then follows from (6.50) that the function $ξ_{A,B,a}(T, z)$ is a root factor for some abelian factors of automorphy if and only if the matrices $A, B ∈ C^{g×g}$ are such that $\frac{1}{2}\text{tr}(NP) ∈ Z$, and the holomorphic line bundles described by these abelian factors of automorphy all have the characteristic class $\frac{1}{2}\text{tr}(NP)$. This does not require that the matrix $N$ be integral, of course. However it is clear from (6.49) that the matrix $N$ is skew-symmetric, and since $P$ also is skew-symmetric it follows that $n_{ij}p_{ji} = (−n_{ji})(−p_{ij}) = n_{ji}p_{ij}$ and consequently that

\[\frac{1}{2}\text{tr}(NP) = \frac{1}{2} \sum_{i,j=1}^{2g} n_{ij}p_{ji} = \frac{1}{2} \sum_{1 ≤ i,j ≤ 2g} n_{ij}p_{ji}\]

\[= \frac{1}{2} \sum_{1 ≤ i,j ≤ 2g} (n_{ij}p_{ji} + n_{ji}p_{ij}) = \sum_{1 ≤ i,j ≤ 2g} n_{ij}p_{ji}.\]

Therefore if $N$ is an integral matrix then $\frac{1}{2}\text{tr}(NP) ∈ Z$ since $P$ is also integral, hence the function $ξ_{A,B,a}(T, z)$ is the root factor for some abelian factors of automorphy and the holomorphic line bundles described by these factors of automorphy all have characteristic class $\frac{1}{2}\text{tr}(NP)$.

If $T_k ∈ Γ$ for $1 ≤ k ≤ 2g$ are generators of the covering translation group that represent a basis for $H_1(M)$ then since $[Γ, Γ] ⊂ Γ$ is the normal subgroup of $Γ$ generated by the commutators $[T_k, T_l]$ and the auxiliary mappings for any root factors restrict to homomorphisms on the commutator subgroup it is evident from Theorem 6.13 that a root factor $ξ_{A,B,a}(T, z)$ is hyperabelian if and only if

\[(6.51) \quad \exp \left( φ_{A,B}(T_k, T_l) - φ_{A,B}(T_l, T_k) \right) = 1 \quad \text{for} \quad 1 ≤ k, l ≤ 2g.\]
The homomorphism $\phi_{A,B}$ is defined as in (6.36) and $\omega(T_k) = \Omega \delta_k$ where $\delta_k$ are the column vectors of the identity matrix, so

$$\phi_{A,B}(T_k, T_l) = 2\pi i \omega(T_l) \left( A\omega(T_k) + B\omega(T_k) \right) = 2\pi i \delta_l \left( A\Omega \delta_k + B\Omega \delta_k \right);$$

consequently (6.51) is equivalent to the condition that the difference

$$\tilde{n}_{lk} = \delta_l \left( A\Omega \delta_k + B\Omega \delta_k \right) - \delta_k \left( A\Omega \delta_l + B\Omega \delta_l \right)$$

is an integer for any indices $k, l$, or in matrix terms that the $2g \times 2g$ matrix $\tilde{N} = \{ \tilde{n}_{lk} \}$ defined by

$$\tilde{N} = \left( \begin{array}{cc} \Omega(A\Omega + B\Omega) & -\Omega(A\Omega + B\Omega) \\ \Omega B\Omega - \Omega A\Omega & -\Omega B\Omega \\ \Omega & -\Omega \end{array} \right)$$

is an integral matrix. Note that if $\Pi$ is the inverse period matrix to the period matrix $\Omega$, the $g \times 2g$ matrix $\Pi$ for which

$$\Pi \Pi = \left( \begin{array}{cc} \Omega & -\Omega \\ \Omega & -\Omega \end{array} \right)$$

as in Theorem F.12 in Appendix F.1, then equation (6.52) can be rewritten

$$(6.53) \begin{pmatrix} B - \Omega B & \Omega \\ -A & 0 \end{pmatrix} = \begin{pmatrix} \Pi & \Pi \end{pmatrix} \tilde{N} \begin{pmatrix} \Pi & \Pi \end{pmatrix} = \begin{pmatrix} \Pi N \Pi & \Pi N \Pi \end{pmatrix};$$

consequently the matrix $\tilde{N}$ always satisfies $\Pi \tilde{N} \Pi = 0$. The defining equation (6.49) for the matrix $N$ can be written

$$N = \left( \begin{array}{cc} \Omega & -\Omega \\ -\Omega & \Omega \end{array} \right) \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \begin{pmatrix} \Omega & -\Omega \\ \Omega & -\Omega \end{pmatrix};$$

and a comparison of (6.52) and (6.54) shows that $N = \tilde{N}$ whenever $B = \Omega B$. If $\xi_{A,B,a}(T, z)$ is a hyperabelian root factor then as just noted the matrix $\tilde{N}$ is integral so it follows from (6.53) that $B - \Omega B = \Pi \tilde{N} \Pi = \Pi N \Pi = 0$ hence that the matrix $B$ is symmetric; therefore $\tilde{N} = N$ so the matrix $N$ also is integral, and consequently the matrices $A$ and $B$ satisfy (6.49). Conversely if the matrices $A, B$ satisfy (6.49) then since $N$ is integral it follows from (6.50) as in the discussion following that equation that $\xi_{A,B,a}(T, z)$ is the root factor for some abelian factors of automorphy, all of which describe holomorphic line bundles of characteristic class $\frac{1}{2} \text{tr}(NP)$; and since $B$ is symmetric then $\tilde{N} = N$ so $\tilde{N}$ is an integral matrix and consequently the abelian factor of automorphy $\zeta_{A,B,a}(T, z)$ is hyperabelian. That suffices to conclude the proof.
Corollary 6.16 Let $M$ be a compact Riemann surface of genus $g > 0$ and $\Omega$ and $P$ be the period matrix and intersection matrix of $M$ in terms of bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$; and let $\Pi$ be the inverse period matrix to $\Omega$. The function $\xi_{A,B,a}(T,z)$ is a hyperabelian root factor if and only if

\[(6.55)\quad A = \Pi N \Pi \quad \text{and} \quad B = ^tB\]

where

\[(6.56)\quad N \in \mathbb{Z}^{2g \times 2g}, \quad ^tN = -N, \quad \text{and} \quad \Pi N \Pi = 0;\]

and the holomorphic line bundles described by the associated hyperabelian factors of automorphy all have characteristic class $\frac{1}{2} \text{tr}(PN)$.

Proof: If matrices $A, B$ and $N$ satisfy (6.49) the integral matrix $N$ defined by that equation clearly is skew-symmetric. Since (6.49) can be written in the form (6.54), which is the same as the equation (6.52) for $\tilde{N}$ with $^tB = B$, it follows that the matrix $N$ satisfies (6.53) with $^tB = B$ as well, hence that $\Pi N \Pi = 0$ and $A = \Pi N \Pi$; thus the matrices $A, B$ and the matrix $N$ defined by (6.49)satisfy (6.55) and (6.56). Conversely if the matrices $A, B$ and $N$ satisfy (6.55) and (6.56) then

\[
\Omega \Omega - \Omega \Omega = \Omega N \Omega - \Omega \Omega \Omega \\
= \Omega N \Omega + \Omega \Omega \Omega \\
= \left( \Omega \Omega + \Omega \Omega \right) N \left( \Omega \Omega + \Omega \Omega \right) \\
= N
\]

since $^tN = -N$ and $\Pi N \Pi = \Pi N \Pi = 0$ by (6.56) while $\Omega \Omega + \Omega \Omega = I$ by (F.7); thus the matrices $A, B$ and $N$ satisfy (6.49). The desired result then follows from Theorem 6.15, and that suffices for the proof.

Since the matrix $B$ in a hyperabelian factor of automorphy is symmetric by the preceding corollary, it follows from Theorem 6.9 that an abelian factor of automorphy $\lambda_0(T)\xi_{A,B,a}(T,z)$ is holomorphically equivalent to the abelian factor of automorphy $\lambda_B(T)\lambda_0(T)\xi_{A,0,a}(T,z)$, where $\lambda_B : \Gamma \to \mathbb{C}^*$ is the mapping (6.32); and since it is evident from (6.32) that $\lambda_B(T) = 1$ whenever $T \in [\Gamma, \Gamma]$, the factor of automorphy $\lambda_B(T)\lambda_0(T)\xi_{A,0,a}(T,z)$ is also hyperabelian, so there is no loss of generality in considering only hyperabelian root factors for which $B = 0$. It follows from the preceding corollary that these root factors are described by matrices $N$ satisfying (6.56), which is just Riemann’s equality for the period matrix $\Pi$, one part of the characterization of a Riemann matrix as discussed in Appendix F.3. By the Riemann Matrix Theorem, Theorem 3.20, the period matrix $\Omega$ of a Riemann surface $M$ in terms of any bases for the homology of $M$ and for the space of holomorphic abelian differentials on $M$ is a Riemann matrix with principal matrix the intersection matrix $P$ of the surface, for which $\det P = 1$. It then follows from Theorem F.20 that the inverse period matrix
Π is also a Riemann matrix, with principal matrix $\text{tr}P^{-1}$; and consequently this principal matrix determines a hyperabelian root factor for the Riemann surface $M$. Thus there do exist hyperabelian root factors for any Riemann surface of genus $g > 1$, and in particular there exists a natural and intrinsic such root factor, which is most conveniently taken to be the root factor described by the matrix $N = -\text{tr}P^{-1} = P^{-1}$.

**Corollary 6.17** The intrinsic root factor $\xi_{g,a}(T, z)$ of a compact Riemann surface $M$ of genus $g > 0$ is a hyperabelian root factor.

**Proof:** As just noted, the matrix $N = P^{-1}$ satisfies (6.56), so by Corollary 6.16 it determines a hyperabelian root factor $\xi_{A,0,a}(T, z)$ where $A = \Pi P^{-1} \overline{\Pi}$. By (F.39)

\begin{equation}
(6.57) \quad P^{-1} = i \left( 4 \overline{H}^{-1} \Omega - 4 \overline{\Omega} \overline{H}^{-1} \Omega \right),
\end{equation}

while $\Pi \Omega = 0$ and $\Pi \overline{\Pi} = \Omega \overline{\Pi} = I$ by (F.6) so

\begin{equation}
(6.58) \quad A = \Pi P^{-1} \overline{\Pi} = i \Pi \left( \overline{H}^{-1} \Omega - \overline{\Omega} \overline{H}^{-1} \Omega \right) \overline{\Pi} = i H^{-1} = iG = i\overline{G}
\end{equation}

where $G = \overline{H}^{-1}$ for the matrix $H = i \Omega P \overline{\Omega}$; and by Theorem 6.6 the root factor $\xi_{A,0,a}(T, z)$ for the matrix $A = i \overline{G}$ is the intrinsic root factor $\xi_{g,a}(T, z)$, which concludes the proof.

If $\xi_{g,a}(T, z)$ is a hyperabelian root factor then so is $\xi_{g,a}(T, z)^{\nu}$ for any integer $\nu \in \mathbb{Z}$; these are the hyperabelian root factors corresponding to the matrices $\nu P^{-1}$ that also satisfy (6.56). For special Riemann matrices $\Pi$ there may be still other matrices satisfying (6.56). By Theorem F.16 (ii) the integral matrices $N$ such that $\Pi N \overline{\Pi} = 0$ are precisely the integral matrices that are part of a Hurwitz relation $(A, N)$ from the period matrix $\Omega$ of the surface $M$ to its inverse period matrix $\Pi$, where $A = \Pi N \overline{\Pi}$; so these matrices are associated to holomorphic mappings $A : J(\Omega) \longrightarrow J(\Pi)$, holomorphic mappings from the Jacobi variety $J(M)$ to the Picard variety $P(M)$ of the surface $M$. The particular mappings $A$ for which the matrix $N$ is skew-symmetric describe all the hyperabelian root factors for the surface $M$. The question of the existence of further hyperabelian root factors thus reduces to the question of the existence of special holomorphic mappings between the Jacobi variety and the Picard variety of a compact Riemann surface. Of course through the biholomorphic mapping between the Picard and Jacobi varieties of Theorem 3.23 these special mappings between the Jacobi and Picard varieties can be described equivalently as special automorphisms of the Jacobi variety. The further exploration of this topic will be postponed to another point, though, and the discussion here will turn instead to the examination of the hyperabelian factors of automorphy associated to the intrinsic hyperabelian root factors.

By Lemma 6.14 hyperabelian factors of automorphy $\zeta(T, z) = \zeta(\tau, z)$ can be viewed as functions of homology classes $\tau \in H^1(M)$ rather than of elements
of the covering translation group $\Gamma$; so the auxiliary mappings $\lambda_0$ for any for any hyperabelian root factor can also be viewed correspondingly as mappings $\lambda_0 : H_1(M) \to \mathbb{C}^*$. The normalization (6.37) consequently can be expressed in terms of the homology group rather than of the covering translation group; thus for any hyperabelian root factor $\xi(T,z)$ for a compact Riemann surface $M$ of genus $g > 0$ there is a uniquely determined auxiliary mapping $\lambda_{0,(\tau_j)}$ normalized by requiring that

$$\lambda_{0,(\tau_j)}(\tau_j) = 1 \quad \text{for} \quad 1 \leq j \leq 2g. \tag{6.59}$$

It is convenient for the subsequent discussion to introduce the symmetrization $\hat{P}$ of the inverse transpose $P^* = tP^{-1}$ of the intersection matrix $P$ of the surface $M$, the symmetric matrix $\hat{P} = \{\hat{p}_{jk}\}$ defined by

$$\hat{p}_{jk} = \begin{cases} p^*_{jk} & \text{if} \ j < k, \\ 0 & \text{if} \ j = k, \\ -p^*_{jk} & \text{if} \ j > k \end{cases} \tag{6.60}$$

where $P^* = \{p^*_{jk}\}$. It is evident that all the entries in the matrix $\hat{P} \pm P^*$ are even integers. As a convenient notation, for any square matrix $R$ let $\text{diag}(R)$ denote the column vector formed by the diagonal entries of the matrix $R$.

**Theorem 6.18** Let $P$ be the intersection matrix and $\Omega$ be the period matrix of a compact Riemann surface $M$ of genus $g > 0$ in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}(1,0))$ and $\tau_j \in H_1(M)$, let $\hat{P}$ be the symmetrization of the matrix $P^* = tP^{-1}$, let $G = tH^{-1} = H^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\Omega$, let $R$ be the $2g \times 2g$ real symmetric matrix $R = 2\pi\Re(\Omega G\Omega)$, and let $r = \text{diag}(R) = \{r_{jj}\} \in \mathbb{R}^{2g}$ be the column vector formed from the diagonal entries of the matrix $R$. The normalized auxiliary mapping for the hyperabelian root factor $\xi_{g,a}(T,z)$ in terms of the basis $\tau_j$ is given by

$$\lambda_{g,(\tau_j)}(\tau) = \epsilon_{\{\tau_j\}}(\tau) \cdot \exp \left( i n\tau^2 \hat{P} \right) \quad \text{for any} \ \tau = \sum_{j=1}^{2g} n_j \tau_j \tag{6.61}$$

where

$$\epsilon_{\{\tau_j\}}(\tau) = \exp \pi i \sum_{1 \leq j < k \leq 2g} p^*_{jk}n_jn_k = \exp \frac{\pi i}{2} n\hat{P}n = \pm 1. \tag{6.62}$$

**Proof:** It follows from Theorem 6.12 that the auxiliary mapping $\lambda_{g,(\tau_j)}$ takes the form (6.38) for the matrices $A = i\hat{G}$ and $B = 0$, where $\lambda_{g,(\tau_j)}(C) = 1$ since the factor of automorphy is hyperabelian; thus if $\tau = \sum_{j=1}^{2g} n_j \tau_j$

$$\lambda_{g,(\tau_j)}(\tau) = \exp \left( \sum_{j=1}^{2g} \frac{n_j}{2} \phi_{i\hat{G},0}(T_j, T_j) \right) + \sum_{1 \leq j < k \leq 2g} n_jn_k \phi_{i\hat{G},0}(T_j, T_k). \tag{6.63}$$
The homomorphism \( \phi_{\mathcal{M},0} \) as defined in (6.36) can be viewed as a homomorphism \( \phi_{\mathcal{M},0} \in \text{Hom}(H_1(M) \times H_1(M), \mathbb{C}^*) \) and takes the values

\[
\phi_{\mathcal{M},0}(\tau_j, \tau_k) = 2\pi \imath \omega(\tau_k) G\omega(\tau_j) = 2\pi \delta_k \Omega G\Omega_j = \phi_{kj}
\]

where \( \phi_{kj} \) are the entries in the \( 2g \times 2g \) complex matrix

\[
\Phi = 2\pi \imath \Omega G\Omega.
\]

Consequently (6.63) can be rewritten

\[
\lambda_{\mathcal{M},(\tau_j)}(\tau) = \exp \left( \sum_{j=1}^{2g} r_{jj} \left( \frac{n_j}{2} \right) + \sum_{1 \leq j < k \leq 2g} r_{kj} n_j n_k + \pi i \sum_{1 \leq j < k \leq 2g} p_{jk}^* n_j n_k \right),
\]

and in these terms (6.57) reduces to

\[
2\pi i P^{-1} = 2\pi \left( \Omega G\Omega - \overline{\Omega G\Omega} \right) = \Phi - \overline{\Phi}.
\]

Thus

\[
\Phi = R + \pi i P^{-1}
\]

where \( R = \Re(\phi) = \{ r_{kj} \} \) is the real part of the matrix \( \Phi \); and \( \imath \Phi = 2\pi \imath \Omega G\Omega = 2\pi \imath \Omega G\Omega = \overline{\Phi} \) so \( \Phi \) is a Hermitian complex matrix, hence its real part \( R \) is a symmetric real matrix. If \( \imath P^{-1} = P^* = \{ p_{kj}^* \} \) then \( \phi_{kj} = r_{kj} + \pi p_{jk}^* \) by (6.67), and since \( P^* \) is skew-symmetric \( p_{jj}^* = 0 \) so (6.66) can be rewritten

\[
\lambda_{g,(\tau_j)}(\tau) = \exp \left( \sum_{j=1}^{2g} r_{jj} \left( \frac{n_j}{2} \right) + \sum_{1 \leq j < k \leq 2g} r_{kj} n_j n_k + \pi i \sum_{1 \leq j < k \leq 2g} p_{jk}^* n_j n_k \right).
\]

Since the matrix \( R \) is symmetric

\[
\sum_{j=1}^{2g} r_{jj} \left( \frac{n_j}{2} \right) + \sum_{1 \leq j < k \leq 2g} r_{kj} n_j n_k = \frac{1}{2} \sum_{1 \leq j, k \leq 2g} r_{jk} n_j n_k - \frac{1}{2} \sum_{j=1}^{2g} r_{jj} n_j = \frac{1}{2} \left( n^T R n - 4 r_n \right).
\]

On the other hand since \( P^* = \hat{P} + 2Q \) for an integral matrix \( Q \), as noted following the definition (6.60) of the symmetrization \( \hat{P} \), while the diagonal terms of the symmetric matrix \( \hat{P} \) vanish, it follows that

\[
\exp \pi i \sum_{1 \leq j < k \leq 2g} p_{jk}^* n_j n_k = \exp \pi i \sum_{1 \leq j < k \leq 2g} \hat{p}_{jk} n_j n_k
\]

\[
= \exp \frac{\pi i}{2} \sum_{1 \leq j < k \leq 2g} \left( \hat{p}_{jk} + \hat{p}_{kj} \right) n_j n_k
\]

\[
= \exp \frac{\pi i}{2} \sum_{j, k=1}^{2g} \hat{p}_{jk} n_j n_k = \exp \frac{\pi i}{2} n^T \hat{P} n = \pm 1.
\]
Substituting the two preceding observations in (6.68) yields (6.61), and that concludes the proof.

The explicit form (6.61) of the auxiliary mapping of the preceding theorem can be simplified slightly by multiplying by a suitable flat factor of automorphy. The canonically parametrized flat factor of automorphy \( \rho_{t, \{ \tau_j \}} \) for the group \( \Gamma \) in terms of the basis \( \tau_j \in H_1(M) \) was defined in (3.26) as that representation \( \rho_{t, \{ \tau_j \}} \in \text{Hom}(\Gamma, \mathbb{C}^*) = \text{Hom}(H_1(M), \mathbb{C}^*) \) for which \( \rho_{t, \{ \tau_j \}}(\tau_j) = \exp 2\pi it \) for \( 1 \leq j \leq 2g \), so

\[
(6.69) \quad \rho_{t, \{ \tau_j \}}(\tau) = \exp 2\pi i \sum_{j=1}^{2g} n_j \tau_j = \exp 2\pi i t n \quad \text{if} \quad \tau = \sum_{j=1}^{2g} n_j \tau_j.
\]

In particular for the vector \( t = \frac{1}{4\pi} r \), where \( r = \text{diag}(R) \)

\[
(6.70) \quad \rho_{\frac{1}{4\pi} r, \{ \tau_j \}}(\tau) = \exp \frac{t}{2} \tau n \quad \text{if} \quad \tau = \sum_{j=1}^{2g} n_j \tau_j;
\]

consequently the product of this flat factor of automorphy with the auxiliary mapping (6.61) is the auxiliary mapping for the hyperabelian root factor \( \xi_{g,a}(T, z) \) given by

\[
(6.71) \quad \lambda_{g,a, \{ \tau_j \}}(\tau) = \rho_{\frac{1}{4\pi} r, \{ \tau_j \}}(\tau) \cdot \lambda_{g, \{ \tau_j \}}(\tau)
\]

\[
= \epsilon_{\{ \tau_j \}}(\tau) \cdot \exp \frac{1}{2} t n R n \quad \text{if} \quad \tau = \sum_{j=1}^{2g} n_j \tau_j \in H_1(M).
\]

This is the auxiliary mapping \textit{intrinsically normalized} by requiring that

\[
(6.72) \quad \lambda_{g,a, \{ \tau_j \}}(\tau_j) = \exp \frac{1}{2} r_{jj} \quad \text{for} \quad 1 \leq j \leq 2g
\]

in place of (6.59), and with this normalization it is indeed rather more intrinsically defined.

\textbf{Lemma 6.19} Let \( \{ \tau_i \} \) and \( \{ \tau'_i \} \) be two bases for the homology group \( H_1(M) \) of a compact Riemann surface \( M \) of genus \( g > 0 \), related by \( \tau_i = \sum_{i=1}^{2g} q_{ij} \tau'_j \) for a matrix \( Q \in \text{Gl}(2g, \mathbb{Z}) \); and let \( P \) be the intersection matrix of the surface in terms of the basis \( \{ \tau_i \} \) and \( P' \) be the intersection matrix of the surface in terms of the basis \( \{ \tau'_i \} \). The symmetrizations \( \hat{P} \) and \( \hat{P}' \) of the matrices \( P = tP^{-1} \) and \( P' = tP'^{-1} \) are symmetric integral matrices for which

\[
(6.73) \quad \hat{P}' = Q^* \hat{P} Q^* \quad (\text{mod} \ 2).
\]
Proof: It follows from (6.25) that $P^* = Q^* P^* tQ^*$, so since the symmetrization $\hat{P}$ of the skew-symmetric matrix $P^*$ is defined by (6.60)

$$p'^*_{ij} = \sum_{k,l=1}^{2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} = \sum_{1 \leq k < l \leq 2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} + \sum_{1 \leq i < k \leq 2g} q^*_{ik} q^*_{ij} \hat{p}_{kl}$$

$$= \sum_{1 \leq k < l \leq 2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} - \sum_{1 \leq i < k \leq 2g} q^*_{ik} q^*_{ij} \hat{p}_{kl}$$

$$= \sum_{k,l=1}^{2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} - 2 \sum_{1 \leq l < k \leq 2g} q^*_{ik} q^*_{jl} \hat{p}_{kl}$$

$$\equiv 2g \sum_{k,l=1}^{2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} \pmod{2}.$$ 

In particular for $i < j$

$$\hat{p}^*_{ij} = p'^*_{ij} \equiv \sum_{k,l=1}^{2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} \pmod{2}$$

while for $i > j$

$$\hat{p}^*_{ij} = -p'^*_{ij} \equiv -\sum_{k,l=1}^{2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} \equiv \sum_{k,l=1}^{2g} q^*_{ik} q^*_{jl} \hat{p}_{kl} \pmod{2};$$

and for $i = j$ by definition $\hat{p}_{ii} = \hat{p}'_{ii} \equiv 0$ while since $\hat{p}_{kl} = \hat{p}_{kl}$ interchanging the variables of summation shows that

$$\sum_{k,l=1}^{2g} q^*_{ik} q^*_{il} \hat{p}_{kl} = \sum_{1 \leq k < l \leq 2g} q^*_{ik} q^*_{il} \hat{p}_{kl} + \sum_{1 \leq i < k \leq 2g} q^*_{ik} q^*_{il} \hat{p}_{kl}$$

$$= \sum_{1 \leq k < l \leq 2g} q^*_{ik} q^*_{il} \hat{p}_{kl} + \sum_{1 \leq l < k \leq 2g} q^*_{il} q^*_{ik} \hat{p}_{kl} \equiv 0 \pmod{2}$$

so in this case also

$$\hat{p}^*_{ii} \equiv \sum_{k,l=1}^{2g} q^*_{ik} q^*_{il} \hat{p}_{kl} \pmod{2}.$$ 

Altogether therefore $\hat{P}^* \equiv Q^* \hat{P} Q^* \pmod{2}$, which was to be proved.

Theorem 6.20 If $M$ is compact Riemann surface $M$ of genus $g > 0$, if $\{\tau_i\}$ and $\{\tau'_i\}$ are two bases for the homology group $H_1(M)$ related by $\tau_i = \sum_{l=1}^{2g} q_{il} \tau'_l$ for a matrix $Q \in \text{GL}(2g, \mathbb{Z})$, and if $\lambda_{g, \{\tau_i\}}$ is the intrinsically normalized auxiliary mapping for the hyperabelian root factor $\xi_{g, \{T, z\}}$ in terms of the basis $\{\tau_i\}$
\[ (6.74) \quad \frac{\lambda_{\ast,g,\left\{ \tau_{i}' \right\} } (\tau)}{\lambda_{\ast,g,\left\{ \tau_{i} \right\} } (\tau)} = \frac{\epsilon_{\left\{ \tau_{i}' \right\} } (\tau)}{\epsilon_{\left\{ \tau_{i} \right\} } (\tau)} = \rho_{\frac{1}{2}}(\hat{P},Q),\left\{ \tau_{i} \right\} (\tau) = \pm 1 \]

where \( \delta(\hat{P}, Q) \) is the integral vector

\[ (6.75) \quad \delta(\hat{P}, Q) = \frac{1}{2} Q \text{ diag}(Q^* \hat{P}^t Q^*) \in \mathbb{Z}^{2g} \]

for the symmetrization \( \hat{P} \) of the inverse transpose \( P^* = \hat{P}^{-1} \) of the intersection matrix \( P \) of the surface \( M \) in terms of the basis \( \left\{ \tau_{i} \right\} \).

**Proof:** When a homology class in \( H_1(M) \) is written \( \tau = \sum_{i=1}^{2g} n_i \tau_i = \sum_{i=1}^{2g} n_i' \tau_i' \) then \( n_i' = \sum_{j=1}^{2g} q_{ij} n_i \) or in vector notation \( n' = Qn \). It follows from (6.25) and (6.26) with \( \hat{C} = I \) that the intersection matrices \( P \) and \( P' \), the period matrices \( \Omega \) and \( \Omega' \), and the Hermitian matrices \( G \) and \( G' \) in terms of the two bases are related by \( P = Q^{-1} P' Q^{-1} \), \( \Omega = \Omega' Q \), and \( G = G' \); consequently the matrices \( R \) and \( R' \) of (6.67) are related by

\[ (6.76) \quad R = 2\pi i \mathfrak{H}(\Omega G \Omega^t) = 2\pi i \mathfrak{H}(\left\{ Q' \Omega' \cdot G' \cdot \Omega^t Q \right\} = \left\{ Q' R' Q \right\} \]

so \( hRn = \left\{ h'Q'R'Qn = \left\{ h'R'n' \right\} \right\} \). Thus in formula (6.71) for the auxiliary mapping \( \lambda_{\ast,g,\left\{ \tau_{i} \right\} } \) only the term \( \epsilon_{\left\{ \tau_{i} \right\} } (\tau) \) depends on the choice of the basis \( \tau_i \), so the two auxiliary mappings are related by

\[ (6.77) \quad \frac{\lambda_{\ast,g,\left\{ \tau_{i}' \right\} } (\tau)}{\lambda_{\ast,g,\left\{ \tau_{i} \right\} } (\tau)} = \frac{\epsilon_{\left\{ \tau_{i}' \right\} } (\tau)}{\epsilon_{\left\{ \tau_{i} \right\} } (\tau)}. \]

Next from (6.62) it follows that

\[ \frac{\epsilon_{\left\{ \tau_{i}' \right\} } (\tau)}{\epsilon_{\left\{ \tau_{i} \right\} } (\tau)} = \exp \frac{\pi i}{2} \left( \left\{ h'P'n' \right\} - \left\{ h\hat{P}n \right\} \right) = \exp \frac{\pi i}{2} \left( \left\{ h'\hat{P}n' \right\} - \left\{ h'Q'Q^* \hat{P}^t Q^* \right\} \right) \]

\[ = \exp \frac{\pi i}{2} \left\{ h'An' \right\} \quad \text{where} \quad A = \hat{P}^t - Q^* \hat{P}^t Q^* \]

Since the matrix \( A \) is symmetric

\[ \left\{ h'An' \right\} = \sum_{j,k=1}^{2g} a_{jk} n'_j n'_k = \sum_{1\leq j < k \leq 2g} a_{jk} n'_j n'_k + \sum_{1\leq k < j \leq 2g} a_{jk} n'_j n'_k + \sum_{1 \leq i \leq 2g} a_{ii} n'_i n'_i \]

\[ = \sum_{1 \leq j < k \leq 2g} 2a_{jk} n'_j n'_k + \sum_{1 \leq i \leq 2g} a_{ii} n'_i n'_i ; \]

and since the entries \( a_{jk} \) are even integers by the preceding Lemma 6.19 and \( n'_i^2 = n'_i \) (mod 2) it follows that

\[ \exp \frac{\pi i}{2} \left\{ h'An' \right\} = \exp \frac{\pi i}{2} \sum_{i=1}^{2g} a_{ii} n'_i n'_i = \exp \frac{\pi i}{2} \sum_{i=1}^{2g} a_{ii} n'_i. \]
However $\hat{p}_{ij} = 0$ by definition so $a_{ii} = -\sum_{j,k=1}^{2g} q_{ij} q_{ik}^* \hat{p}_{jk}$ while $n'_i = \sum_{l=1}^{2g} q_{il} n_l$, hence
\[
\exp \frac{i}{2} \sum_{i=1}^{2g} a_{ii} n'_i = \exp -\pi i \cdot \frac{1}{2} \sum_{i,j,k,l=1}^{2g} q_{ij} q_{ik}^* \hat{p}_{jk} q_{il} \cdot n_l.
\]
The vector
\[
(6.78) \quad \delta(\hat{P}, Q) = \frac{1}{2} \epsilon Q \, \text{diag} (Q^* \hat{P} Q^*)
\]
has the components
\[
(6.79) \quad \delta(\hat{P}, Q)_l = \frac{1}{2} \sum_{i,j,k,l=1}^{2g} q_{il} q_{ij} q_{ik}^* \hat{p}_{jk} q_{il} n_l,
\]
which are integers since the matrix $\hat{P}$ is symmetric with zeros along its diagonal; and in terms of this vector
\[
\exp \frac{i}{2} \sum_{i=1}^{2g} a_{ii} n'_i = \exp -\pi i \cdot \frac{1}{2} \sum_{i=1}^{2g} \delta(\hat{P}, Q)_l n_l = \exp -\pi i \cdot \delta(\hat{P}, Q) \cdot n
\]
for the homology class $\tau = \sum_{l=1}^{2g} n_l \tau_l \in H_1(M)$, and that suffices to conclude the proof.

The abelian and hyperabelian factors of automorphy depend on the choice of a point $a \in \tilde{M}$, which appears in the vector of abelian integrals $\tilde{w}(z, a)$. By definition a pointed Riemann surface is a Riemann surface together with the choice of a base point in its universal covering surface; so on a pointed Riemann surface the base point $a \in \tilde{M}$ is uniquely determined, but on a general Riemann surface that point is an additional parameter for these factors of automorphy. The preceding results about hyperabelian factors of automorphy can be summarized as follows.

**Theorem 6.21** (i) On a compact Riemann surface $M$ of genus $g > 0$, represented as the quotient $M = \tilde{M}/\Gamma$ of its universal covering space $\tilde{M}$ by the covering translation group $\Gamma$, for any choice of a point $a \in \tilde{M}$ there is a hyperabelian factor of automorphy $\zeta_{g,a}(\tau, z)$ of characteristic class $g$ that is uniquely and intrinsically determined up to a specific collection of homomorphisms in $\text{Hom}(\Gamma, \pm 1)$.

(ii) In terms of any bases $\tau_j \in H_1(M)$ and $\omega_i(z) \in \Gamma(M, O^{(1,0)})$, this factor of automorphy has the form
\[
(6.80) \quad \zeta_{g,a}(\tau, z) = e(\tau) \exp \left( \frac{1}{2} \lambda R n + 2\pi I \tilde{w}(z, a) G \Omega n \right)
\]
for a homology class $\tau = \sum_{j=1}^{2g} n_j \tau_j \in H_1(M)$, where $e(\tau) = \pm 1$, $\tilde{w}(z, a)$ is the vector $\{w_i(z, a)\}$ of integrals of the holomorphic abelian differentials $\omega_i(z)$, and if $P$ is the intersection matrix and $\Omega$ is the period matrix of $M$ in terms of these

---

4 See the discussion on page 449 in Appendix D.
bases, \( G = \mathcal{H}^{-1} = \overline{H^{-1}} \) for the positive definite Hermitian matrix \( H = i\Omega P\overline{\Omega} \), and \( R = 2\pi i \mathcal{R}(\Omega \Omega) \). This explicit form for the factor of automorphy is the same for any choices of the bases \( \tau \) and \( \omega(z) \), in terms of the matrices \( R, G, \Omega \) defined in terms of these bases.

(iii) The sign factor \( \epsilon(\tau) \) depends on the choice of a basis for the homology group \( H_1(M) \) of the surface \( M \) but is independent of the choice of a basis for the space \( \Gamma(M,\mathcal{O}^{(1,0)}) \) of holomorphic abelian differentials. The sign factor determined in terms of a basis \( \tau_j \in H_1(M) \) is given explicitly by

\[
\epsilon_{(\tau_j)}(\tau) = \exp \frac{\pi i}{2} \ln \hat{P}
\]

for a homology class \( \tau = \sum_{j=1}^{2g} v_j \tau_j \in H_1(M) \), where \( \hat{P} \) is the symmetrization of the matrix \( P^* = \mathcal{H}^{-1} \) as in \( (6.60) \). If \( \tau_j' \in H_1(M) \) is another homology basis with the same intersection matrix \( P \) then \( \tau_i = \sum_{j=1}^{2g} q_{ji} \tau_j' \) for a matrix \( Q \in \text{GL}(2g,\mathbb{Z}) \) such that \( \mathcal{H} Q P Q = P \), and

\[
\epsilon_{(\tau_j')}(\tau) = \rho_{\frac{1}{2} \delta(\hat{P},Q),\{\tau_j\}}(\tau) \epsilon_{(\tau_j)}(\tau)
\]

where \( \delta(\hat{P},Q) \) is the integral vector

\[
\delta(\hat{P},Q) = \frac{1}{2} Q \text{ diag}(\mathcal{H}^{-1} \hat{P} Q^{-1}) \in \mathbb{Z}^{2g}
\]

in which \( \text{diag}(X) \) denotes the column vector formed from the diagonal entries of the matrix \( X \).

**Proof:** For a choice of bases \( \tau_j \in H_1(M) \) and \( w_i(z,a) \in \Gamma(M,\mathcal{O}^{(1,0)}) \) the factor of automorphy \( (6.80) \) is the hyperabelian factor of automorphy for the root factor \( \xi_{g,a}(T,z) \) of \( (6.23) \) and the auxiliary mapping \( \lambda_{*,g,\{\tau_j\}}(\tau) \), intrinsically normalized as in \( (6.72) \), for which the sign factor is given by \( (6.62) \). By Theorem 6.6 and Corollary 6.20 this factor is intrinsically defined on the surface \( M \), in the sense that it has the same form \( (6.80) \) for any given homology class \( \tau \in H_1(M) \), although for different bases \( \tau_j \) the sign factor changes as in \( (6.74) \); but since this is a factor of automorphy, the sign factor \( \epsilon(\tau) \) is determined up to the product with a homomorphism in \( \text{Hom}(\Gamma,\pm 1) \). The condition that a change of the homology basis preserves the intersection matrix can be read from \( (6.25) \). That suffices for the proof.

The sign factor \( \epsilon(\tau) \) is the subtlest component of the factors of automorphy \( (6.80) \). It is clear from the preceding theorem that what is uniquely and intrinsically associated to a compact Riemann surface \( M \) of genus \( g > 0 \) is not a single hyperabelian factor of automorphy of characteristic class \( g \) but rather a family consisting of a collection of factors of automorphy \( (6.80) \) for some sign factors \( \epsilon(\tau) \). The simplest family of such factors of automorphy and the most useful one for present purposes is the basic family of hyperabelian factors of automorphy of characteristic class \( g \), the collection of \( 2^{2g} \) factors of automorphy derived from any one of the factors of automorphy \( (6.80) \) by multiplying by all of the \( 2^{2g} \) homomorphisms in \( \text{Hom}(\Gamma,\pm 1) \), so the collection of factors of automorphy

\[
\zeta_{g,a,\delta/2}(\tau,z) = \rho_{\delta/2}(\tau) \zeta_{g,a}(\tau,z) \quad \text{for all } \delta \in \mathbb{Z}^{2g} \pmod{2}
\]
where $\zeta_{g,a}(\tau, z)$ is the factor of automorphy (6.80) in terms of any bases $\tau_j \in H_1(M)$ and $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$; and as in (3.26)

\[(6.85) \quad \rho_{g/2}(\tau) = \exp(\pi i \delta \cdot n - (\delta \delta n) \text{ for a homology class } \tau = \sum_{j=1}^{2g} n_j \tau_j).
\]

These factors of automorphy can be written out more explicitly as

\[(6.86) \quad \zeta_{g,a,\delta/2}(\tau, z) = \exp(\frac{1}{2} \nu \Omega n + \frac{1}{2} \pi i \nu \tilde{\Omega} n + \pi i \delta n + 2\pi i \tilde{\omega}(z, a)G\Omega n)\]

for a homology class $\tau = \sum_{j=1}^{2g} n_j \tau_j \in H_1(M)$, where the matrices $\Omega, G, \tilde{\Omega}$ are defined in terms of the bases $\tau_j \in H_1(M)$ and $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ as in Theorem 6.21 and $\delta \in \mathbb{Z}^{2g}$ ranges through the $2^{2g}$ integral vectors modulo 2. The entire collection of these factors of automorphy is uniquely and intrinsically associated to the Riemann surface $M$. The dependence of these factors of automorphy on the parameters $a \in \tilde{M}$ and $\delta \in \mathbb{Z}^{2g}$ can be summarized as follows.

**Theorem 6.22** Let $M$ be a compact Riemann surface of genus $g > 0$, let $P$ be the period matrix and $\Omega$ be the intersection matrix of $M$ in terms of bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, and let $H = i \Omega P\tilde{\Omega}$ and $G = \nu H^{-1} = H^{-1}$.

(i) For any points $a', a'' \in \tilde{M}$ and any vectors $\delta', \delta'' \in \mathbb{Z}^{2g}$

\[(6.87) \quad \zeta_{g,a'',\delta''/2}(\tau, z) = \rho_{t(a'', a'; \delta'' - \delta')}(\tau)\zeta_{g,a',\delta'/2}(\tau, z)
\]

where $\rho_{t(a'', a'; \delta)}(\tau)$ is the canonically parametrized flat factor of automorphy for the parameter

\[(6.88) \quad t(a'', a'; \delta) = \frac{1}{2} \delta + i \nu \tilde{\Omega} \tilde{w}(a'', a') \in \mathbb{C}^{2g}
\]

in terms of the basis $\tau$, and the vector $\tilde{w}(a'', a') = \{w_i(a'', a')\} = \{\int_{\alpha''} \omega_i\} \text{ of integrals of the holomorphic abelian differentials } \omega_i(z).

(ii) The factors of automorphy $\zeta_{g,a'',\delta''/2}(\tau, z)$ and $\zeta_{g,a',\delta'/2}(\tau, z)$ are holomorphically equivalent if and only if

\[(6.89) \quad \frac{1}{2} (\delta'' - \delta') + i \nu \tilde{\Omega} \tilde{w}(a'', a') \in \mathbb{Z}^{2g} + \Omega \mathbb{C}^g.
\]

**Proof:** The factor of automorphy $\zeta_{g,a,\delta/2}(\tau, z)$ is given explicitly by (6.86) for any homology class $\tau = \sum_{j=1}^{2g} n_j \tau_j \in H_1(M)$. Only the terms $\pi i \delta n$ and $2\pi \tilde{\omega}(z, a) G\Omega n$ depend on the point $a \in \tilde{M}$ and the vector $\delta$, and clearly

\[
\zeta_{g,a'',\delta''/2}(\tau, z) = \zeta_{g,a',\delta'/2}(\tau, z) \cdot \exp \left( \pi i \left( \frac{1}{2} (\delta'' - \delta') - 2\pi \tilde{\omega}(a'', a') G\Omega \right) \right) n \\
= \zeta_{g,a',\delta'/2}(\tau, z) \cdot \exp \left( 2\pi i \left( \frac{1}{2} (\delta'' - \delta') + i \nu \tilde{\Omega} \tilde{w}(a'', a') \right) \right) n \\
= \zeta_{g,a',\delta'/2}(\tau, z) \rho_{t(a'', a'; \delta'' - \delta')}(\tau)
\]
as in (6.69) for the vector \( t(a'', a'; \delta) \) given in (6.88), and that is (6.87). Consequently the two factors of automorphy \( \zeta_{g,a'', \delta'/2}(\tau, z) \) and \( \zeta_{g,a', \delta'/2}(\tau, z) \) are holomorphically equivalent if and only if the flat line bundle \( \rho_t(a_2, a_1; \delta'' - \delta') \) is holomorphically trivial; and by Theorem 3.14 that is the case if and only if

\[
t(a'', a'; \delta'' - \delta') \in \mathbb{Z}^{2g} + \frac{i}{2} \Omega \mathbb{C}^g,
\]

which is (6.89), thereby concluding the proof.

The collection of \( 2^{2g} \) holomorphic line bundles \( \zeta_{g,a,\delta/2} \) over the Riemann surface \( M \) described by the basic family of factors of automorphy \( \zeta_{g,a,\delta/2}(\tau, z) \) is called the basic family of holomorphic line bundles of characteristic class \( g \) over \( M \); this collection of holomorphic line bundles is intrinsically and uniquely associated to the Riemann surface with the specified base point \( a \in \tilde{M} \).

**Corollary 6.23** With the notation as in the preceding theorem, the holomorphic line bundles \( \zeta_{g,a',\delta/2} \) and \( \zeta_{g,a'',\delta'/2} \) are holomorphically equivalent if and only if

\[
\nu = n - \frac{1}{2} (\delta'' - \delta') = \frac{1}{2} \Omega t + \frac{i}{2} \Omega \bar{G} \bar{w}(a'', a') = \frac{1}{2} \Omega t + \frac{i}{2} \Omega \bar{G} \bar{w}(a'', a'),
\]

In particular

(i) the line bundles \( \zeta_{g,a,\delta/2} \) and \( \zeta_{g,a,\delta'/2} \) are holomorphically equivalent if and only if \( \delta'' \equiv \delta' \mod 2 \);

(ii) the line bundles \( \zeta_{g,a',\delta/2} \) and \( \zeta_{g,a'',\delta'/2} \) are holomorphically equivalent if and only if \( \bar{w}(a'', a') \in \Omega \mathbb{Z}^{2g} \), and

(iii) the line bundles \( \zeta_{g,a',\delta/2} \) and \( \zeta_{g,a'',\delta'/2} \) are holomorphically equivalent whenever \( a', a'' \in \tilde{M} \) represent the same point of the Riemann surface \( M \).

**Proof:** By the preceding Theorem 6.22 (ii) the holomorphic line bundles \( \zeta_{g,a',\delta'/2} \) and \( \zeta_{g,a'',\delta'/2} \) are holomorphically equivalent if and only if

\[
\frac{1}{2} (\delta'' - \delta') + i \Omega \bar{G} \bar{w}(a'', a') = n - \frac{1}{2} \Omega t
\]

for some \( n \in \mathbb{Z}^{2g} \) and \( t \in \mathbb{C}^g \), and that can be rewritten equivalently as

\[
\nu = n - \frac{1}{2} (\delta'' - \delta') = \frac{1}{2} \Omega t + i \Omega \bar{G} \bar{w}(a'', a').
\]

If (6.91) holds then since \( \Pi \Omega = 0 \) and \( \bar{\Pi} \Omega = I \) by (F.6) it follows upon multiplying (6.91) by \( \Pi \) that \( \Pi \nu = i \bar{G} \bar{w}(a'', a') \) or equivalently

\[
\bar{w}(a'', a') = -i \Pi \nu.
\]

Recall from (6.57) that

\[
P^{-1} = i \bar{\Pi} \bar{G} \Omega - i \Omega \bar{G} \bar{\Pi},
\]

consequently \( \Pi P^{-1} = i \bar{G} \Omega \) or equivalently \( \Pi \Pi = i \Omega P \), and upon substituting this into (6.92) it follows that

\[
\bar{w}(a'', a') = \Omega P \nu = \Omega P n - \frac{1}{2} \Omega P (\delta'' - \delta').
\]
which is equivalent to (6.90), so (6.91) implies (6.90). Conversely if (6.90)
holds then \( \hat{w}(a'', a') = \Omega n - \frac{1}{2} \Omega P(\delta'' - \delta') = \Omega \mu \) for some \( n \in \mathbb{Z}^{2g} \), where
\( \mu = n - \frac{1}{2} P(\delta'' - \delta') \); by (6.93) again it follows that
\[
\begin{align*}
\mu \Omega \mathcal{G} \hat{w}(a'', a') &= \mu \Omega \mathcal{G} \Omega \mu = \left( P^{-1} + i \Omega \mathcal{G} \Omega \right) \mu \\
&= \left( P^{-1} + i \Omega \mathcal{G} \Omega \right) \left( n - \frac{1}{2} P(\delta'' - \delta') \right) \\
&= P^{-1}n - \frac{1}{2}(\delta'' - \delta') - \Omega \mathcal{G}
\end{align*}
\]
where \( t = -i \mathcal{G} \Omega (n - \frac{1}{2} P(\delta'' - \delta')) \in \mathbb{C}^g \), which is (6.91), so (6.90) implies (6.91).
Thus (6.90) and (6.91) are equivalent, which establishes the first assertion of the
corollary. In particular if \( a' = a'' = a \) condition (6.90) reduces to the condition
that \( \frac{1}{2} \Omega P(\delta'' - \delta') = \Omega n' \) for some \( n' \in \mathbb{Z}^{2g} \), or equivalently since \( n = P^{-1}n' \in \mathbb{Z}^{2g} \) to the condition that \( \Omega P \left( n - \frac{1}{2}(\delta'' - \delta') \right) = 0 \); however \( n - \frac{1}{2}(\delta'' - \delta') \) is real
so \( \Omega P \left( n - \frac{1}{2}(\delta'' - \delta') \right) = 0 \) as well, and since the full period matrix \( \left( \frac{\Omega}{P} \right) \) and the
matrix \( P \) are nonsingular it follows that \( n - \frac{1}{2}(\delta'' - \delta') = 0 \), which is equivalent
to (i). On the other hand if \( \delta' = \delta'' = \delta \) then (6.90) reduces immediately to the condition that \( \hat{w}(a'', a') \in \Omega \mathbb{Z}^{2g} \), which is (ii). Finally whenever \( a'' = Ta' \)
for a covering translation \( T \) on the universal covering space then \( \hat{w}(a'', a') = \omega(T) \) where \( \omega(T) \in \Omega \mathbb{Z}^{2g} \) is the period of the abelian differential \( \omega \); and by
(ii) it follows that the line bundles \( \zeta_{g,a',\delta/2} \) and \( \zeta_{g,a'',\delta/2} \) are holomorphically
equivalent, which is (iii), thereby concluding the proof.

The square \( \zeta_{g,a,\delta/2}(\tau, z)^2 \) of any member of the basic family of hyperabelian
factors of automorphy of characteristic class \( g \) of course is a hyperabelian factor
of automorphy of characteristic class \( 2g \); and it is evident from (6.86) that this
factor of automorphy is independent of the value of the vector \( \delta \in \mathbb{Z}^{2g} \), so in
view of Theorem 6.21 it follows that \( \zeta_{g,a,\delta/2}(\tau, z)^2 \) is a uniquely and intrinsically
defined hyperabelian factor of automorphy of characteristic class \( 2g \) on the Riemann
surface \( M \) associated to the base point \( a \in M \), that is independent of the
choice of a basis for the homology of \( M \) or for the space of holomorphic abelian
differentials on \( M \). It is called the intrinsic hyperabelian factor of automorphy
of characteristic class \( 2g \) on \( M \), and is denoted by \( \zeta_{2g,a}(\tau, z) \); so in terms of any
bases \( \tau_j \in H_1(M) \) and \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)}) \)
\[
(6.94) \quad \zeta_{2g,a}(\tau, z) = \exp \left( iHn + 4\pi i \hat{w}(z, a)G\mathcal{G}n \right)
\]
for a homology class \( \tau = \sum_{j=1}^{2g} n_j \tau_j \in H_1(M) \), where \( \hat{w}(z, a) \) is the vector
\( \{ \omega_i(z, a) \} \) of integrals of the abelian differentials \( \omega_i(z) \) and if \( P \) is the
intersection matrix and \( \Omega \) is the period matrix of \( M \) in terms of these bases,
\( G = H^{-1} = \mathcal{G}^{-1} \) for the positive definite Hermitian matrix \( H = i\Omega \mathcal{G} \mathcal{G} \Omega \) and
\( R = 2\pi R(\Omega \mathcal{G} \mathcal{G} \Omega) \). The holomorphic line bundle \( \zeta_{2g,a} \) over the Riemann surface
\( M \) described by this factor of automorphy is also uniquely and intrinsically
defined on \( M \), and is called the intrinsic hyperabelian line bundle of characteristic
class $2g$ on $M$. The dependence of this factor of automorphy and the holomorphic line bundle it describes on the base point $a \in \tilde{M}$ follow immediately from the preceding corollary.

**Corollary 6.24** The holomorphic line bundles $\zeta_{2g,a'}$ and $\zeta_{2g,a''}$ are holomorphically equivalent if and only if

$$2\bar{w}(a'',a') \in \Omega \mathbb{Z}^{2g};$$

in particular the line bundles $\zeta_{2g,a'}$ and $\zeta_{2g,a''}$ are holomorphically equivalent whenever $a',a'' \in \tilde{M}$ represent the same point of the Riemann surface $M$.

**Proof:** The square of (6.87) in Theorem 6.22 for $\delta = \delta''$ is the identity

$$\zeta_{2g,a''}(\tau, z) = \rho_{2i(a'',a';0)}(\tau)\zeta_{2g,a'}(\tau, z),$$

from which it follows that the two holomorphic line bundles $\zeta_{2g,a'}$ and $\zeta_{2g,a''}$ are holomorphically equivalent if and only if the canonically parametrized flat factor of automorphy $\rho_{2i(a'',a';0)}(\tau)$ for the parameter $2t(a'',a';0) = 2i\bar{\Omega} \bar{\varphi}(a'',a')$ is holomorphically trivial; and by Theorem 3.14 that is the case if and only if

$$2i\bar{\Omega} \bar{\varphi}(a'',a')n + \Omega t \quad \text{for some } n \in \mathbb{Z}^{2g}, t \in \mathbb{C}^g. \tag{6.96}$$

If that is the case multiplying (6.96) by $\Pi$ shows that $2i\bar{\Omega} \bar{\varphi}(a'',a') = \Pi n$ and consequently that $2\bar{w}(a'',a') = -iH\Pi n = \Omega Pn \in \Omega \mathbb{Z}^{2g}$, which is (6.95); thus (6.96) implies (6.95). Conversely if (6.95) holds so that $2\bar{w}(a'',a') = \Omega n'$ for some $n' \in \mathbb{Z}^{2g}$ then from (6.93) it follows that

$$2i\bar{\Omega} \bar{\varphi}(a'',a') = i\bar{\Omega} \bar{\varphi} \Omega n' = \left(P^{-1} + i\Omega G \Omega \right)n' \quad \text{where } n = P^{-1}n' \in \mathbb{Z}^{2g} \text{ and } t = iG \Omega \Pi n \in \mathbb{C}^g \tag{6.96}$$

which is (6.96); thus (6.95) implies (6.96), and that suffices for the proof of the first assertion of the corollary. Whenever the points $a',a'' \in \tilde{M}$ represent the same point on the Riemann surface $M$ then $a'' = Ta'$ for some $T \in \Gamma$ and $2\bar{w}(a'',a') = 2\omega(T) \in \Omega \mathbb{Z}^{2g}$, which suffices to conclude the proof.

There is a subfamily of the basic family of hyperabelian factors of automorphy that is also uniquely and intrinsically defined on compact Riemann surfaces $M$ of genus $g > 0$, with the choice of a base point $a \in \tilde{M}$ and the special intersection matrix $P = J M$. As discussed in Appendix D.2, it is always possible to choose a basis $\tau_i \in H_1(M)$ for which the intersection matrix is the basic skew-symmetric matrix $P = J$; and as in that discussion and in Theorem 6.21, any two such bases $\tau_i$ and $\tau'_j$ are related by $\tau_i = \sum_{j=1}^{2g} q_{ji} \tau'_j$ for a matrix $Q \in \text{Gl}(2g,\mathbb{Z})$ such that $QJQ = J$, that is, for a symplectic$^5$ integral matrix $Q$. Conversely of course, any symplectic integral matrix describes a change of bases for the homology group of $M$ preserving the intersection matrix $P = J$. By (6.82)

$^5$The basic properties of symplectic matrices are discussed in Appendix H.
in Theorem 6.21, the sign factors $\epsilon_{\tau_1}(\tau)$ and $\epsilon_{\tau_1}(\tau)$ in terms of these two homology bases $\{\tau_i\}$ and $\{\tau'_i\}$ are related by

$$
\epsilon_{\tau_1}(\tau) = \rho_{\delta(\hat{J},Q)/2}(\tau) \epsilon_{\tau_1}(\tau)
$$

in terms of the canonically parametrized flat factor of automorphy in $\text{Hom}(\Gamma, \pm 1)$ described by the vector $\delta(\hat{J}, Q) \in \mathbb{Z}^g$ given by (6.83). Not all integral vectors in $\mathbb{Z}^{2g}$ are of this form for some symplectic matrix $Q$ though, so not all of the flat factors of automorphy in $\text{Hom}(\Gamma, \pm 1)$ appear in (6.97) for some symplectic matrix $Q$; hence the collection of hyperabelian factors of automorphy of characteristic class $g$ that arise through changes in the basis for $H_1(M)$ preserving the intersection matrix $J$ form a subfamily of the basic family of hyperabelian factors of automorphy of characteristic class $g$. This subfamily is called the canonical family of hyperabelian factors of automorphy of characteristic class $g$ on the surface $M$. It also is a uniquely and intrinsically defined family of hyperabelian factors of automorphy on the Riemann surface $M$, with the choice of a base point $a \in M$ and the intersection matrix $J$, that is independent of the choice of the bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{1,0})$ and $\tau_j \in H_1(M)$.

The simple example of a compact Riemann surface $M$ of genus $g = 1$ with the intersection matrix $P = J = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$ may serve to clarify the preceding discussion. The symmetrization of $P^* = J^* = J^{-1} = J$ is the matrix $P = J = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. A symplectic matrix of rank 2 is a matrix $Q = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ for which $\det Q = ad - bc = 1$, as a consequence of (H.7) in Appendix H; and for this matrix $Q^* = Q^{-1} = (\begin{smallmatrix} d & c \\ -b & a \end{smallmatrix})$. A straightforward calculation shows that in this case (6.75) reduces to

$$
\delta(\hat{J}, Q) = \frac{1}{2} Q \text{diag} \left( Q^* \hat{J} Q \right) = - \left( \begin{array}{cccc}
ac + cd & ab + bd \\
bc + da & ad + dc
\end{array} \right).
$$

The flat factor of automorphy $\rho_{\delta/2}$ for any integral vector $\delta \in \mathbb{Z}^2$ is one of the four homomorphisms in $\text{Hom}(\Gamma, \pm 1)$, and depends only on the image $\delta# \in \mathbb{F}_2^2$ of the integral vector $\delta \in \mathbb{Z}^2$ in the vector space $\mathbb{F}_2^2$, where $\mathbb{F}_2$ is the finite field of order two. The image of the vector (6.98) in the vector space $\mathbb{F}_2^2$ depends only the image $Q# \in \text{GL}(2, \mathbb{F}_2)$ of the matrix $Q$ in the group $\text{GL}(2, \mathbb{F}_2)$. There are only six matrices $Q# \in \text{GL}(2, \mathbb{F}_2)$; and for these matrices the image vector $\delta(\hat{J}, Q)# \in \mathbb{F}_2^2$ and the value $\rho_{\delta(\hat{J}, Q)/2}(\tau)$ for an element $\tau = \sum_{i=1}^2 n_i \tau_i \in H_1(M)$ are as listed in the following table:

$$
Q# = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
$$

$$
\delta(\hat{J}, Q)# = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}
$$

$$
\rho_{\delta(\hat{J}, Q)/2}(\tau) = 1 \quad (-1)^{n_1} \quad 1 \quad (-1)^{n_2} \quad (-1)^{n_1} \quad (-1)^{n_2}.
$$
As this table shows, not all of the four vectors in \( \mathbb{F}_2^2 \) actually arise as vectors of the form \( \delta(J, Q)\), for some symplectic matrix \( Q \). In particular there are no symplectic matrices \( Q \) for which \( \delta(P, Q)\) = \( \frac{1}{2} \), which can be seen directly by noting that if there were such a matrix it would follow from (6.98) that all of the integers \( a, b, c, d, (b + d), (a + c) \) would be odd, which is impossible. Therefore only three of the four flat factors of automorphy in \( \text{Hom}(\Gamma, \pm 1) \) actually arise in the relation (6.97), showing that only three of the four members (6.84) of the basic family of hyperabelian factors of automorphy of characteristic class \( g = 1 \) over \( M \) belong to the canonical family of hyperabelian factors of automorphy of characteristic class \( g = 1 \). Before returning to the general discussion of the canonical family of hyperabelian factors of automorphy, though, it may be worth seeing the explicit forms of these factors of automorphy in this special case. If \( \omega(z) = dw(z, a) \) is a basis for the one-dimensional complex vector space \( \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_1, \tau_2 \) is a basis for the homology group \( H_1(M) \) with the intersection matrix \( P = J \), the period matrix is \( \Omega = (\omega_1, \omega_2) \) where \( \omega_i = \int_{\tau_i} \omega(z) \). A straightforward calculation shows that

\[
H = i\Omega \bar{\Omega} = 2\Im(\omega_1 \omega_2) \in \mathbb{R} \quad \text{and} \quad G = 1/H \in \mathbb{R},
\]

so both are just real numbers in this case; and

\[
R = 2\pi \Re(\Omega \bar{\Omega}) = \frac{\pi}{\Im(\omega_1 \omega_2)} \begin{pmatrix} \Im(\omega_2) & \Re(\omega_1 \omega_2) \\ \Re(\omega_1 \omega_2) & \Im(\omega_2) \end{pmatrix} \in \mathbb{R}^{2 \times 2}.
\]

For an element \( \tau = n_1 \tau_1 + n_2 \tau_2 \in H_1(M) \) since \( \epsilon(\tau) = \exp \frac{\pi i}{2} n \hat{J} N = (-1)^{n_1 n_2} \) by (6.81) it follows from (6.80), (6.82) and the results in Table 6.98 that the four basic hyperabelian factors of automorphy of characteristic class 1 on \( M \) are given by

\[
\begin{align*}
1\zeta_{1,a}(\tau, z) &= (-1)^{n_1 n_2} \exp \left( \frac{1}{2} n Rn + 2\pi w(z, a)G\bar{\Omega}n \right) \\
2\zeta_{1,a}(\tau, z) &= (-1)^{n_1 n_2 + n_1} \exp \left( \frac{1}{2} n Rn + 2\pi w(z, a)G\bar{\Omega}n \right) \\
3\zeta_{1,a}(\tau, z) &= (-1)^{n_1 n_2 + n_2} \exp \left( \frac{1}{2} n Rn + 2\pi w(z, a)G\bar{\Omega}n \right) \\
4\zeta_{1,a}(\tau, z) &= (-1)^{n_1 n_2 + n_1 + n_2} \exp \left( \frac{1}{2} n Rn + 2\pi w(z, a)G\bar{\Omega}n \right).
\end{align*}
\]

These formulas involve the real constants (6.100) and the symmetric real matrix (6.101) in terms of the chosen bases \( \omega(z) \) and \( \tau_1, \tau_2 \). The first factor of automorphy \( 1\zeta_{1,a}(\tau, z) \) is in the standard form (6.80) with the sign factor (6.81); the second and third factors of automorphy have the modified sign factor (6.82) for the two nontrivial representations \( \rho_{\delta(J, Q)/2} \) in Table 6.99, so the first three factors of automorphy are the three canonical hyperabelian factors of automorphy of characteristic class 1. The fourth factor of automorphy is that for which the sign factor is the remaining homomorphism in \( \text{Hom}(\Gamma, \pm 1) \), so the four factors of automorphy together form the basic family of hyperabelian factors of
automorphy of characteristic class 1. It is of course possible to begin with another member of the canonical family of hyperabelian factors of automorphy of characteristic class 1, such as $\zeta_{1,a}(\tau, z)$, and derive the others by applying Theorem 6.21; however the application of that theorem requires a change in the basis for the homology group $H_1(M)$, since the derivation of the formula (6.82) for the change of the sign factor begins with a sign factor that is in the standard form (6.81). What is particularly interesting in the case of Riemann surfaces of genus $g = 1$ is that there is a unique member of the basic family of hyperabelian factors of automorphy, the factor of automorphy $\zeta_{1,a}(\tau, z)$; it is uniquely and intrinsically associated to the Riemann surface $M$, and is independent of the choice of the bases $\omega_i(z) \in \Gamma(M, \mathcal{O}(1, 0))$ and $\tau_j \in H_1(M)$.

For a compact Riemann surface $M$ of arbitrary genus $g > 0$ and for a basis $\tau_i \in H_1(M)$ in terms of which the intersection matrix is the basic skew-symmetric matrix $P = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, any other basis $\tau'_j \in H_1(M)$ with the same intersection matrix is related to the initial basis by $\tau_i = \sum_{j=1}^{2g} q_{ij} \tau'_j$ for an integral symplectic matrix $Q \in \text{Sp}(2g, \mathbb{Z})$, and any integral symplectic matrix describes in this way another basis with the same intersection matrix. If the matrix $Q$ is decomposed into four $g \times g$ matrix blocks

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then as in (H.9) in Appendix H

$$Q^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \quad \text{and} \quad Q^* = Q^{-1} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix};$$

hence

$$\text{diag} \left( Q^* \hat{P} Q^* \right) = \text{diag} \left( \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} D & -C \\ -B & A \end{pmatrix} \right)$$

$$= -2 \begin{pmatrix} \text{diag}(C'D) \\ \text{diag}(A'B) \end{pmatrix}$$

and consequently

$$\delta(\hat{P}, Q) = \frac{1}{2} \text{tr} Q \text{ diag} \left( Q^* \hat{P} Q^* \right) = -\begin{pmatrix} 'A & 'C \\ 'B & 'D \end{pmatrix} \begin{pmatrix} \text{diag}(C'D) \\ \text{diag}(A'B) \end{pmatrix}$$

or finally

$$\delta(\hat{P}, Q) = -\begin{pmatrix} 'A \text{ diag}(C'D) + 'C \text{ diag}(A'B) \\ 'B \text{ diag}(C'D) + 'D \text{ diag}(A'B) \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

where $\delta_1, \delta_2 \in \mathbb{Z}^g$ are the vectors defined in terms of the symplectic matrix $Q$ in the preceding equation.
Lemma 6.25  The vectors $\delta_1, \delta_2 \in \mathbb{Z}^g$ defined in terms of the symplectic matrix $Q$ as in (6.105) satisfy

\begin{equation}
\delta_1 \cdot \delta_2 \equiv 0 \pmod{2};
\end{equation}

and conversely any two vectors $\delta_1, \delta_2 \in \mathbb{Z}^g$ satisfying (6.106) are the vectors defined in terms of some symplectic matrix $Q$ as in (6.105).

Proof: For any symplectic matrix $Q$ decomposed as in (6.103) the vectors $\delta_1, \delta_2$ are defined as in (6.105), so if

\[ \alpha = \text{diag}(A'B) \quad \text{and} \quad \gamma = \text{diag}(C'D) \]

then

\[ \delta_1 = -\gamma A' \gamma \quad \text{and} \quad \delta_2 = -\gamma B' \gamma \]

and consequently

\begin{equation}
\delta_1 \cdot \delta_2 = \gamma A' B' \gamma + \gamma A' D \alpha + \gamma C' B' \gamma + \gamma C' D \alpha.
\end{equation}

The integer $\delta_1 \cdot \delta_2$ is even precisely when its image in the field $\mathbb{F}_2$ vanishes; and that image is given by (6.107) when that equation is viewed as expressed in terms of the images of the entries of the matrix $Q$ in the field $\mathbb{F}_2$. With that in mind,

\[ \gamma A' B' \gamma = \sum_{i,j=1}^{g} \gamma_i (A'B)_{ij} \gamma_j \]

\[ = \sum_{1 \leq i < j \leq g} \gamma_i (A'B)_{ij} \gamma_j + \sum_{1 \leq j < i \leq g} \gamma_i (A'B)_{ij} \gamma_j + \sum_{i=1}^{g} \gamma_i (A'B)_{ii} \gamma_i \]

\[ = \sum_{i=1}^{g} \gamma_i \alpha_i \gamma_i = \sum_{i=1}^{g} \alpha_i \gamma_i \quad \text{in} \quad \mathbb{F}_2 \]

since the matrix $A'B$ is symmetric by (H.7) so the first and second terms in the second line above are equal hence their sum in the field $\mathbb{F}_2$ is zero, and $(A'B)_{ii} = \alpha_i$ by definition while $\gamma_i^2 = \gamma_i$ in the field $\mathbb{F}_2$; similarly

\[ \gamma C' D \alpha = \sum_{i=1}^{g} \alpha_i \gamma_i \quad \text{in} \quad \mathbb{F}_2. \]

On the other hand, also with entries viewed as elements in the field $\mathbb{F}_2$,

\[ \gamma A' D \alpha + \gamma C' B' \gamma = \gamma (A'D + B'C) \alpha \]

\[ = \gamma \alpha = \sum_{i=1}^{g} \alpha_i \gamma_i \quad \text{in} \quad \mathbb{F}_2 \]
since $A^tD - B^tC = I$ by (H.8) so $A^tD + B^tC = 2B^tC + I = I$ in the field $\mathbb{F}_2$. Substituting these observations into (6.107) shows that

$$t\delta_1 \cdot \delta_2 = \sum_{i=1}^{g} (\alpha_i \gamma_i + \alpha_i \gamma_i + \alpha_i \gamma_i) = \sum_{i=1}^{g} \alpha_i \gamma_i \text{ in } \mathbb{F}_2,$$

which is zero in $\mathbb{F}_2$ by Theorem H.1 of Appendix H. For the converse direction, if $B$ and $C$ are any symmetric integral $g \times g$ matrices then

$$Q = \begin{pmatrix} 1 & B \\ 0 & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & I \end{pmatrix} = \begin{pmatrix} 1 + BC & B \\ C & I \end{pmatrix}$$

is an integral symplectic matrix, since the two factors are symplectic matrices as in (H.10) (iii). By (6.105) the vectors $\delta_1$ and $\delta_2$ associated to this symplectic matrix are given by

$$-\delta_1 = (^t(I + BC)\text{diag}(C) + (^tB + BC^tB))$$
$$-\delta_2 = ^tB\text{diag}(C) + \text{diag}(^tB + BC^tB).$$

When the entries are viewed as belonging to the field $\mathbb{F}_2$ the $j$-th coordinate of the vector diag$(BC^tB)$ is explicitly

$$\text{diag}(BC^tB)_j = \sum_{k,l=1}^{g} c_{kl}b_{jk}b_{jl}$$

$$= \sum_{1 \leq k < l \leq g} c_{kl}b_{jk}b_{jl} + \sum_{1 \leq l < k \leq g} c_{kl}b_{jk}b_{jl} + \sum_{1 \leq k \leq g} c_{kk}b_{jk}b_{jk}$$
$$= \sum_{k=1}^{g} c_{kk}b_{jk}$$

since the matrices $B$ and $C$ are symmetric so the first and second terms in the second line are equal and their sum in $\mathbb{F}_2$ consequently is zero and $b_{jk}^2 = b_{jk}$ in $\mathbb{F}_2$. Using this observation, and similar calculations, it follows from (6.109) and the symmetry of the matrices $B$ and $C$ that the $i$-th entries in the vectors $\delta_1 = -\delta_1$ and $\delta_2 = -\delta_2$ in $\mathbb{F}_2$ are

$$\delta_{1i} = c_{ii} + \sum_{j,k=1}^{g} c_{ji}b_{kj}c_{kk} + \sum_{j=1}^{g} c_{ji}b_{jj} + \sum_{j,k=1}^{g} c_{ji}c_{kk}b_{jk}$$
$$= c_{ii} + \sum_{j=1}^{g} c_{ij}b_{jj}$$

and

$$\delta_{2i} = \sum_{j=1}^{g} b_{ji}c_{jj} + b_{ii} + \sum_{j=1}^{g} c_{jj}b_{ij} = b_{ii}.$$
Note incidentally that
\[ g \sum_{i=1}^{g} \delta_{1i} \delta_{2i} = \sum_{i=1}^{g} c_{ii} b_{ii} + \sum_{i,j=1}^{g} c_{ij} b_{ij} b_{jj} = \sum_{i=1}^{g} c_{ii} b_{ii} + \sum_{i=1}^{g} c_{ii} b_{ii} = 0, \]
as was already shown to be the case in general. The final step is to show that for any vectors \( \delta_1, \delta_2 \in \mathbb{F}_2^g \) such that \( \sum_{i=1}^{g} \delta_{1i} \delta_{2i} = 0 \) there are symmetric matrices \( B, C \in \mathbb{F}_2^{g \times g} \) such that \( \delta_1 \) and \( \delta_2 \) can be expressed in terms of the entries of these matrices as in (6.112) and (6.113). By reordering the indices \( i \) to simplify the notation it can be assumed that the entries in the vectors \( \delta_1 \) and \( \delta_2 \) are
\[
\begin{align*}
\delta_{1i} &= 1, \quad \delta_{2i} = 0 \quad \text{for } 1 \leq i \leq \nu_1, \\
\delta_{1i} &= 0, \quad \delta_{2i} = 1 \quad \text{for } \nu_1 + 1 \leq i \leq \nu_2, \\
\delta_{1i} &= 1, \quad \delta_{2i} = 1 \quad \text{for } \nu_2 + 1 \leq i \leq g;
\end{align*}
\]
since \( \sum_{i=1}^{g} \delta_{1i} \delta_{2i} = 0 \) it must then be the case that \( g - \nu_2 \equiv 0 \pmod{2} \). Introduce matrices \( B \) and \( C \) that have the following form when these matrices are decomposed into 9 blocks for row and column indices \( i, j \) lying in the ranges as in (6.114):
\[
B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & E
\end{pmatrix},
\]
where 0 is the zero matrix, I is the identity matrix, and E is the matrix having all entries equal to 1. For the values \( \delta_{1i} \) and \( \delta_{2i} \) defined in terms of these matrices \( B \) and \( C \) by (6.112) and (6.113) it follows immediately that \( \delta_{2i} = b_{ii} \), while for the entries \( \delta_{1i} \)
\[
\begin{align*}
\delta_{1i} &= c_{ii} + \sum_{j=1}^{\nu_1} c_{ij} b_{jj} + \sum_{j=\nu_1+1}^{\nu_2} c_{ij} b_{jj} + \sum_{j=\nu_2+1}^{g} c_{ij} b_{jj} \\
&= 1 + \sum_{j=1}^{\nu_1} \delta_{ji} \cdot 0 + \sum_{j=\nu_1+1}^{\nu_2} 0 \cdot 0 + \sum_{j=\nu_2+1}^{g} 0 \cdot 0 = 1 \quad \text{if } 1 \leq i \leq \nu_1 \\
&= 0 + \sum_{j=1}^{\nu_1} 0 \cdot 0 + \sum_{j=\nu_1+1}^{\nu_2} 0 \cdot 1 + \sum_{j=\nu_2+1}^{g} 0 \cdot 0 = 0 \quad \text{if } \nu_1 + 1 \leq i \leq \nu_2 \\
&= 1 + \sum_{j=1}^{\nu_1} 0 \cdot 0 + \sum_{j=\nu_1+1}^{\nu_2} 0 \cdot 0 + \sum_{j=\nu_2+1}^{g} 1 \cdot 1 = 1 \quad \text{if } \nu_2 + 1 \leq i \leq g
\end{align*}
\]
since \( \sum_{j=\nu_2+1}^{g} 1 = g - \nu_2 = 0 \in \mathbb{F}_2 \). Consequently the vectors \( \delta_1 \) and \( \delta_2 \) do arise from a symplectic matrix \( BC \) as in (6.112) and (6.113), and that suffices to conclude the proof.

The preceding lemma can be applied to yield a complete description of the canonical family of hyperabelian factors of automorphy of characteristic class \( g \).
on a compact Riemann surface $M$ of genus $g > 0$, paralleling the description of the basic family of hyperabelian factors of automorphy of characteristic class $g$ in terms of the parameter $\delta \in \mathbb{Z}^{2g}$ in (6.84).

**Theorem 6.26** The canonical family of hyperabelian factors of automorphy of characteristic class $g$ on a compact Riemann surface $M$ of genus $g > 0$, in terms of a base point $a \in \tilde{M}$ and bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ for which the intersection matrix is the basic skew-symmetric matrix $P = J$, consists of a family of factors of automorphy parametrized by integer vectors $\delta = (\delta_1, \delta_2) \in \mathbb{Z}^{2g}$ (mod 2) where $\delta_1, \delta_2 \in \mathbb{Z}^g$ satisfy $\delta_1 \cdot \delta_2 \equiv 0 \pmod{2}$; a member of this family associates to a homology class $\tau = \sum_{i=1}^{2g} n_i \tau_i \in H_1(M)$ the function

\[ \zeta_{g,a;\delta_1,\delta_2}(\tau, z) = (-1)^{n_1 \cdot n_2 + \delta_1 \cdot n_1 + \delta_2 \cdot n_2} \exp \left( \frac{1}{2} \Im \tau \cdot n + 2\pi \Im \hat{\omega}(z, a) G \Omega n \right) \]

in which $\Omega$ is the period matrix in terms of the chosen bases, $G = \frac{1}{2} \Im H^{-1} = \frac{1}{2} \Im H^{-T}$ for the positive definite Hermitian matrix $H = \i \Omega \overline{\Omega}$ and $R = 2\pi \Im(\i \Omega \overline{\Omega})$.

This collection of factors of automorphy has the same form in terms of any bases $\omega_i$ and $\tau_j$ for which the intersection matrix is the basic skew-symmetric matrix $P = J$, in terms of the matrices $R, G, H$ defined in terms of those bases.

**Proof:** The canonical family of hyperabelian factors of automorphy of characteristic class $g$ on $M$ is defined as the collection of hyperabelian factors of automorphy that arise from the factor of automorphy $\zeta_{g,a}(\tau, z)$ given by (6.80) in Theorem 6.21 in terms of any one choice of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ by all changes of the basis for the homology of $M$ that preserve the intersection matrix of the surface, which is assumed to be the basic skew-symmetric matrix $P = J$. For this intersection matrix the sign factor $\epsilon(\tau)$ in (6.80) for a homology class $\tau = \sum_{j=1}^{2g} n_j \tau_j$ is $\epsilon(\tau) = \exp \frac{\pi}{2} \Im \hat{\omega} n = \exp \pi i n \cdot n = (-1)^{n_1 \cdot n_2}$ when the vector $n \in \mathbb{Z}^{2g}$ is written as $n = (n_1, n_2)$ for $n_i \in \mathbb{Z}^g$, since $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The changes of the homology basis $\tau_j$ that preserve the intersection matrix are described by symplectic matrices $Q \in \text{Sp}(2g, \mathbb{Z})$; and by Theorem 6.21 (iii) the effect of the change of homology basis described by $Q$ is to multiply the sign factor $\epsilon(\tau)$ by $(-1)^{b(3, Q) \cdot n}$ as in (6.82) for the vector (6.83). By the preceding Lemma 6.25 the vectors so arising are just the vectors $\delta = (\delta_1, \delta_2) \in \mathbb{Z}^{2g}$ where $\delta_1, \delta_2 \in \mathbb{Z}^g$ are vectors in $\mathbb{Z}^g$ such that $\delta_1 \cdot \delta_2 \equiv 0 \pmod{2}$, and that suffices for the proof.

**Corollary 6.27** The canonical family of hyperabelian factors of automorphy of characteristic class $g$ on a compact Riemann surface of genus $g > 0$ consists of $2^{g-1}(2^g + 1)$ distinct factors of automorphy.

**Proof:** If $M$ is a Riemann surface of genus $g > 0$ the preceding theorem shows that the number of distinct factors of automorphy in the canonical family of hyperabelian factors of automorphy of characteristic class $g$ is equal to the
number \( \nu_g \) of pairs of vectors \( \delta_1, \delta_2 \in F_g^2 \) such that \( \delta_1 \cdot \delta_2 = 0 \) in \( F_2 \). It will be demonstrated by induction on \( g \) that

\[
(6.117) \quad \nu_g = 2^{g-1}(2^g + 1).
\]

It was already observed in the discussion of the special case \( g = 1 \) that \( \nu_1 = 3 \), which is (6.117) for the case \( g = 1 \). Assume that the result has been demonstrated for a value \( g \geq 1 \), so that there are \( \nu_g = 2^{g-1}(2^g + 1) \) pairs of vectors \( \delta_1, \delta_2 \in F_g^2 \) such that \( \delta_1 \cdot \delta_2 = 0 \). There are altogether \( 2^{2g} \) pairs of vectors \( \delta_1, \delta_2 \in F_g^2 \), and consequently there are \( 2^{2g} - \nu_g = 2^{g-1}(2^g - 1) \) pairs of vectors \( \delta_1, \delta_2 \in F_g^2 \) such that \( \delta_1 \cdot \delta_2 = 1 \). Any vector \( \delta'_i \in F_{g+1}^2 \) can be written \( \delta'_i = (\delta_i, d_i) \) for some \( d_i \in F_2 \), and \( \delta'_1 \cdot \delta'_2 = \delta_1 \cdot \delta_2 + d_1 d_2 \). Therefore \( \delta'_1 \cdot \delta'_2 = 0 \) if and only if either \( \delta_1 \cdot \delta_2 = 0 \) and \( d_1 d_2 = 0 \), for which there are \( 3\nu_g \) possibilities by the induction hypothesis and the result for the special case \( g = 1 \), or alternatively \( \delta_1 \cdot \delta_2 = 1 \) and \( d_1 d_2 = 1 \), for which there are \( 2^{2g} - \nu_g \) possibilities by the induction hypothesis and the result for the special case \( g = 1 \). Therefore altogether there are \( 3\nu_g + 2^{2g} - \nu_g = 3 \cdot 2^{g-1}(2^g + 1) + 2^{g-1}(2^g - 1) = 2^g(2^{g+1} + 1) \) possibilities, which is (6.117) for the case \( g+1 \), thereby completing the inductive step and proving the corollary.
Chapter 7

Families of Holomorphic Line Bundles

When considering the spaces of holomorphic cross-sections of families of holomorphic line bundles over a compact Riemann surface $M$ from an analytic perspective, it is convenient and relevant to the analytic interpretation to represent line bundles by factors of automorphy for the action of the covering translation group $\Gamma$ on the universal covering space $\tilde{M}$ of that Riemann surface and to describe the holomorphic cross-sections of these line bundles by holomorphic relatively automorphic functions for the factors of automorphy. It was demonstrated in the preceding Chapter 6 that every holomorphic line bundle over $M$ actually can be described by a factor of automorphy, indeed by an abelian factor of automorphy. For the discussion in this chapter the specific form of the factor of automorphy generally is not relevant. What is very relevant though is that if $\eta(T, z)$ is a factor of automorphy describing a holomorphic line bundle $\eta$ of characteristic class $c(\eta) = r$ over $M$, all holomorphic line bundles of characteristic class $r$ can be represented by factors of automorphy of the form $\rho_t(T) \eta(T, z)$ for parameters $t \in \mathbb{C}^{2g}$, where $\rho_t$ is the flat line bundle parametrized by the point $t \in \mathbb{C}^{2g}$ under the canonical parametrization (3.26) of flat line bundles associated to generators $T_1, \ldots, T_{2g} \in \Gamma$ representing a basis $\tau_j H_1(M)$; explicitly $\rho_t \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is the representation of the group $\Gamma$ for which $\rho_t(T_j) = \exp 2\pi i t_j$ for $1 \leq j \leq 2g$. Families of holomorphic line bundles of characteristic class $g$ then can be described by factors of automorphy $\rho_t(T) \eta(T, z)$ for parameter values $t \in V$ lying in subsets $V \subset \mathbb{C}^{2g}$; and holomorphic cross-sections of these families of line bundles can be described by relatively automorphic functions $f(z, t) \in \Gamma(M, \mathcal{O}(\rho_t \eta))$ of the variable $z \in \tilde{M}$ that also depend on the variable $t \in \mathbb{C}^{2g}$. When the subset $V \subset \mathbb{C}^{2g}$ is a holomorphic subvariety the factors of automorphy $\rho_t(T) \eta(T, z)$ are holomorphic functions of the parameter $t \in V$, and it is possible to consider relatively automorphic functions $f(z, t)$ for these factors of automorphy that are also holomorphic or meromorphic functions of the variables $(z, t) \in \tilde{M} \times V$. 

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**Theorem 7.1** Let \( \eta \) be a factor of automorphy describing a holomorphic line bundle of characteristic class \( c(\eta) = r \) on a compact Riemann surface \( M \) of genus \( g > 0 \); let \( a_1, \ldots, a_n \) be \( n \) points of the universal covering space \( \tilde{M} \) of the surface \( M \), not necessarily distinct; and let \( t_0 \in \mathbb{C}^{2g} \) be a point in the parameter space for the canonical parametrization of flat line bundles over \( M \) associated to generators \( T_1, \ldots, T_{2g} \in \Gamma \) of the covering translation group of \( M \).

If \( r + n > 2g - 2 \) there are open neighborhoods \( \tilde{U}_j \subset \tilde{M} \) of the points \( a_j \), an open neighborhood \( \tilde{U}_0 \subset \mathbb{C}^{2g} \) of the point \( t_0 \), and \( r + n + 1 - g \) meromorphic functions \( f_i(z, u, t) \) of the variables \( z \in \tilde{M}, u_j \in \tilde{U}_j, t \in \tilde{U}_0 \) with singularities at most simple poles along the subvarieties \( z = T u_j \) for \( T \in \Gamma \) and \( 1 \leq j \leq n \), such that for any fixed points \( u_j \in \tilde{U}_j, t \in \tilde{U}_0 \) these functions are a basis for the vector space

\[
(7.1) \quad \Lambda_\eta(u, t) = \left\{ f \in \Gamma(M, \mathcal{M}(\rho_t \eta)) \left| \partial(f) + u_1 + \cdots + u_n \geq 0 \right. \right\}.
\]

**Proof:** To simplify the calculations slightly, choose a marking of the surface \( M \), that is, a base point \( a \in \tilde{M} \) and generators \( A_j = T_j, B_j = T_{g+j} \in \Gamma \) for \( 1 \leq j \leq g \); and suppose that the canonical parametrization of flat line bundles is expressed in terms of these generators \( T_j \in \Gamma \). Let \( \omega(z) \) be the canonical basis for the holomorphic abelian differentials on the marked Riemann surface \( M \) and let \( w_i(z, a) \) be the associated holomorphic abelian integrals at the base point \( a \in \tilde{M} \). The \( g \times 2g \) period matrix of the differentials \( \omega_i(z) \) has the form \( \Omega = \begin{pmatrix} I & Z \end{pmatrix} \) for a \( g \times g \) symmetric matrix \( Z = \{ z_{ij} \} \) in the Siegel upper half-space \( \mathcal{H}_g \) of rank \( g \), as in Theorem 3.22. Let \( \tilde{U}_j \subset \tilde{M} \) be open coordinate neighborhoods of the points \( a_j \in \tilde{M} \) for \( 1 \leq j \leq n \). Choose \( g \) auxiliary points \( b_k \in \tilde{M} \) that represent distinct points of the surface \( M \) that are also distinct from any of the points of \( \tilde{M} \) represented by the points \( a_j \) and are such that the \( g \times g \) matrix \( W' = \{ w_i'(b_k, a) \} \) is nonsingular, where \( w_i'(b_k, a) = \partial w_i(z_k, a)/\partial z_k |_{z_k = b_k} \) in terms of a local coordinate \( z_k \) centered at the point \( b_k \); and let \( \tilde{V}_k \subset \tilde{M} \) be open coordinate neighborhoods of the points \( b_k \). By shrinking the neighborhoods \( \tilde{U}_j \) and \( \tilde{V}_k \) if necessary it can be supposed that the open subsets \( \tilde{V}_1, \ldots, \tilde{V}_g, \bigcup_{j=1}^n \tilde{U}_j \) of \( \tilde{M} \) represent \( g+1 \) disjoint open subsets of the surface \( M \). The holomorphic line bundles \( \zeta_a = \zeta_{a_1+\cdots+a_n} \) and \( \zeta_b = \zeta_{b_1+\cdots+b_g} \) have characteristic classes \( c(\zeta_a) = n \) and \( c(\zeta_b) = g \) respectively; and since \( c(\eta_{\zeta_a}) = r + n > 2g - 2 \) by assumption it follows from the Riemann-Roch Theorem that \( \gamma(\rho_t, \eta_{\zeta_a}) = r + n + 1 - g \) and \( \gamma(\rho_t, \eta_{\zeta_b}) = r + n + 1 \). Consider then the vector space

\[
X = \left\{ g \in \Gamma(M, \mathcal{M}(\rho_t \eta_{\zeta_a})) \left| \partial(g) + b_1 + \cdots + b_g \geq 0 \right. \right\}
\]

consisting of meromorphic cross-sections of the line bundle \( \rho_t \eta_{\zeta_a} \) with singularities at most simple poles at the points \( \Gamma b_k \subset \tilde{M} \). Since \( X \) and \( \Gamma(M, \mathcal{O}(\rho_t \eta_{\zeta_a})) \) are isomorphic vector spaces under the isomorphism that arises by multiplying a cross-section \( g \in X \) by a nontrivial cross-section \( h \in \Gamma(M, \mathcal{O}(\zeta_b)) \) that vanishes at the points \( b_k \), as in the argument on page 47, it follows that \( \dim X = \gamma(\rho_t, \eta_{\zeta_a} \zeta_b) = r + n + 1 \). Let \( g_i(z) \) be a basis for the vector space.
X for \(0 \leq i \leq r + n\), where the cross-sections \(g_i(z)\) are meromorphic functions on \(\tilde{M}\) that are relatively automorphic functions for the factor of automorphy \(\rho_{t_0}\eta\gamma\alpha\); and if \(\hat{q}(z_1, z_2; z_3, z_4)\) is the canonical cross-ratio function of the marked Riemann surface \(M\) and \(h_{a_j}(z) \in \Gamma(M, \mathcal{O}(\zeta_{a_j}))\) is a nontrivial holomorphic cross-section vanishing at the points \(\Gamma_{a_j}\), set

\[
\hat{g}_i(z, u, x, s) = g_i(z) \cdot \prod_{j=1}^{n} \left( h_{a_j}(z)^{-1} \hat{q}(z, p; a_j, u_j) \right) \cdot \prod_{k=1}^{g} \hat{q}(z, q; b_k, x_k) \cdot \exp 2\pi i \sum_{l=1}^{g} s_l w_l(z, b)
\]

for \(0 \leq i \leq r + n\), where \(z \in \tilde{M}\), \(u_j \in \tilde{U}_j \subset \tilde{M}\), \(x_k \in \tilde{V}_k \subset \tilde{M}\), \(s_l \in \mathbb{C}\), and \(p, q \in \tilde{M}\) are fixed points of \(\tilde{M}\) such that \(p, q \notin \left( \bigcup_{j=1}^{n} \Gamma_{U_j} \right) \cup \left( \bigcup_{k=1}^{g} \Gamma_{V_k} \right)\); thus \(T u_j \neq p\) and \(T x_k \neq q\) for any points \(u_j \in \tilde{U}_j\), \(x_k \in \tilde{V}_k\), and any \(T \in \Gamma\), so in particular \(T a_j \neq p\) and \(T b_k \neq q\). Recall from Theorem 5.11 (i) that the cross-ratio function \(\hat{q}(z_1, z_2; z_3, z_4)\) has simple zeros along the subvarieties \(z_1 = T z_3\) and \(z_2 = T z_4\) and simple poles along the subvarieties \(z_1 = T z_4\) and \(z_2 = T z_3\) for all \(T \in \Gamma\), and no other zeros or poles on the surface \(\tilde{M}\).

It follows that the product \(h_{a_j}(z)^{-1} \hat{q}(z, p; a_j, u_j)\) is a nontrivial meromorphic function of the variables \((z, u_j)\) in \(\tilde{M} \times \tilde{U}_j\) with at most simple poles along the subvarieties \(z = T u_j\) for \(1 \leq j \leq n\) and all \(T \in \Gamma\) and no other singularities, since the poles of the function \(h_{a_j}(z)^{-1}\) at \(z = T a_j\) are cancelled by the zeros of the cross-ratio function there; and the product \(g_i(z) \prod_{k=1}^{g} \hat{q}(z, q; b_k, x_k)\) is a nontrivial meromorphic function of the variables \((z, x)\) in \(\tilde{M} \times \prod_{k=1}^{g} \tilde{V}_k\) with at most simple poles along the subvarieties \(z = T x_k\) in \(\tilde{M}\) for \(1 \leq k \leq g\) and all \(T \in \Gamma\) and no other singularities, since the poles of the function \(g_i(z)\) at the points \(T b_k\) are cancelled by the zeros of the cross-ratio functions at those points.

Altogether the functions \(\hat{g}_i(z, u, x, s)\) are meromorphic functions of the variables \((z, u, x, s)\) in \(\tilde{M} \times \prod_{j=1}^{n} \tilde{U}_j \times \prod_{k=1}^{g} \tilde{V}_k \times \mathbb{C}^2\) with singularities at most simple poles along the subvarieties \(z = T u_j\) and \(z = T x_k\) and no other singularities; and for any fixed points \((u, x, s)\) they are \(r + n + 1\) linearly independent meromorphic functions of the variable \(z \in \tilde{M}\).

Furthermore it follows from Theorem 5.11 (iii) that

\[
\hat{g}_i(A_m z, u, x, s) = \hat{g}_i(z, u, x, s) \rho_{t_0} \eta(A_m, z) \exp 2\pi i s_m
\]

\[
\hat{g}_i(B_m z, u, x, s) = \hat{g}_i(z, u, x, s) \rho_{t_0} \eta(B_m, z) \cdot \exp 2\pi i \left( -\sum_{j=1}^{n} w_m(u_j, a_j) - \sum_{k=1}^{g} w_m(x_k, b_k) + \sum_{l=1}^{g} s_l z_{lm} \right)
\]

for \(1 \leq m \leq g\); hence for each fixed point \((u, x, s)\) these functions are \(r + n + 1\) linearly independent meromorphic relatively automorphic functions for the factor of automorphy \(\rho \eta\) where

(7.2)
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$$t_m = \begin{cases} 
  t_{0m} + s_m & \text{for } 1 \leq m \leq g, \\
  t_{0m} - \sum_{j=1}^g w_{m-g}(u_j, a_j) - \sum_{k=1}^g w_{m-g}(x_k, b_k) + \sum_{l=1}^g s_l z_l m^{-g} & \text{for } g + 1 \leq m \leq 2g,
\end{cases}$$

and $t_{0m}$ are the coordinates of the initial point $t_0 \in \mathbb{C}^g$. In particular since $\hat{q}(z, p; a_j, a_j) = \hat{q}(z, q; b_k, b_k) = 1$ by Theorem 5.11 (i) it follows that

$$\hat{g}_i(z, a, b, 0) = g_i(z) \prod_{j=1}^n h_{a_j}(z)^{-1} \in \Gamma(M, \mathcal{M}(\rho_{a, \eta}));$$

these are $r + n + 1$ linearly independent meromorphic relatively automorphic functions for the factor of automorphy $\rho_{a, \eta}$ with poles at most at the divisors $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_g$ and no other singularities, so they are a basis for the vector space

$$Y = \left\{ g \in \Gamma(M, \mathcal{M}(\rho_{a, \eta})) \mid \hat{d}(g) = a_1 + \cdots + a_n + b_1 + \cdots + b_g = 0 \right\}$$

since as in the earlier argument $\dim Y = \gamma(\rho_{a, \eta} \zeta_a \zeta_b) = r + n + 1$.

The residue of the meromorphic function $\hat{g}_i(z, u, x, s)$ of the variable $z \in \hat{M}$ at the simple pole $z = x_p$ is

$$r_{ip}(u, x, s) = \frac{1}{2\pi i} \int_{z \in \partial V_p} \hat{g}_i(z, u, x, s) dz,$$

which is a holomorphic function of the variables $u_j \in \Gamma \hat{U}_j$, $x_k \in \Gamma \hat{V}_k$, $s_l \in \mathbb{C}$. Let $R(u, x, s) = \{r_{ip}(u, x, s)\}$ be the $(r + n + 1) \times g$ matrix composed of these functions. For any vector $c = (c_0, \ldots, c_{r+n}) \in \mathbb{C}^{r+n+1}$ it is evident that $\sum_{l=0}^{r+n} c_l r_{ip}(a, b, 0) = 0$ for $1 \leq i \leq g$ if and only if the linear combination $\hat{g}_c(z) = \sum_{l=0}^{r+n} c_l \hat{g}_i(z, a, b, 0)$ is a meromorphic relatively automorphic function $\hat{g}_c \in \Gamma(M, \mathcal{M}(\rho_{a, \eta}))$ for which $\hat{d}(\hat{g}_c) + a \geq 0$. Since the functions $\hat{g}_i(z, a, b, 0)$ are a basis for the vector space $Y$ it follows that the functions $\hat{g}_c(z)$ for vectors $c$ for which $\sum_{l=0}^{r+n} c_l r_{ip}(a, b, 0) = 0$ for $1 \leq i \leq g$ span the $(r + n + 1 - g)$-dimensional space of relatively automorphic functions for the factor of automorphy $\rho_{a, \eta}$ with singularities at most along the divisor $a$; consequently there are precisely $r + n + 1 - g$ linearly independent such vectors $c$, so the $(r + n + 1) \times g$ matrix $R(u, x, s)$ must have rank $g$ at the point $(u, x, s) = (a, b, 0)$. This matrix then also has rank $g$ at all nearby points, so it follows from familiar arguments that after shrinking the neighborhoods $\hat{U}_j, \hat{V}_k$ further if necessary and choosing a sufficiently small open neighborhood $\hat{W}$ of the origin in $\mathbb{C}^g$ there will exist $r + n + 1 - g$ holomorphic mappings

$$c^i : \prod_{j=1}^n \hat{U}_j \times \prod_{k=1}^g \hat{V}_k \times \hat{W} \to \mathbb{C}^{r+n+1}$$
that are linearly independent vectors at each point \((u, x, s)\) and are such that 
\[ \sum_{i=0}^{r+n} c_i^p(u, x, s) = 0 \] for \(1 \leq p \leq g\), where \(c_i^p(u, x, s) = \{c_i^p(u, x, s)\}\).

In case the sort of arguments required for this conclusion are not altogether familiar, a detailed proof is included in Lemma 7.2 following the proof of the present theorem. The \(r + n + 1 - g\) linear combinations

\[ f_i(z, u, x, s) = \sum_{l=0}^{r+n} c_i^l(u, x, s) \hat{g}_i(z, u, x, s) \]

therefore are meromorphic functions of the variables \(z \in \tilde{M}, u_j \in \tilde{U}_j, x_k \in \tilde{V}_k, s \in \tilde{W} \) with singularities at most simple poles along the subvarieties \(z = Tu_j\), since the singularities along the subvarieties \(z = Tx_k\) have been eliminated; and for any fixed point \((u, x, s)\) they are linearly independent meromorphic relatively automorphic functions for the factor of automorphy \(\rho_t \eta\), so are a basis for the vector space \(\Lambda(u, t)\) where the parameter \(t \in \mathbb{C}^2\) is the function of the parameters \((u, x, s)\) given explicitly by (7.2).

To examine \(t\) as a function of the variables \((u, x, s)\) consider the holomorphic mapping

\[ F: \prod_{j=1}^n \tilde{U}_j \times \prod_{k=1}^g \tilde{V}_k \times \tilde{W} \rightarrow \prod_{j=1}^n \tilde{U}_j \times \mathbb{C}^{2g} \]

defined by \(F(u, x, s) = (u, t(u, x, s))\), and note that in particular \(F(a, b, 0) = (a, t_0)\). It is natural to write \(t(u, x, s) = (t'(u, x, s), t''(u, x, s)) \in \mathbb{C}^g \times \mathbb{C}^g\), since \(t(u, x, s)\) is defined separately in (7.2) for the first and second \(g\) components; it then follows from (7.2) that the Jacobian matrix \(J\) of this mapping at the point \((a, b, 0)\) is

\[
\begin{pmatrix}
\frac{\partial u}{\partial u} & \frac{\partial u}{\partial x} & \frac{\partial u}{\partial s} \\
\frac{\partial t'}{\partial u} & \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial s} \\
\frac{\partial t''}{\partial u} & \frac{\partial t''}{\partial x} & \frac{\partial t''}{\partial s}
\end{pmatrix}
\begin{pmatrix}
u = a \\
x = b \\
t = 0
\end{pmatrix}
= \begin{pmatrix}
I & 0 & 0 \\
0 & * & 1 \\
-\mathbf{W}' & Z
\end{pmatrix}.
\]

The matrix \(W'\), consisting of the derivatives of the holomorphic abelian integrals at the points \(b_k\), is nonsingular by the choice of the auxiliary points \(b_k\); consequently the Jacobian matrix \(J\) also is nonsingular, so after shrinking the neighborhoods \(\tilde{U}_j, \tilde{V}_k\) and \(\tilde{W}\) further if necessary \(F\) will be a biholomorphic mapping onto an open neighborhood of the point \((a, t_0) \in \prod_{j=1}^n \tilde{U}_j \times \mathbb{C}^{2g}\). Through this mapping the cross-sections \(f_i(z, u, x, s) \in \Gamma(M, \mathcal{O}(\rho_t \eta))\) can be viewed alternatively as cross-sections \(f_i(z, u, t) \in \Gamma(M, \mathcal{O}(\rho_t \eta))\) that are meromorphic functions of the parameters \((u, t)\) rather than of the parameters \((u, x, s)\). Although it was assumed for convenience that the canonical parametrization of
flat line bundles over $M$ was taken with respect to generators for $\Gamma$ arising from a marking of the surface, it is clear once the theorem has been proved for these generators that it is valid for the canonical parametrization of flat line bundles with respect to any generators $T_j \in \Gamma$ representing a basis for the homology $H_1(M)$. That suffices to conclude the proof of the theorem.

The auxiliary result used in the proof of the preceding theorem is demonstrated by the following slightly more general lemma.

**Lemma 7.2** If $F(z)$ is an $m \times n$ matrix of holomorphic functions in an open neighborhood $U$ of a point $z_0$ in a holomorphic variety, and if rank $F(z) = r$ at all points $z \in U$, then after shrinking the neighborhood $U$ if necessary there are $n - r$ holomorphic mappings $g_i : U \to \mathbb{C}^n$ that have linearly independent values at each point $z \in U$ and that satisfy $F(z)g_i(z) = 0$ at each point $z \in U$.

**Proof:** After rearranging rows and columns as necessary it can be assumed that the leading $r \times r$ minor of the matrix $F(z_0)$ is nonsingular; and after shrinking the neighborhood $U$ if necessary that also will be the case for the matrices $F(z)$ at all points $z \in U$. The matrix $F(z)$ thus can be decomposed into matrix blocks

$$F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}$$

where the $r \times r$ submatrix $F_{11}(z)$ is nonsingular for all points $z \in U$. Since rank $F(z) = r$ the last $m - r$ rows of the matrix $F(z)$ are unique linear combinations of the first $r$ rows, where the coefficients in these linear combinations must be holomorphic functions in $U$; thus

$$\begin{pmatrix} F_{21}(z) & F_{22}(z) \end{pmatrix} = A(z) \begin{pmatrix} F_{11}(z) & F_{12}(z) \end{pmatrix}$$

for some $(m - r) \times r$ matrix $A(z)$ of holomorphic functions of the variable $z \in U$. If $c_i \in \mathbb{C}^{n-r}$ for $1 \leq i \leq n - r$ are $n - r$ linearly independent constant vectors then the $n - r$ vectors

$$g_i(z) = \begin{pmatrix} -F_{11}(z)^{-1}F_{12}(z)c_i \\ c_i \end{pmatrix} \in \mathbb{C}^n$$

for $1 \leq i \leq n - r$ are linearly independent holomorphic functions of the variable $z \in U$ such that

$$\begin{pmatrix} F_{11}(z) & F_{12}(z) \end{pmatrix}g_i(z) = 0;$$

and in addition

$$\begin{pmatrix} F_{21}(z) & F_{22}(z) \end{pmatrix}g_i(z) = A(z)\begin{pmatrix} F_{11}(z) & F_{12}(z) \end{pmatrix}g_i(z) = 0$$

as well, so altogether $F(z)g_i(z) = 0$ for $1 \leq i \leq n - r$. That suffices to conclude the proof.
In the definition (7.1) of the vector space $\Lambda_\eta(u,t)$ the meromorphic cross-sections $f \in \Gamma(M, \mathcal{M}(\rho_\eta))$ are required to satisfy the condition that

$$\delta(f) + u_1 + \cdots + u_n \geq 0$$

for points $u_i \in \tilde{M}$, a condition that arose naturally in the course of the proof through the use of cross-ratio functions. However the cross-sections have the same singularities at all points of $\tilde{M}$ representing the same point of $M$, so it is evident that this condition actually can be viewed as a condition involving points $u_i \in M$ rather than points on the universal covering space; that is the most convenient interpretation for most applications of the theorem. The hypothesis that the family of holomorphic line bundles involved and the singularities allowed for the cross-sections are restricted so that $\dim \Lambda_\eta$ is independent of $u$ and $t$. With these modifications the preceding theorem can be restated as follows.

**Corollary 7.3** Let $\eta$ be a factor of automorphy describing a holomorphic line bundle on a compact Riemann surface $M$ of genus $g > 0$; let $W$ be a holomorphic subvariety of an open subset of the complex manifold $M^n$; and consider the family of flat line bundles $\rho_i$ parametrized by a holomorphic subvariety $\tilde{V}$ of an open subset of the parameter space $C^{2g}$ for the canonical parametrization of flat line bundles over $M$ associated to generators $T_1, \ldots, T_{2g} \in \Gamma$ of the covering translation group $\Gamma$ of $M$. If $\dim \Lambda_\eta(u,t) = \nu$ for all points $u = (u_1, \ldots, u_n) \in W$ and $t \in \tilde{V}$ then for any points $u \in W$ and $t_0 \in \tilde{V}$ there are open neighborhoods $U' \subset M^n$ of the point $u$ and $U'' \subset C^{2g}$ of the point $t_0$ and $\nu$ meromorphic functions $f_i(z,u,t)$ on the holomorphic variety $M \times (U' \cap W) \times (U'' \cap V)$ such that for any fixed points $u \in U' \cap W$ and $t \in U'' \cap V$ these functions are a basis for the vector space $\Lambda_\eta(u,t)$.

**Proof:** If $c(\eta) + n > 2g - 2$ the desired result is just that of the preceding theorem, upon recalling that in the conclusions of that theorem it can be assumed that the parameters $u_i$ lie in the Riemann surface $M$ rather than in its universal covering space. If $c(\eta) + n \leq 2g - 2$ choose $m = 2g - c(\eta) - n - 1$ points $b_1, \ldots, b_m$ on $M$ which represent distinct points of $M$ which are also distinct from the points $a_j$; and let $h_b \in \Gamma(M, \mathcal{O}(\zeta_{b_1} + \cdots + b_m))$ be a holomorphic cross-section such that $\delta(h_b) = b_1 + \cdots + b_m$. Since $c(\eta \zeta_{b_k}) = c(\eta) + n + m = 2g - 1$ it follows from the preceding theorem applied to the line bundle $\eta \zeta_{b_k}$ that there are open neighborhoods $U'_j \subset M$ of the points $a_j$, an open neighborhood $U'' \subset C^{2g}$ of the point $t_0$, and $g$ meromorphic functions $g_l(z,u,t)$ on the variety $\tilde{M} \times U'_1 \times \cdots \times U'_n \times \tilde{U}''$ with singularities at most simple poles along the subvarieties $z = Tu_j$ for $T \in \Gamma$, $1 \leq j \leq n$, such that for any fixed points $u_j \in U'_j$, $t \in U''$ these functions are a basis for the $g$-dimensional vector space

$$X = \left\{ g \in \Gamma(M, \mathcal{M}(\rho_\eta \zeta_{b_k})) \mid \delta(g) + u_1 + \cdots + u_n \geq 0 \right\}.$$ 

Assume that the neighborhoods $U'_j$ are sufficiently small that they do not contain any of the points $b_k$, and set $U'' = U'_1 \times \cdots \times U'_n \subset M^n$. If $(u_1, \ldots, u_n) \in$
$U' \cap W$ and $t \in \tilde{U}'' \cap V$ and if $c = (c_1, \ldots, c_g) \in \mathbb{C}^g$ is a vector such that 
\[ \sum_{i=1}^{g} c_i g_i(b_k, u, t) = 0 \] 
for $1 \leq k \leq m$ then $g_c(z, u, t) = \sum_{i=1}^{g} h_b(z)^{-1} c_i g_i(z, u, t)$ is an element of the vector space $\Lambda_g(u, t)$. Conversely whenever $g(z)$ is an element of the vector space $\Lambda_g(u, t)$ where $(u_1, \ldots, u_n) \in U' \cap W$ and $t \in \tilde{U}'' \cap V$ then $h_b(z)g(z)$ is an element of the vector space $X$ that vanishes at the distinct points $b_k$, hence it can be written $g(z) = \sum_{i=1}^{g} c_i g_i(b_k, u, t)$ for some vector $c = (c_1, \ldots, c_g) \in \mathbb{C}^g$ for which $\sum_{i=1}^{g} c_i g_i(b_k, u, t) = 0$ for $1 \leq k \leq m$. Since $\dim \Lambda_g(u, t) = \nu$ it follows that the set of vectors $c$ such that $\sum_{i=1}^{g} c_i g_i(b_k, u, t) = 0$ for $1 \leq k \leq m$ is a vector space of dimension $\nu$; consequently the $g \times m$ matrix $G(u, t) = \{g_i(b_k, u, t)\}$ has rank $g - \nu$ for any points $(u_1, \ldots, u_n) \in U' \cap W$ and $t \in \tilde{U}'' \cap V$. It then follows from Lemma 7.2 that, after shrinking the neighborhoods $U_j'$ and $U''$ sufficiently, there are $\nu$ holomorphic mappings $c^i : (U' \cap W) \times (\tilde{U}'' \cap V) \longrightarrow \mathbb{C}^g$ that have linearly independent values at each point and satisfy $\sum_{i=1}^{g} c^i(u, t) g_i(b_k, u, t) = 0$ for $1 \leq k \leq m$ and for all points $u \in U' \cap W$ and $t \in \tilde{U}'' \cap V$, where $c^i(u, t) = \{c^i(u, t)\}$. The functions $f_i(z, u, t) = \sum_{i=1}^{g} c^i(u, t) h_b(z)^{-1} g_i(z, u, t)$ are meromorphic functions on the variety $M' \times (U' \cap W) \times (\tilde{U}'' \cap V)$, and for any fixed point $(u, t) \in (U' \cap W) \times (\tilde{U}'' \cap V)$ they are a basis for the vector space $\Lambda_g(u, t)$. That suffices to conclude the proof of the corollary.

Two factors of automorphy $\rho_{t'} \eta$ and $\rho_{t'' \eta}$ are holomorphically equivalent whenever $t' - t'' \in \Omega \mathbb{C}^g + \mathbb{Z}^g$, by Corollary 3.14; and in that case the preceding results can be extended somewhat to show that the relatively automorphic functions for these two bundles are related through this holomorphic equivalence, when that equivalence is described fairly explicitly. For this purpose select generators $T_j \in \Gamma$ associated to a homology basis on the Riemann surface $M$ and a basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on $M$; so the associated period matrix of the surface is the $g \times 2g$ complex matrix $\Omega$ with entries $\omega_{ij} = \omega_i(T_j)$. In these terms introduce the auxiliary function $\phi(z, s)$ of points $(z, s) \in \tilde{M} \times \mathbb{C}^g$ defined by

\begin{equation}
\phi(z, s) = \exp 2\pi i \sum_{k=1}^{g} s_k w_k(z, p) \tag{7.3}
\end{equation}

where $w_k(z, p) = \int_{p}^{z} \omega_i$ are the integrals of the abelian differentials $\omega_i$ for a base point $p \in \tilde{M}$; as a function of the variable $z \in \tilde{M}$ this is a holomorphic and nowhere vanishing function on the universal covering surface $\tilde{M}$, and as a function of the variable $s = (s_1, \ldots, s_g) \in \mathbb{C}^g$ it is a homomorphism from the additive group $\mathbb{C}^g$ to the multiplicative group $\mathbb{C}^\ast$. Note that

\begin{align*}
\phi(T_j z, s) &= \exp 2\pi i \sum_{k=1}^{g} s_k w_k(T_j z, p) \\
&= \rho_{t' \eta}(T_j) \phi(z, s)
\end{align*}
for the flat line bundle $\rho_{\Omega s} \in \text{Hom}(\Gamma, \mathbb{C}^g)$ described by vector $\Omega s \subset \mathbb{C}^{2g}$. Since $\rho_{\Omega s}$ is a group homomorphism iterating the preceding formula shows that

$$\phi(T_j T_k z, s) = \rho_{\Omega s}(T_j) \phi(T_k z, s) = \rho_{\Omega s}(T_j) \rho_{\Omega s}(T_k) \phi(z, s)$$

and since any element $T \in \Gamma$ can be written as a product of the elements $T_j$ and their inverses it follows that

$$\phi(T z, s) = \rho_{\Omega s}(T) \phi(z, s)$$

for all $T \in \Gamma$. This exhibits explicitly the holomorphic triviality of the flat line bundles parametrized by these points $t \in \OmegaC^g$, and hence the holomorphic equivalence of the flat line bundles parametrized by points $t, t'' \in \mathbb{C}^{2g}$ for which $t - t'' \in \OmegaC^g$. Of course it is clear from the definition of the canonically parametrized flat line bundle $\rho_t$ is the identity bundle whenever $t \in \mathbb{Z}^{2g}$.

There is a convenient interpretation of this holomorphic equivalence that is worth noting here. The holomorphic equivalence of flat line bundles of Corollary 3.14 was expressed by the exact sequence

$$0 \rightarrow \OmegaC^g + \mathbb{Z}^{2g} \rightarrow \OmegaC^g \xrightarrow{p} P(M) \rightarrow 0$$

where $\iota$ is the inclusion mapping and $p$ is the mapping that associates to a point $t \in \mathbb{C}^{2g}$ the holomorphic line bundle represented by the canonically parametrized flat line bundle $\rho_t \in P(M)$ for the Picard variety $P(M)$ of the Riemann surface $M$. In this exact sequence (7.5) each coset $t + \OmegaC^g + \mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ consists of all the points of $\mathbb{C}^{2g}$ parametrizing an holomorphic equivalence class of flat line bundles; but for present purposes it is convenient to describe these cosets slightly differently. The full period matrix $(\Omega)$ of the Riemann surface $M$ is a nonsingular $2g \times 2g$ matrix, and the inverse of its transpose conjugate is a matrix of the form $(\Omega)^{-1}$ where the $g \times 2g$ matrix $\Pi$ is the inverse period matrix to $\Omega$; thus $\Pi$ is a $g \times 2g$ matrix such that

$$\Pi \Omega = 0, \quad \Pi \overline{\Pi} = I_g, \quad \Omega \Pi + \overline{\Omega \Pi} = I_{2g},$$

where $I_r$ denotes the identity matrix of rank $r$. The last identity in (7.6) shows that any point $s \in \mathbb{C}^{2g}$ can be written $s = \Omega \Pi s + \overline{\Omega \Pi s}$, which is an explicit formula for the direct sum decomposition

$$\mathbb{C}^{2g} = \OmegaC^g \oplus \overline{\OmegaC^g}$$

of the vector space $\mathbb{C}^{2g}$ into the two complementary linear subspaces $\OmegaC^g$ and $\overline{\OmegaC^g}$. Points $t \in \OmegaC^g$ thus can be viewed as representing cosets $t + \OmegaC^g \subset \mathbb{C}^{2g}$; and since parameter values in $\OmegaC^g \subset \mathbb{C}^{2g}$ describe holomorphically trivial flat line bundles it follows that any holomorphic equivalence class of flat line bundles can be parametrized by a point $t \in \overline{\OmegaC^g} \subset \mathbb{C}^{2g}$. Different points of $\overline{\OmegaC^g}$ still may parametrize holomorphically equivalent flat line bundles though. The first
two identities in (7.6) show that the linear mapping $\tilde{\Pi} : \mathbb{C}^{2g} \to \mathbb{C}^{2g}$ defined by the $2g \times 2g$ matrix $\Pi$ is the zero mapping on the linear subspace $\Omega \mathbb{C}^g \subset \mathbb{C}^{2g}$ and is the identity mapping on the linear subspace $\Pi \mathbb{C}^g \subset \mathbb{C}^{2g}$; thus it is the natural projection of the direct sum (7.7) to its second factor. Applying this projection to the exact sequence (7.5) hence yields the exact sequence

\[(7.8) \quad 0 \to \Pi \mathbb{Z}^{2g} \to \Pi \mathbb{C}^g \xrightarrow{p_0} P(M) \to 0\]

in which $\iota$ again is the natural inclusion mapping and $p_0$ is just the restriction $p_0 = p|\Pi \mathbb{C}^g$ of the mapping $p$ in the sequence (7.5) to the subspace $\Pi \mathbb{C}^g \subset \mathbb{C}^{2g}$. The resulting description of the Picard variety $P(M)$ as the quotient

\[P(M) = \Pi \mathbb{C}^g / \Pi \mathbb{Z}^{2g}\]

clearly is equivalent to the customary description as the quotient $P(M) = \mathbb{C}^g / \mathbb{Z}^{2g}$; so the mapping $p_0$ in (7.8) is a covering projection, and two points of $\Pi \mathbb{C}^g$ parametrize holomorphically equivalent flat line bundles precisely when they differ by a point in the lattice subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^g$. If $U \subset P(M)$ is a contractible open subset and $\tilde{U} \subset \Pi \mathbb{C}^g$ is a connected component of the inverse image $p_0^{-1}(U)$ under the covering projection $p_0$ of (7.8) the restriction of the mapping $p_0$ to the set $\tilde{U}$ is a one-to-one mapping $p_0 : \tilde{U} \to U$; so parameters $t \in \tilde{U}$ can be used as local coordinates in the subset $U \subset P(M)$. The complete inverse image of the subset $U$ under the covering projection $p_0$ is the disjoint union

\[(7.9) \quad p_0^{-1}(U) = \bigcup_{\nu \in \mathbb{Z}^{2g}} (\tilde{U} + t \Pi \nu) \subset \Pi \mathbb{C}^g\]

of translates of the subset $\tilde{U} \subset \Pi \mathbb{C}^g$ by points in the lattice subgroup $\Pi \mathbb{Z}^{2g} \subset \Pi \mathbb{C}^g$. The complete inverse image of the subset $U$ under the mapping $p$ of the exact sequence (7.5), the set of all points of $\mathbb{C}^{2g}$ parametrizing holomorphic line bundles in the subset $U \subset P(M)$, is the disjoint union

\[(7.10) \quad p^{-1}(U) = \bigcup_{\nu \in \mathbb{Z}^{2g}} (\tilde{U} + t \Omega \mathbb{C}^g + t \Pi \nu) \subset \mathbb{C}^{2g}\]

of translates of the open subset $\tilde{U} + t \Omega \mathbb{C}^g \subset \mathbb{C}^{2g}$ by points of the lattice subgroup $\Pi \mathbb{Z}^{2g} \subset \Pi \mathbb{C}^g$.

**Corollary 7.4** Let $\eta$ be a factor of automorphy describing a holomorphic line bundle on a compact Riemann surface $M$ of genus $g > 0$, let $W$ be a holomorphic subvariety of an open subset of the complex manifold $M^n$, let $V$ be a holomorphic subvariety of an open subset of the Picard variety $P(M)$ and set $\tilde{V} = p^{-1}(V) \subset \mathbb{C}^{2g}$; and suppose that $\dim \Lambda_\eta(u, t) = \nu$ for all points $u = (u_1, \ldots, u_n) \in W$ and $t \in \tilde{V} \subset \mathbb{C}^{2g}$. For any points $u_0 \in W$ and $\xi_0 \in V$ there are open neighborhoods $U' \subset M^n$ of the point $u_0$ and $U'' \subset P(M)$ of the point $\xi_0$ and $\nu$ meromorphic functions $f_i(z, u, t)$ on the holomorphic variety $\tilde{M} \times (W \cap U') \times (\tilde{V} \cap U'')$ for $1 \leq i \leq \nu$, where $\tilde{U}'' = p^{-1}(U'') \subset \mathbb{C}^{2g}$, such that these functions are a basis for the vector space $\Lambda_\eta(u, t)$ at any point $(u, t) \in (W \cap U') \times (\tilde{V} \cap U'')$ and satisfy

\[(7.11) \quad f_i(z, u, t + \imath \Omega s + \Pi \nu) = \phi(z, s - \Pi \nu)\ f_i(z, u, t)\]
for all points \( z \in \tilde{M} \), \( u \in W \cap U' \), \( t \in \tilde{V} \cap \tilde{U}'' \), \( s \in \mathbb{C}^g \), \( \nu \in \mathbb{Z}^{2g} \).

**Proof:** If \( (u_0, \xi_0) \in W \times V \) and \( t_0 \in p_0^{-1}(\xi_0) \subset \mathcal{P} \mathbb{C}^g \subset \mathbb{C}^{2g} \) it follows from Corollary 7.3 that there are open neighborhoods \( U' \subset \mathbb{C}^g \) of \( u_0 \) and \( U \subset \mathbb{C}^{2g} \) of \( t_0 \) and meromorphic functions \( f_i(z, u, t) \) on the holomorphic variety \( \tilde{M} \times (W \cap U') \times (\tilde{V} \cap U) \) that are a basis for the vector space \( \Lambda(\nu, \eta) \) at any point \( (z, u, t) \in (W \cap U') \times (\tilde{V} \cap U) \); these functions thus satisfy

\[
(7.12) \quad f_i(Tz, u, t) = \rho_i(T) \eta(T, z) f_i(z, u, t)
\]

for all \( T \in \Gamma \) and all points \( (z, u, t) \in \tilde{M} \times (W \cap U') \times (\tilde{V} \cap U) \). By restricting the neighborhood \( U \) suitably it can be assumed that the intersection \( U = \tilde{U} \cap \mathcal{P} \mathbb{C}^g \) is a contractible open neighborhood of \( t_0 \) in the linear subspace \( \mathcal{P} \mathbb{C}^g \subset \mathbb{C}^{2g} \), so is homeomorphic to an open neighborhood \( U'' \subset \mathbb{C}^g \) of the point \( \xi_0 \) under the restriction \( p_0 : \tilde{U} \rightarrow \mathbb{C}^g \) of the covering projection \( p_0 \). The restriction of the function \( f_i(z, u, t) \) then is a meromorphic function \( \hat{f}_i(z, u, t) \) on \( M \times (W \cap U') \times (\tilde{V} \cap U) \). For any points \( z \in \tilde{M} \), \( u \in W \cap U' \), \( t \in \tilde{V} \cap U \), \( s \in \mathbb{C}^g \), \( \nu \in \mathbb{Z}^{2g} \) set

\[
f_i(z, u, t + \nu) = \phi(z, s - \nu) f_i(z, u, t).
\]

If \( U'' = p^{-1}(U'') \subset \mathbb{C}^{2g} \) then since any point \( t \in \tilde{V} \cap \tilde{U}'' \) can be written uniquely as the sum \( t = \hat{t} + \nu \) of \( \hat{t} \) and \( \nu \) for points \( \hat{t} \in \tilde{V} \cap \tilde{U} \), \( s \in \mathbb{C}^g \), \( \nu \in \mathbb{Z}^{2g} \) in view of the decomposition \( (\ref{7.6}) \) it is evident that these functions are meromorphic functions \( \hat{f}_i(z, u, t) \) of the variables \( (z, u, t) \in \tilde{M} \times (W \cap U') \times (\tilde{V} \cap U) \). If \( t = \hat{t} + \nu \) then (7.12) and (7.4) that

\[
f_i(Tz, u, t) = \rho_i(T) \eta(T, z) f_i(z, u, t),
\]

since \( \hat{t} + \nu \) and \( \eta(T, z) f_i(z, u, t) \) are linearly independent they are a basis for the vector space \( \Lambda(\nu, \eta) \). Furthermore for any \( s_0 \in \mathbb{C}^g \) and \( \nu_0 \in \mathbb{Z}^{2g} \)

\[
f_i(z, u, t + \nu_0) = \phi(z, s + \nu) f_i(z, u, t).
\]

That suffices to conclude the proof.

When the subvarieties \( W \subset \mathbb{C}^g \) and \( V \subset \mathbb{P}(\mathbb{C}) \) are sufficiently regular the local result of the preceding corollary can be extended to the following global assertion.

**Theorem 7.5** Let \( \eta \) be a factor of automorphy describing a holomorphic line bundle on a compact Riemann surface \( M \) of genus \( g > 0 \), let \( W \) be a holomorphic submanifold of an open subset of the complex manifold \( \mathbb{C}^g \), let \( V \)
be a holomorphic submanifold of an open subset of the Picard variety $P(M)$ and set $V = p^{-1}(V) \subset \mathbb{C}^{2g}$; and suppose that $\dim \Lambda_\eta(u, t) = \nu$ for all points $u = (u_1, \ldots, u_n) \in W$ and $t \in \hat{V}$. Then the union

$$
\Lambda_\eta(W, \hat{V}) = \bigcup_{u \in W; t \in \hat{V}} \Lambda_\eta(u, t)
$$

has a uniquely determined structure as a holomorphic vector bundle of rank $\nu$ over the complex submanifold $W \times \hat{V}$ such that for any sufficiently fine coordinate coverings \{U'\} of the complex manifold $M^n$ and \{U''\} of the complex manifold $P(M)$ there are meromorphic functions $f_{al,i}(z, u, t)$ on the holomorphic submanifolds $\tilde{M} \times (W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i')$ for $1 \leq i \leq \nu$, where $\tilde{U}_i'' = p^{-1}(U_i'') \subset \mathbb{C}^{2g}$, that are bases for the vector space $\Lambda_\eta(u, t)$ at any point $(u, t) \in (W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'')$ and that for any fixed point $z \in \tilde{M}$ are meromorphic cross-sections of the vector bundle $\Lambda_\eta(W, \hat{V})$. Under translation of the parameter $t \in \hat{V}$ through vectors in $\Omega \mathbb{C}^g + \mathbb{M} \mathbb{Z}^{2g}$ the vector bundle $\Lambda_\eta(W, \hat{V})$ is invariant while the functions $f_{al,i}(z, u, t)$ satisfy 7.11.

Proof: For any sufficiently fine open coverings \{U'_a\} of the product $M^n$ and \{U''_a\} of the Picard variety $P(M)$ the conclusions of Corollary 7.4 hold for the products $U'_a \times \tilde{U}_i''$; thus there are $\nu$ meromorphic functions $f_{al,i}(z, u, t)$ on the holomorphic submanifolds $\tilde{M} \times (W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'')$ for $1 \leq i \leq \nu$, where $\tilde{U}_i'' = p^{-1}(U_i'') \subset \mathbb{C}^{2g}$, that are bases for the vector space $\Lambda_\eta(u, t)$ at any point $(u, t) \in (W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'')$ and that satisfy (7.11). The union of the subspaces $\Lambda_\eta(u, t)$ for points $(u, t) \in (W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'')$ can be identified with the product $(W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'') \times \mathbb{C}^\nu$ by associating to any point $(u, t, x) \in (W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'') \times \mathbb{C}^\nu$ the element $\sum_{i=1}^\nu x_i f_{al,i}(z, u, t) \in \Lambda_\eta(u, t)$; the union (7.13) thus has a local product structure over each subset $(W \cap U'_a) \times (\hat{V} \cap \tilde{U}_i'') \subset W \times \hat{V}$. To combine these local product structures into a vector bundle note that there are two sets of meromorphic functions $f_{al,i}(z, u, t)$ and $f_{bm,j}(z, u, t)$ of the variable $z \in \tilde{M}$ that are bases for the vector space $\Lambda_\eta(u, t)$ at any point $(u, t)$ in an intersection $(W \cap U'_a \cap U'_b) \times (\hat{V} \cap \tilde{U}_i'' \cap \tilde{U}_m'')$; hence there are uniquely determined complex values $\lambda_{ab,lm;ij}(u, t)$ depending on the point $(u, t)$ such that

$$
f_{al,i}(z, u, t) = \sum_{j=1}^\nu \lambda_{ab,lm;ij}(u, t) f_{bm,j}(z, u, t).
$$

It follows from Cramer’s rule that these values $\lambda_{ab,lm;ij}(u, t)$ are determinants of matrices having as entries various of the functions $f_{al,i}(z, u, t)$ and $f_{bm,j}(z, u, t)$, so they are meromorphic functions of the variables $(u, t)$ on the complex manifold $(W \cap U'_a \cap U'_b) \times (\hat{V} \cap \tilde{U}_i'' \cap \tilde{U}_m'')$; and since they take finite values on this complex manifold they are actually holomorphic functions. The nonsingular matrices $\Lambda_{ab,lm}(u, t) = \{\lambda_{ab,lm;ij}(u, t)\}$ are thus holomorphic matrix-valued functions on the complex manifold $(W \cap U'_a \cap U'_b) \times (\hat{V} \cap \tilde{U}_i'' \cap \tilde{U}_m'')$; and it follows immediately from their uniqueness that $\Lambda_{ab,lm}(u, t) \cdot \Lambda_{bc,mn}(u, t) = \Lambda_{ac,ln}(u, t)$.
whenever \((u, t) \in (W \cap U'_a \cap U'_b \cap U'_c) \times (\tilde{V} \cap \tilde{U}'_m \cap \tilde{U}'_n)\), so they are the coordinate transition functions describing a holomorphic vector bundle \(\Lambda_q(W, \tilde{V})\) of rank \(\nu\) over the complex manifold \(W \times \tilde{V}\). Equation (7.14) is just the condition that for any fixed point \(z \in M\) the functions \(f_{a_l, i}(z, u, t)\) are a meromorphic cross-section of the vector bundle \(\Lambda_q(W, \tilde{V})\). Any other local meromorphic functions \(g_{a_l, i}(z, u, t)\) that are bases of the vector spaces \(\Lambda_q(u, t)\) can be expressed in terms of the functions \(f_{a_l, i}(z, u, t)\) as

\[
g_{a_l, i}(z, u, t) = \sum_{j=1}^{\nu} c_{a_l, ij}(u, t) \cdot f_{a_l, j}(z, u, t)
\]

for some nonsingular complex matrices \(C_{al}(u, t) = \{c_{al, ij}(u, t)\}\); again the entries in these matrices are bounded meromorphic functions and consequently holomorphic functions on the submanifolds \((W \cap U'_a) \times (\tilde{V} \cap \tilde{U}'_m)\). The coordinate transition functions for the vector bundle defined in terms of this alternative basis are \(C_{al}(u, t) \Lambda_{ab, lm}(u, t) C_{al}(u, t)^{-1}\), so they describe the same holomorphic vector bundle \(\Lambda_q(W, \tilde{V})\); and that demonstrates the uniqueness of this vector bundle. For any vector \(\Omega s + \Phi \Pi \nu \in \Omega \mathbb{C}^g + \Phi \Pi \Omega \mathbb{Z}^{2g}\) equation (7.14) takes the form

\[
f_{a_l, i}(z, u, t + \Omega s + \Phi \Pi \nu) = \sum_{j=1}^{\nu} \lambda_{ab, lm; ij}(u, t + \Omega s + \Phi \Pi \nu) \cdot f_{bm, j}(z, u, t + \Omega s + \Phi \Pi \nu);
\]

and since the functions \(f_{a_l, i}(z, u, t)\) and \(f_{bm, j}(z, u, t)\) satisfy (7.11) with the same factor \(\phi(z, s - \Pi \nu)\) it follows from the preceding equation that

\[
f_{a_l, i}(z, u, t) = \sum_{j=1}^{\nu} \lambda_{ab, lm; ij}(u, t + \Omega s + \Phi \Pi \nu) f_{bm, j}(z, u, t).
\]

Comparing this last equation with (7.14) and using the uniqueness of the coefficients \(\lambda_{ab, lm; ij}(u, t)\) shows that

\[
(7.15) \quad \lambda_{ab, lm; ij}(u, t + \Omega s + \Phi \Pi \nu) = \lambda_{ab, lm; ij}(u, t),
\]

so the vector bundle \(\Lambda_q(W, \tilde{V})\) is invariant under translation of the variable \(t \in \mathbb{C}^{2g}\) by vectors in \(\Omega \mathbb{C}^g + \Phi \Pi \Omega \mathbb{Z}^{2g}\) and that suffices to conclude the proof.

The proof of the preceding theorem required the subsets \(W \subset M^n\) and \(V \subset P(M)\) to be submanifolds rather than merely holomorphic subvarieties in order to show that the coordinate transition functions for the holomorphic vector bundle \(\Lambda_q(W, \tilde{V})\) are holomorphic functions rather than merely bounded meromorphic functions (weakly holomorphic functions)\(^1\). For most applications this additional hypothesis is not a problem. Without this additional hypothesis the vector bundle \(\Lambda_q(W, \tilde{V})\) at least is a weakly holomorphic vector bundle

\(^1\)Weakly holomorphic functions are discussed on page 403 in Appendix A.
over the subvariety \( W \times \tilde{V} \). The invariance of the vector bundle \( \Lambda_\eta(W, \tilde{V}) \) under translation of the parameter \( t \in \tilde{V} \) through vectors in \( \Omega C^g + \mathbb{Z}^{2g} \), means that this vector bundle induces a holomorphic vector bundle \( \Lambda_\eta(W, V) \) over the quotient submanifold \( W \times V \) when the Picard variety \( P(M) \) is described as in (7.5) as the quotient space of \( \mathbb{C}^{2g} \) under the group of such translations. The meromorphic functions \( f_{\alpha, t}(z, u, t) \) are not invariant under such translations so they do not describe a meromorphic cross-section of the induced vector bundle \( \Lambda_\eta(W, V) \) over \( W \times V \); but they do satisfy the relative invariance condition (7.11), which has a related interpretation. To see this assume further that the open subsets \( U''_t \subset P(M) \) are contractible and that the intersections \( U''_t \cap U''_m \) are connected, and choose a connected component \( \hat{U}_t \subset \Omega C^g \) of the inverse image \( p_0^{-1}(U''_t) \) of each set \( U''_t \) under the covering projection \( p_0 \) of (7.8). In view of the decomposition (7.7) the bundle \( \Lambda_\eta(V, W) \) can be described equivalently as the quotient of the restriction \( \lambda_\eta(V, \tilde{W})|(V \times \tilde{W} \cap \Omega C^g \) under the action of the covering translation group of the covering \( p_0 \) of (7.8), the lattice subgroup \( \Omega \mathbb{H}Z^{2g} \subset \Omega C^g \). The restriction is defined by the coordinate transition functions \( \Lambda_{ab, lm}(u, t) \) in the intersections

\[
(7.16) \quad (W \cap U'_a \cap U'_b) \times (V \cap U''_t \cap U''_m) \cap \Omega C^g = (W \cap U'_a \cap U'_b) \times (V \cap p_0^{-1}(U_t \cap U_m)),
\]

and these coordinate functions are invariant under translation through the lattice subgroup \( \Omega \mathbb{H}Z^{2g} \). In view of (7.9) the intersection (7.16) is a union of disjoint components, translates of the intersection \( (W \cap U'_a \cap U'_b) \times (V \cap \hat{U}_t \cap \hat{U}_m) \) by vectors in the lattice subgroup \( \Omega \mathbb{H}Z^{2g} \); consequently the bundle \( \Lambda_\eta(V, W) \) can be described by the restrictions of the coordinate transition functions \( \Lambda_{ab, lm}(u, t) \) to the intersections \( (W \cap U'_a \cap U'_b) \times (V \cap \hat{U}_t \cap \hat{U}_m) \) where the subsets \( \hat{U}_t \) are viewed as a coordinate covering of the Picard variety \( P(M) \). If \( t_t \in \hat{U}_t \) and \( t_m \in \hat{U}_m \) have the same image \( p_0(t_t) = p_0(t_m) \in U''_t \cap U''_m \) then it follows from the exact sequence (7.8) that

\[
(7.17) \quad t_t - t_m = \Omega \nu_{lm} \quad \text{where} \quad \nu_{lm} \in \mathbb{Z}^{2g},
\]

and the integer \( \nu_{lm} \) is independent of the point \( p_0(t_t) = p_0(t_m) \in U''_t \cap U''_m \) since this intersection is assumed to be connected. If points \( t_t \in \hat{U}_t \), \( t_m \in \hat{U}_m \), \( t_n \in \hat{U}_n \) have the same image \( p_0(t_t) = p_0(t_m) = p_0(t_n) \in U''_t \cap U''_m \cap U''_n \) then the analogue of (7.17) holds for any pair of these points, so

\[
0 = (t_t - t_m) + (t_m - t_n) + (t_n - t_t) = \Omega(\nu_{lm} + \nu_{mn} + \nu_{nl});
\]

and since the linear mapping \( \Omega : \mathbb{C}^g \to \mathbb{C}^{2g} \) is injective it follows that \( \nu_{lm} + \nu_{mn} + \nu_{nl} = 0 \), so taking complex conjugates \( \Omega \nu_{lm} + \Omega \nu_{mn} + \Omega \nu_{nl} = 0 \) since \( \nu_{lm} = \nu_{lm} \). Therefore for any fixed point \( z \in M \)

\[
\phi(z, -\Omega \nu_{lm}) \phi(z, -\Omega \nu_{mn}) \phi(z, -\Omega \nu_{nl}) = \phi(z, 0) = 1,
\]

\[
(7.18)
\]
and the complex constants $\phi(z, -\overline{\nu}_m)$ for that fixed point $z \in \hat{M}$ consequently can be viewed as the coordinate transition functions describing a flat line bundle $\Phi_z$ over the Picard variety $P(M)$ in terms of the coordinate covering $\{U_l\}$.

**Corollary 7.6** The functions $f_{a,l,i}(z, u, t)$ of the preceding theorem when restricted to the subspace $\overline{\mathbb{C}}^g \subset \mathbb{C}^g$ describe a meromorphic cross-section of the vector bundle $\Phi_z \otimes \Lambda_q(W, V)$ over $W \times V$.

**Proof:** If $u \in U'_n \cap U'_m$ and the points $t_l \in \hat{U}_l$ and $t_m \in \hat{U}_m$ have the same image $p_{\eta}(t_l) = p_{\eta}(t_m) \in U''_n \cap U''_m$ then from (7.14), (7.17) and the invariance conditions (7.15) and (7.11) it follows that

$$f_{a,l,i}(z, u, t_l) = \sum_{j=1}^\nu \lambda_{ab,lm;ij}(u, t_l) \ f_{bm,j}(z, u, t_l) = \sum_{j=1}^\nu \lambda_{ab,lm;ij}(u, t_m + i\overline{\mu}_l \nu_m) \ f_{bm,j}(z, u, t_m + i\overline{\mu}_l \nu_m) = \sum_{j=1}^\nu \lambda_{ab,lm;ij}(u, t_m) \cdot \phi(z, -\overline{\mu}_l \nu_m) \ f_{bm,j}(z, u, t_m);$$

that is just the condition that for a fixed point $z \in \hat{M}$ the functions $f_{a,l,i}(z, u, t_l)$ of the variables $(u, t_l) \in (U'_n \times \hat{U}_n) \subset (M^n \times \overline{\mathbb{C}}^g)$ are a meromorphic cross-section of the product bundle $\Phi_z \otimes \Lambda_q(W, V)$ over $W \times V$, which concludes the proof.

For many applications of the results in this section the primary interest is in cross-sections $f \in \Lambda_q(u, t)$ for a compact Riemann surface $M$ of genus $g > 0$ viewed as functions of the parameters $u \in M^n$ and $\xi = \rho_l \eta \in P_r(M)$, where $P_r(M)$ is the connected component of the extended Picard variety consisting of holomorphic line bundles $\xi$ of characteristic class $c(\xi) = r$; and there is somewhat less interest in the explicit description of the line bundles $\xi$ as being represented by canonically parametrized flat line bundles. Thus in these circumstances it is more convenient to describe the vector space $\Lambda_q(u, t)$ in the equivalent form

$$(7.19) \quad \Lambda(u, \xi) = \left\{ f \in \Gamma(M, M(\xi)) \mid \vartheta(f) + u_1 + \cdots + u_n \geq 0 \right\}$$

where $u = (u_1, \ldots, u_n) \in M^n$ and $\xi \in P_r(M)$; these two forms are clearly equivalent under the identification $\xi = \rho_l \eta$. Coordinate coverings $\hat{U}_l$ of the universal covering space $\overline{\mathbb{C}}^g$ of the Picard variety $P(M)$, viewed as coordinate coverings of $P(M)$ itself, can be viewed as coordinate coverings of the component $P_r(M)$ of the extended Picard variety through the same identification $\xi = \rho_l \eta \in P_r(M)$; and unless it is necessary to be more explicit this identification can be ignored. The flat line bundle $\Phi_z$ over the Picard variety $P(M)$ can be identified correspondingly with a flat line bundle over $P_r(M)$. Since all flat line bundles over $M$ can be described by local coordinate bundles in a sufficiently
fine coordinate covering of the surface $M$ the line bundles $\xi$ in any family of line bundles can be represented by coordinate bundles $\xi_{\alpha\beta}$ in a fixed suitably fine coordinate covering $\{U_\alpha\}$ of $M$. Then for coordinate coverings $U'_\alpha$ of $M^n$ and $\tilde{U}_l$ of $P_r(M)$ the meromorphic cross-sections of Theorem 7.5 can be viewed as local meromorphic functions $f_{\alpha\alpha}(z, u, \xi)$ of the variables $z \in U_\alpha \subset M$, $u \in U'_\alpha \subset M^n$ and $\xi \in \tilde{U}_l \subset P_r(M)$ such that

$$(7.20) \quad f_{\alpha\alpha}(z, u, \xi) = \xi_{\alpha\beta}(z)f_{\beta\beta}(z, u, \xi)$$

for $z \in U_\alpha \cap U_\beta$. In these terms Theorem 7.5 and its corollary can be restated as follows.

**Corollary 7.7** If $M$ is a compact Riemann surface of genus $g > 0$, $W$ is a holomorphic submanifold of an open subset of the compact manifold $M^n$, and $V$ is a holomorphic submanifold of an open subset of the component $P_r(M)$ of the extended Picard variety of $M$, and if $\dim \Lambda(u, \xi) = \nu$ for all points $(u, \xi) \in W \times V$, then the union

$$\Lambda(W, V) = \bigcup_{u \in W, \xi \in V} \Lambda(u, \xi)$$

has a uniquely determined structure as a holomorphic vector bundle of rank $\nu$ over the manifold $W \times V$ such that for any sufficiently fine open coordinate coverings $\{U_\alpha\}$ of the surface $M$, $\{U'_\alpha\}$ of the product $M^n$, and $\{U''_l\}$ of the complex manifold $P_r(M)$ there are $\nu$ meromorphic functions $f_{\alpha\alpha, i}(z, u, \xi)$ on the subsets $U_\alpha \times (W \cap U'_\alpha) \times (V \cap U''_l)$ that are a basis for the vector space $\Lambda(u, \xi)$ at any point $(u, \xi) \in (W \cap U'_\alpha) \times (V \cap U''_l)$ and that for any fixed point $z_\alpha \in U_\alpha$ are a meromorphic cross-section of the product bundle $\Phi_{z_\alpha} \otimes \Lambda(W, V)$ over $W \times V$.

**Proof:** This is merely a restatement of the result of the preceding theorem and its corollary, so no further proof is required.

In the special case that $n + r > 2g - 2$ it is possible to take $W = M^n$ and $V = P_r(M)$ in Theorem 7.5 and its corollaries, and the conclusions then take a slightly simpler form.
Chapter 8

The Intrinsic Theta Function

The hyperabelian factors of automorphy of characteristic class \( g \) discussed in Chapter 6 all admit nontrivial holomorphic relatively automorphic functions, as an immediate consequence of the Riemann-Roch Theorem: some of these functions can be described quite explicitly in terms of the hyperabelian factors of automorphy themselves. The formulas involve the intersection matrix \( P \) and the period matrix \( \Omega \) of the surface \( M \) for choices of generators \( T_i \in \Gamma \) of the covering translation group \( \Gamma \) representing a basis \( \tau_i \in H^1(M) \) for the homology of \( M \) and of a basis \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)}) \) for the holomorphic abelian differentials over \( M \); and from these matrices there are derived the symmetrization \( \hat{P} \) of the skew-symmetric matrix \( P^* = TP^{-1} \), as defined in (6.60), and the \( 2g \times 2g \) real symmetric matrix

\[
R = 2\pi \Re((i\Omega G\overline{\Omega})),
\]

where \( G = iHT^{-1} = H^{-\top} \) for the positive definite Hermitian matrix \( H = i\Omega P\overline{\Omega} \). The symmetrization \( \hat{P} \) of the matrix \( P^* = TP^{-1} \) has the property that

\[
\hat{P} - P^{-1} = \hat{P} + P^* = -2Q \quad \text{where} \quad Q \in \mathbb{Z}^{2g \times 2g},
\]

as noted following the definition of the matrix \( \hat{P} \) in (6.60); and it follows immediately from this that

\[
Q + \overline{Q} = -\hat{P} \quad \text{and} \quad Q - \overline{Q} = P^{-1},
\]

so it is possible to use the matrix \( Q \) in place of the matrices \( P^* \) and \( \hat{P} \) as convenient. In particular if the intersection matrix is the basic skew-symmetric matrix \( J \) then

\[
P = P^* = J = \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} 0 & 1 \\ I & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -I \\ 0 & 0 \end{pmatrix}.
\]
It is convenient for the subsequent discussion also to introduce the $2g \times 2g$ complex matrix

$$\tilde{Z} = -\frac{1}{2} \hat{P} + \frac{i}{\pi} R = -\frac{1}{2\pi i} \left( R + \pi i \hat{P} \right).$$

which is a symmetric complex matrix since $R$ and $\hat{P}$ are real symmetric matrices; its complex conjugate matrix clearly satisfies

$$\overline{\tilde{Z}} = -\tilde{Z} - \hat{P}.$$

The matrix $\tilde{Z}$ can be written alternatively

$$\tilde{Z} = -\frac{1}{2\pi i} \left( R + \pi i P^{-1} - 2\pi i Q \right) = i \Omega G \Omega + Q$$

in view of (6.65) and (6.67).

**Lemma 8.1** The symmetric real matrix $R$ is positive definite, so the symmetric complex matrix $\tilde{Z}$ lies in the Siegel upper half-space $\mathcal{H}_{2g}$; it is a rather special matrix in $\mathcal{H}_{2g}$, since its real part is the half-integral symmetric matrix $-\frac{1}{2} \hat{P}$.

**Proof:** It follows from (8.7) that $R + \pi i P^{-1} = 2\pi i \Omega \Omega$, so if $x \in \mathbb{R}^{2g}$ then since $t^x P^{-1} x = 0$ for the skew-symmetric matrix $P^{-1}$

$$t^x R x = t^x \left( R + \pi i P^{-1} \right) x = 2\pi i t^x \Omega G \Omega x.$$

The period matrix $\Omega$ is a nonsingular period matrix, so the vector $\Omega x \in \mathbb{C}^g$ is nonzero for any nonzero real vector $x \in \mathbb{R}^{2g}$; and since the matrix $G$ is positive definite Hermitian

$$g(x) = \frac{t^x R x}{t^x \cdot x} = 2\pi \frac{\Omega x G(\overline{\Omega} x)}{t^x \cdot x} > 0$$

for all nonzero vectors $x \in \mathbb{R}^{2g}$. The function $g(x)$ is continuous, so there is a positive constant $\delta > 0$ such that $g(x) \geq \delta$ for all real vectors in the compact set consisting of vectors in $\mathbb{R}^{2g}$ of length $t^x \cdot x = 1$; the function $g(x)$ also is homogeneous, in the sense that $g(tx) = g(x)$ for all real parameters $t \in \mathbb{R}$, so it follows that $g(x) \geq \delta$ for all nonzero real vectors $x \in \mathbb{R}^{2g}$ and consequently that the matrix $R$ is positive definite. The imaginary part of the Hermitian matrix $\tilde{Z}$ of (8.5) is the positive definite symmetric matrix $\frac{1}{\pi} R$, and consequently $\overline{\tilde{Z}} \in \mathcal{H}_{2g}$; and its real part is the half-integral symmetric matrix $-\frac{1}{2} \hat{P}$. That suffices to conclude the proof.

Any holomorphic line bundle of characteristic class $g$ over a compact Riemann surface $M$ of genus $g > 0$ with bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_i \in H_1(M)$ can be described by a factor of automorphy that is the product of the intrinsic hyperabelian factor of automorphy $\zeta_{g,a}(\tau, z)$ of Theorem 6.21 and a canonically
parametrized flat factor of automorphy $\rho_t$ for some $t \in \mathbb{C}^{2g}$, so as a factor of automorphy of the form

\begin{equation}
\zeta_{g,a,t}(\tau, z) = \rho_t(\tau)\zeta_{g,a}(\tau, z)
\end{equation}

\begin{align*}
&= \exp(2\pi i \eta t) \cdot \exp\left(\frac{\pi i}{\eta} \hat{h} n \hat{P} n\right) \cdot \exp\left(\frac{1}{2} \eta R n + 2\pi \eta G \hat{w}(z, a)\right) \\
&= \exp 2\pi i \left(-\frac{1}{2} \eta z \eta n + \frac{t}{\eta} \hat{Z}_n + \frac{1}{2} \eta n R n + 2\pi \eta G \hat{w}(z, a)\right)
\end{align*}

for any $\tau = \sum_{i=1}^{2g} n_i \tau_i \in H_1(M)$. It is clear from the definition of the canonically parametrized flat factor of automorphy $\rho_t(\tau)$ that

\begin{equation}
\zeta_{g,a,t'}(\tau, z) = \zeta_{g,a,t''}(\tau, z) \quad \text{if and only if} \quad t' - t'' \in \mathbb{Z}^{2g}
\end{equation}

and it follows from the analytic classification of flat factors of automorphy in Theorem 3.14 that

\begin{equation}
\zeta_{g,a,t'}(\tau, z) \sim \zeta_{g,a,t''}(\tau, z) \quad \text{if and only if} \quad t' - t'' \in \mathbb{Z}^{2g} + \eta \Omega C^{2g}
\end{equation}

where $\sim$ denotes the holomorphic equivalence of factors of automorphy.

The explicit formula (8.8) depends on the choice of the bases $\omega_i(z)$ and $\tau_i$, since the matrices $\Omega, G, R$ are determined in terms of these bases and the description of the homology class $\tau$ as the sum $\tau = \sum_{i=1}^{2g} n_i \tau_i$ in terms of the basis $\tau_i$ appears in both $\zeta_{g,a}(\tau, z)$ and $\rho_t(\tau)$. By Theorem 6.21 though the factor of automorphy $\zeta_{g,a}(\tau, z)$ has precisely the same form in terms of any bases $\omega_i(z)$ and $\tau_i$, aside from its sign $\epsilon(\tau)$, when the matrices $\Omega, G, R$ are expressed in terms of the relevant bases; and by (3.30) and Theorem 6.21 the explicit formulas for both the flat factor of automorphy $\rho_t$ and the sign term $\epsilon(\tau)$ in the factor of automorphy $\zeta_{g,a}(\tau, z)$ depend only on the choice of the basis $\tau_i$ and not on the choice of the basis $\omega_i(z)$. When it is necessary or helpful to indicate the basis $\tau_i$ for the explicit expressions (6.80) and (8.8) these factors of automorphy will be denoted by $\zeta_{g,a,\{\tau_i\}}(\tau, z)$ and $\zeta_{g,a,\{\tau_i\}}(\tau, z)$. Any two bases $\{\tau_i\}$ and $\{\tau'_j\}$ for the homology group $H_1(M)$ are related by

\begin{equation}
\tau_i = \sum_{j=1}^{2g} n_{ji} \tau'_j \quad \text{for some} \quad N = \{n_{ji}\} \in \text{Gl}(2g, \mathbb{Z}),
\end{equation}

and it follows from (3.30) that

\begin{equation}
\rho_{t,\{\tau'_j\}}(\tau) = \rho_{\eta N t,\{\tau_i\}}(\tau)
\end{equation}

and from Theorem 6.21 (iii) that

\begin{equation}
\zeta_{g,a,\{\tau'_j\}}(\tau, z) = \rho_{\frac{1}{2} \delta(P, N),\{\tau_i\}}(\tau) \zeta_{g,a,\{\tau_i\}}(\tau, z)
\end{equation}
in terms of the integral vector \( \delta(\hat{P}, N) \in \mathbb{Z}^{2g} \) given by (6.82); consequently

\[
\zeta_{g,a,t;(\tau')}(\tau, z) = \rho_{t;(\tau')};\zeta_{g,a;(\tau')}(\tau) \\
= \rho_{\mathfrak{N},(\tau)}(\tau)\rho_{\frac{1}{2}\delta(\hat{P}, N),(\tau)}(\tau)\zeta_{g,a,(\tau)}(\tau, z) \\
= \zeta_{g,a,(\mathfrak{N} + \frac{1}{2}\delta(\hat{P}, Q),(\tau)}(\tau, z).
\]

**Theorem 8.2** On a compact Riemann surface of genus \( g > 0 \) the series

\[
\theta_g(z, a; t) = \sum_{\tau \in H_1(M)} \zeta_{g,a,t}(\tau, z)^{-1}
\]

is locally absolutely uniformly convergent to a nontrivial holomorphic function of the variables \( (z, a, t) \in \hat{M} \times \hat{M} \times \mathbb{C}^{2g} \); and for any fixed point \( (a, t) \in \hat{M} \times \mathbb{C}^{2g} \) the function \( \theta_g(z, a; t) \) of the variable \( z \in \hat{M} \) is a holomorphic relatively automorphic function for the factor of automorphy \( \zeta_{g,a,t}(T, z) \).

**Proof:** For \( z = a \) in (8.8) the series (8.15) takes the form

\[
\sum_{\tau \in H_1(M)} \zeta_{g,a,t}(\tau, a)^{-1} = \sum_{n \in \mathbb{Z}^{2g}} \epsilon_n \exp -\frac{1}{2} \left( h_n R + 4\pi i h t \right)
\]

where \( \epsilon_n = \exp \frac{\pi i}{2} \delta(\hat{P}, n) = \pm 1 \) since \( \hat{P} \) is a symmetric integral matrix with zeros along its diagonal. The matrix \( R \) is positive definite and symmetric by the preceding lemma, so \( h_n R \geq \delta \sum_{n=1}^{2g} n_j^2 = \delta \|n\|^2 \) for all \( n \in \mathbb{Z}^{2g} \) for some positive constant \( \delta > 0 \), where \( \|t\|^2 = \sum_{j=1}^{2g} t_j^2 \) for \( t = \{t_j\} \in \mathbb{C}^{2g} \). If \( \|t\| \leq M \) then

\[
\left| \epsilon_n \exp -\frac{1}{2} \left( h_n R + 4\pi i h t \right) \right| \leq \exp \left( -\frac{1}{2} \delta \|n\|^2 + 2\pi M \|n\| \right),
\]

and if \( \|n\| \leq N \) then \( |n_i| \leq N \) for \( 1 \leq i \leq 2g \) so there are altogether at most \( (2N + 1)^{2g} \) indices \( n \in \mathbb{Z}^{2g} \) such that \( \|n\| \leq N \); consequently the series (8.16) is absolutely and uniformly convergent in any compact set \( \{t \in \mathbb{C}^{2g} \|t\| \leq M\} \), so its sum is a holomorphic function of the complex variables \( t \in \mathbb{C}^{2g} \).

The series (8.16) is a Fourier series in the variables \( t = (t_1, \ldots, t_{2g}) \in \mathbb{C}^{2g} \) with nonzero coefficients, so the sum is a nontrivial holomorphic function. The series (8.15) arises from the series (8.16) by substituting \( t - i \hat{P} \rho w(z, a) \) in place of the variable \( t \); so its sum is a nontrivial holomorphic function of the variables \( (z, a, t) \in \hat{M} \times \hat{M} \times \mathbb{C}^{2g} \). The hyperabelian factor of automorphy \( \zeta_{g,a,t}(T, z) \) is defined by \( \zeta_{g,a,t}(T, z) = \zeta_{g,a,t}(\tau, z) \) for all covering translations \( T \in \Gamma \) that represent the homology class \( \tau \in H_1(M) \); consequently for any covering translation \( S \in \Gamma \) representing a homology class \( \sigma \in H_1(M) \)

\[
\zeta_{g,a,t}(\tau, S z)^{-1} = \zeta_{g,a,t}(T, S z)^{-1} \\
= \zeta_{g,a,t}(T S, z)^{-1}\zeta_{g,a,t}(S, z) \\
= \zeta_{g,a,t}(\tau + \sigma, z)^{-1}\zeta_{g,a,t}(\sigma, z)
\]
\[ \theta_g(Sz, a; t) = \sum_{\tau \in H_1(M)} \zeta_{g,a,t}(\tau, Sz)^{-1} \]

\[ = \sum_{\tau \in H_1(M)} \zeta_{g,a,t}(\tau + \sigma, z)^{-1} \zeta_{g,a,t}(\sigma, z) \]

\[ = \theta_g(z, a; t) \zeta_{g,a,t}(S, z) \]

since the elements \( \tau + \sigma \) run through the entire group \( H_1(M) \) as the elements \( \tau \) do. Thus \( \theta_g(z, a; t) \) as a function of the complex variable \( z \in \bar{M} \) is a holomorphic relatively automorphic function for the factor of automorphy \( \zeta_{g,a,t}(\tau, z) \), and that suffices to conclude the proof.

When the factor of automorphy \( \zeta_{g,a,t}(\tau, z) \) is written out explicitly as in (8.8), in terms of a choice of bases \( \omega_j(z) \in \Gamma(M, \mathcal{O}^{1,0}) \) and \( \tau_i \in H_1(M) \), the series (8.15) takes the form

\[ (8.17) \quad \theta_g(z, a; t) = \sum_{n \in \mathbb{Z}^{2g}} \exp 2\pi i \left( \frac{1}{2} \bar{\omega}_n - \bar{n}t + i n \frac{\Omega G}{2} \bar{\omega}(z, a) \right). \]

This explicit formula for the function \( \theta_g(z, a; t) \) depends on the choice of the basis \( \tau_i \), which can be indicated by writing \( \theta_{g;\{\tau_i\}}(z, a; t) \) when necessary or useful to consider that dependence. It then follows immediately from the definition (8.15) and (8.14) that if \( \tau_i = \sum_{j=1}^{2g} n_{ji} \tau_j^i \) then

\[ (8.18) \quad \theta_{g;\{\tau_i^j\}}(z, a; t) = \theta_{g;\{\tau_i\}}(z, a; \bar{n}t + i \frac{1}{2} \delta(P, N)). \]

Thus aside from a relatively simple change of the coordinate \( t \in \mathbb{C}^{2g} \) the function \( \theta_g(z, a; t) \) is uniquely and intrinsically associated to the Riemann surface \( M \); consequently it is called the **intrinsic theta function** of the surface \( M \).

**Theorem 8.3** The intrinsic theta function \( \theta_g(z, a; t) \) on a compact Riemann surface \( M \) of genus \( g > 0 \) has the following properties:

(i) \( \theta_g(Tz, a; t) = \zeta_{g,a,t}(T, z)\theta_g(z, a; t) \) for \( T \in \Gamma \);

(ii) \( \theta_g(Tz, a; t) = \theta_g(z, a; t - \bar{\nu}) \) for \( T \in \Gamma \),

where \( \omega(T) = \Omega \nu \in \Omega \mathbb{Z}^{2g} \) and \( \bar{\nu} \) is the matrix (8.5);

(iii) \( \theta_g(z, a; t) = \theta_g(z, z; -t) \).

**Proof** (i) This is equivalent to the assertion that the function \( \theta_g(z, a; t) \) is a relatively automorphic function for the hyperabelian factor of automorphy \( \zeta_{g,a,t}(T, z) \), which was demonstrated in Theorem 8.2.

(ii) If \( \omega(T) = \Omega \nu \) then \( \bar{\omega}(Tz, a) = \bar{\omega}(z, a) + \Omega \nu \), and by (8.7) and (8.6)

\[ (8.19) \quad i n \frac{\Omega G}{2} \left( \bar{\omega}(T, a) - \bar{\omega}(z, a) \right) = i n \frac{\Omega G}{2} \nu = n \left( \bar{Z} + Q \right) \nu = n \left( \bar{Z} + \bar{P} + Q \right) \nu. \]
Substituting this into the series (8.17) yields the series

\[ \theta_g(Tz,a; t) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} h_{\mathbb{Z}} n - \nu t + i^* h_{\mathbb{Z}} \Omega G \hat{w}(z,a) + h_{\mathbb{Z}} (\tilde{Z} + \hat{P} + Q) \nu \right) \]

\[ = \theta_g(z,a; t - \tilde{Z} \nu) \]

since \( \exp 2\pi i h_{\mathbb{Z}} (\hat{P} + Q) \nu = 1 \) for any \( n, \nu \in \mathbb{Z}^g \), and that yields (ii).

(iii) In the series (8.17) it is possible to replace \( n \) by \( -n \), since \( -n \) runs through \( \mathbb{Z}^g \) as \( n \) does, so that

\[ \theta_g(z,a; t) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} h_{\mathbb{Z}} n - \nu t + i^* h_{\mathbb{Z}} \Omega G \hat{w}(z,a) \right) \]

\[ = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} h_{\mathbb{Z}} n - \nu t + h_{\mathbb{Z}} (\tilde{Z} + \hat{P} + Q) \nu \right) \]

\[ = \theta_g(a,z; -t) \]

which demonstrates (iii) and thereby concludes the proof.

What is particularly interesting is that the intrinsic theta function embodies both a relatively automorphic function for the action of the covering translation group \( \Gamma \) on the universal covering space \( \tilde{M} \) of the Riemann surface \( M \) and at the same time a relatively automorphic function for the action of the translation group \( \mathbb{Z}^g \) on the vector space \( \mathbb{C}^g \), as is evident from (i) and (ii) of the preceding theorem. In a natural sense the intrinsic theta function is a manifestation of the role of the classical theta function in the study of compact Riemann surfaces. Indeed for \( z = a \) the series expansion (8.17) reduces to

\[ \theta_g(a,a; t) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} h_{\mathbb{Z}} n - \nu t \right), \]

which is independent of the choice of the base point \( a \in \tilde{M} \). This series is the classical theta series \( \Theta(t; \tilde{Z}) \) associated to the matrix \( \tilde{Z} \in \mathfrak{H}_{2g} \), as in (G.2) in Appendix G; in these terms

\[ \theta_g(a,a; t) = \Theta(t; \tilde{Z}), \]

and conversely the function \( \theta_g(z,a; t) \) can be written

\[ \theta_g(z,a; t) = \Theta(t - i^* \Omega G \hat{w}(z,a); \tilde{Z}), \]

as is evident upon comparing the series expansions (8.17) and (8.20). The function \( \Theta(t; \tilde{Z}) \) is an even function, as follows readily upon replacing the variable of summation \( n \) in (8.20) by \( -n \), so

\[ \Theta(-t; \tilde{Z}) = \Theta(t; \tilde{Z}); \]
consequently the sign of the variable $t$ can be changed as desired, so the classical theta series can be written in either of the two equivalent forms

\[(8.24) \quad \Theta(t; \tilde{Z}) = \sum_{n \in \mathbb{Z}^{2g}} \exp 2\pi i \left( \frac{1}{2} n \tilde{Z} n - 'n t \right) = \sum_{n \in \mathbb{Z}^{2g}} \exp 2\pi i \left( \frac{1}{2} n \tilde{Z} n + 'n t \right) \]

as convenient. The series expansion (8.24) can be written alternatively by using (8.5) as

\[(8.25) \quad \Theta(t; \tilde{Z}) = \sum_{n \in \mathbb{Z}^{2g}} (-1)^{\frac{1}{2} n \tilde{P} n} \cdot \exp - \left( \frac{1}{2} t n R n + 2\pi i 'n t \right), \]

since $\tilde{P}$ is a symmetric integral matrix with zeros along the main diagonal so $'n \tilde{P} n$ is an even integer; from this and (8.23) it follows that

\[(8.26) \quad \overline{\Theta(t; \tilde{Z})} = \Theta(-\overline{t}; \tilde{Z}) = \Theta(\overline{t}; \tilde{Z}) \]

so the function $\Theta(t; \tilde{Z})$ is a real-valued function for real values of the parameter $t \in \mathbb{C}^{2g}$.

Since the function $\theta_g(z, a; t)$ is uniquely and intrinsically defined on the surface $M$ up to the change (8.18) in the parameter $t \in \mathbb{C}^{2g}$, the function $\Theta(t; \tilde{Z})$ must be uniquely and intrinsically defined by the surface $M$ up to the corresponding change in the parameter $t \in \mathbb{C}^{2g}$. It is of some use, and may be of some interest, to see that directly. If $\tau_j$ and $\tau'_j$ are two bases for the homology group $H_1(M)$ related by $\tau_i = \sum_{j=1}^{2g} n_{ij} \tau'_j$ for a matrix $N = \{n_{ij}\} \in \text{Gl}(2g, \mathbb{Z})$, then it follows from (6.76) and Lemma 6.19 that the auxiliary matrices associated to these bases are related by $R = 'N R' N$ and $\tilde{P} = 'N \tilde{P} N - 2E$ for a symmetric integral matrix $E \in \mathbb{Z}^{2g \times 2g}$, so from (8.5) it follows that

\[(8.27) \quad \tilde{Z} = 'N \tilde{Z}' N + E. \]

Consequently if $n' = N n$

\[
\Theta(t; \tilde{Z}') = \sum_{n' \in \mathbb{Z}^{2g}} \exp 2\pi i \left( \frac{1}{2} n' \tilde{Z}' n' - 'n' t \right) \\
= \sum_{n' \in \mathbb{Z}^{2g}} \exp 2\pi i \left( \frac{1}{2} n' 'N^{-1}(\tilde{Z} - E)N^{-1}n' - 'n' t \right) \\
= \sum_{n \in \mathbb{Z}^{2g}} \exp 2\pi i \left( \frac{1}{2} n(\tilde{Z} - E)n - 'n t \right) \\
= \Theta( Nt; \tilde{Z} - E); 
\]
and since \( n^2 \equiv n \pmod{2} \) and \( E = \{ e_{jk} \} \) is a symmetric integral matrix

\[
\exp(-\pi i t n E n) = \exp(-\pi i \left( \sum_{j,k=1}^{2g} e_{jk} n_j n_k \right))
\]

\[
= \exp(-\pi i \left( 2 \sum_{1 \leq j < k \leq 2g} e_{jk} n_j n_k + \sum_{j=1}^{2g} e_{jj} n_j n_j \right))
\]

\[
= \exp(-\pi i \left( \sum_{j=1}^{2g} e_{jj} n_j \right)) = \exp(-2\pi i t n \cdot \frac{1}{2} \text{diag}(E))
\]

or alternatively

\[
\Theta(t; \tilde{Z}') = \Theta \left( t' N t + \frac{1}{2} \text{diag}(E); \tilde{Z} \right).
\]

Comparing this and (8.18) shows that

\[
\text{diag} E = \delta(\hat{P}, N),
\]

which of course can be derived directly through an examination of the proof of Lemma 6.20.

As discussed in Appendix G, and in particular by Theorem G.1, the function \( \Theta(t; \tilde{Z}) \) is determined uniquely up to a constant factor as a holomorphic relatively automorphic function for the theta factor of automorphy (G.7) for the action of the lattice subgroup \( L(\tilde{\Omega}) = \tilde{\Omega} \tilde{Z}^{4g} \subset \mathbb{C}^{2g} \) on the vector space \( \mathbb{C}^{2g} \), where \( \tilde{\Omega} \) is the \( 2g \times 4g \) nonsingular period matrix

\[
(8.30) \quad \tilde{\Omega} = \begin{pmatrix} 1_{2g} & \tilde{Z} \end{pmatrix} \in \mathbb{C}^{2g \times 4g};
\]

so explicitly for any \( \mu, \nu \in \mathbb{Z}^{2g} \)

\[
(8.31) \quad \Theta(t + \mu + \tilde{Z} \nu; \tilde{Z}) = \Xi \tilde{Z}(\mu + \tilde{Z} \nu, t) \cdot \Theta(t; \tilde{Z})
\]

\[
= \exp(-2\pi i \left( \frac{1}{2} t' \mu + \nu t \right) \cdot \Theta(t; \tilde{Z}),
\]

which also follows directly from Theorem 8.3 (i) and (ii) for \( z = a \). The lattice subgroup \( L(\tilde{\Omega}) \) describes a complex torus

\[
(8.32) \quad J(\tilde{\Omega}) = \frac{\mathbb{C}^{2g}}{L(\tilde{\Omega})} = \frac{\mathbb{C}^{2g}}{\Omega \mathbb{Z}^{4g}}
\]

of dimension \( 2g \); the factor of automorphy \( \Xi \tilde{Z}(\mu + \tilde{Z} \nu, t) \) describes a holomorphic line bundle \( \Xi \tilde{Z} \) of rank \( 2g \) over the complex torus \( J(\tilde{\Omega}) \) and the function \( \Theta(t; \tilde{Z}) \) describes a nontrivial holomorphic cross-section of this line bundle. The complex
torus \( J(\tilde{\Omega}) \) is defined in terms of the choice of a basis \( \tau_j \in H_1(M) \). However as in (8.27) the matrices \( \tilde{Z} \) and \( \tilde{Z}' \) corresponding to two bases for the homology group \( H_1(M) \) that are related by \( \tau_i = \sum_{j=1}^{2g} n_{ji} \tau_j \) for a matrix \( N \in \text{Gl}(2g, \mathbb{Z}) \) satisfy (8.27) for a symmetric integral matrix \( E \in \mathbb{Z}^{2g \times 2g} \); consequently

\[
\begin{pmatrix} t^N \tilde{\Omega}' \end{pmatrix} = \begin{pmatrix} t^N & t^N \tilde{Z}' \end{pmatrix} = \begin{pmatrix} t^N & (\tilde{Z} - E)N^{-1} \end{pmatrix} = \begin{pmatrix} t^N & -EN^{-1} \end{pmatrix},
\]

showing that the pair

(8.33) \[
\begin{pmatrix} t^N, & \begin{pmatrix} t^N & -EN^{-1} \end{pmatrix} \end{pmatrix}
\]

is a Hurwitz relation from the period matrix \( \tilde{\Omega}' \) to the period matrix \( \tilde{\Omega} \). The two matrices in this Hurwitz relation are invertible integral matrices, so as in Theorem F.9 (ii) the matrix \( N \) describes a biholomorphic mapping

(8.34) \[
J(\tilde{\Omega}') \longrightarrow J(\tilde{\Omega}).
\]

The biholomorphic equivalence class of the complex tori \( J(\tilde{\Omega}) \) for the period matrices \( \tilde{\Omega} \) associated to all choices of bases \( \tau_j \in H_1(M) \) thus is another complex torus intrinsically associated to the Riemann surface \( M \), called the intrinsic Jacobi variety of the Riemann surface \( M \); when it is not necessary to specify the particular basis \( \tau_j \in H_1(M) \) or the particular period matrix \( \tilde{\Omega} \) this torus can be denoted just by \( J(M) \). Actually the equivalence relation of complex tori \( J(\tilde{\Omega}) \) described in (8.34) is a more restrictive equivalence relation than just biholomorphic equivalence of complex tori since it arises from the integral change of bases described by the matrix \( N \); but that is a point that will not be pursued further here.

As might be expected, the complex torus \( J(M) \) is closely related to the Jacobi variety \( J(M) \) and the Picard variety \( P(M) \) of the surface \( M \). If \( \Omega \) is the period matrix of \( M \) and \( \Pi \) is the inverse period matrix to \( \Omega \) as introduced in Theorem F.12 in Appendix F.1, so that

(8.35) \[
\Pi^T \Omega = 0, \quad \Pi \Omega = I_g, \quad \Pi \Omega + \Omega \Pi = I_{2g},
\]

then the Jacobi variety of \( M \) is the \( g \)-dimensional complex torus

(8.36) \[
J(M) = \frac{\mathbb{C}^g}{\Omega \mathbb{Z}^{2g}} \cong \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + \Omega \mathbb{C}^g};
\]

and the Picard variety of \( M \) is the \( g \)-dimensional complex torus

(8.37) \[
P(M) = \frac{\mathbb{C}^g}{\mathbb{Z}^{2g} + \Pi \mathbb{C}^g};
\]
and since the complex conjugates of the matrices $\Omega$ and $\Pi$ are also period matrices there are in addition the conjugate Jacobi variety

$$J^*(M) = \frac{\mathbb{C}^g}{\Omega \mathbb{Z}^{2g}} \cong \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + \mathbb{Q} \mathbb{C}^g},$$

and the conjugate Picard variety

$$P^*(M) = \frac{\mathbb{C}^g}{\Pi \mathbb{Z}^{2g}} \cong \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + \mathbb{Q} \mathbb{C}^g},$$

both of which are $g$-dimensional complex tori.

**Theorem 8.4** On a compact Riemann surface $M$ of genus $g > 0$, with the intersection matrix $P$ and the period matrix $\Omega$ in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in \mathcal{H}_1(M)$, let $\Pi$ be the inverse period matrix to $\Omega$, $\tilde{P}$ be the symmetrization of the matrix $P^* = tP^{-1}$, and $Q$ be the matrix $Q = -\frac{1}{2}(\tilde{P} + P^*)$.

(i) The pair of matrices

$$(8.40) \quad \left( i \tilde{\Omega} G, \begin{pmatrix} -t^*Q \\ I_{2g} \end{pmatrix} \right)$$

form a Hurwitz relation from the period matrix $\Omega$ to the period matrix $\tilde{\Omega}$.

(ii) The pair of matrices

$$(8.41) \quad \left( i^* \Omega G, \begin{pmatrix} -Q \\ I_{2g} \end{pmatrix} \right)$$

form a Hurwitz relation from the period matrix $\tilde{\Omega}$ to the period matrix $\tilde{\Omega}$.

(iii) The pair of matrices

$$(8.42) \quad \left( \Pi, \begin{pmatrix} I_{2g} \\ Q \end{pmatrix} \right)$$

form a Hurwitz relation from the period matrix $\tilde{\Omega}$ to the period matrix $\Pi$.

(iv) The pair of matrices

$$(8.43) \quad \left( \Pi, \begin{pmatrix} I_{2g} \\ -t^*Q \end{pmatrix} \right)$$

form a Hurwitz relation from the period matrix $\tilde{\Omega}$ to the period matrix $\tilde{\Omega}$.

(v) The pair of matrices

$$(8.44) \quad (i \tilde{G}, P^{-1})$$

form a Hurwitz relation from the period matrix $\Omega$ to the period matrix $\Pi$, and the pair of matrices

$$(8.45) \quad (-i H, P)$$

form a Hurwitz relation from the period matrix $\Pi$ to the period matrix $\Omega$. 
\textbf{Proof:} (i) It follows from (8.7), (8.6) and (8.3) that $i \Omega G \Omega = -\Omega + Q = \tilde{Z} + \Omega + Q = \tilde{Z} - tQ$, so

\begin{equation}
(i \Omega G) \cdot \Omega = \tilde{Z} - tQ = \tilde{\Omega} \cdot \left( -tQ \right) \quad (I_{2g})
\end{equation}

since $\tilde{\Omega} = \begin{pmatrix} I_{2g} & \tilde{Z} \end{pmatrix}$, and that is the Hurwitz relation (8.40).

(ii) It follows from (8.7) that $i \Omega G \Omega = \tilde{Z} - Q$, so

\begin{equation}
(i \Omega G) \cdot \Omega = \tilde{Z} - Q = \tilde{\Omega} \cdot \left( -Q \right) \quad (I_{2g})
\end{equation}

which is the Hurwitz relation (8.41).

(iii) By (8.7) again $\Pi \tilde{Z} = \Pi (i \Omega G \Omega + Q) = \Pi Q$ since $\Pi \Omega = 0$ by (8.35), so

\begin{equation}
\Pi \cdot \tilde{\Omega} = \Pi \begin{pmatrix} I_{2g} & \tilde{Z} \end{pmatrix} = (\Pi \cdot \Pi Q) = \Pi \cdot (I_{2g} \cdot Q),
\end{equation}

and that is the Hurwitz relation (8.42).

(iv) It was noted in the proof of (i) that $\tilde{Z} = i \Omega G \Omega + tQ$ hence $\Pi \tilde{Z} = \Pi (i \Omega G \Omega + tQ) = \Pi \Omega$ since $\Pi \Omega = 0$ by (8.35) again, so

\begin{equation}
\Pi \cdot \tilde{\Omega} = \Pi \begin{pmatrix} I_{2g} & \tilde{Z} \end{pmatrix} = (\Pi \cdot \Pi Q) = \Pi \cdot (I_{2g} \cdot tQ),
\end{equation}

and that is the Hurwitz relation (8.43).

(v) Since $\Pi \Omega = I$ by (8.35) then from (8.46), (8.48) and (8.3) it follows that

\begin{equation}
i \Omega \cdot \Omega = \Pi \cdot i \Omega G \cdot \Omega = \Pi \cdot \tilde{\Omega} \cdot \Omega = \Pi \begin{pmatrix} I_{2g} & Q \end{pmatrix} \begin{pmatrix} -tQ \end{pmatrix} = \Pi \cdot ( -tQ + Q) = \Pi \cdot P^{-1},
\end{equation}

which shows that $(i \Omega, P^{-1})$ is a Hurwitz relation from the matrix $\Omega$ to the matrix $\Pi$; and since $H = G^{-1}$ this can be written equivalently as

\begin{equation}
-i H \cdot \Pi = \Omega \cdot P,
\end{equation}

which shows that $(-i H, P)$ is a Hurwitz relation from the matrix $\Pi$ to the matrix $\Omega$ and thereby suffices to conclude the proof.

\textbf{Corollary 8.5} With the notation of the preceding theorem, the linear mappings described by the matrices $i \Omega G$, $\Pi$ and their complex conjugates define the exact sequences of holomorphic mappings of complex tori

\begin{equation}
0 \longrightarrow J(M) \xrightarrow{i \Omega G} J(M) \xrightarrow{\Pi} P^*(M) \longrightarrow 0
\end{equation}

and

\begin{equation}
0 \longrightarrow J^*(M) \xrightarrow{i \Omega G} J(M) \xrightarrow{\Pi} P(M) \longrightarrow 0.
\end{equation}
Proof: Since Hurwitz relations between nonsingular period matrices define holomorphic mappings between the complex tori described by these matrices as in Theorem F.9 (i) in Appendix F.1, it follows from Theorem 8.4 (i) that the linear mapping described by the matrix \(i\overline{\Omega}G\) determines a holomorphic mapping

\[
i \overline{\Omega}G : J(M) \to J(M)
\]

from the Jacobi variety \(J(M)\) of dimension \(g\) to the complex torus \(J(M)\) of dimension \(2g\); and similarly it follows from Theorem 8.4 (iv) that the linear mapping described by the matrix \(\overline{\Pi}\) yields a holomorphic mapping

\[
\overline{\Pi} : J(M) \to P^*(M)
\]

from the complex torus \(J(M)\) of dimension \(2g\) to the conjugate Picard variety \(P^*(M)\) of dimension \(g\). The composition of these mappings is the zero mapping, since \(\overline{\Pi} \overline{\Omega} = 0\) by (8.35). If \(t \in \mathbb{C}^{2g}\) represents an element in the complex torus \(J(M)\) that is in the kernel of the holomorphic mapping induced by \(\overline{\Pi}\) then \(\overline{\Pi}t = \overline{\Pi}k\) for some \(k \in \mathbb{Z}^{2g}\), and in view of (8.35)

\[
t = (\Omega \overline{\Pi} + \overline{\Omega} \Pi)t = \Omega \overline{\Pi}k + \overline{\Omega} \Pi t = (I_{2g} - \overline{\Omega} \Pi)k + \overline{\Omega} \Pi t = k + \overline{\Omega}t
\]

where \(x = \Pi(-k + t) \in \mathbb{C}^g\); and \(\overline{\Omega}t \in i \overline{\Omega}G\mathbb{C}^g\) since the matrix \(\overline{G}\) is invertible, so \(t = k + \overline{\Omega}t \in \mathbb{Z}^{2g} + i \overline{\Omega}G\mathbb{C}^g\) hence the element of \(J(M)\) represented by \(t\) is contained in the image of the mapping of complex tori determined by the matrix \(i\overline{\Omega}G\) and therefore the sequence (8.51) is exact at the complex torus \(J(M)\). If \(t \in \mathbb{C}^g\) represents an element in the complex torus \(J(M)\) that is in the kernel of the holomorphic mapping induced by \(i\overline{\Omega}G\) then \(i\overline{\Omega}Gt = \Omega k\) for some \(k \in \mathbb{Z}^{2g}\). Since \(\Pi \overline{\Omega} = I_g\) by (8.35) it follows that \(i\overline{\Omega}Gt = \Pi \overline{\Omega}k = \Pi (I_{2g} - Q) k\) in view of (8.48); and since \(H = \overline{G}^{-1}\) it further follows that \(t = -iH \cdot i\overline{\Omega}Gt = -iH \cdot \Pi (I_{2g} - Q) k = \Omega P (I_{2g} - Q) k\) in view of (8.50), so that \(t \in \Omega \mathbb{Z}^{2g}\) represents the zero element in the complex torus \(J(M)\), showing that the sequence (8.51) is exact at the complex torus \(J(M)\). Finally since the torus \(J(M)\) is of dimension \(2g\) and the torus \(J(M)\) is of dimension \(g\) the image of the mapping \(\overline{\Pi}\) is a torus of dimension \(g\), so must be the entire torus \(P^*(M)\), showing the exactness of the entire sequence (8.51). The exactness of the sequence (8.52) follows from the complex conjugate of the argument used to prove the exactness of the sequence (8.52), and that suffices to conclude the proof.

Corollary 8.6 The complex torus \(J(M)\) for a compact Riemann surface \(M\) of genus \(g > 0\) is biholomorphic to the product of the Jacobi variety \(J(M)\) and the conjugate Jacobi variety \(J^*(M)\) of \(M\).

Proof: The two equations (8.46) and (8.47) can be combined as the identity

\[
(i \overline{\Omega}G - i \Omega G) \begin{pmatrix} \Omega & 0 \\ 0 & \overline{\Pi} \end{pmatrix} = \overline{\Omega} \begin{pmatrix} -Q & -Q \\ I_{2g} & I_{2g} \end{pmatrix}.
\]
This identity has the form $A\Omega' = \tilde{\Omega}B$ where $A = \begin{pmatrix} i\Omega G & i\Omega G \end{pmatrix}$ is a $2g \times 2g$ complex matrix, $\Omega' = \begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}$ and $\Omega$ are $2g \times 4g$ nonsingular period matrices, and $B = \begin{pmatrix} -Q & -Q \\ I_{2g} & I_{2g} \end{pmatrix}$ is a $4g \times 4g$ integral matrix; thus the identity (8.53) amounts to the condition that $(A, B)$ is a Hurwitz relation from the period matrix $\Omega'$ to the period matrix $\tilde{\Omega}$. The period matrix $\Omega'$ describes a lattice subgroup of $\mathbb{C}^{2g}$ that is the direct sum of the two lattice subgroups $L(\Omega) \subset \mathbb{C}^g$ and $L(\Omega) \subset \mathbb{C}^g$, so the quotient torus $J(\Omega')$ is the product $J(\Omega) \times J(\Omega) = J(\Omega) \times J^*(\Omega)$; and the quotient torus $J(\Omega)$ is the complex torus $J(M)$. Therefore the Hurwitz relation (8.53) describes a holomorphic mapping

$$J(M) \times J^*(M) \longrightarrow J(M).$$

Since $Q - iQ = P^{-1}$ by (8.3) and $\det P^{-1} = 1$ it follows that

$$\det B = \det \begin{pmatrix} -iQ & -Q \\ I_{2g} & I_{2g} \end{pmatrix} = \det \begin{pmatrix} -Q + Q & -Q \\ 0 & I_{2g} \end{pmatrix} = \det \begin{pmatrix} P^{-1} & -q \\ 0 & I_{2g} \end{pmatrix} = 1,$$

so $B \in \text{Gl}(4g, \mathbb{Z})$; Lemma F.8 shows that $A \in \text{Gl}(2g, \mathbb{C})$, and it then follows from Theorem F.9 (ii) that the holomorphic mapping (8.54) is a biholomorphic mapping, which suffices to conclude the proof.

There is an alternative proof of the preceding corollary; since it is of some independent interest, as providing another way of looking at the complex torus $J(M)$, it will be included here. Readers so inclined can skip the discussion of Corollaries 8.7 and 8.8 and continue with the discussion of the conjugate Jacobi variety $J^*(M)$.

**Corollary 8.7** (i) With the notation of the preceding theorem, the pair of matrices

$$\left( \begin{smallmatrix} \Omega & \Pi \\ \end{smallmatrix} \right), \ T \quad \text{where} \quad T = \begin{pmatrix} -iQP & -iQPQ \\ P & PQ \end{pmatrix} \in \mathbb{Z}^{4g \times 4g}$$

form a Hurwitz relation from the period matrix $\tilde{\Omega}$ to itself, which is the composition of the Hurwitz relation (8.42) from the matrix $\tilde{\Omega}$ to the matrix $\Pi$, the Hurwitz relation (8.45) from the matrix $\Pi$ to the matrix $\Omega$, and the Hurwitz relation (8.40) from the matrix $\Omega$ to the matrix $\tilde{\Omega}$.

(ii) The pair of matrices

$$\left( \begin{smallmatrix} i\Omega & \Pi \\ \end{smallmatrix} \right), \ S \quad \text{where} \quad S = \begin{pmatrix} QP & QP'Q \\ -P & -P'Q \end{pmatrix} \in \mathbb{Z}^{4g \times 4g}$$

also form a Hurwitz relation from the period matrix $\tilde{\Omega}$ to itself, which is a composition of the the Hurwitz relation (8.43) from the matrix $\tilde{\Omega}$ to the matrix
Π, the complex conjugate of the Hurwitz relation (8.45) leading to a Hurwitz relation from the matrix Π to the matrix Ω, and the Hurwitz relation (8.41) from the matrix Ω to the matrix Ω.

**Proof:** (i) The combination of the Hurwitz relation (8.42) from the matrix Ω to the matrix Π, the Hurwitz relation (8.45) from the matrix Π to the matrix Ω, and the Hurwitz relation (8.40) from the matrix Ω to the matrix Ω shows that

$$(i \Omega G) \cdot (-iH) \cdot \Pi \cdot \tilde{\Omega} = (i \Omega G) \cdot (-iH) \cdot \Pi \cdot (I_{2g} \quad Q)$$

$$= (i \Omega G) \cdot \Omega \cdot P \cdot (I_{2g} \quad Q)$$

$$= \tilde{\Omega} \left( \begin{array}{cc} -Q \quad & \quad 0 \\ P \quad & \quad P \end{array} \right) \cdot P \cdot (I_{2g} \quad Q)$$

and since

$$(i \Omega G) \cdot (-iH) \cdot (\Pi) \tilde{\Omega} = \bar{\Omega} \Pi$$

that yields the Hurwitz relation (8.55).

(ii) The combination of the Hurwitz relation (8.43) from the matrix Ω to the matrix Π, the complex conjugate of the Hurwitz relation (8.45) yielding a Hurwitz relation from the matrix Π to the matrix Ω, and the Hurwitz relation (8.41) from the matrix Ω to the matrix Ω shows that

$$(i \Omega G) \cdot (iH) \cdot \bar{\Omega} \cdot \Pi \bar{\Omega} = (i \Omega G) \cdot (iH) \cdot \bar{\Omega} \cdot (I_{2g} \quad Q)$$

$$= (i \Omega G) \cdot \bar{\Omega} \cdot P \cdot (I_{2g} \quad Q)$$

$$= \bar{\Omega} \left( \begin{array}{cc} Q \quad & \quad 0 \\ P \quad & \quad P \end{array} \right) \cdot P \cdot (I_{2g} \quad Q)$$

and since

$$(i \Omega G) \cdot (iH) \cdot \bar{\Omega} \cdot \Pi = -\bar{\Omega} \Pi$$

after a change of sign that yields the Hurwitz relation (8.56), which suffices to conclude the proof.

The last identity in (8.35) when written equivalently

$$(8.57) \quad \bar{\Omega} \Pi + \bar{\Omega} \bar{\Pi} = I_{2g}$$

exhibits the direct sum decomposition

$$(8.58) \quad C^{2g} = \bar{\Omega} \Pi C^{2g} \oplus \bar{\Omega} \bar{\Pi} C^{2g}$$
provided by the linear operators \( \mathbf{\Pi} : \mathbb{C}^{2g} \rightarrow \mathbb{C}^{2g} \) and \( \mathbf{\Pi} : \mathbb{C}^{2g} \rightarrow \mathbb{C}^{2g} \), which are projection operators since it follows from the first two identities in (8.35) that
\[
(8.59) \quad \mathbf{\Pi} \cdot \mathbf{\Pi} = \mathbf{\Pi}, \quad \mathbf{\Pi} \cdot \mathbf{\Pi} = \mathbf{\Pi}, \quad \mathbf{\Pi} \cdot \mathbf{\Pi} = \mathbf{\Pi} \cdot \mathbf{\Pi} = 0.
\]
It follows from the preceding Corollary 8.7 that the integral matrices \( S, T \) satisfy the corresponding relation
\[
(8.60) \quad T + S = I_{4g},
\]
exhibiting the direct sum lattice decomposition
\[
(8.61) \quad \mathbb{Z}^{4g} = T \mathbb{Z}^{4g} \oplus S \mathbb{Z}^{4g}
\]
provided by the linear operators \( T : \mathbb{Z}^{4g} \rightarrow \mathbb{Z}^{4g} \) and \( S : \mathbb{Z}^{4g} \rightarrow \mathbb{Z}^{4g} \), which are projection operators satisfying
\[
(8.62) \quad T^2 = T, \quad S^2 = S, \quad TS = ST = 0.
\]
It is perhaps comforting to verify these identities directly. First
\[
T + S = \begin{pmatrix}
(Q - \bar{Q})P & -QPQ + QP \bar{Q} \\
P - P & P(Q - \bar{Q})
\end{pmatrix}
= \begin{pmatrix}
I_{2g} & 0 \\
0 & I_{2g}
\end{pmatrix}
\]
since \( Q - \bar{Q} = P^{-1} \) as in (8.3) so \( \bar{Q} = Q - P^{-1} \) hence \( -QPQ + QP \bar{Q} = -(Q - P^{-1})PQ + PQ(Q - P^{-1}) = 0 \). Next
\[
T^2 = \begin{pmatrix}
-\bar{Q}P & -QPQ \\
P & P
\end{pmatrix}
\begin{pmatrix}
-\bar{Q}P & -QPQ \\
P & P
\end{pmatrix}
= \begin{pmatrix}
-\bar{Q}PQ & -\bar{Q}PQP \\
-P\bar{Q}P + PQP & -P\bar{Q}PQ + PQP
\end{pmatrix}
= \begin{pmatrix}
-\bar{Q}P & -QPQ \\
P & PQ
\end{pmatrix}
= T
\]
since \( Q - \bar{Q} = P^{-1} \) by (8.3), and similarly for the matrix \( S \), while
\[
T \cdot S = \begin{pmatrix}
-\bar{Q}P & -QPQ \\
P & PQ
\end{pmatrix}
\begin{pmatrix}
QP & QP \bar{Q} \\
-P & -P \bar{Q}
\end{pmatrix}
= 0,
\]
and similarly for the product \( S \cdot T \).

**Corollary 8.8** The complex torus \( J(M) \) for a compact Riemann surface \( M \) of genus \( g > 0 \) is biholomorphic to the product of the Jacobi variety \( J(M) \) and the conjugate Jacobi variety \( J^*(M) \) of \( M \).

**Proof:** In the decomposition (8.58), since the period matrices are nonsingular it follows from (8.59) that the subspace \( \mathbf{\Pi} \mathbb{C}^{2g} \subset \mathbb{C}^{2g} \) can be described as the
kernel of the linear mapping $\Pi : \mathbb{C}^{2g} \to \mathbb{C}^g$ and the subspace $\Omega \Pi \mathbb{C}^{2g} \subset \mathbb{C}^{2g}$ can be described as the kernel of the linear mapping $\Pi : \mathbb{C}^{2g} \to \mathbb{C}^g$. By Corollary 8.7 these projection operators also satisfy

$$ (8.63) \quad \Omega \Pi \cdot \overline{\Omega} = \overline{\Omega} \cdot T \quad \text{and} \quad \Omega \Pi \cdot \overline{\Omega} = \overline{\Pi} \cdot S $$

for the integral $4g \times 4g$ matrices $T$ and $S$; and since the $g \times 2g$ projection matrices $\Omega \Pi$ and $\Omega \Pi$ are of rank $g$ it follows from Lemma F.8 in Appendix F.1 that the matrices $T$ and $S$ are of rank $2g$; and they satisfy (8.62). The image $L_T = T \mathbb{Z}^{4g} \subset \mathbb{Z}^{4g}$ of the mapping $T : \mathbb{Z}^{4g} \to \mathbb{Z}^{4g}$ hence is a subgroup of the lattice $\mathbb{Z}^{4g}$ that spans a 2-dimensional linear subspace of $\mathbb{C}^{4g}$; and since that image is contained in the 2-dimensional linear subspace $\Omega \Pi \mathbb{C}^{2g} \subset \mathbb{C}^{2g}$ it follows that $L_T$ is a lattice subgroup of $\Omega \Pi \mathbb{C}^{2g}$. The same argument holds for the matrix $S$, so

$$ (8.64) \quad L_T = \overline{\Omega} T \mathbb{Z}^{4g} = \overline{\Omega} \Pi \overline{\Omega} \mathbb{Z}^{4g} \quad \text{and} \quad L_S = \overline{\Omega} S \mathbb{Z}^{4g} = \overline{\Pi} \overline{\Omega} \mathbb{Z}^{4g} $$

are lattice subgroups of the respective linear subspaces $\overline{\Omega} \Pi \mathbb{C}^{2g}$ and $\overline{\Omega} \Pi \mathbb{C}^{2g}$ of $\mathbb{C}^{2g}$. It further follows from (8.35) that

$$ (8.65) \quad L(\overline{\Omega}) = \overline{\Omega} \mathbb{Z}^{4g} = (\overline{\Omega} \Pi \oplus \overline{\Omega} \Pi) \overline{\Omega} \mathbb{Z}^{4g} = L_T \oplus L_S \subset \overline{\Omega} \Pi \mathbb{C}^{2g} \oplus \overline{\Omega} \Pi \mathbb{C}^{2g} $$

consequently

$$ (8.66) \quad \mathcal{J}(M) = \frac{\mathbb{C}^{2g}}{L(\overline{\Omega})} = \frac{\overline{\Omega} \Pi \mathbb{C}^{2g}}{L_T} \oplus \frac{\overline{\Omega} \Pi \mathbb{C}^{2g}}{L_S} $$

exhibiting the complex torus $\mathcal{J}(M)$ as the direct sum of two complex tori of dimension $4$. Since the linear subspace $\overline{\Omega} \Pi \mathbb{C}^{2g} \subset \mathbb{C}^{2g}$ is the kernel of the linear mapping $\Pi : \mathbb{C}^{2g} \to \mathbb{C}^g$, the image of the injection mapping $i \Omega G : J(M) \to \mathcal{J}(M)$ of (8.51) in Corollary 8.5 must lie in the subtorus $\overline{\Omega} \Pi \mathbb{C}^{2g} / L_T \subset \mathcal{L}(M)$; consequently the mapping $i \Omega G$ must be a biholomorphic mapping from the complex torus $J(M)$ to the subtorus $\overline{\Omega} \Pi \mathbb{C}^{2g} / L_T \subset \mathcal{L}(M)$. Correspondingly the injection $i \Omega G : J^*(M) \to \mathcal{J}(M)$ of (8.52) in Corollary 8.5 must be a biholomorphic mapping from the complex torus $J^*(M)$ to the subtorus $\overline{\Omega} \Pi \mathbb{C}^{2g} / L_S \subset \mathcal{L}(M)$, and that suffices for the proof.

The product decomposition of the complex torus $\mathcal{J}(\overline{\Omega})$ of Corollary 8.6 or Corollary 8.8 leads to a corresponding product formula for the associated theta functions. Actually it is easier not to use the reduction of the period matrix directly but instead just to analyze the effect of the direct sum decomposition (8.58). From the identity $\overline{\Pi} H = I_g$ it follows that

$$ (8.67) \quad \overline{\Omega} \Pi \mathbb{C}^{2g} = (i \Omega G) \cdot (\overline{\Pi} H \Pi) \mathbb{C}^{2g} = i \Omega G \mathbb{C}^g $$

and similarly that

$$ (8.68) \quad \overline{\Omega} \Pi \mathbb{C}^{2g} = (i \Omega G) \cdot (\overline{\Pi} H \Pi) \mathbb{C}^{2g} = i \Omega G \mathbb{C}^g $$

consequently the direct sum decomposition (8.58) can be rewritten

$$ (8.69) \quad \mathbb{C}^{2g} = i \overline{\Omega} G \mathbb{C}^g \oplus i \Omega G \mathbb{C}^g. $$
Theorem 8.9 On a compact Riemann surface \( M \) of genus \( g > 0 \) with the period matrix \( \Omega \) and the intersection matrix \( P \) in terms of some bases \( \omega_i \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and \( \tau_j \in H_1(M) \), let \( G = \hat{H}^{-1} = H^{-1} \) for the positive definite Hermitian matrix \( H = i \Omega P \overline{\Omega} \) and let \( \tilde{Z} = -\frac{1}{2\pi i} (R + \pi i \tilde{P}) \) where \( \tilde{P} \) is the symmetrization of the matrix \( P \) and \( R = 2\pi \Re(\hat{\Omega} \overline{\Omega}) \). In these terms

\[
\Theta(s; \tilde{Z}) \Theta \left( s + i \overline{\Omega} t_1 + i^\nu G t_2; \tilde{Z} \right) = \exp 2\pi \left( t_1 G t_2 \right) \cdot \Theta \left( s + i \overline{\Omega} G t_1; \tilde{Z} \right) \cdot \Theta \left( s + i \overline{\Omega} G t_2; \tilde{Z} \right)
\]

for all \( t_1, t_2 \in \mathbb{C}^g, \ s \in \mathbb{C}^{2g} \).

Proof: As in (8.69), any point \( t \in \mathbb{C}^{2g} \) can be written uniquely as

\[
t = i \overline{\Omega} t_1 + i^\nu G t_2 \in \mathbb{C}^{2g} \text{ for some } t_1, t_2 \in \mathbb{C}^g.
\]

In terms of this decomposition, for a fixed point \( s \in \mathbb{C}^{2g} \) introduce the function

\[
f_s(t) = f_s(t_1, t_2) = \exp -2\pi \left( t_1 G t_2 \right) \cdot \Theta(s + t; \tilde{Z})
\]

of the variables \( t_1, t_2 \in \mathbb{C}^g \), or equivalently of the variable \( t \in \mathbb{C}^{2g} \). It follows from (8.46) and (8.31) that

\[
f_s(t_1 + \Omega \nu, t_2) = \exp -2\pi \left( t_1 G t_2 + \nu \overline{\Omega} G t_2 \right) \cdot \Theta \left( s + t + i \overline{\Omega} G \nu; \tilde{Z} \right) = \exp -2\pi \left( t_1 G t_2 + \nu \overline{\Omega} G t_2 \right) \cdot \Theta \left( s + t + \nu \Omega \nu; \tilde{Z} \right)
\]

\[
= \exp -2\pi i \left( \frac{1}{2} \nu \overline{Z} \nu + \nu (s + i \overline{\Omega} G t_1 + i^\nu \Omega G t_2) \right) \cdot \Theta(s + t; \tilde{Z})
\]

\[
= \exp -2\pi i \left( \frac{1}{2} \nu \overline{Z} \nu + \nu s + i^\nu \overline{\Omega} G t_1 \right) f_s(t_1, t_2);
\]

and since \( t_1 G g_2 = t_2 G t_1 \) it follows similarly from (8.47) and (8.31) that

\[
f_s(t_1, t_2 + \Omega \nu) = \exp -2\pi \left( t_2 G t_1 + \nu \overline{\Omega} G t_1 \right) \cdot \Theta \left( s + t + i \overline{\Omega} G \nu; \tilde{Z} \right)
\]

\[
= \exp -2\pi \left( t_2 G t_1 + \nu \overline{\Omega} G t_1 \right) \cdot \Theta(s + t - \nu \Omega \nu; \tilde{Z})
\]

\[
= \exp -2\pi i \left( \frac{1}{2} \nu \overline{Z} \nu + \nu s + i^\nu \overline{\Omega} G t_2 \right) f_s(t_1, t_2).
\]

It is evident from the preceding two equations that the function

\[
g_s(t) = g_s(t_1, t_2) = f_s(t_1, 0) f_s(0, t_2)
\]

satisfies the same equations (8.73) and (8.74) as does the function \( f_s(t) = f_s(t_1, t_2) \); consequently the functions \( \exp 2\pi \left( t_1 G t_2 \right) g_s(t) \) and \( \Theta(s + t; \tilde{Z}) = \exp 2\pi \left( t_1 G t_2 \right) f_s(t_1, t_2) \) satisfy the same equations under the translations

\[
t_1 \mapsto t_1 + \Omega \nu_1 \text{ and } t_2 \mapsto t_2 + \Omega \nu_2.
\]
As in the proofs of (8.73) and (8.74), when expressed in terms of the parameter \( t \in \mathbb{C}^{2g} \) the translations (8.76) take the form

\[
(8.77) \quad t \longrightarrow t - tQ\nu_1 + \tilde{Z}\nu_1 \quad \text{and} \quad t \longrightarrow t - Q\nu_2 + \tilde{Z}\nu_2
\]

for all \( \nu_1 \in \mathbb{Z}^{2g} \). These are special cases of the translations \( t \longrightarrow t + \mu + \tilde{Z}\nu \) for all \( \mu, \nu \in \mathbb{Z}^{2g} \), the translations generating the lattice subgroup \( L(\Omega) \); but actually these special cases serve to generate the full set of translations in the lattice subgroup \( L(\Omega) \). Indeed the composition of the two translations in (8.77) is the translation

\[
(8.78) \quad t \longrightarrow t - tQ\nu_1 - Q\nu_2 + \tilde{Z}(\nu_1 + \nu_2);
\]

for \( \nu_2 = -\nu_1 \) this is the translation \( t \longrightarrow t - tQ\nu_1 + Q\nu_1 = t + P^{-1}\nu_1 \) in view of (8.3), and since \( \det P^{-1} = 1 \) the translations \( t \longrightarrow t + \mu \) for all \( \mu \in \mathbb{Z}^{2g} \) arise in this way, while for \( \nu_2 = 0 \) this is the translation \( t \longrightarrow t - tQ\nu_1 + \tilde{Z}\nu_1 \), which when combined with the translation \( t \longrightarrow t + tQ\nu_1 \) yields the translation \( t \longrightarrow t + \tilde{Z}\nu \) for all \( \nu \in \mathbb{Z}^{2g} \). Consequently the functions \( \exp 2\pi(tGt_1)g_s(t) \) and \( \Theta(s + t; \tilde{Z}) = \exp 2\pi(tGt_1)f_s(t) \) satisfy the same equations under all the translations in the lattice subgroup \( L(\Omega) \). The translate \( \Theta(s + t; \tilde{Z}) \) of the theta function of the variable \( t \in \mathbb{C}^{2g} \) is characterized uniquely up to a constant multiple by these equations, as in Theorem G.1 in Appendix G, and therefore

\[
(8.79) \quad \exp 2\pi(tGt_1)g_s(t) = c(s) \Theta(s + t; \tilde{Z})
\]

for some uniquely determined factor \( c(s) \) that is independent of the parameter \( t \in \mathbb{C}^{2g} \). Here \( g_s(t) = f_s(t_1, 0)f_s(0, t_2) = \Theta(s + i\Omega Gt_1; \tilde{Z})\Theta(s + i\Omega Gt_1; \tilde{Z}) \), and setting \( t = 0 \) shows that \( c(s) = \Theta(s; \tilde{Z}) \); hence (8.79) reduces to (8.70), thereby concluding the proof.

**Corollary 8.10** (i) For any \( s \in \mathbb{C}^{2g} \) there is up to a constant factor a unique holomorphic function \( \vartheta_{s, \Omega}(t) \) of \( t \in \mathbb{C}^g \) such that

\[
(8.80) \quad \vartheta_{s, \Omega}(t + \Omega \nu) = \zeta_{g, s}(\Omega \nu, t) \cdot \vartheta_{s, \Omega}(t) \quad \text{for all} \quad \nu \in \mathbb{Z}^{2g}
\]

for the factor of automorphy

\[
(8.81) \quad \zeta_{g, s}(\Omega \nu, t) = \exp -2\pi i \left( \frac{1}{2} \nu \tilde{Z} \nu - \nu s + i \nu \Omega G t \right)
\]

for the action of the lattice subgroup \( L(\Omega) \) on \( \mathbb{C}^g \); and there is up to a constant factor a unique holomorphic function \( \vartheta_{s, \Omega}(t) \) of \( t \in \mathbb{C}^g \) such that

\[
(8.82) \quad \vartheta_{s, \Omega}(t + \Omega \nu) = \zeta_{g, s}(\Omega \nu) \cdot \vartheta_{s, \Omega}(t) \quad \text{for all} \quad \nu \in \mathbb{Z}^{2g}
\]

for the factor of automorphy

\[
(8.83) \quad \zeta_{g, s}(\Omega \nu, t) = \exp -2\pi i \left( \frac{1}{2} \nu \tilde{Z} \nu - \nu s + i \nu \Omega G t \right)
\]
for the action of the lattice subgroup $L(\Omega)$ on $\mathbb{C}^g$.

(ii) For any point $s \in \mathbb{C}^{2g}$ the function

\[ \vartheta_{s,\Omega}(t) = \Theta \left( s + i \Omega G t; \tilde{Z} \right) \]

satisfies (8.80), so if it is not identically zero it is determined uniquely up to a constant factor by that equation; and also for any point $s \in \mathbb{C}^{2g}$ the function

\[ \vartheta_{s,\tilde{\Omega}}(t) = \Theta \left( s + i \tilde{\Omega} G t; \tilde{Z} \right) \]

satisfies (8.82), so if it is not identically zero it is determined uniquely up to a constant factor by that equation.

**Proof:** The proof of the preceding theorem used only the conditions that the function $f_s(t_1, 0)$ satisfies (8.73), which is just (8.80), and that the function $f_s(0, t_2)$ satisfies (8.74), which is just (8.82), in order to show that the product $g_s(t) = f_s(t_1, 0) f_s(0, t_2)$ has the form (8.79) for some value $c(s)$ that is independent of $t$; hence that argument shows that the product of any holomorphic function satisfying (8.80) and any holomorphic function satisfying (8.82) also has the form (8.79) for some $c(s)$ that is independent of $t$. That was applied in the preceding theorem to the functions (8.84) and (8.85), the product of which has the form (8.79) for some $c(s)$ that is independent of $t$. If $h(t)$ is any holomorphic function satisfying (8.80) this argument shows that $h(t) \vartheta_{s,\Omega}(t) = c(s) \vartheta_{s,\Omega}(t) \vartheta_{s,\tilde{\Omega}}(t)$ for some $c(s)$ that is independent of $t$ and consequently that $h(t) = c(s) \vartheta_{s,\Omega}(t)$. The same argument of course applies to a holomorphic function satisfying (8.82), and that suffices for the proof.

The function $\tilde{\vartheta}_{s,\Omega}(t)$ is defined in (8.84) in terms of the classical theta series for the period matrix $\tilde{\Omega} = \begin{pmatrix} I & \tilde{Z} \end{pmatrix}$, as is the factor of automorphy (8.80); but the function $\vartheta_{s,\Omega}(t)$ actually is a relatively automorphic function for the action of the lattice subgroup $L(\Omega)$ on $\mathbb{C}^g$. It might be expected that the construction of this function would be expressed as a theta series in terms of the period matrix $\Omega$, although of course that would only be possible if that period matrix was in the standard form $\Omega = \begin{pmatrix} I & \tilde{Z} \end{pmatrix}$. The advantage of the approach taken here is that it involves more intrinsic constructions on the Riemann surface $M$ and does not require that the period matrix $\Omega$ have the standard form. The expression (8.22) of the intrinsic theta function in terms of the classical theta series can be rewritten in these terms as

\[ \theta_g(z, a; t) = \vartheta_{-t,\Omega}(\tilde{w}(z, a)) \]

since the classical theta series is an even function; this is actually a rather simpler form than (8.22), although not so intrinsic since it is expressed in terms of the period matrix $\Omega$. The product formula (8.70) can be expressed in these terms as well, as

\[ \Theta(s; \tilde{Z}) \Theta \left( s + i \Omega G t_1 + i \tilde{\Omega} G t_2; \tilde{Z} \right) = \exp 2\pi \left( t_1 G t_2 \right) \cdot \vartheta_{s,\Omega}(t_1) \cdot \vartheta_{s,\tilde{\Omega}}(t_2) \]
for all \( t_1, t_2 \in \mathbb{C}^g, s \in \mathbb{C}^2g \).

An alternate form of the product formula (8.70) expressed in terms of the decomposition (8.58) of the vector space \( \mathbb{C}^2g \) is

\[
(8.88) \quad \Theta(s; \tilde{Z}) \Theta(s + t; \tilde{Z}) = \exp -2\pi \left( t' \Pi' \Pi t \right) \cdot \Theta \left( s + \Pi t; \tilde{Z} \right) \cdot \Theta \left( s + \Omega \Pi t; \tilde{Z} \right)
\]

for all \( s, t \in \mathbb{C}^2g \), since \( t_1 = -i \Pi t \) and \( t_2 = -i \Pi \Pi t \) in the decomposition (8.71); this product formula can be written more symmetrically in terms of the points \( s, u = s + t \) as

\[
(8.89) \quad \Theta(s; \tilde{Z}) \Theta(u; \tilde{Z}) = \exp -2\pi \left( u - s \right) \Pi' \Pi (u - s) \cdot \Theta \left( \Omega \Pi s + \Pi \Pi u; \tilde{Z} \right) \cdot \Theta \left( \Omega \Pi s + \Omega \Pi u; \tilde{Z} \right)
\]

since \( \Pi' \Pi + \Omega \Pi = I \) as in (8.35). The matrix \( H \) is positive definite Hermitian so the function \( \delta(t) = \exp -2\pi \left( t' \Pi' \Pi t \right) \) satisfies \( 0 < \delta(t) \leq 1 \) for all \( t \in \mathbb{C}^2g \); and if \( s \) and \( t \) in (8.88) are real it follows from the reality (8.26) of the theta function that \( \Theta \left( s + \Pi \Pi t; \tilde{Z} \right) = \Theta \left( s + \Omega \Pi t; \tilde{Z} \right) \) and therefore the product formula (8.88) takes the form

\[
(8.90) \quad \Theta(s; \tilde{Z}) \Theta(s + t; \tilde{Z}) = \delta(t) \left| \Theta \left( s + \Pi \Pi t; \tilde{Z} \right) \right|^2 \quad \text{if } s, t \in \mathbb{R}^2g
\]

where \( 0 < \delta(t) \leq 1 \).

The zero locus

\[
(8.91) \quad \tilde{V}_\Theta = \{ \ t \in \mathbb{C}^2g \mid \Theta(t; \tilde{Z}) = 0 \ \}
\]

of the classical theta function is a holomorphic subvariety of \( \mathbb{C}^2g \) of pure codimension 1 that is invariant under translation by any vector in the lattice subgroup \( L(\Omega) \) since the theta function is a relatively automorphic function for the action of that subgroup; the subvariety \( \tilde{V}_\Theta \subset \mathbb{C}^2g \) therefore represents a holomorphic subvariety \( V_\Theta \subset J(\Omega) \) of the complex torus \( J(\Omega) \) described by the lattice subgroup \( L(\tilde{\Omega}) \), as in (8.32). The product formula of the preceding Theorem 8.9 implies that the subvarieties \( \tilde{V}_\Theta \) and \( V_\Theta \) are reducible.

**Corollary 8.11** The holomorphic subvariety \( \tilde{V}_\Theta \subset \mathbb{C}^2g \) is the union

\[
(8.92) \quad \tilde{V}_\Theta = \tilde{V}_1 \cup \tilde{V}_2
\]

of the holomorphic subvarieties \( \tilde{V}_1, \tilde{V}_2 \) in \( \mathbb{C}^2g \) defined by

\[
(8.93) \quad \tilde{V}_1 = \bigcap_{a \in \mathbb{C}^2g} \left( \tilde{V}_\Theta - \Pi \Pi a \right) \quad \text{and} \quad \tilde{V}_2 = \bigcap_{a \in \mathbb{C}^2g} \left( \tilde{V}_\Theta - \Omega \Pi a \right).
\]
The subvarieties $\tilde{V}_1, \tilde{V}_2$ in $\mathbb{C}^{2g}$ are invariant under the action of the lattice subgroup $L(\Omega)$ and satisfy

\[(8.94) \quad \tilde{V}_1 + \mathfrak{T} \Pi \mathbb{C}^{2g} = \tilde{V}_1 \quad \text{and} \quad \tilde{V}_2 + \mathfrak{T} \Pi \mathbb{C}^{2g} = \tilde{V}_2.\]

For any point $a \in \mathbb{C}^{2g}$ such that $\Theta(a; \tilde{Z}) \neq 0$ these subvarieties can be described alternatively by

\[(8.95) \quad \tilde{V}_1 = \left\{ t \in \mathbb{C}^{2g} \mid \Theta \left( t, \mathfrak{T} \Pi t + \mathfrak{T} \Pi a; \tilde{Z} \right) = 0 \right\}, \]
\[\tilde{V}_2 = \left\{ t \in \mathbb{C}^{2g} \mid \Theta \left( \mathfrak{T} \Pi t + \mathfrak{T} \Pi a; \tilde{Z} \right) = 0 \right\}.

**Proof:** Each of the subsets (8.93) is an intersection of holomorphic subvarieties of $\mathbb{C}^{2g}$ that are invariant under the action of the lattice subgroup $L(\Omega)$, so each is itself a holomorphic subvariety of $\mathbb{C}^{2g}$ that is invariant under the action of the lattice subgroup $L(\Omega)$. The definition (8.93) of the set $\tilde{V}_1$ can be rephrased as

\[(8.96) \quad \tilde{V}_1 = \left\{ t \in \mathbb{C}^{2g} \mid t + \mathfrak{T} \Pi a \in \tilde{V}_\Theta \quad \text{for all} \quad a \in \mathbb{C}^{2g} \right\}.

Thus if $t \in \tilde{V}_1$ then in particular for $a = 0$ in (8.96) it follows that $t \in \tilde{V}_\Theta$, so $\tilde{V}_1 \subset \tilde{V}_\Theta$; and for $a = a_1 + a_2$ it further follows that $t + \mathfrak{T} \Pi (a_1 + a_2) \in \tilde{V}_\Theta$ for all $a_2 \in \mathbb{C}^{2g}$ so that $t + \mathfrak{T} \Pi a_1 \in \tilde{V}_1$ for all $a_1 \in \mathbb{C}^{2g}$ hence $\tilde{V}_1 + \mathfrak{T} \Pi \mathbb{C}^{2g} \subset \tilde{V}_1$.

The corresponding argument for the subvariety $\tilde{V}_2$ shows that $\tilde{V}_2 \subset \tilde{V}_\Theta$ and $\tilde{V}_2 + \mathfrak{T} \Pi \mathbb{C}^{2g} \subset \tilde{V}_2$. On the other hand if $t \in \tilde{V}_\Theta$ then by definition $\Theta(t; \tilde{Z}) = 0$ so it follows from (8.88) that for any $a \in \mathbb{C}^{2g}$

\[0 = \Theta(t; \tilde{Z})\Theta(t + a; \tilde{Z}) = \epsilon \cdot \Theta \left( t + \mathfrak{T} \Pi a; \tilde{Z} \right) \cdot \Theta \left( t + \mathfrak{T} \Pi a; \tilde{Z} \right),\]

where $\epsilon \neq 0$; thus the product $\Theta \left( t + \mathfrak{T} \Pi a; \tilde{Z} \right) \cdot \Theta \left( t + \mathfrak{T} \Pi a; \tilde{Z} \right)$ vanishes identically in the variable $a \in \mathbb{C}^{2g}$, so at least one of the two factors must vanish identically in the variable $a$. If $\Theta \left( t + \mathfrak{T} \Pi a; \tilde{Z} \right) = 0$ identically in $a \in \mathbb{C}^{2g}$ then $t \in \tilde{V}_1$ by (8.96), while if $\Theta \left( t + \mathfrak{T} \Pi a; \tilde{Z} \right) = 0$ identically in $a \in \mathbb{C}^{2g}$ then $t \in \tilde{V}_2$; thus $\tilde{V}_\Theta \subset \tilde{V}_1 \cup \tilde{V}_2$, so since the reversed inclusion already has been demonstrated it follows that $\tilde{V}_\Theta = \tilde{V}_1 \cup \tilde{V}_2$. From (8.88) again it follows that for $t, a, b \in \mathbb{C}^{2g}$

\[(8.97) \quad \Theta(a; \tilde{Z})\Theta(a + t + \mathfrak{T} \Pi b; \tilde{Z}) = \epsilon \cdot \Theta \left( a + \mathfrak{T} \Pi(t + \mathfrak{T} \Pi b); \tilde{Z} \right) \cdot \Theta \left( a + \mathfrak{T} \Pi(t + \mathfrak{T} \Pi b); \tilde{Z} \right),\]

where $\epsilon \neq 0$, since $\mathfrak{T} \Pi \mathfrak{T} = 1$ and $\mathfrak{T} \Pi \mathfrak{T} = 0$ by (8.35). Now $a + t \in \tilde{V}_1$ if and only if $\Theta(a + t + \mathfrak{T} \Pi b) = 0$ for all $b \in \mathbb{C}^{2g}$, as a consequence of (8.96); so...
if $\Theta(a; \tilde{Z}) \neq 0$ then in view of (8.97) it follows that $a + t \in \tilde{V}_1$ if and only if

$$\Theta(a + \Omega \Pi(t + b); \tilde{Z}) - \Theta(a + \Omega \Pi t; \tilde{Z}) = 0$$

identically in $b \in \mathbb{Z}^2g$, which is equivalent to the condition that $\Theta(a + \Omega \Pi(t + b); \tilde{Z}) = 0$ since $\Theta(a + \Omega \Pi(t + b); \tilde{Z}) = \Theta(a; \tilde{Z}) \neq 0$ for $b = -t$. So $a + t \in \tilde{V}_1$ if and only if $\Theta(a + \Omega \Pi t; \tilde{Z}) = 0$, hence $t \in \tilde{V}_1$ if and only if $0 = \Theta(a + \Omega \Pi(t - a); \tilde{Z}) = \Theta(\Omega \Pi a + \Omega \Pi t; \tilde{Z})$ since $a = \Omega \Pi a + \Omega \Pi a$ by (8.35). The corresponding argument for $\tilde{V}_2$ yields the other part of (8.95), and that suffices to conclude the proof.

The requirement that $\Theta(a; \tilde{Z}) \neq 0$ in (8.95) may seem somewhat inessential. However if $\Theta(a; \tilde{Z}) = 0$ it follows from (8.97) that

$$\Theta(a + \Omega \Pi(t + b); \tilde{Z}) \cdot \Theta(a + \Omega \Pi t; \tilde{Z}) = 0$$

for all $b, t \in \mathbb{C}^{2g}$; hence if $\Theta(a; \tilde{Z}) = 0$ then either $\Theta(a + \Omega \Pi t; \tilde{Z}) = 0$ for all $t \in \mathbb{C}^{2g}$ or $\Theta(a + \Omega \Pi t; \tilde{Z}) = 0$ for all $t \in \mathbb{C}^{2g}$, so that (8.95) then does not describe describe the subvariety $\tilde{V}_1$. That really reflects the decomposition (8.92) of the zero locus $V_0$.

**Corollary 8.12** The intersections

$$\tilde{V}_1' = \tilde{V}_1 \cap \Omega \Pi \mathbb{C}^{2g} \quad \text{and} \quad \tilde{V}_2' = \tilde{V}_2 \cap \Omega \Pi \mathbb{C}^{2g}$$

are holomorphic subvarieties of the $g$-dimensional linear subspaces $\Omega \Pi \mathbb{C}^{2g}$ and $\Omega \Pi \mathbb{C}^{2g}$ of $\mathbb{C}^{2g}$ respectively, for which

$$\tilde{V}_1 = \tilde{V}_1' + \Omega \Pi \mathbb{C}^{2g} \quad \text{and} \quad \tilde{V}_2 = \tilde{V}_2' + \Omega \Pi \mathbb{C}^{2g}.$$  

These subvarieties can be described by

$$\tilde{V}_1' = \left\{ t \in \Omega \Pi \mathbb{C}^{2g} \mid \Theta \left(t + \Omega \Pi a; \tilde{Z}\right) = 0 \right\}$$

$$\tilde{V}_2' = \left\{ t \in \Omega \Pi \mathbb{C}^{2g} \mid \Theta \left(t + \Omega \Pi a; \tilde{Z}\right) = 0 \right\}$$

for any point $a \in \mathbb{C}^{2g}$ for which $\Theta(a; \tilde{Z}) \neq 0$.

**Proof:** The intersections of holomorphic subvarieties are again holomorphic subvarieties, so in particular that is the case for $\tilde{V}_1'$ and $\tilde{V}_2'$. If $t \in \tilde{V}_1$ then $t = \Omega \Pi t + \Omega \Pi t$ in view of the direct sum decomposition of (8.58) and it follows from (8.94) that $\Omega \Pi t = t - \Omega \Pi t \in \tilde{V}_1 \cap \Omega \Pi \mathbb{C}^{2g} = \tilde{V}_1'$ and consequently that $t = \Omega \Pi t + \Omega \Pi t \in \tilde{V}_1' + \Omega \Pi \mathbb{C}^{2g}$ so that $\tilde{V}_1 \subset \tilde{V}_1' + \Omega \Pi \mathbb{C}^{2g}$, while the inverse containment is trivial in view of (8.94) since $\tilde{V}_1' \subset \tilde{V}_1$. The corresponding argument demonstrates the second assertion of (8.99). The subvariety $\tilde{V}_1'$ then consists of points $t = \Omega \Pi s$ for some $s \in \mathbb{C}^{2g}$ such that $t \in \tilde{V}_1$; and by (8.95) the latter condition is just that $0 = \Theta \left(\Omega \Pi t + \Omega \Pi a; \tilde{Z}\right) = \Theta \left(\Omega \Pi \Omega \Pi s + \Omega \Pi a; \tilde{Z}\right) = \Theta \left(\Omega \Pi s + \Omega \Pi a; \tilde{Z}\right) = \Theta \left(t + \Omega \Pi a; \tilde{Z}\right)$ since $\Omega \Pi \Omega \Pi = \Omega \Pi$ by (8.59), and
that is the first equation in (8.100). The second equation is demonstrated correspondingly, and that suffices for the proof.

Since \( \mathcal{H}^{\mathcal{L}} \mathcal{C}^{2g} = i \mathcal{H}^{\mathcal{C}} \mathcal{C}^{g} \) as in (8.67), points \( t \in \mathcal{H}^{\mathcal{L}} \mathcal{C}^{2g} \) can be parametrized by points \( s \in \mathcal{C}^{g} \) by setting \( t = i \mathcal{H}^{\mathcal{C}} s \); and in these terms the equations (8.100) can be rewritten

\[
\begin{align*}
\tilde{V}_1' &= \left\{ i \mathcal{H}^{\mathcal{C}} s \mid \Theta \left( i \mathcal{H}^{\mathcal{C}} s + \mathcal{H}^{\mathcal{C}} a; \tilde{Z} \right) = 0 \right\}, \\
\tilde{V}_2' &= \left\{ i \mathcal{H}^{\mathcal{C}} s \mid \Theta \left( i \mathcal{H}^{\mathcal{C}} s + \mathcal{H}^{\mathcal{C}} a; \tilde{Z} \right) = 0 \right\}
\end{align*}
\]  

(8.101)

for any point \( a \in \mathcal{C}^{2g} \) for which \( \Theta(a; \tilde{Z}) \neq 0 \). In these terms it follows from Corollary 8.10 that the function \( h_1(s) = \Theta \left( i \mathcal{H}^{\mathcal{C}} s + \mathcal{H}^{\mathcal{C}} a; \tilde{Z} \right) = 0 \) defining the subvariety \( \tilde{V}_1' \) and the function \( h_2(s) = \Theta \left( i \mathcal{H}^{\mathcal{C}} s + \mathcal{H}^{\mathcal{C}} a; \tilde{Z} \right) = 0 \) defining the subvariety \( \tilde{V}_2' \) are relatively automorphic functions of the variable \( s \in \mathcal{C}^{g} \) under the action of the lattice subgroups \( \mathcal{L}(\Omega) \) and \( \mathcal{L}(\Omega) \) on \( \mathcal{C}^{g} \), and that these functions are determined uniquely up to a constant factor as relatively automorphic functions. The factors of automorphy in (8.80) and (8.82) are derived from the theta factor of automorphy for the lattice subgroup \( \mathcal{L}(\mathcal{C}) \) acting on \( \mathcal{C}^{2g} \), rather than from the theta factor for the period matrices \( \Omega \) and \( \Omega \) determining the two lattice subgroups of \( \mathcal{C}^{g} \) for which these functions are relatively automorphic. The descriptions in terms of the subvarieties \( \tilde{V}_1' \) and \( \tilde{V}_2' \) provide what is probably the clearest and most useful description of the subvariety \( \tilde{V}_\Theta \).

Corollary 8.13 The intersection \( \tilde{V}_1 \cap \tilde{V}_2 \) is a holomorphic subvariety of \( \mathcal{C}^{2g} \) of pure dimension \( 2g - 2 \) that can be described alternatively as

\[
\tilde{V}_1 \cap \tilde{V}_2 = \tilde{V}_1' \oplus \tilde{V}_2'.
\]  

(8.102)

Proof: If \( t \in \tilde{V}_1 \cap \tilde{V}_2 \) then since \( t \in \tilde{V}_1 \) it follows from (8.99) that

\[
t = t_1 + x_1 \quad \text{where} \quad t_1 \in \tilde{V}_1' \subset \mathcal{H}^{\mathcal{C}} \mathcal{C}^{2g} \quad \text{and} \quad x_1 \in \overline{\mathcal{H}^{\mathcal{C}} \mathcal{C}^{2g}}
\]

while since \( t \in \tilde{V}_2 \) it also follows from (8.99) that

\[
t = t_2 + x_2 \quad \text{where} \quad t_2 \in \tilde{V}_2' \subset \mathcal{H}^{\mathcal{C}} \mathcal{C}^{2g} \quad \text{and} \quad x_2 \in \overline{\mathcal{H}^{\mathcal{C}} \mathcal{C}^{2g}};
\]

and since

\[
\mathcal{C}^{2g} = \mathcal{H}^{\mathcal{C}} \mathcal{C}^{2g} \oplus \overline{\mathcal{H}^{\mathcal{C}} \mathcal{C}^{2g}}
\]

as in (8.58) it must be the case that \( x_1 = t_2 \), which demonstrates (8.102). In view of the properties of the dimension of holomorphic varieties as sketched in Appendix A.3, the direct sum \( \tilde{V}_1' \oplus \tilde{V}_2' \) is a holomorphic subvariety of dimension at most \( 2g - 2 \), since it is the image of the holomorphic variety \( \tilde{V}_1' \times \tilde{V}_2' \) of dimension \( 2g - 2 \) under the injective holomorphic mapping that takes a point \( (t_1, t_2) \in \tilde{V}_1' \times \tilde{V}_2' \) to the point \( t_1 + t_2 \in \tilde{V}_1' \oplus \tilde{V}_2' \), while each component of the
intersection $\widetilde{V}_1 \cap \widetilde{V}_2$ is a subvariety of dimension at least $2g-2$; hence the set $\widetilde{V}_1 \cap \widetilde{V}_2 = \widetilde{V}_1' \oplus \widetilde{V}_2'$ is of pure dimension $2g-2$, which suffices for the proof.

The product decomposition of the complex torus $\mathcal{J}(M)$ and the associated product formula for the theta series $\Theta(t; \tilde{Z})$ yield some information about the intrinsic theta function $\theta_g(z; t) = \Theta(t - i \Omega G \tilde{w}(z; a))$.

**Theorem 8.14** (i) The intrinsic theta function $\theta_g(z; a; t_0)$ vanishes identically in the variables $(z, a) \in \widetilde{M}^2$ for a fixed point $t_0 \in \mathbb{C}^{2g}$ if $t_0 \in \widetilde{V}_1$.

(ii) If the intrinsic theta function $\theta_g(z; a; t_0)$ for a fixed point $t_0 \in \mathbb{C}^{2g}$ vanishes identically in the variables $(z, a) \in \widetilde{M}^2$ then either $t_0 \in \widetilde{V}_1$ or $\Theta(t; \tilde{Z})$ vanishes to second order at the point $t_0 \in \mathbb{C}^{2g}$.

**Proof:** (i) If $t_0 \in \widetilde{V}_1$ it follows from (8.67) and (8.94) that $t_0 - i \Omega G \mathbb{C}^g = t_0 - \mathbb{M} \Omega \mathbb{C}^g \subset \widetilde{V}_1 \subset \widetilde{V}_2$ and consequently that $\Theta(t_0 - i \Omega G s; \tilde{Z}) = 0$ for all $s \in \mathbb{C}^g$; in particular for $s = \tilde{w}(z, a)$ it follows that

$$\theta_g(z; a; t_0) = \Theta(t_0 - i \Omega G \tilde{w}(z, a); \tilde{Z}) = 0$$

for all $(z, a) \in \widetilde{M}^2$.

(ii) If $\theta_g(z; a; t_0) = \Theta(t_0 - i \Omega G \tilde{w}(z, a); \tilde{Z}) = 0$ for all $(z, a) \in \widetilde{M}$ then in particular $\Theta(t_0; \tilde{Z}) = \theta_g(a, a; t_0) = 0$ so $t_0 \in \widetilde{V}_1 \cup \widetilde{V}_2$. If $t_0 \in \widetilde{V}_1$ then $\theta_g(z, a; t_0) = 0$ identically in $(z, a) \in \widetilde{M}$ as in (i). On the other hand if $t_0 \in \widetilde{V}_2$ it then follows from (8.68) and (8.94) that $t_0 - i \Omega G \mathbb{C}^g = t_0 - \mathbb{M} \Omega \mathbb{C}^g \subset \widetilde{V}_2 \subset \widetilde{V}_2$ and consequently that $\Theta(t_0 - i \Omega G s; \tilde{Z}) = 0$ for all $s \in \mathbb{C}^g$. Then for all $s = \{s_i\} \in \mathbb{C}^g$ and for any $l$ for which $1 \leq l \leq 2g$

$$0 = \frac{\partial}{\partial s_l} \Theta(t_0 - i \Omega G s; \tilde{Z})$$

$$= -i \sum_{j=1}^{2g} \sum_{k=1}^{g} \partial_j \Theta(t_0 - i \Omega G s; \tilde{Z}) \omega_{l j} g_{k j};$$

in particular for $s = 0$ and since the matrix $G$ is nonsingular it follows that

$$0 = \sum_{j=1}^{2g} \omega_{k j} \partial_j \Theta(t_0; \tilde{Z})$$

(8.103) for all indices $k$ for which $1 \leq k \leq g]$. In addition since

$$\Theta(t_0 - i \Omega G \tilde{w}(z, a); \tilde{Z}) = 0 \quad \text{for all } z, a \in \widetilde{M}$$

it follows that

$$0 = \frac{\partial}{\partial z} \theta_g(z, a; t) \, dz = \frac{\partial}{\partial z} \Theta(t_0 - i \Omega G \tilde{w}(z, a); \tilde{Z}) \, dz$$

$$= -i \sum_{j=1}^{2g} \sum_{k, l=1}^{g} \partial_j \Theta(t_0 - i \Omega G \tilde{w}(z, a); \tilde{Z}) \omega_{l j} \tilde{g}_{k l} \omega_l(z)$$
for all \( z,a \in \tilde{M} \); in particular for \( a = z \) and since the holomorphic abelian differentials \( \omega_l(z) \) are linearly independent and the matrix \( G \) is nonsingular

\[
(8.104) \quad 0 = \sum_{j=1}^{2g} \omega_{kj} \partial_j \Theta \left( t_0; \tilde{Z} \right)
\]

for all \( k \) for which \( 1 \leq k \leq g \). The full period matrix \( \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \) is nonsingular so it follows from (8.103) and (8.104) that \( \partial_j \Theta(t_0; \tilde{Z}) = 0 \) for all indices \( j \) for which \( 1 \leq j \leq 2g \); and since \( \Theta(t_0; \tilde{Z}) = 0 \) the function \( \Theta(t; \tilde{Z}) \) vanishes to at least the second order at the point \( t_0 \), to conclude the proof.

The set of points at which \( \Theta(t; \tilde{Z}) \) vanishes to second order form the proper holomorphic subvariety \( \tilde{V}_2^2 \subset C^{2g} \) defined by the \( g+1 \) equations

\[
(8.105) \quad \tilde{V}_2^2 = \{ t \in C^{2g} \mid \Theta(t; \tilde{Z}) = \partial_j \Theta(t; \tilde{Z}) = 0 \text{ for } 1 \leq j \leq g \};
\]

for general Riemann surfaces this is a subvariety that can be expected to have codimension at least 2, but it is a nontrivial holomorphic subvariety for any Riemann surface.

**Corollary 8.15** For any Riemann surface \( \tilde{V}_1 \cap \tilde{V}_2 \subset \tilde{V}_2^2 \).

**Proof:** In the proof of the preceding theorem (8.103) was a consequence of the condition that \( t_0 \in \tilde{V}_1 \); the corresponding argument shows that (8.104) is a consequence of the condition that \( t_0 \in \tilde{V}_1 \), and the desired corollary then follows as in the end of the proof of the preceding theorem.

There is also an alternative interpretation of the conjugate Jacobi variety \( J^*(M) \) and the inclusion mapping (8.52). The complex structure on the Riemann surface \( M \) is described by a coordinate covering \( \{ U_\alpha \} \) with homeomorphisms \( z_\alpha : U_\alpha \rightarrow V_\alpha \) from each open subset \( U_\alpha \subset M \) to an open subset \( V_\alpha \subset \mathbb{C} \), taking a point \( p \in U_\alpha \) to the point \( z_\alpha(p) \in V_\alpha \), such that for each nonempty intersection \( U_\alpha \cap U_\beta \) the composition

\[
f_{\alpha \beta} = z_\alpha \circ z_\beta^{-1} : z_\beta(U_\alpha \cap U_\beta) \rightarrow z_\alpha(U_\alpha \cap U_\beta)
\]

is a biholomorphic mapping, for which of course \( z_\alpha(p) = f_{\alpha \beta}(z_\beta(p)) \) whenever \( p \in U_\alpha \cap U_\beta \). Complex conjugation is a homeomorphism taking the subset \( V_\alpha \subset \mathbb{C} \) to the homeomorphic subset \( \overline{V_\alpha} \subset \mathbb{C} \); and whenever \( p \in U_\alpha \) the complex conjugate of the image \( z_\alpha(p) \in V_\alpha \) is a point \( t_\alpha(p) = \overline{z_\alpha(p)} \in \overline{V_\alpha} \). If \( p \in U_\alpha \cap U_\beta \) then

\[
t_\alpha(p) = \overline{z_\alpha(p)} = \overline{f_{\alpha \beta}(z_\beta(p))} = f_{\alpha \beta}(\overline{z_\beta(p)}) = g_{\alpha \beta}(t_\beta(p))
\]

where \( g_{\alpha \beta} \) is a holomorphic function of the point \( t_\beta(p) \in \overline{V_\beta} \); and these functions \( g_{\alpha \beta} \) describe a biholomorphic mapping

\[
g_{\alpha \beta} = t_\alpha \circ t_\beta^{-1} : t_\beta(U_\alpha \cap U_\beta) \rightarrow t_\alpha(U_\alpha \cap U_\beta).
\]
The coordinate covering \( \{ U_\alpha \} \) and the homeomorphisms \( t_\alpha : U_\alpha \rightarrow \overline{V}_\alpha \) describe another complex structure on the topological space \( M \), exhibiting it as another compact Riemann surface of genus \( g \), the conjugate Riemann surface \( M^* \) of the Riemann surface \( M \). A complex valued function \( h \) on the subset \( U_\alpha \) is holomorphic in terms of the complex structure \( M \) precisely when \( h_\alpha = h \circ z_\alpha^{-1} \) is holomorphic in \( V_\alpha \), and is holomorphic in terms of the complex structure \( M^* \) precisely when \( h_\alpha^* = h \circ t_\alpha^{-1} \) is holomorphic in \( \overline{V}_\alpha \); and since \( h_\alpha(z_\alpha(p)) = h(p) = h_\alpha^*(t_\alpha(p)) = h_\alpha^*(z_\alpha(p)) \) for each point \( p \in U_\alpha \), it follows that \( h \) is holomorphic in \( M^* \) precisely when \( h \) is holomorphic in \( M \). A holomorphic abelian differential \( \omega \) in \( M \) can be written \( \omega = h_\alpha(z_\alpha)dz_\alpha \) in each coordinate neighborhood \( U_\alpha \), where \( h_\alpha \) is holomorphic in \( V_\alpha \), and a holomorphic abelian differential \( \omega^* \) in \( M^* \) can be written \( \omega^* = h_\alpha^*(z_\alpha)dz_\alpha \), where \( h_\alpha^* \) is holomorphic in \( \overline{V}_\alpha \); thus \( \omega \) is holomorphic in \( M \) precisely when \( \overline{\omega} \) is holomorphic in \( M^* \). In terms of a choice of bases \( \omega_i \in \Gamma(M, \Omega^{0,1}) \) and \( t_j \in H_1(M) \) for the Riemann surface \( M \) and the corresponding bases \( \overline{\omega_i} \in \Gamma(M^*, \Omega^{0,1}) \) and \( t_j \in H_1(M) \), the period matrix \( \Omega^* \) for \( M^* \) is just the complex conjugate \( \overline{\Omega} \) of the period matrix \( \Omega \) for \( M \); hence the conjugate Jacobi variety \( J^*(M) \) can be identified with the Jacobi variety of the conjugate Riemann surface \( M^* \), or \( J^*(M) = J(M^*) \). Since the orientation of \( M \) is described by \( i \ dz_\alpha \wedge \overline{dz_\alpha} \) while the orientation of \( M^* \) is described by \( i \ dt_\alpha \wedge dt_\alpha = i \overline{dz_\alpha} \wedge dz_\alpha = -idz_\alpha \wedge \overline{dz_\alpha} \), it follows that the orientations of \( M \) and \( M^* \) are reversed so the intersection matrix \( P_\alpha \) of \( M^* \) is the negative \( P_\alpha = -P_\alpha \) of the intersection matrix of \( M \). The other auxiliary matrices for the conjugate Riemann surface \( M^* \) are

\[
H^* = -i \overline{P} \Omega = \overline{H}, \quad \text{and} \quad G^* = (H^*)^{-1} = \overline{H}^{-1} = \overline{G};
\]

consequently

\[
R^* = 2\pi \Re(\overline{\Omega} \Omega) = 2\pi \Re(\Omega \overline{\Omega}) = R
\]

and

\[
\widetilde{Z}^* = \frac{1}{2} \overline{\Omega} + \frac{i}{2\pi} R = \overline{Z} - \overline{P}.
\]

The two period matrices \( \widetilde{\Omega} = \left( I_{2g} \overline{Z} \right) \) and \( \widetilde{\Omega}^* = \left( I_{2g} \overline{Z} - \overline{P} \right) \) generate the same lattice subgroup of \( \mathbb{C}^{2g} \), since \( \overline{P} \) is a symmetric integral matrix, so they determine the same complex torus \( \mathcal{J}(M) = \mathcal{J}(M^*) \).

In parallel to the description of the intrinsic theta function \( \theta_\theta(z, a; t) \) on \( M \) in terms of the classical theta series (8.22) or of the function \( \vartheta_{s, \Omega}(t) \) of (8.85) introduce the conjugate intrinsic theta function

\[
(8.106) \quad \theta_\theta(z, a; t) = \Theta \left( t - i \Omega \overline{\omega}(z, a); \widetilde{Z} \right) = \vartheta_{-1, \overline{\Omega}}(\overline{\omega}(z, a)),
\]

This is a holomorphic function of the variable \( t \in \mathbb{C}^{2g} \) but a conjugate holomorphic function of the variables \( z, a \in \overline{M} \); however it is a holomorphic function of the variables \( \overline{z}, \overline{a} \in \overline{M}^* \), where \( \overline{M}^* \) is the universal covering space of the conjugate Riemann surface \( M^* \).
Theorem 8.16 The conjugate intrinsic theta function $\theta_g^*(z, a; t)$ on a compact Riemann surface $M$ of genus $g > 0$ has the following properties:

(i) $\theta_g^*(Tz, a; t) = \zeta_{g,a,t}^*(T, z) \theta_g^*(z, a; t)$ for $T \in \Gamma$ where $\zeta_{g,a,t}^*(T, z)$ is the holomorphic factor of automorphy for the action of $\Gamma$ on $\tilde{M}^*$ given by

$$\zeta_{g,a,t}^*(T, z) = \exp 2\pi i \left(-\frac{1}{2} \nu \tilde{Z} \nu + \nu t - i \nu \Omega \tilde{w}(z, a)\right) \text{ if } \omega(T) = \Omega \nu;$$

(ii) $\theta_g^*(Tz, a; t) = \theta_g^*(z, a; t - \tilde{Z} \nu)$ where $\omega(T) = \Omega \nu \in \Omega \mathbb{Z}^{2g}$;

(iii) $\theta_g^*(z, a; t) = \theta_g^*(a, z; -t)$.

Proof: If $T \in \Gamma$ is a covering translation for which $\omega(T) = \Omega \nu$ where $\nu \in \mathbb{Z}^{2g}$, it follows from (8.47) that

$$i \nu \Omega G \tilde{w}(Tz, a) = i \nu \Omega G \tilde{w}(z, a) + \Omega \nu$$

so since the function $\Theta(t; \tilde{Z})$ is invariant under translation through the integral vector $Q \nu \in \mathbb{Z}^{2g}$

$$\theta_g^*(Tz, a; t) = \Theta \left(t - i \nu \Omega G \tilde{w}(Tz, a); \tilde{Z}\right)$$

$$= \Theta \left(t - i \nu \Omega G \tilde{w}(z, a) - \tilde{Z} \nu; \tilde{Z}\right) = \theta_g^*(z, a; t - \tilde{Z} \nu),$$

which demonstrates (ii). On the other hand by (8.31) the preceding equation can be rewritten

$$\theta_g^*(Tz, a; t) = \exp -2\pi i \left(\frac{1}{2} \nu \tilde{Z} \nu - \nu t + i \nu \Omega G \tilde{w}(z, a)\right) \Theta \left(t - i \nu \Omega G \tilde{w}(z, a); \tilde{Z}\right)$$

$$= \zeta_{g,a,t}^*(T, z) \theta_g^*(z, a; t),$$

which shows that $\zeta_{g,a,t}^*(T, z)$ is a factor of automorphy for the action of $\Gamma$ on $\tilde{M}^*$ and demonstrates (i). Finally since $\Theta(t; \tilde{Z})$ is an even function of $t \in \mathbb{C}^{2g}$

$$\theta_g^*(z, a; t) = \Theta \left(t - i \nu \Omega G \tilde{w}(z, a); \tilde{Z}\right) = \Theta \left(t + i \nu \Omega G \tilde{w}(a, z); \tilde{Z}\right)$$

$$= \Theta \left(-t - i \nu \Omega G \tilde{w}(a, z); \tilde{Z}\right) = \theta_g^*(a, z; -t),$$

which demonstrates (iii) and thereby concludes the proof.

The use of the matrix $\tilde{Z}$ rather than the matrix $\Omega$ to define the relevant theta function was impelled by the desire for an intrinsically defined function. Several of the preceding results really reflect how the matrix $\tilde{Z}$ was defined in terms of the period matrix $\Omega$; the product formulas and the decomposition of the theta locus are good examples. Properties of the intrinsic theta function that reflect anything beyond the Riemann relations require a later and more detailed examination.
Chapter 9

Prym Differentials and Prym Cohomology

A holomorphic Prym differential for a flat line bundle $\rho$ over a Riemann surface $M$ is a holomorphic cross-section $\sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho))$. Holomorphic differential forms that are cross-sections of an arbitrary holomorphic line bundle were introduced on page 14; holomorphic Prym differentials are just the special case in which the line bundle is a flat line bundle. A holomorphic Prym differential can be viewed as a collection of holomorphic differential forms $\sigma_\alpha$ in the open neighborhoods of a coordinate covering $\mathcal{U} = \{U_\alpha\}$ of the surface $M$ such that $\sigma_\alpha = \rho_{\alpha\beta} \sigma_\beta$ in any nonempty intersection $U_\alpha \cap U_\beta$, when the bundle $\rho$ is described by a flat cocycle $\rho_{\alpha\beta} \in Z^1(\mathcal{U}, \mathbb{C}^*)$. When these local differential forms are written $\sigma_\alpha = f_\alpha(z_\alpha) dz_\alpha$ in terms of local coordinates $z_\alpha$ in the coordinate neighborhoods $U_\alpha$, the coefficients $f_\alpha(z_\alpha)$ describe a holomorphic cross-section $f \in \Gamma(M, \mathcal{O}(\rho_\kappa))$ where $\kappa$ is the canonical bundle of the surface $M$; thus holomorphic Prym differentials can be viewed equivalently as holomorphic cross-sections of the holomorphic line bundle $\rho_\kappa$ of characteristic class $c(\rho_\kappa) = 2g - 2$, and the sheaf $\mathcal{O}^{(1,0)}(\rho)$ of germs of holomorphic Prym differentials for the flat line bundle $\rho$ can be identified in this way with the sheaf $\mathcal{O}(\rho_\kappa)$. Correspondingly a meromorphic Prym differential is a cross-section $\sigma \in \Gamma(M, \mathcal{M}^{(1,0)}(\rho))$ and can be viewed as a meromorphic cross-section of the holomorphic line bundle $\rho_\kappa$. On the other hand a $C^\infty$ Prym differential is defined to be a cross-section $\sigma \in \Gamma(M, \mathcal{E}_c^1(\rho))$, where $\mathcal{E}_c^1(\rho)$ is the sheaf of germs of closed complex-valued $C^\infty$ differential forms of total degree 1 that are cross-sections of $\rho$. Holomorphic and meromorphic Prym differentials are automatically closed differential forms on $M$; $C^\infty$ Prym differentials are closed by definition, and do not correspond simply to $C^\infty$ cross-sections of the line bundle $\rho_\kappa$. There are no nontrivial flat line bundles over the Riemann sphere $\mathbb{P}^1$ since it is simply connected. There are nontrivial flat line bundles over a compact Riemann surface $M$ of genus $g = 1$; but since the canonical bundle of $M$ is trivial by Corollary 2.20, Prym differentials are just cross-sections of a flat line bundle over $M$. For these reasons
the discussion of Prym differentials generally is limited to Riemann surfaces of genus \( g > 1 \).

Holomorphic Prym differentials have well defined period classes, analogous to the period classes of holomorphic abelian differentials. There are two equivalent definitions of these period classes, a sheaf-theoretic definition which will be discussed first and a group-theoretic definition which in some ways is more convenient. Since the coordinate transition functions \( \rho_{\alpha\beta} \) of a flat line bundle \( \rho \) over a Riemann surface \( M \) are constant, exterior differentiation leads to the exact sequence of sheaves

\[
0 \longrightarrow \mathbb{C}(\rho) \longrightarrow \mathcal{O}(\rho) \xrightarrow{d} \mathcal{O}^{(1,0)}(\rho) \longrightarrow 0
\]

over \( M \), where \( \mathbb{C}(\rho) \subset \mathcal{O}(\rho) \) is the subsheaf of locally constant cross-sections of the flat line bundle \( \rho \). The associated exact cohomology sequence includes the segment

\[
\Gamma(M, \mathcal{O}(\rho)) \xrightarrow{d} \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)),
\]

which can be rewritten equivalently

\[
(9.1) \quad 0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)).
\]

The image \( \delta_s \sigma \in H^1(M, \mathbb{C}(\rho)) \) of a Prym differential \( \sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \) is the sheaf-theoretic period class of that differential. If the bundle \( \rho \) is not analytically trivial, indicated by writing \( \rho \not\approx 1 \), then \( \Gamma(M, \mathcal{O}(\rho)) = 0 \) by Corollary 1.5 so the exact sequence \((9.1)\) reduces to the exact sequence

\[
(9.2) \quad 0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)) \quad \text{if} \quad \rho \not\approx 1.
\]

Correspondingly for \( C^\infty \) Prym differentials exterior differentiation leads to the exact sequence of sheaves

\[
0 \longrightarrow \mathbb{C}(\rho) \longrightarrow \mathcal{E}(\rho) \xrightarrow{d} \mathcal{E}_c^1(\rho) \longrightarrow 0
\]

over \( M \), where \( \mathcal{E}(\rho) \) is the sheaf of germs of complex-valued \( C^\infty \) cross-sections of the flat line bundle \( \rho \); it is to obtain this exact sequence, paralleling the corresponding sequence for holomorphic Prym differentials, that \( C^\infty \) Prym differentials are defined to be closed differential forms. The exact cohomology sequence associated to this exact sequence of sheaves contains the segment

\[
\Gamma(M, \mathcal{E}(\rho)) \xrightarrow{d} \Gamma(M, \mathcal{E}_c^1(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)) \longrightarrow H^1(M, \mathcal{E}(\rho));
\]

but \( H^1(M, \mathcal{E}(\rho)) = 0 \) since \( \mathcal{E}(\rho) \) is a fine sheaf\(^1\), so this exact sequence can be written as the isomorphism

\[
(9.3) \quad \delta_s : \Gamma(M, \mathcal{E}_c^1(\rho)) \xrightarrow{\cong} H^1(M, \mathbb{C}(\rho)).
\]

\(^1\)Fine sheaves and their cohomological properties are discussed in Appendix C.2.
The image \( \delta \sigma \in H^1(M, \mathbb{C}(\rho)) \) of a \( C^\infty \) Prym differential \( \sigma \in \Gamma(M, \mathcal{E}_1^1(\rho)) \) is the sheaf-theoretic period class of that differential. It follows from the isomorphism (9.3) that every cohomology class in \( H^1(M, \mathbb{C}(\rho)) \) is the period class of a \( C^\infty \) Prym differential; that is not the case for holomorphic Prym differentials though, as will be seen as the discussion continues.

For a more explicit description of the exact cohomology sequence leading to the sheaf-theoretic period class, a \( C^\infty \) Prym differential \( \sigma \) for a flat line bundle \( \rho \) is represented by closed differential forms \( \sigma_\alpha(z) \) in the open subsets of a coordinate covering \( \mathcal{U} = \{ U_\alpha \} \) of \( M \); and \( \sigma_\alpha(z) = \rho_{\alpha\beta}\sigma_\beta(z) \) in any intersection \( U_\alpha \cap U_\beta \), where the cocycle \( \rho_{\alpha\beta} \in Z^1(\mathcal{U}, \mathbb{C}^*) \) is a coordinate bundle describing the flat line bundle \( \rho \). After a refinement of the coordinate covering if necessary, the local differentials \( \sigma_\alpha(z) \) can be written as the exterior derivatives \( \sigma_\alpha(z) = df_\alpha(z) \) of \( C^\infty \) functions \( f_\alpha(z) \) in the coordinate neighborhoods \( U_\alpha \); and these functions form a zero-cochain \( f \in C^0(\mathcal{U}, \mathcal{E}(\rho)) \). Since \( d(f_\beta(z) - \rho_{\beta\alpha}f_\alpha(z)) = \sigma_\beta(z) - \rho_{\beta\alpha}\sigma_\alpha(z) = 0 \) in any intersection \( U_\alpha \cap U_\beta \) it follows that

\[
\sigma_{\alpha\beta} = f_\beta(z) - \rho_{\beta\alpha}f_\alpha(z)
\]

is a complex constant; these constants form the one-cochain \( \sigma \in C^1(\mathcal{U}, \mathbb{C}(\rho)) \) for which \( \sigma = \delta f \), as in (1.33), so this one-cochain is a one-cocycle \( \sigma \in Z^1(\mathcal{U}, \mathbb{C}(\rho)) \).

The cohomology class of this cocycle is independent of the choice of the local integrals \( f_\alpha(z) \), since adding constants \( c_\alpha \) to the functions \( f_\alpha(z) \) adds the coboundary \( \beta_\alpha - \rho_{\beta\alpha}c_\alpha \) to the cocycle (9.4). This cohomology class is the period class \( \delta \sigma \) of the Prym differential \( \sigma \). For a holomorphic Prym differential \( \sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \subset \Gamma(M, \mathcal{E}_1^1(\rho)) \) the same construction with holomorphic functions \( f_\alpha \) yields the corresponding period class.

The period classes of Prym differentials can be interpreted alternatively and more conveniently in terms of the cohomology\(^2\) of the covering translation group \( \Gamma \) of the Riemann surface \( M \). Since the universal covering space \( \tilde{M} \) of the surface \( M \) is simply connected, any holomorphic differential form on \( \tilde{M} \) can be written as the exterior derivative of a holomorphic function defined on all of \( \tilde{M} \); thus there is the exact sequence of complex vector spaces

\[
0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(\tilde{M}, \mathcal{O}) \overset{d}{\longrightarrow} \Gamma(\tilde{M}, \mathcal{O}^{(1,0)}) \longrightarrow 0.
\]

A flat line bundle \( \rho \) over the Riemann surface \( M \) can be represented by a flat factor of automorphy \( \rho \in \text{Hom}(\Gamma, \mathbb{C}^*) \), that is, by a one-dimensional representation of the covering translation group \( \Gamma \), as in Theorem 3.11. The group \( \Gamma \) acts as a group of operators on the right on the vector space \( \Gamma(\tilde{M}, \mathcal{O}^{(1,0)}) \) by associating to an element \( T \in \Gamma \) and a holomorphic differential form \( \sigma \in \Gamma(\tilde{M}, \mathcal{O}^{(1,0)}) \) the holomorphic differential form \( \sigma|_\rho T \in \Gamma(\tilde{M}, \mathcal{O}^{(1,0)}) \) defined by

\[
(\sigma|_\rho T)(z) = \rho(T)^{-1}\sigma(Tz);
\]

for it is readily verified that \( \sigma|_\rho(ST) = (\sigma|_\rho S)|_\rho T \) for any \( S, T \in \Gamma \). The group \( \Gamma \) acts as a group of operators on the right on the vector space \( \Gamma(\tilde{M}, \mathcal{O}) \) in

\(^2\)The machinery of the cohomology of groups used here is discussed in Appendix E.
the same way, with the group operation (9.6) on holomorphic functions rather than on holomorphic differential forms. The subspace $\mathbb{C} \subset \Gamma(\tilde{M}, \mathcal{O})$ of constant functions under this group operation will be denoted by $\mathbb{C}_\rho$ for clarity in the subsequent discussion; this is just the complex numbers under the group operation that associates to an element $T \in \Gamma$ and a complex number $c \in \mathbb{C}$ the complex number

\[(9.7) \quad c|_\rho T = \rho(T)^{-1} c.\]

It is evident that these group actions commute with the mappings in the exact sequence (9.5); so, as in the discussion on page 373 in Appendix E.1, associated to this exact sequence is the exact sequence of cohomology groups of the group $\Gamma$, which contains the segment

\[H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O})) \xrightarrow{d} H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O}(1,0))) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho).\]

By (E.15) the cohomology group $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O}))$ is the subspace of $\Gamma(\tilde{M}, \mathcal{O})$ consisting of elements $f \in \Gamma(\tilde{M}, \mathcal{O})$ that are invariant under the action of the group $\Gamma$, hence that satisfy $f(z) = (f|_\rho T)(z) = \rho(T)^{-1} f(Tz)$ for each $T \in \Gamma$; consequently there is the natural identification $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O})) \cong \Gamma(M, \mathcal{O}(\rho))$, and correspondingly $H^0(\Gamma, \Gamma(M, \mathcal{O}(1,0))) \cong \Gamma(M, \mathcal{O}(1,0)(\rho))$. Thus the preceding segment of the exact cohomology sequence can be rewritten as the exact sequence

\[\Gamma(M, \mathcal{O}(\rho)) \xrightarrow{d} \Gamma(M, \mathcal{O}(1,0)(\rho)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho),\]

or equivalently as the exact sequence

\[(9.8) \quad 0 \longrightarrow \Gamma(M, \mathcal{O}(1,0)(\rho)) \xrightarrow{d\Gamma(M, \mathcal{O}(\rho))} \Gamma(M, \mathcal{O}(1,0)(\rho)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho).\]

The image $\delta \sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ of a Prym differential $\sigma \in \Gamma(M, \mathcal{O}(1,0)(\rho))$ under the coboundary mapping $\delta$ is the group-theoretic period class of that differential. Again if the bundle $\rho$ is not analytically trivial, that is if $\rho \not\approx 1$, then $\Gamma(M, \mathcal{O}(\rho)) = 0$ by Corollary 1.5 so the preceding exact sequence takes the form

\[(9.9) \quad 0 \longrightarrow \Gamma(M, \mathcal{O}(1,0)(\rho)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho) \quad \text{if} \quad \rho \not\approx 1.\]

For $C^\infty$ Prym differentials paralleling (9.5) is the exact sequence of complex vector spaces

\[(9.10) \quad 0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(\tilde{M}, \mathcal{E}) \xrightarrow{d} \Gamma(\tilde{M}, \mathcal{E}_c^1) \longrightarrow 0\]

on which the group $\Gamma$ acts in the same way, and hence there is the associated exact cohomology sequence containing the segment

\[H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E})) \xrightarrow{d} H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E}_c^1)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho).\]
It follows from (E.15) again that there are the isomorphisms $H^0(\Gamma, \Gamma(M, E)) \cong \Gamma(M, E(\rho))$ and $H^0(\Gamma, \Gamma(M, E^1)) \cong \Gamma(M, E^1_c(\rho))$, so the preceding exact sequence can be rewritten as the exact sequence

$$0 \to \frac{\Gamma(M, E^1(\rho))}{d\Gamma(M, E(\rho))} \to H^1(\Gamma, C_\rho).$$

(9.11)

The image $\delta\sigma \in H^1(\Gamma, C_\rho)$ of a $C^\infty$ Prym differential $\sigma \in \Gamma(M, E^1_c(\rho))$ under the coboundary mapping $\delta$ is its group-theoretic period class, and the exact sequence (9.8) is just the restriction of the exact sequence (9.11) to holomorphic Prym differentials.

The cohomology group $H^1(\Gamma, C_\rho)$, called the Prym cohomology group of the Riemann surface $M$ for the flat line bundle $\rho$, is the quotient

$$H^1(\Gamma, C_\rho) = \frac{Z^1(\Gamma, C_\rho)}{B^1(\Gamma, C_\rho)},$$

(9.12)

where as in (E.17) the group $Z^1(\Gamma, C_\rho)$ of cocycles consists of those mappings $\sigma : \Gamma \to \mathbb{C}$ for which

$$\sigma(ST) = \sigma(S)|_\rho T + \sigma(T) = \rho(T)^{-1}\sigma(S) + \sigma(T)$$

(9.13)

for all $S, T \in \Gamma$ and as in (E.18) the subgroup $B^1(\Gamma, C_\rho) \subset Z^1(\Gamma, C_\rho)$ of coboundaries consists of cocycles of the form

$$(\delta c)(T) = c|_\rho T - c = c(\rho(T)^{-1} - 1)$$

(9.14)

for all $T \in \Gamma$ and a complex constant $c$. For a more explicit description of the exact cohomology sequence leading to the group-theoretic period class, a $C^\infty$ Prym differential $\sigma$ for the flat line bundle $\rho$ is represented by a closed differential form $\sigma(z)$ on the universal covering space $\tilde{M}$ such that $(\sigma|_\rho T)(z) = \sigma(z)$, or equivalently such that $\sigma(Tz) = \rho(T)\sigma(z)$, for all $T \in \Gamma$. Since $\tilde{M}$ is simply connected $\sigma(z)$ is the exterior derivative of a $C^\infty$ function $f(z)$ on $\tilde{M}$; the function $f(z)$ is called a Prym integral, and is determined uniquely up to an arbitrary additive constant. If $f(z)$ is any choice of a Prym integral then

$$d((f|_\rho T)(z) - f(z)) = (\sigma|_\rho T)(z) - \sigma(z) = 0$$

for any $T \in \Gamma$, so

$$\sigma(T) = (f|_\rho T)(z) - f(z) = \rho(T)^{-1}f(Tz) - f(z)$$

(9.15)

is a complex constant, and clearly $\sigma(I) = 0$ for the identity $I \in \Gamma$; equivalently

$$f(Tz) = \rho(T)(f(z) + \sigma(T))$$

(9.16)

for any $T \in \Gamma$ for a complex constant $\sigma(T)$, and $\sigma(I) = 0$ for the identity $I \in \Gamma$. The mapping $\sigma : \Gamma \to \mathbb{C}$ that associates to any $T \in \Gamma$ the value $\sigma(T)$ is a one-cochain $\sigma \in C^1(\Gamma, C_\rho)$; and since (9.15) amounts to the condition that $\sigma = \delta f$, when the function $f(z)$ is viewed as a zero-cochain $f \in C^0(\Gamma, \Gamma(M, O(\rho)))$, it follows that $\sigma$ actually is a one-cocycle $\sigma \in Z^1(\Gamma, C_\rho)$. When the Prym integral
Lemma 9.1  Every cohomology class in $H^1(\Gamma, \mathcal{C}_p)$ is the period class of a $C^\infty$ Prym differential, so the period mapping

\begin{equation}
(9.17) \quad \delta : \frac{\Gamma(M, \mathcal{E}_c^1(\rho))}{\mathcal{E}_c(\rho)} \rightarrow H^1(\Gamma, \mathcal{C}_p)
\end{equation}

is an isomorphism of complex vector spaces.

Proof: Choose a $C^\infty$ partition of unity $\{r_\alpha(z)\}$ on the Riemann surface $M$ subordinate to a finite covering $\{U_\alpha\}$ of $M$ by contractible open subsets, and view the functions $r_\alpha(z)$ as $\Gamma$-invariant functions on the universal covering surface $\tilde{M}$. For each set $U_\alpha$ choose a connected component $\tilde{U}_\alpha \subset \tilde{M}$ of the inverse image $\pi^{-1}(U_\alpha)$ under the universal covering $\pi : \tilde{M} \rightarrow M$; the restrictions $\pi : \tilde{U}_\alpha \rightarrow U_\alpha$ then are biholomorphic mappings, the sets $S\tilde{U}_\alpha$ for $S \in \Gamma$ are pairwise disjoint, and the full inverse image of the set $U_\alpha$ is the union $\pi^{-1}(U_\alpha) = \bigcup_{S \in \Gamma} S\tilde{U}_\alpha \subset \tilde{M}$. For any cocycle $\sigma \in Z^1(\Gamma, \mathcal{C}_p)$ and for each index $\alpha$ introduce the $C^\infty$ function $f_\alpha(z)$ on $\tilde{M}$ defined by

\[ f_\alpha(z) = \begin{cases} 
\rho(S)\sigma(S)r_\alpha(z) & \text{if } z \in S\tilde{U}_\alpha, \\
0 & \text{if } z \notin \bigcup_{S \in \Gamma} S\tilde{U}_\alpha;
\end{cases} \]

it is evident that this is a well defined $C^\infty$ function on $\tilde{M}$ that vanishes outside the sets $S\tilde{U}_\alpha$. If $z \in S\tilde{U}_\alpha$ and $T \in \Gamma$ then $Tz \in T\tilde{U}_\alpha$ so it follows from the definition of the functions $f_\alpha(z)$ and (9.13) that

\[ f_\alpha(Tz) = \rho(TS)\sigma(TS)r_\alpha(Tz) = \rho(TS)(\sigma(S) + \rho(S)^{-1}\sigma(T))r_\alpha(z) = \rho(T)f_\alpha(z) + \rho(T)\sigma(T)r_\alpha(z); \]

and of course this holds trivially if $z \notin \bigcup_{S \in \Gamma} S\tilde{U}_\alpha$ since all the terms vanish. The sum $f(z) = \sum_\alpha f_\alpha(z)$ is a $C^\infty$ function on $\tilde{M}$; and since $1 = \sum_\alpha r_\alpha(z)$, summing the preceding identity shows that $f(Tz) = \rho(T)(f(z) + \sigma(T))$. Then $\sigma(z) = df(z)$ is a $C^\infty$ Prym differential on $M$, and it is evident from (9.16) that the period class of this Prym differential is represented by the cocycle $\sigma$; that suffices to conclude the proof.

The sheaf-theoretic and group-theoretic period classes of Prym differentials can be identified through the isomorphisms $\delta_\alpha$ of (9.3) and $\delta$ of (9.17), which when combined provide the commutative diagram
\[\phi \quad \delta \quad \delta_s\]

in which \(\delta\) and \(\delta_s\) are isomorphisms and \(\phi = \delta_s \cdot \delta^{-1}\) consequently also is an isomorphism. When the period mappings \(\delta\) and \(\delta_s\) are restricted to the subspace \(\Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \subset \Gamma(M, \mathcal{E}^1(\rho))\) it follows that the restriction of the isomorphism \(\phi\) is an isomorphism

\[\phi : \delta \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\sim} \delta_s \Gamma(M, \mathcal{O}^{(1,0)}(\rho))\]

between the subspaces of the cohomology groups \(H^1(\Gamma, \mathbb{C}_\rho)\) and \(H^1(M, \mathbb{C}(\rho))\) consisting of the group-theoretic and sheaf-theoretic period classes of holomorphic Prym differentials. The isomorphism \(\phi\) of (9.18) can be described alternatively and rather more explicitly. Choose a coordinate covering of the Riemann surface \(M\) by finitely many contractible coordinate neighborhoods \(U_\alpha\) such that the intersections \(U_\alpha \cap U_\beta\) are connected; and for each subset \(U_\alpha\) choose a connected component \(\tilde{U}_\alpha \subset M\) of the inverse image \(\pi^{-1}(U_\alpha) \subset M\) under the covering projection \(\pi : \tilde{M} \rightarrow M\). The restricted covering projection \(\pi_\alpha : \tilde{U}_\alpha \rightarrow U_\alpha\) then is a biholomorphic mapping between these two coordinate neighborhoods. For any point \(z \in U_\alpha \cap U_\beta\) the two points \(\pi_\alpha^{-1}(z) \in \tilde{U}_\alpha\) and \(\pi_\beta^{-1}(z) \in \tilde{U}_\beta\) are related by \(\pi_\alpha^{-1}(z) = T_{\alpha\beta} \cdot \pi_\beta^{-1}(z)\) for a uniquely determined covering translation \(T_{\alpha\beta} \in \Gamma\) that is independent of the choice of the point \(z \in U_\alpha \cap U_\beta\). If \(\sigma(z)\) is a closed \(C^\infty\)-differential form on the universal covering surface \(\tilde{M}\) representing a Prym differential \(\sigma \in \Gamma(M, \mathcal{E}^1(\rho))\) then \(\sigma(Tz) = \rho(T)\sigma(z)\) for each \(T \in \Gamma\). Introduce the associated differential forms \(\sigma_\alpha\) in the coordinate neighborhoods \(U_\alpha\) defined by \(\sigma_\alpha(z) = \sigma(\pi_\alpha^{-1}(z))\) for \(z \in U_\alpha\). If \(z \in U_\alpha \cap U_\beta\) then \(\sigma_\alpha(z) = \sigma(\pi_\alpha^{-1}(z)) = \sigma(T_{\alpha\beta} \cdot \pi_\beta^{-1}(z)) = \rho(T_{\alpha\beta})\sigma(\pi_\beta^{-1}(z)) = \rho(T_{\alpha\beta})\sigma_\beta(z)\); thus the local differentials \(\sigma_\alpha(z)\) also represent the Prym differential \(\sigma\) when the line bundle \(\rho\) is represented by the cocycle \(\rho_{\alpha\beta} = \rho(T_{\alpha\beta})\) for the coordinate covering \(\mathcal{U} = \{U_\alpha\}\). If \(f(z)\) is a Prym integral of the Prym differential \(\sigma(z)\), so that \(df(z) = \sigma(z)\), then the group-theoretic period class \(\delta\sigma\) of the Prym differential is represented by the cocycle \(\sigma(T) = \rho(T)^{-1}f(Tz) - f(z) \in Z^1(\Gamma, \mathbb{C}_\rho)\) as in (9.15); and if \(f_\alpha(z) = f(\pi_\alpha^{-1}(z))\) then these local functions satisfy \(df_\alpha(z) = \sigma_\alpha(z)\) so the sheaf-theoretic period class \(\delta_s\sigma\) of the Prym differential \(\sigma\) is represented by the cocycle \(\sigma_{\alpha\beta} = \rho_{\beta\alpha}f_\alpha(z) - f_\beta(z) \in Z^1(\mathcal{U}, \mathbb{C}(\rho))\) as in (9.4). Thus

\[
\sigma_{\alpha\beta} = \rho_{\beta\alpha}f(\pi_\alpha^{-1}(z)) - f(\pi_\beta^{-1}(z)) = \rho(T_{\beta\alpha})f(T_{\alpha\beta} \cdot \pi_\beta^{-1}(z)) - f(\pi_\beta^{-1}(z)) \\
= \rho(T_{\beta\alpha})\rho(T_{\alpha\beta})(f(\pi_\beta^{-1}(z)) + \sigma(T_{\alpha\beta})) - f(\pi_\beta^{-1}(z)) = \sigma(T_{\alpha\beta});
\]
consequently the image $\phi(\sigma) \in H^1(\Gamma, \mathbb{C}(\rho))$ under the isomorphism (9.18) of the cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ represented by a cocycle $\sigma(T) \in Z^1(\Gamma, \mathbb{C}_\rho)$ is the cohomology class represented by the cocycle $\sigma_{\alpha\beta} \in Z^1(M, \mathbb{C}(\rho))$ for which

$$\sigma_{\alpha\beta} = \sigma(T_{\alpha\beta}).$$

Before continuing with a more detailed discussion of the Prym cohomology group it is convenient first to list some general properties of Prym cocycles, properties that hold for an arbitrary group $\Gamma$.

**Lemma 9.2** If $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a representation of a group $\Gamma$ then for any cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ and any elements $S, T \in \Gamma$

(i) $\sigma(I) = 0$ for the identity $I \in \Gamma$,

(ii) $\sigma(T^{-1}) = -\rho(T)\sigma(T)$,

(iii) $\sigma(STS^{-1}) = \rho(S)\sigma(T) + (\rho(T)^{-1} - 1)\rho(S)\sigma(S)$,

(iv) $\sigma([S, T]) = (1 - \rho(T))\rho(S)\sigma(S) - (1 - \rho(S))\rho(T)\sigma(T)$

for the commutator $[S, T] = STS^{-1}T^{-1}$; and

(v) if $R \in \Gamma$ is an element for which $\rho(R) \neq 1$ the cocycle $\sigma$ is cohomologous to a unique cocycle $\sigma_R$ for which $\sigma_R(R) = 0$.

**Proof:** The first four results follow from the defining equation (9.13) by straightforward calculations, so no details need be given here. As for (v), the cocycle $\sigma$ is cohomologous to the cocycle $\sigma_R$ defined by

$$\sigma_R(T) = \sigma(T) - \frac{\sigma(R)}{\rho(R)^{-1} - 1} (\rho(T)^{-1} - 1),$$

as is evident from (9.14), and $\sigma_R(R) = 0$. A cocycle $\tau \in Z^1(\Gamma, \mathbb{C}_\rho)$ cohomologous to $\sigma$ must be of the form $\tau(T) = \sigma_R(T) + c (\rho(T)^{-1} - 1)$; and if $\tau(R) = c (\rho(R)^{-1} - 1) = 0$ then $c = 0$ so $\tau = \sigma_R$, which suffices for the proof.

It follows from (v) of the preceding lemma that if $\rho(R) \neq 1$ for an element $R \in \Gamma$ then any cohomology class in $H^1(\Gamma, \mathbb{C}_\rho)$ can be represented by a unique cocycle $\sigma_R$ for which $\sigma_R(R) = 0$, a cocycle called a normalized cocycle with respect to the element $R \in \Gamma$; thus if $Z^1_R(\Gamma, \mathbb{C}_\rho) \subset Z^1(\Gamma, \mathbb{C}_\rho)$ is the subgroup of normalized cocycles with respect to $R$ then

$$H^1(\Gamma, \mathbb{C}_\rho) \cong Z^1_R(\Gamma, \mathbb{C}_\rho).$$

This provides an explicit description of the Prym cohomology group $H^1(\Gamma, \mathbb{C}_\rho)$ for a nontrivial representation $\rho$, and it is useful in various circumstances; but this description involves the choice of a particular element $R \in \Gamma$ for which $\rho(R) \neq 1$, so to that extent it is not intrinsic. However there is a more intrinsic description of the Prym cohomology group $H^1(\Gamma, \mathbb{C}_\rho)$ for a nontrivial representation $\rho$ of an arbitrary group $\Gamma$. Since the representation $\rho$ is trivial on the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ it is clear from the cocycle condition (9.13) that the restriction of a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ to the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ is a group homomorphism $\sigma|[\Gamma, \Gamma] \in \text{Hom}([\Gamma, \Gamma], \mathbb{C})$; and it follows.
from Lemma 9.2 (iii) that this homomorphism satisfies $\sigma(TCT^{-1}) = \rho(T)\sigma(C)$
for any elements $T \in \Gamma$, $C \in \Gamma, \Gamma$ since $\rho(C) = 1$. That suggests introducing
for an arbitrary homomorphism $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ of a group $\Gamma$ the set of homomorphisms
\begin{equation}
\text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C}) = \left\{ \sigma \in \text{Hom}(\Gamma, \Gamma, \mathbb{C}) \middle| \begin{array}{l}
\sigma(TCT^{-1}) = \rho(T)\sigma(C) \\
\text{for all } T \in \Gamma, \ C \in \Gamma, \Gamma
\end{array} \right\}.
\end{equation}
It is clear that the set $\text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ has the natural structure of a complex
vector space, since if $\sigma_1, \sigma_2 \in \text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ and $c_1, c_2 \in \mathbb{C}$ then $c_1\sigma_1 + c_2\sigma_2 \in \text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ as well. The individual homomorphisms in $\text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ have the following properties.

**Lemma 9.3** If $\sigma \in \text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ then for any $R, S, T \in \Gamma$
(i) $\sigma([ST, R]) = \sigma([S, R]) + \rho(S)\sigma([T, R])$,
(ii) $\sigma(T^{-1}, R]) = -\rho(T^{-1})\sigma([T, R])$,
(iii) $(1 - \rho(R))\sigma([S, T]) + (1 - \rho(T))\sigma([R, S]) + (1 - \rho(S))\sigma([T, R]) = 0$,
(iv) if $\rho(R) \neq 1$ then $\sigma$ is the restriction $\sigma = \sigma_R|\Gamma, \Gamma$ to the commutator
subgroup of a unique normalized cocycle $\sigma_R \in Z^2_R(\Gamma, \mathbb{C}_\rho)$, that given by
\begin{equation}
\sigma_R(T) = \frac{\sigma([T, R])}{\rho(T)(1 - \rho(R))}.
\end{equation}

**Proof:**
(i) From the standard commutator identity
\begin{equation}
[ST, R] = STRT^{-1}S^{-1}R^{-1}
= S \cdot TRT^{-1}R^{-1} \cdot S^{-1} \cdot SRS^{-1}R^{-1}
= S[T, R]S^{-1} \cdot [S, R]
\end{equation}
and the assumption that $\sigma \in \text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ it follows that
\[\sigma([ST, R]) = \sigma(S[T, R]S^{-1}) + \sigma([S, R]) = \rho(S)\sigma([T, R]) + \sigma([S, R]).\]
(ii) This follows immediately from (i) upon setting $S = T^{-1}$, since $\sigma([I, R]) = \sigma(I) = 0$.
(iii) From the commutator identity
\[R[S, T]R^{-1} = RSTS^{-1}T^{-1}R^{-1} = [R, S]SR \cdot TS^{-1} \cdot R^{-1}T^{-1}|T, R] = [R, S]T[S, R]T^{-1} \cdot [S, T] \cdot T[S, R]T^{-1} \cdot [T, R]\]
and the assumption that $\sigma \in \text{Hom}_\rho(\Gamma, \Gamma, \mathbb{C})$ it follows that
\[\rho(R)\sigma([S, T]) = \sigma([R, S]) + \rho(S)\sigma([R, T]) + \sigma([S, T]) + \rho(T)\sigma([S, R]) + \sigma([T, R]);\]
and since \(\sigma([S, R]) = -\sigma([R, S])\) and \(\sigma([T, R]) = -\sigma([R, T])\) this is equivalent to (iii).

(iv) For the mapping \(\sigma_R : \Gamma \rightarrow \mathbb{C}\) defined by (9.23) it follows from (i) that
\[
\sigma_R(ST) = \frac{\sigma([ST, R])}{\rho(ST)(1 - \rho(R))} = \frac{\rho(S)\sigma([T, R]) + \sigma([S, R])}{\rho(ST)(1 - \rho(R))}
\]
\[
= \sigma_R(T) + \rho(T)^{-1}\sigma_R(S);
\]
hence \(\sigma_R\) is a cocycle, and since it is clear from (9.23) that \(\sigma_R(R) = 0\) it is even a normalized cocycle with respect to \(R\). Next for any commutator \(C \in [\Gamma, \Gamma]\)
\[
\sigma([C, R]) = \sigma(C \cdot RC^{-1}R^{-1}) = \sigma(C) + \sigma(RC^{-1}R^{-1})
\]
\[= (1 - \rho(R))\sigma(C),\]
or equivalently
\[
\sigma(C) = \frac{\sigma([C, R])}{\rho(C)(1 - \rho(R))} = \sigma_R(C)
\]

since \(\rho(C) = 1\); thus the normalized cocycle \(\sigma_R\) restricts to the homomorphism \(\sigma\) on commutators. Finally if \(\sigma'_R \in Z^1_R(\Gamma, C_\rho)\) is a normalized cocycle that vanishes on commutators then \(\sigma'(R) = \sigma'([R, T]) = 0\) for any \(T \in \Gamma\) so by Lemma 9.2 (iv)
\[
0 = \sigma'_R([R, T]) = (1 - \rho(T))\rho(R)\sigma'_R(R) - (1 - \rho(R))\rho(T)\sigma'_R(T)
\]
\[= 0 - (1 - \rho(R))\rho(T)\sigma'_R(T);
\]
and since \(\rho(R) \neq 1\) it follows that \(\sigma'_R(T) = 0\). Thus there is a unique normalized cocycle that restricts to \(\sigma\) on the commutator subgroup \([\Gamma, \Gamma] \subset \Gamma\), and that suffices to conclude the proof.

The symmetry expressed in part (iii) of the preceding lemma is a form of Lie identity for cocycles in the group \(Z^1(\Gamma, C_\rho)\). The result in part (iv) leads to the following intrinsic description of the cohomology group \(H^1(\Gamma, C_\rho)\).

**Theorem 9.4** If \(\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)\) is a nontrivial representation of a group \(\Gamma\), the mapping that associates to a cocycle \(\sigma \in Z^1(\Gamma, C_\rho)\) its restriction to the commutator subgroup \([\Gamma, \Gamma] \subset \Gamma\) induces an isomorphism
\[
(9.25)\quad H^1(\Gamma, C_\rho) \cong \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})
\]
of complex vector spaces.

**Proof:** It was already observed that the restriction of a cocycle \(\sigma \in Z^1(\Gamma, C_\rho)\) to the commutator subgroup is a homomorphism \(\sigma|[\Gamma, \Gamma] \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})\); so restricting normalized cocycles to the commutator subgroup is a homomorphism of complex vector spaces
\[
Z^1_R(\Gamma, C_\rho) \rightarrow \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}).
\]
By Lemma 9.3 (iv) this homomorphism actually is an isomorphism; and since $Z_1^R(\Gamma, \mathbb{C}_\rho) \cong H^1(\Gamma, \mathbb{C}_\rho)$ by (9.21) that suffices to conclude the proof of the theorem.

Let $\Gamma_\rho \subset \Gamma$ be the kernel of the representation $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$, so that

$$
\Gamma_\rho = \left\{ T \in \Gamma \mid \rho(T) = 1 \right\}.
$$

Of course $[\Gamma, \Gamma] \subset \Gamma_\rho$ since any homomorphism to a commutative group vanishes on the commutator subgroup. For general representations $\Gamma_\rho = [\Gamma, \Gamma]$; but for special representations such as those for which the image $\rho(\Gamma)$ is a finite group the subgroup $\Gamma_\rho \subset \Gamma$ is of finite index. In analogy with (9.22) let

$$
\text{Hom}_\rho(\Gamma_\rho, \mathbb{C}) = \left\{ \sigma \in \text{Hom}(\Gamma_\rho, \mathbb{C}) \mid \sigma(TS^{T^{-1}}) = \rho(T)\sigma(S) \right\}.
$$

Clearly this too has the natural structure of a complex vector space. It follows from the cocycle condition (9.13) that the restriction of a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ to the subgroup $\Gamma_\rho$ is a homomorphism $\sigma|_{\Gamma_\rho} \in \text{Hom}(\Gamma_\rho, \mathbb{C})$; and it follows from Lemma 9.2 (iii) that this homomorphism satisfies $\sigma(TS^{T^{-1}}) = \rho(T)\sigma(S)$ so it belongs to the subgroup $\text{Hom}_\rho(\Gamma_\rho, \mathbb{C})$. In addition the restriction of a homomorphism $\sigma \in \text{Hom}_\rho(\Gamma_\rho, \mathbb{C})$ to the subgroup $[\Gamma, \Gamma] \subset \Gamma_\rho$ is an element $\phi(\sigma) = \sigma|[\Gamma, \Gamma] \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$.

**Theorem 9.5** If $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a nontrivial representation of a group $\Gamma$ the restriction mapping

$$
\phi : \text{Hom}_\rho(\Gamma_\rho, \mathbb{C}) \longrightarrow \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})
$$

is an isomorphism of complex vector spaces, and consequently

$$
\text{Hom}_\rho(\Gamma_\rho, \mathbb{C}) \cong \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}) \cong H^1(\Gamma, \mathbb{C}_\rho).
$$

**Proof:** If $\sigma \in \text{Hom}_\rho(\Gamma_\rho, \mathbb{C})$ and $\sigma|[\Gamma, \Gamma] = 0$ then for any $T \in \Gamma_\rho$ and $R \notin \Gamma_\rho$

$$
0 = \sigma([T, R]) = \sigma(T \cdot RT^{-1}R^{-1}) = \sigma(T) + \sigma(RT^{-1}R^{-1}) = (1 - \rho(R))\sigma(T);
$$

and since $\rho(R) \neq 1$ it follows that $\sigma(T) = 0$, so the restriction mapping (9.28) is an injective linear mapping. On the other hand if $\sigma \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ then it follows from Lemma 9.3 (iv) that $\sigma$ is the restriction $\sigma = \sigma_R|[\Gamma, \Gamma]$ of a normalized cocycle $\sigma_R \in Z_1^R(\Gamma, \mathbb{C}_\rho)$; and since $\sigma_R|_{\Gamma_\rho} \in \text{Hom}_\rho(\Gamma_\rho, \mathbb{C})$ that shows that the restriction mapping (9.28) also is surjective, and consequently is an isomorphism. In view of Theorem 9.4 that suffices to conclude the proof.

The further study of the Prym cohomology groups requires more use of the detailed structure of the covering translation group $\Gamma$ of a compact Riemann
surface $M$ of genus $g > 0$; this structure can be described most conveniently in terms of a marking\(^3\) of the surface. For the purposes of the present discussion a marking of a compact Riemann surface $M$ of genus $g > 1$ is a set $\mathcal{T} = (T_1, T_2, \ldots, T_{2g})$ of generators of the group $\Gamma$, often labeled alternatively $A_i = T_i$, $B_i = T_{g+i}$ for $1 \leq i \leq g$, subject to the relation $C_1 \cdots C_g = 1$ where $C_i = [A_i, B_i] = [T_i, T_{g+i}]$. A representation $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ can be described fully in terms of this marking by the parameters $\zeta_i = \rho(T_i)$ for $1 \leq i \leq 2g$, since any representation is determined uniquely by its values on the generators $T_i$; and all the relations among these generators are contained in the commutator subgroup so these values can be specified arbitrarily, leading to the identification

\begin{equation}
\text{Hom}(\Gamma, \mathbb{C}^*) = (\mathbb{C}^*)^{2g}.
\end{equation}

**Theorem 9.6** If $M$ is a marked Riemann surface of genus $g > 1$ with the marking $\mathcal{T} = (T_1, \ldots, T_{2g})$, and if $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a representation described by parameters $\zeta_i = \rho(T_i)$, then for any cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ the values $z_i = \sigma(T_i)$ satisfy

\begin{equation}
\sum_{i=1}^g \left((1 - \zeta_{g+i}) \zeta_i z_i - (1 - \zeta_i) \zeta_{g+i} z_{g+i}\right) = 0.
\end{equation}

Conversely if $z_i$ are any $2g$ complex numbers satisfying this identity there is a uniquely determined cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ for which $z_i = \sigma(T_i)$.

**Proof:** If $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ then since $\sigma([\Gamma, \Gamma]) \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ and the generators $T_i$ of the marking satisfy $\prod_{i=1}^g [T_i, T_{g+i}] = 1$ it follows from Lemma 9.2 (iv) that

\[0 = \sigma\left(\prod_{i=1}^g [T_i, T_{g+i}]\right) = \sum_{i=1}^g \sigma([T_i, T_{g+i}])\]

\[= \sum_{i=1}^g \left((1 - \zeta_{g+i}) \zeta_i \sigma(T_i) - (1 - \zeta_i) \zeta_{g+i} \sigma(T_{g+i})\right),\]

which is (9.31). For the converse assertion suppose that $z_i$ are complex constants satisfying (9.31). The covering translation group $\Gamma$ is the quotient of the free group $F$ on $2g$ generators $\tilde{T}_1, \ldots, \tilde{T}_{2g}$ by the normal subgroup $K \subset F$ generated by the product $\mathcal{C} = \mathcal{C}_1 \cdots \mathcal{C}_g$ where $\mathcal{C}_i = [\tilde{T}_i, \tilde{T}_{g+i}]$. The composition of the representation $\rho$ and the natural homomorphism $F \longrightarrow \Gamma$ is a representation of the free group $F$, which to simplify the notation also will be denoted by $\rho$. There is a cocycle $\tilde{\sigma} \in Z^1(F, \mathbb{C}_\rho)$ such that $\tilde{\sigma}(\tilde{T}_i) = z_i$; to avoid a digression in the proof here, this will be established in the following Lemma 9.7. Since the restriction of this cocycle is an element $\tilde{\sigma}([F, F]) \in \text{Hom}_\rho([F, F], \mathbb{C})$ it also

\(^3\)The definition and properties of markings of surfaces are discussed in Appendix D.1.
follows from Lemma 9.2 (iv) and (9.31) that

$$\hat{\sigma}(\tilde{C}) = \hat{\sigma}\left(\prod_{i=1}^{g}[[\tilde{T}_i, \tilde{T}_{g+i}]]\right) = \sum_{i=1}^{g} \hat{\sigma}([[\tilde{T}_i, \tilde{T}_{g+i}]]$$

$$= \sum_{i=1}^{g} \left( (1 - \zeta_{g+i})\zeta_i\hat{\sigma}(\tilde{T}_i) - (1 - \zeta_i)\zeta_{g+i}\hat{\sigma}(\tilde{T}_{g+i}) \right)$$

$$= \sum_{i=1}^{g} \left( (1 - \zeta_{g+i})\zeta_i z_i - (1 - \zeta_i)\zeta_{g+i} z_{g+i} \right) = 0.$$

Moreover it follows from Lemma (9.2) (iii) that $\hat{\sigma}(\tilde{T}\tilde{C}\tilde{T}^{-1}) = \hat{\rho}(\tilde{T})\hat{\sigma}(\tilde{C}) = 0$ for all $\tilde{T} \in \Gamma$ as well, since $\hat{\rho}(\tilde{C}) = 1$; so since $K \subset F$ is the normal subgroup generated by $\tilde{C}$ then $\hat{\sigma}(\tilde{S}) = 0$ for all $\tilde{S} \in K$. From this and the cocycle condition (9.13) it follows that $\hat{\sigma}(ST) = \hat{\rho}(T)^{-1}\hat{\sigma}(S) + \hat{\sigma}(T)$ for all $S \in K$ and $\tilde{T} \in F$, which means that $\hat{\sigma}(T_1) = \hat{\sigma}(T_2)$ for any elements $T_1, T_2 \in F$ that represent the same element of $\Gamma$ under the natural homomorphism $F \rightarrow \Gamma$; it is thus possible to define a mapping $\sigma : \Gamma \rightarrow \mathbb{C}$ by setting $\sigma(T) = \hat{\sigma}(\tilde{T})$ for any $\tilde{T} \in F$ representing $T \in \Gamma$. The result is a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$, since whenever $\tilde{S}, \tilde{T} \in F$ represent $S, T \in \Gamma$ then $\tilde{S}\tilde{T} \in F$ represents $ST \in \Gamma$ so from the cocycle condition (9.13) for the group $F$ it follows that $\sigma(ST) = \hat{\sigma}(\tilde{ST}) = \rho(\tilde{T})^{-1}\hat{\sigma}(\tilde{S}) + \hat{\sigma}(\tilde{T}) = \rho(\tilde{T})^{-1}\sigma(S) + \sigma(T)$. The cocycle thus constructed satisfies $\sigma(T_i) = z_i$, and is uniquely determined by this condition; and that suffices to conclude the proof.

**Lemma 9.7** If $F$ is a free group generated by finitely many elements $T_i$, and if $\rho \in \text{Hom}(F, \mathbb{C}^\ast)$ is a representation of that group, then for any complex constants $z_i$ there is a unique cocycle $\sigma \in Z^1(F, \mathbb{C}_\rho)$ such that $\sigma(T_i) = z_i$.

**Proof:** To any word $T$ in the formal symbols $T_i$ and $T_i^{-1}$, that is, to any finite sequence of these symbols with possible repetitions but no cancellation of terms, associate a value $\sigma(T)$ by setting $\sigma(T_i) = z_i$ and $\sigma(T_i^{-1}) = -\rho(T_i)z_i$ and then inductively setting $\sigma(ST) = \rho(T)^{-1}\sigma(S) + \sigma(T)$ and $\sigma(T_i^{-1}T) = \rho(T)^{-1}\sigma(T_i^{-1}) + \sigma(T)$ for any word $T$. It is easy to see by induction on the length of the word $ST$, the number of symbols in that word, that $\sigma$ satisfies the cocycle condition $\sigma(ST) = \rho(T)^{-1}\sigma(S) + \sigma(T)$ for any words $S$ and $T$. Indeed that follows immediately from the definition of the mapping $\sigma$ if the word $ST$ is of length 2; and if it true for the word $ST$ then

$$\sigma(T_iS \cdot T) = \rho(ST)^{-1}\sigma(T_i) + \sigma(ST)$$

$$= \rho(ST)^{-1}\sigma(T_i) + \rho(T)^{-1}\sigma(S) + \sigma(T)$$

$$= \rho(T)^{-1}(\rho(S)^{-1}\sigma(T_i) + \sigma(S)) + \sigma(T)$$

$$= \rho(T)^{-1}\sigma(T_iS) + \sigma(T),$$
and similarly for $T_i^{-1}$ in place of $T_i$. It further follows that the value $\sigma(T)$ is unchanged when the pairs $T_iT_i^{-1}$ and $T_i^{-1}T_i$ are deleted from any word. Indeed $\sigma(T_iT_i^{-1}) = \rho(T_i)\sigma(T_i) + \sigma(T_i^{-1}) = \rho(T_i)z_i - \rho(T_i)z_i = 0$ and correspondingly for the other order $T_i^{-1}T_i$; then from the cocycle condition it follows that $\sigma(\rho(T_i)T_i^{-1}B) = \rho(T_iT_i^{-1}B)^{-1}\sigma(A) + \rho(B)^{-1}\sigma(T_iT_i^{-1}) + \sigma(B) = \rho(B)^{-1}\sigma(A) + \sigma(B) = \sigma(AB)$. Thus the mapping $\sigma$ can be viewed as defined on the free group $F$, and is a cocycle $\sigma \in Z^1(F,\mathbb{C}_\rho)$; and that suffices for the proof.

For the trivial representation $\rho = 1$ condition (9.31) is vacuous and Theorem 9.6 reduces to the familiar assertion that for any complex numbers $z_i$ there is a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}) = \text{Hom}(\Gamma, \mathbb{C})$ such that $\sigma(T_i) = z_i$.

**Corollary 9.8** Let $M$ be a marked Riemann surface $M$ of genus $g > 1$ with the marking $T = (T_1, \ldots, T_{2g})$, let $\rho \in \text{Hom}(\Gamma, \mathbb{C})$ be a representation described by parameters $\zeta_i = \rho(T_i)$, and assume that $\rho(T_i) = \zeta_i \neq 1$ for some index $i$. Then the linear mapping

\begin{equation}
Z_i : \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}) \longrightarrow \mathbb{C}^{2g}
\end{equation}

that associates to a homomorphism $\sigma \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ the vector

\begin{equation}
Z_i(\sigma) = \left\{ z_{i,l} = \sigma([-T_i, T_i]) \mid 1 \leq i \leq 2g \right\} \in \mathbb{C}^{2g}
\end{equation}

is an injective linear mapping; its image is the linear subspace $H_{\rho,l} \subset \mathbb{C}^{2g}$ consisting of vectors $\{z_{i,l}\} \in \mathbb{C}^{2g}$ such that

(i) $z_{l, l} = 0$

(ii) $\sum_{i=1}^{2g} \left( (1 - \zeta_{g+i})z_{i,l} - (1 - \zeta_i)z_{g+i,l} \right) = 0$, and

\begin{equation}
\dim H_{\rho,l} = 2g - 2.\end{equation}

**Proof:** A homomorphism $\sigma \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ is determined completely by the values $z_{i,j} = \sigma([-T_i, T_j])$ for $1 \leq i, j \leq 2g$, since any elements $S, T \in \Gamma$ can be written as words in the generators $T_i$ and their inverses and $\sigma([S,T])$ then can be expressed in terms of the values $z_{i,j} = \sigma([-T_i, T_j])$ by repeated use of Lemma 9.3 (i) and (ii). By Lemma 9.3 (iii)

\begin{equation}
(1 - \zeta_i)z_{j,i} + (1 - \zeta_i)z_{i,j} + (1 - \zeta_j)z_{i,l} = 0,
\end{equation}

and since $\zeta_i \neq 1$ by assumption the values $z_{i,j}$ for all $i, j$ are determined completely by the values $z_{i,l}$ for $1 \leq i \leq 2g$. It follows from these two observations that the linear mapping $Z_i$ is injective. Clearly $z_{l, l} = \sigma([-T_i, T_i]) = \sigma(I) = 0$, so the vectors in the image of the linear mapping $Z_i$ satisfy (i). Since $C_1 \cdots C_g = I$
then for any mapping $\sigma \in \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$

$$0 = \sigma(C_1 \cdots C_g) = \sum_{i=1}^{g} \sigma(C_i) = \sum_{i=1}^{g} \sigma([T_i, T_{g+i}]) = \sum_{i=1}^{g} z_{i,g+i}$$

$$= \frac{1}{1-\zeta_l} \sum_{i=1}^{g} \left( (1-\zeta_{g+i})z_{i,l} - (1-\zeta_i)z_{g+i,l} \right).$$

by (9.35), so the vectors in the image of the linear mapping $Z_l$ also satisfy (ii).

On the other hand for any vector $\{z_{i,l}\} \in H_{p,l}$ it follows from (ii) that the values $z_i = z_{i,l}/\zeta_l(1-\zeta_l)$ satisfy (9.31), so by Theorem 9.6 there is a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ such that

$$\sigma(T_i) = z_i = \frac{z_{i,l}}{\zeta_l(1-\zeta_l)};$$

it then follows from (i) and Lemma 9.2 (iv) that

$$\sigma([T_i, T_l]) = (1-\zeta_l)\sigma(T_i) - (1-\zeta_i)\sigma(T_l) = z_{i,l} - z_{l,t} = z_{i,t},$$

hence $\{z_{i,t}\}$ is in the image of the linear mapping $Z_l$ so the image of $Z_l$ is the full linear subspace $H_{p,t}$. Finally since

$$(9.36)$$

$$(1-\zeta_{g+i})z_{i,t} - (1-\zeta_i)z_{g+i,t} = \begin{cases} 
- (1-\zeta_l)z_{l+g,l} & \text{for } i = l \text{ if } 1 \leq l \leq g, \\
(1-\zeta_l)z_{l-g,l} & \text{for } i = l - g \text{ if } g + 1 \leq l \leq 2g
\end{cases}$$

and $\zeta_l \neq 1$, it is clear that the linear equations (i) and (ii) are linearly independent, and consequently that $\dim H_{p,t} = 2g - 2$, which suffices to conclude the proof.

For any index $l$ in the range $1 \leq l \leq 2g$, the unique index $l'$ in that range such that $|l' - l| = g$ is called the dual index to $l$; thus if $1 \leq l \leq g$ then $l' = l + g$ while if $g + 1 \leq l \leq 2g$ then $l' = l - g$. If $\{z_{i,l}\} \in H_{p,t} \subset \mathbb{C}^{2g}$ it is evident from (9.36) that equation (ii) of Corollary 9.8 can be used to express the entry $z_{l,l}$ as a linear function of the remaining entries of that vector, while $z_{l,t} = 0$ by (i); so since $\dim H_{p,t} = 2g - 2$ a vector in $H_{p,t}$ is determined uniquely by the entries $z_{i,t}$ for all indices $i \neq l, l'$, and these values can be assigned arbitrarily. If $\rho$ is a representation for which $\rho(T_l) \neq 1$, the composition of the isomorphism $H^1(\Gamma, \mathbb{C}_\rho) \xrightarrow{\cong} \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ of Theorem 9.4 and the isomorphism $Z_l : \text{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}) \xrightarrow{\cong} H_{p,l}$ of Corollary 9.8 is an isomorphism

$$(9.37)$$

$$\tilde{Z}_l : H^1(\Gamma, \mathbb{C}_\rho) \xrightarrow{\cong} H_{p,l},$$

so a cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ can be described uniquely by the values

$$(9.38)$$

$$\{z_{i,l}\} = \{\sigma([T_i, T_l])\} \in H_{p,l} \quad \text{for } 1 \leq i \leq 2g, \; i \neq l, l'.$$
for the image $\hat{Z}_i(\sigma) \in H_{\rho, l}$; these values are called the canonical coordinates of the cohomology class $\sigma$ with respect to the generator $T_i$ in the marking $T$, and any set of $2g - 2$ complex numbers $z_{i,l}$ for $1 \leq i \leq 2g$, $i \neq l, l'$ are the canonical coordinates for some Prym cohomology class. This coordinatization of the Prym cohomology classes of course depends on the choice of the marking $T$ and of a generator $T_i \in T$ for which $\rho(T_i) \neq 1$; it is convenient just to say that the canonical coordinates depend on the indexed marking $T(l)$ of the surface $M$, where an indexed marking is defined to be a marking $T = (T_1, \ldots, T_{2g})$ together with the choice of a particular generator $T_i \in T$. Of course only those indexed markings for which $\rho(T_i) \neq 1$ can yield a coordinatization of the Prym cohomology group for the representation $\rho$.

To describe the relations between different systems of canonical coordinates, it is useful to introduce some auxiliary algebraic observations. For any nontrivial representation $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ of the covering translation group $\Gamma$ of the Riemann surface $M$ let $\mathbb{Z}[\rho] \subset \mathbb{C}$ be the subring defined by

$$Z[\rho] = \mathbb{Z} \left[ \rho(T), \frac{1 - \rho(T)}{1 - \rho(S)} \right] \text{ for all } S, T \in \Gamma, \ \rho(S) \neq 1;$$

thus $\mathbb{Z}[\rho]$ is the ring generated by $\rho(T)$ and $(1 - \rho(T))/(1 - \rho(S))$ for all $S, T \in \Gamma$ for which $\rho(S) \neq 1$. It is an integral domain, as a subring of the field $\mathbb{C}$, and its field of quotients is denoted by $\mathbb{Q}(\rho)$. A representation $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is said to be finite of order $q$ if its image $\rho(\Gamma) \subset \mathbb{C}^*$ is a finite subgroup of order $q$ in the multiplicative group $\mathbb{C}^*$; the image $\rho(\Gamma)$ then is the cyclic group of order $q$ generated by a primitive $q$-th root of unity. For finite representations the ring $\mathbb{Z}(\rho)$ takes a particularly simple form.

**Lemma 9.9** (i) If $\rho$ is a finite representation of order $q$ of a group $\Gamma$ and $\epsilon$ is a primitive $q$-th root of unity then the ring $\mathbb{Z}[\rho]$ is generated by $\epsilon$ and the quotients $(1 - \epsilon)(1 - \epsilon^d)^{-1}$ for all integers $d$ such that $1 < d < q$ and $d | q$, and the field of quotients $\mathbb{Q}(\rho)$ of the ring $\mathbb{Z}[\rho]$ is the cyclotomic field $\mathbb{Q}(\epsilon)$ of $q$-th roots of unity;

(ii) $\mathbb{Z}[\rho] = \mathbb{Z}[\epsilon]$ if $q = p$ is prime; and

(iii) $\mathbb{Z}[\rho] = \mathbb{Z}$ if $q = 2$.

**Proof:** If $\rho$ is a finite representation of order $q$ and $\epsilon$ is a primitive $q$-th root of unity then for any $T \in \Gamma$ it is the case that $\rho(T) = \epsilon^n$ for some integer $n$ in the range $1 \leq n \leq q$; so it follows immediately from the definition (9.39) that the ring $\mathbb{Z}[\rho]$ is generated by $\epsilon$ and the quotients $(1 - \epsilon)(1 - \epsilon^m)^{-1} \epsilon^m$ for integers $m, n$ in the range $1 \leq m, n < q$. If $m > 1$ then since

$$\frac{1 - \epsilon^m}{1 - \epsilon^n} = P(\epsilon) \cdot \frac{1 - \epsilon}{1 - \epsilon^n}$$

where $P(\epsilon) = 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{m-1} \in \mathbb{Z}(\epsilon)$ it follows that $(1 - \epsilon^m)/(1 - \epsilon^n)$ can be replaced by $(1 - \epsilon)/(1 - \epsilon^n)$ in the list of generators of $\mathbb{Z}[\epsilon]$. If $n = 1$ then
\[
\frac{1 - \epsilon}{1 - \epsilon^n} = 1, \text{ so it can be assumed that } 1 < n < q. \text{ Then since } \epsilon^q = 1
\]

\[
0 = 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{q-1}
\]

\[
= (1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}) + \epsilon^n(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}) + \cdots + \epsilon^{kn}(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{r-1})
\]

\[
= (1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1})(1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n})
\]

\[
+ \epsilon^{kn}(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{r-1})
\]

where \( r \leq n < q \) and \( q = kn + r, \) so that \( \epsilon^{kn} = \epsilon^{-r}; \) hence

\[
\frac{1 - \epsilon}{1 - \epsilon^n} = \frac{1}{1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}}
\]

\[
= \frac{1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n}}{(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1})(1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n})}
\]

\[
= \frac{1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n}}{-\epsilon^{kn}(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{r-1})} = -\epsilon^{r} P(\epsilon) \frac{(1 - \epsilon)}{(1 - \epsilon^r)}
\]

where \( P(\epsilon) = 1 + \epsilon^n + \cdots + \epsilon^{(k-1)n} \in \mathbb{Z}[\epsilon]. \) If \( r = n \) then \( q = kn + r = (k+1)n \) so that \( n|q; \) hence if \( n \) is not a divisor of \( q \) then \( r < n \) and the quotient \( \frac{1 - \epsilon}{1 - \epsilon^n} \) can be replaced in the list of generators of the ring \( \mathbb{Z}[\rho] \) by the quotient \( \frac{1 - \epsilon}{1 - \epsilon^r}. \) The argument can be repeated, and eventually the only generators left of the form \( \frac{1 - \epsilon}{1 - \epsilon^n} \) are those for which \( 1 < n < q \) and \( n|q, \) as desired. All of these generators are contained in the cyclotomic field \( \mathbb{Q}(\epsilon), \) which therefore must be the field of quotients of the ring \( \mathbb{Z}[\rho]. \) If \( q = p \) is a prime it follows immediately from (i) that the ring \( \mathbb{Z}[\rho] \) is generated by \( \epsilon \) alone, so that \( \mathbb{Z}[\rho] = \mathbb{Z}[\epsilon]; \) and if \( p = 2 \) then \( \epsilon = -1 \) and \( \mathbb{Z}[\rho] = \mathbb{Z}. \) That suffices to conclude the proof.

Part (iii) of the preceding lemma can be demonstrated more directly by noting that if \( \rho \) is a finite representation of order two then \( \rho(T) = \pm 1 \in \mathbb{Z} \) for all \( T \in \Gamma; \) consequently \( (1 - \rho(T))(1 - \rho(S))^{-1} \in \mathbb{Z} \) for all \( S, T \in \Gamma \) for which \( \rho(S) \neq 1, \) since this quotient obviously is either \( 0 \) or \( 1. \) If \( q \) is not a prime the ring \( \mathbb{Z}[\rho] \) can be properly larger than just the ring \( \mathbb{Z}[\epsilon] \) generated by a primitive \( q \)-th root of unity. For example if \( \rho \) is a finite representation of order 4 and \( \epsilon = i \) then

\[
\mathbb{Z}[\rho] = \mathbb{Z} \left[ i, \frac{1 - i}{1 - i^2} \right] = \mathbb{Z} \left[ i, \frac{1 - i}{2} \right] = \mathbb{Z} \left[ \frac{i}{2} \right]
\]

since \( \left( \frac{1 - i}{2} \right)^2 = -\frac{i}{2}; \) and if \( \rho \) is a finite representation of order 6 and \( \epsilon = \exp(2\pi i/6) = \frac{1}{2}(1 + i\sqrt{3}) \) then

\[
\mathbb{Z}[\rho] = \mathbb{Z} \left[ \epsilon, \frac{1 - \epsilon}{1 - \epsilon^2}, \frac{1 - \epsilon}{1 - \epsilon^3} \right] = \mathbb{Z} \left[ \epsilon, \frac{1 + \epsilon}{3}, \frac{1 - \epsilon}{2} \right] = \mathbb{Z} \left[ \frac{i}{2\sqrt{3}} \right] = \mathbb{Z} \left[ \frac{\epsilon}{3} \right]
\]

since \( \left( \frac{1 + i}{2} \right)(\frac{1 - i}{2}) = \frac{i}{2\sqrt{3}}. \) In both of these cases, and in general, the ring \( \mathbb{Z}[\rho] \) is not a finite \( \mathbb{Z} \)-module; but an evident consequence of the preceding lemma is
that ring $\mathbb{Z}[\rho]$ is a finite $\mathbb{Z}$-module whenever $\rho$ is finite of prime order, and that is one of the reasons that special case is so interesting.

For the application of these algebraic observations, consider first the various systems of indexed markings $T(l)$ for a fixed marking $T$ of a compact Riemann surface $M$ of genus $g > 0$. If $\rho : \Gamma \to \mathbb{C}^\ast$ is a homomorphism for which $\rho(T_l) = \zeta_l \neq 1$ and $\rho(T_k) = \zeta_k \neq 1$ equation (9.35) can be rewritten

\[(9.40) \quad z_{i,k} = \frac{1 - \zeta_k}{1 - \zeta_l} z_{i,l} - \frac{1 - \zeta_i}{1 - \zeta_l} z_{k,l},\]

which expresses the canonical coordinates $z_{i,k}$ of Prym cohomology classes in terms of the indexed marking $T(k)$ as linear functions of the canonical coordinates $z_{i,l}$ of Prym cohomology classes in terms of the indexed marking $T(l)$; but this formula also involves the additional variable $z_{l',l}$ if either $i = l'$ or $k = l'$, and that variable is not one of the canonical coordinates with respect to the indexed marking $T(l)$. A straightforward calculation shows that equation (ii) of Theorem 9.8 is equivalent to

\[(9.41) \quad z_{l',l} = \epsilon(l) \sum_{1 \leq i \leq g, i \neq l, l'} \left( \frac{1 - \zeta_{i+g}}{1 - \zeta_l} z_{i,l} - \frac{1 - \zeta_i}{1 - \zeta_l} z_{i+g,l} \right)\]

where

\[(9.42) \quad \epsilon(l) = \begin{cases} +1 & \text{if } l < l' \\ -1 & \text{if } l > l' \end{cases},\]

and this expresses $z_{l',l}$ as a linear function of the canonical coordinates in terms of the indexed marking $T(l)$. Equations (9.40) and (9.41) taken together therefore express the canonical coordinates $z_{i,k}$ of a Prym cohomology class in terms of the indexed marking $T(k)$ as linear functions of the canonical coordinates $z_{i,l}$ of that Prym cohomology class in terms of the indexed marking $T(l)$.

The coefficients of these linear equations are rational functions of the parameters $\zeta_i = \rho(T_i)$ describing the representation $\rho$, so they can be viewed as elements $R^{ij}_{kl}(\zeta) \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g})$; and at the same time it is also clear from the equations (9.40) and (9.41) that for any fixed value of the parameters $\zeta_i$ these coefficients are contained in the ring $\mathbb{Z}[\rho]$ for the representation $\rho$ described by those values of the parameters $\zeta_i$. Thus the systems of canonical coordinates for Prym cohomology classes in terms of the indexed markings $T(k)$ and $T(l)$ are related by linear equations of the form

\[(9.43) \quad z_{i,k} = \sum_{1 \leq j \leq g, j \neq l, l'} R^{ij}_{kl}(\zeta) z_{j,l} \quad \text{for } 1 \leq i \leq 2g, \ i \neq k, k'\]

where $R^{ij}_{kl}(\zeta) \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g}) \cap \mathbb{Z}[\rho]$. If the vectors $z_k = \{z_{i,k}\}$ and $z_l = \{z_{i,l}\}$ are viewed as column vectors of length $2g - 2$ these linear equations can be written more succinctly in matrix form as $z_k = R_{kl}(\zeta) z_l$ where $R_{kl}(\zeta) = \{ R^{ij}_{kl}(\zeta) \}$. 

When these coefficients are viewed as rational functions in $Q(\zeta_1, \ldots, \zeta_{2g})$ they are uniquely determined by these linear equations, since the canonical coordinates are independent variables; and from the uniqueness it follows that $R_{kl}(\zeta)R_{lk}(\zeta) = I$, so that $R_{kl}(\zeta) \in \text{GL}(2g - 2, \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g}))$, and further that $R_{kl}(\zeta)R_{lm}(\zeta) = R_{km}(\zeta)$.

The set of all values $\zeta_i \neq 0$ for $1 \leq i \leq 2g$ parametrize the full group $H^1(M, \mathbb{C}^*)$ of flat line bundles over the Riemann surface $M$, and the subset of values $\zeta_i$ not all of which are equal to 1 parametrize the subset $U \subset H^1(M, \mathbb{C}^*)$ of all nontrivial flat line bundles, those flat line bundles other than the identity flat line bundle. The set $U$ can be written as the union

$$U = \bigcup_{l=1}^{2g} U_l \quad \text{where} \quad U_l = \left\{ (\zeta_1, \zeta_2, \ldots, \zeta_{2g}) \in \mathbb{C}^*^{2g} \mid \zeta_l \neq 1 \right\}. \quad (9.44)$$

The matrix $R_{kl}(\zeta)$ takes well defined numerical values on the intersection $U_k \cap U_l$ so is actually a holomorphic nonsingular matrix-valued function in that subset. The condition that $R_{kl}(\zeta)R_{lm}(\zeta) = R_{km}(\zeta)$ means that these matrices describe a holomorphic vector bundle of rank $2g - 2$ over the complex manifold $U$, a bundle called the Prym cohomology bundle of the marked Riemann surface $M$, an interesting if rather simple bundle that will not be examined further just here.

**Theorem 9.10** If $M$ is a marked Riemann surface with the marking $\mathcal{T} = (T_1, \ldots, T_{2g})$, and if $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a representation for which $\rho(T_1) \neq 1$, then for any commutator $[R, S] \in [\Gamma, \Gamma]$ there are complex numbers $\epsilon_j(R, S) \in \mathbb{Z}[\rho]$ depending only on the elements $R, S \in \Gamma$ such that

$$\sigma([R, S]) = \sum_{j=1}^{2g} \epsilon_j(R, S)\sigma([T_j, T_l]) \quad (9.45)$$

for every cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$.

**Proof:** When a covering translation $S \in \Gamma$ is expressed in terms of the generators $T_i$ as $S = T_{k_1}^{\nu_1}T_{k_2}^{\nu_2}T_{k_3}^{\nu_3} \cdots$ it follows from Lemma 9.3 (i) that

$$\sigma([S, T]) = \sigma([T_{k_1}^{\nu_1}, T_l]) + \rho(T_{k_1}^{\nu_1})\sigma([T_{k_2}^{\nu_2}, T_l]) + \rho(T_{k_1}^{\nu_1}T_{k_2}^{\nu_2})\sigma([T_{k_3}^{\nu_3}, T_l]) + \cdots = \sum_{i} \epsilon_i(S)\sigma([T_{k_i}^{\nu_i}, T_l])$$

where $\epsilon_i(S) \in \mathbb{Z}[\rho]$ depends only on the element $S \in \Gamma$. Next it follows inductively from Lemma 9.3 (i) that if $\nu > 0$ then

$$\sigma([T_{k}^{\nu}, T_l]) = (1 + \rho(T_k) + \cdots + \rho(T_k^{\nu-1}))\sigma([T_k, T_l]) = \epsilon'(T_k^{\nu})\sigma([T_k, T_l])$$

where $\epsilon'(T_k^{\nu}) \in \mathbb{Z}[\rho]$ depends only on the element $T_{k}^{\nu} \in \Gamma$; and if $\nu < 0$ it follows from this and Lemma 9.3 (ii) that

$$\sigma([T_{k}^{-\nu}, T_l]) = -\rho(T_k^{-\nu})\sigma([T_k^{-\nu}, T_l]) = -\rho(T_k^{-\nu})\epsilon'(T_k^{-\nu})\sigma([T_k, T_l])$$

$$= \epsilon'(T_k^{\nu})\sigma([T_k, T_l])$$
where $\epsilon''(T_k^n) = -\rho(T_k^n)\epsilon''(T_k^{-n}) \in \mathbb{Z}[\rho]$ depends only on the element $T_k^n$. Combining these observations shows that

$$
\sigma([S, T]) = \sum_{i} \epsilon_i(S) \epsilon''(T_{k_i}^n) \sigma([T_{k_i}, T])
= \sum_{j=1}^{2g} \epsilon_j(S) \sigma([T_j, T])
$$

where $\epsilon_j(S) \in \mathbb{Z}[\rho]$ depends only on the element $S \in \Gamma$. Finally for any two covering translations $R, S \in \Gamma$ it follows from Lemma 9.3 (iii) and the preceding observation that

$$
\sigma([R, S]) = \frac{1 - \rho(S)}{1 - \rho(T_l)} \sigma([R, T_l]) - \frac{1 - \rho(R)}{1 - \rho(T_l)} \sigma([S, T_l])
= \sum_{j=1}^{2g} \left( \frac{1 - \rho(S)}{1 - \rho(T_l)} \epsilon_j(R) - \frac{1 - \rho(R)}{1 - \rho(T_l)} \epsilon_j(S) \right) \sigma([T_j, T_l])
= \sum_{j=1}^{2g} \epsilon_j(R, S) \sigma([T_j, T_l])
$$

where $\epsilon_j(R, S) \in \mathbb{Z}[\rho]$ depends only on the elements $R, S \in \Gamma$, and that suffices to conclude the proof.

**Corollary 9.11** If $S = (S_1, \ldots, s_{2g})$ and $T = (T_1, \ldots, T_{2g})$ are two markings of a compact Riemann surface $M$ of genus $g > 1$, and if $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a representation for which both $\rho(S_l) \neq 1$ and $\rho(T_m) \neq 1$, then there is a uniquely determined $(2g - 2) \times (2g - 2)$ matrix

$$
E(\rho) = \{ \epsilon_{ij} \} \in \text{Gl}(2g - 2, \mathbb{Z}[\rho])
$$

with entries $\epsilon_{ij} \in \mathbb{Z}[\rho]$ such that for any Prym cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ the canonical coordinates $w_{i,l} = \sigma([S_i, S_l])$ and $z_{j,m} = \sigma([T_j, T_m])$ of this class in terms of the generators $S_l$ and $T_m$ in these two markings are related by

$$
w_{i,l} = \sum_{1 \leq j \leq 2g} \sum_{j \neq m, m'}^\epsilon_{ij} z_{j,m}
$$

for all indices $1 \leq i \leq 2g, i \neq l, l'$.

**Proof:** By the preceding theorem there are values $\epsilon_{ij}^* \in \mathbb{Z}[\rho]$ for all indices $1 \leq i, j \leq 2g$ such that

$$
\sigma([S_i, S_l]) = \sum_{1 \leq j \leq 2g} \epsilon_{ij}^* \sigma([T_j, T_m])
$$
for every cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_p)$; thus in terms of the canonical coordinates $w_{i,l} = \sigma([S_i, S_l])$ and $z_{j,m} = \sigma([T_j, T_m])$ of that cohomology class

$$w_{i,l} = \sum_{1 \leq j \leq 2g} \epsilon_{ij}^* \sigma([T_j, T_m])$$

$$= \sum_{1 \leq j \leq g} \epsilon_{ij}^* z_{j,m} + \epsilon_{im}^* \sigma([T_m, T_m]) + \epsilon_{im'}^* \sigma([T_{m'}, T_m])$$

for $1 \leq i \leq 2g$, $i \neq l, l'$. Of course $\sigma([T_m, T_m]) = 0$, while by (9.41)

$$\sigma([T_{m'}, T_m]) = \epsilon(m) \sum_{1 \leq j \leq 2g} \sum_{j \neq m, m'} \left( \frac{1 - \rho(T_{j+g})}{1 - \rho(T_j)} z_{j,m} - \frac{1 - \rho(T_j)}{1 - \rho(T_m)} z_{j+g,m} \right)$$

and the coefficients of the canonical coordinates $z_{i,m}$ in this equation also lie in the ring $\mathbb{Z}[\rho]$. These observations taken together yield (9.47). Since the canonical coordinates are linearly independent the complex matrix $E(\rho)$ is invertible. The same arguments hold when the roles of the two markings are interchanged, so the inverse matrix also must have coefficients in the ring $\mathbb{Z}[\rho]$ and hence $E(\rho) \in \text{GL}(2g - 2, \mathbb{Z}[\rho])$. That suffices to conclude the proof.

The matrix $E(\rho)$ of the preceding corollary can be viewed as a function of the flat line bundle $\rho \in U$. To be explicit, choose a marking of the surface $M$ by generators $R_i \in \Gamma$ and use the values $\zeta_i = \rho(R_i)$ to parametrize all nontrivial flat representations $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ by points in the complex manifold $U$ defined as on page 247 in terms of this marking, so that the representations $\rho$ can be viewed as functions $\rho(\zeta)$. Since any element $T \in \Gamma$ can be written in terms of the generators $R_i$ as a product $T = \prod R_{\nu(i)}^k$ where $1 \leq \nu(i) \leq 2g$ and $\nu_i \in \mathbb{Z}$, it is evident that any element $\epsilon \in \mathbb{Z}[\rho]$ can be written as a quotient of polynomials in the variables $\zeta_i$ with integer coefficients, and consequently can be viewed as an element $\epsilon \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g})$.

Corollary 9.12 If $\{S_i\}$ and $\{T_j\}$ are two sets of generators of the covering translation group $\Gamma$ of a compact Riemann surface $M$ of genus $g > 1$, describing two markings of $M$, there is a uniquely determined $(2g - 2) \times (2g - 2)$ matrix

$$(9.48) \quad E(\zeta) = \{e_{ij}(\zeta)\} \in \text{GL}(2g - 2, \mathbb{Q}(\zeta))$$

of rational functions $e_{ij}(\zeta)$ of the variables $\zeta = (\zeta_1, \ldots, \zeta_{2g}) \in U$ parametrizing flat line bundles over $M$ such that for any flat line bundle $\rho = \rho(\zeta) \in \text{Hom}(\Gamma, \mathbb{C}^*)$ for which $\rho(S_i) \neq 1$ and $\rho(T_m) \neq 1$ and for any Prym cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_p)$ the canonical coordinates $w_{i,l} = \sigma([S_i, S_l])$ and $z_{j,m} = \sigma([T_j, T_m])$ of $\sigma$ in terms of the generators $S_i$ and $T_m$ in these two
markings are related by

\[ w_{i,l} = \sum_{1 \leq j \leq 2g \atop j \neq m,m'} e_{ij}(\zeta) \ z_{j,m} \]

for \( 1 \leq i \leq 2g, i \neq l, l' \).

**Proof:** This is merely a restatement of the conclusion of the preceding corollary, since the coefficients \( \epsilon_{ij} \) of that corollary can be written \( \epsilon_{ij} = e_{ij}(\zeta) \) for rational functions \( e_{ij}(\zeta) \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g}) \), so no further proof is required.

Thus the matrix \( E(\rho) \) of Corollary 9.11 when viewed as a function of the flat line bundle \( \rho \) described by the parameters \( \zeta_i = \rho(R_i) \) is a rational function \( E(\zeta) \) of the variables \( \zeta_i \); and the values taken by this matrix function for a fixed value of the parameters \( \zeta_i \) lie in the ring \( \mathbb{Z}[\rho] \) for the representation \( \rho \) described by the parameters \( \zeta_i \). It is evident from this that

\[ E(\overline{\rho}) = \overline{E(\rho)} \]

since the rational functions \( e_{ij}(\zeta) \) have rational integral coefficients.
Addendum: Higher Prym Cohomology Groups

The discussion in this chapter so far has involved only the first Prym cohomology groups. It is perhaps natural to ask about the higher-dimensional Prym cohomology groups, so a brief discussion of these groups is included here; but since they play only a very limited role in the study of Riemann surfaces the discussion is somewhat condensed and is relegated to this addendum, which can be ignored altogether if desired. If \( M \) is a compact Riemann surface of genus \( g > 0 \) with covering translation group \( \Gamma \) then by using a \( C^\infty \) partition of unity on \( M \), as in the demonstration of (E.33) in Appendix E.2 or the natural extension of the proof of Lemma 9.1, it is easy to see that \( H^q(\Gamma, \Gamma(\tilde{M}, E)) = 0 \) whenever \( q > 0 \); consequently from the exact cohomology sequence associated to the exact sequence (9.10) of vector spaces on which the group \( \Gamma \) acts as in (9.6) it follows that

\[
H^q(\Gamma, C_\rho) \cong H^{q-1}(\Gamma, \Gamma(\tilde{M}, E^1)) \quad \text{for} \quad q \geq 2.
\]

On the other hand there is also the exact sequence of vector spaces

\[
0 \longrightarrow \Gamma(\tilde{M}, E^1) \longrightarrow \Gamma(\tilde{M}, E) \longrightarrow \Gamma(\tilde{M}, E^2) \longrightarrow 0
\]

on which the group \( \Gamma \) acts in the same way; and from the associated exact cohomology sequence it follows first that

\[
H^1(\Gamma, \Gamma(\tilde{M}, E^1)) \cong \frac{H^0(\Gamma, \Gamma(\tilde{M}, E^2))}{dH^0(\Gamma, \Gamma(M, E^1))} \cong \frac{\Gamma(M, E^2(\rho))}{d\Gamma(M, E^1(\rho))}
\]

and second that

\[
H^q(\Gamma, \Gamma(\tilde{M}, E^1)) = 0 \quad \text{for} \quad q \geq 2.
\]

Combining these observations shows that

\[
(9.51) \quad H^q(\Gamma, C_\rho) \cong \begin{cases} 
\Gamma(M, E^2(\rho)) & \text{for } q = 2, \\
\frac{d\Gamma(M, E^1(\rho))}{d\Gamma(M, E^2(\rho))} & \text{for } q \geq 3;
\end{cases}
\]

consequently for the study of Riemann surfaces only the second cohomology group \( H^2(\Gamma, C_\rho) \) really is of interest.

To trace through the isomorphism (9.51) for \( q = 2 \) explicitly, recall the description of the coboundary operators on inhomogeneous cochains in the cohomology of groups in (E.13) in Appendix E.1 and consider a differential form \( \phi \in \Gamma(M, E^2(\rho)) \), viewed as a \( C^\infty \) differential form \( \phi(z) \) on the universal covering surface \( \tilde{M} \) such that \( \phi(Tz) = \rho(T)\phi(z) \) for all \( T \in \Gamma \). This differential form can be written as the exterior derivative \( \phi(z) = d\psi(z) \) of a \( C^\infty \) one-form \( \psi(z) \) on \( \tilde{M} \) since \( \tilde{M} \) is contractible. Then for any \( T \in \Gamma \) it follows that

\[
d(\rho(T)^{-1}\psi(Tz) - \psi(z)) = \rho(T)^{-1}\phi(Tz) - \phi(z) = 0,
\]

so since \( \tilde{M} \) is contractible there is a \( C^\infty \) function \( f(T, z) \) on \( \tilde{M} \), depending also on the element \( T \in \Gamma \), such
that $\rho(T)^{-1}\psi(Tz) - \psi(z) = df(T, z)$. In particular $df(I, z) = 0$ for the identity element $I \in \Gamma$, so it always can be assumed that $f(I, z) = 0$. For these functions and for any elements $T_1, T_2 \in \Gamma$

$$d\left(\rho(T_2)^{-1}f(T_1, T_2z) + f(T_2, z) - f(T_1T_2, z)\right)$$

$$= \rho(T_2)^{-1}\left(\rho(T_1)^{-1}\psi(T_1T_2z) - \psi(T_2z)\right)$$

$$+ \left(\rho(T_2)^{-1}\psi(T_2z) - \psi(z)\right)$$

$$- \left(\rho(T_1T_2)^{-1}\psi(T_1T_2z) - \psi(z)\right)$$

$$= 0;$$

consequently the expression

$$v(T_1, T_2) = \rho(T_2)^{-1}f(T_1, T_2z) + f(T_2, z) - f(T_1T_2, z)$$

is a constant in the variable $z \in \tilde{M}$. In particular since $f(I, z) = 0$ by assumption it follows that

$$v(I, T_2) = \rho(T_2)^{-1}f(I, T_2z) + f(T_2, z) - f(T_2, z) = 0,$$

$$v(T_1, I) = f(T_1, z) + f(I, z) - f(T_1, z) = 0,$$

so the mapping $v : \Gamma \times \Gamma \rightarrow \mathbb{C}$ is an inhomogeneous two-cochain $v \in C^2(\Gamma, \mathbb{C}_\rho)$ as defined in equation (E.9) of Appendix E.2. Equation (9.52) exhibits the two-cochain $v$ as the coboundary of the one-cochain $f(T, z) \in C^1(\Gamma, \Gamma(M, \mathcal{E}))$, so it must be a two-cocycle $v \in Z^2(\Gamma, \mathbb{C}_\rho)$. Actually it is a straightforward calculation to verify directly from (9.52) that the cocycle $v(T_1, T_2)$ satisfies the cocycle condition

$$\rho(T_2)^{-1}v(T_1, T_2) - v(T_2, T_3) + v(T_1T_2, T_3) - v(T_1, T_2T_3) = 0,$$

(9.53)

keeping in mind that the expression (9.52) is independent of the variable $z \in \tilde{M}$. This cocycle of course depends on the choices of the differential form $\psi(z)$ and of the functions $f(T, z)$. If $d\psi_1(z) = \phi(z)$ and $df_1(T, z) = \rho(T)^{-1}\psi_1(Tz) - \psi_1(z)$ then since $d(\psi_1(z) - \psi(z)) = 0$ it must be the case that $\psi_1(z) - \psi(z) = dg(z)$ for some $C^\infty$ function $g(z)$ on $\tilde{M}$, so $\rho(T)^{-1}\psi_1(Tz) - \psi_1(z) = \rho(T)^{-1}\psi(Tz) - \psi(z) + d(\rho(T)^{-1}g(Tz) - g(z)) = df(T, z) + \rho(T)^{-1}g(Tz) - g(z)$ and consequently $f_1(T, z) = f(T, z) + \rho(T)^{-1}g(Tz) - g(z) + c(T)$ for some constants $c(T)$; in order that $f_1(I, z) = 0$ it is necessary that $c(I) = 0$, so the mapping $c : \Gamma \rightarrow \mathbb{C}$ can be viewed as an inhomogeneous one-cochain $c \in C^0(\Gamma, \mathbb{C}_\rho)$. It is a straightforward calculation to verify that for these functions $f_1(T, z)$ the definition (9.52) yields the expression $v_1(T_1, T_2) = v(T_1, T_2) + \rho(T_2)^{-1}c(T_1) + c(T_2) - c(T_1T_2)$, which is a cocycle cohomologous to $v(T_1, T_2)$. Thus the cohomology class in $H^2(\Gamma, \mathbb{C}_\rho)$ represented by the cocycle $v(T_1, T_2)$ is independent of these choices so depends only on the initial differential form $\phi(z)$; this cohomology class is the period class $\delta \phi \in H^2(\Gamma, \mathbb{C}_\rho)$ of the differential form $\phi$. 

CHAPTER 9. PRYM DIFFERENTIALS
The second cohomology group $H^2(\Gamma, \mathbb{C}_p)$ has an explicit description given by the theorem of H. Hopf, which for the simpler situation in which the group $\Gamma$ acts trivially is Theorem E.2 in Appendix E.2. This description does not involve any special properties of the group $\Gamma$ as the covering translation group of a compact Riemann surface, so will be demonstrated in Theorem 9.14 for more general groups as well; its application to Riemann surfaces of course does depend on the special structure of the covering translation group of a surface, so will be discussed separately in the subsequent Corollary 9.15. Note that any representation $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ of any group $\Gamma$ can be used to exhibit that group as a group of operators on the complex numbers as in (9.7), in terms of which the cohomology groups $H^p(\Gamma, \mathbb{C}_p)$ are well defined.

**Lemma 9.13** If $F$ is a finitely generated free group and $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ then $H^2(F, \mathbb{C}_p) = 0$.

**Proof:** The proof is a straightforward extension of the proof of Lemma E.25 in Appendix E.2. First setting $T_1 = T_3 = T$ and $T_2 = T^{-1}$ in the cocycle condition (9.53) shows that $v(T^{-1}, T) = \rho(T)^{-1}v(T, T^{-1})$ for any cocycle $v \in Z^2(F, \mathbb{C}_p)$, since $v(T_1, T_2) = 0$ if $T_1 = I$ or $T_2 = I$. Then for any cocycle $v \in Z^2(F, \mathbb{C}_p)$ and any free generator $T_i$ of the group $F$ choose a value $\sigma(T_i) \in \mathbb{C}$, set $\sigma(T_i^{-1}) = v(T_i, T_i^{-1}) - \rho(T_i)\sigma(T_i) = \rho(T_i)v(T_i^{-1}, T_i) - \rho(T_i)\sigma(T_i)$, and define $\sigma(T)$ for any formal product of the generators $T_i$ and their inverses, without cancellation, by

\[
\sigma(ST) = \rho(T)^{-1}\sigma(S) + \sigma(T) - v(S, T).
\]

It follows readily from these definitions that $\sigma(T_iT_i^{-1}) = \sigma(T_i^{-1}T_i) = 0$, and it is a straightforward calculation to verify from the definition (9.54) and the cocycle condition (9.53) that $\sigma(R \cdot ST) = \sigma(RS \cdot T)$; consequently $\sigma$ is a well defined mapping $\sigma : F \longrightarrow \mathbb{C}$, and since $\sigma(I) = 0$ it actually is a one-cochain $\sigma \in C^0(F, \mathbb{C}_p)$. Equation (9.54) shows that the cocycle $v$ is the coboundary of the cochain $\sigma$, and consequently that $H^1(F, \mathbb{C}_p) = 0$. That suffices to conclude the proof.

Next suppose that $\Gamma$ is any finitely generated group, so can be described by an exact sequence

\[
0 \longrightarrow K \overset{\iota}{\longrightarrow} F \overset{p}{\longrightarrow} \Gamma \longrightarrow 0
\]

in which $F$ is a finitely generated free group, $\iota$ is an inclusion mapping and $p$ is the projection to the quotient group. A representation $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ lifts to the representation $\rho \circ p \in \text{Hom}(F, \mathbb{C}^*)$, which to simplify the notation also will be denoted just by $\rho$; so if $\tilde{T} \in F$ then by definition $\rho(\tilde{T}) = \rho(p(\tilde{T}))$. For any cocycle $v \in Z^2(\Gamma, \mathbb{C}_p)$ the composition $v \circ p = \rho^*(v)$ is a cocycle $\rho^*(v) \in Z^2(F, \mathbb{C}_p)$, so by the preceding lemma is the coboundary of a one-cochain $\sigma \in C^1(F, \mathbb{C}_p)$; thus $\sigma$ is a mapping $\sigma : F \longrightarrow \mathbb{C}$ such that $\sigma(I) = 0$ and

\[
v(\tilde{T}_1, \tilde{T}_2) = \rho(\tilde{T}_2)^{-1}\sigma(\tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{T}_1\tilde{T}_2)
\]
for all $\tilde{T}_i \in F$. Since $\nu(T_1, T_2) = 0$ if either $T_1 = I$ or $T_2 = I$ then $p^*(\nu)(\tilde{T}_1, \tilde{T}_2) = 0$ if either $\tilde{T}_1 \in K$ or $\tilde{T}_2 \in K$. It follows from this and from the cocycle condition (9.56) that if $\tilde{S} \in K$ and $\tilde{T} \in F$ then
\[
\sigma(\tilde{S}\tilde{T}) = \rho(\tilde{T})^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T}),
\]
\[
\sigma(\tilde{T}\tilde{S}) = \sigma(\tilde{T}) + \sigma(\tilde{S});
\]
consequently the restriction $\sigma|K$ is a homomorphism from the group $K$ to the additive group $\mathbb{C}$ and
\[
\sigma(T) + \sigma(S) = \sigma(TS) = \sigma(T\tilde{S}\tilde{T}^{-1})
\]
so that $\sigma(T\tilde{S}\tilde{T}^{-1}) = \rho(T)\sigma(\tilde{S})$. Therefore the restriction $\sigma|K$ is an element of the group
\[
(9.57) \quad \text{Hom}_\rho(K, \mathbb{C}) = \left\{\sigma \in \text{Hom}(K, \mathbb{C}) \mid \sigma(T\tilde{S}\tilde{T}^{-1}) = \rho(p(\tilde{T}))\sigma(\tilde{S}) \right\}
\]
for all $\tilde{S} \in K, \tilde{T} \in F$ analogous to (9.22). On the other hand for any cocycle $\tau \in Z^1(F, \mathbb{C}_\rho)$ the cocycle condition shows that $\tau(\tilde{S}\tilde{T}) = \rho(\tilde{T})^{-1}\tau(\tilde{S}) + \tau(\tilde{T})$ for all $\tilde{S}, \tilde{T} \in F$, and consequently that $\tau|K \in \text{Hom}(K, \mathbb{C})$; and it follows from Lemma 9.2 that $\tau(T\tilde{S}\tilde{T}^{-1}) = \rho(T)\tau(\tilde{S}) + (\rho(\tilde{S})^{-1} - 1)\rho(T)\tau(\tilde{T})$ for all $\tilde{S}, \tilde{T} \in F$, so if $\tilde{S} \in K$ then $\rho(\tilde{S}) = 1$ and $\tau(T\tilde{S}\tilde{T}^{-1}) = \rho(T)\tau(\tilde{S})$. Thus
\[
Z^1(F, \mathbb{C}_\rho)|K \subset \text{Hom}_\rho(K, \mathbb{C}),
\]
so it is possible to introduce the quotient space
\[
(9.58) \quad \text{Hom}_{\rho, Z}(K, \mathbb{C}) = \frac{\text{Hom}_\rho(K, \mathbb{C})}{Z^1(F, \mathbb{C}_\rho)|K}.
\]

**Theorem 9.14** If $\Gamma$ is a finitely generated group described by the exact sequence (9.55) for a finitely generated free group $F$, and if $\rho \in \text{Hom}(\Gamma, \mathbb{C}^*)$ is a representation of the group $\Gamma$, then the mapping $\phi$ that associates to the cohomology class represented by a cocycle $\nu \in Z^2(\Gamma, \mathbb{C}_\rho)$ the restriction $\sigma|K$ of any cocycle $\sigma \in C^1(F, \mathbb{C}_\rho)$ with the coboundary $\delta\sigma = p^*(\nu) \in Z^2(F, \mathbb{C}_\rho)$ determines an isomorphism
\[
H^2(\Gamma, \mathbb{C}_\rho) \cong \text{Hom}_{\rho, Z}(K, \mathbb{C}).
\]

**Proof:** If $\nu \in B^2(\Gamma, \mathbb{C}_\rho)$ then $\nu = \delta\sigma$ for a cocycle $\sigma \in C^1(\Gamma, \mathbb{C}_\rho)$ and $p^*(\nu) = \delta p^*(\sigma)$ where $p^*(\sigma)|K = 0$. If $\nu \in Z^2(\Gamma, \mathbb{C}_\rho)$ is the coboundary of two cochains $\sigma_1, \sigma_2 \in C^1(F, \mathbb{C}_\rho)$ then $\sigma_1 - \sigma_2 \in Z^1(F, \mathbb{C}_\rho)$ so that $(\sigma_1 - \sigma_2)|K \in Z^1(F, \mathbb{C}_\rho)|K$ and consequently $\sigma_1$ and $\sigma_2$ determine the same element in the
quotient $\text{Hom}_\rho,\mathbb{Z}(K,\mathbb{C})$. It follows from these observations that the mapping $\phi$ yields a well defined homomorphism

$$\phi^* : H^2(\Gamma,\mathbb{C}_\rho) \longrightarrow \text{Hom}_\rho,\mathbb{Z}(K,\mathbb{C}).$$

If a cocycle $v \in Z^2(\Gamma,\mathbb{C}_\rho)$ represents an element in the kernel of the homomorphism $\phi^*$ then $p^*(v) = \delta \sigma$ for a cochain $\sigma \in C^1(F,\mathbb{C}_\rho)$ for which $\sigma|K \in Z^1(F,\mathbb{C}_\rho)|K$, so there is a cocycle $\tau \in Z^1(F,\mathbb{C}_\rho)$ such that $\sigma|K = \tau|K$; thus $\sigma - \tau \in C^1(\Gamma,\mathbb{C}_\rho)$, and since $p^*(v) = \delta(\sigma - \tau)$ it must be the case that $v \in B^2(\Gamma,\mathbb{C}_\rho)$ and therefore the mapping $\phi^*$ is injective. Finally consider any element $\sigma \in \text{Hom}_\rho(K,\mathbb{C})$. Choose a coset decomposition $F = \bigcup_i K L_i$ for some elements $\tilde{L}_i \in F$ representing elements $L_i \in \Gamma$, select arbitrary values $\sigma(\tilde{L}_i) \in \mathbb{C}$, and in terms of these values define the mapping $\sigma : F \longrightarrow \mathbb{C}$ by setting $\sigma(\tilde{S} \tilde{L}_i) = \rho(\tilde{L}_i)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{L}_i)$ for all $\tilde{S} \in K$. For any elements $\tilde{R}, \tilde{S} \in K$ and for $\tilde{T} = \tilde{S} \tilde{L}_i \in F$ it follows that

$$\sigma(\tilde{T} \tilde{R}) = \sigma(\tilde{R} \tilde{S} \tilde{L}_i) = \rho(\tilde{L}_i)^{-1}\sigma(\tilde{R} \tilde{S}) + \sigma(\tilde{L}_i)$$

$$= \rho(\tilde{L}_i)^{-1}\sigma(\tilde{R}) + \rho(\tilde{L}_i)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{L}_i)$$

$$= \rho(\tilde{T})^{-1}\sigma(\tilde{R}) + \sigma(\tilde{T})$$

and

$$\sigma(\tilde{T} \tilde{R}^{-1}) = \sigma(\tilde{T} \tilde{R}^{-1} \cdot \tilde{T}) = \rho(\tilde{T})^{-1}\sigma(\tilde{T} \tilde{R}^{-1}) + \sigma(\tilde{T})$$

$$= \rho(\tilde{T})^{-1}\rho(\tilde{T})\sigma(\tilde{R}) + \sigma(\tilde{T})$$

$$= \sigma(\tilde{R}) + \sigma(\tilde{T}).$$

The mapping $v : F \times F \longrightarrow \mathbb{C}$ defined by

$$v(\tilde{T}_1, \tilde{T}_2) = \rho(\tilde{T}_2)^{-1}\sigma(\tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{T}_1 \tilde{T}_2)$$

is a cocycle $v \in Z^2(F,\mathbb{C}_\rho)$ that is the coboundary of the cochain $\sigma \in C^1(F,\mathbb{C}_\rho)$. For any $\tilde{S} \in K$ it follows from the preceding observations that

$$v(\tilde{S} \tilde{T}_1, \tilde{T}_2) = \rho(\tilde{T}_2)^{-1}\sigma(\tilde{S} \tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{S} \tilde{T}_1 \tilde{T}_2)$$

$$= \rho(\tilde{T}_2)^{-1}(\rho(\tilde{T}_1)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T}_1)) + \sigma(\tilde{T}_2)$$

$$- \rho(\tilde{T}_1 \tilde{T}_2)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T}_1 \tilde{T}_2)$$

$$= \rho(\tilde{T}_2)^{-1}\sigma(\tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{T}_1 \tilde{T}_2) = v(\tilde{T}_1, \tilde{T}_2)$$

and similarly $v(\tilde{T}_1, \tilde{S} \tilde{T}_2) = v(\tilde{T}_1, \tilde{T}_2)$; thus $v \in Z^2(\Gamma,\mathbb{C}_\rho)$, and since $\sigma = \phi^*(v)$ that shows that the mapping $\phi^*$ is surjective and thereby concludes the proof.

**Corollary 9.15** If $\Gamma$ is the covering translation group of a compact Riemann surface $M$ of genus $g > 1$ then $H^2(\Gamma,\mathbb{C}_\rho) = 0$ for any nontrivial representation $\rho \in \text{Hom}(\Gamma,\mathbb{C}^*)$. 
 Proof: Choose a marking of $M$ described by $2g$ generators $T_i = A_i$, $T_{g+i} = B_i$ of the group $\Gamma$ for $1 \leq i \leq g$. The group $\Gamma$ then is described by an exact sequence (9.55) in which $F$ is the free group generated by $2g$ symbols $\tilde{T}_i$ for which $p(\tilde{T}_i) = T_i$; and the kernel $K \subset F$ is the normal subgroup generated by the single element $C \in \{A_iB_i\}$ for $C_i = [A_iB_i]$. Thus an element $\sigma \in \text{Hom}_\rho(K, \mathbb{C})$ is determined fully by the value $\sigma(C)$ alone, and consequently $\dim \text{Hom}_\rho(K, \mathbb{C}) \leq 1$. On the other hand there is a cocycle $\sigma \in Z^1(F, C_\rho)$ taking any specified values $\sigma(\tilde{T}_i)$ on the generators $\tilde{T}_i$; for given any values $\sigma(\tilde{T}_i)$ the cocyle condition $\sigma(\tilde{S}\tilde{T}) = \rho(\tilde{T})^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T})$ can be used to define the value $\sigma(\tilde{T})$ for any word $\tilde{T} \in F$. Then as in Lemma 9.2

$$\sigma(\tilde{S}, \tilde{T}) = (1 - \rho(\tilde{T}))\rho(\tilde{S})\sigma(\tilde{S}) - (1 - \rho(\tilde{S}))\rho(\tilde{T})\sigma(\tilde{T}).$$

It is clear from this that if the representation $\rho$ is nontrivial there is a cocycle $\sigma$ such that $\sigma(C) \neq 0$, and consequently that $Z^1(\Gamma, C_\rho)|K \neq 0$; and then $0 < \dim Z^1(\Gamma, C_\rho)|K \leq \dim \text{Hom}_\rho(K, \mathbb{C}) \leq 1$ and therefore $\text{Hom}_{\rho,Z}(K, \mathbb{C}) = 0$. It then follows from the preceding theorem that $H^2(\Gamma, C_\rho) = 0$, and that suffices for the proof.

For the trivial representation $\rho = 1$ of course $Z^1(F, C_1) = \text{Hom}(F, \mathbb{C})$; and since any homomorphism $\sigma \in \text{Hom}(F, \mathbb{C})$ is trivial on the commutator subgroup $[F, F] \subset F$ necessarily $Z^1(F, C_1)|K = 0$ so $\text{Hom}_{1,Z}(K, \mathbb{C}) = \text{Hom}(K, \mathbb{C}) \cong \mathbb{C}$. Thus in this special case Theorem 9.14 reduces to the isomorphism $H^2(\Gamma, \mathbb{C}) \cong \mathbb{C}$, as expected since $H^2(\Gamma, \mathbb{C}) = \mathbb{C}$. 


Chapter 11

Mappings to the Riemann Sphere

[PRELIMINARY]

The Riemann sphere $\mathbb{P}^1$ and some of its basic properties were discussed at the beginning of Chapter 2; in particular it was observed there that a nonconstant meromorphic function on an arbitrary Riemann surface $M$ can be viewed as a holomorphic mapping $f : M \rightarrow \mathbb{P}^1$. To discuss such mappings it may be best to begin with an examination of some general results about holomorphic mappings between Riemann surfaces. Explicitly a holomorphic mapping $\phi : M \rightarrow N$ between Riemann surfaces $M$ and $N$ is a continuous mapping between these topological manifolds that is holomorphic when expressed in terms of the local coordinates on $M$ and $N$. If $U_\alpha \subset M$ is a coordinate neighborhood of a point $a \in M$ with local coordinate $z_\alpha$ and if $V_\beta \subset N$ is a coordinate neighborhood of the image $b = \phi(a) \in N$ with local coordinate $w_\beta$, the image of a point $z_\alpha \in U_\alpha$ sufficiently near $a$ under the mapping $\phi : M \rightarrow N$ is a point $\phi(z_\alpha) = w_\beta \in V_\beta$; in this way the coordinate $w_\beta$ is expressed as a complex-valued function $w_\beta(z_\alpha)$ of the complex variable $z_\alpha$, and the mapping $\phi$ is holomorphic if all of these local representations are holomorphic. If the function $w_\beta(z_\alpha)$ is constant the entire neighborhood $U_\alpha$ is mapped to the single point $b \in N$, and it follows from the identity theorem for holomorphic functions that the entire surface $M$ is mapped to the point $b$; such trivial mappings generally will be excluded from consideration. A nonconstant holomorphic function of a complex variable is an open mapping, so the image of a nontrivial holomorphic mapping $\phi : M \rightarrow N$ between Riemann surfaces is an open subset of $N$; and if $M$ is compact its image is a compact and hence closed subset of $N$, so since $N$ is connected $\phi(M) = N$. Therefore a nontrivial holomorphic mapping between compact Riemann surfaces is always surjective. Of course that is not the case if $M$ is a noncompact Riemann surface; for instance the inclusion mapping of an open subset $M \subset N$ is a holomorphic mapping $\phi : M \rightarrow N$ between two Riemann surfaces. If the local coordinates $z_\alpha$ and $w_\beta$ are chosen so that $a \in U_\alpha$
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corresponds to the origin \( z_\alpha = 0 \) and \( b = \phi(a) \in V_\beta \) corresponds to the origin \( w_\beta = 0 \), then the local holomorphic function \( w_\beta(z_\alpha) \) vanishes at the origin \( z_\alpha = 0 \) and is determined uniquely up to nonsingular holomorphic changes of coordinates in its domain and range preserving the origin; since the non-negative integer \( \text{ord}_0(w_\beta(z_\alpha)) \) − 1 is invariant under such changes of coordinates it is an intrinsic property of the mapping \( \phi \), called the ramification order of the mapping \( \phi \) at the point \( a \in M \) and denoted by \( r_a(\phi) \).

For a nontrivial holomorphic mapping \( \phi \) the local coordinates \( z_\alpha \) and \( w_\beta \) can be chosen so that

\[
(11.1) \quad w_\beta(z_\alpha) = z_\alpha^{r+1} \text{ where } r = r_a(\phi) \geq 0,
\]

thus providing a local normal form of the mapping \( \phi : M \rightarrow N \). A point \( a \in M \) at which \( r_a(\phi) > 0 \) is called a ramification point of the mapping \( \phi \), or equivalently the mapping \( \phi \) is said to be ramified at the point \( a \). The ramification points are a discrete set of points on \( M \), since they are just the points at which the derivative of the mapping \( \phi \) in terms of any local coordinate is zero; consequently the divisor

\[
(11.2) \quad (\mathcal{t}(\phi)) = \sum_{a \in M} r_a(\phi) \cdot a
\]

is a well defined positive divisor on \( M \), called the ramification divisor of the holomorphic mapping \( \phi \). The mapping \( \phi \) is said to be simply ramified at a point \( a \in M \) if \( r_a(\phi) = 1 \), and such a ramification point is called a simple ramification point; the mapping \( \phi \) itself is said to be simply ramified if all of its ramification points are simple ramification points. A point \( a \in M \) that is not a ramification point, a point at which \( r_a(\phi) = 0 \), is called a regular point or an unramified point of the mapping \( \phi \); the regular points are those points of \( M \) at which the mapping \( \phi \) is locally biholomorphic.

There are only finitely many ramification points \( a_i \in M \) of a nontrivial holomorphic mapping \( \phi : M \rightarrow N \) between compact Riemann surfaces; the images \( \phi(a_i) \) of these points are finitely many points of \( N \) called the branch points of the mapping \( \phi \), and the set of branch points is called the branch locus of the mapping \( \phi \) and is denoted by \( B \) or by \( B(\phi) \) when it is useful to be more specific. The inverse image \( \phi^{-1}(b) \subset N \) of any point \( b \in N \) consists of a finite number of points of \( M \). If \( b \) is not a branch point none of these points are ramification points, so for a suitably small open neighborhood \( W \) of the point \( b \) in \( N \) the inverse image \( \phi^{-1}(W) \) is a collection of \( \delta \) disjoint open subsets of \( M \), each of which is mapped biholomorphically to \( W \) under the mapping \( \phi \); thus the mapping \( \phi \) exhibits the inverse image \( \phi^{-1}(W) \) as a covering space of \( \delta \) sheets over \( W \). The number \( \delta \) is independent of the point \( b \notin B \), since it is a locally constant function on the connected manifold \( N \sim B \); it is called the degree of the mapping \( \phi \), and it is denoted by \( \text{deg} \phi \).

The restriction

\[
(11.3) \quad \phi : (M \sim \phi^{-1}(B)) \rightarrow (N \sim B)
\]

of a holomorphic mapping \( \phi : M \rightarrow N \) of degree \( \delta \) with branch locus \( B \subset N \) thus is a \( \delta \)-sheeted covering space in the usual sense. On the other hand if \( b \in B \)
then at least some of the points \( a_i \in \phi^{-1}(b) \subset M \) are ramification points. However for each point \( a_i \in \phi^{-1}(b) \) there are local coordinates \( z_i \) centered at \( a_i \) and \( w_i \) centered at \( b \) such that the mapping \( \phi \) is described locally by the holomorphic function \( w_i(z_i) = z_i^{r_i+1} \) where \( r_i = r_{a_i}(\phi) \); so if \( U_i \) is a sufficiently small open neighborhood of \( a_i \) the restriction of the mapping \( \phi \) to the complement \( U_i \sim a_i \) is a covering mapping of \( r_i + 1 \) sheets and \( a_i \) is the only point in \( U_i \) that has the point \( b \) as its image under \( \phi \). It is evident from this that

\[
\text{(11.4)} \quad \deg \phi = \sum_{a \in \phi^{-1}(b)} (r_a(\phi) + 1) \quad \text{for any point } b \in N,
\]

so the branch points of the mapping \( \phi \) can be characterized as those points \( b \in N \) at which \( \phi^{-1}(b) \) consists of fewer than \( \deg \phi \) distinct points; the difference between \( \deg \phi \) and the number of distinct points in \( \phi^{-1}(b) \) is called the local branch order of the mapping \( \phi \) over the point \( b \) and is denoted by \( b_b(\phi) \), so

\[
\text{(11.5)} \quad b_b(\phi) = \sum_{a \in \phi^{-1}(b)} r_a(\phi).
\]

The branch points are precisely those points \( b \in N \) at which \( b_b(\phi) > 0 \), and the branch divisor is defined to be the divisor

\[
\text{(11.6)} \quad b(\phi) = \sum_{b \in N} b_b(\phi) \cdot b
\]

on the image surface \( N \); thus the branch locus of the mapping \( \phi \) is the support \( B(\phi) = |b(\phi)| \) of the branch divisor of that mapping. The branch order of the mapping \( \phi \) is the integer \( \text{br}(\phi) \) defined by

\[
\text{(11.7)} \quad \text{br}(\phi) = \sum_{b \in N} b_b(\phi) = \sum_{a \in M} r_a(\phi),
\]

and it is evident from (11.2) and (11.6) that

\[
\text{(11.8)} \quad \text{br}(\phi) = \deg \tau(\phi) = \deg b(\phi).
\]

The mapping \( \phi \) is said to be simply branched over a point \( b \in N \) if \( b_b(\phi) = 1 \), and is said to be simply branched if it is simply branched over each point of \( N \); clearly \( \phi \) is simply branched precisely when it has \( \text{br} \phi \) branch points altogether, and in that case there is precisely one simply ramified point of \( M \) over each of these branch points in \( N \). The mapping \( \phi \) is said to be fully branched over a point \( b \in N \) if \( \phi^{-1}(b) = a \) is a single point \( a \in M \), and in that case it is also said to be fully ramified at the point \( a \in M \); thus these are equivalent notions, and it is clear that the mapping \( \phi \) is fully ramified at a point \( a \in M \) whenever \( r_a(\phi) = \deg \phi - 1 \) and it is fully branched over a point \( b \in N \) whenever \( b_b(\phi) = \deg \phi - 1 \). A mapping \( \phi : M \rightarrow N \) for which \( \deg \phi = 2 \) is both simply branched and fully branched over each point of \( N \), while these notions are quite distinct if \( \deg \phi \geq 3 \).
**Theorem 11.1 (Riemann-Hurwitz Formula)** If $\phi : M \longrightarrow N$ is a nontrivial holomorphic mapping from a compact Riemann surface $M$ of genus $g$ to a compact Riemann surface $N$ of genus $h$ then

$$2g - 2 = (2h - 2) \deg \phi + \br (\phi).$$

**Proof:** It is always possible to triangulate the manifold $N$ in such a way that the branch points of the mapping $\phi$ are among the vertices of the triangulation. If there are $n_i$ simplices of dimension $i$ in the triangulation of $N$ and $m_i$ simplices of dimension $i$ in the induced triangulation of $M$ then

$$m_0 = \delta n_0 - \beta, \quad m_1 = \delta n_1, \quad m_2 = \delta n_2,$$

where $\delta = \deg \phi$ and $\beta = \br (\phi)$, since each simplex of $N$ of dimension 1 or 2 is covered by $\delta$ simplices of $M$ while each simplex of $N$ of dimension 0 is covered by $\delta$ simplices of $M$ except for the branch points, at which the number of simplices in $M$ is reduced by the local branch order of the mapping $\phi$ at that point of $N$. Euler’s formula (D.11) asserts that the Euler characteristic $\chi (M)$ of the surface $M$, the alternating sum of the ranks of the homology groups, is equal to the alternating sum of the numbers of simplices of each dimension; for the surface $M$ with Betti numbers $b_i = \text{rank} H_i (M)$ the Euler characteristic is $\chi (M) = b_0 - b_1 + b_2 = 2 - 2g$, so by Euler’s formula $2 - 2g = m_0 - m_1 + m_2$, and correspondingly $2 - 2h = n_0 - n_1 + n_2$ for the surface $N$. Consequently

$$2 - 2g = m_0 - m_1 + m_2 = (\delta n_0 - \beta) - \delta n_1 + \delta n_2 = \delta (2 - 2h) - \beta,$$

which yields the desired formula and thereby concludes the proof.

**Corollary 11.2**

(i) If there is a nontrivial holomorphic mapping $\phi : M \longrightarrow N$ from a compact Riemann surface $M$ of genus $g$ to a compact Riemann surface $N$ of genus $h$ the branch order of $\phi$ is an even integer and $g \geq h$.

(ii) If $g = h > 1$ the mapping $\phi$ is a biholomorphic mapping between the two surfaces. If $g = h = 1$ the mapping $\phi$ is a holomorphic covering mapping in the usual sense; there are holomorphic covering mappings of $\delta$ sheets for any $\delta > 0$.

**Proof:** It is clear from the Riemann-Hurwitz formula (11.9) that the branch order $\br (\phi)$ is an even integer. That formula can be rewritten

$$g - h = (h - 1)(\delta - 1) + \frac{\beta}{2},$$

where $\delta = \deg \phi$ and $\beta = \br (\phi)$. If $h = 0$ then of course $g \geq h$ for any $g \geq 0$. If $h > 0$ then both terms on the right-hand side are non-negative, hence $g - h \geq 0$; and $g = h$ if and only if $(h - 1)(\delta - 1) = \beta = 0$, so if and only if $\phi$ is an unbranched covering and either $\delta = 1$ or $h = 1$. If $g = h > 1$ then $\delta = 1$ and the mapping $\phi$ is a biholomorphic mapping. If $g = h = 1$ the mapping $\phi$
is an unbranched covering mapping between two Riemann surfaces of genus 1.
To see that there are unbranched coverings of δ sheets between two Riemann surfaces of genus 1 for any δ ≥ 1, by Corollary 10.10 (iii) a compact Riemann surface N of genus g = 1 can be identified with a complex torus $N = \mathbb{C}/\Omega_2\mathbb{Z}^2$ for a 1 × 2 period matrix $\Omega = (\omega_1 \ \omega_2)$, and for any integer δ ≥ 1 the period matrix $\Omega' = (\delta_\omega_1 \ \omega_2)$ describes another complex torus $M = \mathbb{C}/\Omega'\mathbb{Z}$ and the identity mapping $z \rightarrow z$ induces a holomorphic mapping $M \rightarrow N$ that is an unbranched covering of degree δ. That suffices for the proof.

To return to the discussion at the beginning of Chapter 2 of mappings from compact Riemann surfaces to the Riemann sphere $P^1$, but in somewhat more detail, the Riemann sphere $P^1$ was described as the union $P^1 = V_0 \cup V_1$ of two coordinate neighborhoods $V_0 = \{ w_0 \in \mathbb{C} \}$ and $V_1 = \{ w_1 \in \mathbb{C} \}$, where the local coordinates are related by

$$w_1 = 1/w_0 \quad \text{in} \quad V_0 \cap V_1 = \{ w_0 \in \mathbb{C} \mid w_0 \neq 0 \} = \{ w_1 \in \mathbb{C} \mid w_1 \neq 0 \}.$$  

The Riemann surface $P^1$ is often described as the surface arising from the complex plane $\mathbb{C} = V_0$ by the addition of the point $\infty \in P^1$ corresponding to the origin $0 \in V_1$; that identification is quite common and when made in the subsequent discussion it will be indicated by writing $P^1 = \mathbb{C} \cup \infty$. If $f$ is a meromorphic function $f$ on a Riemann surface $M$ and $U_0 \subset M$ is the set of points $z \in M$ at which $f(z)$ is holomorphic while $U_1 \subset M$ is the set of points $z \in M$ at which $f(z) \neq 0$ then the function $f$ determines a holomorphic mapping $\phi_f : M \rightarrow P^1$ by associating to any point $z \in U_0$ the point $\phi_f(z) \in V_0$ with the coordinate $w_0(z) = f(z)$ and to any point $z \in U_1$ the point $\phi_f(z) \in V_1$ with the coordinate $w_1(z) = 1/f(z)$; clearly these definitions are compatible in the intersection $U_0 \cap U_1$ and define a holomorphic mapping of $M$ to $P^1$. Conversely if $\phi : M \rightarrow P^1$ is a holomorphic mapping and $U_0 = \phi^{-1}(V_0)$ while $U_1 = \phi^{-1}(V_1)$ then the function $f$ on $M$ defined by setting $f(z) = w_0(z)$ if $z \in U_0$ and $f(z) = 1/w_1(z)$ if $z \in U_1$ is clearly a well defined meromorphic function on $M$ for which $\phi = \phi_f$. This establishes a one-to-one correspondence between meromorphic functions $f$ on $M$ and holomorphic mappings $\phi_f : M \rightarrow P^1$, which will be used systematically in the subsequent discussion.

Since the mapping $\phi_f$ associated to a meromorphic function is described locally by either the function $f(z)$ or the function $(1/f)(z)$ it follows that the ramification divisor of the mapping $\phi_f$ can be described by

$$r_\alpha(\phi_f) = \begin{cases} 
\deg_a f'(z) & \text{if } a \in U_0, \\
\deg_a (1/f)'(z) & \text{if } a \in U_1,
\end{cases}$$

where these two definitions are easily seen to be compatible for points in $U_0 \cap U_1$. When the divisor of a meromorphic function $f$ is decomposed as the difference $\delta(f) = \delta_+(f) - \delta_-(f)$ of two disjoint positive divisors, the degree of the function $f$ is the common value $\deg f = \deg \delta_+ f = \deg \delta_- f$ as on page 5. The function $f - c$ for any complex number $c$ has the same degree as the function $f$, so its
zero divisor $\vartheta_{+}(f-c)$ is a divisor of degree $\deg f$ and consists of those points at which the function $f$ takes the value $c$, where these points are counted with the appropriate multiplicities. So long as $c$ is not one of the finitely many values taken by the function $f$ at points of the ramification divisor $\tau(\phi_f)$ the divisor $\vartheta_{+}(f-c)$ consists of $\deg f$ distinct points; hence the mapping $\phi_f$ is a mapping of degree $\deg f$, so that
\begin{equation}
\deg \phi_f = \deg f.
\end{equation}

Consequently by the Riemann-Hurwitz Theorem the branch order of the mapping $\phi_f: M \rightarrow \mathbb{P}^1$ from a compact Riemann surface $M$ of genus $g$ to the Riemann sphere described by a meromorphic function $f$ on the surface $M$ is
\begin{equation}
br (\phi_f) = 2 \deg f + 2g - 2.
\end{equation}

For example, there are rational functions of any degree $\delta > 0$ on $\mathbb{P}^1$ so it follows that there are nontrivial holomorphic mappings $\phi_f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of any positive degree $\delta$, for which $\br (\phi_f) = 2(\delta - 1)$ by (11.14). A holomorphic mapping $\phi_f: \mathbb{P}^1 \rightarrow \mathbb{P}^1$ of degree 1 is a biholomorphic mapping, also called an automorphism, of the Riemann sphere; and the set of automorphisms clearly form a group under the composition of mappings. An automorphism $\phi_f$ is described by a meromorphic function $f$ of degree 1, a function with a single simple pole and a single simple zero on $\mathbb{P}^1$, so a function of the form $f(z) = (az + b)/(cz + d)$ for an invertible matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ when the Riemann sphere is viewed as the union $\mathbb{P}^1 = \mathbb{C} \cup \infty$; such functions traditionally are called linear fractional mappings. It is perhaps worth noting incidentally that a linear fractional mapping $f(z)$ actually is just the cross-ratio function $q(z, \alpha; -b/a - d/c)$ for the Riemann sphere, as characterized in Theorem 5.6; in this case it is the uniquely determined meromorphic function on $\mathbb{P}^1$, which is its own universal covering space, having a simple zero at the point $z = -b/a$ and a simple pole at the point $z = -d/c$ and taking the value 1 at the point $\alpha = (d - b)/(a - c)$. At any rate, it is evident from this that there is a uniquely determined automorphism of $\mathbb{P}^1$ that takes any three distinct points to the three points 1, 0, $\infty$ on $\mathbb{P}^1$, or equivalently that takes any three distinct points to any other three distinct points of $\mathbb{P}^1$. Clearly two matrices $A$ and $B$ determine the same automorphism if and only if $A = \epsilon B$ for some nonzero complex constant $\epsilon$; so the group of automorphisms of the Riemann sphere $\mathbb{P}^1$ can be identified with the projective linear group $\text{PGL}(2, \mathbb{C}) = \text{GL}(2, \mathbb{C})/\mathbb{C}^*$, the quotient of the general linear group by the subgroup of diagonal matrices. A rational function $f(z) = (az + b)/(cz + d)$ on $\mathbb{P}^1$ can be written as the quotient $f(z) = f_0(z)/f_1(z)$ of the two affine functions $f_0(z) = az + b$ and $f_1(z) = cz + d$ of the variable $z$, which are unique up to a common constant factor; but this decomposition is really valid just on the coordinate neighborhood of the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$ defined by the variable $z$. In terms of the variable $z_1 = 1/z$ in the other coordinate neighborhood the function $f(z)$ has the corresponding representation $f(z) = f(1/z_1) = (a + bz_1)/(c + dz_1)$ on $\mathbb{P}^1$, and it too can be written as the
quotient \( f(1/z_1) = g_0(z_1)/g_1(z_1) \) of two affine functions \( g_0(z_1) = bz_1 + a \) and 

\( g_1(z_1) = dz_1 + c \). For these functions \( f_0(z) = az + b = z(bz_1 + a) = zg_0(z_1) \) and similarly \( f_1(z) = zg_1(z_1) \), so the pairs of functions \((f_0(z), g_0(z_1))\) and the pair of functions \((f_1(z), g_1(z_1))\) really describe two holomorphic cross-sections of a holomorphic line bundle over \( \mathbb{P}^1 \). That is also the case for mappings of the Riemann sphere \( \mathbb{P}^1 \) to itself of degree \( \delta > 1 \) as well as for mappings from compact Riemann surfaces of genus \( g > 1 \) to the Riemann sphere.

**Theorem 11.3** A meromorphic function \( f \) of degree \( \delta > 0 \) on a compact Riemann surface \( M \) can be written as the quotient \( f = f_{a_1}/f_{a_0} \) of two holomorphic cross-sections with no common zeros for the holomorphic line bundle \( \zeta_{\delta-}(f) \) of the polar divisor of the function \( f \), where \( c(\zeta_{\delta-}(f)) = \delta \); and the cross-sections \( f_{a_0} \) and \( f_{a_1} \) are determined uniquely up to a common constant factor. The line bundle \( \zeta_{\delta-}(f) \) is base-point-free, and any base-point-free holomorphic line bundle on \( M \) of characteristic class \( \delta > 0 \) arises in this way.

**Proof:** If \( f \) is a meromorphic function of degree \( \delta > 0 \) on the surface \( M \) and \( f \) has the divisor \( \vartheta(f) = \vartheta_+(f) - \vartheta_-(f) \), where \( \vartheta_+(f) \) and \( \vartheta_-(f) \) are disjoint positive divisors, these two divisors are linearly equivalent and \( \zeta_{\vartheta_+}(f) = \zeta_{\vartheta_-}(f) \) as in Theorem 1.2. The holomorphic line bundle \( \zeta_{\vartheta_-}(f) \) has characteristic class \( c(\zeta_{\vartheta_-}(f)) = \deg \vartheta_-(f) = \delta \) as in (1.10); and it has a holomorphic cross-section \( f_{a_+} \) with the divisor \( \vartheta(f_{a_+}) = \vartheta_-(f) \), where that cross-section is determined uniquely up to a constant factor. The product \( f_{a_-} f_{a_+} \) is another holomorphic cross-section of the line bundle \( \zeta_{\vartheta_-}(f) \) and has the divisor \( \vartheta(f_{a_-}) = \vartheta_+(f) \); and the function \( f \) is the quotient \( f = f_+/f_- \). Since the divisors \( \vartheta_+(f) \) and \( \vartheta_-(f) \) are disjoint it follows from Lemma 2.9 that the line bundle \( \zeta_{\vartheta_-(f)} \) is base-point-free. Conversely if \( \lambda \) is a base-point-free holomorphic line bundle over \( M \) with \( c(\lambda) = \delta \) then \( \lambda \) has two holomorphic cross-sections \( f_{a_0}, f_{a_1} \) with no common zeros by Lemma 2.9: the quotient \( f = f_{a_0}/f_{a_1} \) is a meromorphic function with polar divisor \( \vartheta_-(f) = \vartheta(f_{a_1}) \), where \( \deg f = \deg \vartheta(f_{a_1}) = c(\lambda) = \delta \), and \( f_{a_0} \) is a holomorphic cross-section of the line bundle \( \zeta_{\vartheta_-(f)} \). That suffices for the proof.

When a meromorphic function \( f \) on a Riemann surface \( M \) is written as the quotient \( f = f_{a_0}/f_{a_1} \) of two holomorphic cross-sections with no common zeros for a holomorphic line bundle \( \lambda \) then \([f_{a_0}(z), f_{a_1}(z)]\) is a well-defined point in the projective space \( \mathbb{P}^1 \) for each point \( z \in U_\alpha \) for the coordinate neighborhood \( U_\alpha \). In an intersection \( U_\alpha \cap U_\beta \) of two coordinate neighborhoods 

\([f_{a_0}(z), f_{a_1}(z)] = [\lambda_{a\beta}(z)f_{a_0}(z), \lambda_{a\beta}(z)f_{a_1}(z)] = [f_{a_0}(z), f_{a_1}(z)]\); therefore the mapping that associates to any point \( z \in U_\alpha \) the point \( F_\lambda(z) = [f_{a_0}(z), f_{a_1}(z)] \) is a well defined holomorphic mapping \( F_\lambda : M \longrightarrow \mathbb{P}^1 \). If \( f_{a_1} \neq 0 \) then 

\([f_{a_0}, f_{a_1}] = [f_{a_0}/f_{a_1}, 1] \) so the mapping \( F_\lambda \) is really the same as the mapping \( \phi_f \) defined by the meromorphic function \( f = f_{a_0}/f_{a_1} \); this alternative description of the mapping is sometimes quite convenient. In analogy with the Brill-Noether matrix (2.36) introduce the holomorphic functions

\[
D_\alpha(z) = \det \begin{pmatrix} f_{a_0}(z) & f'_{a_0}(z) \\ f_{a_1}(z) & f'_{a_1}(z) \end{pmatrix}
\]
in the coordinate neighborhoods \( U_\alpha \subset M \). Since \( f_\alpha = \lambda_{\alpha,\beta} f_{\beta} \) in an intersection \( U_\alpha \cap U_\beta \) it follows as in \( (2.39) \) that
\[
f'_\alpha = \kappa_{\alpha,\beta} \lambda_{\alpha,\beta} f'_{\beta} + \kappa_{\alpha,\beta} \lambda'_{\alpha,\beta} f_{\beta}
\]
in an intersection \( U_\alpha \cap U_\beta \); consequently as in \( (2.40) \)
\[
D_\alpha(z) = \lambda^2_{\alpha,\beta}(z) \kappa_{\alpha,\beta}(z) D_\beta(z)
\]
in an intersection \( U_\alpha \cap U_\beta \), where \( \lambda_{\alpha,\beta}(z) \) are the coordinate transition functions for the line bundle \( \lambda \) and \( \kappa_{\alpha,\beta}(z) \) are the coordinate transition functions for the canonical bundle \( \kappa \). The functions \( \{D_\alpha\} \) thus form a holomorphic cross-section of the line bundle \( \kappa \lambda^2 \).

**Corollary 11.4** On a compact Riemann surface \( M \) the ramification divisor of the holomorphic mapping \( F_\lambda : M \rightarrow \mathbb{P}^1 \) described by two holomorphic cross-sections \( f_{\alpha_0}, f_{\alpha_1} \in \Gamma(M, \mathcal{O}(\lambda)) \) with no common zeros is the divisor \( \tau(F_\lambda) = \mathfrak{d}(D_\alpha) \) of the holomorphic cross-section \( D_\alpha \in \Gamma(M, \mathcal{O}(\kappa \lambda^2)) \).

**Proof:** The mapping \( F_\lambda \) is just the mapping \( \phi_f \) described by the meromorphic function \( f = f_{\alpha_0}/f_{\alpha_1} \), so for any point \( a \in M \) the ramification order of the mapping \( \phi_f \) is given by \( (11.12) \); hence \( \tau_a = \text{ord}_a f'(z) \) if \( f(z) \) is holomorphic at the point \( a \in M \), or equivalently if \( f_{\alpha_1}(a) \neq 0 \), and \( \tau_a = \text{ord}_a (1/f)'(z) \) if \( (1/f)(z) \) is holomorphic at the point \( a \in M \), or equivalently if \( f_{\alpha_0}(a) \neq 0 \). Therefore if \( f_{\alpha_1}(a) \neq 0 \) then since
\[
f'(z) = \left( \frac{f_{\alpha_0}}{f_{\alpha_1}} \right)' = \frac{f_{\alpha_1} f'_{\alpha_0} - f_{\alpha_0} f'_{\alpha_1}}{f_{\alpha_1}^2} = -f_{\alpha_1}^{-1} D_\alpha(z)
\]
it follows that \( \tau_a = \text{ord}_a f'(z) = \text{ord}_a D_\alpha(z) \); on the other hand if \( f_{\alpha_0}(a) \neq 0 \) then since
\[
(1/f(z))' = \left( \frac{f_{\alpha_1}}{f_{\alpha_0}} \right)' = \frac{f_{\alpha_0} f'_{\alpha_1} - f_{\alpha_1} f'_{\alpha_0}}{f_{\alpha_0}^2} = f_{\alpha_0}^{-1} D_\alpha(z)
\]
it follows that \( \tau_a = \text{ord}_a f'(z) = \text{ord}_a D_\alpha(z) \), in that case as well, which suffices for the proof.

Since \( D_\alpha \in \Gamma(M, \mathcal{O}(\kappa \lambda^2)) \) it follows that
\[
\text{deg } \mathfrak{d}(D_\alpha) = c(\kappa \lambda^2) = 2c(\lambda) + 2g - 2,
\]
and since \( f_{\alpha_1} \in \Gamma(M, \mathcal{O}(\lambda)) \) it follows that
\[
c(\lambda) = \text{deg } \mathfrak{d}(f_{\alpha_1}) = \text{deg } F_\lambda
\]
where again \( F_\lambda \) is just the mapping \( \phi_f \) described by the meromorphic function \( f = f_{\alpha_0}/f_{\alpha_1} \); so altogether \( \text{deg } \mathfrak{d}(D_\alpha) = 2 \text{deg } F_\lambda + 2g - 2 \). It then follows from the preceding theorem that
\[
\text{deg } \tau(F_\lambda) = \text{deg } \mathfrak{d}(D_\alpha) = 2 \text{deg } F_\lambda + 2g - 2,
\]
which provides an alternative proof of (11.14) in view of (11.8). The preceding theorem also ties holomorphic mappings \( \phi : M \rightarrow \mathbb{P}^1 \) to the Lüroth semigroup \( \mathcal{L}(M) \) of the compact Riemann surface \( M \), which was defined on page 37 as the additive semigroup of nonnegative integers consisting of the characteristic classes of base-point-free holomorphic line bundles over \( M \). The identity bundle is a base-point free holomorphic line bundle of characteristic class 0, so the Lüroth semigroup begins with the integer 0; and the integers in the Lüroth semigroup can be labeled \( 0 = \delta_0(M) < \delta_1(M) < \delta_2(M) < \cdots \).

**Corollary 11.5** There exists a meromorphic function of degree \( \delta > 0 \) on a compact Riemann surface \( M \), or equivalently there exists a nontrivial holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree \( \delta > 0 \), if and only if \( \delta \) is an integer in the Lüroth semigroup \( \mathcal{L}(M) \) of the surface \( M \).

**Proof:** Theorem 11.3 shows that the base-point-free holomorphic line bundles \( \lambda \) on \( M \) are precisely the line bundles of the polar divisors of meromorphic functions on \( M \), where the degree \( \delta \) of the meromorphic function is the characteristic class \( \delta = c(\lambda) \) of the line bundle; and these are precisely the meromorphic functions describing holomorphic mappings \( \phi : M \rightarrow \mathbb{P}^1 \) of degree \( \delta \), which suffices for the proof.

The least integer \( \delta > 0 \) such that there is a meromorphic function of degree \( \delta \) on the compact Riemann surface \( M \), or equivalently such that there is a nontrivial holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree \( \delta \), is known rather uneuphoniously as the **gonality** of the Riemann surface \( M \); so by Corollary 11.5 the gonality of the surface \( M \) is the integer \( \delta = \delta_1(M) \) in the Lüroth semigroup of \( M \). By Corollary 2.26 (ii) there are base-point-free holomorphic line bundles \( \lambda \) with \( c(\lambda) = 2 \) on elliptic surfaces, Riemann surfaces of genus \( g = 1 \), but none of characteristic class 1 on this or on any compact Riemann surface of genus \( g > 0 \); so \( \delta_1(M) = 2 \) if \( M \) is an elliptic surface. More generally a compact Riemann surface \( M \) of genus \( g > 1 \) is said to be **hyperelliptic** if it admits a nontrivial holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree 2, so if \( \delta_1(M) = 2 \). Hyperelliptic Riemann surfaces have a number of quite special properties, and were the first general class of surfaces beyond elliptic surfaces to be examined in considerable detail. A compact Riemann surface \( M \) of genus \( g > 1 \) is said to be **trigonal** if it admits a nontrivial holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree 3 but is not hyperelliptic, so if \( \delta_1(M) = 3 \); and \( M \) is said to be **quadrigonal** if it admits a nontrivial holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree 4 but is not trigonal or hyperelliptic, so if \( \delta_1(M) = 4 \). The term gonality is thus a rather natural extension of this classical terminology. An alternative characterization is occasionally useful.

**Corollary 11.6** Let \( M \) be a compact Riemann surface of genus \( g > 0 \).

(i) The gonality \( \delta = \delta_1(M) \) of \( M \) is the least integer \( \delta \) such that there is a holomorphic line bundle \( \lambda \) over \( M \) with \( c(\lambda) = \delta \) and \( \gamma(\lambda) = 2 \).

(ii) If \( \lambda \) is a holomorphic line bundle over \( M \) with \( c(\lambda) = \delta_1(M) \) then \( \gamma(\lambda) \leq 2 \), and \( \gamma(\lambda) = 2 \) if and only if \( \lambda \) is base-point-free.
**Proof:** (i) Let $\lambda$ be a holomorphic line bundle over $M$ such that $\gamma(\lambda) = 2$ and $\gamma(\sigma) \leq 1$ whenever $c(\sigma) < c(\lambda)$. Then $c(\lambda) > 0 = \delta_0(M)$ since $\gamma(\lambda) = 2$; and $\gamma(\lambda\zeta_a^{-1}) \leq 1$ for any point $a \in M$, so it follows from Lemma 2.10 that $\lambda$ is base-point-free hence $c(\lambda) \geq \delta_1(M)$. On the other hand no line bundle $\sigma$ over $M$ for which $0 < c(\sigma) < c(\lambda)$ is base-point-free, since no holomorphic line bundle $\sigma$ with $c(\sigma) > 0$ and $\gamma(\sigma) \leq 1$ can be base-point-free on a compact Riemann surface of genus $g > 0$; hence $\delta_1(M) \geq c(\lambda)$ as well, so actually $\delta_1(M) = c(\lambda)$.

(ii) It follows from (i) that $\gamma(\sigma) \leq 1$ whenever $c(\sigma) < \delta_1(M)$. Therefore if $\lambda$ is a line bundle for which $c(\lambda) = \delta_1(M)$ then for any point $a \in M$ it follows that $c(\lambda\zeta_a^{-1}) < c(\lambda) = \delta_1(M)$ so $\gamma(\lambda\zeta_a^{-1}) \leq 1$; and it then follows from Lemma 2.6 that $\gamma(\lambda) \leq \gamma(\lambda\zeta_a^{-1}) + 1 = 2$. If $\gamma(\lambda) = 2$ then $\gamma(\lambda\zeta_a^{-1}) = 1 = \gamma(\lambda) - 1$ for any point $a \in M$ and it follows from Lemma 2.10 that $\lambda$ is base-point-free; and of course if $\gamma(\lambda) = 1$ the bundle $\lambda$ is not base-point-free. That suffices for the proof.

The simplest holomorphic mapping $f : M \rightarrow \mathbb{P}^1$ from a compact Riemann surface $M$ to the Riemann sphere is that associated to a point bundle $\zeta_a^r$ over $M$ as in Theorem 11.3; it is the mapping defined by a meromorphic function $f$ on $M$ that has as its only singularity a pole of order $r$ at a single point $a \in M$, so it is a mapping $\phi_f : M \rightarrow \mathbb{P}^1$ of degree $r$ that is fully ramified at the point $a \in M$ and fully branched over the point $\infty$. A convenient tool in examining such mappings is the local maximal function of the surface $M$ at the point $a \in M$, the mapping $\mu_a : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$
(11.20) \quad \mu_a(r) = \gamma(\zeta_a^r) - 1 \quad \text{for any} \quad r \in \mathbb{Z}
$$

for the point bundle $\zeta_a$.

**Theorem 11.7** The local maximal function $\mu_a(r)$ of a compact Riemann surface $M$ of genus $g > 0$ at a point $a \in M$ satisfies the following conditions:

$$
(11.21) \quad \mu_a(r) \leq \mu_a(r + 1) \leq \mu_a(r) + 1 \quad \text{for any} \quad r \in \mathbb{Z}.
$$

$$
(11.22) \quad \mu_a(r) = \begin{cases} 
-1 & \text{for} \quad r < 0, \\
0 & \text{for} \quad r = 0,1 \\
 r - g & \text{for} \quad r > 2g - 2.
\end{cases}
$$

**Proof:** Lemma 2.6 asserts that $\gamma(\lambda) \leq \gamma(\zeta_a^r) \leq \gamma(\lambda) + 1$ for any holomorphic line bundle $\lambda$ and point bundle $\zeta_a$ on $M$: in particular for $\lambda = \zeta_a^r$ that yields (11.21). Since $\gamma(\zeta_a^r) = 0$ for $r < 0$ it follows that $\mu_a(r) = -1$ for $r < 0$; and since $\gamma(\zeta_a^0) = \gamma(1) = 1$ for any compact Riemann surface and $\gamma(\zeta_a^1) = 1$ for any compact Riemann surface of genus $g > 0$ by Theorem 2.7 it follows that $\mu_a(0) = 0$. By the Riemann-Roch Theorem $\gamma(\zeta_a^r) = \gamma(\kappa\zeta_a^{-r}) + c(\zeta_a^r) + 1 - g$; hence $c(\zeta_a^r) = r$ hence $c(\kappa\zeta_a^{-r}) = 2g - 2 - r < 0$ for $r > 2g - 2$ so $\gamma(\zeta_a^r) = r + 1 - g$ for $r > 2g - 2$ and $\mu_a(r) = r - g$ for $r > 2g - 2$, which suffices for the proof.
For a surface of genus $g = 1$ the local maximal function is fully determined by the preceding theorem and is actually independent of the choice of the point $a \in M$, since Theorem 11.7 shows that

$$\mu_a(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r = 0, \\ r - 1 & \text{if } r > 0, \end{cases}$$

thus in this case $\mu_a(r)$ is actually even independent of the particular Riemann surface of genus $g = 1$ being considered. For $g = 2$ the preceding Theorem 11.7 only shows that

$$\mu_a(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r = 0,1, \\ 0 \text{ or } 1 & \text{if } r = 2, \\ r - 1 & \text{if } r \geq 3, \end{cases}$$

By the Riemann-Roch Theorem in this case $\gamma(\zeta_a^2) = \gamma(\kappa \zeta_a^{-2}) + 1$, and since $c(\kappa \zeta_a^{-2}) = 0$ then $\gamma(\kappa \zeta_a^{-2}) = 1$ if $\kappa \zeta_a^{-2}$ is the trivial line bundle while otherwise $\gamma(\kappa \zeta_a^{-2}) = 0$, so

$$\mu_a(2) = \begin{cases} 1 & \text{if } \kappa = \zeta_a^2, \\ 0 & \text{if } \kappa \neq \zeta_a^2, \end{cases}$$

thus generally $\mu_a(2) = 0$, but if $M$ is a Riemann surface for which the canonical bundle has the form $\kappa = \zeta_a^2$ for some point $a \in M$ then $\mu_a(2) = 1$ for that particular point $a$. This is a model for the values of the local maximal function $\mu_a(r)$ for Riemann surfaces of genus $g > 2$ in the range $2 \leq r \leq 2g - 2$ in which the local critical values are not fully determined by Theorem 11.7. A discussion of the essential properties of mappings satisfying the basic property (11.21) of the local maximal sequence will be taken up in Chapter 13 in connection with the Brill-Noether diagram. For the simple case of the local maximal function the essential results can be derived directly quite easily though, as will be done here. The basic tool in the more explicit description of the local maximal function is the collection of indices

$$r_i(a) = \inf \left\{ r \in \mathbb{Z} \mid \mu_a(r) \geq i \right\},$$

called the local critical values of the Riemann surface $M$ at the point $a \in M$.

**Theorem 11.8** The local critical values at a point $a$ of a compact Riemann surface $M$ of genus $g > 0$ take the following values:

$$r_i(a) = \begin{cases} -\infty & \text{for } i < 0, \\ 0 & \text{for } i = 0, \\ g + i & \text{for } i \geq g, \end{cases}$$
so all indices \( r \geq 2g \) are local critical values; and
\[
(11.28) \quad r_1(a) \geq 2.
\]

For any index \( i \geq 0 \)
\[
(11.29) \quad r_i(a) < r_{i+1}(a) \quad \text{and} \quad \mu_a(r) = i \quad \text{for} \quad r_i(a) \leq r < r_{i+1}(a).
\]

The local critical values at a point \( a \in M \) are characterized by the condition that
\[
(11.30) \quad \mu_a(r) - \mu_a(r - 1) = \begin{cases} 
1 & \text{if } r \text{ is a local critical value at } a \\
0 & \text{otherwise}
\end{cases}
\]

**Proof:** Since \( \mu_a(r) = -1 \) for \( r < 0 \) by (11.22) it follows that \( r_1(a) = -\infty \) whenever \( i < 0 \); and since \( \mu_a(-1) = -1 \) and \( \mu_a(0) = 0 \) by (11.22) it further follows that \( r_0(a) = 0 \). The equation \( \mu_a(r) = r - g \) for all \( r \geq 2g - 1 \) in (11.22) can be rewritten by setting \( i = r - g \) as the equation \( \mu_a(g + i) = i \) for all \( i \geq g - 1 \); hence \( \mu_a(2g - 1) = g - 1, \mu_a(2g) = g, \mu_a(2g + 1) = g + 1, \ldots \), which shows that \( r_1(a) = 2g \) and more generally that \( r_i(a) = g + i \) for all \( i \geq g \), thus demonstrating (11.29). Since \( r_0(a) = 0 \) then \( r_1(a) \geq 1 \); but if \( r_1(a) = 1 \) then \( \gamma(\zeta_1) = \mu_a(1) + 1 = 2 \) and by Theorem 2.4 the Riemann surface \( M \) would be the Riemann sphere \( \mathbb{P}^1 \) of genus \( g = 0 \), which is excluded. From the definition of the local critical values it is evident that \( \mu_a(r_i(a)) \geq i \) and \( \mu_a(r_i(a) - 1) < i \) for any finite values of \( r_i(a) \), so for \( i \geq 0 \); and from (11.24) it is evident that
\[
\mu_a(r_i(a)) \leq \mu_a(r_i(a) - 1) + 1,
\]

which shows that \( \mu_a(r_i(a)) = i \) and \( \mu_a(r_i(a)) < \mu_a(r_{i+1}(a)) \) hence that \( r_i(a) < r_{i+1}(a) \). In view of this it is apparent that \( \mu_a(r) = i \) for \( r_i(a) \leq r < r_{i+1}(a) \), thus demonstrating (11.29). Finally if \( \mu_a(r) = \mu_a(r - 1) \) it is clear from the definition of the local critical values that \( r \) cannot be a local critical value; on the other hand it follows from (11.29) that \( \mu_a(r_i(a)) - \mu_a(r_i(a) - 1) = 1 \) for any local critical value \( r_i(a) \), and that suffices for the proof.

The preceding Theorem 11.8 shows that the finite local critical values at a point \( a \in M \) are nonnegative integers, beginning with
\[
(11.31) \quad 0 = r_0(a) < 1 < r_1(a) < \cdots < r_{g-1}(a) < r_g(a) = 2g
\]

and continuing with all integers \( r > 2g \); but the theorem does not determine the \( g - 1 \) local critical values \( r_1(a), \ldots, r_{g-1}(a) \) in the range \([2, 2g - 1]\) explicitly. The integers that are not local critical values at the point \( a \in M \) are called the local gap values at the point \( a \in M \). The local gap values thus are all integers \( \nu < 0 \) together with the \( g \) integers \( \nu \) in the range \([1, 2g - 1]\) that are not local critical values; the positive local gap values also are called the Weierstrass gaps at the point \( a \) of the Riemann surface \( M \). Since \( \mu_a(r) = 0 \) for \( r = 0 \) and \( r = 1 \)
it follows that $\nu_1(a) = 1$ is a gap value; thus the Weierstrass gaps are $g$ integers in the range

$$1 = \nu_1(a) < \nu_2(a) < \ldots < \nu_g(a) \leq 2g - 1,$$

and $g - 1$ of these are not specified explicitly by the preceding theorem. It follows immediately from these observations that the characterization (11.30) of the local critical values can be extended to assert that

$$\mu_a(r) - \mu_a(r - 1) = \begin{cases} 1 & \text{if } r \text{ is a local critical value at } a \\ 0 & \text{if } r \text{ is a local gap value at } a. \end{cases}$$

The local maximal function at the point $a \in M$ thus can be described either in terms of the local gap values or in terms of the local critical values at the point $a \in M$.

**Theorem 11.9** The holomorphic line bundle $\zeta^*_a$ on a compact Riemann surface $M$ is base-point-free if and only if $r$ is a local critical value at the point $a$; therefore the line bundle $\zeta^*_a$ is not base-point-free if and only if $r$ is a local gap value at the point $a$.

**Proof:** The line bundle $\zeta^*_a$ is base-point-free if and only if $\gamma(\zeta^*_a \zeta^{-1}_x) = \gamma(\zeta^*_a) - 1$ for all $x \in M$, by Lemma 2.10; hence $\zeta^*_a$ fails to be base-point-free if and only if $\gamma(\zeta^*_a \zeta^{-1}_x) = \gamma(\zeta^*_a)$ for some point $x \in M$, and by Lemma 2.6 that is equivalent to the condition that all holomorphic cross-sections of the bundle $\zeta^*_a$ vanish at the point $x$. There is always at least one holomorphic cross-section of $\zeta^*_a$ that vanishes only at the point $a$, the $r$-th power of a holomorphic cross-section of the bundle $\zeta^*_a$; hence the only point $x$ at which all the holomorphic cross-sections of the bundle $\zeta^*_a$ can possibly vanish is the point $x = a$, and the condition that all the holomorphic cross-sections of the line bundle $\zeta^*_a$ vanish at the point $a$ is that $\gamma(\zeta^*_a) = \gamma(\zeta^*_a^{-1})$, or equivalently that $\mu_a(r) = \mu_a(r - 1)$. Therefore the line bundle $\zeta^*_a$ is not base-point-free if and only if $\mu_a(r) = \mu_a(r - 1)$, which by (11.33) is just the condition that $r$ is a local gap value at the point $a$. That suffices for the proof.

**Corollary 11.10** The local critical values of a compact Riemann surface $M$ form an additive semigroup of the integers $\mathbb{Z}$ that is contained in the Lüroth semigroup $\mathcal{L}(M) \subset \mathbb{Z}$ of the surface $M$.

**Proof:** This follows immediately from the preceding theorem and the definition of the Lüroth semigroup, so no further proof is required.

The semigroup of local critical values at a point $a$ of a compact Riemann surface $M$ is called the Weierstrass semigroup\(^1\) at the point $a$ and is denoted

---

\(^{1}\)Hurwitz pointed out the problem of describing precisely which semigroups of nonnegative integers can be such a semigroup $\mathcal{W}_a(M)$ for some point $a$ on some compact Riemann surface $M$. That not all semigroups of nonnegative integers can be such semigroups was first established by R.-O. Buchweitz in his Hanover PhD thesis, 1976; the problem has been studied extensively but is still not fully solved.
by \( \mathcal{W}_a(M) \), so that there is the chain of semigroups
\[
0 \in \mathcal{W}_a(M) \subset \mathcal{L}(M) \subset \mathbb{Z}
\]
at any point \( a \in M \).

Another approach to the local critical values is through the holomorphic abelian differentials and their integrals. For each point \( a \in M \) of a compact Riemann surface \( M \) of genus \( g > 0 \) it is possible to choose a basis \( w_{i,a}(z) \) for the holomorphic abelian integrals normalized to vanish at the point \( a \) and such that
\[
\text{ord}_a w_{i,a}(z) = \rho_i(a) \quad \text{where} \quad 1 = \rho_1(a) < \rho_2(a) < \cdots < \rho_g(a).
\]
Indeed since not all the holomorphic abelian differentials vanish at any point there is an abelian integral \( w_{1,a}(z) \) of order 1 at the point \( a \). After subtracting suitable multiples of \( w_{1,a}(z) \) from the remaining integrals it can be assumed that they all vanish to at least order 2 at \( a \); so let \( w_{2,a}(z) \) be one of the integrals of the least order \( \rho_2(a) \) among them. After subtracting suitable multiples of \( w_{2,a}(z) \) from the remaining integrals it can be assumed that they all vanish to at least order \( \rho_2(a) + 1 \); so let \( w_{3,a}(z) \) can one of the integrals of least order \( \rho_3(a) \) among them, and so on. The integers \( \rho_i(a) \) thus are well defined analytic invariants intrinsically attached to the point \( a \in M \); they are called the normalized orders of the holomorphic abelian integrals at the point \( a \in M \).

It follows immediately from (11.35) that the holomorphic abelian differentials \( \omega_{i,a}(z) = dw_{i,a}(z) \) correspondingly satisfy
\[
\text{ord}_a dw_{i,a}(z) = \rho_i(a) - 1 \quad \text{where} \quad 1 = \rho_1(a) < \rho_2(a) < \cdots < \rho_g(a).
\]
When the \( r \times g \) Brill-Noether matrix \( \Omega(r \cdot a) \) for the divisor \( r \cdot a \), as defined on page 48, is written in terms of the basis \( \omega_{i,a}(z) \) for the holomorphic abelian differentials this matrix has the form
\[
\Omega(r \cdot a) = \begin{pmatrix}
c_{1,1} & * & \cdots & * & \cdots & * & \cdots & * & \cdots \\
0 & 0 & \cdots & c_{2,\rho_2(a)} & \cdots & * & \cdots & * & \cdots \\
0 & 0 & \cdots & 0 & \cdots & c_{3,\rho_3(a)} & \cdots & * & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\
0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots
\end{pmatrix}
\]
where \( c_{i,j} \) denotes a nonzero constant at row \( i \) and column \( j \); for if \( \omega_{i,a}(z) = f_{i,a}(z)dz \) the entry in row \( i \) and column \( j \) of the Brill-Noether matrix is the derivative \( f_{i,a}^{(j-1)}(a) \) and \( f_{i,a}^{(j-1)}(a) = 0 \) for \( j < \rho_i(a) \) while \( f_{i,a}^{(j-1)}(a) \neq 0 \) for \( j = \rho_i(a) \). It is evident from (11.37) that the rank of the matrix \( \Omega(r \cdot a) \) is the number of columns numbered \( \rho_1(a), \rho_2(a), \rho_3(a) \cdots \) among the \( r \) columns of the matrix, so that
\[
\text{rank} \Omega(r \cdot a) = \# \{ \rho_i(a) \leq r \}
\]
and consequently
\[
\text{rank} \Omega(r \cdot a) = \begin{cases} 
\text{rank} \Omega((r - 1) \cdot a) + 1 & \text{if} \quad r = \rho_i(a) \quad \text{for some} \ i, \\
\text{rank} \Omega((r - 1) \cdot a) & \text{otherwise.}
\end{cases}
\]
The Riemann-Roch Theorem for the holomorphic line bundle $\xi_a^r$ can be expressed in terms of the Brill-Noether matrix $\Omega(r \cdot a)$ as in Theorem 2.24, so

\[(11.40) \quad \mu_a(r) = \gamma(\xi_a^r) - 1 = r - \text{rank} \Omega(r \cdot a).\]

**Theorem 11.11** If $\nu_i(a)$ are the Weierstrass gaps and $\rho_i(a)$ are the normalized orders of the holomorphic abelian differentials at a point $a$ of a compact Riemann surface $M$ of genus $g > 0$ then

\[(11.41) \quad \rho_i(a) = \nu_i(a) \quad \text{for} \quad 1 \leq i \leq g.\]

**Proof:** From (11.40) and (11.39) it follows that

\[
\mu_a(r) - \mu_a(r - 1) = \left( r - \text{rank} \Omega(r \cdot a) \right) - \left( r - 1 - \text{rank} \Omega((r - 1) \cdot a) \right)
= 1 - \left( \text{rank} \Omega(r \cdot a) - \text{rank} \Omega((r - 1) \cdot a) \right)
= \begin{cases} 
0 & \text{if } r = \rho_i(a) \text{ for some } i \\
1 & \text{otherwise.}
\end{cases}
\]

Comparing the preceding equation with (11.33) shows that the values $\rho_i(a)$ are precisely the finite local gap values $\nu_i(a)$ at the point $a$, the Weierstrass gaps at the point $a$, and that suffices for the proof.

Any basis $\omega_i(z)$ for the holomorphic abelian differentials on $M$ can be written $\omega_i(z) = f_i(z)dz$ for some holomorphic functions $f_i(z)$ in coordinate neighborhoods $U_\alpha \subset M$ with the local coordinates $z_\alpha$; and for any point $z_\alpha \in U_\alpha$ the $g \times r$ Brill-Noether matrix $\Omega(r \cdot z_\alpha)$ can be viewed as the holomorphic matrix valued function $\Omega(z) = \left\{ f_i^{-1}(z_\alpha) \right\}$ in $U_\alpha$. A point in an intersection $U_\alpha \cap U_\beta$ can be described in terms of either the local coordinate $z_\alpha$ or the local coordinate $z_\beta$. The coefficients of the holomorphic abelian differentials at such a point are related by $f_\alpha(z) = \kappa_{\alpha\beta}(z)f_\beta(z)$ where $\{U_\alpha, \kappa_{\alpha\beta}\}$ is the holomorphic coordinate bundle describing the canonical bundle $\kappa$ of the Riemann surface $M$ as discussed on page 43; and as in equations (2.40) and (2.41) the Brill-Noether matrices in terms of these two local coordinates are related by

\[(11.42) \quad \Omega_\alpha(r \cdot z_\alpha) = \Omega_{\beta}(r \cdot z_\beta)K_{\alpha\beta}(z)\]

for a nonsingular $r \times r$ matrix $K_{\alpha\beta}(z)$ for which

\[(11.43) \quad \det K_{\alpha\beta}(z) = \kappa_{\alpha\beta}(z)^{\frac{1}{2}r(r+1)}.\]

For the special case of the divisor $g \cdot z_\alpha$ of degree $g$ on a Riemann surface $M$ of genus $g$ the Brill-Noether matrix is a $g \times g$ square matrix, so its determinant is a well defined holomorphic function; and it follows from (11.42) and (11.43) that

\[(11.44) \quad \det \Omega_\alpha(g \cdot z_\alpha) = \kappa_{\alpha\beta}^{g(g+1)/2}(z)\det \Omega_{\beta}(g \cdot z_\beta) \quad \text{in } U_\alpha \cap U_\beta,\]
so the determinants of these local Brill-Noether matrices describe a holomorphic cross-section

\[ \det \Omega(g \cdot z) \in \Gamma\left( M, \mathcal{O}(n(g+1)/2) \right). \]

The Brill-Noether matrix in this case is essentially the Wronskian matrix of the local functions \( f_{i\alpha}(z_\alpha) \); indeed it differs from the Wronskian matrix only by the factor \( 1/j! \) in column \( j+1 \), so the function \( \det \Omega_\alpha(g \cdot z_\alpha) \) vanishes if and only if the Wronskian of the local functions \( f_{i\alpha}(z_\alpha) \) vanishes. Since these local functions are linearly independent holomorphic functions the determinant of their Wronskian matrix does not vanish identically\(^2\) and consequently \( \det \Omega(g \cdot z) \) is a nontrivial holomorphic cross-section.

The orders of the functions \( f_{i\alpha}(z) \) at a point \( a \in M \) are as in (11.36), so the Taylor expansions of these functions in a local coordinate \( z_\alpha \) centered at the point \( a \) begin \( f_{i\alpha}(z) = z_\alpha^{\rho_i(a)} + \cdots \). The Taylor expansion of the function \( \det \Omega_\alpha(g \cdot z_\alpha) \) at the point \( a \in M \) can be calculated by replacing the functions \( f_{i\alpha}(z) \) by their Taylor expansions; so the initial term of the Taylor expansion of the function \( \det \Omega_\alpha(g \cdot z_\alpha) \) is given aside from a nonzero constant factor by

\[
\begin{pmatrix}
\frac{1}{z_\alpha^{\rho_1(a)-1}} & \frac{1}{z_\alpha^{\rho_1(a)-2}} & \frac{1}{z_\alpha^{\rho_1(a)-3}} & \ldots & \frac{1}{z_\alpha^{\rho_1(a)-g}} \\
\frac{1}{z_\alpha^{\rho_2(a)-1}} & \frac{1}{z_\alpha^{\rho_2(a)-2}} & \frac{1}{z_\alpha^{\rho_2(a)-3}} & \ldots & \frac{1}{z_\alpha^{\rho_2(a)-g}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{z_\alpha^{\rho_g(a)-1}} & \frac{1}{z_\alpha^{\rho_g(a)-2}} & \frac{1}{z_\alpha^{\rho_g(a)-3}} & \ldots & \frac{1}{z_\alpha^{\rho_g(a)-g}}
\end{pmatrix},
\]

where any terms involving a negative power of the variable \( z_\alpha \) are replaced by 0. The matrix in (11.46) is essentially the Wronskian of the polynomials \( z_\alpha^{\rho_i(a)-1} \), and since these functions are linearly independent the determinant (11.46) does not vanish identically. The determinant (11.46) explicitly is the sum of products of one element from each separate row and column; so since the entry in row \( i \) and column \( j \) is \( z_\alpha^{\rho_i(a)-j} \) it follows that the determinant is a nonzero constant multiple of the variable \( z_\alpha \) to the power \( \sum_{i=1}^{g} \rho_i(a) - \sum_{j=1}^{g} j = \sum_{i=1}^{g} (\rho_i(a) - i) \).

That exponent then is the order of the Brill-Noether matrix \( \Omega_\alpha(g \cdot z_\alpha) \); and since \( \rho_i(a) = \nu_i(a) \), the order is given by

\[ \text{ord}_a \det \Omega_\alpha(g \cdot z_\alpha) = \omega(a) \]

for the integer

\[ \omega(a) = \sum_{i=1}^{g} (\rho_i(a) - i) = \sum_{i=1}^{g} (\nu_i(a) - i). \]

This integer is called the Weierstrass weight of the point \( a \in M \), and a point \( a \in M \) is called a Weierstrass point of \( M \) if \( \omega(a) > 0 \). The Weierstrass points

form a particularly interesting intrinsically defined set of points on any compact Riemann surface. For some purposes it is convenient to have an alternative characterization of these points in terms of the local critical values of the Riemann surface.

**Lemma 11.12** A point \( a \in M \) on a compact Riemann surface \( M \) of genus \( g > 0 \) is a Weierstrass point if and only if \( r_1(a) < g + 1 \); equivalently a point \( a \) is not a Weierstrass point if and only if \( r_1(a) = g + 1 \).

**Proof:** Since the Weierstrass gaps are in the range (11.32) it follows that \( \nu_i(a) \geq i \) for \( 1 \leq i \leq g \), hence that the Weierstrass weight satisfies \( \omega(a) \geq 0 \); therefore the point \( a \) is not a Weierstrass point if and only if \( \omega(a) = 0 \), or equivalently if and only if \( \nu_i(a) = i \) for \( 1 \leq i \leq g \). The local critical values are the complement of the local gap values and \( r_1(a) > 1 \) so \( \nu_i(a) = i \) for \( 1 \leq i \leq g \) if and only if \( r_1(a) = g + 1 \); therefore that is the condition that \( a \) is not a Weierstrass point.

It follows from (11.31) that \( r_1(a) \leq g + 1 \), and therefore \( a \) is a Weierstrass point if and only if \( r_1(a) < g + 1 \), and that suffices for the proof.

**Theorem 11.13** There are only finitely many Weierstrass points on a compact Riemann surface \( M \) of genus \( g > 0 \), and the Weierstrass weights at these points satisfy

\[
\sum_{a \in M} \omega(a) = (g - 1)g(g + 1).
\]

**Proof:** The functions \( \det \Omega_{\alpha}(g \cdot z_{\alpha}) \) describe a holomorphic cross-section (11.45), so the degree of the divisor of this cross-section is equal to the characteristic class \( c(k^{g+1}/2) = (2g - 2)k^{g+1}/2 = (g - 1)g(g + 1) \) of the holomorphic line bundle \( k^{g+1}/2 \). By (11.47) the degree of the divisor of the function \( \det \Omega_{\alpha}(g \cdot z_{\alpha}) \) at a point \( a \in M \) is just the Weierstrass weight \( \omega(a) \) of the point \( a \), so the sum of the weights \( \omega(a) \) at all points \( a \in M \) is equal to \( c(k^{g+1}/2) \), and that yields equation (11.49). It follows from this equation that there are only finitely many points \( a \in M \) at which \( \omega(a) > 0 \), that is, there are only finitely many Weierstrass points, and that suffices for the proof.

For a compact Riemann surface \( M \) of genus \( g = 1 \) it follows from (11.49) that there are no Weierstrass points on \( M \); of course that is obvious, since the holomorphic abelian differential on \( M \), which is uniquely determined aside from a constant factor, has no zeros. At any point \( a \in M \) that is not a Weierstrass point, so at any point at which \( \omega(a) = 0 \), it follows directly from (11.48) that \( \rho_i(a) = \nu_i(a) = i \) for \( 1 \leq i \leq g \); thus for all but finitely many points of the Riemann surface the normalized orders of the holomorphic abelian integrals, or equivalently the sequence of Weierstrass gaps, at that point is \((1, 2, \ldots, g)\), the lowest possible values.
Theorem 11.14 The Weierstrass weight $\omega(a)$ of a Weierstrass point $a \in M$ on a compact Riemann surface $M$ of genus $g > 1$ satisfies

\begin{equation}
1 \leq \omega(a) \leq \frac{1}{2}g(g-1);
\end{equation}

the minimal value $\omega(a) = 1$ is attained when the Weierstrass gap sequence at the point $a$ is

\begin{equation}
(1, 2, \ldots, g-1, g+1),
\end{equation}

while the maximal value $\omega(a) = \frac{1}{2}g(g-1)$ is attained when the Weierstrass gap sequence at the point $a$ is

\begin{equation}
(1, 3, 5, \ldots, 2g-3, 2g-1).
\end{equation}

Proof: By (11.48) the Weierstrass weight at a point $a \in M$ is given by $\omega(a) = \sum_{i=1}^{g} (\nu_i(a) - i)$. If $\nu_j(a) \geq j + 1$ for some index $1 \leq j \leq g$ then $\nu_i(a) \geq i + 1$ for all $i \geq j$ so that $\omega(a) \geq g - j$; the least Weierstrass weight at a Weierstrass point thus is $\omega(a) = 1$, which is the first inequality in (11.50), and that occurs when $\nu_j = j$ for $1 \leq j \leq g$ and $\nu_g = g + 1$, which shows that the associated Weierstrass gap sequence is (11.51).

If $r = r_1(a) \geq 2$ is the least local critical value at a Weierstrass point $a \in M$ then $2r, 3r, \ldots$ are also local critical values at $a \in M$, since the local critical values form a semigroup in $\mathbb{Z}$ by Corollary 11.10. Each integer $i$ in the range $1 \leq i \leq r - 1$ then is a Weierstrass gap at the point $a \in M$; and for each of these integers there is a further integer $\lambda_i \geq 0$ such that

\begin{equation}
i, (i + r), (i + 2r), \ldots, (i + \lambda_i r) \quad \text{are Weierstrass gaps at } a
\end{equation}

while

\begin{equation}i + (\lambda_i + 1)r, i + (\lambda_i + 2)r, \ldots \quad \text{are local critical values at } a.
\end{equation}

Any integer is congruent to one of the integers $1 \leq i \leq r - 1$ modulo $r$ so all the Weierstrass gaps are included in the lists (11.53); hence the total number of Weierstrass gaps at $a$ is

\begin{equation}g = \sum_{i=1}^{r-1} (\lambda_i + 1) = r - 1 + \sum_{i=1}^{r-1} \lambda_i.
\end{equation}
By (11.48) the Weierstrass weight of the point \( a \) is

\[
\omega(a) = \sum_{i=1}^{g} (\nu_i(a) - i) = \sum_{i=1}^{g} \nu_i(a) - \frac{1}{2} g(g + 1)
\]

\[
= \sum_{i=1}^{r-1} \sum_{j=0}^{r-1} (i + jr) - \frac{1}{2} g(g + 1)
\]

\[
= \sum_{i=1}^{r-1} \left( i(\lambda_i + 1) + \frac{1}{2} i \lambda_i (\lambda_i + 1) \right) - \frac{1}{2} g(g + 1)
\]

\[
= \frac{1}{2} r(r - 1) + \sum_{i=1}^{r-1} \frac{1}{2} \lambda_i (2i + r \lambda_i + r) - \frac{1}{2} g(g + 1).
\]

Here \( i \leq r - 1 \), and since the largest Weierstrass gap is \( \nu_g(a) \leq 2g - 1 \) by (11.32) then \( i + \lambda_i r \leq 2g - 1 \); consequently

\[
2i + r \lambda_i + r = i + (i + r \lambda_i) + r 
\]

\[
\leq (r - 1) + (2g - 1) + r = 2(g + r - 1)
\]

so recalling (11.55)

\[
\sum_{i=1}^{r-1} \frac{1}{2} \lambda_i (2i + r \lambda_i + r) \leq (g + r - 1) \sum_{i=1}^{r-1} \lambda_i = (g + r - 1) (g - r + 1) = g^2 - (r - 1)^2.
\]

Substituting this inequality in (11.56) shows that

\[
\omega(a) \leq \frac{1}{2} r(r - 1) + g^2 - (r - 1)^2 - \frac{1}{2} g(g + 1)
\]

\[
\leq \frac{1}{2} g(g - 1) - \frac{1}{2} (r - 1)(r - 2).
\]

It is evident from this inequality that the largest Weierstrass weight at a Weierstrass point is \( \frac{1}{2} g(g - 1) \), which is the second inequality in (11.50); and since \( r \geq 2 \) that occurs when \( r = 2 \), so that all even numbers are local critical values at the point \( a \) and hence the Weierstrass gap sequence consists just of the odd integers (11.58). That suffices for the proof.

If all of the Weierstrass points on \( M \) have the minimal Weierstrass weight \( \omega(a) = 1 \) it is evident from (11.49) in Theorem 11.13 that the number of Weierstrass points on \( M \) is \( (g - 1) g(g + 1) \), and that of course is the largest possible number of Weierstrass points on \( M \). On the other hand if all the Weierstrass points on \( M \) have the maximal Weierstrass weight \( \frac{1}{2} g(g - 1) \) it is evident from (11.49) that the number of Weierstrass points on \( M \) is \( 2g + 2 \), and that is the minimal number of Weierstrass points on \( M \). Altogether therefore the number \( N \) of Weierstrass points on \( M \) satisfies

\[
2(g + 1) \leq N \leq (g - 1) g(g + 1),
\]
where the largest value is taken at a Riemann surface for which all the Weierstrass gap sequences are (11.51) while the least value is taken at a Riemann surface for which all the Weierstrass gap sequences are (11.52). A slight refinement of the preceding theorem is occasionally useful.

**Corollary 11.15** On a compact Riemann surface $M$ of genus $g > 1$ the Weierstrass weight at a point $a \in M$ at which the first local critical value is $r_1(a)$ satisfies

$$
\omega(a) \leq \frac{1}{2} g(g - 1) - \frac{1}{2} (r_1(a) - 1)(r_1(a) - 2).
$$

**Proof:** The inequality (11.59) is just the inequality (11.57) in the proof of the preceding theorem, so no further proof is required here.

The collection of Weierstrass points on a compact Riemann surface $M$ of genus $g > 1$ is an intrinsically defined finite set of exceptional points of the Riemann surface $M$. Actually the Weierstrass points can be distinguished by their weights, or rather more precisely by their Weierstrass gap sequences; and each of these separate subsets of Weierstrass points is an intrinsically defined set of exceptional points on the Riemann surface $M$. Weierstrass points play a variety of roles in the study of compact Riemann surfaces; that can be illustrated by examining one of the classical special classes of Riemann surfaces, the hyperelliptic Riemann surfaces as defined on page 297.

**Theorem 11.16** Hyperelliptic Riemann surfaces of genus $g > 1$ are characterized by any of the following equivalent conditions:

(i) There is a point $a$ at which the first critical value is $r_1(a) = 2$.
(ii) The surface has a point at which the Weierstrass gap sequence is (11.52).
(iii) The surface has a Weierstrass point of the maximal possible Weierstrass weight $\omega(a) = \frac{1}{2} g(g - 1)$.
(iv) The surface has the minimal possible number $2(g+1)$ of Weierstrass points.

**Proof:**

(i) If $M$ is a hyperelliptic then by definition there is a holomorphic mapping $\phi : M \to \mathbb{P}^1$ of degree 2, which as on page 293 is the mapping $\phi_f$ described by a meromorphic function $f$ of degree 2. The mapping $\phi_f$ is not a homeomorphism since $g > 1$ so it must have at least one ramification point $a \in M$, which must be of ramification order 1; and after replacing the function $f$ by $1/(f - f(a))$ if necessary it can be assumed that the function $f$ has a pole at the ramification point $a$, which must be a double pole. Then by Theorem 11.3 the function $f$ can be written as the quotient of two holomorphic cross-sections of the holomorphic line bundle $\mathcal{O}_a^2$ of the polar divisor of $f$; so $\gamma(\mathcal{O}_a^2) \geq 2$ hence $r_1(a) = 2$. Conversely if $r_1(a) = 2$ then 2 is in the Lüroth semigroup by Corollary 11.10 hence $M$ is hyperelliptic.

(ii) If the Weierstrass gap sequence at a point $a$ is (11.52) then the first local critical value at the point $a$ is $r_1(a) = 2$, so the surface is hyperelliptic by (i). Conversely if the surface is hyperelliptic then $r_1(a) = 2$ for some point $a \in M$ by (ii); and since the local critical values form a semigroup under addition, all
even values are local critical values hence the Weierstrass gap sequence must be (11.52).

(iii) If the surface has a Weierstrass point of the maximal possible Weierstrass weight then by Theorem 11.14 the Weierstrass gap sequence at that point is (11.52), and conversely; and by (ii) that is equivalent to the surface being hyperelliptic.

(iv) If the surface has the minimal possible number of Weierstrass points then each Weierstrass point has the maximal Weierstrass weight, and conversely; so by (iii) that is equivalent to the surface being hyperelliptic. That suffices for the proof.

**Theorem 11.17** A hyperelliptic Riemann surface \( M \) of genus \( g > 1 \) has a unique representation as a two sheeted branched covering of \( \mathbb{P}^1 \), up to biholomorphic mappings of \( \mathbb{P}^1 \) to itself; the \( 2g + 2 \) ramification points are precisely the Weierstrass points of \( M \), each of which has the Weierstrass gap sequence (11.52).

**Proof:** By definition a hyperelliptic Riemann surface \( M \) has a holomorphic mapping \( \phi : M \to \mathbb{P}^1 \) of degree 2, which can be taken as the mapping \( \phi = \phi_f \) defined by a meromorphic function \( f \) of degree 2 on the surface \( M \) as in the discussion on page 293. For a mapping \( \phi_f \) of degree 2 all ramification points \( a \in M \) clearly have the ramification order \( r_a(\phi_f) = 1 \); and since the Riemann-Hurwitz formula takes the form (11.14), when \( \deg f = 2 \) the total branch order is \( \text{br}(\phi_f) = 2g + 2 \) so there are altogether \( 2g + 2 \) ramification points of the mapping. At any ramification point \( a \in M \) the function \( f \) defining the mapping \( \phi = \phi_f \) can be assumed to have a double pole, by replacing \( f \) by \( 1/(f - f(a)) \) if necessary; that amounts to composing the mapping \( \phi_f \) with the biholomorphic mapping of \( \mathbb{P}^1 \) to itself defined by \( t \to 1/(t - f(a)) \). As in Theorem 11.3 the function \( f \) then can be written as a quotient \( f = f_{\alpha_0}/f_{\alpha_1} \) of two holomorphic cross-sections of the holomorphic line bundle of the polar divisor of the function \( f \), the line bundle \( \zeta^2_a \), which is base-point free and for which \( \gamma(\zeta^2_a) = 2 \) since the bundle has the two holomorphic cross-sections \( f_{\alpha_0}, f_{\alpha_1} \) and \( \gamma(\zeta^2_a) \leq 2 \) by Theorem 2.7. On the one hand that means that the first local critical value at the point \( a \) is \( r_1(a) = 2 \), so \( a \) is a Weierstrass point with the Weierstrass gap sequence (11.52). That is the case for any of the \( 2g + 2 \) ramification points of the mapping \( f \), so all the ramification points are Weierstrass points; and since by (11.49) there are also altogether \( 2g + 2 \) Weierstrass points, all of which must be ramification points. On the other hand the cross-sections \( f_{\alpha_0}, f_{\alpha_1} \) are a basis for the holomorphic cross-sections of the line bundle \( \zeta^2_a \), and \( f_{\alpha_1} \) has a double zero at the point \( a \in M \). Any other meromorphic function \( g \) on the surface \( M \) having a double pole at the point \( a \in M \) and defining a mapping \( \phi_g : M \to \mathbb{P}^1 \) of degree 2 can also be expressed in the same way as a quotient \( g = g_{\alpha_0}/g_{\alpha_1} \) of two holomorphic cross-sections of the line bundle \( \zeta^2_a \), where \( g_{\alpha_1} = cf_{\alpha_1} \) for some constant \( c \neq 0 \) since \( g_{\alpha_1} \) also has a double zero at the point \( a \in M \) and \( g_{\alpha_0} = c_0f_{\alpha_0} + c_1f_{\alpha_1} \) for some further constants \( c_0, c_1 \); thus \( g = (c_0/c)f + (c_1/c) \),
so the mapping \( \phi_g \) is the composition of the mapping \( \phi_f \) and the biholomorphic mapping \( t \rightarrow (c_0/c)t + (c_1/c) \) of \( \mathbb{P}^1 \) to itself. That suffices for the proof.

If \( M \) is a hyperelliptic Riemann surface and \( \phi : M \rightarrow \mathbb{P}^1 \) is the unique holomorphic mapping exhibiting \( M \) as a two-sheeted branched covering of \( \mathbb{P}^1 \) then for each point \( z \in M \) that is not a ramification point of the mapping \( \phi \) there is a unique point \( Tz \in M \) other than \( z \) for which \( \phi(Tz) = \phi(z) \); the mapping \( T : z \rightarrow Tz \), extended by setting \( Tz = z \) for each ramification point \( z \in M \), is called the hyperelliptic involution on \( M \), which thus is uniquely defined on any hyperelliptic Riemann surface. It is clear from its definition that the hyperelliptic involution is a one-to-one mapping of the surface \( M \) to itself, that its square is the identity mapping \( T^2 = I \) on \( M \), and that \( T \) is locally biholomorphic except possibly for the ramification points; but it follows from the Riemann removable singularities theorem that the mapping is also holomorphic at the ramification points, since it is continuous at these points and holomorphic except at these points. Thus the mapping \( T : M \rightarrow M \) is an automorphism of the Riemann surface \( M \), a biholomorphic mapping of the surface \( M \) to itself. The quotient of the Riemann surface \( M \) under the group of transformations \( \{ T, I \} \) is just the Riemann sphere; and conversely any compact Riemann surface \( M \) with a biholomorphic involution \( T \) such that \( M/\{ T, I \} = \mathbb{P}^1 \) is obviously a hyperelliptic Riemann surface. The description of other possible automorphisms rests on their action on the Weierstrass points of the surface.

**Corollary 11.18** An automorphism of a hyperelliptic Riemann surface \( M \) of genus \( g > 1 \) permutes the Weierstrass points of \( M \); and an automorphism fixes all the Weierstrass points if and only if the automorphism is the hyperelliptic involution or the identity mapping.

**Proof:** Since the Weierstrass points on \( M \) are intrinsically determined they are clearly mapped to themselves by any automorphism of the surface. If \( f : M \rightarrow \mathbb{P}^1 \) is the unique mapping exhibiting the Riemann surface \( M \) as a two-sheeted cover of the Riemann sphere and if \( \phi : M \rightarrow M \) is an automorphism of the Riemann surface \( M \) then the composite mapping \( f \circ \phi^{-1} : M \rightarrow \mathbb{P}^1 \) also exhibits the surface \( M \) as a two-sheeted cover of the Riemann sphere, so by the preceding theorem \( f \circ \phi^{-1} = \theta \circ f \) for an automorphism \( \theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \), a linear fractional transformation of the projective space \( \mathbb{P}^1 \). If the automorphism \( \phi \) preserves the Weierstrass points, the ramification points of the mapping \( f \), then the branch points of \( f \) must be preserved by the mapping \( \theta : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \); but there are \( 2g + 2 > 3 \) of these points, so the mapping \( \theta \) is linear fractional transformation leaving at least 3 points fixed hence \( \theta \) is the identity transformation. Consequently the automorphism \( \phi \) at most interchanges the two points \( f^{-1}(a) \) over each point \( a \in \mathbb{P}^1 \) so \( \phi \) is either the identity mapping or the hyperelliptic involution, and that suffices for the proof.

**Corollary 11.19** If \( M \) is a hyperelliptic Riemann surface of genus \( g \) with the hyperelliptic involution \( T : M \rightarrow M \) then \( \omega(Tz) = -\omega(z) \) for any holomorphic
abelian differential on $M$; hence the canonical divisors on $M$ are precisely the divisors of the form $\ell = \sum_{j=1}^{g-1} (1 \cdot a_j + 1 \cdot Ta_j)$ for any points $a_1, \ldots, a_{g-1} \in M$.

**Proof:** If $\omega$ is a holomorphic abelian differential on $M$ then $\omega(Tz)$ also is a holomorphic abelian differential on $M$, since the hyperelliptic involution $T : M \to M$ is a biholomorphic mapping. The sum $\omega(z) + \omega(Tz)$ then is a holomorphic abelian differential on $M$ that is invariant under the hyperelliptic involution $T$, so determines a holomorphic abelian differential on the quotient space $M/I = \mathbb{P}^1$; but there are no nontrivial holomorphic abelian differentials on $\mathbb{P}^1$ and consequently $\omega(z) + \omega(Tz) = 0$. It follows from this that if $\omega(a) = 0$ for a point $a \in M$ then $\omega(Ta) = 0$ as well, so the divisor of the holomorphic abelian differential $\omega(z)$ must be of the form $\mathfrak{d}(\omega) = \sum_{j=1}^{g-1} (1 \cdot a_j + 1 \cdot Ta_j)$ for some points $a_1, \ldots, a_{g-1}$ on $M$. Conversely for any divisor of the form $\mathfrak{d} = \sum_{j=1}^{g-1} (1 \cdot a_j + 1 \cdot Ta_j)$ there is a nontrivial holomorphic abelian differential $\omega(z)$ such that $\omega(a_j) = 0$ for $1 \leq j \leq g-1$, since there are $g$ linearly independent holomorphic abelian differentials on $M$; and since $\omega(Tz) = \omega(z)$ the divisor of this differential is $\mathfrak{d}(\omega) = \sum_{j=1}^{g-1} (1 \cdot a_j + 1 \cdot Ta_j)$, and that suffices for the proof.

The Weierstrass points on any general compact Riemann surface are also useful in the examination of automorphisms of the surface.

**Theorem 11.20** An automorphism of a compact Riemann surface of genus $g > 1$ that is not hyperelliptic permutes the Weierstrass points of the surface; and an automorphism fixes all the Weierstrass points if and only if the automorphism is the identity mapping.

**Proof:** The Weierstrass points on any compact Riemann surface $M$ are intrinsically defined so they are mapped to themselves by any automorphism $T : M \to M$ of the surface. Suppose $T$ is a nontrivial automorphism of the Riemann surface $\mathcal{R}$ that fixes the Weierstrass points of $M$. There must be a point $a \in M$ that is not a Weierstrass point and that is not preserved by the mapping $T$, since there are only finitely many Weierstrass points so the complement of the set of Weierstrass points is a dense subset of $M$ and any automorphism that is the identity on a dense subset is the identity automorphism. Since $a$ is not a Weierstrass point $r_1(a) = g + 1$ by Lemma 11.12, so $\gamma_a = 2$. One cross-section $f_0$ of $\zeta_a$ can be taken to be a power of a nontrivial cross-section of $\zeta_a$, so it has a zero of order $g + 1$ at the point $a$, while the other cross-section $f_1$ is necessarily nonzero at the point $a$, since $\zeta_a^{g+1}$ is base-point-free; the quotient $f = f_1/f_0$ is a nontrivial meromorphic function on $M$ with a pole of order $g + 1$ at the point $a$ and no other singularities. Since $Ta \neq a$ the function $g = f - f \circ T$ is a nontrivial meromorphic function on $M$ with poles of order $g + 1$ at the points $a$ and $Ta$ but no other singularities on $M$, so it is a meromorphic function of degree $2g + 2$; its zero divisor then is a positive divisor of degree $2g + 2$, so consists of at most $2g + 2$ points of $M$. However, for each Weierstrass point $a \in M$ so the the function $g = f - f \circ T$ must vanish at each Weierstrass point of $M$; and if $M$ is not hyperelliptic then $g > 1$ and by Theorem 11.16 (iii) there are $N > 2g + 2$ Weierstrass points on
$M$, so the function $g$ has at least $N > 2g + 2$ zeros. That is a contradiction, so any automorphism $\theta$ of $M$ that preserves the Weierstrass points must be the identity mapping, which concludes the proof.

**Corollary 11.21** The group of automorphisms of a compact Riemann surface of genus $g > 1$ is finite.

**Proof:** For a hyperelliptic Riemann surface of genus $g > 1$ this follows from Corollary 11.18, since every automorphism preserves the finite set of Weierstrass points and the only automorphisms that fix the Weierstrass points are the identity mapping and the hyperelliptic involution. For a non-hyperelliptic Riemann surface of genus $g > 1$ this follows from Theorem 11.20, since every automorphism preserves the finite set of Weierstrass points and the only automorphism that fixes the finitely many Weierstrass points is the identity mapping. That suffices for the proof.

The determination of those compact Riemann surfaces that have nontrivial groups of automorphisms and the classification of the finite groups that can be groups of automorphisms of compact Riemann surfaces are topics that have been investigated extensively; but the discussion here will be limited to an examination of the quotient mapping for Riemann surfaces with nontrivial automorphisms.

**Theorem 11.22** If $G$ is a group of order $N$ that is the group of automorphisms of a compact Riemann surface $M$ of genus $g > 1$ the quotient space $M/G$ has the natural structure of a compact Riemann surface of genus $h \geq 0$ for which the quotient mapping $\pi : M \rightarrow M/G$ is a holomorphic mapping of degree $N$. If $B(\pi)$ is the branch locus of the mapping $\pi$ and $b(\pi) = \sum_{q \in B(\pi)} \nu_q \cdot q$ is the branch divisor of the mapping $\pi$ then $\nu_q$ is the order of the subgroup of $G$ that fixes any one of the points $p \in \pi^{-1}(q)$; the total branch order of the mapping $\pi$ is

$$
(11.60) \quad \text{br}(\pi) = N \sum_{q \in B(\pi)} \left(1 - \frac{1}{\nu_q}\right),
$$

and these invariants are related by

$$
(11.61) \quad 2g - 2 = N(2h - 2) + N \sum_{q \in B(\pi)} \left(1 - \frac{1}{\nu_q}\right).
$$

**Proof:** The group $G$ of automorphisms of any compact Riemann surface $M$ of genus $g > 1$ is a finite group by Corollary 11.21. If $\pi : M \rightarrow M/G$ is the natural mapping from $M$ to the quotient space $M/G$ let $X \subset M$ be the set of those points of $M$ that are fixed under an automorphism in $G$. Each automorphism is a holomorphic mapping so its fixed points are a discrete subset of the compact manifold $M$, hence a finite set of points of $M$; and since the group $G$ is finite the set $X$ also is finite. For a sufficiently small open neighborhood $U_p$ of a point
$p \in (G \sim X)$ each automorphism $T \in G$ determines a biholomorphic mapping $T : U_p \to TU_p$ and the images $TU_p$ for distinct automorphisms $T \in G$ are disjoint subsets of $M$; therefore the neighborhood $U_p$ can be identified with an open neighborhood $V_p$ of the quotient space $M/G$, thus providing the structure of a Riemann surface on the quotient $(M \sim X)/G$. The quotient mapping $\pi$ is then a holomorphic mapping exhibiting $M \sim G$ as a covering map of $N$ sheets over the quotient manifold $(M \sim G)/G$. On the other hand if $p \in X$ the set of automorphisms $T \in G$ such that $Tp = p$ form a finite subgroup $G_p \subset G$; and if $G_p$ is of order $\nu_p$ then $\nu_p|N$. If $S \in G$ and $S \notin G_p$ then $STS^{-1} : Sp = STp = Sp$ so $Sp \in X$ and the subgroup $G_Sp = SG_pS^{-1} \subset G$ of those automorphisms leaving the point $Sp$ fixed has the same order $\nu_p$ as the subgroup $G_p$. If $U_p$ is an open neighborhood of the point $p$ the intersection $U_p' = \bigcap_{T \in G_p} TU_p \subset U_p$ is an open subneighborhood of the point $p$ that is mapped to itself by any automorphism in the subgroup $G_p$; so it can be assumed that there are arbitrarily small open neighborhoods $U_p$ of the point $p$ that are preserved by the group $G_p$. If $z$ is a local coordinate centered at the point $p$ in such a neighborhood $U_p$ the product $h_p(z) = \prod T_i z$ is a holomorphic function in $U_p$ that is invariant under the action of the subgroup $G_p$; and if the neighborhood $U_p$ is sufficiently this function determines a bijective holomorphic mapping $h_p : U_p/G_p \to V_p$ to an open subset $V_p \subset \mathbb{C}$, and in this way the set $V_p$ can be identified with an open neighborhood of the point $\pi(p) \in M/G_p$. Further if $G = \bigcup T_i G_p$ is the coset decomposition of the group $G$ where $1 \leq i \leq N/\nu_p$ and if $U_p$ is sufficiently small then each holomorphic automorphism $T_i$ determines a biholomorphic mapping $T_i : U_p \to T_iU_p$ and the images $T_iU_p$ are disjoint subsets of $M$; therefore the neighborhood $V_p$ actually can be identified with an open neighborhood of the point $\pi(p)$ in the quotient space $M/G$. These neighborhoods then provide an extension of the Riemann surface structure on $(M \sim X)/G$ to a Riemann surface structure on $M/G$ and the quotient mapping $\pi : M \to M/G$ is a holomorphic mapping between these two surfaces. Each point $p \in X$ is a ramification point of order $\nu_p - 1$ for this mapping and there are $N/\nu_p$ of these ramification points that have the same image $\pi(p) = q \in M/G$ and the same ramification order; the branch locus $B(\pi)$ of the mapping $\pi$ therefore is the image $B(\pi) = \pi(X)$, and at each point $q \in B(\pi)$ the local branch order is $b_q(\pi) = \sum_{\nu_q} (nu_q - 1) = N \left(1 - \frac{1}{\nu_q}\right)$, which is (11.60). Substituting these observations into the Riemann-Hurwitz equation (11.9) then yields (11.61), and that suffices for the proof.

The invariants $\nu_q$ are the orders of the subgroups of $G$ that fix a point $p \in \pi^{-1}(q)$, so these invariants are closely related to the structure of the group $G$. If the group $G$ has no fixed points then (11.61) reduces to the relation

$$g - 1 = N(h - 1)$$

(11.62)

relating the order of the group $G$ and the genus of the quotient space $M/G$.

Since $\nu_q \geq 2$ it follows that $\left(1 - \frac{1}{\nu_q}\right) \geq \frac{1}{2}$ If $h \geq 2$ it follows from (11.9) that $2g - 2 \geq 2N$ so $N \leq g - 1$
Theorem 11.23 If \( M \) is a compact Riemann surface of genus \( g > 0 \) then for any point \( a \in M \) there is a holomorphic mapping \( \phi_a : M \rightarrow \mathbb{P}^1 \) of degree \( r > 1 \) that is fully ramified at that point if and only if \( r \) is a local critical value of the Riemann surface \( M \) at the point \( a \).

Proof: If \( \phi : M \rightarrow \mathbb{P}^1 \) is a holomorphic mapping of degree \( r \) that is fully ramified at the point \( a \) it can be assumed by choosing suitable coordinates on \( \mathbb{P}^1 \) that \( \phi(a) = \infty \) where \( \mathbb{P}^1 = \mathbb{C} \cup \infty \); so \( \phi = \phi_f \) actually is the holomorphic mapping described by a meromorphic function \( f \) on \( M \) that is of degree \( r \) and that has a pole of order \( r \) at the point \( a \) as its sole singularity. By Theorem 11.3 this function can be written as the quotient \( f = f_{\alpha_1}/f_{\alpha_0} \) of two holomorphic cross-sections with no common zeros for the holomorphic line bundle \( \mathcal{O}_a^r \) of the polar divisor \( \cdot a \) of the function \( f \). The line bundle \( \mathcal{O}_a^r \) therefore is a base-point-free holomorphic line bundle, and it follows from Theorem 11.9 that \( r \) is a local critical value of the Riemann surface \( M \) at the point \( a \). That suffices for the proof.

Corollary 11.24 If \( M \) is a compact Riemann surface of genus \( g > 0 \) there is a holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree \( 1 < r < g + 1 \) that is fully ramified at a point \( a \in M \) if and only if \( a \) is one of the finitely many Weierstrass points of \( M \).

Proof: The preceding theorem shows that there is a holomorphic mapping \( \phi : M \rightarrow \mathbb{P}^1 \) of degree \( r < g + 1 \) that is fully ramified at the point \( a \in M \) if and only if \( r < g + 1 \) is a local critical point of the the surface \( M \); at such a point \( r_1(a) \leq r < g + 1 \), and since a point \( a \in M \) is a Weierstrass point of \( M \) if and only if \( r_1(a) < g + 1 \) by Lemma 11.12 it follows that these points \( a \) are precisely the Weierstrass points of \( M \). That suffices for the proof.

Lemma 11.25 If \( M \) is a compact Riemann surface of genus \( g > 0 \) then for any point \( a \in M \) there are holomorphic cross-sections \( f_{i,a} \in \Gamma(M, \mathcal{O}(\mathcal{O}_a^r(a))) \) for which \( f_{i,a}(a) \neq 0 \) if \( i > 0 \) while \( f_{0,a}(a) \) has a simple zero at \( a \); and for any integer \( r_1(a) \leq r < r_{i+1}(a) \) for \( i > 0 \) the vector space \( \Gamma(M, \mathcal{O}(\mathcal{O}_a^r)) \) has dimension \( i + 1 \) and has a basis consisting of the holomorphic cross-sections \( f_{0,a} \) and \( f_{j,a} \) for \( 1 \leq j \leq i \).

Proof: The vector space \( \Gamma(M, \mathcal{O}(\mathcal{O}_a^r)) \) of holomorphic cross-sections of the line bundle \( \mathcal{O}_a^r \) is one-dimensional and is spanned by a holomorphic cross-section \( f_{0,a} \in \Gamma(M, \mathcal{O}(\mathcal{O}_a^r)) \) that has a simple zero at the point \( a \in M \). On the other hand for any local critical value \( r_i(a) \) for \( i > 0 \) the line bundle \( \mathcal{O}_a^{r_i(a)} \) is base-point-free, by Theorem 11.9; so for any index \( i > 0 \) there is a holomorphic cross-section \( f_{i,a} \in \Gamma(M, \mathcal{O}(\mathcal{O}_a^{r_i(a)})) \) for which \( f_{i,a}(a) \neq 0 \). It follows from Theorem 11.8 that \( \dim \Gamma(M, \mathcal{O}(\mathcal{O}_a^r)) = i + 1 \) if \( r_i(a) < r < r_{i+1}(a) \), and it is
apparent that the $i + 1$ cross-sections $f_{0,a}^r$ and $f_{j,a}f_{0,a}^{r - r_j(a)}$ for $1 \leq j \leq i$, hence for which $r_j(a) \leq r$, are contained in that vector space; these cross sections are linearly independent, since their orders at the point $a$ are distinct, so they form a basis, and that suffices for the proof.

By the preceding Lemma the holomorphic cross-sections of the line bundle $\mathcal{O}_a^{r_i(a)}$ for $i > 0$ form the complex vector space of dimension $i + 1$ consisting of the sections

\begin{equation}
(11.63) \quad f_{t,a} = t_0f_{0,a}^{r_i(a)} + \sum_{j=1}^{i} t_jf_{j,a}f_{0,a}^{r_i(a) - r_j(a)} \in \Gamma(M, \mathcal{O}_a^{r_i(a)})
\end{equation}

for vectors $t = (t_0, t_1, \ldots, t_i) \in \mathbb{C}^{i+1}$, where $f_{i,a} \in \Gamma(M, \mathcal{O}_a^{r_i(a)})$ and also $f_{i,a}(a) \neq 0$ for $i > 0$ while $f_{0,a}(a) = 0$ and $f_{0,a}(z) \neq 0$ at any point $z \neq a$. The quotients $f_t = f_{t,a}/f_{0,a}$ thus are meromorphic functions on $M$ of degree at most $r_i(a)$ and with a pole only at the point $a \in M$; and by Theorem 11.3 all such meromorphic functions $f$ on $M$ are of this form so all of them can be written

\begin{equation}
(11.64) \quad f_t = t_0 + t_1f_{1,a}f_{0,a}^{r_i(a)} + \cdots + t_{i-1}f_{i-1,a}f_{0,a}^{r_i(a) - 1(a)} + t_if_{i,a}f_{0,a}^{r_i(a)}
\end{equation}

for vectors $t = (t_0, t_1, \ldots, t_i) \in \mathbb{C}^{i+1}$. These are the meromorphic functions that describe holomorphic mappings $\phi_{f_t} : M \rightarrow \mathbb{P}^1$ of degree at most $r_i(a)$ that are fully ramified at the point $a \in M$ and for which $\phi_{f_t}(a) = \infty \in \mathbb{P}^1 = \mathbb{P}^1 \cup \mathbb{C}$. If $t_i \neq 0$ the mapping $\phi$ is of degree precisely $r_i$; but if $t_i = 0$ the meromorphic function $f_{t,a}$ has a pole of degree strictly less than $r_i(a)$ at the point $a$, indeed of degree $r_j(a)$ if $t_j \neq 0$ but $t_{j+1} = t_{j+2} = \cdots = t_i = 0$. Thus the set of those holomorphic mappings $\phi : M \rightarrow \mathbb{P}^1$ of degree $r_i(a)$ that are fully ramified at the point $a \in M$ and for which $\phi_{f_t}(a) = \infty \in \mathbb{P}^1 = \mathbb{P}^1 \cup \mathbb{C}$ is parametrized by the set of vectors $t = (t_0, t_1, \ldots, t_i) \in \mathbb{C}^{i+1}$ for which $t_i \neq 0$. In particular there are meromorphic functions $f_t$ of degree $r_i(a)$ that have a pole only at the point $a \in M$ and all of them can be written

\begin{equation}
(11.65) \quad f_t = t_0 + t_1f_{1,a}/f_{0,a}^{r_i(a)} \quad \text{for constants} \quad t_0, t_1 \in \mathbb{C}, \quad t_1 \neq 0.
\end{equation}

These functions describe holomorphic mappings $\phi_{f_t} : M \rightarrow \mathbb{P}^1$ of degree $r_i(a)$ that are fully ramified at the point $a \in M$ and for which $\phi_{f_t}(a) = \infty \in \mathbb{P}^1$.

**Theorem 11.26** If $M$ is a compact Riemann surface of genus $g > 0$ then for any two points $a, a_0 \in M$ there are meromorphic functions $f_{a,a_0}$ on $M$ that have a pole of order $r_1(a)$ at the point $a \in M$ as their only singularity and have a zero at the point $a_0$; and these functions are unique up to multiplication by a complex constant.

**Proof:** The meromorphic functions $f_t$ on $M$ that have a pole of order $r_1(a)$ at the point $a \in M$ as their only singularity can be written in the form (11.65) for constants $t_0, t_1$ where $t_1 \neq 0$, as just observed. The value $t_0$ can be chosen so
that \( f_t(a_0) = 0 \), and all such functions then are of the form \( f_t = t_1 f_{1,a}/f_{0,a}^{r_1(a)} \) for nonzero constants \( t_1 \); that suffices for the proof.

For an alternative interpretation of the preceding theorem, a doubly pointed Riemann sphere is defined to be the Riemann surface \( \mathbb{P}^1 \) together with the choice of a point \( \infty \in \mathbb{P}^1 \), hence of an identification \( \mathbb{P}^1 = \infty \cup \mathbb{C} \), and of a point that is identified with the origin \( 0 \in \mathbb{C} \subset \mathbb{P}^1 \). For such a surface the complex coordinate \( t \in \mathbb{C} \) where \( \mathbb{P}^1 = \infty \cup \mathbb{C} \) is determined uniquely up to multiplication by a nonzero complex number, which is a biholomorphic mapping \( c : t \rightarrow ct \) of \( \mathbb{P}^1 \) that fixes the points \( \infty \) and \( 0 \).

**Corollary 11.27** If \( M \) is a compact Riemann surface of genus \( g > 0 \) then for any two points \( a,a_0 \in M \) there are holomorphic mappings \( \phi_{a,a_0} : M \rightarrow \mathbb{P}^1 \) of degree \( r_1(a) \) over the doubly pointed Riemann sphere such that \( \phi_{a,a_0} \) is fully ramified at the point \( a \in M \), that \( \phi_{a,a_0}(a) = \infty \in \mathbb{P}^1 \) and that \( \phi_{a,a_0}(a_0) = 0 \). The mappings \( \phi_{a,a_0} \) are uniquely determined up to the composition with biholomorphic mappings \( c : t \rightarrow ct \) of \( \mathbb{P}^1 \) that fix the points \( 0,\infty \in \mathbb{P}^1 \).

**Proof:** By the preceding theorem the mapping \( \phi_{a,a_0} = \phi_{f_{a,a_0}} \) defined by the meromorphic function \( f_{a,a_0} \) has the properties listed in the present corollary, so no further proof is required.

The mappings \( \phi_{a,a_0} \) of the preceding corollary are called the basic mappings of the Riemann surface \( M \) to the Riemann sphere \( \mathbb{P}^1 \) for points \( a,a_0 \in M \); these mappings play an interesting role in the study of compact Riemann surfaces, so it is worth including here a list of their essential properties. If \( \phi_{a,a_0}^* \) is any one of the basic mappings for the points \( a,a_0 \) then the set of all basic mappings for the points \( a,a_0 \) is the set

\[
(11.66) \quad \mathcal{B}(a,a_0) = \left\{ \phi_{a,a_0} = c \circ \phi_{a,a_0}^* \mid c \in \mathbb{C}, c \neq 0 \right\},
\]

the set consisting of the composition of the mapping \( \phi_{a,a_0}^* : M \rightarrow \mathbb{P}^1 \) and biholomorphic mappings \( c : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) defined by \( c : t \rightarrow ct \) for all points \( t \in \mathbb{C} \subset \mathbb{P}^1 \). The inverse image \( \phi_{a,a_0}^{-1}(\infty) \) is the single point \( \phi_{a,a_0}^{-1}(\infty) = a_0 \subset M \) while the inverse image \( \phi_{a,a_0}^{-1}(0) \) is a set of points

\[
(11.67) \quad Z(a,a_0) = \phi_{a,a_0}^{-1}(0) \subset M
\]

that is independent of the particular basic mapping since the biholomorphic mappings \( c : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) preserve the origin; of course \( a_0 \in Z(a,a_0) \), but there are other points in \( Z(a,a_0) \) unless all the mappings \( \phi_{a,a_0} \) are fully ramified at the point \( a_0 \). All of the basic mappings for points \( a,a_0 \in M \) have the same degree

\[
(11.68) \quad \deg \phi_{a,a_0} = r_1(a) \quad \text{for all } \phi_{a,a_0} \in \mathcal{B}(a,a_0),
\]

and by the Riemann-Hurwitz formula (11.14) they therefore all have the same branching order

\[
(11.69) \quad \text{br}(\phi_{a,a_0}) = 2r_1(a) + 2g - 2 \quad \text{for all } \phi_{a,a_0} \in \mathcal{B}(a,a_0).
\]
Each basic mapping $\phi_{a,a_0}$ is fully ramified at the point $a \in M$, so the ramification order at that point is $r_a(\phi_{a,a_0}) = r_1(a) - 1$. The mappings $\phi_{a,a_0}$ are those defined by the meromorphic functions $f_{a,a_0}$ of Theorem 11.26; and since the functions $f_{a,a_0}$ are holomorphic in $M \sim a$ then as in (11.12) their ramification order at any point $p \in (M \sim a)$ is $r_p(f_{a,a_0}) = \deg_{p} f_{a,a_0}$. All the meromorphic functions $f_{a,a_0}$ are nonzero scalar multiples of one another so they all have the same ramification order at any point $p \in M$. The common ramification divisor of all of the basic mappings thus is a divisor on $M$ of the form

$$\tau(\phi_{a,a_0}) = (r_1(a) - 1) \cdot a + \tau^*(a, a_0) \quad \text{for all } \phi_{a,a_0} \in B(a, a_0)$$

where $\tau^*(a, a_0) = \sum_{i=1}^{m} \mu_i \cdot a_i$ is a divisor on $M$ with

$$1 \leq \mu_i \leq r_1(a) - 1 \quad \text{and} \quad \sum_{i=1}^{m} \mu_i = r_1(a) + 2g - 1;$$

explicitly $\tau^*(a, a_0) = \sum_{p \in M \sim a} \deg_{p} f'_{a,a_0} \cdot p$, which is independent of the particular basic mapping $\phi_{a,a_0}$. The branch divisors of the mappings $\phi_{a,a_0}$ are the images of their ramification divisors under these mappings $\phi_{a,a_0}$ as in (11.5) and (11.6); although the ramification divisors are the same for all basic mappings $\phi_{a,a_0}$, their images under the mappings $\phi_{a,a_0}$ are not the same but differ by biholomorphic mappings $c : \mathbb{P}^1 \rightarrow \mathbb{P}^1$. If $\phi_{a,a_0}^0 \in B(a, a_0)$ is a particular basic mapping for the points $a, a_0$ then its branch divisor has the form

$$b(\phi_{a,a_0}) = (r_1(a) - 1) \cdot \infty + b^0(a, a_0)$$

where $b^0(a, a_0) = \sum_{i=1}^{n} \nu_i^0 \cdot b_i$ is a divisor on $\mathbb{P}^1$ with

$$1 \leq \nu_i^0 \leq r_1(a) - 1 \quad \text{and} \quad \sum_{i=1}^{n} \nu_i^0 = r_1(a) + 2g - 1.$$

The image of the divisor $b(\phi_{a,a_0}) = (r_1(a) - 1) \cdot \infty + \sum_{i=1}^{n} \nu_i^0 \cdot b_i$ of (11.71) under the biholomorphic mapping $c : t \rightarrow ct$ of $\mathbb{P}^1$ is the divisor

$$cb(\phi_{a,a_0}) = (r_1(a) - 1) \cdot \infty + \sum_{i=1}^{n} \nu_i^0 \cdot cb_i$$

since the point $\infty \in \mathbb{P}^1 = \infty \cup \mathbb{C}$ is fixed under this biholomorphic mapping; the origin $0 \in \mathbb{C}$ is also a fixed point, so also plays a somewhat special role.

The branch divisors of all the basic mappings $\phi_{a,a_0} \in B(a, a_0)$ form the family of divisors (11.72) on $\mathbb{P}^1$, a family indexed by the parameter $c \in \mathbb{C}^* = (\mathbb{C} \sim 0)$. To specify in a natural way a single one of these branch divisors choose a third point $a_1 \in M$ and select the uniquely defined basic mapping $\phi_{a,a_0,a_1} \in B(a, a_0)$ for which $\phi_{a,a_0,a_1}(a_1) = 1$; this mapping is called the normalized basic mapping of the Riemann surface $M$ for points $a, a_0, a_1 \in M$. The branch divisor of the normalized basic mapping $\phi_{a,a_0,a_1}$ then is a uniquely defined divisor on $\mathbb{P}^1$ of the form

$$b(\phi_{a,a_0,a_1}) = (r_1(a) - 1) \cdot \infty + b^*(a, a_0, a_1)$$

where $b^*(a, a_0, a_1) = \sum_{i=1}^{n} \nu_i \cdot b_i$ is a divisor on $\mathbb{P}^1$ with

$$1 \leq \nu_i \leq r_1(a) - 1 \quad \text{and} \quad \sum_{i=1}^{n} \nu_i = r_1(a) + 2g - 1.$$
Of course it is necessary to assume that \( a, a_0, a_1 \) are three distinct points on \( M \), and in addition to require that \( a_1 \notin Z(a, a_0) \). The slight complication here is that the set \( Z(a, a_0) \) must be defined first in order to specify an appropriate third point \( a_1 \in M \); so the three points \( a, a_0, a_1 \) are not just any three distinct points on \( M \). The construction in (11.73) associates to any choice of such points \( a, a_0, a_1 \) the divisor \( b^*(a, a_0, a_1) \) of degree \( r_1(a) + 2g - 1 \) in the doubly pointed Riemann sphere \( \mathbb{P}^1 \), where the degree depends on the choice of the point \( a \in M \); but if \( a \) is not a Weierstrass point then \( r_1(a) = g + 1 \) by Lemma 11.12 so the degree of the divisor is \( 3g \) for any choice of such a point \( a \in M \). This calls to mind the observation of Riemann that the set of all Riemann surfaces of genus \( g \) can be described by \( 3g - 3 \) moduli; the coordinates on the \( 3g \)-dimensional complex manifold \( (\mathbb{P}^1)^{(3g)} \) are \( 3g \) parameters associated to the surface \( M \) itself and the three additional points \( a, a_0, a_1 \) that are involved in the description of the normalized basic mapping, so this construction suggests one approach to the classical problem of describing all compact Riemann surfaces of a fixed genus by finitely many parameters or moduli.

With this in mind, define a \textit{triply pointed Riemann surface} \( M_{a, a_0, a_1} \) to be a compact Riemann surface \( M \) together with the choice of three distinct points \( a, a_0, a_1 \in M \) such that \( a \) is not a Weierstrass point of the surface \( M \) and \( a_1 \notin Z(a, a_0) \); and let \( \mathcal{M}_{g,3} \) denote the set of triply pointed compact Riemann surfaces \( M_{a, a_0, a_1} \). The mapping

\[
\phi : \mathcal{M}_{g,3} \longrightarrow (\mathbb{P}^1)^{(3g)}
\]

that associates to a triply pointed Riemann surface \( M_{a, a_0, a_1} \in \mathcal{M}_{g,3} \) the divisor \( b^*(a, a_0, a_1) \) as in (11.73) is called the \textit{Hurwitz mapping}. The investigation of the Hurwitz mapping will occupy the remainder of this chapter. As a preliminary observation, the divisor \( b^*(a, a_0, a_1) = \sum_{i=1}^{n} \nu_i \cdot b_i \) of (11.73) satisfies the additional condition that \( 1 \leq \nu_i \leq r_1(a) - 1 = g \); the image of the Hurwitz mapping (11.74) therefore must lie in the proper subset

\[
\mathcal{P} = \left\{ \emptyset = \sum_i \nu_i \cdot b_i \in (\mathbb{P}^1)^{(3g)} \mid \nu_i \leq g \right\} \subset (\mathbb{P}^1)^{(3g)}.
\]

**Lemma 11.28** The subset \( \mathcal{P} \subset (\mathbb{P}^1)^{(3g)} \) for \( g > 0 \) is the complement of a proper holomorphic subvariety of dimension at most \( 2g \) in the compact complex manifold \( (\mathbb{P}^1)^{(3g)} \).

**Proof:** For any integer \( \nu \) in the range \( g + 1 \leq \nu \leq 3g \) introduce the mapping

\[
\phi_{\nu} : (\mathbb{P}^1)^{(3g+1-\nu)} \longrightarrow (\mathbb{P}^1)^{(3g)}
\]

defined by

\[
\phi_{\nu}(z_0, z_1, \ldots, z_{3g-\nu}) = \nu \cdot z_0 + z_1 + \cdots + z_{3g-\nu} \in (\mathbb{P}^1)^{(3g)}.
\]

This is a well defined holomorphic mapping, since it factors through the holomorphic mapping that associates to the point \( (z_0, z_1, \ldots, z_{3g-\nu}) \in (\mathbb{P}^1)^{(3g+1-\nu)} \)
the point \((z_0, \ldots, z_0, z_1, \ldots z_{3g-\nu})\) ∈ \((\mathbb{P}^1)^{3g}\) and the natural holomorphic mapping \(\pi_{3g} : (\mathbb{P}^1)^{3g} \to (\mathbb{P}^1)^{(3g)}\) of Theorem 10.5. The domain of the mapping \(\phi_{\nu}\) is a compact manifold so the mapping is a proper holomorphic mapping; hence by Remmert’s proper mapping theorem\(^3\) the image is a holomorphic subvariety of \((\mathbb{P}^1)^{(3g)}\) of dimension at most \(3g + 1 - \nu\); in particular its dimension is at most \(2g\) since \(\nu \geq g + 1\). The union of the images of the finitely many mappings \(\phi_{\nu}\) for \(g + 1 \leq \nu \leq 3g\) is the complement of the subset \(\mathcal{P} \subset (\mathbb{P}^1)^{(3g)}\), since it consists of divisors for which at least one point appears with multiplicity at least \(g + 1\); and that union is also a holomorphic subvariety of \((\mathbb{P}^1)^{(3g)}\) of dimension at most \(2g\), which suffices for the proof.

The normalized basic mapping \(\phi_{a,a_0,a_1} : M \to \mathbb{P}^1\) is of degree \(r_1(a) = g + 1\) as in (11.68) and has branching order \(2r_1(a) + 2g - 2 = 4g\) as in (11.69). By (11.73) its branch locus \(B\) consists of the point \(\infty \in \mathbb{P}^1 = \infty \cup \mathbb{C}\), over which it is fully branched, hence has local branch order \(g\), together with \(n\) further points \(b_i \in \mathbb{P}^1\), for which the sum of the local branch orders is \(3g\); thus \(B = b_0 \cup b_1 \cup \cdots \cup b_n\) where \(b_0 = \infty \in \mathbb{P}^1\). The restriction of the mapping \(\phi_{a,a_0,a_1}\) to the complement of the branch locus \(B\) is an unbranched covering of degree \(g + 1\)

\[
\pi : (M \sim \pi^{-1}(B)) \to (\mathbb{P}^1 \sim B),
\]

denoted by \(\pi\) to simplify the notation. To describe this covering, the fundamental group \(\pi_1(\mathbb{P}^1 \sim B, p)\) of the complement \(\mathbb{P}^1 \sim B\) at a base point \(p \in \mathbb{P}^1 \sim B\) is a free group on \(n\) generators, as discussed in Appendix D.1; the generators can be represented by \(n\) paths \(\lambda_1, \ldots, \lambda_n\) from a base point \(p \in \mathbb{P}^1 \sim B\) to the points \(b_1, \ldots, b_n\) and back in the complement of \(B\), encircling but avoiding the points \(b_i\) as sketched in the accompanying Figure 11.1. If \(\lambda_0\) is the corresponding path to the point \(b_0 = \infty\) as in the Figure the homotopy classes of the paths

\(^3\)Remmert’s proper mapping theorem is discussed on page 409 in Appendix A.3.
$\lambda_i \in \pi_1(\mathbb{P}^1 \sim B, p)$ satisfy

\begin{equation}
\lambda_0 \cdot \lambda_1 \cdot \lambda_2 \cdots \lambda_n = 1 \in \pi(\mathbb{P}^1 \sim B, p),
\end{equation}

since the homotopy class of the product of these paths is represented by a path from $p$ that encircles all the points $b_i$ and that path is homotopic to a point. Label the $g + 1$ points $\pi^{-1}(p) \subset M$ by the integers $1, 2, \ldots, g + 1$ to simplify the notation. As a consequence of a basic property of covering mappings each path $\lambda_j \subset \mathbb{P}^1$ lifts uniquely to a path $\tilde{\lambda}_j \subset M$ beginning at the point in $M$ labeled $i$, a path for which $\pi(\tilde{\lambda}_j) = \lambda_j$. The end point of the lifted path $\tilde{\lambda}_j$ is not necessarily the initial point $i$ but is another of the points over $p$, say the point $\sigma_j(i)$ for some permutation $\sigma_j$ of the $g + 1$ integers $1, 2, \ldots, g + 1$, an element $\sigma_j \in \mathfrak{S}_{g+1}$, the permutation group on $g + 1$ points, as in the following Figure 11.2. These permutations describe the branching of the mapping $\pi$ over the points $b_j$, and in that way fully describe the covering mapping (11.78). For instance since the mapping $\pi$ is fully branched over the point $b_0 = \infty$ that mapping can be described locally by the holomorphic function $z^{g+1}$ in terms of a local coordinate $z$ at the point $a \in M$; continuation once around the origin is a transitive permutation, so by numbering the points over $p$ suitably it can be assumed that $\sigma_0$ is the permutation

\begin{equation}
\sigma_0 = (1, 2, \ldots, g + 1) \in \mathfrak{S}_{g+1}
\end{equation}

in the permutation group $\mathfrak{S}_{g+1}$ on $g + 1$ points, the permutation for which $\sigma_0(i) = i + 1$ for $1 \leq i \leq g$ and $\sigma_0 = 1$ for $i = g + 1$. If there is a single ramification point over $b_1$ with ramification order $\nu$ then the permutation $\sigma_1$ will be a cyclic permutation of order $\nu$ among $\nu$ of the integers $1, 2, \ldots, g + 1$ but will leave the remaining integers unchanged. If there are two ramification points over $b_2$ then similarly the permutation $\sigma_2$ will be two cyclic permutations among two distinct subsets of the integers $1, 2, \ldots, g + 1$ but will leave the remaining integers unchanged. As a consequence of (11.79) these permutations satisfy

\begin{equation}
\sigma_n \cdot \sigma_{n-1} \cdots \sigma_1 \cdot \sigma_0 = 1,
\end{equation}

the identity permutation, where permutations are viewed as mappings so $\sigma_2 \sigma_1$ denotes the mapping arising from first applying the permutation $\sigma_2$ then apply-

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Figure 11.2: Monodromy from lifting of paths in $\mathbb{P}^1$
ing the permutation \(\sigma_1\). Conversely given any collection of permutations \(\sigma_i\) of the \(g + 1\) integers \(1, 2, \ldots, g + 1\) satisfying (11.81) there is a uniquely determined covering mapping (11.78) for which these permutations describe the branching behavior, a result often called the monodromy theorem\(^4\). For the applications here it is required that \(\sigma_0 = (1, 2, \ldots, g + 1)\) as in (11.80); this is a transitive permutation, so any choice of permutations \(\sigma_1, \ldots, \sigma_n\) satisfying (11.81) describes a connected unbranched covering (11.78) of degree \(g + 1\) over the complement \(\mathbb{P}^1 \sim B\). The covering can be extended to be a branched covering over the set \(B\), with the branching structure over the point \(b_i\) described by the permutation \(\sigma_i\) as just discussed. The set \(M\) then has the natural structure of a compact Riemann surface induced by the complex structure on the space \(\mathbb{P}^1\); and the mapping \(\pi\) is described by a meromorphic function \(f\) on \(M\), which can be assumed to have as its sole singularity a pole of order \(g\) at a single point \(a \in M\) for which \(\pi(a) = \infty \in \mathbb{P}^1\). The genus of the surface \(M\) depends on the branching over the points \(b_j \in B\), which is determined by the particular permutations \(\sigma_i\); so there are some restrictions on these permutations necessary to ensure that the surface \(M\) is of genus \(g\). If \(a_0 \in M\) is a point for which \(\pi(a_0) = 0 \in \mathbb{P}^1\) then the mapping \(\pi\) is one of the basic mappings \(\phi_{a,a_0}\) of the Riemann surface \(M\); if the surface \(M\) is of genus \(g\) then the mapping \(\pi\) is the normalized basic mapping \(\phi_{a,a_0,a_1}\) for some point \(a_1 \in M\) for which \(\pi(a_1) = 1 \in \mathbb{P}^1\), unless \(a\) is a Weierstrass point of \(M\) of the special form having a Weierstrass gap of \(g + 1\).

The branching configurations in Figure 11.1 can be modified continuously by moving the branch points \(b_j\) and even allowing mergers or separations of the branch points. For example if two branch points \(b_j\) and \(b_k\) are near one another and are to be merged choose circles around these two points and touching at a single point \(q\), as sketched in Figure 11.3. The path \(\lambda_j\) can be taken to proceed from the point \(p\) to the point \(q\), then along the circles around \(b_j\) back to the point \(p\) and then back to the point \(p\), and correspondingly for the path \(\lambda_k\). In the product path \(\lambda_j \lambda_k\) the segment of the path \(\lambda_j\) proceeding from \(q\) to \(p\) cancels the segment of the path \(\lambda_k\) from \(p\) to \(q\), so the resulting path \(\lambda_{jk}\) proceeds from \(p\) to \(q\), then follows the circle around \(b_j\) then the circle around \(b_k\) then back to \(p\); so that path really amounts to a path \(\lambda_{jk}\) from \(p\) encircling both \(b_j\) and \(b_k\) before proceeding back to \(p\). The permutation \(\sigma_{jk}\) associated to the path \(\lambda_{jk}\) is just the product \(\sigma_{jk} = \sigma_j \sigma_k\), so the result is to merge the branch points \(b_j\) and \(b_k\) to a single point with the associated permutation \(\sigma_{jk}\). The process can be reversed, replacing a single branch point \(b_{jk}\) with an associated permutation \(\sigma_{jk}\) to a pair of branch points \(b_j\) and \(b_k\) with associated permutations \(\sigma_j\) and \(\sigma_k\) so long as \(\sigma_{jk} = \sigma_j \sigma_k\) in \(\mathfrak{S}_{g+1}\). In particular since any cyclic permutation of order \(\nu\) can be written as a product of simple transpositions (cyclic permutations of order 2) it follows that any branched covering can be deformed in this way to a branched covering that is fully branched over the point \(\infty \in \mathbb{P}^1 = \mathbb{C} \cup \infty\) but

\(^4\)A description of covering mappings through monodromy is quite classical, going back in some sense to Riemann and in the present context to A. Hurwitz, in his paper “Uber Riemann’sche Flächen mit gegebenen Verzweigungspunkten”, *Math. Annalen*, vol. 39 (1899), pp. 1 – 60; it is treated in more detail for example in the book by H. Seifert and W. Threlfall, “Lehrbuch der Topologie”.\n
CHAPTER 11. MAPPINGS TO THE RIEMANN SPHERE

Figure 11.3: Merger of branch points for the covering mapping $\phi_a$

where all the other ramification points are simply ramified; such a branched
covering is called a branched covering of the *Hurwitz standard form*.

To construct branched coverings in the Hurwitz standard form consider a
set $B = \{b_0 \cup b_1 \cup \cdots \cup b_{3g}\}$ of $3g+1$ distinct points $b_i \in \mathbb{C} \subset \mathbb{P}^1$ and the associated
divisor $b = g \cdot b_0 + \sum_{i=1}^{3g} 1 \cdot b_i$ of total order $4g$ where $b_0 = \infty \in \mathbb{P}^1$. For this
to be the branch divisor of a mapping $\pi : M \rightarrow \mathbb{P}^1$ from a compact Riemann
surface $M$ in the Hurwitz standard form it is necessary and sufficient to find
permutations $\sigma_i \in \mathfrak{S}_{g+1}$ for which $\sigma_0 = (1, 2, \ldots, g+1)$, $\sigma_i$ is a transposition for
$1 \leq i \leq 3g$ and $\sigma_{3g} \cdot \sigma_1 \cdot \sigma_0 = 1$. It is clear that there do exist transpositions
$\sigma_i$ for which the preceding holds. Indeed by direct calculation

$$(g, g+1)(g-1, g) \cdots (3, 4)(2, 3)(1, 2, 3, \ldots, g+1) = 1,$$

for $g$ transpositions $\sigma_i = (i, i+1)$ for $1 \leq i \leq g$; then for any $g$ transpositions $\tau_i$
since $\tau_i^2 = 1$ it follows that

$$\prod_{i=1}^{g}(\tau_i \cdot \tau_i) \cdot \prod_{i=1}^{g} \sigma_i \cdot (1, 2, 3, \ldots, g+1) = 1$$

as desired. It is clear though that there are actually a great many different ways
in which it is possible to find transpositions $\sigma_i$ such that $\sigma_1 \cdot \sigma_{3g} \cdot \sigma_1 \cdot \sigma_0 = 1$;
that leaves the problem of determining the number $H_g$ of possibilities, called
the *Hurwitz number*. This has been investigated extensively, beginning with the
work of Hurwitz and a great many extensions and relations to other subjects.
For present purposes the explicit value of $H_g$ is not needed though.

For any divisor $b^* = \sum_{i=1}^{3g} 1 \cdot b_i \in (\mathbb{P})^{3g}$ for distinct points $b_i \in \mathbb{P}$ there are
$H_g$ mappings

(11.82) $\phi_{\nu, b^*} : M_{\nu, b^*} \rightarrow \mathbb{P}^1$

in Hurwitz standard form for the divisor $b = g \cdot \infty + b$ for compact Riemann
surfaces $M_{\nu, b^*}$ of genus $g$. This set of Riemann surfaces can be viewed as a
covering space \( \mathcal{M}_g^0 \) of degree \( H_g \) over the open subset of the complex manifold \( \mathcal{P} \), as defined in (11.75), formed by the set divisors consisting of \( 3g \) distinct points of \( \mathbb{P}^1 \). When a divisor consisting of \( 3g \) distinct points is modified by mergers of some of the branch points then the number of distinct permutations \( \sigma_i \) satisfying (11.81) is reduced, some permutations that involve an interchange of two equal points are not distinct; any divisor \( b^* \in \mathcal{P} \subset (\mathbb{P}^1)^{3g} \) can be attained by mergers of points of divisors of distinct points, so for these divisors too there are holomorphic mappings (11.82) for Riemann surfaces \( M_{\nu, b^*} \); that yields an extension of the complex manifold \( \mathcal{M}_g^0 \) to a branched covering space over \( \mathcal{P} \), branched over the subset of divisors with multiple points, a subset of \( \mathfrak{calP} \) of codimension 1. This extension is at least a holomorphic variety of dimension \( 3g \). For any choice of points \( a_0, a_1 \in M_{\nu, b^*} \) for which \( \phi_{\nu, b^*}(a_0) = 0 \in \mathbb{P}^1 \) and \( \phi_{\nu, b^*}(a_1) = 1 \in \mathbb{P}^1 \) the mapping \( \phi_{\nu, b^*} \) is the normalized basic mapping \( \phi_{a, a_0, a_1} \) for the Riemann surface \( M_{\nu, b^*} \); so all such mappings are among the Riemann surfaces in the set \( \mathcal{M}_g \). The holomorphic variety \( \mathcal{M}_g \) is called the Hurwitz moduli space for compact Riemann surfaces of genus \( g \).

It is worth noting that an alternative approach is to consider in place of triply marked Riemann surfaces the equivalent classes of doubly marked Riemann surfaces, where all the divisors (11.72) are viewed as equivalent divisors on \( \mathbb{P}^1 \); this gives an essentially equivalent Hurwitz moduli space, but as a branched covering of the set of equivalence classes of divisors in \( \mathcal{P} \), a complex variety of dimension \( 3g - 1 \). Indeed the set of divisors (11.72) for all \( c \in \mathbb{C}^* \) is an equivalence class of divisors under the action of the multiplicative group \( \mathbb{C}^* \) on the points of the quotient space \( \mathbb{C}^*/\mathfrak{S}_r \). The action of the permutation group \( \mathfrak{S}_r \) and the group \( \mathbb{C}^* \) on the product \( \mathbb{C}^*/\mathfrak{S}_r \) commute, so the quotient space \( \left( \mathbb{C}^*/\mathfrak{S}_r \right)/\mathbb{C}^* \) can be identified with the quotient space \( \left( \mathbb{C}^*/\mathbb{C}^* \right)/\mathfrak{S}_r \), which is just the quotient of the projective space \( \mathbb{P}^{r-1} \) under the action of the permutation group \( \mathfrak{S}_r \), so is naturally a holomorphic variety; there may be singularities at those points that are fixed under some permutations, namely those points corresponding to some divisors with multiple points.

The complex projective space of dimension \( r - 1 \) similarly is the quotient \( \mathbb{P}^{r-1} = (\mathbb{C}^* \sim 0)/\mathbb{C}^* \) of the complement of the origin \( 0 \in \mathbb{C}^* \) under the action of the multiplicative group \( \mathbb{C}^* \) of nonzero complex numbers.
Chapter 12

The Role of the Intrinsic Functions

[PRELIMINARY]

Another approach to the Abel-Jacobi mapping is through an application of the cross-ratio function, defined in (5.23). By Theorem 5.6 the intrinsic cross-ratio function for a compact Riemann surface $M$ with the universal covering surface $\tilde{M}$ is the meromorphic function $q(z, a; z^+, z^-)$ on the product $\tilde{M}^4$ that as a function of the variable $z \in \tilde{M}$ is meromorphic on $\tilde{M}$, takes the value 1 at the point $z = a$, has simple zeros at the points $Tz_+$ and simple poles at the points $Tz_-$ for all covering translations $T \in \Gamma$, and is a meromorphic relatively automorphic function for the flat factor of automorphy $\rho_{T(z^+, z^-)}$ that in terms of a basis $\omega_i(z) \in \Gamma(M, \mathcal{O}((1,0))$ and generators $T_j \in \Gamma$ of the covering translation group of $M$ has the explicit form

\begin{equation}
\rho_{z^+, z^-}(T_j) = \exp \left[ -2\pi \sum_{m,n=1}^{g} (w_m(z^+) - w_m(z^-)) g_{mn} \bar{\omega}_{nj} \right],
\end{equation}

where $\Omega$ is the period matrix, $P$ is the intersection matrix and $G = iH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P \bar{\Omega}$ in terms of these bases. For any two ordered sets of $r$ points $A^+ = \{a_1^+, a_2^+, \ldots, a_r^+\}$ and $A^- = \{a_1^-, a_2^-, \ldots, a_r^-\}$ on $\tilde{M}$ the product cross-ratio function of degree $r$ is defined to be the meromorphic function

\begin{equation}
Q(z, a; A^+, A^-) = \prod_{\nu=1}^{r} q(z, a; a^+_\nu, a^-\nu)
\end{equation}

as a function of the variable $z \in \tilde{M}$ is a relatively automorphic function for the
flat factor of automorphy

\[
\rho_{A^+, A^-}(T_j) = \prod_{\nu=1}^{r} \rho_{a^+_{\nu}, a^-_{\nu}}(T_j)
\]

\[
= \exp -2\pi \sum_{m,n=1}^{g} \sum_{\nu=1}^{r} (w_m(a^+_{\nu}) - w_m(a^-_{\nu}))g_{mn}\omega_{nj},
\]

and the divisor of the relatively automorphic function \( Q(z, a; A^+, A^-) \) is

\[
\delta Q(z, a; A^+, A^-) = \delta^+ - \delta^-
\]

where \( \delta^+ = \sum_{\nu=1}^{r} \pi(a^+_{\nu}) \), \( \delta^- = \sum_{\nu=1}^{r} \pi(a^-_{\nu}) \)

in terms of the universal covering projection \( \pi: \tilde{M} \rightarrow M \).

**Lemma 12.1** There is a meromorphic function on the compact Riemann surface \( M \) with the divisor \( \delta^+ - \delta^- \) if and only if there is a holomorphic abelian differential \( \omega(z) \) on \( M \) with the period class \( \omega(T) \) for which

\[
\rho_{A^+, A^-}(T) = \exp \omega(T);
\]

and if that condition is satisfied then

\[
f(z) = Q(z, a; A^+, A^-) e^{-\omega(z)}
\]

is the unique meromorphic function on \( M \) with the divisor \( \delta^+ - \delta^- \), up to an arbitrary nonzero constant factor.

**Proof**: If there is a meromorphic function \( f(z) \) on the Riemann surface \( M \) with the divisor \( \delta = \delta^+ - \delta^- \) then the quotient \( Q(z, a; A^+, A^-)/f(z) \) is a holomorphic and nowhere vanishing relatively automorphic function for the flat factor of automorphy \( \rho_{A^+, A^-} \); consequently that factor of automorphy represents the trivial holomorphic line bundle so as in Corollary 3.10 has the form \( \rho_{A^+, A^-}(T) = \exp \omega(T) \) where \( \omega(T) \) is the period class of a holomorphic abelian differential \( \omega(z) \). Conversely if \( \rho_{A^+, A^-}(T) = \exp \omega(T) \) for the period class of the holomorphic abelian differential \( \omega(z) \) and if \( w(z) = \int_{z_0}^{z} \omega \) then \( f(z) = Q(z, a; A^+, A^-) \exp -w(z) \) is a relatively automorphic function for the factor of automorphy \( \rho_{A^+, A^-}(T) \exp -\omega(T) = 1 \) and that function has the divisor \( \delta^+ - \delta^- \); and any meromorphic function on \( M \) with this divisor of course is a constant multiple of the function \( f(z) \), and that suffices for the proof.

This simple lemma leads directly to the following rather more explicit formulation of Abel’s Theorem than that given earlier in Corollary 10.2.

**Theorem 12.2** Let \( M \) be a compact Riemann surface of genus \( g > 0 \), let \( \Omega \) be the period matrix of \( M \) in terms of a basis \( \omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)}) \) and generators \( T_j \) of the covering translation group \( \Gamma \), let \( \tilde{w}: \tilde{M} \rightarrow \mathbb{C}^g \) be the holomorphic
mapping described by the integrals \( w_i(z) = \int_a^z \omega_i \), and let \( \pi: \tilde{M} \rightarrow M \) be the universal covering space of the surface \( M \). Further let \( \partial^+ \) and \( \partial^- \) be two positive divisors of degree \( r \) on \( M \), described by \( \partial^+ = \sum_{i=1}^r \pi(a^+_i) \) and \( \partial^- = \sum_{i=1}^r \pi(a^-_i) \) for two ordered sets \( A^+ = \{a^+_1, a^+_2, \ldots, a^+_r\} \) and \( A^- = \{a^-_1, a^-_2, \ldots, a^-_r\} \) of points of \( M \).

(i) The necessary and sufficient condition that there exists a meromorphic function on \( M \) with divisor \( \partial = \partial^+ - \partial^- \) is that

\[
(12.7) \quad \sum_{j=1}^r (\wtilde{w}(a^+_j) - \wtilde{w}(a^-_j)) = \Omega n \quad \text{for a vector} \; n \in \mathbb{Z}^{2g}.
\]

(ii) If (12.7) is satisfied then the function

\[
(12.8) \quad f(z) = Q(z; a; A^+, A^-) \exp -2\pi i \sum_{k,l=1}^g \sum_{s=1}^{2g} w_k(z) g_{kl} \wtilde{\omega}_{ls} n_s,
\]

where \( G = \iota H^{-1} \) for the matrix \( H = i\Omega P^T \Omega \) expressed in terms of the period matrix \( \Omega \) and the intersection matrix \( P \) of the surface in terms of the given bases, is a meromorphic function on \( M \) with the divisor \( \partial = \partial^+ - \partial^- \) and this function is unique up to a nonzero constant factor.

**Proof:** (i) The proof just amounts to interpreting the condition of the preceding Lemma 12.1. A holomorphic abelian differential \( \omega(z) \) can be written as the sum \( \omega(z) = \sum_{k=1}^g c_k \omega_k(z) \) in terms of the basis \( \omega_k(z) \in \Gamma(M, \mathcal{O}^{1,0}) \); and then \( \exp \omega(T_j) = \exp \sum_{k=1}^g c_k \omega_k(z) \) in terms of the period matrix \( \Omega = \{\omega_{kj}\} \). To simplify the notation set \( w_m(A^+) = \sum_{i=1}^r w_m(a^+_i) \), \( w_m(A^-) = \sum_{i=1}^r w_m(a^-_i) \) and \( w_m(A) = w_m(A^+) - w_m(A^-) \). By (12.3)

\[
(12.9) \quad \rho_{A^+, A^-}(T_j) = \exp -2\pi i \sum_{m,n=1}^g w_m(A) g_{mn} \wtilde{\omega}_{nj}.
\]

Condition (12.5) that \( \rho_{A^+, A^-}(T_j) = \exp \omega(T_j) = \exp \sum_{k=1}^g c_k \omega_k(z) \) can be written as the condition that

\[
(12.10) \quad -2\pi i \sum_{m,n=1}^g w_m(A) g_{mn} \wtilde{\omega}_{nj} = 2\pi i N_j + \sum_{k=1}^g c_k \omega_k(z)
\]

for some integers \( N_j \). The preceding equation can be viewed as a system of linear equations \( v_j = 2\pi i N_j + \sum_{k=1}^g c_k \omega_k(z) \) in the unknowns \( N_j \) and \( c_k \), where \( v_j \) denotes the left-hand side of (12.10); and in terms of the column vectors \( v = \{v_j\}, N = \{N_j\} \in \mathbb{C}^{2g} \) and \( c = \{c_k\} \in \mathbb{C}^g \) this equation can be written

\[
(12.11) \quad v - 2\pi i N = \iota \Omega c.
\]

The inverse period matrix to \( \Omega \) as defined in Theorem F.12 in Appendix F.1 is the \( g \times 2g \) complex matrix \( \Pi \) for which

\[
(12.12) \quad \Pi \iota \Omega = 0, \quad \Pi \Omega = I, \quad \iota \Omega \Pi^T + \iota \Pi \Omega^T = I.
\]
In view of these properties of the inverse period matrix, if for some \( N \) there is a solution \( c \) to the linear equation (12.11) then \( \Pi(v - 2\pi i N) = \Pi t \Omega c = 0; \) and conversely if \( \Pi(v - 2\pi i N) = 0 \) then

\[
v - 2\pi i N = (t \Omega + t \Pi)(v - 2\pi i N) = t \Omega \Pi (v - 2\pi i N) = t \Omega c.
\]

Thus there is a solution \( c \) to (12.11) if and only if

\[
(12.13) \quad \Pi(v - 2\pi i N) = 0,
\]

and a solution is given explicitly by

\[
(12.14) \quad c = \Pi(v - 2\pi i N).
\]

Condition (12.13) is just that

\[
\sum_{g} \Pi_{kj}(v_j - 2\pi i N_j) = 0,
\]

and upon replacing \( v_j \) by its explicit value it becomes

\[
(12.15) \quad -2\pi \sum_{m,n=1}^{2g} w_m(\delta) g_{mn} \Pi_{kj} - 2\pi i \sum_{j=1}^{2g} N_j \Pi_{kj} = 0
\]

for \( 1 \leq k \leq g \). If this condition is satisfied then a solution \( c_k \) is given by

\[
(12.16) \quad c_k = -2\pi \sum_{m,n=1}^{2g} w_m(\delta) g_{mn} \Pi_{kj} - 2\pi i \sum_{j=1}^{2g} N_j \Pi_{kj}
\]

for \( 1 \leq k \leq g \). However \( \sum_{j=1}^{2g} N_j \Pi_{kj} = \delta_k^g \) by (12.12) so equation (12.15) reduces to

\[
(12.17) \quad \sum_{m=1}^{2g} w_m(\delta) g_{mk} = -2i \sum_{j=1}^{2g} N_j \Pi_{kj}
\]

for \( 1 \leq k \leq g \); and \( \sum_{k=1}^{2g} g_{mk} h_{rk} = \delta_r^m \) so multiplying the preceding equation by \( h_{rk} \) and adding the result for \( 1 \leq k \leq g \) yields the equation

\[
(12.18) \quad w_r(\delta) = -2i \sum_{j=1}^{2g} N_j \Pi_{kj} h_{rk}
\]

for \( 1 \leq r \leq g \). In this equation though \( H \Pi = i\Omega P t \Omega \Pi = i\Omega P (I - t \Omega \Pi) = i\Omega P \) by (12.12) and Riemann’s equality \( \Omega P t \Omega = 0 \), so (12.17) can be rewritten

\[
(12.19) \quad w_r(\delta) = \sum_{s,j=1}^{2g} \Omega_{rs} P_s N_j
\]

for \( 1 \leq r \leq g \). The intersection matrix \( P \) is an integral matrix of determinant \( \det P = 1 \), so as the entries \( N_j \) vary over all integral values in \( \mathbb{Z}^{2g} \) so do the
entries \( n_s = \sum_{j=1}^{2g} P_{sj} N_j \); consequently the preceding equation (12.19) is just the assertion that

\[
(12.20) \quad w_r(\delta^+) - w_r(\delta^-) = \sum_{s=1}^{2g} \Omega_{rs} n_s
\]

for some integers \( n_s \), which is (12.7).

(ii) If the condition (i) is satisfied then as in the discussion preceding the statement of the theorem the function \( f(z) \) of (12.6) is a meromorphic function with the divisor \( \delta^+ - \delta^- \) where \( w(z) = \sum_{k=1}^{g} c_k w_k(z) \) for the constants \( c_k \) of (12.16). Since \( \Omega \Pi = 0 \) by (12.12) equation (12.16) reduces to the simpler form

\[
(12.21) \quad c_k = -2\pi i \sum_{l=1}^{2g} N_l \Pi_{kl}.
\]

Since condition (12.20) is expressed in terms of the constants \( n_s = \sum_{l=1}^{2g} P_{sl} N_l \) it is natural to use those same constants in the expression for the function \( f(z) \) hence to rewrite the preceding equation as

\[
(12.22) \quad c_k = -2\pi i \sum_{l,s=1}^{2g} n_s P_{ls}^{-1} \Pi_{kl}.
\]

Equation (F.35) in Appendix F shows that \( \overline{G} = i\Pi^{-1} \overline{P} \Pi \) where \( G = \overline{H}^{-1} \); hence

\[
\overline{G\Omega} = i\Pi^{-1} \overline{P} \Pi \overline{\Omega} = i\Pi^{-1} (I - \overline{\Pi \Omega}) = i\Pi^{-1}
\]

from (12.12) and (F.35); substituting this into (12.22) shows that

\[
(12.23) \quad c_k = -2\pi \sum_{k=1}^{g} \sum_{s=1}^{2g} g_{ks} \overline{\omega}_{ks} n_s,
\]

and substituting these values of the coefficients \( c_k \) in the formula for the integral \( w(z) \) yields (12.8) to conclude the proof.

To examine the product cross-ratio function further it is necessary to consider in somewhat more detail the complex manifolds \( \tilde{M}^{(r)} \) and various quotients of these manifolds. Since the commutator subgroup \([\Gamma, \Gamma] \subset \Gamma \) is a normal subgroup, the universal covering projection \( \tilde{\pi} : \tilde{M} \longrightarrow M = \hat{M}/\Gamma \) can be decomposed as the composition \( \tilde{\pi} = \hat{\pi} \circ \pi_a \) of the two mappings in the chain of covering projections

\[
(12.24) \quad \tilde{M} \xrightarrow{\pi_a} \hat{M} = \tilde{M}/[\Gamma, \Gamma] \xrightarrow{\hat{\pi}} M = \hat{M}/\Gamma_a
\]

where \( \Gamma_a = \Gamma/[\Gamma, \Gamma] \) is the abelianization of the group \( \Gamma \). The group \( \Gamma \) can be generated by \( 2g \) generators with the single relation (D.4), as discussed in Appendix D.1; consequently its abelianization \( \Gamma_a \) is a free abelian group on \( 2g \).
generators. The intermediate Riemann surface \( \tilde{M} \) is a useful auxiliary surface in studying function theory on compact Riemann surfaces. The subgroup \( \Gamma, \Gamma \subset \Gamma \) is not of finite index, since the quotient \( \Gamma_a = \Gamma/[\Gamma, \Gamma] \) is an infinite group, so \( \tilde{M} \) is not a compact Riemann surface. The surface \( \tilde{M} \) is not simply connected; indeed its fundamental group is isomorphic to \( [\Gamma, \Gamma] \). The fundamental group of any noncompact connected surface is a free group\(^1\); so the group \( [\Gamma, \Gamma] \) actually is a free group, a result which though interesting will not be used here. The Riemann surface \( \tilde{M} \) can be identified with the unit disc, through the general uniformization theorem. The Riemann surface \( \tilde{M} \) however appears to be an example of a non-continuable\(^2\) Riemann surface, a noncompact Riemann surface that cannot be realized as a proper subset of another Riemann surface; but that topic will not be pursued further here.

The holomorphic abelian differentials on \( M \) are represented by \( \Gamma \)-invariant holomorphic differential 1-forms \( \omega_i \) on \( \tilde{M} \), and their integrals \( w_i(z) = \int_{z_0}^{z} \omega_i \) are holomorphic functions on \( \tilde{M} \) such that

\[
(12.25) \quad w_i(Tz) = w_i(z) + \omega_i(T) \quad \text{for all} \quad T \in \Gamma.
\]

The set of period vectors \( \omega(T) = \{\omega_i(T)\} \in \mathbb{C}^g \) for all \( T \in \Gamma \) form the lattice subgroup denoted by \( L(\Omega) \subset \mathbb{C}^g \); and the set of integrals \( w_i(z) \) describe a holomorphic mapping

\[
(12.26) \quad \bar{w}_{z_0} : \tilde{M} \rightarrow \mathbb{C}^g \quad \text{where} \quad \bar{w}_{z_0}(z) = \{w_i(z)\} \in \mathbb{C}^g.
\]

It follows from (12.25) that the mapping (12.26) commutes with the covering projections \( \tilde{\pi} : \tilde{M} \rightarrow M \) and \( \pi : \mathbb{C}^g \rightarrow J(M) = \mathbb{C}^g/L(\Omega) \), so it induces the Abel-Jacobi mapping \( w_{z_0} : M \rightarrow J(M) \) as in the commutative diagram (3.4). Recall from the earlier discussion that the Abel-Jacobi mapping is a nonsingular biholomorphic mapping from the Riemann surface \( M \) to its image \( \tilde{W}_1 = w_{z_0}(M) \subset J(M) \), which is an irreducible holomorphic submanifold of the complex torus \( J(M) \). The holomorphic mapping (12.26) and the Abel-Jacobi mapping have the same local expression; so if the image of the mapping (12.26) is denoted by

\[
(12.27) \quad \tilde{W}_1 = \bar{w}_{z_0}(\tilde{M}) \subset \mathbb{C}^g
\]

then the mapping (12.26) is a nonsingular holomorphic mapping, hence is a locally biholomorphic mapping

\[
(12.28) \quad \bar{w}_{z_0} : \tilde{M} \rightarrow \tilde{W}_1.
\]

This situation can be summarized in the commutative diagram of holomorphic

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\(^1\)See the discussion the book *Riemann Surfaces* by Lars Ahlfors and Leo Sario, section 44.

mappings

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{w}_{z_0}} & \tilde{W}_1 \\
\pi & \downarrow & \pi \\
M = \tilde{M}/\Gamma & \xrightarrow{w_{z_0}} & W_1 = \tilde{W}_1/\mathcal{L}(\Omega) \cong \mathbb{C}^g
\end{array}
\]

(12.29)

Although the subset \(\tilde{W}_1\) is defined as the image (12.27), it also can be characterized by

\[
\tilde{W}_1 = \pi^{-1}(W_1) \quad \text{so} \quad \tilde{W}_1 + \lambda = \tilde{W}_1 \quad \text{for all} \quad \lambda \in \mathcal{L}(\Omega).
\]

(12.30)

Indeed if \(t \in \tilde{W}_1 \subset \mathbb{C}^g\) then by definition \(t = \tilde{w}_{z_0}(z)\) for some point \(z \in \tilde{M}\), and if \(\lambda \in \mathcal{L}(\Omega)\) then \(\lambda = \omega(T)\) for some \(T \in \Gamma\); it then follows from (12.25) that \(w_{z_0}(Tz) = w_{z_0}(z) + \lambda = t + \lambda\), so \(t + \lambda \in \tilde{W}_1\). Consequently as the inverse image of the holomorphic variety \(W_1\) by the holomorphic mapping \(\pi\) it follows that \(\tilde{W}_1\) is at least a holomorphic subvariety of \(\mathbb{C}^g\).

Although the holomorphic mapping (12.28) is locally biholomorphic it is not globally biholomorphic. Indeed if \(\tilde{w}_{z_0}(z_1) = \tilde{w}_{z_0}(z_2)\) for two points \(z_1, z_2 \in \tilde{M}\) then by the commutativity of the diagram (12.29) the images \(p_1 = \tilde{\pi}(z_1)\) and \(p_2 = \tilde{\pi}(z_2)\) in \(M\) have the same image under the Abel-Jacobi mapping \(w_{z_0}\); and since the mapping \(w_{z_0}\) is injective it follows that \(p_1 = p_2\). Consequently \(z_1 = Tz_2\) for some \(T \in \Gamma\); and then \(w_{z_0}(z_1) = w_{z_0}(Tz_2) = w_{z_0}(z_1) + \omega(T)\) so that \(\omega(T) = 0\), which by Corollary 3.6 is equivalent to the condition that \(T \in [\Gamma, \Gamma]\). The converse clearly holds, so

\[
\tilde{w}_{z_0}(z_1) = \tilde{w}_{z_0}(z_2) \quad \text{if and only if} \quad z_1 = Tz_2 \quad \text{where} \quad T \in [\Gamma, \Gamma].
\]

(12.31)

That means that the mapping \(\tilde{w}_{z_0}\) in the diagram (12.29) can be factored through the quotient surface \(\tilde{M} = \tilde{M}/[\Gamma, \Gamma]\), so the entire diagram (12.29) can be factored into the commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{w}_{z_0}} & \tilde{W}_1 \\
\pi_a \downarrow & & \downarrow \pi \\
M = \tilde{M}/\Gamma_a & \xrightarrow{w_{z_0}} & W_1 = \tilde{W}_1/\mathcal{L}(\Omega) \cong \mathbb{C}^g
\end{array}
\]

(12.32)

where \(\tilde{\pi} \circ \pi_a = \tilde{\pi} : \tilde{M} \longrightarrow M\). The holomorphic mapping \(\tilde{w}_{z_0}\) is surjective by definition of the set \(\tilde{W}_1\), it is injective as a consequence of (12.31), and it is locally biholomorphic since it has the same local expression as the Abel-Jacobi
mapping $w_{z_0}$; hence it is a biholomorphic mapping, as indicated in the diagram. The image $\hat{w}_{z_0}(\hat{M}) = \hat{W}_1 = \hat{W}_1$ thus is an irreducible holomorphic submanifold of $\mathbb{C}^g$ that is biholomorphic to $\hat{M}$. Since $\hat{W}_1 = \hat{W}_1$ it follow from (12.30) that

$$
(12.33) \quad \hat{W}_1 = \pi^{-1}(W_1) \quad \text{so} \quad \hat{W}_1 + \lambda = \hat{W}_1 \quad \text{for all} \ \lambda \in \mathcal{L}(\Omega).
$$

The holomorphic mapping $\hat{w}_{z_0}$ is defined as the mapping induced by the mapping $\hat{w}_{z_0}$; but it also can be described somewhat independently. Indeed it follows from (12.25) that the holomorphic abelian integrals $w_i(z)$ are invariant under the covering translation group $[\Gamma, \Gamma]$ so they can be viewed as holomorphic functions $\hat{w}_i(\hat{z})$ of points $\hat{z}$ in the complex manifold $\hat{M}$. Of course the holomorphic abelian differentials can be viewed as holomorphic differential forms on the Riemann surface $\hat{M}$, which is not simply connected; but their integrals nonetheless are well defined global holomorphic functions $\hat{w}_i(\hat{z})$ on the manifold $\hat{M}$. In terms of these integrals the mapping $\hat{w}_{z_0}$ can be viewed as the mapping defined by

$$
(12.34) \quad \hat{w}_{z_0}(\hat{z}) = \{\hat{w}_i(\hat{z})\} \in \mathbb{C}^g;
$$

and the integrals $\hat{w}_{z_0}$ satisfy

$$
(12.35) \quad \hat{w}_i(\hat{T}\hat{z}) = \hat{w}_i(\hat{z}) + \hat{\omega}_i(\hat{T}) \quad \text{for all} \ \hat{T} \in \Gamma_a
$$

where $\hat{\omega}_i(\hat{T}) \in \mathbb{C}$ is the period $\omega_i(T)$ for any $T \in \Gamma$ representing $\hat{T} \in \Gamma_a$. The set of period vectors $\hat{\omega}(\hat{T})$ for all $\hat{T} \in \Gamma_a$ also form the lattice subgroup $\mathcal{L}(\Omega) \subset \mathbb{C}^g$.

The chain of covering projections (12.24) naturally induces a chain of covering projections

$$
(12.36) \quad \hat{M}^r \xrightarrow{\pi_a} \hat{M}^r \xrightarrow{\hat{\pi}^r} M^r
$$

between the Cartesian products of these Riemann surfaces, with the composition

$$
(12.37) \quad \hat{\pi}^r = \hat{\pi}^r \circ \pi_a^r : \hat{M}^r \longrightarrow M^r.
$$

The symmetric group $\mathfrak{S}_r$ of permutations of $r$ entities acts naturally on these products; and the quotients have the structures of complex manifolds of dimension $r$, as in Theorem 10.5 for the case of the surface $M$ itself. The quotient $M^r/\mathfrak{S}_r$ was denoted by $M^{(r)}$ and was identified with the set of positive divisors of degree $r$ on $M$ in the discussion at the beginning of Chapter 10; the corresponding assertions and notation can be applied to surfaces $\hat{M}$ and $\hat{M}$ as well. The holomorphic mappings in (12.36) commute with the action of the symmetric group $\mathfrak{S}_r$, so there results the corresponding chain of holomorphic mappings

$$
(12.38) \quad \hat{M}^{(r)} \xrightarrow{\pi_a^{(r)}} \hat{M}^{(r)} \xrightarrow{\hat{\pi}^{(r)}} M^{(r)}.
$$
with the composition

\[(12.39) \quad \tilde{\pi}^{(r)} = \tilde{\pi}^{(r)} \circ \pi_a^{(r)} : \tilde{M}^{(r)} \longrightarrow M^{(r)}\]

The holomorphic mappings \(\tilde{w}_{z_0}\) and \(\tilde{w}_{z_0}\) in the diagram (12.32) can be extended to the symmetric products, in analogy with the extension of the holomorphic mapping \(w_{z_0}\) to the Abel-Jacobi mapping (10.26); thus there is the holomorphic mapping

\[(12.40) \quad \tilde{w}_{z_0}^{(r)} : \tilde{M}^{(r)} \longrightarrow \mathbb{C}^g\]

defined by

\[(12.41) \quad \tilde{w}_{z_0}^{(r)}(z_1 + \cdots + z_r) = \tilde{w}_{z_0}(z_1) + \cdots + \tilde{w}_{z_0}(z_r) \in \mathbb{C}^g\]

for any divisor \(z_1 + \cdots + z_r \in \tilde{M}^{(r)}\), where \(\tilde{w}_{z_0}(z)\) is the mapping (12.26), and this induces the corresponding holomorphic mapping \(\tilde{w}_{z_0}^{(r)} : \tilde{M}^{(r)} \longrightarrow \mathbb{C}^g\) defined by the restricted abelian integrals (12.34) on \(\tilde{M}\). The image of the Abel-Jacobi mapping \(w_{z_0}^{(r)} : M^{(r)} \longrightarrow J(M)\) is the irreducible holomorphic subvariety \(W_r \subset J(M)\), as in Theorem 10.9; and if the images of the holomorphic mappings \(\tilde{w}_{z_0}^{(r)}\) and \(\tilde{w}_{z_0}^{(r)}\) are denoted correspondingly by \(\tilde{W}^{(r)}\) and \(\tilde{W}^{(r)}\) there results the commutative diagram of holomorphic mappings

\[
\begin{array}{cccc}
\tilde{M}^{(r)} & \overset{\tilde{w}_{z_0}^{(r)}}{\longrightarrow} & \tilde{W}_r & \overset{\iota}{\longrightarrow} & \mathbb{C}^g \\
\pi_a^{(r)} & \downarrow & \| & \downarrow & \|
\end{array}
\]

\[
\begin{array}{cccc}
\tilde{M}^{(r)} & \overset{\tilde{w}_{z_0}^{(r)}}{\longrightarrow} & \tilde{W}_r & \overset{\iota}{\longrightarrow} & \mathbb{C}^g \\
\bar{\pi}^{(r)} & \downarrow & \pi & \downarrow & \pi
\end{array}
\]

\[
\begin{array}{cccc}
M^{(r)} & \overset{w_{z_0}^{(r)}}{\longrightarrow} & W_r & \overset{\iota}{\longrightarrow} & J(M).
\end{array}
\]

It follows from (12.25) just as in the proof of (12.30) that

\[(12.43) \quad \tilde{W}_r = \tilde{\pi}^{-1}(W_r) \quad \text{so} \quad \tilde{W}_r + \lambda = \tilde{W}_r \quad \text{for all} \quad \lambda \in \mathcal{L}(\Omega);\]

indeed if \(t \in \tilde{W}_r\) then \(t = \tilde{w}_{z_0}^{(r)}(z_1 + \cdots + z_r)\) for some divisor \(z_1 + \cdots + z_r \in \tilde{M}^{(r)}\), and since any lattice vector \(\lambda \in \mathcal{L}(\Omega)\) is the period \(\lambda = \omega(T)\) for some covering translation \(T \in \Gamma\) it follows that \(t + \lambda = \tilde{w}_{z_0}(Tz_1 + \cdots + z_r) \in \tilde{W}_r\). As the inverse image of the holomorphic subvariety \(W_r \subset J(M)\) under the holomorphic mapping \(\tilde{\pi}\) the subset \(\tilde{W}_r \subset \mathbb{C}^g\) is a holomorphic subvariety; and as the image of a connected complex manifold under the holomorphic mapping \(\tilde{w}_{z_0}^{(r)}\) it is an irreducible holomorphic subvariety.

The diagram (12.42) for \(r = 1\) reduces to the diagram (12.32) in which both \(w_{z_0}\) and \(\tilde{w}_{z_0}\) are biholomorphic mappings which identify the Riemann surfaces \(M\) and \(\tilde{M}\) with holomorphic submanifolds of \(J(M)\) and \(\mathbb{C}^g\) respectively;
but for \( r > 1 \) the situation is a bit more complicated. Subsets \( G^1_r \subset M^{(r)} \) for \( 1 < r \leq g \) were defined in (10.41) and (10.42); they are holomorphic subvarieties by Theorem 10.20, and can be identified with the proper holomorphic subvarieties \( \text{sp} M^{(r)} \subset M^{(r)} \) of special positive divisors as in (10.43). Their images \( W^1_r = w_\infty(G^1_r) \subset J(M) \) are holomorphic subvarieties of \( J(M) \) by Remmert’s Proper Mapping Theorem, as in Theorem 10.22. By Theorem 10.9 (iv) with the interpretation (10.43) the restriction

\[
(12.44) \quad w^{(r)}_{\infty} : (M^{(r)} \sim G^1_r) \longrightarrow (W_r \sim W^1_r)
\]

is a biholomorphic mapping. The inverse images \( \tilde{G}^1_r = (\tilde{\pi}^{(r)})^{-1}(G^1_r) \subset \tilde{M}^{(r)} \) and \( \tilde{W}^1_r = \pi^{-1}(W^1_r) \subset \tilde{W}_r \) then are holomorphic subvarieties for which \( \tilde{W}^1_r = \tilde{w}_\infty(\tilde{G}^1_r) \), and the restriction of the holomorphic mapping \( \tilde{w}^{(r)}_{\infty} \) is a surjective holomorphic mapping

\[
(12.45) \quad \tilde{w}^{(r)}_{\infty} : (\tilde{M}^{(r)} \sim \tilde{G}^1_r) \longrightarrow (\tilde{W}_r \sim \tilde{W}^1_r).
\]

The mappings \( \tilde{\pi}^{(r)} = \tilde{\pi}^{(r)} \circ \pi^{(r)}_a \) and \( \pi \) in (12.42) are covering mappings so it follows from (12.44) that the mapping (12.45) is a locally biholomorphic mapping. The inverse image

Now introduce an equivalence relation on the divisors in \( \tilde{M}^{(r)} \) by setting

\[
(12.46) \quad (z_1 + \cdots + z_r) \sim (T_1 \tilde{z}_1 + \cdots + T_r \tilde{z}_r)
\]

for any \( T_1 \in \Gamma \) for which \( \omega(T_1) + \cdots + \omega(T_r) = \omega(T_1 \cdots T_r) = 0 \),

and let \( \hat{M}^{(r)} = \tilde{M}^{(r)} / \sim \) be the quotient of \( \tilde{M}^{(r)} \) by this equivalence relation. This equivalence relation is a weaker equivalence relation than that defined by the quotient mapping to \( M^{(r)} \), in the sense that any divisors equivalent under the relation (12.46) have the same image in \( M^{(r)} \); and since \( \omega(T) = 0 \) for all \( T \in [\Gamma, \Gamma] \) it is a stronger equivalence relation than that defined by the quotient mapping to \( \tilde{M}^{(r)} \), in the sense that any two divisors that have the same image in \( \tilde{M}^{(r)} \) are equivalent under the relation (12.46). Consequently the covering projection \( \hat{\pi}^{(r)} : \hat{M}^{(r)} \longrightarrow M^{(r)} \) can be factored into the composition \( \hat{\pi}^{(r)} = \tilde{\pi}^{(r)} \circ \pi^{(r)}_0 \) of covering projections

\[
(12.47) \quad \hat{M}^{(r)} \xrightarrow{\hat{\pi}^{(r)}} \tilde{M}^{(r)} \xrightarrow{\tilde{\pi}^{(r)}} M^{(r)},
\]

so the quotient space \( \hat{M}^{(r)} \) has the structure of a complex manifold for which the covering projections in (12.47) are holomorphic and locally biholomorphic mappings. If \( \hat{G}^1_r = (\hat{\pi}^{(r)})^{-1}(G^1_r) \subset \hat{M}^{(r)} \) then \( \hat{G}^1_r \) is a holomorphic subvariety of \( \hat{M}^{(r)} \) and \( \hat{G}^1_r = \hat{\pi}^{(r)}_a(\hat{G}^1_r) \).

**Lemma 12.3** (i) If two divisors \( \mathcal{D}', \mathcal{D}'' \in \hat{M}^{(r)} \) are equivalent under the equivalence relation (12.46) then \( \tilde{w}^{(r)}_{\infty}(\mathcal{D}') = \tilde{w}^{(r)}_{\infty}(\mathcal{D}'') \in \tilde{W}_r \).
(ii). If \( \tilde{w}_{z_0}^{(r)}(\mathbf{v}') = \tilde{w}_{z_0}^{(r)}(\mathbf{v}'') \in \tilde{W}_r \) for two divisors \( \mathbf{v}', \mathbf{v}'' \in \tilde{M}^{(r)} \sim \tilde{G}_r^1 \) then the divisors \( \mathbf{v}' \) and \( \mathbf{v}'' \) are equivalent under the equivalence relation (12.46).

**Proof:**

(i) If \( \mathbf{v}' = z_1' + \cdots + z_p' \) and \( \mathbf{v}'' = z_1'' + \cdots + z_p'' \) are two equivalent divisors in \( \tilde{M}^{(r)} \) then after reordering the points \( z''_i \) if necessary there will be covering translations \( T_i \in \Gamma \) such that \( z''_i = T_i z'_i \) where \( \sum_{i=1}^r \omega(T_i) = 0 \). Then \( \tilde{w}_{z_0}^{(r)}(\mathbf{v}'') = \sum_{i=1}^r \tilde{w}_{z_0}^{(r)}(z''_i) = \sum_{i=1}^r \tilde{w}_{z_0}^{(r)}(T_i z'_i) = \sum_{i=1}^r \left( \tilde{w}_{z_0}^{(r)}(z'_i) + \omega(T_i) \right) = \tilde{w}_{z_0}^{(r)}(\mathbf{v}') \) as desired.

(ii) If \( \tilde{w}_{z_0}^{(r)}(\mathbf{v}') = \tilde{w}_{z_0}^{(r)}(\mathbf{v}'') \) for two divisors \( \mathbf{v}', \mathbf{v}'' \in \tilde{M}^{(r)} \sim \tilde{G}_r^1 \) then from the commutativity of the diagram (12.42) it follows that \( w_z^{(r)}(\tilde{\pi}^{(r)}(\mathbf{v}')) = w_z^{(r)}(\tilde{\pi}^{(r)}(\mathbf{v}'')) \) \( \in J(M) \). Since the mapping (12.44) is injective it must be the case that \( \tilde{\pi}^{(r)}(\mathbf{v}') = \tilde{\pi}^{(r)}(\mathbf{v}'') \); thus the divisors \( \mathbf{v}' \) and \( \mathbf{v}'' \) in \( \tilde{M}^{(r)} \) represent the same divisor in \( M^{(r)} \), so after reordering the points in these divisors as necessary there will be covering translations \( T_i \in \Gamma \) such that \( T_i z'_i = z''_i \) for each index \( i \). Then \( \tilde{w}_{z_0}^{(r)}(\mathbf{v}'') = \sum_{i=1}^r \tilde{w}_{z_0}^{(r)}(T_i z'_i) = \sum_{i=1}^r \left( \tilde{w}_{z_0}^{(r)}(z'_i) + \omega(T_i) \right) = \tilde{w}_{z_0}^{(r)}(\mathbf{v}') \), and since \( w_z^{(r)}(\tilde{\pi}^{(r)}(\mathbf{v}'')) = \tilde{w}_{z_0}^{(r)}(\mathbf{v}'') \) by assumption it follows that \( \sum_{i=1}^r \omega(T_i) = 0 \) so the two divisors \( \mathbf{v}' \) and \( \mathbf{v}'' \) are equivalent. That suffices for the proof.

The preceding Lemma 12.3 (i) shows that equivalent divisors in \( \tilde{M}^{(r)} \) have the same image under the holomorphic mapping \( \tilde{w}_{z_0}^{(r)} : \tilde{M}^{(r)} \rightarrow \tilde{W}_r; \) hence that mapping induces a holomorphic mapping

\[
(12.48) \quad \tilde{w}_{z_0}^{(r)} : \tilde{M}^{(r)} \rightarrow \tilde{W}_r.
\]

It follows from Lemma 12.3 (i) that the restriction of the mapping (12.48) is actually an injective and hence biholomorphic mapping

\[
(12.49) \quad \tilde{w}_{z_0}^{(r)} : \tilde{M}^{(r)} \sim \tilde{G}_r^1 \rightarrow \tilde{W}_r \sim \tilde{W}_r^1.
\]

From these observations it follows that the commutative diagram (12.42) can be extended to the further commutative diagram of holomorphic mappings

\[
\begin{array}{c}
\tilde{M}^{(r)} \sim \tilde{G}_r^1 \xrightarrow{\tilde{w}_{z_0}^{(r)}} \tilde{W}_r \sim \tilde{W}_r^1 \xrightarrow{\iota} \mathbb{C}^g \\
\downarrow \pi^{(r)} \quad \downarrow \\
\tilde{M}^{(r)} \sim \tilde{G}_r^1 \xrightarrow{\tilde{w}_{z_0}^{(r)}} \tilde{W}_r \sim \tilde{W}_r^1 \xrightarrow{\iota} \mathbb{C}^g
\end{array}
\]

\[
\begin{array}{c}
\tilde{M}^{(r)} \sim \tilde{G}_r^1 \xrightarrow{\tilde{w}_{z_0}^{(r)}} \tilde{W}_r \sim \tilde{W}_r^1 \xrightarrow{\iota} \mathbb{C}^g \\
\downarrow \pi \quad \downarrow \\
M^{(r)} \sim G_r^1 \xrightarrow{w_{z_0}^{(r)}} W_r \sim W_r^1 \xrightarrow{\iota} J(M)
\end{array}
\]
where the biholomorphic mappings are indicated by \( \cong \).

To return to the product cross-ratio function, that function was defined in (12.2) as a meromorphic function

\[
Q(z; a; z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_r^-)
\]

of the ordered set of variables

\[
(z; a; z_1^+, \ldots, z_r^+, z_1^-, \ldots, z_r^-) \in \tilde{M} \times \tilde{M} \times \tilde{M}^r \times \tilde{M}^r;
\]

but really it is symmetric in the variables \( z_1^+, \ldots, z_r^+ \) and in the variables \( z_1^-, \ldots, z_r^- \).

**Theorem 12.4** The product cross-ratio function of degree \( r \) can be viewed as a meromorphic function

\[
Q(z; a; z_1^+ + \cdots + z_r^+, z_1^- + \cdots + z_r^-)
\]

of the variables

\[
(z, a; z_1^+ + \cdots + z_r^+, z_1^- + \cdots + z_r^-) \in \tilde{M} \times \tilde{M} \times \tilde{M}^{(r)} \times \tilde{M}^{(r)}.
\]

**Proof:** As a function of the variable \( z \in \tilde{M} \) the quotient

\[
g(z) = \frac{q(z; a; z_1^+, z_1^-)q(z; a; z_2^+, z_2^-)}{q(z; a; z_1^+, z_1^-)q(z; a; z_1^+, z_2^-)}
\]

is a nowhere vanishing holomorphic function in \( \tilde{M} \), since the zero divisor of the numerator is the same as the zero divisor of the denominator and correspondingly for the polar divisors. The function \( g(z) \) is also a relatively automorphic function for the factor of automorphism

\[
\rho(T) = \frac{\rho_{z_1^+, z_1^-}(T)\rho_{z_2^+, z_2^-}(T)}{\rho_{z_1^+, z_1^-}(T)\rho_{z_2^+, z_2^-}(T)}
\]

which in view of the defining equation (12.3) is easily seen to reduce to \( \rho(T) = 1 \); hence the function \( g(z) \) is really a function on the compact Riemann surface \( M \), so it is actually a constant in the variable \( z \). Since \( q(z; a; z_1^+, z_1^-) = 1 \) for any values \( z_1^+, z_1^- \) it follows that \( g(z) = 1 \) for all \( z \in \tilde{M} \). That is the case for any values of the auxiliary parameters \( z_1^+, z_1^- \) so the function (12.55) is identically equal to 1 in all variables, or equivalently the product \( q(z; a; z_1^+, z_1^-)q(z; a; z_2^+, z_2^-) \) is symmetric in the parameters \( z_1^+, z_2^+ \). This argument can be applied to any pair of points among those in \( z_1^+, \ldots, z_r^+ \) or \( z_1^-, \ldots, z_r^- \), showing that the product cross-ratio function is symmetric in the parameters \( z_1^+, \ldots, z_r^+ \) as well as in the parameters \( z_1^-, \ldots, z_r^- \) and consequently can be viewed as a function of the variables \( z_1^+ + \cdots + z_r^+ \in \tilde{M}^{(r)} \) and \( z_1^- + \cdots + z_r^- \in \tilde{M}^{(r)} \), which suffices for the proof.
The factor of automorphy (12.1) for the cross-ratio function \( q(z, a; z^+, z^-) \) as a function of the variable \( z \in \hat{M} \) can be written alternatively as in (5.24), thus as the function

\[
\rho_{z^+, z^-}(T) = \exp - 2\pi \sum_{m,n=1}^{q} (w_m(z^+) - w_m(z^-)) g_{mn} \omega_n(T)
\]

for any \( T \in \Gamma \). In view of the symmetry \( q(z, a; z^+, z^-) = q(z^+, z^-; z, a) \) of the cross-ratio function as in Theorem 5.28 (ii) it follows that

\[
q(z, a; Tz^+, z^-) = \rho_{z,a}(T)q(z, a; z^+, z^-)
\]

for any \( T \in \Gamma \) and consequently that

\[
Q(z, a; T_1z_1^+ + \cdots + T_rz_r^+, \ z_1^- + \cdots + z_r^-) = \rho_{z,a}(T_1 \cdots T_r)Q(z, a; z_1^+ + \cdots + z_r^+, \ z_1^- + \cdots + z_r^-)
\]

for any covering translations \( T_1, \ldots, T_r \in \Gamma \). In particular

\[
Q(z, a; T_1z_1^+ + \cdots + T_rz_r^+, \ z_1^- + \cdots + z_r^-) = Q(z, a; z_1^+ + \cdots + z_r^+, \ z_1^- + \cdots + z_r^-)
\]

if \( \omega(T_1 \cdots T_r) = \omega(T_1) + \cdots + \omega(T_r) = 0 \)

since \( \rho_{z,a}(T_1 \cdots T_r) = 1 \) if \( \omega(T_1 \cdots T_r) = 0 \); thus the product cross-ratio function \( Q(z, a; z_1^+ + \cdots + z_r^+, \ z_1^- + \cdots + z_r^-) \) as a function of the divisor \( \hat{Z}^+ = z_1^+ + \cdots + z_r^+ \in \hat{M}^{(r)} \) is invariant under the equivalence relation (12.46) on the manifold \( \hat{M}^{(r)} \) and therefore it can be viewed as a meromorphic function of the point \( \hat{Z}^+ \in \hat{M}^{(r)} \) represented by the divisor \( \hat{Z}^+ \in \hat{M}^{(r)} \). The same argument can be applied to the divisor \( z_1^- + \cdots + z_r^- \); so altogether the product cross-ratio function can be viewed as a meromorphic function \( Q(z, a; \hat{Z}^+, \hat{Z}^-) \) on the complex manifold \( \hat{M} \times \hat{M} \times \hat{M}^{(r)} \times \hat{M}^{(r)} \).

For the special case \( r = g \) the subvariety \( \hat{W}_g \subset \mathbb{C}^g \) is the entire space \( \mathbb{C}^g \) and the subvariety \( \hat{W}_g^1 \subset \hat{W} = \mathbb{C}^g \) is a holomorphic subvariety of dimension \( g - 2 \) in \( \mathbb{C}^g \), since \( \hat{W}_g^1 \) is a covering space of the subvariety \( W_g^1 \subset J(M) \) as in the diagram (12.50) and \( W_g^1 = k - W_g - 2 \) by Theorem 10.24; and the mapping (12.49) is a biholomorphic mapping

\[
\hat{w}_g^{(g)} : (\hat{M}^{(g)} \sim \hat{G}_g^1) \longrightarrow (\mathbb{C}^g \sim \hat{W}_g^1).
\]

The composition

\[
Q \left( z, a; \hat{w}_g^{(g)}(t^+), \hat{w}_g^{(g)}(t^-) \right) = Q(z, a; t^+, t^-)
\]

of the product cross-ratio function \( Q(z, a; \hat{Z}^+, \hat{Z}^-) \) with the biholomorphic mappings (12.60) applied to the variables \( \hat{Z}^+ \) and \( \hat{Z}^- \) then is a well defined meromorphic function of points

\[
(z, a; t^+, t^-) \in \left( \hat{M} \times \hat{M} \times (\mathbb{C}^g \sim \hat{W}_g^1) \times (\mathbb{C}^g \sim \hat{W}_g^1) \right) \subset (\hat{M} \times \hat{M} \times \mathbb{C}^g \times \mathbb{C}^g).
\]
Since for each fixed point \((z, a; t^-)\) this is a meromorphic function of the variable \(t^+ \in \mathbb{C}^g \sim \hat{W}_g^1\) and the subvariety \(\hat{W}_g^1\) is of codimension 2 it follows from the Theorem of Levi\(^3\) that the function \(Q(z, a; t^+, t^-)\) extends uniquely to a meromorphic function of the variable \(t^+\) on the entire space \(\mathbb{C}^g\). The same holds as a function of the variable \(t^-\), so by Rothstein’s Theorem\(^4\) the extended function \(Q(z, a; t^+, t^-)\) is a meromorphic function on the complex manifold \(\hat{M} \times \hat{M} \times \mathbb{C}^g \times \mathbb{C}^g\); this function is called the extended cross-ratio function of the Riemann surface \(\hat{M}\).

**Theorem 12.5** (i) The extended cross-ratio function \(Q(z, a; t^+, t^-)\) of degree \(g\) for a compact Riemann surface \(M\) of genus \(g > 0\) is a meromorphic function \(Q(z, a; t^+, t^-)\) on the complex manifold \(\hat{M} \times \hat{M} \times \mathbb{C}^g \times \mathbb{C}^g\) that has the symmetries

\[
Q(z, a; t^+, t^-) = Q(a, z; t^+, t^-)^{-1} = Q(z, a; t^-, t^+)^{-1}
\]

and for any lattice vector \(\lambda \in \mathcal{L}(\Omega)\) satisfies the functional equation

\[
Q(z, a; t^+, \lambda, t^-) = \rho_{z, a}(T) Q(z, a; t, t^-) \quad \text{if} \quad \lambda = \omega(T).
\]

(ii) The divisor of the extended cross-ratio function \(Q(z, a; t^+, t^-)\) of degree \(g\) as a meromorphic function of the variable \(t^+ \in \mathbb{C}^g\) for any fixed values \((z, a, t^-)\) is the inverse image under the covering projection \(\pi : \mathbb{C}^g \rightarrow J(M)\) of the divisor

\[
(w_{z_0}(z) + W_{g-1}) - (w_{z_0}(a) + W_{g-1}) \subset J(M).
\]

**Proof:** (i) Since the product cross-ratio function is defined by (12.2) while the cross-ratio function \(q(z, a; z^+, z^-)\) has the symmetries (5.28) of Theorem 5.6 it follows that the product cross-ratio function has the symmetries corresponding to (12.63); and that is also the case when the product cross-ratio function is viewed as a function on the complex manifold \(\hat{M} \times \hat{M} \times \hat{M}(g) \times \hat{M}(g)\). These symmetries in turn are preserved under the equivalence relation (12.46), hence they carry over to the extended cross-ratio function under the biholomorphic mapping (12.60), and that suffices to demonstrate (12.63). The product cross-ratio function as a function on the complex manifold \(\hat{M} \times \hat{M} \times \hat{M}(g) \times \hat{M}(g)\) satisfies the functional equation (12.58), which is preserved under the equivalence relation (12.46) in view of the observation (12.59); and since

\[
\tilde{w}^{(g)}_{z_0}(T_1 z^+_1 + \cdots + T_g z^+_g) = \tilde{w}^{(g)}_{z_0}(z^+_1 + \cdots + z^+_g) + \omega(T_1 \cdots T_g)
\]

the functional equation (12.58) carries over to the functional equation (12.64) for the extended cross-ratio function \(Q(z, a; t^+, t^-)\) under the biholomorphic mapping (12.60).

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\(^3\)See the discussion of extension properties of meromorphic functions on page 395 in Appendix A.

\(^4\)Rothstein’s theorem is the analogue of Hartogs’s Theorem for meromorphic functions, the assertion that a function on \(\mathbb{C}^n\) that is meromorphic in each variable separately is meromorphic in all variables; see the discussion on page 395 in Appendix A.
(ii) The extended cross-ratio function was defined in terms of the product cross-ratio function

\[(12.66) \quad Q(z, a; z_1^+, \ldots, z_g^+, z_1^-, \ldots, z_g^-) = \prod_{j=1}^{g} q(z, a; z_j^+, z_j^-)\]

on \(\tilde{M}^{2g+2}\). When the function (12.66) is viewed as a function of the variables \(z_1^+, \ldots, z_g^+\), with all the other variables held fixed, its zero locus consists of those points \((z_1^+, \ldots, z_g^+)\) in \(\tilde{M}^{g}\) such that \(z = z_j\) for some index \(1 \leq j \leq g\), since the cross-ratio function \(q(z, a; z_j^+, z_j^-)\) as a function of the variable \(z_j^+\) has a simple zero just at the point \(z_j = z\). Consequently when the product cross-ratio function is viewed as a function (12.53) of the divisor \(z_1^+ + \cdots + z_g^+\), with all the other variables held fixed, its zero locus consists of those divisors for which one of the points \(z_j\) is equal to \(z\), so consists of divisors of the form \(z + M^{(g-1)}\). In particular if all the points \(a, z_1^+, \ldots, z_g^+, z_1^-, \ldots, z_g^-\) are distinct the separate points \(z_1^+, \ldots, z_g^+\) are local coordinates in an open neighborhood in \(\tilde{M}^{(g)}\); and the equation of the zero locus of the function \(Q(z, a; z_1^+ + \cdots + z_g^+, z_1^- + \cdots + z_g^-)\) of the divisor \(z_1^+ + \cdots + z_g^+\) in an open neighborhood of the point \((z_1^1, \ldots, z_g^{g-1}, z)\) at which \(z_g^+ = z\) really is just the function \(q(z, a; z_g^+, z_g^-)\), which has a simple zero at the point \(z_g = z\). Consequently the cross-ratio function (12.53) vanishes at the divisor \(z + \tilde{M}^{(g-1)}\) to first order. The local biholomorphic mapping (12.49) takes the subvariety \(\tilde{w}_{z_0}(z) + \tilde{M}^{(g-1)} \subset \tilde{M}^{(g)}\) defined by the meromorphic function \(Q(z, a; \tilde{Z}^+, \tilde{Z}^-)\) of the variable \(\tilde{Z}^+\) to the subvariety \(\tilde{w}_{z_0}(z) + \tilde{W}_{g-1} \subset \mathbb{C}^g\) defined by the meromorphic function \(Q(z, a; t^+, t^-)\) of the variable \(t^+ \in \mathbb{C}^g\), and \((\tilde{w}_{z_0}(z) + \tilde{W}_{g-1}) = \pi^{-1}(w_{z_0}(z) + W_{g-1})\). The functional equation (12.63) identifies the polar locus of the function \(Q(z, a; \tilde{Z}^+, \tilde{Z}^-)\) of the variable \(\tilde{Z}^+\) with the zero locus of the function \(Q(a, z; \tilde{Z}^+, \tilde{Z}^-)\) of the variable \(\tilde{Z}^+\), which yields the formula (12.65) and thereby concludes the proof.

Another of the intrinsic functions associated to a compact Riemann surface of genus \(g > 0\), the intrinsic theta function \(\theta_g(z; a; t)\), is defined in terms of the hyperabelian factors of automorphy discussed in Chapter 6. A hyperabelian factor of automorphy \(\zeta(T, z)\) for the action of the covering translation group \(\Gamma\) on the universal covering space \(\tilde{M}\) is a factor of automorphy that depends only on the equivalence class \(\tau \in \Gamma \simeq \Gamma / [\Gamma, \Gamma]\) represented by a covering translation \(T \in \Gamma\) and on the point \(\tilde{z} \in \tilde{M} = M / [\Gamma, \Gamma]\) represented by the point \(z \in \tilde{M}\); so any such factor of automorphy can be viewed as a factor of automorphy \(\zeta(\tau, \tilde{z})\) for the action of the abelian covering translation group \(\Gamma \simeq \Gamma / [\Gamma, \Gamma]\) on the quotient manifold \(\tilde{M} = M / [\Gamma, \Gamma]\) of the diagram (12.32). The basic family of hyperabelian factors of automorphy on \(\tilde{M}\) is a collection of \(2^{2g}\) hyperabelian factors of automorphy automorphy of characteristic class \(g\) associated to a point \(\tilde{a} \in \tilde{M}\), as discussed on page 173 in Chapter 6; it consists of the products

\[(12.67) \quad \zeta_{\tilde{a}, \tilde{a}, \tilde{a}}(\tau, \tilde{z}) = \rho_{g/2}(\tau) \zeta_{\tilde{a}, \tilde{a}}(\tau, \tilde{z})\]
of an arbitrarily chosen hyperabelian factor of automorphy \( \zeta_{g,\tilde{a}}(\tau, \tilde{z}) \) of characteristic class \( g \) and the \( 2^{2g} \) canonically parametrized\(^5\) flat factors of automorphy \( \rho_{2g}(\tau) = \pm 1 \) indexed by the \( 2^{2g} \) parameter values \( \delta \in \mathbb{Z}^{2g} \mod 2 \). The family (12.67) of factors of automorphy is uniquely and intrinsically defined on the surface \( M \), and is independent of the choice of bases for the holomorphic abelian differentials on \( M \) and for the homology of \( M \); changes in the homology basis have the effect of permuting the various individual factors of automorphy, but otherwise the explicit formula (6.86) for these factors of automorphy has the same form in terms of any choices of bases \( \omega_i(z) \) for the holomorphic abelian differential forms on \( M \) and \( \tau_j \) for the homology group \( H_1(M) \). If two points \( \tilde{a}', \tilde{a}'' \in \tilde{M} \) represent the same point in the Riemann surface \( M = \tilde{M}/\Gamma_a \) the holomorphic line bundles \( \zeta_{g,\tilde{a}'}, \delta/2 \) and \( \zeta_{g,\tilde{a}'}, \delta/2 \) are holomorphically equivalent by Corollary 6.23 (iii), so they describe the same line bundle over \( M \). Consequently associating to a point \( a \in \tilde{M} \) the holomorphic line bundle \( \widehat{\zeta}_{g,\delta/2}(a) \in P_g(M) \) described by the factor of automorphy \( \zeta_{g,\tilde{a}}, \delta/2 \) for any point \( \tilde{a} \in \tilde{M} \) that represents the point \( a \in M \) yields a well defined mapping

\[
(12.68) \quad \widehat{\zeta}_{g,\delta/2} : M \longrightarrow P_g(M)
\]

from the Riemann surface \( M \) to the set \( P_g(M) \) of holomorphic line bundles of characteristic class \( g \) over the surface \( M \). The image of this mapping is a subset

\[
(12.69) \quad \widehat{W}_{1,g,\delta/2} = \widehat{\zeta}_{g,\delta/2}(M) \subset P_g(M);
\]

and the collection of all of these subsets for \( \delta \in \mathbb{Z}^{2g} \) (mod 2) is uniquely and intrinsically defined. The notation is suggested by that used in Chapter 10.

The set \( P_g(M) \) and its subsets \( \widehat{W}_{1,g,\delta/2} \) have natural complex structures. The set \( P_g(M) \) can be identified with the Picard group \( P(M) = P_0(M) \) of topologically trivial holomorphic line bundles over \( M \) by associating to any line bundle \( \lambda \in P_g(M) \) the product bundle \( \psi(\lambda) = \lambda(\zeta_{g,0})^{-1} \in P_0(M) \) for any fixed choice of a base line bundle \( \zeta_{g,0} \in P_g(M) \). The mapping

\[
(12.70) \quad \psi : P_g(M) \longrightarrow P(M)
\]

is a bijective mapping and defines on \( P_g(M) \) the structure of a complex manifold induced from the complex structure of \( P(M) \); the induced structure on \( P_g(M) \) is easily seen to be independent of the choice of the base bundle \( \zeta_{g,0} \). The complex structure of \( P(M) \) was defined for any choice of bases for the holomorphic abelian differentials on \( M \) and for the homology of \( M \), in terms of which the period matrix \( \Omega \) and the intersection matrix \( P \) of the surface are defined. Holomorphic line bundles in the Picard group \( P(M) \) can be represented by canonically parametrized flat factors of automorphy \( \rho_t \) for parameters \( t \in \mathbb{C}^g \) as in Theorem 3.14, where two flat factors of automorphy \( \rho_{t'} \) and \( \rho_{t''} \) represent the same holomorphic line bundle in \( P(M) \) if and only if \( t' - t'' \in \mathbb{Z}^{2g} + \Omega \mathbb{C}^g \); that

\(^5\)The canonically parametrized flat factor of automorphy \( \rho_t \) for any \( t \in \mathbb{C}^g \) in terms of a choice of a basis for \( H_1(M) \) is defined in equation (3.26).
identifies the Picard group $P(M)$ with the quotient torus $\mathbb{C}^{2g}/(\mathbb{Z}^{2g} + \mathfrak{N}\mathbb{C}^{g})$. The Picard group $P(M)$ in turn can be identified with the Jacobi variety $J(M)$ of the Riemann surface $M$. The linear mapping $\Omega P : \mathbb{C}^{2g} \to \mathbb{C}^{g}$ establishes a biholomorphic mapping

$$\tag{12.71} (\Omega P)^* : P(M) \to J(M)$$

between the Picard variety $P(M) = \mathbb{C}^{2g}/(\mathbb{Z}^{2g} + \mathfrak{N}\mathbb{C}^{g})$ and the Jacobi variety $J(M) = \mathbb{C}^{g}/\Omega\mathbb{Z}^{2g}$ as in Theorem 3.23. The composition of the mappings (12.70) and (12.71) then is the biholomorphic mapping

$$\tag{12.72} \phi = (\Omega P)^* \circ \psi : P_g(M) \to J(M) \text{ defined by}$$

$$\phi(\rho_t \zeta^g) = \Omega \rho_t \in \mathbb{C}^{g}/\Omega\mathbb{Z}^{2g} = J(M) \text{ for any } \rho_t \zeta^g \in P_g(M)$$

in terms of any choice of a base line bundle $\zeta^g \in P_g(M)$. The further composition $\zeta_{g,\delta/2} = \phi \circ \tilde{\zeta}_{g,\delta/2}$ then is a mapping

$$\tag{12.73} \zeta_{g,\delta/2} : M \to J(M)$$

from the Riemann surface $M$ to the Jacobi variety $J(M)$. The images

$$\tag{12.74} W_{1,g,\delta/2} = \phi\left(\tilde{W}_{1,g,\delta/2}\right) = \zeta_{g,\delta/2}(M) \subset J(M)$$

form a collection of subsets of the Jacobi variety $J(M)$ that are intrinsically defined up to translation, since the correspondence between the Picard variety $P(M)$ and the Jacobi variety $J(M)$ is determined uniquely only up to a translation in $J(M)$.

**Theorem 12.6** (i) If $M$ is a compact Riemann surface $M$ of genus $g > 0$ then, for the mapping $\phi : P_g(M) \to J(M)$ defined in terms of the base line bundle $\zeta_{g,\delta/2}(a_0) \in P_g(M)$ for a choice of a base point $\tilde{a}_0 \in \tilde{M}$, the mapping $\zeta_{g,\delta/2} = \phi \circ \tilde{\zeta}_{g,\delta/2}$ is just the Abel-Jacobi mapping $w_{a_0} : M \to J(M)$.

(ii) The mappings $\tilde{\zeta}_{g,\delta/2} : M \to P_g(M)$ and $\zeta_{g,\delta/2} = \phi \circ \tilde{\zeta}_{g,\delta/2} : M \to J(M)$ are nonsingular holomorphic mappings.

(iii) The images $\zeta_{g,\delta/2}(M) = \tilde{W}_{1,g,\delta/2} \subset P_g(M)$ and $\zeta_{g,\delta/2}(M) = W_{1,g,\delta/2} \subset J(M)$ for parameter values $\delta \in \mathbb{Z}^{2g}$ (mod 2) are uniquely and intrinsically defined families of one-dimensional connected submanifolds, up to arbitrary translations in the Jacobi variety $J(M)$; and the mappings $\zeta_{g,\delta/2} : M \to \tilde{W}_{1,g,\delta/2}$ and $\zeta_{g,\delta/2} : M \to W_{1,g,\delta/2}$ are biholomorphic mappings.

(iv) The operation of multiplication by the line bundle $\rho_{(\delta''-\delta')/2} \in P_0(M)$ represented by the flat factor of automorphy $\rho_{(\delta''-\delta')/2}(\tau)$ is a biholomorphic mapping $\rho_{(\delta''-\delta')/2} : P_g(M) \to P_g(M)$ for which

$$\tag{12.75} \rho_{(\delta''-\delta')/2}\left(\tilde{W}_{1,g,\delta/2}\right) = \tilde{W}_{1,g,\delta''/2}.$$ 

(v) The subvarieties $W_{1,g,\delta/2} \subset J(M)$ satisfy

$$\tag{12.76} W_{1,g,\delta''/2} = W_{1,g,\delta'} + \Omega P(\delta'' - \delta')/2.$$
(vi) If $M$ is not a hyperelliptic Riemann surface the holomorphic submanifolds $\hat{W}_{1,g,\delta/2} \subset P_g(M)$ for distinct parameters $\delta \in \mathbb{Z}^{2g}$ (mod 2) are disjoint, as are the submanifolds $W_{1,g,\delta/2} \subset J(M)$.

(vii) If $M$ is a hyperelliptic Riemann surface then for any distinct parameters $\delta'/2, \delta''/2 \in \mathbb{Z}^{2g}$ (mod 2) the intersection $\hat{W}_{1,g,\delta'/2} \cap \hat{W}_{1,g,\delta''/2}$ consists of points $\lambda \in P_g(M)$ that are the images $\lambda = \hat{\zeta}_{1,\delta/2}(a)$ of Weierstrass points $a \in M$ under the mappings $\hat{\zeta}_{1,\delta/2}$ for all $\delta \in \mathbb{Z}^{2g}$.

**Proof:** (i) For any point $a \in M$ the line bundle $\hat{\zeta}_{g,\delta/2}(a) \in P_g(M)$ is represented by a factor of automorphy $\zeta_{g,\hat{a},\delta/2}(\tau, \bar{z})$ for any choice of a point $\hat{a} \in \hat{M}$ representing the point $a \in M$. For any fixed point $\hat{a}_0 \in \hat{M}$ it follows from (6.87) in Theorem 6.22, with $\delta'' = \delta' = \delta$, that

$$
(12.77) \quad \zeta_{g,\hat{a},\delta/2}(\tau, \bar{z}) = \rho(\hat{a}, \hat{a}_0)(\tau) \zeta_{g,\hat{a}_0,\delta/2}(\tau, \bar{z}),
$$

where $t(\hat{a}, \hat{a}_0) = i \Omega \hat{w}(\hat{a}, \hat{a}_0)$. If the base line bundle $\zeta_g \in P_g(M)$ is taken to be the line bundle described by the factor of automorphy $\hat{\zeta}_{g,\hat{a},\delta/2}(\tau, \bar{z})$ then (12.77) implies that $\zeta_{g,\delta/2}(a) = \rho(\hat{a}, \hat{a}_0)\zeta_g^o$, and it follows from (12.72) since $i\Omega P \Omega G = H\hat{G} = I$ that

$$
(12.78) \quad \zeta_{g,\delta/2}(a) = \phi(\hat{\zeta}_{g,\delta/2}(\hat{a})) = \phi(\rho(\hat{a}, \hat{a}_0)\zeta_g^o) = \Omega P t(\hat{a}, \hat{a}_0) = i\Omega P \Omega G \hat{w}(\hat{a}, \hat{a}_0) = w_{a_0}(a) \in J(M)
$$

hence that $\zeta_{g,\delta/2}$ is the Abel-Jacobi mapping $w_{a_0} : M \rightarrow J(M)$.

(ii) The Abel-Jacobi mapping is a holomorphic mapping; and by Corollary 10.7 for the case $r = 1$ the derivative of the Abel-Jacobi mapping has rank 1 so the Abel-Jacobi mapping $w_{a_0}$ is a nonsingular holomorphic mapping. Since $\zeta_{g,\delta/2} = w_{a_0}$ by (i) it follows that $\zeta_{g,\delta/2}$ is a nonsingular holomorphic mapping; and since $\hat{\zeta}_{g,\delta/2} = \phi^{-1} \circ \zeta_{g,\delta/2}$ and $\phi^{-1} : J(M) \rightarrow P_g(M)$ is a biholomorphic mapping it follows that $\hat{\zeta}_{g,\delta/2}$ also is a nonsingular biholomorphic mapping.

(iii) The mappings $\hat{\zeta}_{g,\delta/2}$ and $\zeta_{g,\delta/2}$ are proper holomorphic mappings so it follows from Remmert’s proper mapping theorem\(^6\) that their images are holomorphic subvarieties of the complex tori $P_g(M)$ and $J(M)$; and since the mappings are nonsingular these subvarieties are actually complex submanifolds. They are intrinsically and uniquely defined up to translations in the Jacobi variety since the mappings $\hat{\zeta}_{g,\delta/2}$ are uniquely and intrinsically defined and the mappings $\zeta_{g,\delta/2}$ are uniquely and intrinsically defined up to translations in the Jacobi variety $J(M)$. The Abel-Jacobi mapping $\hat{w} : M \rightarrow \hat{w}(M)$ is a biholomorphic mapping by Corollary 10.10 (ii), hence so is the mapping $\hat{\zeta}_{g,\delta/2}$.

(iv) For any two parameter values $\delta', \delta'' \in \mathbb{Z}^{2g}$ and any point $a \in M$ the line bundles $\zeta_{g,\delta'/2}(a)$ and $\zeta_{g,\delta''/2}(a)$ in $P_g(M)$ are represented by the factors of automorphy $\zeta_{g,\hat{a},\delta'/2}(\tau, \bar{z})$ and $\zeta_{g,\delta''/2}(\tau, \bar{z})$ It follows from Theorem 6.22 (i), with

---

\(^6\) Remmert’s proper mapping theorem is discussed on page 409 in Appendix A.3
a'' = a' = a, then

\[
(12.79) \quad \zeta_{g,\hat{a},\hat{\delta}'/2}(\tau, z) = \rho_{(\delta' - \delta)/2}(\tau) \zeta_{g,\hat{a},\delta'/2}(\tau, z),
\]

hence that

\[
(12.80) \quad \hat{\zeta}_{g,\delta''/2}(a) = \rho_{(\delta'' - \delta)/2} \hat{\zeta}_{g,\delta'/2}(a) \in P_g(M).
\]

That is the case for all points \( a \in M \), which implies (12.75).

(v) If the base line bundle \( \zeta'' \in P_g(M) \) is taken to be the line bundle \( \hat{\zeta}_{g,\delta'/2}(a) \) described by the factor of automorphy \( \zeta_{g,\hat{a},\delta'/2}(\tau, z) \) then (12.80) is just the representation \( \hat{\zeta}_{g,\delta''/2}(a) = \rho_{(\delta'' - \delta)/2} \zeta'' \in P_g(M) \) so it follows from (12.72) that

\[
(12.81) \quad \zeta_{g,\delta''/2}(a) = \phi \left( \hat{\zeta}_{g,\delta''/2} \right) = \phi \left( \rho_{(\delta'' - \delta)/2} \zeta'' \right) = \Omega P(\delta'' - \delta')/2;
\]

but \( \zeta_{g,\delta'/2}(a) = \phi \left( \zeta_{g,\delta'/2} \right) = \phi \left( \zeta'' \right) = \phi \left( \rho_0 \zeta'' \right) = \Omega P0 = 0 \) by (12.72) so the result in equation (12.81) can be written as

\[
(12.82) \quad \zeta_{g,\delta''/2}(a) = \zeta_{g,\delta'/2}(a) + \Omega P(\delta'' - \delta')/2.
\]

The same argument can be applied to any point \( a \in M \) and any parameters \( \delta', \delta'' \), and that implies (12.76).

(vi) and (vii) If \( \hat{\zeta} \in P_g(M) \) is a holomorphic line bundle on the Riemann surface \( M \) and if \( \hat{\zeta} \in \hat{W}_{1,1,\delta'/2} \subset \hat{W}_{1,1,\delta''/2} \) for distinct parameters \( \delta', \delta'' \in \mathbb{Z}^g \) (mod 2) then \( \hat{\zeta} = \hat{\zeta}_{1,\delta'/2}(a') = \hat{\zeta}_{1,\delta''/2}(a'') \) for some points \( a', a'' \in M \), hence by the definition of the mapping (12.68) the two factors of automorphy \( \hat{\zeta}_{g,\hat{a},\delta'/2}(\tau, z) \) and \( \hat{\zeta}_{g,\hat{a},\delta''/2}(\tau, z) \) for points \( \hat{a}', \hat{a}'' \in \hat{M} \) that represent the points \( a', a'' \in M \) are holomorphically equivalent. By Corollary 6.23 the two factors of automorphy \( \zeta_{g,\hat{a},\delta'/2}(\tau, z) \) and \( \zeta_{g,\hat{a},\delta''/2}(\tau, z) \) are holomorphically equivalent if and only if

\[
(12.83) \quad \hat{w}(\hat{a}'', \hat{a}') + \frac{1}{2} \Omega P(\delta'' - \delta') \in \Omega \mathbb{Z}^g;
\]

that equality therefore holds in this case and consequently \( \hat{w}(2\hat{a}'', 2\hat{a}') = 2\hat{w}(\hat{a}'', \hat{a}') \in \Omega \mathbb{Z}^g \) since \( \Omega P(\delta'' - \delta') \in \Omega \mathbb{Z}^g \), so by Abel's Theorem in the form of Theorem 12.2 it follows that there is a meromorphic function \( f \) on the Riemann surface \( M \) with the divisor \( 2 \cdot a'' - 2 \cdot a' \). That function describes a holomorphic mapping \( f : M \to \mathbb{P}^1 \) that exhibits \( M \) as a two-sheeted branched covering of \( \mathbb{P}^1 \), where \( a' \) and \( a'' \) are branch points of the mapping; and by Theorem 11.17 the surface \( M \) is hyperelliptic and the points \( a' \) and \( a'' \) are Weierstrass points of \( M \). Thus if \( M \) is not hyperelliptic there can be no intersection point \( \hat{\zeta} \) so \( \hat{W}_{1,1,\delta'/2} \cap \hat{W}_{1,1,\delta''/2} = \emptyset \); but if \( M \) is hyperelliptic the intersection point \( \hat{\zeta} \) is the image \( \hat{\zeta} = \zeta_{1,\delta'/2}(a') \) of a Weierstrass point \( a' \in M \) under the mapping \( \zeta_{1,\delta'/2} \) for some parameter \( \delta' \in \mathbb{Z}^g \). Conversely suppose that \( M \) is hyperelliptic and \( \hat{\zeta} = \zeta_{1,\delta'/2}(a') \) for a Weierstrass point \( a' \in M \) under the mapping \( \zeta_{1,\delta'/2} \)
for some parameter $\delta' \in \mathbb{Z}^g$. By Theorem 11.17 again there is a meromorphic function $f$ on $M$ that exhibits $M$ as a two-sheeted branched covering of $\mathbb{P}^1$ where $a'$ is a branch point; if $a''$ is another Weierstrass point of $M$, hence another branch point of the mapping $f$, it can be assumed by choosing suitable coordinates on $\mathbb{P}^1$ that $f(a') = \infty$ and $f(a'') = 0$, so by Abel's theorem again

$$\hat{w}(2a'' - 2a') = 2\hat{w}(a'', a') \in \Omega \mathbb{Z}^g$$

and consequently $\hat{w}(a'', a') = \Omega \frac{1}{2}\delta$ for some $\delta \in \mathbb{Z}^g$. The vector $P^{-1}\delta$ is also integral, as is the sum $\delta'' = P^{-1}\delta + \delta'$, and since $\delta = P(\delta'' - \delta')$ it follows that

$$\hat{w}(a'' - a') = \frac{1}{2}\Omega P(\delta'' - \delta'),$$

which is a form of equation (12.76) and consequently the holomorphic line bundles $\zeta_{g,a',\delta'/2}(\tau, \hat{z})$ and $\zeta_{g,a'',\delta''/2}(\tau, \hat{z})$ are holomorphically equivalent. Since the points $a', a''$ by assumption do not represent the same point of $M$ then in (12.84) the vector $\frac{1}{2}\Omega P(\delta'' - \delta')$ is not contained in the lattice subgroup $\Omega \mathbb{Z}^g$ so the parameters $\delta', \delta''$ must be distinct mod 2 and hence the submanifolds $\hat{W}_{1,g,\delta'/2}$ and $\hat{W}_{1,g,\delta''/2}$ are distinct. Thus the point $\hat{z} = \zeta_{1,\delta'/2}(a') = \zeta_{1,\delta''/2}(a'')$ is in the nontrivial intersection $\hat{W}_{1,g,\delta'/2} \cap \hat{W}_{1,g,\delta''/2}$, and that suffices to conclude the proof.

The preceding theorem exhibits a uniquely and intrinsically defined family of nonsingular biholomorphic imbeddings $\hat{\zeta}_{1,\delta/2} : M \rightarrow P_g(M)$ of the Riemann surface $M$ in the complex torus $P_g(M)$, through the basic family of hyperabelian factors of automorphy. If $M$ is not hyperelliptic the image submanifolds $\hat{\zeta}_{1,\delta/2}(M) = \hat{W}_{1,g,\delta/2} \subset P_g(M)$ are disjoint and symmetrically arranged in $P_g(M)$, in the sense that they are permuted transitively upon multiplying by the holomorphic line bundles $\rho \in P_0(M)$ for which $\rho^2 = 1$, the identity bundle. The biholomorphic mapping $\phi : P_g(M) \rightarrow J(M)$, which is defined uniquely only up to translations in the Jacobi variety, identifies the mappings $\hat{\zeta}_{1,\delta/2}$ with the Abel-Jacobi mapping $w_{\mu_0} : M \rightarrow J(M)$ so identifies the image submanifolds $W_{1,g,\delta/2} = \phi \left( \hat{W}_{1,g,\delta/2} \right) \subset J(M)$ with translates of the image submanifold $W_1 = w_{\mu_0}(M) \subset J(M)$ of the Abel-Jacobi mapping $w_{\mu_0} : M \rightarrow J(M)$. Therefore if the Riemann surface $M$ is not hyperelliptic the translates $W_1 + \frac{1}{2}\Omega n$ of the submanifold $W_1 \subset J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^g$ by inequivalent half-periods for $n \in \mathbb{Z}^g$ are disjoint submanifolds of the Jacobi variety $J(M)$.

The further discussion here requires a slight modification of the Abel-Jacobi mapping. Since the points $2\hat{w}(\hat{z}', \hat{z}_0)$ and $2\hat{w}(\hat{z}'', \hat{z}_0)$ in $\mathbb{C}^g$ represent the same point in the Jacobi variety $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^g$ whenever $\hat{z}'$ and $\hat{z}''$ represent the same point of $M$, the mapping that associates to any point $z \in M$ the point of $J(M)$ represented by the point $2\hat{w}(\hat{z}, \hat{z}_0) \in \mathbb{C}^g$ for any point $\hat{z} \in \hat{M}$ that represents $z \in M$ is a well defined holomorphic mapping

$$2w_{\mu_0} : M \rightarrow J(M)$$

from the Riemann surface $M$ into its Jacobi variety $J(M)$; naturally it is called the double Abel-Jacobi mapping.
Lemma 12.7 (i) The double Abel-Jacobi mapping is a nonsingular holomorphic mapping, the image of which is the one-dimensional irreducible holomorphic subvariety

\[(12.86) \quad 2w_{z_0}(M) = 2W_1 = \left\{ 2t \mid t \in W_1 \subset J(M) \right\}.\]

(ii) If \( M \) is not hyperelliptic the double Abel-Jacobi mapping is a biholomorphic mapping \( 2w_{z_0} : M \rightarrow 2W_1 \) and its image is a submanifold of the torus \( J(M) \).

(iii) If \( M \) is hyperelliptic then \( 2w_{z_0}(z') = 2w_{z_0}(z'') \) if and only if \( z', z'' \) are Weierstrass points of \( M \); so except for the common image of all the Weierstrass points, the subvariety \( 2W_1 \) is a holomorphic submanifold of \( J(M) \).

Proof: (i) The derivative of the double Abel-Jacobi mapping is just twice the derivative of the Abel-Jacobi mapping, hence the double Abel-Jacobi mapping also is a nonsingular holomorphic mapping. The double Abel-Jacobi mapping is a proper holomorphic mapping so its image is a holomorphic subvariety of the complex torus \( J(M) \) by Remmert’s proper mapping theorem; and as the image of a connected complex manifold the image is an irreducible holomorphic subvariety. The characterization (12.86) of the image is an immediate consequence of its definition and the definition of the subvariety \( W_1 \subset J(M) \).

(ii) and (iii) If \( 2w_{z_0}(z') = 2w_{z_0}(z'') \) for two distinct points \( z', z'' \in M \) then for any points \( \hat{z}', \hat{z}'' \in \hat{M} \) that represent the points \( z', z'' \) it follows that

\[
\hat{w}(2\hat{z}', 2\hat{z}_0) = 2\hat{w}(\hat{z}', \hat{z}_0) - 2\hat{w}(\hat{z}'', \hat{z}_0) \in \Omega \mathbb{Z}^{2g};
\]

so by Abel’s Theorem in the form of Theorem 12.2 (i) there is a meromorphic function \( f \) on \( M \) with the divisor \( \delta(f) = 2 \cdot z' - 2 \cdot z'' \). Then, as in the latter part of the proof of assertions (v) and (vi) of Theorem 12.6, the function \( f \) exhibits \( M \) as a two-sheeted branched covering of \( \mathbb{P}^1 \) where \( z' \) and \( z'' \) are branch points of the mapping; and by Theorem 11.17 the surface \( M \) is hyperelliptic and the points \( z' \) and \( z'' \) are Weierstrass points of \( M \). Thus if \( M \) is not hyperelliptic this situation cannot arise so the double Abel-Jacobi mapping is injective; and since it is a nonsingular holomorphic mapping its image is a holomorphic submanifold of the torus \( J(M) \). However if \( M \) is hyperelliptic and if \( 2w_{z_0}(z') = 2w_{z_0}(z'') \) then \( z', z'' \) are Weierstrass points of \( M \). Conversely if \( M \) is hyperelliptic and \( z', z'' \in M \) are Weierstrass points of \( M \) then there exists a meromorphic function \( f \) on \( M \) with the divisor \( \delta(f) = 2 \cdot z' - 2 \cdot z'' \) so by Abel’s Theorem in the form of Theorem 12.2 (i) again \( \hat{w}(2\hat{z}', 2\hat{z}_0) \in \Omega \mathbb{Z}^{2g} \) hence \( 2w_{z_0}(z') = 2w_{z_0}(z'') \). Since that is the case for any two Weierstrass points it follows that all Weierstrass points of \( M \) have the same image in \( 2W_1 \subset J(M) \). The mapping \( 2w_{z_0} \) is an injective nonsingular holomorphic mapping otherwise, so the complement of the common image of the Weierstrass points is a submanifold of \( J(M) \), and that suffices for the proof.

Since the individual members \( \zeta_{g,a,\delta/2}(\tau, \bar{z}) \) of the basic family of hyperabelian factors of automorphy for all parameter values \( \delta \in \mathbb{Z}^{2g} \) differ merely by sign,
their squares coincide to form the single intrinsic hyperabelian factor of automorphy of characteristic class 2g, the factor of automorphy

\[(12.87) \quad \zeta_{2g,a}(\tau, \bar{\tau}) = \zeta_{g,a,\delta/2}(\tau, \bar{\tau})^2\]

for any parameter value \(\delta \in \mathbb{Z}^{2g}\), as on page 175. The explicit formula (6.94) for this factor of automorphy is the same for any choice of bases \(\omega_i(\tau)\) for the holomorphic abelian differentials on \(M\) and \(\tau_j\) for the homology of \(M\); and it follows from (12.77) that for all \(\tilde{a} \in \tilde{M}\)

\[(12.88) \quad \zeta_{2g,\tilde{a}}(\tau, \bar{\tau}) = \rho_{2t(\tilde{a}, \tilde{a}_0)}(\tau) \zeta_{g,\tilde{a}_0}(\tau, \bar{\tau})\]

where \(2t(\tilde{a}, \tilde{a}_0) = 2t(\Omega G \tilde{w}(\tilde{a}, \tilde{a}_0))\). If two points \(\tilde{a}', \tilde{a}'' \in \tilde{M}\) represent the same point in the Riemann surface \(M = \tilde{M}/\Gamma_a\) the two factors of automorphy \(\zeta_{g,\tilde{a}',\delta/2}(\tau, \bar{\tau})\) and \(\zeta_{g,\tilde{a}'',\delta/2}(\tau, \bar{\tau})\) then are holomorphically equivalent by Corollary 6.23 (iii), so from (12.87) it is clear that the two factors of automorphy \(\zeta_{2g,\tilde{a}',\delta/2}(\tau, \bar{\tau})\) and \(\zeta_{2g,\tilde{a}'',\delta/2}(\tau, \bar{\tau})\) also are holomorphically equivalent hence they describe the same holomorphic line bundle over \(M\). The mapping that associates to any point \(a \in M\) the holomorphic line bundle described by the factor of automorphy \(\zeta_{2g,\tilde{a}}(\tau, \bar{\tau})\) for any point \(\tilde{a} \in \tilde{M}\) that represents the point \(a \in M\) thus is a well defined mapping

\[(12.89) \quad \widehat{\zeta}_{2g} : M \longrightarrow P_{2g}(M).\]

Indeed it is a holomorphic mapping since it is the composition \(\widehat{\zeta}_{2g} = s \circ \widehat{\zeta}_{g,\delta/2}\) of the holomorphic mapping \(\zeta_{g,\delta/2} : M \longrightarrow P_g(M)\) for any choice of \(\delta \in \mathbb{Z}^{2g}\) and the natural holomorphic mapping \(s : P_g(M) \longrightarrow P_{2g}(M)\) that takes a line bundle \(\lambda \in P_g(M)\) to the line bundle \(s(\lambda) = \lambda^2 \in P_{2g}(M)\); and it is uniquely and intrinsically defined since the factor of automorphy \(\zeta_{2g,\tilde{a}',\delta/2}(\tau, \bar{\tau})\) is uniquely and intrinsically defined. The image

\[(12.90) \quad \widehat{W}_{1,2g} = \widehat{\zeta}_{2g}(M) \subset P_{2g}(M)\]

is a holomorphic subvariety of the complex torus \(P_{2g}(M)\) by Remmert’s proper mapping theorem again, and it is uniquely and intrinsically defined. If the line bundle \(\zeta_{2g}(a_0) \in P_{2g}(M)\) for a choice of a base point \(a_0 \in M\) is taken as the base bundle for the mapping \(\psi_{2g} : P_{2g}(M) \longrightarrow P(M)\), where \(\psi_{2g}(\lambda) = \lambda \zeta_{2g}(a_0)^{-1}\) for any line bundle \(\lambda \in P_{2g}(M)\) as in (12.70), then the composition \(\phi_{2g} = (\Omega P)^* \circ \psi_{2g}\) for the mapping \((\Omega P)^*\) as in (12.71) is the biholomorphic mapping

\[(12.91) \quad \phi_{2g} = (\Omega P)^* \circ \psi_{2g} : P_{2g}(M) \longrightarrow J(M)\]

defined by

\[\phi_{2g}(\rho_{\zeta_{2g}}(a_0)) = \Omega Pt \in \mathbb{C}^g/\Omega \mathbb{Z}^{2g} = J(M)\]

for any \(\rho_{\zeta_{2g}}(a_0) \in P_{2g}(M)\).

The further composition \(\zeta_{2g} = \phi_{2g} \circ \widehat{\zeta}_{2g}\) is a holomorphic mapping

\[(12.92) \quad \zeta_{2g} = \phi_{2g} \circ \widehat{\zeta}_{2g} : M \longrightarrow J(M),\]
the image of which is the holomorphic subvariety

\[(12.93) \quad W_{1,2g} = \phi_{2g} \left( \widehat{W}_{1,2g} \right) = \zeta_{2g}(M) \subset J(M)\]

that is biholomorphic to the subvariety \(\widehat{W}_{1,2g} \subset P_{2g}(M)\) through the biholomorphic mapping \((12.91)\) and that is uniquely and intrinsically defined up to arbitrary translation in the Jacobi variety \(J(M)\).

**Theorem 12.8** (i) If \(M\) is a compact Riemannian surface of genus \(g > 0\) then, for the mapping \(\phi_{2g} : P_{2g}(M) \rightarrow J(M)\) defined in terms of the base line bundle \(\zeta_{2g}(a_0) \in P_{2g}(M)\) for a base point \(a_0 \in M\), the mapping \(\zeta_{2g} = \phi \circ \zeta_{2g}\) is just the double Abel-Jacobi mapping \(2w_{a_0} : M \rightarrow J(M)\).

(ii) The mappings \(\zeta_{2g} : M \rightarrow P_2(M)\) and \(\zeta_{2g} = \phi_{2g} \circ \zeta_{2g} : M \rightarrow J(M)\) are nonsingular holomorphic mappings.

(iii) If \(M\) is not hyperelliptic the two subvarieties \(\widehat{W}_{1,2g}(M) \subset P_{2g}(M)\) and \(W_{1,2g} \subset J(M)\) are uniquely and intrinsically defined holomorphic submanifolds, up to arbitrary translations in the Jacobi variety \(J(M)\); and the two mappings \(\zeta_{2g} : M \rightarrow \widehat{W}_{1,2g,\delta/2}\) and \(\zeta_{2g} : M \rightarrow W_{1,2g,\delta/2}\) are biholomorphic mappings.

(iv) If \(M\) is hyperelliptic the subvarieties \(\widehat{W}_{1,2g}(M) \subset P_{2g}(M)\) and \(W_{1,2g} \subset J(M)\) are submanifolds aside from a single singular point, which is the common image of the Weierstrass points on \(M\); and the mappings \(\zeta_{2g} : M \rightarrow \widehat{W}_{1,2g}\) and \(\zeta_{2g} = \phi_{2g} \circ \zeta_{2g} : M \rightarrow \widehat{W}_{1,2g}\) are bijective mappings from the complement of the Weierstrass points of \(M\) but map all of the Weierstrass points to the same image point.

**Proof:** (i) For any point \(a \in M\) the line bundle \(\zeta_{2g}(a) \in P_{2g}(M)\) is represented by a factor of automorphy \(\zeta_{2g}(a) = \rho(t, a_0) \zeta_{2g}(a)\) for any choice of a point \(a \in M\) representing the point \(a \in M\). It follows from \((12.88)\) that \(\zeta_{2g}(a) = \rho(t, a_0) \zeta_{2g}(a_0)\) where \(a_0 \in \hat{M}\) represents \(a_0\) and \(t(a, a_0) = i \Omega_{\hat{G}} \omega(a, a_0)\). In terms of the base bundle \(\zeta_{2g}(a_0) \in P_{2g}(M)\) it follows from \((12.91)\) since \(i \Omega_{\hat{G}} = H_{\hat{G}} = I\) that

\[(12.94) \quad \zeta_{2g}(a) = \phi_{2g} \left( \zeta_{2g}(a) \right) = \phi_{2g} \left( \rho(t, a_0) \zeta_{2g}(a_0) \right) = \Omega \rho(t, a_0) = i \Omega \omega(a, a_0) = 2w_{a_0}(a) \in J(M)\]

hence that \(\zeta_{2g}\) is the double Abel-Jacobi mapping \(2w_{a_0} : M \rightarrow J(M)\).

(ii) Since the double Abel-Jacobi mapping is a nonsingular holomorphic mapping by Lemma 12.7 (i) it follows from the preceding part (i) of this theorem that the holomorphic mapping \(\zeta_{2g}\) is a nonsingular holomorphic mapping; and since \(\zeta_{2g} = \phi_{2g}^{-1} \circ \zeta_{2g}\) while \(\phi_{2g}\) is a biholomorphic mapping if follows that \(\zeta_{2g}\) also is a nonsingular holomorphic mapping.

(iii) and (iv) Since the mapping \(\zeta_{2g}\) is just the double Abel-Jacobi mapping the asserted properties of this mapping follow from Lemma 12.7 (ii) and (iii); and since \(\phi_{2g} \left( \widehat{W}_{1,2g} \right) = W_{1,2g}\) while \(\phi_{2g} : P_{2g}(M) \rightarrow J(M)\) is a biholomorphic
mapping the the asserted properties of the mapping $\hat{\zeta}_{2g}$ follow from those of $\zeta_{2g}$, which suffices for the proof.

There are other closely related uniquely and intrinsically defined holomorphic mappings of a Riemann surface $M$ into the complex torus $P_{2g}(M)$, in particular the point bundle mapping

$$\hat{\varpi}_{2g} : M \rightarrow P_{2g}(M)$$

defined by

$$\hat{\varpi}_{2g}(a) = \zeta_{2g}^2 \in P_{2g}(M)$$

and the canonical bundle mapping

$$\hat{\kappa}_{2g} : M \rightarrow P_{2g}(M)$$

defined by

$$\hat{\kappa}_{2g}(a) = \kappa \zeta_{2g}^2 \in P_{2g}(M)$$

for the canonical bundle $\kappa$ on the Riemann surface $M$. For any choice of a base bundle $\zeta_{2g}^o \in P_{2g}(M)$ the compositions of the mappings (12.95) and (12.96) with the mapping (12.72) are the mappings

$$\varpi_{2g} = \phi \circ \hat{\varpi}_{2g} : M \rightarrow J(M)$$

and

$$\kappa_{2g} = \phi \circ \hat{\kappa}_{2g} : M \rightarrow J(M);$$

both are defined uniquely and intrinsically up to arbitrary translations in the Jacobi variety $J(M)$.

**Theorem 12.9** Let $M$ be a compact Riemann surface of genus $g > 0$.

(i) The canonical bundle mapping to the Jacobi variety $J(M)$ is the double Abel-Jacobi mapping, so that for a suitable normalization of the double Abel-Jacobi mapping

(12.99) $\hat{\kappa}_{2g} = 2w_{a_0}$.

(ii) The intrinsic hyperabelian bundle mapping of degree $2g$ and the canonical bundle mapping to the Picard variety $P(M)$ differ by a uniquely and intrinsically defined line bundle $\rho_0 \in P(M)$, so that

(12.100) $\hat{\zeta}_{2g} = \rho_0 \cdot \hat{\kappa}_{2g}$.

**Proof:** (i) For any base point $a_0 \in M$ the line bundle $\eta = \kappa \zeta_{2g}^2 \in P_{2g}(M)$ can be described alternatively as the line bundle $\eta = \zeta_{t+2a_0} \in P_{2g}(M)$ for a canonical divisor $t \in M^{(2g-2)}$; so the line bundle $\eta$ can be described by a factor of automorphy $\eta(T, z)$ for the action of the covering translation group $\Gamma$ on the universal covering space $\tilde{M}$, where $\eta(T, z) = h(Tz)/h(z)$ for a meromorphic function $h(z)$ on $\tilde{M}$ and the divisor of the function $h(z)$ is invariant under $\Gamma$ and represents the divisor $t + 2a_0^2$ on $M$. The cross-ratio function $q(z, z_0; \tilde{a}, a_0)$, where $\tilde{a}, a_0 \in \tilde{M}$ represent the points $a, a_0 \in M$, is a relatively automorphic function for the canonically parametrized flat factor of automorphy $\rho_t(a, a_0)$ for
the parameter \( t(a,a_0) = \rho(a,a_0) \); and its divisor represents the divisor \( a - a_0 \) on \( M \), as in Theorem 5.6. The product \( h(z) \exp 2q(z,z_0;\tilde{a},\tilde{a}_0) \) then is a relatively automorphic function for the factor of automorphy \( \rho_{2t(a,a_0)} \eta(T,z) \), and the divisor of this function on \( M \) is the divisor \( (t+2a_0)+2(a-a_0) = t+2a \); consequently the line bundle \( \rho_{2t(a,a_0)} \eta \) represented by the factor of automorphy \( \rho_{2t(a,a_0)}(T)\eta(T,z) \) is the line bundle \( \rho_{2t(a,a_0)} \eta = \kappa \zeta_0^2 \). Therefore when the line bundle \( \eta \) is taken as the base line bundle in \( P_{2g}(M) \) it follows that for any point \( a \in M \)

\[
x_{2g}(a) = \phi \circ \tilde{x}_{2g}(a) = \Omega P2t(a,a_0) = 2w_{a_0}(a)
\]

so the mapping \( x_{2g} \) is just the double Abel-Jacobi mapping.

(ii) By part (i) of the present theorem and part (i) of Theorem 12.8 each of the mappings \( \tilde{x}_{2g} : M \rightarrow P_{2g}(M) \) and \( \tilde{\zeta}_{2g} : M \rightarrow P_{2g}(M) \) when composed with a suitable normalization of the biholomorphic mapping \( \phi_{2g} : P_{2g}(M) \rightarrow J(M) \) is just the double Abel-Jacobi mapping; the normalization in each case amounts to multiplying the image of the mapping by the inverse of a base line bundle in \( P_{2g}(M) \) to obtain a line bundle in the Picard variety \( P_0(M) \) and then applying the biholomorphic mapping \( \phi : P_0 \rightarrow J(M) \) to any representative of the line bundle in \( P_0(M) \) by a canonically parametrized flat line bundle. Thus at any point \( a \in M \)

\[
\phi(\tilde{x}_{2g}(a_0)^{-1}\tilde{x}_{2g}(a)) = \phi(\tilde{\zeta}_{2g}(a_0)^{-1}\tilde{\zeta}_{2g}(a)) = w_{a_0}(a),
\]

and since \( \phi \) is a biholomorphic mapping

\[(12.101)\]

\[
\tilde{x}_{2g}(a_0)^{-1}\tilde{x}_{2g}(a) = \tilde{\zeta}_{2g}(a_0)^{-1}\tilde{\zeta}_{2g}(a).
\]

Any two line bundles in \( P_{2g}(M) \) differ by the product with a line bundle in \( P_0(M) = P(M) \), so in particular \( \tilde{\zeta}_{2g}(a_0) = \rho_0 \cdot \tilde{\zeta}_{2g}(a_0) \) for some holomorphic line bundle \( \rho_0 \in P(M) \); and substituting this in (12.101) shows that \( \tilde{\zeta}_{2g}(a) = \rho_0 \cdot \tilde{\zeta}_{2g}(a) \) for all \( a \in M \), which demonstrates (12.100). The line bundle \( \rho_0 \) is uniquely and intrinsically defined, since the line bundles \( \tilde{\zeta}_{2g}(a_0) \) and \( \tilde{x}_{2g}(a_0) \) are uniquely and intrinsically defined, and that concludes the proof.

The point bundle mapping \( x_{2g} \) to the Jacobi variety can be identified with the 2g multiple of the Abel-Jacobi mapping by the same argument as in the proof of the preceding theorem. What is perhaps more interesting is that the preceding theorem provides an almost explicit intrinsic formula for the Jacobi bundle of a compact Riemann surface; for from (12.100) and the defining equation (12.96) it follows that \( \tilde{\zeta}_{2g}(a) = \rho_0 \cdot \tilde{x}_{2g}(a) = \rho_0 \cdot \kappa \cdot \zeta_0^2 \) for any point \( a \in M \) and consequently that

\[(12.102)\]

\[
\kappa = \rho_0^{-1} \cdot \zeta_0^{-2} \cdot \tilde{\zeta}_{2g}(a) \quad \text{for any} \quad a \in M,
\]

where the intrinsic hyperabelian factor of automorphy \( \tilde{\zeta}_{2g}(a) \) is given by the explicit formula (6.94). The line bundle \( \rho_0 \) is uniquely and intrinsically defined, although its identification remains to be discussed; in particular there is the question whether it is just the identity line bundle.
The more general hyperabelian factors of automorphy of characteristic class $g$ considered in Chapter 6 are the products
\begin{equation}
(12.103) \quad \zeta_{g,a,t}(\tau, z) = \rho_t(\tau) \zeta_{g,a}(\tau, z)
\end{equation}
for the canonically parametrized flat factors of automorphy $\rho_t(\tau)$ for any parameter $t \in \mathbb{C}^{2g}$, and the members of the basic family of hyperabelian factors of automorphy of (12.67) are a special case; but as in (8.8) each such factor of automorphy represents a holomorphic line bundle $\hat{\mathcal{Q}}_{g,t}(a) \in \mathcal{P}_g(M)$ of characteristic class $g$ over the Riemann surface $M$, and that bundle depends only on the point $a \in M$ represented by the point $\hat{a} \in \hat{M}$. It follows immediately from the Riemann-Roch Theorem that $\gamma(\hat{\mathcal{Q}}_{g,t}(a)) \geq 1$, so there exist relatively automorphic functions for these factors of automorphy, including the intrinsic theta functions considered in Chapter 8. These are the nontrivial holomorphic functions $\theta_g(z,a;t)$ of the variables $z,a \in \hat{M}, t \in \mathbb{C}^{2g}$ defined by the locally uniformly convergent series
\begin{equation}
(12.104) \quad \theta_g(z,a;t) = \sum_{\tau \in H_1(M)} \zeta_{g,a,t}(\tau, z)^{-1},
\end{equation}
where the summation is extended over representatives of the covering translation elements $T \in \Gamma$ in the abelianized group $H_1(M) = \Gamma/[\Gamma,\Gamma]$. By Theorem 8.3
\begin{equation}
(12.105) \quad \theta_g(Tz,a;t) = \theta(z,a;t - \bar{Z}\nu) = \zeta_{g,a,t}(T, z)\theta_g(z,a;t)
\end{equation}
and
\begin{equation}
(12.106) \quad \theta_g(z,a;t) = \Theta \left( t - i \bar{G}\bar{w}(z,a); \bar{Z} \right),
\end{equation}
as in (8.22). Since the classical theta function is an even function it follows that $\theta_g(z,a;t) = \Theta \left( -t + i \bar{G}\bar{w}(z,a); \bar{Z} \right) = \Theta \left( -t - i \bar{G}\bar{w}(a,z); \bar{Z} \right)$ hence that the intrinsic theta function has the symmetry
\begin{equation}
(12.107) \quad \theta_g(z,a;t) = \theta_g(a,z;-t).
\end{equation}

If $t \in \mathbb{C}^{2g}$ is a point for which $\Theta(t) \neq 0$ then $\theta_g(a,a;t) = \Theta(t) \neq 0$ so the function $\theta_g(z,a;t)$ is a nontrivial holomorphic function of the variables $(z,a) \in \hat{M} \times \hat{M}$. 

The intrinsic theta function $\theta_g(z, a; t)$ is a nontrivial holomorphic function of the variables $(z, a, t) \in \widetilde{M} \times \widetilde{M} \times \mathbb{C}^{2g}$, and as a function of the variable $z \in \widetilde{M}$ for any fixed values $(a, t) \in \widetilde{M} \times \mathbb{C}^{2g}$ it is a relatively automorphic function for the factor of automorphy $\zeta_{g, a, t}(\tau, z)$ by Theorem 8.3 (i). This function can be written in terms of the classical theta series as

\begin{equation}
\theta_g(z, a; t) = \Theta \left( t - i \overline{\Gamma G} \overline{w}(z, a); \overline{Z} \right)
\end{equation}

for the $2g \times 2g$ complex matrix $\overline{Z}$ as in (8.22). The zero locus $\overline{V}_0 \subset \mathbb{C}^g$ of the holomorphic function $\Theta(t)$ is a union of two subvarieties $\overline{V}_0 = \overline{V}_1 \cup \overline{V}_2$; and so long as $t \notin \overline{V}_1$ or the theta function $\Theta(t)$ does not vanish at the second order at $t$ the intrinsic theta function $\theta_g(z, a; t)$ is a nontrivial relatively automorphic function in the variable $z \in \widetilde{M}$; but if $t \notin \overline{V}_1$ then $\theta_g(z, a; t) = 0$ for all points $z, a \in \widetilde{M}$, by Theorem 8.14.
The Brill-Noether Diagram

[PRELIMINARY]

The Riemann-Roch Theorem is the basic result about the dimensions \( \gamma(\lambda) = \dim \Gamma(M, \mathcal{O}(\lambda)) \) of the spaces of holomorphic cross-sections of holomorphic line bundles \( \lambda \) over a compact Riemann surface \( M \). For many purposes it is more convenient to focus on the difference \( \gamma(\lambda) - 1 \), the dimension of the complex projective space \( \mathbb{P}\Gamma(M, \mathcal{O}(\lambda)) \) associated to the vector space \( \Gamma(M, \mathcal{O}(\lambda)) \). Of course \( \gamma(\lambda) \geq 0 \) while from the Riemann-Roch Theorem it follows that

\[
\gamma(\lambda) = \gamma(\kappa \lambda^{-1}) + c(\lambda) + 1 - g \geq c(\lambda) + 1 - g,
\]

and consequently

\[
\gamma(\lambda) - 1 \geq \max(-1, c(\lambda) - g).
\]

On the other hand \( \gamma(\lambda) = 0 \) if \( c(\lambda) < 0 \) by Corollary 1.4 and \( \gamma(\lambda) = 0 \) or 1 if \( c(\lambda) = 0 \) by Corollary 1.5, while \( \gamma(\lambda) \leq c(\lambda) + 1 \) if \( c(\lambda) > 0 \) by Theorem 2.7, so

\[
\gamma(\lambda) - 1 \leq \max(-1, c(\lambda)).
\]

More precise upper bounds can be described in terms of the maximal function of the compact Riemann surface \( M \), the function \( \mu(r) \) of integers \( r \in \mathbb{Z} \) defined by

\[
\mu(r) = \sup \left\{ \gamma(\lambda) - 1 \left| \lambda \in P_r(M) \right. \right\}
\]

where \( P_r(M) \) is the set of holomorphic line bundles over \( M \) of characteristic class \( r \). As will become clear in the later discussion, the maximal function shares some of the basic properties of the local maximal function \( \mu_a(r) \) defined in (11.20) in the preceding chapter, and is a somewhat related invariant. Some general properties of the maximal function, special cases of which were demonstrated for the local maximal function in Theorem 11.7, can be established quite easily.

**Theorem 13.1** The maximal function of a compact Riemann surface satisfies

\[
\mu(r) \leq \mu(r + 1) \leq \mu(r) + 1
\]
for all \( r \in \mathbb{Z} \), while

\[
(13.6) \quad \mu(r) = -1 \quad \text{for} \quad r < 0 \quad \text{and} \quad \mu(r) = r - g \quad \text{for} \quad r > 2g - 2.
\]

In particular

\[
(13.7) \quad \mu(0) = 0 \quad \text{and} \quad \mu(2g - 2) = g - 1
\]

and if \( g > 0 \)

\[
(13.8) \quad \mu(1) = 0 \quad \text{and} \quad \mu(2g - 3) = g - 2.
\]

**Proof:** First let \( \lambda_r \) be a holomorphic line bundle for which \( c(\lambda_r) = r \) and \( \gamma(\lambda_r) - 1 = \mu(r) \). For any point bundle \( \zeta_p \) clearly \( c(\lambda_r \zeta_p) = r + 1 \) while \( \gamma(\lambda_r \zeta_p) \geq \gamma(\lambda_r) \) by Lemma 2.6, so \( \mu(r + 1) \geq \gamma(\lambda_r \zeta_p) - 1 \geq \gamma(\lambda_r) - 1 = \mu(r) \), which is the first inequality in (13.4). On the other hand \( c(\lambda_r \zeta_p^{-1}) = r - 1 \) while \( \gamma(\lambda_r \zeta_p^{-1}) \geq \gamma(\lambda_r) - 1 \) by Lemma 2.6 again, so \( \mu(r - 1) \geq \gamma(\lambda_r \zeta_p^{-1}) - 1 \geq \gamma(\lambda_r) - 2 = \mu(r) - 1 \), which is equivalent to the second inequality in (13.4). By the Riemann-Roch Theorem \( \gamma(\lambda) = \gamma(\kappa \lambda^{-1}) + r + 1 - g \) so

\[
\mu(r) = \sup \left\{ \gamma(\lambda) - 1 \mid c(\lambda) = r \right\} \\
= \sup \left\{ \gamma(\kappa \lambda^{-1}) + (r + 1 - g) - 1 \mid c(\lambda) = r \right\} \\
= (r + 1 - g) + \sup \left\{ \gamma(\lambda') - 1 \mid c(\lambda') = 2g - 2 - r \right\} \\
= (r + 1 - g) + \mu(2g - 2 - r)
\]

where \( \lambda' = \kappa \lambda^{-1} \), thus yielding (13.5). The first part of (13.6) follows immediately from (13.2) while from the Riemann-Roch Theorem again \( \gamma(\lambda) = \gamma(\kappa \lambda^{-1}) + c(\lambda) + 1 - g = c(\lambda) + 1 - g \) if \( c(\lambda) > c(\kappa) = 2g - 2 \), which yields the second part of (13.6). If \( c(\lambda) = 0 \) then \( \gamma(\lambda) \leq 1 \) by (13.2) and \( \gamma(\lambda) = 1 \) when \( \lambda \) is the trivial bundle, so \( \mu(0) = 0 \); and it then follows from (13.5) that \( \mu(2g - 2) = \mu(0) + g - 1 = g - 1 \), which yields (13.7). Finally if \( g > 0 \) and \( c(\lambda) = 1 \) then \( \gamma(\lambda) \leq 1 \) by Theorem 2.7 while \( \gamma(\lambda) = 1 \) if \( \lambda \) is a point bundle, so \( \mu(1) = 0 \); and it then follows from (13.5) that \( \mu(2g - 3) = \mu(1) + g - 2 = g - 2 \), which suffices to conclude the proof.

For Riemann surfaces of small genus the maximal function is fully determined by the preceding theorem. Indeed if \( g = 0 \) it follows immediately from (13.6) that

\[
(13.9) \quad \mu(r) = \begin{cases} 
-1 & \text{for} \quad r < 0 \\
r & \text{for} \quad r \geq 0
\end{cases} \quad \text{if} \quad g = 0;
\]

if \( g = 1 \) it follows immediately from (13.6) and (13.7) that

\[
(13.10) \quad \mu(r) = \begin{cases} 
-1 & \text{for} \quad r < 0 \\
0 & \text{for} \quad r = 0 \\
r - 1 & \text{for} \quad r \geq 1
\end{cases} \quad \text{if} \quad g = 1;
\]
and if \( g = 2 \) it follows immediately from (13.6), (13.7) and (13.8) that

\[
\mu(r) = \begin{cases} 
-1 & \text{for } r < 0 \\
0 & \text{for } r = 0, 1 \\
1 & \text{for } r = 2 \\
r - 2 & \text{for } r \geq 3 
\end{cases}
\]

if \( g = 2 \).

For surfaces of genus \( g \geq 3 \) the value of the maximal function in the interval \( 2 \leq g \leq 2g - 2 \) depends on the particular Riemann surface while outside that range it follows from (13.6), (13.7) and (13.8) that

\[
\mu(r) = \begin{cases} 
-1 & \text{for } r < 0 \\
0 & \text{for } r = 0, 1 \\
r - g & \text{for } r > 2g - 2 
\end{cases}
\]

if \( g \geq 3 \).

Any integral-valued function \( \mu(r) \) of the integers that satisfies (13.4), that is, that satisfies

\[
\mu(r) \leq \mu(r + 1) \leq \mu(r) + 1 \quad \text{for all } r \in \mathbb{Z},
\]

can be described fully by the parameters

\[
\begin{align*}
n_+ &= \sup \left\{ \mu(r) \mid r \in \mathbb{Z} \right\}, \\
n_- &= \inf \left\{ \mu(r) \mid r \in \mathbb{Z} \right\}, \\
r_i &= \inf \left\{ r \in \mathbb{Z} \mid \mu(r) \geq i \right\} \quad \text{for } i \leq n_+ \\
\end{align*}
\]

for it is clear from the preceding equation that

\[
\mu(r) = i \quad \text{for } r_i \leq r < r_{i+1}
\]

and that

\[
\mu(r) - \mu(r - 1) = \begin{cases} 
1 & \text{if } r = r_i \text{ for some } i \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

It is convenient to set \( r_i = +\infty \) for \( i > n_+ \), while it follows from the preceding definitions that \( r_i = -\infty \) for \( i \leq n_- \). It is also clear that \( r_i < r_{i+1} \) for \( n_- \leq i < n_+ \) since \( \mu(r_{i+1}) = i + 1 \) while \( \mu(r_{i+1}) - 1 = i \) so \( r_i \leq r_{i+1} - 1 \).

For some purposes it is also useful to consider the dual function

\[
\mu^*(s) = s - \mu(s),
\]

for which

\[
\mu^*(s + 1) - \mu^*(s) = 1 - (\mu(s + 1) - \mu(s))
\]

and consequently

\[
\mu^*(s) \leq \mu^*(s + 1) \leq \mu^*(s) + 1;
\]
thus the dual function $\mu^*(s)$ satisfies the same basic equation as does the function $\mu(r)$, so in parallel with the preceding discussion introduce the corresponding basic parameters

\begin{align}
(13.20) & \quad n_+^* = \sup \left\{ \mu^*(s) \mid s \in \mathbb{Z} \right\}, \\
& \quad n_-^* = \inf \left\{ \mu^*(s) \mid s \in \mathbb{Z} \right\}, \\
& \quad s_j = \inf \left\{ s \in \mathbb{Z} \mid \mu^*(s) \geq j \text{ for } j \leq n_+^* \right\};
\end{align}

it is clear from the preceding equation that

\begin{align}
(13.21) & \quad \mu^*(s) = j \text{ for } s_j \leq s < s_{j+1} \\
(13.22) & \quad \mu^*(s) - \mu^*(s - 1) = \begin{cases} 1 & \text{if } s = s_j \text{ for some } j \\ 0 & \text{otherwise.} \end{cases}
\end{align}

It is convenient to set $s_j = +\infty$ for $j > n_+^*$, while it follows from the preceding equation that $s_j = -\infty$ for $j \leq n_-^*$. It is clear that $s_j < s_{j+1}$ for $n_-^* \leq j < n_+^*$ as before. Furthermore for any integer $n \in \mathbb{Z}$ it follows immediately from (13.18) that

\begin{align}
(13.23) & \quad \text{either } n = r_i \text{ for some } i \text{ or } n = s_j \text{ for some } j \text{ but not both,}
\end{align}

so the sets $\{r_i\}$ and $\{s_j\}$ are disjoint and cover $\mathbb{Z}$.

The maximal function of a compact Riemann surface satisfies (13.13) so it can be described fully by the parameters (13.14). The Riemann sphere is a somewhat anomalous Riemann surface in many ways, and its maximal function is fully determined by (13.9); so to avoid considering too many special cases the subsequent discussion in this chapter generally will be limited to compact Riemann surfaces of genus $g > 0$. It is evident from Theorem 13.1 that

\begin{align}
(13.24) & \quad n_+ = +\infty \quad \text{and} \quad n_- = -1;
\end{align}

the parameters $r_i$ are called the critical values of the Riemann surface $M$. The dual function $\mu^*(r) = r - \mu(r)$ is called the dual maximal function of the Riemann surface $M$, and its invariants $s_j$ are called the gap values of the Riemann surface $M$. From (13.23) it follows that the critical values and gap values are disjoint, and any integer is either a critical value or a gap value. When it is necessary or convenient to specify the Riemann surface $M$ explicitly the maximal function and dual maximal function will be denoted by $\mu_M(r)$ and $\mu_M^*(r)$, and the critical values and gap values will be denoted correspondingly by $r_i(M)$ and $s_j(M)$. The maximal function of the surface $M$ is determined fully by either the critical values $r_i$ or the gap values $s_j$ of that surface. Since $\mu(r) = -1$ for $r < 0$ it follows from the definition of the critical values that

\begin{align}
(13.25) & \quad r_i = -\infty \quad \text{for } i < 0;
\end{align}
and since \( \mu(r) = r - g \) for \( r > 2g - 2 \) it further follows from the definition of the critical values that

\[(13.26) \quad r_i = g + i \quad \text{for} \quad i \geq g.\]

It was already observed in the preceding general discussion of consequences of the basic inequality (13.4) that \( r_i < r_{i+1} \) for \( n_- \leq i < n_+ \), so for the critical values of the Riemann surface

\[(13.27) \quad r_i < r_{i+1} \quad \text{for} \quad 0 \leq i < +\infty,\]

while of course \( r_i = r_{i+1} = -\infty \) for \( i < -1 \) as in (13.25). Since \( \mu(0) = \mu(1) = 0 \) by (13.7) and (13.8) it follows that

\[(13.28) \quad r_0 = 0, \quad \text{and} \quad r_1 > 1;\]

while since \( \mu(2g - 3) = g - 2 \) and \( \mu(2g - 2) = g - 1 \) by the same equations it also follows that

\[(13.29) \quad r_{g-1} = 2g - 2.\]

The gap values of a compact Riemann surface of genus \( g > 0 \) are the complement of the critical values; so since all integers \( r \geq 2g \) are critical values while no integers \( r < 0 \) are critical values it follows that all integers \( s < 0 \) are gap values and the remaining gap values are just those integers in the interval \((0, 2g - 1)\) that are not critical values. In more detail, since \( \mu(r) = -1 \) for \( r < 0 \) it follows that \( \mu^*(r) = r - \mu(r) = r + 1 \) for \( r < 0 \) hence by the definition of the gap values

\[(13.30) \quad s_j = j - 1 \quad \text{for} \quad j \leq 0.\]

and since all integers \( r \geq 2g \) are critical values by (13.26) none of these integers are gap values so

\[(13.31) \quad s_j = +\infty \quad \text{for} \quad j > g.\]

Further since \( \mu(r_i) = i \) and \( \mu(r_i - 1) = i - 1 \) for \( i \geq 0 \) substituting these values into (13.5) shows that

\[\mu(2g - 2 - r_i) = i + g - 1 - r_i \quad \text{and} \quad \mu(2g - 1 - r_i) = i + g - 1 - r_i,\]

or in terms of the dual maximal function

\[\mu^*(2g - 2 - r_i) = g - i - 1 \quad \text{and} \quad \mu^*(2g - 1 - r_i) = g - i\]

for \( i \geq 0 \), and consequently \( s_{g-i} = 2g - 1 - r_i \) for \( i \geq 0 \) or equivalently

\[(13.32) \quad s_j = 2g - 1 - r_{g-j} \quad \text{for} \quad j \leq g.\]

In particular since \( r_0 = 0 \) and \( r_{g-1} = 2g - 2 \)

\[(13.33) \quad s_1 = 1 \quad \text{and} \quad s_g = 2g - 1.\]

For convenience the preceding results about the ranges of the critical and gap values are summarized as follows.
**Theorem 13.2** For a compact Riemann surface $M$ of genus $g > 0$ the critical values satisfy

\[(13.34)\]

$$0 = r_0 \leq 1 < r_1 < r_2 < \cdots < r_{g-1} = 2g - 2$$
while

\[(13.35)\]

$$r_i = -\infty \text{ for } i < 0 \text{ and } r_i = g + i \text{ for } i \geq g.$$ 

The complementary gap values $s_j$ satisfy

\[(13.36)\]

$$1 = s_1 < s_2 < \cdots < s_g = 2g - 1$$
while

\[(13.37)\]

$$s_j = +\infty \text{ for } j > g \text{ and } s_j = j - 1 \text{ for } j \leq 0.$$ 

**Proof:** Since this is just a summary of the preceding discussion no further proof is required.

It is traditional to call the $g$ positive gap values (13.36) the *Weierstrass gap values* of the Riemann surface $M$. The maximal function of a compact Riemann surface $M$ can be represented conveniently and usefully by the *Brill-Noether diagram*, as in the example in Figure 13.1. In this figure the characteristic classes $r = c(\lambda)$ of holomorphic line bundles over $M$ extend along the horizontal axis while the projective dimensions $\nu = \gamma(\lambda) - 1$ of the spaces of holomorphic cross-sections of these line bundles extend along the vertical axis. The upper heavy broken line is the graph of the maximal function itself, consisting of the line segments connecting points $\left(r, \mu(r)\right)$; for short it is called the *maximal curve* of the Brill-Noether diagram. The lower heavy broken line is the graph of the lower bound $\mu_-(r) = \max(-1, r - g)$ for the values $\gamma(\lambda) - 1$ for holomorphic line bundles $\lambda$ with $c(\lambda) = r$, as given in (13.1); for short this broken line is called the *minimal curve* of the Brill-Noether diagram, and the function $\mu_-(r)$ is called the *minimal function* of the Riemann surface $M$. The shaded region in the diagram, lying between the maximal and minimal curves, thus consists of those points $(r, \nu)$ for which there may be line bundles $\lambda$ for which $r = c(\lambda)$ and $\nu = \gamma(\lambda) - 1$. The maximal and minimal curves coincide with the horizontal straight line of height $-1$ for $r < 0$, and coincide with the straight line of slope $1$ if $r > 2g - 2$; those are the ranges of values the characteristic classes $c(\lambda)$ of holomorphic line bundles $\lambda$ for which the dimension $\gamma(\lambda)$ is determined completely by the value $c(\lambda)$ through the Riemann-Roch Theorem, as in Theorem 13.2. The critical value $r_i$ of $M$ is that point on the horizontal axis at which the maximal curve first takes the value $i$; the critical values are indicated explicitly on the diagram in Figure 13.1, while the points on the horizontal axis that are not the critical values are the gap values of $M$. The Riemann-Roch Theorem takes the form of a symmetry of the Brill-Noether diagram about the axis $r = g - 1$, as is evident upon examining Figure 13.1 more closely. Indeed formula (13.5), which is a direct consequence of the Riemann-Roch Theorem, can be rewritten

\[(13.38)\]

$$\mu(2g - 2 - r) - \frac{2g - 2 - r}{2} = \mu(r) - \frac{r}{2}. $$
so it asserts that the difference $\mu(r) - \frac{r}{2}$ is symmetric about the axis $r = g - 1$. Similarly the minimal function $\mu_-(r) = \max(-1, r - g)$ for $0 \leq r \leq g - 1$ satisfies

$$
\mu_-(2g - 2 - r) - \frac{2g - 2 - r}{2} = g - 2 - r - \frac{2g - 2 - r}{2} = -1 - \frac{r}{2} = \mu_-(r) - \frac{r}{2}
$$

so it too is symmetric about the axis $r = g - 1$. Consequently the difference $\mu(r) - \mu_-(r)$, the height of the maximal curve above the minimal curve, also is symmetric about the axis $r = g - 1$. The pattern of increases in the height of the maximal curve above the minimal curve to the left of this axis as $r$ increases is reflected in a corresponding pattern of increases in the height of the maximal curve above the minimal curve to the right of that axis as $r$ decreases. Horizontal line segments of the maximal curve to the left of the axis are reflected in line segments parallel to the minimal curve to the right of the axis. In terms of the critical values this symmetry takes the form

(13.39) $r \geq 0$ is a critical value if and only if $2g - 1 - r$ is a gap value.

These symmetries hold in general cases as well as in the special case considered in Figure 13.1.

To each point $(r, \nu)$ in the Brill-Noether diagram of a compact Riemann surface $M$ of genus $g > 0$ there can be associated the set of holomorphic line bundles $\lambda$ for which $c(\lambda) = r$ and $\gamma(\lambda) - 1 = \nu$, the set

(13.40) $\hat{X}_r^\nu = \{ \lambda \in P_r(M) \mid \gamma(\lambda) - 1 = \nu \} \subset P_r(M)$
where \( P_r(M) \) is the complex torus consisting of those holomorphic line bundles of characteristic class \( r \). It is evident that

\[
\hat{X}_r^{\nu} = \hat{W}_r^{\nu} \sim \hat{W}_r^{\nu+1}
\]

where \( \hat{W}_r^{\nu} = \{ \lambda \in P_r(M) \mid \gamma(\lambda) - 1 \geq \nu \} \) are the holomorphic subvarieties of \( P_r(M) \) defined in (10.45). The subset \( \hat{X}_r^{\nu} \subset P_r(M) \) thus is not necessarily a holomorphic subvariety of the complex torus \( P_r(M) \); but as the subset described by (13.41) in terms of the holomorphic subvarieties \( \hat{W}_r^{\nu+1} \subset \hat{W}_r^{\nu} \) the set \( \hat{X}_r^{\nu} \) at least has the structure of a holomorphic variety, since it is a holomorphic subvariety of the complex torus \( P_r(M) \) in an open neighborhood of each of its points. The sets \( \hat{X}_r^{\nu} \) thus are examples of sets that have natural complex analytic structures but do not have such natural structures as algebraic varieties. For convenience the sets of line bundles associated to points on the maximal curve are called the \textit{maximal line bundles} and are denoted by \( \hat{X}_r^{\text{MAX}} \); while the sets of line bundles associated to points on the minimal curve are called the \textit{minimal line bundles} and are denoted by \( \hat{X}_r^{\text{MIN}} \); thus

\[
\hat{X}_r^{\text{MAX}} = \hat{X}_r^{\mu(r)} \quad \text{and} \quad \hat{X}_r^{\text{MIN}} = \hat{X}_r^{\mu-(r)} = \hat{X}_r^{\text{max}(-1,r-g)}
\]

while

\[
P_r(M) = \bigcup_{\nu = \mu-(r)}^{\mu(r)} \hat{X}_r^{\nu} = \hat{X}_r^{\text{MIN}} \cup \cdots \cup \hat{X}_r^{\text{MAX}}
\]

and the holomorphic subvariety \( \hat{W}_r^{\nu} \subset P_r(M) \) is the union

\[
\hat{W}_r^{\nu} = \bigcup_{\sigma = \nu}^{\mu(r)} \hat{X}_r^{\sigma} = \hat{X}_r^{\nu} \cup \cdots \cup \hat{X}_r^{\text{MAX}}.
\]

The set \( \hat{X}_r^{\text{MAX}} = \hat{W}_r^{\mu(r)} \) hence actually is a holomorphic subvariety of the complex torus \( P_r(M) \) for any index \( r \). Of course

\[
\hat{X}_r^{\text{MAX}} = \hat{X}_r^{\text{MIN}} = P_r(M) \quad \text{if} \quad r < 0 \quad \text{or} \quad r > 2g - 2.
\]

It follows from Corollary 1.5 that \( \hat{X}_0^{\text{MAX}} = \hat{X}_0^{\text{MIN}} \) consists of the identity bundle alone; and it follows from the Canonical Bundle Theorem, Theorem 2.18, that \( \hat{X}_g^{\text{MAX}} = \hat{X}_g^{\text{MIN}} \) consists of the canonical bundle \( \kappa \) alone. For some simple examples or Brill-Noether diagrams, the maximal functions for compact Riemann surfaces for \( g = 0, 1, 2 \) were given explicitly in (13.9), (13.10) and (13.11); so the associated Brill-Noether diagrams are as in Figure 13.2, in which the subsets \( \hat{X}_r^{\nu} \) are as indicated.

The symmetry about the axis \( r = g-1 \) of the maximal and minimal curves in the Brill-Noether diagram can be extended to a corresponding symmetry of the varieties \( \hat{X}_r^{\nu} \). Indeed from the symmetric form of the Riemann-Roch Theorem,
the Brill-Noether formula of Corollary 2.21 stating that $C(\lambda) = C(\kappa\lambda^{-1})$ where $C(\lambda) = c(\lambda) - 2(\gamma(\lambda) - 1)$ is the Clifford Index of a holomorphic line bundle $\lambda$, it follows that

$$\gamma(\lambda) - 1 - \frac{1}{2} c(\lambda) = -\frac{1}{2} C(\lambda) = -\frac{1}{2} C(\kappa\lambda^{-1}) = \gamma(\kappa\lambda^{-1}) - 1 - \frac{1}{2} c(\kappa\lambda^{-1}).$$

Hence whenever $\lambda \in \hat{X}_r^\nu$, so that $c(\lambda) = r$ and $\gamma(\lambda) - 1 = \nu$, then $c(\kappa\lambda^{-1}) = 2g - 2 - r$ and $\gamma(\kappa\lambda^{-1}) = g - 1 - (r - \nu)$ or equivalently $\kappa\lambda^{-1} \in \hat{X}_{2g-2-r}^{g-1-(r-\nu)}$. This observation can be expressed conveniently as the symmetry

$$\kappa \cdot \{\hat{X}_r^\nu\}^{-1} = \hat{X}_{2g-2-r}^{g-1-(r-\nu)},$$

where $\kappa\{\hat{X}_r^\nu\}^{-1}$ denotes the set of line bundles $\kappa\lambda^{-1}$ for all $\lambda \in \hat{X}_r^\nu$; the symmetry (13.46) actually is equivalent to the Riemann-Roch Theorem.

The symmetry relation (13.46) among holomorphic varieties in the complex manifold $P_r(M)$ appears more natural when these varieties are viewed as subsets of the Jacobi variety $J(M)$. It is possibly worth digressing here to discuss equivalent formulations of the Brill-Noether diagram and the varieties $\hat{X}_r^\nu$ in terms of the Jacobi manifold $J(M)$ or the manifold $M^{(r)}$ of positive divisors. The mappings in the Abel-Jacobi diagram (10.44) of Theorem 10.22 associate to the varieties $\hat{X}_r^\nu$ contained in the complex manifold $P_r(M)$ corresponding varieties contained in the complex manifolds $J(M)$ and $M^{(r)}$; and these varieties can be grouped in analogues of the Brill-Noether diagram. Thus set

$$X_r^\nu = W_r^\nu \sim W_r^{\nu+1} \subset J(M) \quad \text{for } r, \nu \in \mathbb{Z}$$

and correspondingly set

$$Y_r^\nu = G_r^\nu \sim G_r^{\nu+1} \subset M^{(r)} \quad \text{for } r, \nu \in \mathbb{Z}, \ r > 0,$$
CHAPTER 13. THE BRILL-NOETHER DIAGRAM

noting that the sets $Y_r^\nu$ are defined only for integers $r > 0$. As the complements of holomorphic subvarieties of holomorphic varieties all of these sets at least have the structures of holomorphic varieties. In analogy to (13.42) introduce the holomorphic varieties $X_r^{MAX} = X_r^{\mu(r)} \subset J(M)$ and $X_r^{MIN} = X_r^{\mu - (r)} \subset J(M)$, and note that as in (13.43) and (13.44)

\[(13.49)\]

\[J(M) = \bigcup_{\nu = \mu - (r)} X_r^\nu = X_r^{MIN} \cup \cdots \cup X_r^{MAX}\]

and

\[(13.50)\]

\[W_r^\nu = \bigcup_{\sigma = \nu} X_r^\sigma = X_r^\nu \cup \cdots \cup X_r^{MAX} \subset J(M).\]

Similarly for positive divisors let $Y_r^{MAX} = Y_r^{\mu(r)} \subset M(r)$. However the varieties $Y_r^\nu$ are empty for points on the minimal curve for which $\nu = -1$, so when considering the varieties $Y_r^\nu \subset M(r)$ it is more natural to consider in place of the minimal curve the \textit{general curve} consisting of points in the Brill-Noether diagram associated to the sets of \textit{general positive divisors} on the surface $M$ as defined in (2.56); thus set

\[(13.51)\]

\[Y_r^{GEN} = Y_r^{\max(0, r - g)} \subset M(r) \quad \text{for } r > 0.\]

The general curve differs from the minimal curve in that it is the horizontal line of height 0 for $r \leq g - 1$ rather than the horizontal line of height $-1$; but it coincides with the minimal curve for $r \geq g$. All points above the general curve in the Brill-Noether diagrams for the manifold $M(r)$ then are \textit{special positive divisors}, as defined in (2.56). It follows that

\[(13.52)\]

\[M(r) = \bigcup_{\nu = \max(0, r - g)} X_r^\nu = Y_r^{GEN} \cup \cdots \cup Y_r^{MAX} \quad \text{for } r > 0\]

and

\[(13.53)\]

\[G_r^\nu = \bigcup_{\sigma = \nu} Y_r^\sigma = Y_r^\nu \cup \cdots \cup X_r^{MAX} \quad \text{for } r > 0 \text{ and } \nu \geq 0.\]

The relations between these three families of sets follow from the Abel-Jacobi diagrams (10.19) or (10.44), which are expressed in terms of the holomorphic mappings

\[(13.54)\]

\[\zeta = \hat{\zeta} \circ \psi : M(r) \longrightarrow P_r(M) \quad \text{and} \quad w_{z_0} = \hat{w}_{z_0} \circ \psi : M(r) \longrightarrow J(M),\]

where $\zeta : M(r) \longrightarrow P_r(M)$ takes the holomorphic variety $Y_r^\nu \subset M(r)$ to the holomorphic variety $X_r^\nu \subset P_r(M)$ and $w_{z_0} : M(r) \longrightarrow J(M)$ takes the subvariety
$Y^\nu_r \subset M^{(r)}$ to the holomorphic variety $X^\nu_r \subset J(M)$. This is summarized in the following commutative diagram.

\[
\begin{array}{ccc}
Y^\nu_r \subset M^{(r)} & \xrightarrow{\zeta} & \hat{X}^\nu_r \subset P_r(M) \\
\downarrow{w_{z_0}} & & \downarrow{\phi_{a_0}} \\
X^\nu_r \subset J(M) & & \\
\end{array}
\]

for $r > 0$.

**Theorem 13.3** The diagram (13.55) is a commutative diagram of surjective holomorphic mappings between holomorphic varieties. The mapping $\phi_{a_0}$ is biholomorphic, but the inverse image of a point under either of the holomorphic mappings $\zeta$ or $w_{z_0}$ is a complex submanifold of $Y^\nu_r$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$ and consequently

\[
\dim X^\nu_r = \dim \hat{X}^\nu_r = \dim Y^\nu_r - \nu \quad \text{for } r > 0 \text{ and } \nu \geq 0.
\]

**Proof:** That (13.55) is a commutative diagram of holomorphic mappings in which $\phi_{a_0}$ is a biholomorphic mapping is clear from the diagrams (10.19) or (10.44), in which $\phi_{a_0}$ is a biholomorphic mapping. By Theorem 10.22 the inverse image $w_{z_0}^{-1}(t)$ of a point $t \in X^\nu_r$ is a complex submanifold of $M^{(r)}$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$; and the commutativity of the diagram (13.55) together with the fact that $\phi_{a_0}$ is a biholomorphic mapping show that the the inverse image $\zeta^{-1}(\lambda)$ of a point $\lambda \in \hat{X}^\nu_r$ also is a complex submanifold of $M^{(r)}$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$. If $V^\nu_r \subset Y^\nu_r$ is an irreducible component of the holomorphic variety $Y^\nu_r$ its image $w_{z_0}(V^\nu_r)$ is an irreducible component of the holomorphic variety $X^\nu_r$, and since the fibres of this mapping have dimension $\nu$ it follows from Remmert’s Proper Mapping Theorem that

\[
\dim w_{z_0}(V^\nu_r) = \dim V^\nu_r - \nu \quad \text{for } r > 0 \text{ and } \nu \geq 0.
\]

Since $\dim Y^\nu_r$ is the largest of the dimensions of its irreducible components, and correspondingly for $\dim X^\nu_r$, it follows that

\[
\dim X^\nu_r = \dim Y^\nu_r - \nu \quad \text{for } r > 0 \text{ and } \nu \geq 0.
\]

Of course $\dim \hat{X}^\nu_r = \dim X^\nu_r$ since the varieties $\hat{X}^\nu_r$ and $X^\nu_r$ are biholomorphic, and that suffices for the proof.

In view of the isomorphisms of the preceding theorem, it is generally sufficient to state and prove results about the varieties $\hat{X}^\nu_r$, $X^\nu_r$ and $Y^\nu_r$ just in terms of one of the three sets of varieties; hence much of the subsequent discussion will continue to be phrased in terms of the sets $\hat{X}^\nu_r \subset P_r(M)$ of holomorphic line bundles, although when another interpretation is more useful or more convenient it will be used. It is always possible to translate the results back and forth among these sets of varieties through Theorem 13.3.
Actually the symmetry relation (13.46) is one example in which an alternative description is more natural; for in terms of points of the Jacobi variety \( J(M) \) the symmetry relation takes the form

\[
(13.57) \quad k - X_r^\nu = X_{2g-2-r}^{g-1-(r-\nu)} \quad \text{for all } \nu, r
\]

where \( k = \phi_\alpha(\kappa) \in J(M) \) is the image of the canonical line bundle \( \kappa \). This is a simple relation between two subsets of the same complex torus \( J(M) \). Indeed the mapping of the Jacobi variety to itself that sends a point \( t \in J(M) \) to the point \( k - t \in J(M) \) is a biholomorphic mapping of the complex torus \( J(M) \) to itself which takes the holomorphic variety \( X_r^\nu \) to the holomorphic variety \( X_{2g-2-r}^{g-1-(r-\nu)} \); so these two holomorphic varieties are biholomorphic, and consequently

\[
(13.58) \quad \dim X_{2g-2-r}^{g-1-(r-\nu)} = \dim X_r^\nu \quad \text{for all } \nu, r.
\]

The sets \( X_r^\nu \) and \( X_{2g-2-r}^{g-1-(r-\nu)} \) are associated to symmetric points in the diagram of the maximal function, in the sense that these points have coordinates \( r \) that are symmetric with respect to the axis \( r = g-1 \) and have the same height above the minimal curve, or equivalently below the maximal curve. In particular

\[
(13.59) \quad k - X_r^{\text{MAX}} = X_{2g-2-r}^{\text{MAX}} \quad \text{and} \quad k - X_r^{\text{MIN}} = X_{2g-2-r}^{\text{MIN}},
\]

showing that \( \dim X_r^{\text{MAX}} = \dim X_{2g-2-r}^{\text{MAX}} \) and \( \dim X_r^{\text{MIN}} = \dim X_{2g-2-r}^{\text{MIN}} \).

It is a useful preliminary to the further discussion to note that the varieties \( X_r^{\text{MIN}} \subset J(M) \) on the minimal curve of the Brill-Noether diagram can be described quite explicitly for all Riemann surfaces. First

\[
(13.60) \quad X_r^{\text{MIN}} = J(M) \sim W_r \quad \text{for } 0 \leq r \leq g-1
\]

since if \( 0 \leq r \leq g-1 \) then \( X_r^{\text{MIN}} = X_r^{-1} = W_r^{-1} \sim W_r^0 = J(M) \sim W_r \); and by the symmetry relation (13.59) the preceding equation implies that

\[
(13.61) \quad X_r^{\text{MIN}} = J(M) \sim (k - W_{2g-2-r}) \quad \text{for } g-1 \leq r \leq 2g-2.
\]

Since \( X_r^{\text{MIN}} = J(M) \) for \( r > 2g-2 \) as in (13.45) it follows from this and the two preceding two equations that

\[
(13.62) \quad \dim X_r^{\text{MIN}} = g \quad \text{for all } g \geq 0.
\]

This observation shows incidentally that the lower bound (13.1) actually is an effective lower bound; of course the upper bound (13.3) is effective by definition. More precisely the observations (13.60) and (13.61) show that the lower bound (13.1) is attained by most holomorphic line bundles, indeed by all holomorphic line bundles except those in a proper holomorphic subvariety of the complex torus \( P_r(M) \). Since the varieties \( X_r^\nu \subset J(M) \) for a fixed value of \( r \) and values of \( \nu \) in the range \( \mu_{-}(r) \leq \nu \leq \mu(r) \) are disjoint it follows from (13.49), (13.60) and (13.61) that

\[
(13.63) \quad X_r^{\mu_{-}(r)+1} \cup \cdots \cup X_r^{\mu(r)} \subset \begin{cases} W_r \subset J(M) & \text{if } 0 \leq r \leq g-1, \\ k - (W_{2g-2-r}) & \text{if } g-1 \leq r \leq 2g-2. \end{cases}
\]
which bounds the varieties $X^r_w$. Of course the corresponding observation holds in the other versions of the Brill-Noether diagram.

The first critical value $r_1$ of a compact Riemann surface of genus $g > 0$ is in many ways the most significant of the critical values of the surface $M$; and bounds on its possible values are of considerable interest. The basic result about these bounds is the following theorem, which has an extensive history\(^1\) and for which there are a variety of proofs.

**Theorem 13.4** The first critical value $r_1$ of a compact Riemann surface $M$ of genus $g > 0$ satisfies

\[(13.64) \quad 2 \leq r_1 \leq \left[ \frac{g}{2} \right] + 1\]

where as usual $\left[ \frac{g}{2} \right]$ is the integer part of $\frac{g}{2}$.

**Proof:** The lower bound is just that of Theorem 13.2. For the upper bound, suppose to the contrary that $r_1 > \left[ \frac{g}{2} \right] + 1$. If $g = 2h$ let $r = h + 1$, $s = h$, and if $g = 2h + 1$ let $r = h + 1$, $s = h + 1$, so in either case $r = \left[ \frac{g}{2} \right] + 1$ and consequently $r < r_1$; therefore $\gamma(\lambda) < 2$ for any holomorphic line bundle $\lambda$ for which $c(\lambda) = r$, or equivalently $W^r_\lambda = \emptyset$. It is more convenient for the rest of the argument to work with the Jacobi variety $J(M)$; thus $W^r_\lambda = \emptyset$, and then of course $W^1_\lambda = \emptyset$ also since $s \leq r$. Choose a base point $z_0 \in M$ for the Abel-Jacobi mapping $w_{z_0} : M \longrightarrow J(M)$ such that the point $a \in M$ it represents is not a Weierstrass point of $M$. It then follows from Lemma 11.12 that $r_1(a) = g + 1$ hence that $\zeta_a^{g+1}$ is base-point-free and $\gamma(\zeta_a^{g+1}) = 2$. Since $0 \in W_r \cap (-W_s)$ and $\dim W_r + \dim (-W_s) - g = r + s - g = 1$ the intersection $(W_r \cap (-W_s))$ is a holomorphic subvariety of the Jacobi variety of dimension at least 1 containing the origin 0; that is a general property of the intersection of holomorphic subvarieties of a complex manifold as discussed on page 404 in Appendix A.3. Consequently there is an irreducible one-dimensional holomorphic subvariety $W \subset J(M)$ for which $0 \in W \subset W_r \cap (-W_s)$. Since $W^1_s = W^1_r = \emptyset$, and hence $G^1_s = G^1_r = \emptyset$ as well, the surface $M$ has no special positive divisors of degrees $r$ or $s$, as in (10.43); consequently by Corollary 10.10 (i) the Abel-Jacobi mappings

\[(13.65) \quad w_{z_0} : M^{(r)} \longrightarrow W_r \quad \text{and} \quad w_{z_0} : M^{(s)} \longrightarrow W_s\]

are biholomorphic mappings. Any point $t \in W \subset W^{(r)} \cap W^{(s)}$ consequently can be written uniquely in the form

\[(13.66) \quad t = w_{z_0}(a_1 + \cdots + a_r) = -w_{z_0}(b_1 + \cdots + b_s)\]

for some divisors $a_1 + \cdots + a_r \in M^{(r)}$ and $b_1 + \cdots + b_s \in M^{(s)}$. Then

\[(13.67) \quad w_{z_0}(a_1 + \cdots + a_r + b_1 + \cdots + b_s) = 0,\]

\(^{1}\)See for example the discussion in the book *Geometry of Algebraic Curves, I* by E. Arbarello, M. Cornalba, P. Griffiths and J. Harris.
so since \( r + s = g + 1 \) and it is also the case that \( w_{z_0}((g + 1) \cdot a_0) = 0 \) it follows from Abel’s Theorem, Corollary 10.2, that

\[
(13.68) \quad a_1 + \cdots + a_r + b_1 + \cdots + b_s \sim (g + 1) \cdot a_0;
\]

the divisors \( d_t = a_1 + \cdots + a_r + b_1 + \cdots + b_s \) thus are the divisors of holomorphic cross-sections of the holomorphic line bundle \( \zeta_a^{r+1} \) for any point \( t \in W \). If \( X \subset M^{(r+s)} \) is the set of divisors of holomorphic cross-sections of the line bundle \( \zeta_a^{r+1} \) then since \( \gamma(\zeta_a^{r+1}) = 2 \) and the line bundle \( \zeta_a^{r+1} \) is base-point-free it follows that any divisor \( d \in X \) is uniquely determined by specifying any of its points, and any point of \( M \) is in the divisor of some holomorphic cross-section of \( \zeta_a^{r+1} \); for if \( f_0(z), f_1(z) \in \gamma(M, O(\zeta_a^{r+1})) \) is a basis for this space of holomorphic cross-sections then \( f_0(z) \) and \( f_1(z) \) have no common zeros and \( f_g(z) = f_1(p)f_0(z) - f_0(p)f_1(z) \) is the unique cross-section that vanishes at a point \( p \in M \). The mapping that associates to a nontrivial cross-section \( f_{x_0,x_1} = x_0f_0(z) + x_1f_1(z) \in \gamma(M, O(\zeta_a^{r+1})) \) its divisor \( d(f_{x_0,x_1}) \in M^{(r+s)} \) is a holomorphic mapping from the nonzero points of \( \mathbb{C}^2 \) to \( M^{(r+s)} \); and since this mapping is the same for any pairs \( x_0, x_1 \) that represent the same point in \( \mathbb{P}^1 \) it induces a holomorphic mapping from \( \mathbb{P}^1 \) to \( M^{(r+s)} \), a proper mapping since \( \mathbb{P}^1 \) is a compact manifold. It then follows from Remmert’s Proper Mapping Theorem\(^2\) that the image of this mapping, the set of divisors of holomorphic cross-sections of \( \zeta_a^{r+1} \), is an irreducible holomorphic subvariety \( X \subset M^{(r+s)} \) and \( \dim X = 1 \). The mapping that associates to any point \( t \in W \) the divisor \( d_t = a_1 + \cdots + a_r + b_1 + \cdots + b_s \in X \) is then a well defined proper holomorphic mapping \( d : W \longrightarrow X \); and since its image contains more than a single point the image must be a holomorphic subvariety of dimension 1 in \( X \), so actually \( \phi(W) = X \). Consequently any divisor \( d_t \) also is uniquely determined by specifying any one of its points, and that can be an arbitrary point of \( M \); hence the decomposition of the divisor \( d_t \) as the sum of the two divisors \( a_1 + \cdots + a_r \) and \( b_1 + \cdots + b_s \) also is unique. On the other hand the divisors \( d_t \) can be deformed continuously by moving the point \( a_1 \) along any path in \( M \), and the decomposition of the divisors \( d_t \) is preserved in this motion; but moving the point \( a_1 \) along a continuous path to the point \( b_1 \) cannot preserve the decomposition of the divisors \( d_t \), and that contradiction serves to conclude the proof.

Part of the significance of the first critical value on a compact Riemann surface \( M \) follows from the fact that it is also the smallest positive integer in the Lüroth semigroup of \( M \); more generally though there is the following simple observation.

**Theorem 13.5** On a compact Riemann surface \( M \) of genus \( g > 0 \) all the holomorphic line bundles in \( X^{\text{max}} \) for any critical value \( r_i \) are base-point free. Consequently the critical values \( r_i \) of \( M \) belong to the Lüroth semigroup \( \mathcal{L}(M) \) of the Riemann surface \( M \), and in particular the critical value \( r_1 \) is the smallest positive integer in the Lüroth semigroup.

\(^2\)Remmert’s Proper Mapping Theorem is discussed on page 409 of Appendix A.3.
Proof: If $\lambda \in X_{r_i}^{\text{max}}$ for $i \geq 0$ then $c(\lambda) = r_i$ and $\gamma(\lambda) - 1 = \mu(r_i) = i$. Since $c(\lambda \zeta^{-1}) = r_i - 1$ for any point $a \in M$ it follows that $\gamma(\lambda \zeta^{-1}) - 1 \leq \mu(r_i - 1) = \mu(r_i) - 1 = i - 1 < \gamma(\lambda) - 1$, hence $\lambda$ is base-point-free by Lemma 2.10. By definition then its characteristic class $c(\lambda) = r_i$ belongs to the Lüroth semigroup. If $0 < r < r_i$ and $\lambda$ is a holomorphic line bundle with $c(\lambda) = r$ then $\gamma(\lambda) = 1$ so the bundle $\lambda$ cannot be base-point-free; therefore there are no base-point-free line bundles $\lambda$ such that $0 < c(\lambda) < r_i$, and that suffices for the proof.

A basic property of the Brill-Noether diagram is a convexity determined by the base-point-free holomorphic line bundles on $M$, a consequence of the following simple observation.

**Theorem 13.6** If $\tau$ is a base-point-free holomorphic line bundle on a compact Riemann surface $M$ and if $h_0, h_1 \in \Gamma(M, \mathcal{O}(\tau))$ are two holomorphic cross-sections of $\tau$ with no common zeros then for any holomorphic line bundle $\lambda$ on $M$ for which $\gamma(\lambda) \neq 0$ there is the exact function of sheaves

\[(13.69) \quad 0 \to \mathcal{O}(\lambda \tau^{-1}) \xrightarrow{p_1} \mathcal{O}(\lambda)^2 \xrightarrow{p_2} \mathcal{O}(\lambda \tau) \to 0\]

in which the sheaf homomorphisms $p_1, p_2$ are defined by

\[
p_1(g) = (h_0g, h_1g) \in \mathcal{O}_p(\lambda)^2 \quad \text{for all} \quad g \in \mathcal{O}_p(\lambda \tau^{-1}),
\]

\[
p_2(g_0, g_1) = h_1g_0 - h_0g_1 \in \mathcal{O}_p(\lambda \tau) \quad \text{for all} \quad g_0, g_1 \in \mathcal{O}_p(\lambda)
\]

for any point $p \in M$.

**Proof:** It is evident that the sheaf homomorphisms $p_1$ and $p_2$ are well defined, that $p_1$ is injective, and that $p_2 p_1 = 0$. If $(g_0, g_1) \in \mathcal{O}_p(\lambda)^2$ and if $0 = p_2(g_0, g_1) = h_1g_0 - h_0g_1$ then $g_0/h_0 = g_1/h_1 = g \in \mathcal{M}_p(\lambda \tau^{-1})$; however either $h_0(p) \neq 0$ or $h_1(p) \neq 0$ so $g$ actually is a holomorphic germ $g \in \mathcal{O}_p(\lambda \tau^{-1})$, and $(g_0, g_1) = (h_0g, h_1g) = p_1(g)$. Thus the kernel of $p_2$ is contained in the image of $p_1$, so the sheaf sequence is exact at the sheaf $\mathcal{O}(\lambda)^2$. If $f \in \mathcal{O}_p(\lambda \tau)$ and if for instance $h_0(p) \neq 0$ then $f = h_0 \cdot (f/h_0) = p_2(f/h_0, 0)$ so $f$ is in the image of the homomorphism $p_2$. The corresponding argument holds if $h_1(p) \neq 0$, so the sheaf homomorphism $p_2$ is surjective, and that concludes the proof of the theorem.

The exact cohomology sequence arising from the exact sequence of sheaves (13.70) of the preceding theorem begins

\[(13.70) \quad 0 \to \Gamma(M, \mathcal{O}(\lambda \tau^{-1})) \xrightarrow{p_1} \Gamma(M, \mathcal{O}(\lambda))^2 \xrightarrow{p_2} \Gamma(M, \mathcal{O}(\lambda \tau)).\]

An immediate consequence of this exact sequence is the following corollary.

**Corollary 13.7** If $\tau$ is a base-point-free holomorphic line bundle on a compact Riemann surface $M$ then for any holomorphic line bundle $\lambda$ on $M$

\[(13.71) \quad \gamma(\lambda) - \gamma(\lambda \tau^{-1}) \leq \gamma(\lambda \tau) - \gamma(\lambda).
\]

and this is an equality if and only if the homomorphism $p_2$ in the exact sequence (13.70) is surjective.
Proof: If \( \gamma(\lambda) = 0 \) the corollary holds trivially. If \( \gamma(\lambda) \neq 0 \) then since
\[
\gamma(\lambda \tau) = \dim \Gamma(M, \mathcal{O}(\lambda \tau)) \text{ is at least equal to the dimension of the image of the homomorphism } p_2 \text{ in (13.70)}
\]
it follows from the exactness of the cohomology sequence (13.70) that
\[
\gamma(\lambda \tau) \geq \dim \Gamma(M, \mathcal{O}(\lambda))^2 - \dim \Gamma(M, \mathcal{O}(\lambda \tau^{-1}))) = 2\gamma(\lambda) - \gamma(\lambda \tau^{-1});
\]
and this is an equality if and only if the sheaf homomorphism \( p_2 \) in the exact sequence (13.70) is surjective. That suffices for the proof.

Corollary 13.8 (Convexity Theorem for the Brill-Noether Diagram)
For any integer \( t \in \mathcal{L}(M) \) in the Lüroth semigroup of a compact Riemann surface \( M \) of genus \( g > 0 \) the maximal function of \( M \) satisfies the convexity condition
\[
(13.72) \quad \mu(r) - \mu(r-t) \leq \mu(r+t) - \mu(r)
\]
for all \( r \).

Proof: If \( t = 0 \) the inequality is trivial. If \( t \in \mathcal{L}(M) \) and \( t \neq 0 \) then by definition of the Lüroth semigroup \( t = c(\tau) \) for a base-point-free holomorphic line bundle \( \tau \). For any integer \( r \) choose a maximal bundle \( \lambda \in X^\max \), so that \( c(\lambda) = r \) and \( \mu(r) = \gamma(\lambda) - 1 \). The inequality of the preceding Corollary can be rewritten \( \gamma(\lambda \tau) + \gamma(\lambda \tau^{-1}) \geq 2\gamma(\lambda); \) and since by the definition of the maximal function \( \mu(r+t) \geq \gamma(\lambda \tau) - 1 \) and \( \mu(r-t) \geq \gamma(\lambda \tau^{-1}) - 1 \) it follows that
\[
\mu(r + t) + \mu(r - t) \geq (\gamma(\lambda \tau) - 1) + (\gamma(\lambda \tau^{-1}) - 1) \geq 2\gamma(\lambda) - 2 = 2\mu(r),
\]
which suffices to prove the corollary.

To examine some of the consequences of the preceding convexity theorem introduce the successive differences
\[
(13.73) \quad \delta_i = r_i - r_{i-1} > 0 \quad \text{for } i \geq 1
\]
and it follows from Corollary 13.8 that
\[
(13.74) \quad r_i = \delta_i + \delta_{i-1} + \cdots + \delta_1 \quad \text{for } i \geq 1
\]
since \( r_0 = 0 \), so in particular \( \delta_1 = r_1 \); and it follows from Corollary 13.8 that
\[
(13.75) \quad \delta_g = 2, \quad \text{and} \quad \delta_i = 1 \quad \text{for } i > g.
\]

The convexity condition of the preceding corollary then can be rephrased as follows.

Corollary 13.9 The successive differences of the critical values of a compact Riemann surface of genus \( g > 0 \) satisfy
\[
\delta_i \leq r_1 \quad \text{for all } i \geq 1;
\]
moreover if
$$\delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1} < r_1$$
for some indices \(i \geq \nu \geq 1\) then
$$\delta_{i+1} + \delta_{i+2} + \cdots + \delta_{i+\nu+1} \leq r_1.$$ 

As in Corollary ?? the difference \(\mu(r) - \mu(s)\) for any integers \(r > s\) is equal to the number of critical values \(r_i\) in the half-open half-closed interval \((s, r]\), that is to say, such that \(s < r_i \leq r\). If \(r_{i-\nu} > r_i - r_1\) where \(i \geq \nu \geq 0\) then there are at least the \(\nu + 1\) critical values \(r_{i-\nu}, r_{i-\nu+1}, \ldots, r_i\) in the interval \((r_i - r_1, r_i]\) so that \(\mu(r_i) - \mu(r_i - r_1) \geq \nu + 1\). It follows from the preceding theorem that \(\mu(r_i + r_1) - \mu(r_i) \geq \nu + 1\), and consequently that there are at least \(\nu + 1\) critical values in the interval \((r_i, r_i + r_1]\); these of course must be the critical values \(r_{i+1}, r_{i+2}, \ldots, r_{i+\nu+1}\), so that \(r_{i+\nu+1} \leq r_i + r_1\). The first conclusion of the corollary is the result just demonstrated for the case \(\nu = 0\), since \(r_i - r_i < r_1\) for all indices \(i \geq 1\), while the second conclusion is that for the case \(\nu \geq 1\), and that suffices to conclude the proof of the corollary.

The property of the successive differences of the critical values described in the preceding lemma can be handled conveniently by introducing the characteristics of the maximal function of the Riemann surface \(M\), the integers \(k_\nu\) defined by

\[
\text{char}(13.76)\]

\[
k_\nu = \min \left\{ i \mid i \geq \nu, \quad \delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1} < r_1 \right\} \quad \text{for } 1 \leq \nu < r_1.
\]

Since \(\delta_i = 1\) for all sufficiently large indices \(i\) it is clear that these characteristics are well defined. For some purposes it is convenient to extend this definition and to set \(k_0 = 0\); although this extension is somewhat anomalous, it does reflect what is the most significant property of the characteristics, that they describe the intervals in the parameter \(i\) of the critical values \(r_i\) in which the maximal function \(\mu(r)\) increases at different rates. This will become more apparent during the subsequent discussion. Initially though this will be taken as a reason for the terminology, since with this interpretation the characteristics are analogous to the relative characteristics \(k_\nu(\lambda; \tau)\). However the behavior of the maximal function \(\mu(r)\) is rather more complicated than that of the simple dimensions \(\gamma(\lambda\tau^i)\), so that there are some further subtleties involved. All of this probably can be clarified best through a discussion of some illustrative examples. First, though, it is useful to note the following.

**Lemma 13.10** The characteristics of the maximal function of a compact Riemann surface of genus \(g > 0\) satisfy

\[
1 < k_1 < k_2 < \cdots < k_{r_1-1} = r_1 + g - 1.
\]
**Proof:** By definition $k_1$ is the least integer $i \geq 1$ such that $\delta_i < r_1$; so since $\delta_1 = r_1 \geq 2$ it follows that $k_1 > 1$. If $i < k_{\nu}$ for some integer $\nu < 1$ then

$$
\delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1} \geq r_1;
$$

and since $\delta_{i+1} \geq 1$ it is also the case that

$$
\delta_{i+1} + \delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1} \geq r_1
$$

hence that $i+1 < k_{\nu+1}$. In particular for $i = k_{\nu} - 1$ it follows that $k_{\nu} < k_{\nu+1}$. Finally since $\delta_i \geq 1$ for all indices $i$

$$
k_{r_1-1} = \min \left\{ i \mid i \geq r_1 - 1 \text{ and } \delta_i + \delta_{i-1} + \cdots + \delta_{i-r_1+2} < r_1 \right\}
$$

Now $\delta_g = 2$ and $\delta_i = 1$ for $i \geq g + 1$ by equation (13.74), so it is clear that $k_{r_1-1} = r_1 + g - 1$ and that suffices to conclude the proof of the lemma.

Possibly the clearest illustration of the significance of the characteristics is provided by examining the first few characteristics more closely. Since $\delta_1 = r_1$ and $k_1$ is the first index $i$ for which $\delta_i < r_1$ then clearly

$$
\delta_1 = \delta_2 = \cdots = \delta_{k_1-1} = r_1, \quad \delta_{k_1} < r_1;
$$

and

$$
r_2 = 2r_1, \quad r_3 = 3r_1, \quad \cdots, \quad r_{k_1-1} = (k_1 - 1)r_1, \quad r_{k_1} < k_1r_1.
$$

Now since $\delta_{k_i} < r_1$ then from Corollary 13.9 for the case $\nu = 1$ it follows that $\delta_{k_{i+1}} + \delta_{k_{i+2}} < r_1$; and since $\delta_i \geq 1$ necessarily $\delta_{k_{i+1}} < r_1$ and $\delta_{k_{i+2}} < r_1$ as well. The argument can be repeated for $k_{i+1} + 1$ and $k_{i+2}$ in place of $k_1$, so that $\delta_{k_{i+3}} < r_1$, $\delta_{k_{i+4}} < r_1$, and $\delta_{k_{i+5}} < r_1$, and so on. Altogether, since $k_2$ is the least index $i$ such that $\delta_i + \delta_{i+1} < r_1$, it follows that

$$
\delta_i < r_1 \text{ for } i \geq k_1 \quad \text{and} \quad \delta_i + \delta_{i-1} = r_1 \text{ for } k_1 + 2 \leq i < k_2.
$$

Of course it may be the case that $k_2 \leq k_1 + 2$, and then the last half of the preceding equation is vacuous; otherwise the successive differences $\delta_i$ in the range $k_1 + 2 \leq i < k_2$ are determined uniquely by the value $\delta_{k_1+1}$ and the recursion relation $\delta_i = r_{i-1} - \delta_{i-1}$. Thus the value of $\delta_{k_1+1}$ is a parameter that describes all the other successive differences up to $\delta_{k_2-1}$, while the value of $\delta_{k_1}$ is a transitional term between the two ranges in which the successive differences behave quite regularly. The sequence of critical values $r_i$ up to $r_{k_2-1}$ is also determined.
by these parameters, although the explicit form is slightly more complicated
so it is really more convenient just to consider the successive differences. It
may be worth noting in passing that $\delta_i + \delta_{i-1} = 2r_1$ for $1 \leq i < k_1$ while
$\delta_{k_1} + \delta_{k_1-1} < 2r_1$ and $\delta_{k_1} + \delta_{k_1} < 2r_1 - 1$.

The argument can be repeated but with increasing complication; it may be
sufficient here just to describe the next stage in detail, to indicate the general
pattern. Since $\delta_{k_2} + \delta_{k_2-1} < r_1$ then from Corollary 13.9 for the case $\nu = 2$
it follows that $\delta_{k_2+1} + \delta_{k_2+2} + \delta_{k_2+3} \leq r_1$; and since $\delta_i \geq 1$ necessarily
$\delta_{k_2+2} + \delta_{k_2+1} < r_1$ and $\delta_{k_2+3} + \delta_{k_2+2} < r_1$. Consequently the argument can
be repeated with $k_2 + 2$ and $k_2 + 3$ in place of $k_2$, although not with $k_2 + 1$
in place of $k_2$; more gaps of this sort arise as the process continues, explaining
part of the increase in complication. However it is at least the case here that
$\delta_{k_2+4} + \delta_{k_2+3} < r_1$, $\delta_{k_2+5} + \delta_{k_2+4} < r_1$ and $\delta_{k_2+6} + \delta_{k_2+5} \leq r_1$, and that is
enough to continue the argument to all the rest of the successive differences.
Altogether, since $k_3$ is the least index $i$ such that $\delta_i + \delta_{i-1} + \delta_{i-2} < r_1$, it follows that

\begin{align}
\delta_i + \delta_{i-1} &< r_1 \quad \text{for } i \geq k_2, \ i \neq k_2 + 1, \ \text{and} \\
\delta_i + \delta_{i-1} + \delta_{i-2} &= r_1 \quad \text{for } k_2 + 3 \leq i < k_3, \ i \neq k_2 + 4.
\end{align}

(13.80)

Again the last half of the preceding equation is vacuous if $k_3 \leq k_2 + 3$; otherwise
the successive differences $\delta_i$ in the range $k_2 + 5 \leq i < k_3$ are determined uniquely
by the values $\delta_{k_2+4}$ and $\delta_{k_2+3}$ and the recursion relation $\delta_i = r_1 - \delta_{i-1} - \delta_{i-2}$. Thus the values of $\delta_{k_2+3}$ and $\delta_{k_2+4}$ are parameters that describe all the
remaining successive differences in this range, while the values $\delta_{k_2}$, $\delta_{k_2+1}$, and
$\delta_{k_2+2}$ describe the transitional range, subject to the relation $\delta_{k_2+3} + \delta_{k_2+4} + \delta_{k_2+1} = 0$. The process can be continued to describe the remaining successive
differences, with corresponding patterns in subsequent cases.

To see the significance of these observations for the behavior of the maximal
function itself, note as a consequence of (13.77) that each interval $(r, r + r_1]$ of
length $r_1$ for $0 \leq r < r_{k_1} - r_1$ includes precisely one critical value of the maximal
function, so that

\begin{align}
\mu(r + r_1) - \mu(r) &= 1 \quad \text{for } 0 \leq r < r_{k_1} - r_1.
\end{align}

(13.81)

Similarly as a consequence of (13.79) each interval $(r, r + r_1]$ of length $r_1$ for
$r_{k_1+2} \leq r < r_{k_2} - r_1$ includes precisely two critical values, so that

\begin{align}
\mu(r + r_1) - \mu(r) &= 2 \quad \text{for } r_{k_1+2} \leq r < r_{k_2} - r_1.
\end{align}

(13.82)

There is a transitional region between these two intervals, in which the increase
of the maximal function modulates between 1 and 2. Of course $\mu(r+r_1) - \mu(r) = 0$
ever $r < r_{k_0} - r_1 = -r_1$, so that the separately defined characteristic
$k_0 = 0$ also describes a point at which the rate of increase of the maximal
function changes. There are corresponding results for higher characteristics,
until finally $\mu(r + r_1) - \mu(r) = r_1$ whenever $r > 2g - 2$. 

Rather than continuing this process, though, it may be better just to turn to the end of the sequence of successive differences of the critical values. Of course \( \delta_i = 1 \) whenever \( i > g \) as in (13.74). The differences \( \delta_i \) for \( i \leq g \) but \( i \) near \( g \) can be determined readily from the duality between the critical values and the gap values of the maximal function. By equation (13.77) it is evident that there is a string of \( r_1 - 1 \) consecutive gap values beginning with \( s_1 = 1 \) and continuing through \( s_{r_1-1} \), followed by the critical value \( r_1 \) for a gap of 2 between the gap values \( s_{r_1-1} \) and \( s_{r_1} \), followed by another string of \( r_1 - 1 \) consecutive gap values from \( r_1 + 1 \) to \( r_2 - 1 = 2r_1 - 1 - 1 \), and so on; consequently from the duality between the gap values and the critical values, as expressed in Corollary ??, it follows that the string of successive differences of the critical values ending with \( \delta_g = 2 \) has the form

\[
1, 1, \ldots, 1, 2, 1, 1, \ldots, 1, 2, \ldots, 1, 1, \ldots, 1, 2
\]

where there are \( k_1 - 1 \) such blocks. This duality of course can be continued to the next string (13.79) of successive differences, and so on, although again with more complication than justifies any detailed treatment here. It may be more useful at this stage to examine some particular cases that arise naturally and in which the results obtained so far can be applied usefully.

Base-point free holomorphic line bundles play a significant role in the Brill-Noether diagram, beyond their appearance in the preceding theorem. The base divisor of a holomorphic line bundle \( \lambda \) on \( M \) was defined in (2.8), and by Theorem 2.11 any holomorphic line bundle can be written uniquely as the product \( \lambda = \lambda_0 \zeta_{b(\lambda)} \) of a base-point-free holomorphic line bundle \( \lambda_0 \) and the line bundle \( \zeta_{b(\lambda)} \) associated to the base divisor \( b(\lambda) \) of \( \lambda \), where \( \gamma(\lambda_0) = \gamma(\lambda) \) and \( \gamma(\zeta_{b(\lambda)}) = 1 \); this is the base decomposition of the holomorphic line bundle \( \lambda \). In this context the base degree of a holomorphic line bundle \( \lambda \) is defined by

\[
\deg_b(\lambda) = \deg(\lambda);
\]

so in the base decomposition \( \lambda = \lambda_0 \zeta_{b(\lambda)} \) of a line bundle, \( \deg_b(\lambda) = c(\zeta_{b(\lambda)}) = c(\lambda) - c(\zeta_0) \), hence \( \deg_b(\lambda) = 0 \) if and only if \( \lambda \) is base-point-free. Let \( \hat{X}^{\nu,o}_r \) denote the set of base-point-free line bundles in \( \hat{X}^\nu_r \) and let \( \hat{X}^{\nu,\sharp}_r \) denote the complementary set of line bundles in \( \hat{X}^\nu_r \) with a nontrivial base divisor, so that

\[
\hat{X}^{\nu,o}_r = \left\{ \lambda \in \hat{X}^\nu_r \mid b(\lambda) = 0 \right\}
\]

and

\[
\hat{X}^{\nu,\sharp}_r = \left\{ \lambda \in \hat{X}^\nu_r \mid b(\lambda) > 0 \right\}.
\]

By definition then there is the decomposition

\[
\hat{X}^\nu_r = \hat{X}^{\nu,o}_r \cup \hat{X}^{\nu,\sharp}_r \quad \text{where} \quad \hat{X}^{\nu,o}_r \cap \hat{X}^{\nu,\sharp}_r = \emptyset.
\]

The basic result about this decomposition is the following.
Theorem 13.11 For any compact Riemann surface $M$ of genus $g > 0$ the subset

$$B^\nu = \left\{ (z, \lambda) \in M \times \hat{X}^\nu_r \mid z \in b(\lambda) \right\}$$

is a holomorphic subvariety of the holomorphic variety $M \times \hat{X}^\nu_r$. The natural projection $\pi : M \times \hat{X}^\nu_r \longrightarrow \hat{X}^\nu_r$ induces a finite proper surjective holomorphic mapping

$$\pi : B^\nu \longrightarrow \hat{X}^\nu_r$$

so $\hat{X}^\nu_r$ is a holomorphic subvariety of $\hat{X}^\nu_r$ and $\dim \hat{X}^\nu_r = \dim B^\nu$.

**Proof:** As in the discussion in Chapter 7, holomorphic line bundles $\lambda \in \hat{X}^\nu_r$ can be described by factors of automorphy of the form $\rho_t \eta$, where $\eta$ is a fixed factor of automorphy describing a fixed holomorphic line bundle of characteristic class $r$ and $\rho_t$ are canonically parametrized flat factors of automorphy for parameter values $t \in V$ for a suitable holomorphic subvariety $V \subset \mathbb{C}^{2g}$ of an open subset of the parameter space for flat line bundles; the subvariety $V$ in this way parametrizes the variety $\hat{X}^\nu_r$ of holomorphic line bundles. Holomorphic cross-sections of a line bundle $\lambda \in \hat{X}^\nu_r$ correspond to holomorphic relatively automorphic functions for the factor of automorphy $\rho_t \eta$ for the parameter value $t \in V$ parametrizing the line bundle $\lambda$. The condition that $\lambda \in \hat{X}^\nu_r$ means that the dimension of the space of relatively automorphic functions is $\nu + 1$ for all $t \in V$. It then follows from Corollary 7.3, for the special case that the relatively automorphic functions are holomorphic so the auxiliary parameter variety $W$ is empty, that for any bundle $\lambda_0$ described by a parameter $t_0 \in V$ there is an open neighborhood $U \subset V$ of the point $t_0$ and there are $\nu + 1$ holomorphic relatively automorphic functions $f_{i,t}$ for the factor of automorphy $\rho_t \eta$ that are holomorphic functions of the parameter $t \in U$ and are a basis for space of relatively automorphic functions for for the factor of automorphy $\rho_t \eta$ for all $t \in U$. The subset

$$Y = \left\{ (z,t) \in M \times U \mid f_{i,t}(z) = 0 \text{ for all } 1 \leq i \leq \nu + 1 \right\}$$

consequently is a holomorphic subvariety of $M \times U$. If $(z,t) \in Y$ then all the relatively automorphic functions $f_{i,t}(z)$ vanish at the point $z \in M$; that means that all the holomorphic cross-sections of the line bundles $\lambda$ parametrized by values $t \in U$ vanish at the point $z$, so by definition $z$ is a point in the divisor $b(\lambda)$. Conversely if $z$ is a point in the divisor $b(\lambda)$ then by definition all the holomorphic cross-sections of the line bundle $\lambda$ vanish at the point $z \in M$ so all the relatively automorphic functions $f_{i,t}(z)$ vanish at the point $z$ and consequently $(z,t) \in Y$. Thus locally the set $B^\nu$ is just the set $Y$; and since $Y$ is a holomorphic variety that shows that the set $B^\nu$ is a holomorphic subvariety in an open neighborhood of each of its points, so $B^\nu$ itself is a holomorphic variety. The natural projection mapping $\pi : M \times \hat{X}^\nu_r \longrightarrow \hat{X}^\nu_r$ is a proper
holomorphic mapping, since $M$ is compact; so there is the commutative diagram of holomorphic mappings

$$
\begin{array}{ccc}
B_{r}^{\nu} & \xrightarrow{\iota} & M \times \hat{X}_{r}^{\nu} \\
\pi \downarrow & & \downarrow \pi \\
\pi(B_{r}^{\nu}) & \xrightarrow{\iota} & \hat{X}_{r}^{\nu}
\end{array}
$$

(13.91)

where in both cases $\pi$ is a proper holomorphic mapping and $\iota$ is the natural inclusion mapping. By Remmert’s Proper Mapping Theorem, as discussed on page 409 in Appendix A.3, the image $\pi(B_{r}^{\nu})$ is a holomorphic subvariety of $\hat{X}_{r}^{\nu}$, so it is a holomorphic variety. If $\lambda \in \pi(B_{r}^{\nu})$ there is at least one point $z \in M$ for which $(z, \lambda) \in B_{r}^{\nu}$ hence for which $z$ is a point of the divisor $b(\lambda)$, so the bundle $\lambda$ is not base-point-free, while on the other hand if $\lambda \notin \pi(B_{r}^{\nu})$ then there is no point $z \in M$ that is a base point for $\lambda$, so $\lambda$ is base-point-free; therefore $\pi(B_{r}^{\nu}) = \hat{X}_{r}^{\nu,2}$, the subset of non-base-point-free holomorphic line bundles in $\hat{X}_{r}^{\nu}$.

The set of points $(z, \lambda) \in \hat{X}_{r}^{\nu}$ that have the same image $\pi(z, \lambda) = \lambda$ consists of those points $z \in M$ that are points in the base divisor of the line bundle $\lambda$ so is a finite set of points; the mapping (13.89) thus also is a finite mapping. It then follows from the more detailed version (A.20) of Remmert’s Proper Mapping Theorem that $\dim \hat{X}_{r}^{\nu,2} = \dim B_{r}^{\nu}$, and that suffices for the proof.

For an alternative to the decomposition (13.87) introduce the set $\hat{X}_{r}^{\nu,*0}$ of those holomorphic line bundles in $X_{r}^{\nu}$ such that the line bundle $\kappa \lambda^{-1} \in \hat{X}_{2g_{r}-(r-\nu-1)}^{\nu}$ is base-point-free and the set $\hat{X}_{r}^{\nu,*2}$ of those holomorphic line bundles in $X_{r}^{\nu}$ such that the line bundle $\kappa \lambda^{-1} \in \hat{X}_{2g_{r}-(r-\nu-1)}^{\nu}$ has a nontrivial base divisor; of course these are defined only if $\hat{X}_{2g_{r}-(r-\nu-1)}^{\nu} \neq 0$. There is then the decomposition

Some basic relations between the various varieties $\hat{X}_{r}^{\nu}$ rest on the decomposition (13.87). Recall from the discussion in Chapter 2 that for any holomorphic line bundle $\lambda$ with base divisor $b(\lambda)$ and for any point $a \in M$

$$
(13.92) \quad \gamma(\lambda \zeta_{a}^{-1}) = \begin{cases} 
\gamma(\lambda) & \text{if } a \in b(\lambda), \\
\gamma(\lambda) - 1 & \text{if } a \notin b(\lambda); 
\end{cases}
$$

Thus for any line bundle $\lambda \in \hat{X}_{r}^{\nu}$ and for any point $a \in \lambda$ it follows that $\gamma(\lambda) - 1 \leq \gamma(\lambda \zeta_{a}^{-1}) \leq \gamma(\lambda)$, so the mapping that associates to any pair $(a, \lambda) \in M \times \hat{X}_{r}^{\nu}$ the line bundle $\psi(a, \lambda) = \lambda \zeta_{a}^{-1}$ is a well defined holomorphic mapping

$$
(13.93) \quad \psi : M \times \hat{X}_{r}^{\nu} \longrightarrow \hat{X}_{r_{-1}}^{\nu,1} \cup \hat{X}_{r_{-1}}^{\nu}.
$$

If $(a, \lambda) \in B_{r}^{\nu}$ then $a \in b(\lambda)$ so $\gamma(\lambda \zeta_{a}^{-1}) = \gamma(\lambda)$, while if $(a, \lambda) \notin B_{r}^{\nu}$ then $\gamma(\lambda \zeta_{a}^{-1}) = \gamma(\lambda) - 1$; thus

$$
(13.94) \quad \psi(B_{r}^{\nu}) \subset \hat{X}_{r_{-1}}^{\nu,1} \quad \text{while} \quad \psi((M \times \hat{X}_{r}^{\nu}) \sim B_{r}^{\nu}) \subset \hat{X}_{r_{-1}}^{\nu,1}.
$$
Note that from the definitions it follows that

\[(13.95)\]

\[B^\nu_r \subset M \times \hat{X}_r^{\nu} \quad \text{while} \quad M \times X_r^{\nu, o} \subset (M \times \hat{X}_r^{\nu}) \sim B^\nu_r.\]

On the other hand if \(\gamma(\kappa \lambda^{-1}) \neq 0\) and \(b(\kappa \lambda^{-1})\) is the base divisor of the line bundle \(\kappa \lambda^{-1}\) then by Corollary 2.28

\[(13.96)\]

\[\gamma(\lambda \zeta_a) = \begin{cases} \gamma(\lambda) + 1 & \text{if } a \in b(\kappa \lambda^{-1}), \\ \gamma(\lambda) & \text{if } a \notin b(\kappa \lambda^{-1}), \end{cases}\]

hence for any line bundle \(\lambda \in \hat{X}_r^{\nu}\) and for any point \(a \in \lambda\) it follows that \(\gamma(\lambda) \leq \gamma(\lambda \zeta_a) \leq \gamma(\lambda) + 1\). Therefore the mapping that associates to any pair \((a, \lambda) \in M \times \hat{X}_r^{\nu}\) the line bundle \(\phi(a, \lambda) = \lambda \zeta_a\) is a well defined holomorphic mapping

\[(13.97)\]

\[\phi : M \times \hat{X}_r^{\nu} \longrightarrow \hat{X}_r^{\nu+1} \cup \hat{X}_r^{\nu+1}.\]

Through the isomorphism \((13.46)\) the decomposition \((13.87)\) for a nonempty variety \(\hat{X}_r^{\nu+g-1-r}\) can be carried over to a different decomposition of the variety \(\hat{X}_r^{\nu}\) by setting

\[(13.98)\]

\[\hat{X}_r^{\nu, o} = \left\{ \lambda \in \hat{X}_r^{\nu} \mid b(\kappa \lambda^{-1}) = 0 \right\}\]

and

\[(13.99)\]

\[\hat{X}_r^{\nu, s} = \left\{ \lambda \in \hat{X}_r^{\nu} \mid b(\kappa \lambda^{-1}) > 0 \right\};\]

this provides an alternative disjoint union decomposition

\[(13.100)\]

\[\hat{X}_r^{\nu} = \hat{X}_r^{\nu, o} \cup \hat{X}_r^{\nu, s} \quad \text{where} \quad \hat{X}_r^{\nu, o} \cap \hat{X}_r^{\nu, s} = \emptyset\]

and the alternative subset

\[(13.101)\]

\[B^\nu_r = \left\{ (z, \lambda) \in M \times \hat{X}_r^{\nu} \mid z \in b(\kappa \lambda^{-1}) \right\}\]

**Corollary 13.12** For any compact Riemann surface \(M\) of genus \(g > 0\) the subset \(B^\nu_r\) is a holomorphic subvariety of the holomorphic variety \(M \times \hat{X}_r^{\nu}\). The natural projection \(\pi : M \times \hat{X}_r^{\nu} \longrightarrow \hat{X}_r^{\nu}\) induces a finite proper surjective holomorphic mapping

\[(13.102)\]

\[\pi : B^\nu_r \longrightarrow \hat{X}_r^{\nu, s},\]

so \(\hat{X}_r^{\nu, s}\) is a holomorphic subvariety of \(\hat{X}_r^{\nu}\) and \(\dim \hat{X}_r^{\nu, s} = \dim B^\nu_r\).

This follows immediately from Theorem 13.11 applied to the variety \(\hat{X}_r^{2g-2-r}\) if it is nonempty, so no further proof is required.

It follows from the preceding observations that in terms of the subvariety \((13.88)\)

\[(13.103)\]

\[\phi(B^\nu_r) \subset \hat{X}_r^{\nu+1} \quad \text{while} \quad \phi(M \times \hat{X}_r^{\nu}) \sim B^\nu_r \subset \hat{X}_r^{\nu+1}.\]
For line bundles on the maximal and minimal curves of the Brill-Noether diagram this decomposition of the holomorphic varieties \( \hat{X}_r^\nu \) can be determined readily.

**Theorem 13.13** Let \( M \) be a compact Riemann surface of genus \( g > 0 \).

(i) For \( g < r \leq 2g - 1 \)

\[
(13.104) \quad \hat{X}_{r}^{MIN, \beta} = \left\{ \kappa\zeta_a\zeta_b^{-1} \mid a \in M, \ z_b \in \hat{X}_{2g-1-r}^0 \text{ and } a \notin b \right\}.
\]

(ii) For \( r > 2g - 1 \)

\[
(13.105) \quad \hat{X}_{r}^{MIN, \beta} = \emptyset
\]

so all line bundles in \( \hat{X}_r^\nu \) for \( r > 2g - 1 \) are base-point-free.

**Proof:** (i) If \( g < r < 2g - 1 \) and \( \lambda \in \hat{X}_r^{MIN} = \hat{X}_r^{g-\gamma} \) is not base-point-free then there is a point \( a \in M \) such that \( \gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) = r - g + 1 \); and by the Riemann-Roch Theorem that is equivalent to \( \gamma(\kappa\lambda\zeta_a^{-1}) = 1 \). Since \( c(\kappa\lambda\zeta_a^{-1}) > 0 \) then \( \kappa\lambda\zeta_a = \zeta_b \in \hat{X}_{2g-r-1}^0 \) for a uniquely determined positive divisor \( b \in M^{(2g-1-r)} \); thus \( \lambda = \kappa\zeta_a\zeta_b^{-1} \) where \( \zeta_b \in \hat{X}_{2g-r-1}^0 \). There is a unique holomorphic cross-section of the line bundle \( \zeta_b \), up to a constant factor, and if \( a \not\in b \) that cross-section vanishes at \( a \) hence \( 1 = \gamma(\zeta_b\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}) \); but then by the Riemann-Roch theorem \( \gamma(\lambda) = r - g + 2 \), a contradiction. Consequently \( a \not\in b \), which shows that the holomorphic line bundles \( \lambda \in \hat{X}_r^{MIN} \) that are not base-point-free satisfy (13.104) in this case. Actually if \( r = 2g - 1 \) it follows from Theorem 2.25 that the holomorphic line bundles \( \lambda \in \hat{X}_{2g-1}^0 \) that are not base-point-free are precisely those of the form \( \lambda = \kappa\zeta_a \), which is the special case of (13.104) for which the divisor \( b \) is the empty set.

Conversely to show that for \( g < r \leq 2g - 1 \) all the bundles of the form (13.104) actually are not base-point-free, suppose that \( \lambda \) is a holomorphic line bundle such that \( \lambda = \kappa\zeta_a\zeta_b^{-1} \) where \( \zeta_b \in \hat{X}_{2g-1-r}^0 \) and \( a \not\in b \). Then \( \gamma(\lambda) = r \) and by the Riemann-Roch Theorem \( \gamma(\lambda) = \gamma(\zeta_b\zeta_a^{-1}) + r + 1 - g \) and \( \gamma(\zeta_a^{-1}) = \gamma(\zeta_b) + r - g = r + 1 - g \). Since \( a \not\in b \) and \( \gamma(\zeta_b) = 1 \) it must be the case that \( \gamma(\zeta_b\zeta_a^{-1}) = 0 \); the preceding equations then show \( \gamma(\lambda) = r + 1 - g \) so \( \lambda \in \hat{X}_r^{g-\gamma} = \hat{X}_r^{MIN} \) and \( \gamma(\zeta_a^{-1}) = \gamma(\lambda) \) hence \( \lambda \) is not base-point-free.

(ii) Finally if \( r > 2g - 1 \) then \( c(\kappa\lambda\zeta_a^{-1}) < 0 \) so \( \gamma(\kappa\lambda\zeta_a^{-1}) = 0 \), which contradicts the condition that \( \gamma(\kappa\lambda\zeta_a^{-1}) = 1 \); thus there can be no holomorphic line bundles in \( \hat{X}_r^{MIN} \) that are not base-point free if \( r > 2g - 1 \). That suffices to conclude the proof.

**Corollary 13.14** For a compact Riemann surface \( M \) of genus \( g > 0 \) all integers \( r > g \) belong to the Lüroth semigroup \( \mathcal{L}(M) \) of \( M \).

**Proof:** Since the Lüroth semigroup \( \mathcal{L}(M) \) by definition is the set of integers \( r \) such that there is a base-point-free holomorphic line bundle \( \lambda \) with \( c(\lambda) = r \),
the corollary will be proved by showing that the holomorphic varieties $\hat{X}^{MIN}_r$ for $r > g$ contain base-point-free holomorphic line bundles. That is of course the case for $r > 2g - 1$ by part (ii) of the preceding theorem, so it suffices to consider the varieties $\hat{X}^{MIN}_r$ for $g < r \leq 2g - 1$. The subset $V \subset \hat{X}^{MIN}_r \subset P_r(M)$ consisting of holomorphic line bundles that are not base-point-free is a holomorphic subvariety by Theorem 78; and if $g < r \leq 2g - 1$ the subvariety $V$ consists of those holomorphic line bundles of the form $\lambda = \kappa \zeta_a \zeta_b^{-1}$ where $\zeta_b \in \hat{X}^0_{2g-r-1}$ and $a \notin b$ by (i) of the preceding theorem, so $V$ is contained in the image of the holomorphic mapping $\phi: M \times \hat{X}^0_{2g-r-1} \rightarrow P(M)$ that associates to any point $a \in M$ any line bundle $\zeta_b \in \hat{X}^0_{2g-r-1}$ the line bundle $\kappa \zeta_a \zeta_b^{-1} \in P_r(M)$. The product $M \times \hat{X}^0_{2g-r-1}$ is a compact complex manifold of dimension $2g - r$, since $\dim M = 1$ and $\dim \hat{X}^0_{2g-r-1} = 2g - r - 1$; and by the detailed form (A.20) of Remmert’s Proper Mapping Theorem it follows that $\dim \phi(M \times \hat{X}^0_{2g-r-1}) \leq 2g - r$ hence $\dim V \leq 2g - r$. The holomorphic variety $\hat{X}^{MIN}_r$ has dimension $g$ by (13.62), and since $2g - r < g$ for $r > g$ it follows that $\dim V < \dim \hat{X}^{MIN}_r$ so the complement $\hat{X}^{MIN}_r \sim V$, the set of base-point-free holomorphic line bundles in $\hat{X}^{MIN}_r$, is nonempty. That suffices for the proof.

For $g < r < 2g - 1$ the proof of the preceding corollary really amounted to showing that both the subvariety $\hat{X}^{MIN,\flat}_r$ and its complement $\hat{X}^{MIN,o}_r$ are nonempty, so in that range the variety $\hat{X}^{\flat}_r$ contains both base-point-free line bundles and line bundles with nontrivial base divisors.

**Theorem 13.15** If $M$ is a compact Riemann surface of genus $g > 0$ with the critical values $r_i$ then for any integer $r$ in the range $r_i < r < r_{i+1}$ for $i \geq 0$ a holomorphic line bundle $\lambda = \lambda_i \zeta_0$ where $\lambda_i \in X^{MAX}_{r_i}$ and $\delta \in M^{(r-r_i)}_r$ is a maximal bundle $\lambda \in X^{MAX}_r$ and $\lambda = \lambda_i \zeta_0$ is its base decomposition; so $\bdeg(\lambda) = \deg \delta$ and $\lambda$ is not base-point-free.

**Proof:** If $0 \leq r_i < r < r_{i+1}$ and $\lambda_i \in \hat{X}^{MAX}_{r_i}$ then $\lambda_i$ is base-point-free by the preceding theorem. If $\lambda = \lambda_i \zeta_0$ where $\delta$ is a positive divisor with $\deg \delta = r - r_i$ then $\gamma(\lambda) \geq \lambda_i$ by Lemma 2.6 and $c(\lambda) = c(\lambda_i) + \deg \delta < r_{i+1}$ so $\gamma(\lambda) < \gamma(\lambda_i)$ as well and consequently $\gamma(\lambda) = \gamma(\lambda_i)$ so $\lambda \in \hat{X}^{MAX}_r$; and by Theorem 2.12 (i) in addition $\lambda = \lambda_i \zeta_0$ is the base decomposition of the line bundle $\lambda$. That suffices for the proof.

Theorem 78 shows that the varieties $\hat{X}^{MAX}_r$ for the critical values $r_i$ contain only base-point-free holomorphic line bundles, while Theorem 13.15 shows that the varieties $\hat{X}^{MAX}_r$ contain line bundles that are not base-point-free if $r$ is not a critical value; thus the critical values of a compact Riemann surface $M$ of genus $g > 0$ can be characterized as those integers $r \geq 0$ such that $\hat{X}^{MAX}_r$ consists entirely of base-point-free holomorphic line bundles. For the general varieties $\hat{X}^r$, recall from the discussion in Chapter 2 that for any holomorphic line bundle $\lambda$ with base divisor $\delta(\lambda)$ on a compact Riemann surface $M$ of genus $g > 0$ and
any point \( a \in M \)

\[
\gamma(\lambda(z_a)^{-1}) = \begin{cases} 
\gamma(\lambda) & \text{if } a \in b(\lambda), \\
\gamma(\lambda) - 1 & \text{if } a \notin b(\lambda);
\end{cases}
\]

(13.106)

and if \( \gamma(\kappa\lambda^{-1}) \neq 0 \) and \( b(\kappa\lambda^{-1}) \) is the base divisor of the line bundle \( \kappa\lambda^{-1} \) then by Corollary 2.28

\[
\gamma(\lambda z_a) = \begin{cases} 
\gamma(\lambda) + 1 & \text{if } a \in b(\kappa\lambda^{-1}), \\
\gamma(\lambda) & \text{if } a \notin b(\kappa\lambda^{-1}).
\end{cases}
\]

(13.107)

For a special case of the preceding discussion, if \( \lambda \in \hat{X}^1_r \) is base-point-free then for any point \( z \in M \) clearly \( \lambda(z^{-1}) \in \hat{X}^0_{r-1} \) so

\[
\lambda(z)^{-1} = \zeta(z) \text{ for a unique divisor } \zeta(z) \in M^{r-1}
\]

(13.108)

thus describing a holomorphic mapping \( \phi : M \to M^{r-1} \). Assume first that the divisor \( \zeta(z) \) consists of \( r-1 \) distinct points \( z_i \) for all points \( z \in M \sim E \) for a finite subset \( E \subset M \). Then in an open neighborhood \( U \) of any point \( z_0 \in M \sim E \) there are well defined holomorphic mappings \( \phi_i : U \to M \) such that \( \zeta(z) = \sum_{i=1}^{r-1} 1 : \phi_i(z) \). In any simply connected intersection of such neighborhoods \( U \) the local holomorphic mappings match so continue analytically to holomorphic mappings in the union of the neighborhoods. For any paths in the universal covering space \( \tilde{M} \sim E \) the local mappings continue analytically to single valued holomorphic mappings \( \tilde{\phi}_1 : M \sim E \to M \); but for any covering translation \( T \in \Gamma_E \) of the universal covering space over \( M \sim E \) the local mappings satisfy \( \phi_i(Tz) = \sum_{j=1}^{r-1} a_{ij}(T)\phi_j(z) \) for some complex constants \( a_{ij}(T) \); describing a permutation matrix \( A_T \); thus if \( \phi = \{ \tilde{\phi}_i \} \) is the mapping \( \tilde{\phi} : M \sim E \to M^{r-1} \) then \( \tilde{\phi}(z) = A_{\phi(z)} \) for some nonsingular matrices \( A_T \) describing a permutation representation \( A \in \text{Hom}(\Gamma_E, \text{Gl}(r-1, \mathbb{C})) \). This describes a branched covering space \( \tilde{M} \) of degree \( r-1 \) over \( M \), branched at the points of \( E \). (The abelian differentials on \( M \) lift fo this covering in two ways; more Riemann relations?)

If \( M \) is a compact Riemann surface of genus \( g > 0 \) then for any point \( a \in M \) there is the local maximal function \( \mu_\alpha(a) \) defined by \( \mu_\alpha(a) = \gamma(\zeta_a^\alpha) - 1 \), as discussed in Chapter 11. It is evident from the definitions that

\[
\mu(r) \geq \mu_\alpha(a) \quad \text{for all } a \in M.
\]

(13.109)

The local critical values \( r_\alpha(a) \) are defined in terms of the local maximal function \( \mu_\alpha(a) \) just as the critical values \( r_i \) are defined in terms of the maximal function \( \mu(r) \), both as special cases of the general discussion of functions similar to the maximal functions in (13.14); thus

\[
r_\alpha(a) = \inf \left\{ r \in \mathbb{Z} \middle| \mu_\alpha(r) \geq i \right\}.
\]

(13.110)
Since \( \mu(r_i(a)) \geq \mu_a(r_i(a)) = i \) it follows from the definition of the critical value \( r_i \) that

\[
(13.111) \quad r_i \leq r_i(a) \quad \text{for all } a \in M.
\]

The sequence \( r_i(a) \) for any point \( a \in M \) is also an additive semigroup in \( \mathbb{Z} \) and a subsemigroup of of the Lüroth semigroup \( \mathcal{L}(M) \) of the Riemann surface, by Corollary 11.10, properties which are shared with the sequence \( r_i \). A somewhat different local version of the maximal function is also of interest.

The sequence \( r_i(a) \) for any point \( a \in M \) is also an additive semigroup in \( \mathbb{Z} \) and a subsemigroup of the Lüroth semigroup \( \mathcal{L}(M) \) of the Riemann surface, by Corollary 11.10, properties which are shared with the sequence \( r_i \). A somewhat different local version of the maximal function is also of interest.

The semilocal maximal function of \( M \) for a positive divisor \( a \in M^{(n)} \) of degree \( n \) is defined as the function of integers

\[
(13.112) \quad \mu_a(r) = \max \left\{ \gamma(\zeta_a') - 1 \mid a = a' + a'', \quad a' \in M(r), \ a'' \in M(n-r) \right\}
\]

for integers \( 1 \leq r \leq n - 1 \), extended to be a function of all integers \( r \in \mathbb{Z} \) by setting

\[
(13.113) \quad \mu_a(r) = \begin{cases} 
-1 & \text{for } r < 0, \\
0 & \text{for } r = 0, \\
\gamma(\zeta_a) - 1 & \text{for } r \geq n.
\end{cases}
\]

As a word of caution, the semilocal maximal function \( \mu_a(r) \) differs significantly from both the maximal function \( \mu(r) \) and the local maximal function \( \mu_a(r) \) since unlike the latter two the function \( \mu_a(r) \) involves the dimensions of the spaces of holomorphic cross-sections of only finitely many holomorphic line bundles so it is bounded above. Thus although the local maximal function for a point \( a \in M \) and the semilocal maximal function for the divisor \( n \cdot a \) for \( n > 0 \) coincide initially, since as is evident from their definitions \( \mu_a(r) = \mu_{n \cdot a}(r) \) for \( 0 \leq r \leq n \), nonetheless \( \mu_a(r) > \mu_{n \cdot a}(r) = \gamma(\zeta_n^a) - 1 \) for \( r > n \).

**Theorem 13.16** The semilocal maximal function of a compact Riemann surface \( M \) of genus \( g > 0 \) for a positive divisor \( a \) satisfies

\[
(13.114) \quad \mu_a(r) \leq \mu_a(r+1) \leq \mu_a(r) + 1,
\]

and in particular

\[
(13.115) \quad \mu_a(1) = 0.
\]

For any two positive divisors \( a_1 \) and \( a_2 \)

\[
(13.116) \quad \text{if } a_1 \geq a_2 \text{ then } \mu_{a_1}(r) \geq \mu_{a_2}(r).
\]

The semilocal maximal function and the maximal function are related by

\[
(13.117) \quad \mu_a(r) \leq \mu(r) \quad \text{for all } r
\]

and

\[
(13.118) \quad \mu(r) = \sup \left\{ \mu_a(r) \mid \deg a \geq r \right\} \quad \text{for all } r
\]
Proof: First since $\gamma(\zeta_p) = 1$ for any point $p \in M$ it is evident that $\mu_a(1) = 0$, which is (13.115).

If $1 \leq r \leq n - 1$ and $a = a' + a''$ where $a \in M^{(n)}$ and $a' \in M^{(r)}$ then for any point $p \in a''$ it follows from Lemma 2.6 that $\gamma(a') \leq \gamma(a)$ and

$$\gamma(a') - 1 \leq \gamma(a'+p) - 1 \leq \gamma(a') \leq \gamma(a).$$

(13.119)

If $r \leq n - 2$ then $\gamma(\zeta_a+p) - 1 \leq \mu_a(r+1)$ by Definition (13.112), while if $r = n - 1$ then $\zeta_a = \zeta_a + a$ so $\gamma(\zeta_a+a) - 1 = \gamma(\zeta_a) - 1 = \mu_a(r+1)$ by (13.113); thus in either case $\gamma(\zeta_a) - 1 \leq \mu_a(r+1)$, and since that is the case for all divisors $a' \in M^{(r)}$ it follows from (13.119) that $\mu_a(r) \leq \mu_a(r+1) \leq \mu_a(r+1)$, which is (13.114) for $1 \leq r \leq n - 1$. If $r = 0$ then since $\mu_a(0) = \mu_a(1) = 0$ by (13.113) and (13.115) that is enough to demonstrate (13.114) for $r = 0$; if $r = 1$ since also $\mu_a(1) = -1$ by (13.113) that is enough to demonstrate (13.114) for $r = -1$; and finally since also $\mu_a(r) = -1$ for $r < -1$ by (13.113) that is enough to demonstrate (13.114) for $r < -1$. Since $\mu_a(r) = \gamma(a) - 1$ for $r \geq n$ by (13.113) that is enough to demonstrate (13.114) for $r \geq n$, which establishes (13.114) for all $r$.

If $a_1 \geq a_2$ then $a_1 = a_2 + a_3$ for another positive divisor $a_3$, so if $n_i = \deg a_i$ then $n_1 = n_2 + n_3$. If $1 \leq r \leq n_2 - 1$ and if $a_2 = a_2^{(1)} + a_2^{(2)}$ for positive divisors $a_2^{(1)}$ and $a_2^{(2)}$ for which $\deg a_2^{(1)} = r$ then $a_1 = a_2^{(1)} + (a_2^{(2)} + a_3)$ and consequently it follows from the definition (13.112) for the divisor $a_1$ that $\gamma(a_2^{(1)}) - 1 \leq \mu_r(a_1)$; and since that is the case for any choice of the divisor $a_2^{(1)}$ it follows from the definition (13.112) for the divisor $a_2$ that $\mu_{a_2}(r) \leq \mu_{a_1}(r)$. On the other hand if $n_2 \leq r \leq n_1 - 1$ and if $a_3 = a_3^{(1)} + a_3^{(2)}$ for positive divisors $a_3^{(1)}$ and $a_3^{(2)}$ for which $\deg a_3^{(1)} = r - n_2$ then $a_1 = (a_2 + a_3^{(1)}) + a_3^{(2)}$ where $\deg(a_2 + a_3^{(1)}) = r$ so from the definition (13.112) it follows that $\gamma(a_2 + a_3^{(1)}) \leq \mu_r(a_1)$; but from (13.113) and Lemma 2.6 it further follows that $\mu_{a_2}(r) = \gamma(\zeta_a) - 1 \leq \gamma(\zeta_a + a_3^{(1)})$, and these two inequalities show that $\mu_{a_2}(r) \leq \mu_{a_1}(r)$ also for $n_2 \leq r \leq n_1 - 1$. Finally if $r \leq 0$ then $\mu_{a_1}(r) = \mu_{a_2}(r) = 0$ by (13.113) while if $r \geq n_1$ then from (13.112) and Lemma 2.6 it follows that $\mu_{a_2}(r) = \gamma(\zeta_a) - 1 \leq \gamma(\zeta_a) - 1 = \mu_{a_1}(r)$, and that suffices to demonstrate (13.116).

Finally (13.117) is quite obvious from the definitions of the two maximal functions; and if $\mu(r) = \gamma(\zeta) - 1$ for some divisor $a \in M^{(r)}$ then it follows from definition (13.113) that $\mu_a(r) = \gamma(\zeta_a) - 1 = \mu_a(r)$, and that suffices to establish (13.118) and thereby to conclude the proof.

Since the semilocal maximal function satisfies (13.13) it has all the properties discussed on page 355 and the following pages. For a positive divisor $a \in M^{(n)}$ of degree $n$ the maximum and minimum values of the semilocal maximal function are $n_+(a) = \gamma(\zeta_a) - 1$ and $n_-(a) = -1$. The invariants $r_i$ for the semilocal maximal function are defined by

$$r_i(a) = \inf \left\{ r \in \mathbb{Z} \mid \mu_a(r) \geq i \right\},$$

(13.120)

following the general definition in (13.14), and are called the the semilocal critical values for the divisor $a$; there are altogether just the $n_+(a) + 1 = \gamma(\zeta_a)$ finite
Figure 13.3: Example of a semilocal maximal function for a positive divisor \(a\) of degree 13.

**Semilocal Critical Values**

(13.121) \[0 = r_0(a) < r_1(a) < \cdots < r_{\gamma(a)}-1(a)\]

Extended for convenience in use in subsequent formulas by setting \(r_{-1}(a) = -\infty\) and \(r_{\gamma(a)}(a) = +\infty\). The complement of the set of semilocal critical values for a divisor \(a\) is the set of **semilocal gap values** \(s_j(a)\) for the divisor \(a\). An example of the graph of a semilocal maximal function, indicating the semilocal critical values, is sketched in Figure 13.3.

The basic properties of the semilocal critical values follow the pattern of the basic properties of the critical values and local critical values. It follows from (13.15) that

(13.122) \[\mu_a(r) = i \text{ for } r_i(a) \leq r < r_{i+1}(a),\]

and it follows from (13.16) that

(13.123) \[\mu_a(r) - \mu_a(r-1) = \begin{cases} 1 & \text{if } r = r_i(a) \text{ for some } i \text{ and} \\ 0 & \text{otherwise.} \end{cases}\]

The local maximal functions for different divisors are related as in (13.116), so if \(a_1 \geq a_2\) then \(\mu_{a_1}((r_i(a_2))) \geq \mu_{a_2}((r_i(a_2))) = i\) and it follows from the definition (13.120) of the semilocal critical value \(r_i(a_1)\) that

(13.124) \[\text{if } a_1 \geq a_2 \text{ then } r_i(a_1) \leq r_i(a_2) \text{ for any index } i.\]

In particular \(r_i(a) \leq r_i(a)\) for any point \(a\) in the divisor \(a\) since then \(1 \cdot a \leq a\). In view of (??) the corresponding argument shows that \(r_i \leq r_i(a)\) for any positive divisor \(a\). In summary, the semilocal critical values \(r_i(a)\), the local critical values \(r_i(a)\) and the critical values \(r_i\) of a Riemann surface \(M\) are related by

(13.125) \[r_i \leq r_i(a) \leq r_i(a) \text{ for any point } a \in \text{a positive divisor } a.\]

The interest of these critical values lies in part in the following observation.
Theorem 13.17  The semilocal critical values at a divisor $a$ on a compact Riemann surface $M$ of genus $g > 0$ belong to the Lüroth semigroup of $M$.

Proof: Since $r_0(a) = 0$ does belong to the Lüroth semigroup of $M$ it is enough just to demonstrate the theorem for strictly positive semilocal critical values. If $r = r_i(a) > 0$ is a semilocal critical value for the divisor $a$ then $\mu_a(r) = i$ while $\mu_a(s) < i$ if $s < r$. By the definition (13.112) of the semilocal maximal function there are positive divisors $a'$ and $a''$ of degrees $\deg a' = r$ and $\deg a'' = n - r$ such that $a = a' + a''$ and $\gamma(\zeta_{a'}) - 1 = i$. The theorem will be proved by showing that the line bundle $\zeta_{a'}$ of characteristic class $c(\zeta_{a'}) = r$ is base-point-free. If to the contrary $\zeta_{a'}$ is not base-point-free then there is some point $x \in M$ for which $\gamma(\zeta_{a'}/\zeta_x^{-1}) = \gamma(\zeta_{a'}) = i$; and in that case it follows from Lemma 2.6 that all the holomorphic cross-sections of the line bundle $\zeta_{a'}$ vanish at the point $x$. If $a' = \sum_k \mu_k \cdot a_k$ and $h_k$ is a nontrivial holomorphic cross-section of the bundle $\zeta_{a_k}$ then $h_k$ has a simple zero at the point $a_k$ as its sole zero; the product $h = \prod_k h_k^{a_k}$ then is a holomorphic cross-section of the bundle $\zeta_{a'}$ that vanishes only at points of $a'$, and since $h(x) = 0$ it follows that $x$ is a point of the divisor $a'$. If $a' = a'' + 1 \cdot x$ for another positive divisor $a''$ then $a = a' + a'' = a'' + (a'' + 1 \cdot x)$ where $\deg a'' = r - 1$, so from the definition (13.112) of the semilocal maximal function again it follows that $\mu_a(r - 1) \geq \gamma(\zeta_{a''}) = i$, which contradicts the assumption that $r$ is the local critical value $r = r_i(a)$. That contradiction suffices to conclude the proof.

The semilocal maximal function for a divisor $a$ can be read directly from the Brill-Noether matrix $\Omega(a)$ of $a$, extending to the semilocal maximal function the treatment of the local maximal function discussed on page 302 and the following pages. If $a = \nu_1 \cdot p_1 + \cdots + \nu_m \cdot p_m$ is a divisor of degree $n = \nu_1 + \cdots + \nu_m$ for distinct points $p_1, \ldots, p_m$ the Brill-Noether matrix is the $g \times n$ complex matrix with the rows as in (2.36). Explicitly if $\omega_i = f_{i,a_j}(z_{a_j}) dz_{a_j}$ for $1 \leq i \leq g$ is a basis for the holomorphic abelian differentials, expressed in terms of local coordinates $z_{a_j}$ at the points $p_{j}$, the entries in row $i$ of the matrix $\Omega_{\alpha_1, \ldots, \alpha_m}(a)$ are the functions $f_{i,\alpha_j}(p_j)/k_j!$ for $1 \leq j \leq m$ and $0 \leq k_j \leq \nu_j - 1$; the columns are indexed by $j$ and $k_j$. The Riemann-Roch Theorem in terms of the Brill-Noether has the form (2.24) so

$$\gamma(\zeta_a) - 1 = n - \text{rank} \, \Omega_{\alpha_1, \ldots, \alpha_m}(a).$$

The divisors $a'$ for which there is a decomposition $a = a' + a''$ into a sum of positive divisors where $\deg a' = r$ are just the divisors $a' = \mu_1 \cdot p_1 + \cdots + \mu_m \cdot p_m$ for which $0 \leq \mu_1 \leq \nu_1$ and $\mu_1 + \cdots + \mu_m = r$; and the Brill-Noether matrix for the divisor $a'$ is just the matrix $\Omega_{\alpha_1, \ldots, \alpha_m}(a)$ formed by the $r = \mu_1 + \cdots + \mu_m$ columns of the matrix $\Omega_{\alpha_1, \ldots, \alpha_m}(a)$ for the column parameters restricted to the values $1 \leq j \leq m$, $0 \leq k_j \leq \mu_j - 1$. Consequently from definition (13.112) it follows that

$$\mu_a(r) = \left\{ \begin{array}{ll} \max \left\{ r - \text{rank} \, \Omega_{\alpha_1, \ldots, \alpha_m}(a) \mid 0 \leq \mu_j \leq \nu_j \right\} & \leq \nu_j \\ r - \min \left\{ \text{rank} \, \Omega_{\alpha_1, \ldots, \alpha_m}(a) \mid 0 \leq \mu_j \leq \nu_j \right\} & \end{array} \right.$$
for integers $1 \leq r \leq n - 1$.

[refer to chap max2 ]

**Theorem 13.18** ??? If $M$ is a compact Riemann surface with the first critical value $r_1$ then

$$
(13.128) \quad \text{rank}\{f_i(z_j)\} \geq r_1 - 1
$$

for any divisor $d = \sum_{j=1}^{r_1-1} z_j \in M^{(r_1-1)}$, where the holomorphic abelian differentials on $M$ are written $\omega_i(z) = f_i(z)dz$.

**Proof:** Since $\gamma(\zeta_\varphi) < 1$ for any divisor $d$ of degree $\deg d < r_1$ it follows from (??) in the proof of the preceding theorem that $\text{rank}\{f_i(z_j)\} \geq r_1 - 1$, which suffices for the proof.

When there are at least 2 linearly independent holomorphic cross-sections of a holomorphic line bundle $\lambda$ it is possible to use these cross-sections to obtain some further information about the spaces of holomorphic cross-sections of the line bundles $\lambda^n$ for any $n > 0$. Since the first critical value $r_1$ is the least integer for which there are line bundles $\lambda$ of characteristic class $c(\lambda) = r_1$ such that $\gamma(\lambda) \geq 2$, the value $r_1$ plays a particularly significant role in the study of the maximal function for Riemann surfaces.

**Theorem 13.19** If $f_0, f_1 \in \Gamma(M, \mathcal{O}(\lambda))$ are linearly independent holomorphic cross-sections of a holomorphic line bundle $\lambda$ over a compact Riemann surface $M$ then for any $n > 0$ the $n+1$ products $f_1^i f_2^{n-1}$ for $0 \leq i \leq n$ are linearly independent holomorphic cross-sections of the line bundle $\lambda^n$.

**Proof:** If there is a nontrivial linear relation $\sum_{i=0}^{n} c_i f_0^i f_1^{n-1} = 0$ and if $g = f_0/f_1$ then $\sum_{i=0}^{n} c_i g^i = 0$, so $g$ is a constant, contradicting the assumption that the cross-sections $f_0, f_1$ are linearly independent; and that suffices for the proof.

**Corollary 13.20** The critical values $r_n$ of a compact Riemann surface $M$ of genus $g > 0$ satisfy $r_n \leq nr_1$ for all $n > 0$.

**Proof:** There is a holomorphic line bundle $\lambda$ for which $c(\lambda) = r_1$ and $\gamma(\lambda) = 2$, and that line bundle has two linearly independent holomorphic cross-sections $f_0, f_1$. The preceding theorem shows that the $n+1$ cross-sections $f_1^i f_2^{n-1} \in \Gamma(M, \mathcal{O}(\lambda^n))$ for $0 \leq i \leq n$ are linearly independent, hence $\gamma(\lambda^n) - 1 \geq n$; and consequently $nr_1 = c(\lambda^n) \geq r_n$, which suffices for the proof.
Theorem 13.21 If \( f_0, f_1 \in \Gamma(M, \mathcal{O}() \) and \( g_0, g_1, \ldots, g_n \in \Gamma(M, \mathcal{O}(\sigma)) \) are linearly independent holomorphic cross-sections of the holomorphic line bundles \( \lambda, \sigma \) over a compact Riemann surface \( M \), where \( f_0, f_1 \) have no common zeros, then either the 2n + 2 holomorphic cross-sections \( f_i g_j \in \Gamma(M, \mathcal{O}(\sigma)) \) for the indices \( 0 \leq i \leq 1 \) and \( 0 \leq j \leq n \) are linearly independent or \( \sigma = \lambda \zeta_3 \) for the line bundle \( \zeta_3 \) of a positive divisor \( \mathfrak{d} \) on \( M \).

Proof: Any nontrivial linear relation \( \sum_{i=0}^{1} \sum_{j=0}^{n} c_{i,j} f_i g_j = 0 \) among the cross-sections \( f_i g_j \) can be rewritten as the identity \( f_0 g_0' = f_1 g_1' \) for the nontrivial holomorphic cross-sections \( g_0' = \sum_{j=0}^{n} c_{0,j} g_j \) and \( g_1' = -\sum_{j=0}^{n} c_{1,j} g_j \) of the line bundle \( \sigma \), or equivalently as the equality \( f_0/f_1 = g_1'/g_0' \) of two meromorphic functions on \( M \). If \( \mathfrak{d} = \mathfrak{d}(g_0', g_1') \) is the divisor of common zeros of the cross-sections \( g_0', g_1' \) and \( h \in \Gamma(M, \mathcal{O}(\zeta_3)) \) is a cross section for which \( \mathfrak{d}(h) = \mathfrak{d} \) then \( g_0' = h g_0'' \) and \( g_1' = h g_1'' \) for holomorphic cross-sections \( g_0'', g_1'' \in \Gamma(M, \mathcal{O}(\sigma \zeta_3^{-1})) \) which have no common zeros; and \( f_0/f_1 = g_1''/g_0'' \). Since neither the cross-sections \( g_0'', g_1'' \) nor the cross-sections \( f_0, f_1 \) have any common zeros, the polar divisor of the meromorphic function \( f_0/f_1 = g_1''/g_0'' \) is \( \mathfrak{d}(f_1) = \mathfrak{d}(g_0'') \); and since \( \zeta_3(f_1) = \lambda \) and \( \zeta_3(g_0'') = \sigma \zeta_3^{-1} \) it follows that \( \lambda = \sigma \zeta_3^{-1} \), which suffices for the proof.

Corollary 13.22 If \( \lambda_1 \in \hat{X}_{MAX}^{1} \) and \( \lambda_i \in \hat{X}_{MAX}^{1} \) for any \( i > 0 \) then either (i) \( r_{2i+1} - r_i < r_1 \) or (ii) \( \lambda_i = \lambda_1 \zeta_3 \) for a positive divisor \( \mathfrak{d} \).

Proof: If \( \lambda_1 \in \hat{X}_{MAX}^{1} \) then \( \lambda_1 \) has two linearly independent holomorphic cross-sections, which have no common zeros since \( \lambda_1 \) is base-point-free by Theorem ??; and if \( \lambda_i \in \hat{X}_{MAX}^{1} \) then \( \lambda_i \) has \( i + 1 \) linearly independent holomorphic cross-sections. It follows from the preceding theorem that either \( \gamma(\lambda_1 \lambda_i) \geq 2i + 2 \) or \( \lambda_i = \lambda_1 \zeta_3 \) for some positive divisor \( \mathfrak{d} \). If \( \gamma(\lambda_1 \lambda_i) = 1 > 2i + 1 \) then by the definition of the critical values \( r_1 + r_i = c(\lambda_1 \lambda_i) = r_{2i+1} \). That suffices for the proof.

If \( \lambda \) and \( \sigma \) are inequivalent base-point-free holomorphic line bundles of characteristic classes \( c(\lambda) = r \) and \( c(\sigma) = s \) then \( r_3 = r + s \).

(ii) If \( \lambda, \sigma \) are base-point-free holomorphic line bundles then by Lemma 2.9 there are linearly independent holomorphic cross-sections \( f_1, f_2 \in \Gamma(M, \mathcal{O}(\lambda)) \) with no common zeros and \( g_1, g_2 \in \Gamma(M, \mathcal{O}(\sigma)) \) also with no common zeros; the products of these cross-sections are the four holomorphic cross-sections \( f_1 g_1, f_1 g_2, f_2 g_1, f_2 g_2 \in \Gamma(M, \mathcal{O}(\sigma \lambda)) \). If there were a nontrivial linear relation between these four cross-sections it could be written as an identity of the form \( f_1 g_1' + f_2 g_2' = 0 \) where \( g_1' = a_1 g_1 + a_2 g_2 \) and \( g_2' = b_1 g_1 + b_2 g_2 \) for some constants \( a_i, b_i \), not all of which are zero; and the cross-sections \( g_1' \) and \( g_2' \) would have to be linearly independent, since the cross-sections \( f_1 \) and \( f_2 \) are, so the cross-sections \( g_1' \) and \( g_2' \) also have no common zeros. But then \( f_2/f_1 = -g_1'/g_2' \) is a meromorphic function with the polar divisor \( \mathfrak{d}(f_1) = \mathfrak{d}(g_2') \) and consequently \( \lambda = \zeta_3 = \sigma \), contradicting the assumption that the line bundles \( \lambda \) and \( \sigma \) are distinct. Therefore the four cross-sections \( f_1 g_1, f_1 g_2, f_2 g_1, f_2 g_2 \in \Gamma(M, \mathcal{O}(\sigma \lambda)) \) are
linearly independent, so $\gamma(\lambda \sigma) - 1 \geq 3$ and therefore $r + s = \gamma(\lambda \sigma) \geq r_3$, which suffices for the proof.

The preceding results can be used to describe fully the first part of the interesting region of the Brill-Noether diagram, at least for some cases.

**Theorem 13.23** If $M$ is a compact Riemann surface of genus $g > 0$ for which $X^1_{1,2}$ is a finite set then whenever $r_1 < r < r_2$ the holomorphic variety $X^1_r$ is a finite union of holomorphic varieties of dimension $r - r_1$ and contains no base-point-free holomorphic line bundles.

**Proof:** If $r_1 < r < r_2$ then since $r_2 < 2r_1$ by Lemma ?? it follows that $0 < r - r_1 < r_2 - r_1 < r_1$ so $\gamma(\zeta_0) = 1$ for any divisor $\mathfrak{d}$ of degree $\deg \mathfrak{d} = r - r_1$. Any holomorphic line bundle $\lambda_0 \in X^1_{1,2}$ is base-point-free by Theorem ??, and if $\deg \mathfrak{d} = r - r_1$ the product $\lambda = \lambda_0 \zeta_0$ for a divisor $\mathfrak{d}$ of degree $\deg \mathfrak{d} = r - r_1$ is a line bundle of characteristic class $c(\lambda) = r$, and $\gamma(\lambda) \geq \gamma(\lambda_0) = 2$ while $\gamma(\lambda) \leq 2$ since $c(\lambda) < r_2$, so actually $\gamma(\lambda) = \gamma(\lambda_0) = 2$. Therefore the decomposition $\lambda = \lambda_0 \zeta_0$ is the base decomposition of that line bundle, so $\lambda$ is not base-point-free. Conversely if $\lambda \in X^1_r$ is not base-point-free it must be a product $\lambda = \lambda_0 \zeta_0$ for some base-point-free line bundle $\lambda_0$ with $c(\lambda_0) < r$, and the only possibility is $c(\lambda_0) = r_1$. For a fixed such line bundle $\lambda_0$ the set of line bundles $\lambda_0 \zeta_0$ for divisors $\mathfrak{d}$ with $\deg \mathfrak{d} = r - r_1$ is the image of the compact complex manifold $M^{(r-r_1)}$ under the mapping $\pi$.

Of interest in connection with the base-decomposition of holomorphic line bundles are relations between the subvarieties $r$, induced by the mappings

\begin{equation}
\pi_r : M \times M^{(r-1)} \longrightarrow M^{(r)} \text{ for which } \pi_r(a, \mathfrak{d}) = a + \mathfrak{d}.
\end{equation}

For any divisor $\mathfrak{d}$ in the open subset $M^{(r)} \subset M^{(r)}$ consisting of divisors of $r$ distinct points of $M$ it is evident that $\pi^{-1}_r(\mathfrak{d})$ consists of $r$ distinct points of $M \times M^{(r-1)}$ and that the restriction of the mapping $\pi_r$ is a covering projection of $r$ sheets over $M^{(r)}$. On the other hand for any divisor $\mathfrak{d}$ in the complementary holomorphic subvariety $(M^{(r)} \sim M^{(r)}_*) \subset M^{(r)}$ the inverse image $\pi^{-1}_r(\mathfrak{d})$ consists of strictly fewer than $r$ points. The mapping $\pi_r$ thus is a finite branched holomorphic covering of $r$ sheets over $M^{(r)}$, branched over the subvariety $M^{(r)} \sim M^{(r)}_*$. In particular the mapping $\pi_r$ is a finite proper surjective holomorphic mapping: so by Remmert’s Proper Mapping Theorem the image under this mapping of the subvariety $M \times G^{r-1}_r \subset M \times M^{(r-1)}$ is a well defined holomorphic subvariety $\pi_r(M \times G^{r-1}_r) \subset M^{(r)}$.

**Theorem 13.24** If $M$ is a compact Riemann surface of genus $g > 0$

\begin{equation}
G^{r+1}_r \subset \pi_r(M \times G^{r-1}_r) \subset G^\nu_r \text{ for } r \geq 2 \text{ and all } \nu;
\end{equation}

and if $\nu > 0$ and $G^{r+1}_r \neq G^\nu_r$ then $\pi_r(M \times G^{r-1}_r) = G^\nu_r$ if and only if none of the line bundles $\zeta_0 \in P_r(M)$ is base-point-free for any divisor $\mathfrak{d} \in (G^\nu_r \sim G^{r+1}_r)$. 

Proof: If $\nu \leq 0$ the inclusion (13.130) reduces to $G_{r+1}^\nu \subset \pi_r(M \times M^{(r-1)}) \subset M^{(r)}$ in view of (??), and that holds quite trivially; so it can be assumed for the remainder of the proof that $\nu > 0$. If $d \in G_r^\nu$ and if $a \in M$ is a point in the divisor $d$, so that $d = a + d' = \pi_r(a, d')$ for a divisor $d' \in M^{(r-1)}$, then

$$\gamma(\zeta_a) - 1 \geq \nu + 1$$

and it follows from Lemma 2.6 that $\gamma(\zeta_a) - 1 = \gamma(\zeta_a - 1) - 1 \geq \nu$. That demonstrates both inclusions in (13.130). Next if $d \in (G_r^\nu \sim G_r^{\nu+1})$ for $\nu > 0$ then $\gamma(\zeta_a) = \nu + 1 > 1$. If the line bundle $\zeta_a$ is not base-point-free then all the holomorphic cross-sections of the line bundle $\zeta_a$ vanish at some point $a$, which must be a point of the divisor $d$, so $d = a + d' = \pi_r(a, d')$ for some divisor $d' \in M^{(r-1)}$; and since all the holomorphic cross-sections of the bundle $\zeta_a$ vanish at the point $a$ it follows from Lemma 2.6 that $\gamma(\zeta_a) - 1 = \gamma(\zeta_a - 1) - 1 = \gamma(\zeta_a) - 1 \geq \nu$, hence that $d' \in G_{r-1}^\nu$ so $d \in \pi_r(M \times G_{r-1}^\nu)$. Thus if none of the line bundles $\zeta_a$ is base-point-free for any divisor $d \in (G_r^\nu \sim G_r^{\nu+1})$ then $G_r^\nu \subset \pi_r(M \times G_{r-1}^\nu)$ by the first part of the proof; in fact altogether $G_r^\nu \subset \pi_r(M \times G_{r-1}^\nu)$, and this inclusion must be an equality since the reversed inclusion was demonstrated in the first part of the proof. Conversely if $G_r^\nu = \pi_r(M \times G_{r-1}^\nu)$ and if $d \in G_r^\nu \sim G_r^{\nu+1}$ then $\gamma(\zeta_a) - 1 = \nu$ and $d = a + d'$ for some divisor $d' \in G_{r-1}^\nu$. Thus $\nu \leq \gamma(\zeta_a) - 1 = \gamma(\zeta_a - 1) - 1 \leq \gamma(\zeta_a) - 1 \leq \nu$ by Lemma 2.6, and consequently $\gamma(\zeta_a) - 1 = \gamma(\zeta_a)$; so by Lemma 2.6 yet again all the holomorphic cross-sections of the bundle $\zeta_a$ must vanish at the point $a$, so the bundle $\zeta_a$ is not base-point-free. That suffices to conclude the proof of the theorem.

Corollary 13.25 If $M$ is a compact Riemann surface of genus $g > 0$

(13.131) \[ \dim G_{r+1}^\nu \leq 1 + \dim G_{r-1}^\nu \leq \dim G_r^\nu \] for $r \geq 2$ and all $\nu$; and if $\nu > 0$ and $G_{r+1}^\nu \neq G_r^\nu$ and none of the line bundles $\zeta_a \in P_r(M)$ is base-point-free for any divisor $d \in G_r^\nu \sim G_{r+1}^\nu$

(13.132) \[ 1 + \dim G_{r-1}^\nu = \dim G_r^\nu. \]

Proof: Since the mapping $\pi_r$ in (13.129) is finite and proper, Remmert’s Proper Mapping Theorem implies not only that the image $\pi_r(M \times G_{r-1}^\nu)$ is a holomorphic subvariety of $M^{(r)}$ but also that $\dim \pi_r(M \times G_{r-1}^\nu) = \dim (M \times G_{r-1}^\nu)$; and of course $\dim (M \times G_{r-1}^\nu) = 1 + \dim G_{r-1}^\nu$ whenever $G_{r-1}^\nu$ is nonempty. The corollary follows immediately from these observations and the inclusion relations of the preceding theorem; and that suffices for the proof.

Corollary 13.26 If $M$ is a compact Riemann surface of genus $g < 0$ with the maximal function $r_1$ and if $r_1 \leq r < r_{i+1}$ and there are no base-point-free holomorphic line bundles in $\dim X_r^\text{MAX}$ then $\dim X_r^\text{MAX} = \dim X_{r-1}^\text{MAX} + 1$ (ii) If $r_1 \leq r < r_{i+1}$ then $\dim X_r^\text{MAX} > \dim X_{r-1}^\text{MAX} + 1$;
Proof: If \( r_i \leq r \leq r_{i+1} \) and there are no base-point-free holomorphic line bundles in \( \dim X_{r_{i+1}}^{MAX} \) it follows from Corollary 13.25 that \( \dim G_{r_{i+1}}^{MAX} = \dim G_{r_{i+1}}^{MAX} + 1 \) and from this in view of Corollary ?? it follows that \( \dim X_{r_{i+1}}^{MAX} = \dim X_{r_{i+1}}^{MAX} + 1 \); and that suffices for the proof.

[REMARK:] Deduce the consequences for the maximal function from this.

\[
\epsilon_k : J(M) \rightarrow J(M) \quad \text{defined by} \quad \epsilon_k(t) = k - t,
\]

a biholomorphic mapping of the Jacobi variety \( J(M) \) to itself of period 2, and consequently of course that \( \epsilon_k(W_{d_{g-1}}) = W_{d_{g-1}} \) as well. The automorphism \( \epsilon_k \) also has individual fixed points, which are just those points \( t \in J(M) \) such that \( t = k - t \) so which are the \( 2^{2g} \) points of the quotient torus \( J(M) = \mathbb{C}^g/\Omega \mathbb{Z}^{2g} \) represented by the half-periods \( \frac{1}{2} \Omega \mathbb{Z}^{2g} \) modulo the periods \( \Omega \mathbb{Z}^{2g} \); they are called either the points of order 2 or the half-periods of the torus \( J(M) \). These fixed points are distributed among the disjoint fixed varieties \( X_{d_{g-1}} \subset J(M) \); so if there are \( v_{\nu} \) fixed points in \( X_{d_{g-1}} \) for \( -1 \leq \nu \leq \mu(g-1) \) then

\[
\sum_{\nu=-1}^{\mu(g-1)} v_{\nu} = 2^{2g}.
\]

Under the biholomorphic mapping \( \phi_{a_{g}} : P_{g-1}(M) \rightarrow J(M) \) the fixed points correspond to holomorphic line bundles \( \lambda \in \mathcal{P}_{g-1}(M) \) for which \( \lambda^2 = \kappa \); they are called the semicanonical bundles of the Riemann surface \( M \), or alternatively the theta characteristics of \( M \) in view of their natural appearance in another form in the study of theta functions on Riemann surfaces. Perhaps the most interesting of the semicanonical line bundles are those that admit holomorphic cross-sections, so those that are contained in the varieties \( X_{d_{g-1}} \) for indices \( \nu \geq 0 \).

**Theorem 13.27** The semicanonical bundles \( \lambda \in \mathcal{X}_{d_{g-1}} \) for \( \nu \geq 0 \) are the line bundles \( \lambda = \zeta_0 \) of positive divisors \( d \in M^{(g-1)} \) of degree \( g-1 \) such that \( 2d = \mathfrak{f} \), the canonical divisor on \( M \).

**Proof:** A semicanonical line bundle \( \lambda \) such that \( \gamma(\lambda) > 0 \) has a nontrivial holomorphic cross-section \( f \in \Gamma(M, \mathcal{O}(\lambda)) \) with a divisor \( \mathcal{O}(f) \); and since \( f^2 \in \Gamma(M, \mathcal{O}(\lambda^2)) = \Gamma(M, \mathcal{O}(\kappa)) \) it follows that \( 2\mathcal{O}(f) = \mathcal{O}(f^2) \) is a canonical divisor on the Riemann surface \( M \). Conversely if there is a canonical divisor on \( M \) of the form \( \mathfrak{f} = 2\mathcal{O} \) for a positive divisor \( d \in M^{(g-1)} \) then there is a holomorphic cross-section \( h \in \Gamma(M, \mathcal{O}(\kappa)) \) of the canonical bundle \( \kappa \) of the Riemann surface \( M \) with the divisor \( \mathcal{O}(h) = 2\mathcal{O} \). When the canonical line bundle \( \kappa \) on \( M \) is
represented by a holomorphic factor of automorphy $\kappa(T, z)$ for the action of the covering translation group of $M$ on the universal covering space $\tilde{M}$ of $M$, the cross-section $h$ corresponds to a holomorphic function $h(z)$ on $\tilde{M}$ that is a relatively automorphic function for this factor of automorphy. Since the function $h(z)$ has a divisor of even order it has a well defined square root in an open neighborhood of each point of $\tilde{M}$; and since $\tilde{M}$ is simply connected any choice of a local square root at one point can be continued to the entire Riemann surface $\tilde{M}$ as a well defined holomorphic function $f(z) = \sqrt{h(z)}$ on $M$. The divisor $\delta(f(z))$ of this function is invariant under the action of the covering translation group $\Gamma$, so the quotients $\lambda(T, z) = f(Tz)/f(z)$ are well defined holomorphic and nowhere vanishing functions on $M$; and it follows from their definition that they satisfy $\lambda(ST, z) = \lambda(S, Tz)\lambda(T, z)$ for any two covering translations $T \in \Gamma$, so they form a factor of automorphy for the action of the group $\Gamma$ on $\tilde{M}$. This factor of automorphy describes a holomorphic line bundle $\lambda$ over $M$, and $f(z)$ represents a holomorphic cross-section $f \in \Gamma(M, O(\lambda))$ of this bundle. Since $f^2 = h \in \Gamma(M, O(\kappa))$ it follows that $\lambda^2 = \kappa$ and consequently that $\lambda$ is a semicanonical bundle over $M$, which has the nontrivial holomorphic cross-section $f$. That suffices for the proof.

[Alternative lemma for proof of Theorem 13.4.]

Lemma 13.28 On a compact Riemann surface $M$ of genus $g \geq 0$ the line bundle $\zeta^{g+1}_a$ is base-point-free and $\gamma(\zeta^{g+1}_a) = 2$ for all but at most finitely many points $a \in M$.

Proof: For any point $a \in M$ it follows from the Riemann-Roch Theorem in the form of Theorem 2.18 that $\gamma(\zeta^{g}_a) = \gamma(\kappa\zeta^{-g}_a) + 1 \geq 1$ and $\gamma(\zeta^{g+1}_a) = \gamma(\kappa\zeta^{-g-1}_a) + 2 \geq 2$. On the other hand it follows from the Riemann-Roch Theorem in the form of Theorem 2.24 that $\gamma(\zeta_a^{g}) = g+1-\text{rank } \Omega(g \cdot a)$ where $\Omega(g \cdot a)$ is the Brill-Noether matrix of the divisor $g \cdot a$. When the holomorphic abelian differentials on $M$ are written in terms of local coordinates $z_a$ as $\omega_i = f_{i\alpha}(z_a)$, the determinant of the Brill-Noether matrix $\Omega(g \cdot a)$ is just the Wronskian of the functions $f_{i\alpha}(z_a)$, as in (2.38). Since the abelian differentials are linearly independent holomorphic functions their Wronskian does not vanish identically\(^3\); hence rank $\Omega(g \cdot a) = g\) and $\gamma(\zeta^{g}_a) = 1$ at all but the finitely many points $a \in M$ at which $\text{det } \Omega(g \cdot a) = 0$. If $\gamma(\zeta^{g}_a) = 1$ then $\gamma(\zeta^{g+1}_a) \leq 2$ by Lemma 2.6, and since it was already noted that $\gamma(\zeta^{g+1}_a) \geq 2$ it follows that $\gamma(\zeta^{g+1}_a) = 2$. Furthermore if $\gamma(\zeta^{g}_a) = 1$ the line bundle $\zeta^{g}_a$ is not base-point-free, indeed it has the base decomposition $\zeta^{g}_a = 1 \cdot \zeta^{g}_a$ for the identity bundle 1; and since $\gamma(\zeta^{g}_a) = 2$ it follows from Theorem 2.12 (iii) that $\zeta^{g}_a$ is base-point-free for some $r$ in the range $1 \leq r \leq g+1$, which can only be the case for $r = g+1$. That suffices for the proof.

\(^3\)It is obvious that if a finite number of functions are linearly dependent their Wronskian determinant is zero. The converse was long known to be false for $C^\infty$ functions but it is true for holomorphic functions; see for instance the paper by M. Bôcher, The theory of linear dependence, Annals of Math. vol 2 (1900), pages 81-96; or more recently the paper by Alin Bostan and Philippe Dumas, Wronskians and Linear Independence, Amer. Math. Monthly, vol.117, (2010), pp. 722-727.
Part III

Appendices
Appendix A

Manifolds and Varieties

A.1 Holomorphic Functions

This appendix contains a survey of some general properties of complex manifolds and holomorphic varieties, an acquaintance with which is presupposed in the present book. The emphasis is on those properties that are relevant to the study of Riemann surfaces. The discussion here is rather abbreviated and generally does not include complete proofs; references for more detailed treatments of particular topics will be included along the way. Any investigation of Riemann surfaces of course presupposes familiarity with the standard properties of holomorphic functions of a single variable; but many topics also involve some properties of holomorphic functions of several variables, and since these properties may not be quite so familiar the appendix will begin with a survey of some of the results that are used in this book\footnote{For more extensive treatments of the general properties of holomorphic functions of several variables see for instance R. C. Gunning, Introduction to Holomorphic Functions of Several Variables, (Wadsworth and Brooks/Cole, 1990), (references to which for short will be given in the form G-IIC12 for Theorem/Corollary/Definition 12, section C, volume III), or L. and B. Kaup, Holomorphic Functions of Several Variables, (deGruyter, 1983), or S. Krantz, Function Theory of Several Complex Variables, (Wadsworth and Brooks/Cole, 1992).}

A complex-valued function $f(z)$ defined in an open subset $U$ of the $n$-dimensional complex vector space $\mathbb{C}^n$ is \textit{holomorphic} in $U$ if in an open neighborhood of any point $a \in U$ it has a convergent power series expansion

\begin{equation}
A.1 \quad f(z_1, \ldots, z_n) = \sum_{i_1, \ldots, i_n=0}^{\infty} c_{i_1 \ldots i_n} (z_1 - a_1)^{i_1} \cdots (z_n - a_n)^{i_n}.
\end{equation}

The set of holomorphic functions in $U$ form a ring $\mathcal{O}_U$ under pointwise addition and multiplication of functions; the units or invertible elements in this ring are the nowhere vanishing holomorphic functions, which form a multiplicative group $\mathcal{O}_U^\times$. The series (A.1) is absolutely convergent in an open neighborhood of the point $(a_1, \ldots, a_n)$ so it can be rearranged as a convergent series in any
one of the variables when the remaining variables are held constant; thus a holomorphic function of several is holomorphic in each variable separately. A basic and nontrivial result is Hartogs’s Theorem\(^2\) that conversely any function that is holomorphic in each variable separately is holomorphic in all variables, without any additional hypothesis of continuity or measurability in all variables; thus holomorphic functions of several complex variables can be characterized by separate conditions in each complex variable \(z_j = x_j + iy_j\) or each pair of real variables \((x_j, y_j)\). For instance a function of several complex variables that is continuously differentiable in each pair of variables \((x_j, y_j)\) and that satisfies the Cauchy-Riemann equations in each pair of variables \((x_j, y_j)\) is a holomorphic function of all variables. It is convenient to write the Cauchy-Riemann equations in terms of the linear partial differential operators

\[
\begin{align*}
\frac{\partial f}{\partial z_j} &= \frac{1}{2} \left( \frac{\partial f}{\partial x_j} + \frac{1}{i} \frac{\partial f}{\partial y_j} \right), \\
\frac{\partial f}{\partial \overline{z}_j} &= \frac{1}{2} \left( \frac{\partial f}{\partial x_j} - \frac{1}{i} \frac{\partial f}{\partial y_j} \right)
\end{align*}
\]

where \(z_j = x_j + iy_j\); in these terms if \(f\) is a continuously differentiable function in an open subset \(U \subset \mathbb{C}^n\), or just a function that is continuously differentiable in each pair of variables \((x_j, y_j)\), then

\[
\left( \begin{array}{c}
\frac{\partial f}{\partial z_j} = 0 \\
\frac{\partial f}{\partial \overline{z}_j}
\end{array} \right)
\]

for \(1 \leq j \leq n\).

If \(f\) is holomorphic then \(\partial f/\partial z_j\) is just the ordinary complex derivative of the holomorphic function \(f(z_j)\) of the complex variable \(z_j\) when the remaining variables are held constant.

The zero locus of a holomorphic function of a single complex variable is a discrete of points, but the situation is rather more complicated for holomorphic functions of several complex variables. A holomorphic subvariety of an open subset \(U \subset \mathbb{C}^n\) is a subset \(V \subset U\) with the property that for each point \(a \in U\) there are an open neighborhood \(U_a\) of that point and finitely many holomorphic functions \(f_{ai}\) in \(U_a\), not all of which vanish identically, such that

\[
V \cap U_a = \left\{ z \in U_a \mid f_{a1}(z) = f_{a2}(z) = \cdots = 0 \right\}
\]

It is not required that a holomorphic subvariety \(V \subset U\) be the set of common zeros of a collection of functions defined and holomorphic in all of \(U\); the notion of a holomorphic subvariety is essentially local in nature. It is evident from this definition that a holomorphic subvariety of \(U\) is a closed subset of \(U\). If \(V \subset U\) is a holomorphic subvariety of a connected open subset \(U \subset \mathbb{C}^n\) then the complement \(U \sim V\) is a connected dense open subset \(^3\). The analogue for functions of several complex variables of the Riemann Removable Singularities Theorem\(^4\) for functions of a single complex variable is the theorem that if \(f\) is a bounded holomorphic function in the complement \(U \sim V\) of a holomorphic

\(^2\)Theorem G-IB6
\(^3\)Corollary G-IA9 and Corollary G-ID3
\(^4\)Theorem G-ID2
subvariety \( V \) of a connected open subset \( U \subset \mathbb{C}^n \) then \( f \) has a unique extension to a holomorphic function on the entire set. There is actually a stronger removable singularities theorem, a special case of a number of extension theorems that arise only for functions of more than one variable. This theorem\(^5\) asserts that if \( V \) is a holomorphic subvariety of an open subset \( D \subset \mathbb{C}^n \) and if \( \dim V \leq n - 2 \) then any holomorphic function in \( D \sim V \) extends uniquely to a holomorphic function in \( D \). If \( f \) and \( g \) are two holomorphic functions in a connected open subset \( U \subset \mathbb{C}^n \) and if they agree on a subset of \( U \) that is not a holomorphic subvariety of \( U \), such as an open subset of \( U \), then clearly they must agree at all points of \( U \).

If \( f \) and \( g \) are two holomorphic functions in a connected open subset \( U \subset \mathbb{C}^n \) and if the function \( g \) does not vanish identically then its zero locus is a holomorphic subvariety \( V_g \subset U \) and the quotient \( m = f/g \) is a well defined complex-valued function on the connected dense open subset \( U \sim V_g \subset U \). A complex-valued function \( m \) that is defined in the complement of a holomorphic subvariety \( V_m \subset U \) of an open subset \( U \subset \mathbb{C}^n \) and that can be represented in an open neighborhood of each point of \( U \) as such a quotient of holomorphic functions is called a meromorphic function in \( U \). Clearly the set of meromorphic functions in a connected open subset \( U \subset \mathbb{C}^n \) form a field under pointwise addition and multiplication of functions; this field is denoted by \( \mathcal{M}_U \). If the open subset \( U \) is not connected \( \mathcal{M}_U \) is not a field, since meromorphic functions that vanish in a connected component of \( U \) but not in all of \( U \) are nontrivial but do not have multiplicative inverses. Of course any holomorphic function in an open subset \( U \subset \mathbb{C}^n \) is also meromorphic, so \( \mathcal{O}_U \subset \mathcal{M}_U \); and it follows from the Riemann Removable Singularities Theorem that a bounded meromorphic function in \( U \) actually is holomorphic in \( U \). It is evident that if \( m \) is meromorphic in an open subset \( U \subset \mathbb{C}^n \) then it is a meromorphic function in each variable separately in \( U \) when the remaining variables are held constant, except when all such points lie in the holomorphic subvariety \( V_m \) where \( m \) is not necessarily well defined. An analogue of Hartogs’s Theorem for meromorphic functions is Rothstein’s Theorem\(^6\) that conversely a complex valued function in the complement of a holomorphic subvariety \( V \) of an open subset \( U \subset \mathbb{C}^n \) that is a meromorphic function in each variable separately in \( U \sim V \) is a meromorphic function in \( U \). There is also an analogue for meromorphic functions of the extension theorem for holomorphic functions. The Theorem of Levi\(^7\) asserts that if \( V \) is holomorphic subvariety of an open subset \( D \subset \mathbb{C}^n \) and if \( \dim V \leq n - 2 \) then any meromorphic function in \( D \sim V \) extends uniquely to a meromorphic function in \( D \).

A holomorphic mapping from an open subset \( U \subset \mathbb{C}^n \) into \( \mathbb{C}^m \) is a mapping that sends a point \( z = (z_1, \ldots, z_n) \in U \) to the point \( w = (w_1, \ldots, w_m) \in \mathbb{C}^m \).

---

\(^5\)Theorem G-IIK1. The theorem requires the notion of the dimension of a holomorphic subvariety, which will be taken up later in Section A.3; but it is more convenient to include the statement here in the discussion of functions.


\(^7\)Theorem G-II06
where \( w_j = f_j(z_1, \ldots, z_n) \) for some holomorphic functions \( f_j \in \mathcal{O}_U \). A holomorphic function \( f \) in an open subset \( U \subset \mathbb{C}^n \) can be viewed as a holomorphic mapping \( f : U \rightarrow \mathbb{C} \). It is familiar that a holomorphic mapping \( f : U \rightarrow \mathbb{C} \) defined in an open subset \( U \subset \mathbb{C} \) is an open mapping; but trivial examples show that is not the case for holomorphic mappings from open subsets of \( \mathbb{C}^n \) into \( \mathbb{C}^n \) for \( n > 1 \). However if \( F : U \rightarrow V \) is a one-to-one holomorphic mapping from an open subset \( U \subset \mathbb{C}^n \) onto a subset \( V \subset \mathbb{C}^n \) then \( V \) is necessarily an open subset of \( \mathbb{C}^n \) and the mapping \( F \) is an open mapping with a holomorphic inverse \(^8\). A holomorphic mapping \( F : \mathbb{C}^n \rightarrow \mathbb{C}^n \) for which \( \det \{\partial f_j / \partial z_k\} \neq 0 \) at a point \( a \in \mathbb{C}^n \) describes a biholomorphic mapping from an open neighborhood of the point \( a \in \mathbb{C}^n \) to an open neighborhood of the image point \( F(a) \in \mathbb{C}^n \).

Differential forms play a more useful role in several complex variables than in one variable. A complex-valued differential form \( \phi \) in an open subset \( U \subset \mathbb{C}^n \) can be written either in terms of the differentials \( dx_j, dy_j \) of the real coordinates in \( \mathbb{C}^n \) or in terms of the complex linear combinations

\[
\begin{align*}
\text{(A.4)} & \\
dz_j = dx_j + idy_j, \quad d\bar{z}_j = dx_j - idy_j
\end{align*}
\]

of these differentials; a differential form of degree \( r \) that can be written

\[
\text{(A.5)} \quad \phi = \sum_{j,k} f_{j_1 \ldots j_p, k_1 \ldots k_q} dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\bar{z}_{k_1} \wedge \cdots \wedge d\bar{z}_{k_q}
\]

in terms of the complex differentials \( dz_j \) and \( d\bar{z}_j \) it is said to be of type \( (p, q) \) and degree \( r = p + q \). The vector space of complex-valued \( \mathcal{C}^\infty \) differential forms of degree \( r \) in \( U \) is denoted by \( \mathcal{E}_U^r \), and the vector space of complex-valued \( \mathcal{C}^\infty \) differential forms of type \( (p, q) \) in \( U \) is denoted by \( \mathcal{E}_U^{(p, q)} \), so there is the direct sum decomposition

\[
\text{(A.6)} \quad \mathcal{E}_U^r = \bigoplus_{p+q=r} \mathcal{E}_U^{(p, q)}.
\]

The exterior derivative of a differentiable function \( f \) in \( U \) is the differential 1-form

\[
\text{(A.7)} \quad df = \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \right).
\]

A straightforward calculation shows that when written in terms of the complex differentials \( dz_j \) and \( d\bar{z}_j \) the exterior derivative takes the form

\[
\text{(A.7)} \quad df = \sum_{j=1}^n \left( \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right).
\]

\(^8\)Corollary G-HIE10
in terms of the differential operators (A.2). The separate differential forms
\[ \partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} \, dz_j \quad \text{and} \quad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} \, d\overline{z}_j \]
are the components of the differential \( df \) of type \((1, 0)\) and type \((1, 0)\) respectively; thus the exterior derivative of a \( C^\infty \) function \( f \in \mathcal{E}_U \) can be written as the sum
\[ df = \partial f + \overline{\partial} f \]
of a differential form \( \partial f \in \mathcal{E}_{U}^{(1, 0)} \) of type \((1, 0)\) and a differential form \( \overline{\partial} f \in \mathcal{E}_{U}^{(0, 1)} \) of type \((0, 1)\). If \( f \) is holomorphic then \( df = \partial f \) since \( \overline{\partial} f = 0 \); and conversely if \( f \) is a continuously differentiable function such that \( df = \partial f \) then \( \overline{\partial} f = 0 \) so the Cauchy-Riemann equations show that \( f \) is holomorphic. Under a biholomorphic mapping \( w_k = f_k(z_j) \) between open subsets of \( \mathbb{C}^n \)
\[ dw_k = \sum_{j=1}^{n} \frac{\partial w_k}{\partial z_j} \, dz_j \quad \text{and} \quad d\overline{w}_k = \sum_{j=1}^{n} \frac{\partial \overline{w}_k}{\partial \overline{z}_j} \, d\overline{z}_j; \]
it is evident from this that the type of a differential form is unchanged under biholomorphic changes of coordinates in \( \mathbb{C}^n \), so to that extent the decomposition (A.6) is intrinsic. The exterior derivative of the differential form (A.5) is the differential form
\[ d\phi = \sum_{j,k} df_{j_1 \ldots j_p, k_1 \ldots k_q} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \]
\[ = \sum_{j,k} \partial f_{j_1 \ldots j_p, k_1 \ldots k_q} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \]
\[ + \sum_{j,k} \overline{\partial} f_{j_1 \ldots j_p, k_1 \ldots k_q} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}; \]
thus if \( \phi \in \mathcal{E}^{(p,q)} \) then \( d\phi = \partial \phi + \overline{\partial} \phi \) where \( \partial \phi \in \mathcal{E}^{(p+1,q)}_{U} \) and \( \overline{\partial} \phi \in \phi \mathcal{E}^{(p,q+1)}_{U} \), so there is the direct sum decomposition \( d = \partial \oplus \overline{\partial} \) of exterior differentiation of arbitrary differential forms in terms of linear differential operators
\[ \partial : \mathcal{E}^{(p,q)}_{U} \longrightarrow \mathcal{E}^{(p+1,q)}_{U} \quad \text{and} \quad \overline{\partial} : \mathcal{E}^{(p,q)}_{U} \longrightarrow \mathcal{E}^{(p,q+1)}_{U}. \]
In particular if \( \phi \) is a differential form of type \((p,0)\) then \( d\phi = 0 \) if and only if both \( \partial \phi = 0 \) and \( \overline{\partial} \phi = 0 \). When the differential form \( \phi \) is written
\[ \phi = \sum_{j} f_{j_1 \ldots j_p} \, dz_{j_1} \wedge \cdots \wedge dz_{j_p} \]
the condition that \( \overline{\partial} \phi = 0 \) clearly is equivalent to the condition that the coefficients \( f_{j_1 \ldots j_p} \) are holomorphic functions; a differential form of type \((p,0)\)
satisfying this condition is called a holomorphic differential form of type \((p,0)\), and the space of such differential forms is denoted by \(\mathcal{O}^{(p,0)}\). Exterior differentiation satisfies \(dd = 0\); the kernel of the linear operator \(d\) is the subspace of closed differential forms in \(U\), the image of \(d\) is the subspace of exact differential forms in \(U\), and every exact form is closed since \(dd = 0\). When exterior differentiation is written as the sum \(d = \partial + \overline{\partial}\) the identity \(dd = 0\) is equivalent to the identities
\[
\partial \partial = \partial \overline{\partial} + \overline{\partial} \partial = 0,
\]
so \(\partial d = \partial \overline{\partial} = d \overline{\partial}\). It is familiar that any closed differential form is at least locally exact. If \(\phi\) is a holomorphic differential form of type \((p,0)\) that is closed then it too is locally the exterior derivative \(\phi = dv\) of a differential form \(v\) of degree \(p - 1\), indeed clearly a differential form \(v\) of type \((p - 1,0)\); and since \(\phi = \partial v + \overline{\partial}v\) it is evident that \(v\) must be a holomorphic differential form of type \((p - 1,0)\). Thus if a holomorphic differential form \(\phi\) of type \((p,0)\) is closed then locally it is the exterior derivative of a holomorphic differential form \(v\) of type \((p - 1,0)\).

### A.2 Manifolds

A manifold or topological manifold of dimension \(n\) is a second countable Hausdorff topological space \(M\) such that each point of \(M\) has an open neighborhood homeomorphic to an open subset of the \(n\)-dimensional Euclidean space \(\mathbb{R}^n\). A coordinate covering \(\{U_\alpha, x_\alpha\}\) of the manifold \(M\) is a covering of \(M\) by open subsets \(U_\alpha \subset M\), for each of which there is a homeomorphism \(x_\alpha : U_\alpha \rightarrow W_\alpha\) between \(U_\alpha\) and an open subset \(W_\alpha \subset \mathbb{R}^n\). The subsets \(U_\alpha\) are called the coordinate neighborhoods, and the mappings \(x_\alpha\) are called the coordinate mappings or the local coordinates of the coordinate covering. In the intersections \(U_\alpha \cap U_\beta\) of coordinate neighborhoods there are two homeomorphisms to subsets of \(\mathbb{R}^n\), the restrictions of \(x_\alpha\) and of \(x_\beta\); the compositions
\[
f_{\alpha\beta} = x_\alpha \circ x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \rightarrow x_\alpha(U_\alpha \cap U_\beta)
\]
are homeomorphisms called the coordinate transition mappings of the coordinate covering, and the two local coordinates in an intersection \(U_\alpha \cap U_\beta\) are related by \(x_\alpha = f_{\alpha\beta}(x_\beta)\). The manifold \(M\) is determined completely by the open subsets \(W_\alpha = x_\alpha(U_\alpha) \subset \mathbb{R}^n\) and the coordinate transition mappings \(f_{\alpha\beta}\) of a coordinate covering, since \(M\) can be recovered from the disjoint union of the sets \(W_\alpha\) by identifying points \(x_\alpha \in W_\alpha\) and \(x_\beta \in W_\beta\) whenever \(x_\alpha = f_{\alpha\beta}(x_\beta)\). If \(\{U_\alpha, x_\alpha\}\) and \(\{V_\beta, y_\beta\}\) are two coordinate coverings of the manifold \(M\) their union also is a coordinate covering of \(M\), consisting of the total collection of coordinate neighborhoods and local coordinates from the two separate coordinate coverings. The set of coordinate transition mappings for the union is properly larger than the union of the sets of coordinate transition mappings for the two separate coverings, though, since it must include the coordinate transition mappings
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relating the local coordinates \(x_\alpha\) and \(y_\beta\) in the intersections \(U_\alpha \cap V_\beta\) of coordinate neighborhoods from the two separate coverings.

Coordinate coverings with special properties can be used to describe additional structures on a topological manifold. A collection \(\mathcal{G}\) of homeomorphisms between open subsets of \(\mathbb{R}^n\) that is determined by local conditions, that includes with any homeomorphism its restrictions to open subsets and its inverse, and that includes with any two homeomorphisms their composition wherever it is defined, is called a pseudogroup\(^9\). One example is the pseudogroup \(\mathcal{G}_1\) of all continuously differentiable or \(C^1\) homeomorphisms between open subsets of \(\mathbb{R}^n\); a second example is the pseudogroup \(\mathcal{G}_2\) of all infinitely differentiable or \(C^\infty\) homeomorphisms between open subsets of \(\mathbb{R}^n\); a third example is the pseudogroup \(\mathcal{G}_3\) of all holomorphic homeomorphisms between open subsets of \(\mathbb{C}^m\), when the real vector space is of dimension \(n = 2m\) and is identified with the complex vector space \(\mathbb{C}^m\); a fourth example is the pseudogroup \(\mathcal{G}_4\) of nonsingular complex linear mappings between open subsets of \(\mathbb{C}^m\). This last example is actually a group, since the composition of any two nonsingular complex linear mappings is again a nonsingular complex linear mapping; in the previous examples only those mappings with suitably overlapping ranges and domains can be composed, hence the terminology pseudogroup rather than group. These four examples are increasingly restrictive, in the obvious sense that \(\mathcal{G}_4 \subset \mathcal{G}_3 \subset \mathcal{G}_2 \subset \mathcal{G}_1\). A coordinate covering \(\{U_\alpha, x_\alpha\}\) is called a \(\mathcal{G}\) coordinate covering if all of its coordinate transition mappings \(f_{\alpha\beta}\) belong to the pseudogroup \(\mathcal{G}\). Two \(\mathcal{G}\) coordinate coverings are called equivalent if their union is again a \(\mathcal{G}\) coordinate covering; this is an equivalence relation in the usual sense, as a simple consequence of the definition of a pseudogroup, and is actually a nontrivial equivalence relation, since there are more coordinate transition mappings in the union of two coordinate coverings than just the union of the two sets of coordinate transition mappings. An equivalence class of \(\mathcal{G}\) coordinate coverings is called a \(\mathcal{G}\) structure on the manifold \(M\), and a manifold \(M\) with a fixed \(\mathcal{G}\) structure is called a \(\mathcal{G}\) manifold. Thus for the four examples of pseudogroups just considered there are continuously differentiable or \(C^1\) manifolds, infinitely differentiable or \(C^\infty\) manifolds, complex analytic manifolds, usually called just complex manifolds, and flat complex linear manifolds. A complex manifold also is a \(C^\infty\) manifold, since any complex analytic coordinate covering is also a \(C^\infty\) coordinate covering and any two equivalent complex analytic coordinate coverings are equivalent as \(C^\infty\) coordinate coverings; thus a complex manifold can be viewed as a \(C^\infty\) manifold by ignoring some of the structure, or alternatively a complex structure is an additional structure that can be imposed on an underlying \(C^\infty\) manifold. Similar considerations of course apply to any pseudogroups \(\mathcal{G}' \subset \mathcal{G}\).

Complex manifolds\(^10\) are of particular interest in the present book. As a

\(^9\)There is an extensive literature devoted to pseudogroups and pseudogroup structures following the initial treatment by E. Cartan, which can be found in his *Oeuvres Complètes*, partie II, vol. 2. (Gauthier-Villars, 1953). See for instance the discussion in S. Sternberg, *Lectures on Differential Geometry*, (Prentice-Hall, 1964).

\(^10\)A more detailed discussion of complex manifolds can be found in K. Kodaira and J. Morrow, *Complex Manifolds*, (Holt, Rhinehart and Winston, 1971) or R. O. Wells, *Differential
matter of convention, a complex manifold of topological dimension \( n = 2m \) customarily is referred to as a complex manifold of dimension \( m \), viewing the complex dimension rather than the real dimension as the more significant index. A complex-valued function \( f \) defined in an open subset \( U \) of a complex manifold \( M \) is \textit{holomorphic} if for each intersection \( U \cap U_\alpha \) of the set \( U \) with a coordinate neighborhood of a holomorphic coordinate covering \( \{ U_\alpha, x_\alpha \} \) of \( M \) the composition \( f \circ x_\alpha^{-1} : x_\alpha(U \cap U_\alpha) \to \mathbb{C} \) is a holomorphic function in the open subset \( x_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n \). This condition clearly is independent of the choice of a complex coordinate covering representing the complex structure of \( M \), so depends only on the complex structure of \( M \). The same notation is used for functions on complex manifolds as for functions on open subsets of \( \mathbb{C}^n \); thus the ring of holomorphic functions in the subset \( U \subset M \) is denoted by \( \mathcal{O}_U \), the multiplicative group of nowhere vanishing holomorphic functions is denoted by \( \mathcal{O}^*_U \), and if \( U \) is connected the field of meromorphic functions is denoted by \( \mathcal{M}_U \) and the multiplicative group of not identically vanishing meromorphic functions is denoted by \( \mathcal{M}^*_U \). The ring \( \mathcal{C}_U \) of continuous complex-valued functions on a topological manifold, the ring \( \mathcal{E}_U \) of \( \mathcal{C}^\infty \) complex-valued functions on a \( \mathcal{C}^\infty \) manifold, and the ring \( \mathcal{F}_U \) of locally constant complex-valued functions on a flat manifold are defined correspondingly. For a connected open subset \( U \subset M \) of a complex manifold \( M \) there are the natural inclusions \( \mathcal{O}_U \subset \mathcal{M}_U \) and \( \mathcal{O}^*_U \subset \mathcal{M}^*_U \); and \( \mathcal{F}_U \subset \mathcal{O}_U \subset \mathcal{E}_U \subset \mathcal{C}_U \), but of course \( \mathcal{M}_U \) is not a subset of \( \mathcal{E}_U \) or \( \mathcal{C}_U \).

In a coordinate neighborhood \( U_\alpha \) of a complex manifold \( M \) with local coordinates \( z_{\alpha_j} = x_{\alpha_j} + iy_{\alpha_j} \) it follows readily from (A.4) that

\[
(A.15) \quad \left( \frac{i}{2} \right)^n dz_{\alpha_1} \wedge dz_{\alpha_2} \wedge \cdots \wedge dz_{\alpha_n} \wedge d\bar{z}_{\alpha_1} = dx_{\alpha_1} \wedge dy_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_n} \wedge dy_{\alpha_n},
\]

so this differential form can be used as an element of volume in the coordinate neighborhood \( U_\alpha \subset M \); in particular in a coordinate neighborhood \( U_\alpha \) on a Riemann surface \( M \) with local coordinate \( z_\alpha = x_\alpha + iy_\alpha \) the differential form \( \frac{i}{2} dz_\alpha \wedge d\bar{z}_\alpha = dx_\alpha \wedge dy_\alpha \) can be taken as an element of area. For another local coordinate \( z_\beta = x_\beta + iy_\beta \)

\[
(A.16) \quad dx_\alpha \wedge dy_\alpha = \frac{i}{2} dz_\alpha \wedge d\bar{z}_\alpha = \left| \frac{dz_\alpha}{dz_\beta} \right|^2 \frac{i}{2} dz_\beta \wedge d\bar{z}_\beta
\]

so this element of area remains positive under any complex analytic change of coordinates on the Riemann surface; equivalently the Jacobian determinant of a complex analytic change of coordinates is everywhere positive. The analogous result holds for \( n \)-dimensional complex manifolds as well, so complex manifolds are orientable topological spaces. In this book the positive orientation of a

\begin{flushright}
\textit{Analysis on Complex Manifolds}, (Prentice Hall, 1973).
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complex manifold is taken to be that for which (A.15) is the positive volume element on the manifold; in particular the orientation of a Riemann surface is that for which (A.16) is the positive element of area.

A continuous mapping \( F : M \rightarrow N \) between complex manifolds \( M \) and \( N \) of dimensions \( m \) and \( n \) with coordinate coverings \( \{ U_\alpha, x_\alpha \} \) and \( \{ V_\beta, y_\beta \} \) respectively is holomorphic if for any point \( p \in U_\alpha \subset M \) for which \( F(p) \in V_\beta \subset N \) the composition \( y_\beta \circ F \circ x_\alpha^{-1} \) is a holomorphic mapping from an open neighborhood of the point \( x_\alpha(p) \in \mathbb{C}^m \) into the space \( \mathbb{C}^n \). Two complex manifolds \( M \) and \( N \) are said to be analytically equivalent or biholomorphic if there is a homeomorphism \( F : M \rightarrow N \) such that both \( F \) and \( F^{-1} \) are holomorphic mappings; the mapping \( F \) is called an analytic equivalence or a biholomorphic mapping. Any one-to-one holomorphic mapping between two complex manifolds of the same dimension is a biholomorphic mapping since as noted in the preceding section a one-to-one holomorphic mapping from an open subset of \( \mathbb{C}^n \) into \( \mathbb{C}^n \) is a biholomorphic mapping. For the most part it is only the analytic equivalence classes or biholomorphic equivalence classes of complex manifolds that are of primary interest.

Riemann surfaces are defined as one-dimensional connected complex manifolds, and are the main topic of this book; however various complex manifolds of higher dimension, such as complex projective spaces and complex tori, arise naturally in the discussion of Riemann surfaces. The \( n \)-dimensional complex projective space \( \mathbb{P}^n \) is defined to be the set of equivalence classes of nonzero points \( (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \), where the equivalence relation is defined by \( (z_0, z_1, \ldots, z_n) \sim (t z_0, t z_1, \ldots, t z_n) \) for any nonzero complex number \( t \in \mathbb{C}^* \); alternatively \( \mathbb{P}^n \) can be defined to be the set of one-dimensional linear subspaces of \( \mathbb{C}^{n+1} \), since any such subspace is an equivalence class as just defined. The space \( \mathbb{P}^n \) is topologized with the natural quotient topology, so the open subsets of \( \mathbb{P}^n \) are the equivalence classes of points in open subsets of \( \mathbb{C}^{n+1} \). The equivalence class containing a point \( (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1} \) is denoted by \( [z_0, z_1, \ldots, z_n] \in \mathbb{P}^n \), and the point \( (z_0, z_1, \ldots, z_n) \) is called the set of homogeneous coordinates for the point of \( \mathbb{P}^n \) that it represents. In the open subset \( U_i \subset \mathbb{P}^n \) consisting of points with homogeneous coordinates \( (z_0, z_1, \ldots, z_n) \) for which \( z_i \neq 0 \), where \( 0 \leq i \leq n \), any point is represented by unique homogeneous coordinates of the form \((z_0^{i}, z_{i-1}^{i}, 1, z_{i+1}^{i}, \ldots, z_{n}^{i})\), which are called the inhomogeneous coordinates of that point; these provide local coordinates in \( U_i \), identifying that subset of \( \mathbb{P}^n \) with the complex vector space \( \mathbb{C}^n \). Points in the intersection \( U_i \cap U_j \) for \( i \neq j \) then are described by two sets of inhomogeneous coordinates which are related by \( (z_0^{i}, \ldots, z_{i-1}^{i}, 1, z_{i+1}^{i}, \ldots, z_{n}^{i}) = t(z_0^{j}, \ldots, z_{j-1}^{j}, 1, z_{j+1}^{j}, \ldots, z_{n}^{j}) \), where clearly \( t = z_j^i \) so that

\[
(A.17) \quad z_k^i = z_j^i z_k^j \quad \text{for} \quad k \neq i, j;
\]

that is a nonsingular linear, hence holomorphic, change of coordinates, so the inhomogeneous coordinates describe on the space \( \mathbb{P}^n \) the structure of a complex manifold of dimension \( n \). The unit sphere \( S^{2n-1} \subset \mathbb{C}^{n+1} \) consists of points \( Z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \) for which \( \sum_{i=0}^{n} |z_i|^2 = 1 \), and is of course a compact subset
of $\mathbb{C}^{n+1}$. The natural mapping $\mathbb{C}^{n+1} \to \mathbb{P}^n$ restricts to a continuous mapping $S^{2n-1} \to \mathbb{P}^n$ with image all of $\mathbb{P}^n$, and consequently $\mathbb{P}^n$ is a compact complex manifold. The inverse image of a point $w \in \mathbb{P}^n$ is the circle $\{ t Z \mid |t| = 1 \}$; it is not difficult to see that the mapping $\mathbb{C}^{n+1} \to \mathbb{P}^n$ is a fibration over the projective space $\mathbb{P}^n$ with fibres the unit circle, and that determines the topology of $\mathbb{P}^n$.

A subset $V \subset M$ of a complex manifold such as $\mathbb{C}^n$ is a submanifold if in an open neighborhood $U_a$ of any point $a \in M$ there are local coordinates in $M$ such that $U_a \cap V$ is a linear subspace in terms of these coordinates. It is easy to see that a submanifold has the natural structure of a complex manifold; the dimension of the submanifold is the dimension of that manifold. If a subset $V$ of an open neighborhood $U_a$ of a point $a \in \mathbb{C}^n$ is the set of common zeros of $k \leq n$ holomorphic functions $f_1, \ldots, f_k$ in $U_a$ for which the $n \times k$ matrix $\{ \partial_i f_j(a) \}$ is of rank $k \leq n$ then $V$ is a submanifold of dimension $n - k$ near $a$; indeed if $f_{k+1}, \ldots, f_n$ are any holomorphic functions in $U_a$ such that the $n \times n$ matrix $\{ \partial_i f_j(a) \}$ has rank $n$ then these functions can be taken as local coordinates near $a$ and in terms of these coordinates the subset $V$ is the linear subspace as the set of zeros of the coordinates $f_1, \ldots, f_k$. Similarly if $f_1, \ldots, f_n$ are $n \geq k$ holomorphic functions in an open neighborhood $U_a$ of a point $a \in \mathbb{C}^k$ for which the $k \times n$ matrix $\{ \partial_i f_j(a) \}$ is of rank $k \leq n$ then the image $f(U_a) \subset \mathbb{C}^k$ of a subneighborhood of the point $a$ under the mapping $F : U_a \to \mathbb{C}^n$ defined by $F(z) = (f_1(z), \ldots, f_n(z))$ is a submanifold of an open neighborhood of $F(a)$ of dimension $k$; for if $\mathbb{C}^k$ is viewed as the subspace consisting of the first $k$ variables $z_1, \ldots, z_k$ in the space $\mathbb{C}^n$ with the coordinates $z_1, \ldots, z_n$ then the functions $f_1, \ldots, f_k, z_{k+1}, \ldots, z_n$ are local coordinates in $\mathbb{C}^n$ for which $\mathbb{C}^k$ is the linear subspace defined as the set of zeros of the coordinates $z_{k+1}, \ldots, z_n$, while the mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ for which $F(z_1, \ldots, z_n) = (f_1(z), \ldots, f_k(z), z_{k+1}, \ldots, z_n)$ is a locally biholomorphic mapping which takes the linear subspace $z_{k+1} = \cdots = z_n = 0$ to the image of the mapping $F$.

### A.3 Holomorphic Varieties

Almost any consideration of complex manifolds eventually leads to more general entities as well. A holomorphic subvariety of an open subset $U \subset \mathbb{C}^n$ is a subset $V \subset U$ with the property that for each point $a \in U$ there exist an open neighborhood $U_a$ and finitely many holomorphic functions $f_{a_i}$ in $U_a$ such that

$$V \cap U_a = \{ z \in U_a \mid f_{a_i}(z) = 0 \}. $$

If a holomorphic subvariety is defined locally by a finite number of holomorphic functions having a nonsingular Jacobian determinant at each point then that subvariety has the natural structure of a complex manifold; thus a complex submanifold of an open subset $U \subset \mathbb{C}^n$ is a special case of a holomorphic subvariety of $U$. A holomorphic subvariety of $U$ always is a closed subset of $U$, as an immediate consequence of this definition. It is not required that the subvariety be the set of common zeros of a collection of functions defined and holomorphic...
in all of $U$; the notion of a holomorphic subvariety is essentially local in nature. For this reason it is of course possible to consider a holomorphic subvariety of an arbitrary complex manifold. More generally if $V_1, V_2 \subset U$ are holomorphic subvarieties of a complex manifold $U$ and $V_1 \subset V_2$ then $V_1$ is called a holomorphic subvariety of $V_2$. A function $f$ on a holomorphic subvariety $V \subset U$ in a complex manifold $U$ is holomorphic on $V$ if in an open neighborhood of each point of $V$ it is the restriction to $V$ of a holomorphic function in an open neighborhood of that point in the manifold $U$; and a function $f$ on $V$ is meromorphic if it can be represented in an open neighborhood of each point of $V$ as a quotient of holomorphic functions on $V$. On a complex manifold any bounded meromorphic function actually is holomorphic; but that is not the case for meromorphic functions on holomorphic subvarieties, as is illustrated in the examples in the discussion of singular points on page 405. The bounded meromorphic functions on a holomorphic subvariety are known as weakly holomorphic functions; and a holomorphic subvariety for which all weakly holomorphic functions are actually holomorphic is known as a normal holomorphic variety.

A mapping $F : V_1 \rightarrow V_2$ between two holomorphic subvarieties $V_1 \subset U_1$ and $V_2 \subset U_2$ of complex manifolds $U_1$ and $U_2$ is holomorphic if for any holomorphic function $f$ in an open neighborhood of a point $a \in V_2$ the composition $f \circ F$ is a holomorphic function in an open neighborhood of the point $F^{-1}(a) \in V_1$; this is readily seen to be equivalent to the condition that in an open neighborhood of each point $a \in V_1$ the mapping $F$ is the restriction to $V_1$ of a holomorphic mapping of an open neighborhood of $a$ in $U_1$ into $U_2$. Two holomorphic subvarieties are analytically equivalent or biholomorphic if there are holomorphic mappings $F : V_1 \rightarrow V_2$ and $G : V_2 \rightarrow V_1$ that are inverse to one another; and a holomorphic variety is a biholomorphic equivalence class of holomorphic subvarieties. A holomorphic variety thus is an abstract version of a holomorphic subvariety, independent of a particular representation as a subvariety of a complex manifold; a complex manifold is a special case of a holomorphic variety. A holomorphic variety $V$ is reducible if it can be written as a nontrivial union of holomorphic varieties, and otherwise is irreducible; in particular a complex manifold is an irreducible holomorphic variety if and only if it is connected. A holomorphic variety $V$ is locally reducible at a point $a \in V$ if the restriction of $V$ to any sufficiently small open neighborhood of the point $a$ is reducible, and otherwise is locally irreducible at that point. A complex manifold is locally irreducible at each of its points; but a holomorphic variety may be locally reducible at some of its points. Any holomorphic variety $V$ can be written uniquely as a union of irreducible subvarieties, called its irreducible components; and somewhat less trivially, an open neighborhood of any point of a holomorphic variety can be written uniquely as a finite union of locally irreducible varieties at that point.

Holomorphic varieties are generalizations of complex manifolds, but actually are complex manifolds at most points; for an arbitrary holomorphic variety $V$ is a complex manifold outside a proper holomorphic subvariety $\mathcal{S}(V) \subset V$ called the singular locus of $V$ and consisting of precisely those points at which $V$ fails to be a complex manifold. An irreducible holomorphic variety $V$ is a connected complex manifold outside its singular locus $\mathcal{S}(V)$; the dimension of the manifold...
APPENDIX A. COMPLEX MANIFOLDS AND VARIETIES

$V \sim \mathcal{S}(V)$ is considered to be the dimension of the holomorphic variety $V$ and is denoted by $\dim V$. The dimension of a reducible holomorphic variety is defined to be the largest of the dimensions of its irreducible components. If all irreducible components have the same dimension $n$ the variety is said to be of pure dimension $n$. For some purposes it is more useful to consider the local dimension of a holomorphic variety $V$ at a point $p \in V$, the dimension of arbitrarily small open neighborhoods of that point, rather than the global dimension of the variety $V$; the local dimension is denoted by $\dim_p V$, and may vary from point to point unless the variety $V$ is irreducible.

A few more detailed properties\(^{11}\) of the dimension of a holomorphic variety are also needed. If $V_1$ is a holomorphic subvariety of a holomorphic variety $V_2$ then $\dim V_1 \leq \dim V_2$, and this is a strict inequality unless $V_1$ and $V_2$ have a common irreducible component of the common dimension; in particular $\dim \mathcal{S}(V) < \dim V$ for the singular locus $\mathcal{S}(V)$ of an irreducible holomorphic subvariety $V$. If $f$ is a nontrivial holomorphic function on an irreducible holomorphic variety $V$ of dimension $n$ then the zero locus of the function $f$ is a holomorphic subvariety of pure dimension $n - 1$ in $V$. Consequently if $f_1, \ldots, f_k$ are holomorphic functions on an irreducible holomorphic variety $V$ of dimension $n$ then the locus of common zeros of these functions is a holomorphic subvariety $W \subset V$ for which $\dim W \geq n - k$. It is not generally true that conversely a holomorphic subvariety $W$ of dimension $n - k$ of an irreducible holomorphic variety $V$ of dimension $n$ can be defined as the set of common zeros of precisely $k$ holomorphic functions on $V$; the minimal number of functions required to describe such a holomorphic subvariety even locally can exceed $k$. However a holomorphic subvariety $W$ of dimension $n - 1$ in a complex manifold $V$ of dimension $n$ always is locally the set of zeros of a single holomorphic function. In general if $W_1, W_2$ are holomorphic subvarieties of a complex manifold of dimension $n$ and $W$ is an irreducible component of the intersection $W_1 \cap W_2$ then $\dim W \geq \dim W_1 + \dim W_2 - n$.

The singular locus $\mathcal{S}(V)$ of a one-dimensional holomorphic variety $V$ is a discrete set of points, called the singular points of $V$, and the complement $V \sim \mathcal{S}(V)$ has the natural structure of a union of Riemann surfaces. It can be shown that if $a \in \mathcal{S}(V)$ is a singular point of the one-dimensional holomorphic variety $V$ and if $V_i$ are the local irreducible components of $V$ in a neighborhood of the point $a$ then to each separate irreducible component $V_i$ there can be associated a Riemann surface $\hat{V_i}$ and a holomorphic mapping $f_i : \hat{V_i} \rightarrow V_i$ such that $f_i^{-1}(a)$ is a single point of $\hat{V_i}$ and the restriction $f_i : \hat{V_i}f_i^{-1}(a) \rightarrow V$ is an analytic equivalence of Riemann surfaces. This construction can be carried out at each singular point, yielding a union of Riemann surfaces $\hat{V}$ called the normalization of the variety $V$ or the nonsingular model of the variety $V$. The local normalization mappings lead to a global normalization mapping $f : \hat{V} \rightarrow V$ that is a biholomorphic mapping between $V \sim \mathcal{S}(V)$ and $f^{-1}(V \sim \mathcal{S}(V)) \subset \hat{V}$; both $V \sim (V \sim \mathcal{S}(V)) = \mathcal{S}(V) \subset V$ and $\hat{V} \sim f^{-1}(V \sim \mathcal{S}(V)) \subset \hat{V}$ are

\(^{11}\)These properties are discussed and proved for instance in R. C. Gunning, Introduction to Holomorphic Functions of Several Variables (Wadsworth & Brooks/Cole, 1990), vol. II.
discrete sets of points. For example, if $V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$, so that $V$ is the union of the two irreducible components consisting of the two coordinate axes in $\mathbb{C}^2$, then the singular locus $\mathcal{S}(V)$ consists of the origin itself; and $V$ is the disjoint union of two copies of $\mathbb{C}^1$ corresponding to the two irreducible components of $V$. In this case the singularity arises as the intersection of two separate manifolds. For another example, if $V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 - z_2^3 = 0\}$ then the singular locus $\mathcal{S}(V)$ again consists of the origin itself, $V$ is a copy of $\mathbb{C}^1$, and the mapping $f : \hat{V} \rightarrow V$ is given explicitly by $z \rightarrow (z^3, z^2)$. In this case the holomorphic subvariety $V$ is globally irreducible, is locally irreducible at each point, and has a definite singularity at the origin, a point at which $V$ fails to be a submanifold even topologically. The singularities of a one-dimensional subvariety are just those points at which there are locally weakly holomorphic functions (bounded meromorphic functions) that fail to be holomorphic. In the first of the two preceding examples the function $f(z_1, z_2) = (b z_1 + a z_2) (z_1 + z_2)^{-1}$ is a meromorphic function on $V$ that takes the value $a$ on the component $z_1 = 0$ and takes the value $b$ on the component $z_2 = 0$, so is not continuous hence not holomorphic; in the second example the function $f(z_1, z_2) = z_1 / z_2$ is a meromorphic function that is bounded, since $z_1 / z_2 = z$ is the value of the normalization mapping, but that is not holomorphic. There is an extensive literature dealing with the classification of the singular points of one-dimensional holomorphic subvarieties, for the most part in the context of algebraic geometry when these subvarieties are viewed as algebraic curves.\footnote{A discussion of the singularities of algebraic curves in $\mathbb{C}^2$ from a geometric point of view can be found in E. Brieskorn and H. Knörrer, Plane Algebraic Curves, (Birkhäuser, 1986); a discussion of the classical results can be found in R. J. Walker Algebraic Curves, (Princeton University Press, 1950).}

There are considerably more complicated results for subvarieties of higher dimensions, where the singular loci can be proper holomorphic subvarieties of various dimensions and the singularities can be resolved only by much more complicated mappings; nothing further about the resolution of singularities of higher dimensional varieties is needed in the discussion in the body of the book, but some familiarity with a few general properties of the singularities of holomorphic varieties is required\footnote{For the proofs of these assertion and a more detailed discussion see for instance R. C. Gunning, Introduction to Holomorphic Functions of Several Variables (Wadsworth & Brooks/Cole, 1990), vol. II.}. For any point $p \in V$ of a holomorphic subvariety $V \subset U$ of an open subset $U \subset \mathbb{C}^n$ the \textit{ideal} $\text{id}_p V \in \mathcal{O}_p$ of the subvariety $V$ at a point $p \in V$ is the ideal in the local ring $\mathcal{O}_p$ of germs of holomorphic functions of $n$ complex variables at the point $p$ consisting of the germs of those holomorphic functions that vanish on $V$ near the point $p$. It can be shown that this ideal always is finitely generated, so has a \textit{basis} consisting of finitely many germs in $\mathcal{O}_p$. It is not necessarily the case that a collection of holomorphic functions in $U$ having $V$ as their set of common zeros generate the ideal of that subvariety at any point $p \in V$. However if $V$ is a holomorphic subvariety of dimension $n - 1$ any holomorphic function $f$ in $U$ that vanishes to the first order at the regular points of $V$ does generate the ideal of the subvari-
ety $V$ at each of its points. If $p$ is a regular point of the subvariety $V$, a point at which $V$ is a submanifold of dimension $r$, then the set of differentials at $p$ of a basis for the ideal $i_d_p V$ has dimension $n - r$; the locus of zeros of these differentials, which are just the linear approximations at $p$ of the holomorphic functions in the basis, is an $r$-dimensional linear subspace of $C^n$ that can be identified with the complex tangent space $T_p(V)$ to the manifold $V$ at the point $p$. However if $p$ is a singular point of the subvariety $V$ the differentials of this basis at $p$ span a linear subspace of dimension strictly less than $r$; indeed it may be the case that all the differentials vanish at the point $p$. The common zero locus of these differentials still form a linear subspace of $C^n$, defined to be the complex tangent space $T_p(V)$ of the holomorphic subvariety $V$ at the point $p$. Thus for any $r$-dimensional subvariety $V \subset U \subset C^n$ it is always the case that $r \leq \dim T_p(V) \leq n$; and $p$ is a singular point precisely when $r < \dim T_p(V)$. The dimension of the tangent space is called the tangential dimension of the subvariety $V$ at the point $p$ and is denoted by $t\dim_p V$; it can be characterized alternatively as the least dimension of a complex submanifold of an open neighborhood of $p$ in $C^n$ containing the subvariety $V$ in that neighborhood, so sometimes is called the imbedding dimension of the subvariety $V$ at the point $p$. The point $p$ is a regular point of the subvariety $V$, a point at which $V$ is a submanifold, precisely when $t\dim_p V = \dim_p V$; equivalently $p \in \mathcal{S}(V)$ precisely when $t\dim_p V > \dim_p V$.

The tangential dimension is a measure of the singularity of the subvariety $V$ at the point $p$, the greater the tangential dimension the worse the singularity. An additional measure of the singularity of a point $p \in V$ for a proper holomorphic subvariety $V$ of an open subset of $C^n$ is the multiplicity of the subvariety $V$ at the point $p$, defined as the least integer $\mu$ such that

\[ \frac{\partial^k f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \bigg|_p = 0 \quad \text{whenever} \quad k = k_1 + \cdots + k_n < \mu \tag{A.18} \]

for all functions $f$ in a basis for the ideal $i_d_p V$ of the holomorphic subvariety $V$ near $p$; the multiplicity of the subvariety $V$ at the point $p$ is denoted by $\mu V$. If $p \not\in V$ then by this definition $\mu V = 0$. On the other hand $\mu V \geq 1$ at all points $p \in V$, and $\mu V = 1$ if and only if in an open neighborhood of the point $p$ the subvariety $V$ is contained in a proper complex submanifold of $C^n$, since that is just the condition that there is a nontrivial holomorphic function near $p$ that vanishes on the subvariety $V$ but has a nonzero differential at the point $p$. The points $p \in V$ at which $\mu V > 1$ are singular points $p \in V \in C^n$ at which $t\dim_p V = n$. The multiplicity consequently distinguishes between singularities at which the tangential dimension is maximal, that is, singularities at which the differentials of all functions in the ideal of the subvariety vanish; it is in this sense a finer measure of the nature of these somewhat extreme singularities. It should be noted that if $V$ is not a proper subvariety but actually coincides with $C^n$ then the local defining basis consists just of the function $0$ and the multiplicity as defined by (A.18) would be infinite; that is the reason for restricting this invariant to proper subvarieties of $C^n$. 

It is useful to note that both the tangential dimension and the multiplicity of a holomorphic subvariety are monotonic, in the sense that for holomorphic subvarieties $V$ and $W$ of an open subset of $\mathbb{C}^n$

\begin{equation}
\text{if } p \in V \subset W \text{ then } \begin{cases} 
tdim_p V \leq tdim_p W, \\
\text{mult}_p V \leq \text{mult}_p W.
\end{cases}
\end{equation}

To see that this is the case, suppose that $f_1, \ldots, f_r$ is a basis for the ideal $\text{id}_p V$ of the holomorphic subvariety $V$ at the point $p \in V$ and that $g$ is a holomorphic function in an open neighborhood of the point $p$ in $\mathbb{C}^n$ that is part of a defining basis for the ideal $\text{id}_p W$ of the subvariety $W$ at $p$. Since the function $g$ vanishes on the subvariety $W$ it must also vanish on $V$, so its germ is in the ideal $\text{id}_p V$ generated by the germs of the functions $f_i$ and consequently $g = \sum_{i=1}^r h_i f_i$ for some holomorphic functions $h_i$ in an open neighborhood of $p$. For any vector $t \in T_p(V)$ it then follows that

$$d_p g(t) = \sum_{i=1}^r h_i(p)d_p f_i(t) = 0;$$

therefore $t \in T_p(W)$, so that $T_p(V) \subset T_p(W)$ and consequently $tdim_p(V) \leq tdim_p(W)$. Furthermore

$$\frac{\partial^k g}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}\bigg|_p = \sum_{i=1}^r h_i(p) \frac{\partial^k f_i}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}}\bigg|_p + \text{lower derivatives of } f_i,$$

and in view of the definition (A.18) it is evident that $\text{mult}_p V \leq \text{mult}_p W$.

A refinement of the notion of the tangent space of a holomorphic subvariety, providing a more precise description of the singularities of holomorphic subvarieties, is the tangent cone\textsuperscript{14} of a holomorphic subvariety, which can be defined in a number of equivalent ways. Geometrically the tangent cone $C_p(V)$ of a holomorphic subvariety $V$ of an open subset $U \subset \mathbb{C}^n$ at a point $p \in V$ is defined to be the set of all vectors $v \in \mathbb{C}^n$ such that there exist a sequence of points $p_i \in V$ tending to the point $p \in V$ and a sequence of complex numbers $c_i \in \mathbb{C}$ such that $c_i(p_i - p) \longrightarrow v$; thus the tangent cone $C_p(V)$ is the set of limits of the secant lines joining points of $V$ to the point $p$. Although far from obvious, the tangent cone $C_p(V)$ can be described as the set of all tangent vectors to smooth curves through $V$ at the point $p$, that is, as the set of derivatives $v = \phi'(0)$ of $C^1$ mappings $\phi : (-\epsilon, \epsilon) \longrightarrow V$ from an open neighborhood of the origin in the real line into the subvariety $V$ such that $\phi(0) = p$. On the other hand the tangent cone can be defined algebraically as the zero locus of the initial polynomials $f^*_i(z)$ of a basis $f_i$ for the ideal $\text{id}_p V \subset O_p$ of the subvariety $V$ at the point $p$; here the \textit{initial polynomial} $f^*_i(z)$ at the point $p$ of a holomorphic function $f^*$(z)\textsuperscript{14}For a further discussion of tangent cones, and proofs of the results described here, see for instance H. Whitney, \textit{Complex Analytic Varieties} (Addison Wesley, 1972), particularly Chapter 7.

\[14\]
an open neighborhood of the point \( p \) is the homogeneous polynomial consisting of the terms of lowest degree in the power series expansion of the function \( f \) in terms of local coordinates \( z_i \) centered at the point \( p \). That too requires proof and is not trivial. It can be shown also that \( \text{mult}_p V = \min_i \deg f_i \) for a proper holomorphic subvariety in \( \mathbb{C}^n \); so the multiplicity of the subvariety \( V \) at a point \( p \in V \) is one of the properties of the tangent cone \( C_p(V) \). It is evident from any of these equivalent definitions that the tangent cone is a cone at the origin in \( \mathbb{C}^n \) in the usual sense, namely that if \( v \in C_p(V) \subset \mathbb{C}^n \) then \( cv \in C_p(V) \) for every complex number \( c \in \mathbb{C} \); consequently the tangent cone determines a well defined subset \( \mathbb{P}C_p(V) \subset \mathbb{P}^{n-1} \) in the complex projective space of dimension \( g - 1 \), the projective tangent cone of the subvariety \( V \) at the point \( p \in V \). It is frequently more convenient to describe the projective tangent cone rather than the tangent cone itself; the tangent cone then can be described as the set of all vectors in \( \mathbb{C}^n \) that represent points in the subset \( \mathbb{P}C_p(V) \subset \mathbb{P}^{n-1} \). The tangent cone \( C_p(V) \) is a holomorphic subvariety of \( \mathbb{C}^n \), and the projective tangent cone is a holomorphic hence an algebraic subvariety of \( \mathbb{P}^{n-1} \). It can be shown that \( C_p(V) \subset T_p(V) \) and that \( \dim C_p(V) = \dim_p V \). Moreover the tangent cone also is monotonic, in the sense that if \( p \in V \subset \bar{W} \) for some holomorphic subvarieties \( V \) and \( W \) in an open neighborhood of the point \( p \in \mathbb{C}^n \) then \( C_p(V) \subset C_p(W) \). The tangent cone \( C_p(V) \) may be reducible even though the holomorphic subvariety \( V \) is irreducible at the point \( p \in V \).

There are other possible notions of the tangent cone to a holomorphic subvariety at a point, although the preceding is the commonly used notion and is almost inevitably what is meant by the term “tangent cone”. One alternative notion that is useful for some purposes is the extended tangent cone \( C^*_p(V) \) of a holomorphic subvariety \( V \) of an open neighborhood of a point \( p \) in \( \mathbb{C}^n \), defined as the set of all vectors \( v \in \mathbb{C}^n \) that are the limits of tangent vectors to the regular part of the variety \( V \) at points approaching \( p \); the extended tangent cone coincides with the tangent space at any regular point of \( V \), and is a natural extension of the tangent space of the regular part of \( V \) to the singular points. The extended tangent cone is a holomorphic cone containing the usual tangent cone, so that \( C_p(V) \subset C^*_p(V) \) at any point \( p \) of a holomorphic subvariety \( V \); but this may be a strict inclusion, and indeed the dimension of the extended tangent cone may exceed the dimension of the subvariety \( V \) at the point \( p \).

In addition to the preceding properties of the singularities of holomorphic varieties, some acquaintance with the properties\(^{16}\) of some special classes of holomorphic mappings between holomorphic varieties also will be required at some points in the discussion in the body of the book. Particularly important

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\(^{15}\)The extended tangent cone is the cone \( C_4 \) in Whitney’s terminology, while the usual tangent cone is \( C_3 \).

\(^{16}\)For the proofs and further discussion of these topics see for instance R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables*, Vol. II, (Wadsworth and Brooks/Cole, 1990), especially Sections I and N. Remmert’s proper mapping theorem was proved in the paper by R. Remmert, “Holomorphe und meromorphe Abbildungen komplexer Räume”, *Math. Ann.*, vol 133(1957), pp. 328-370; that result and the local properties of holomorphic mappings have been discussed extensively in the literature.
A.3. HOLOMORPHIC VARIETIES

are proper holomorphic mappings, those holomorphic mappings such that the inverse image of any compact set is compact, and finite holomorphic mappings, those holomorphic mappings such that the inverse image of any point is a finite set of points. The simplest finite proper holomorphic mappings are finite branched holomorphic coverings, holomorphic mappings $F : V \to W$ between two holomorphic varieties $V$ and $W$ with the properties that

(i) $F$ is a finite, proper, surjective holomorphic mapping;
(ii) there are dense open subsets $V_0 \subset V$ and $W_0 \subset W$ such that $V_0 = F^{-1}(W_0)$ and the restriction $F|_{V_0} : V_0 \to W_0$ is a locally biholomorphic covering mapping;
(iii) the complement $W - W_0$ is a holomorphic subvariety of $W$.

Any holomorphic mapping between one-dimensional holomorphic varieties is locally a finite branched holomorphic covering, as is quite familiar; finite branched holomorphic coverings are those holomorphic mappings between holomorphic varieties of arbitrary dimensions that are most like holomorphic mappings between one-dimensional holomorphic varieties. The local parametrization theorem asserts that any irreducible holomorphic variety of dimension $n$ can be represented locally as a finite branched holomorphic covering of an open subset of $\mathbb{C}^n$; that provides a particularly convenient local representation for the study of holomorphic varieties. A holomorphic mapping $F : V \to W$ between two holomorphic varieties $V$ and $W$ is said to be finite if $F^{-1}(p)$ is a finite subset of $V$ for each point $p \in W$; it can be shown that a holomorphic mapping $F : V \to W$ is finite if and only if for each irreducible component $V_i$ of $V$ the restriction $F|_{V_i} : V_i \to F(V_i)$ is locally a finite branched holomorphic covering. More general proper holomorphic mappings arise quite frequently. One of their most important properties is given in Remmert’s proper mapping theorem, which asserts that if $F : V \to W$ is a proper holomorphic mapping between holomorphic varieties $V$ and $W$ then the image $F(V)$ is a holomorphic subvariety of $W$; and if $V$ is irreducible then so is its image, and

$$(A.20) \quad \dim F(V) = \sup_{p \in V} \left( \dim V - \dim_p F^{-1}(F(p)) \right).$$

For a finite proper holomorphic mapping $\dim_p F^{-1}(F(p)) = 0$ for all points $p \in V$ so the preceding formula reduces to

$$(A.21) \quad \dim F(V) = \dim V.$$ 

In general the fibres $F^{-1}(q)$ over points $q \in W$ need not be irreducible, and their dimensions may vary from point to point. However there is at least some regularity to the behavior of the dimension, as a consequence of Remmert’s semi-continuity theorem, which asserts that if $F : V \to W$ is a holomorphic mapping between holomorphic varieties $V$ and $W$, not necessarily a proper holomorphic mapping, then for any integer $\nu$ the subset $\{ p \in V \mid \dim_p F^{-1}(F(p)) \geq \nu \}$ is a holomorphic subvariety of $V$; this is an extension of the condition that $\dim_p F^{-1}(F(p))$ is an upper semi-continuous function of the point $p \in V$. Images of holomorphic varieties under holomorphic mappings that are not proper
also may be holomorphic subvarieties. The local mapping theorem asserts that
if $F: V \rightarrow W$ is a holomorphic mapping between two holomorphic varieties $V$
and $W$, not necessarily a proper holomorphic mapping, and if $\dim_p F^{-1}(F(p)) = \nu$
is independent of the point $p \in V$, then each point $p \in V$ has arbitrarily small
open neighborhoods $V_p$ such that $F(V_p)$ is a holomorphic subvariety of an open
neighborhood of $F(p)$ in $W$ and $\dim_{F(p)} F(V_p) = \dim_p V - \nu$. If the subvariety
$V$ is irreducible at the point $p \in V$ then the converse also holds: if there are
arbitrily small open neighborhoods $V_p$ of the point $p \in V$ such that $F(V_p)$ is a
holomorphic subvariety of an open neighborhood of the point $f(p) \in W$ then
$\dim_q F^{-1}(F(q))$ is a constant independent of the point $q \in V$ in some open
neighborhood of the point $p \in V$. Thus the fact that the fibres of a holomorphic
mapping have constant dimension really is almost equivalent to the condition
that the image of the mapping is locally a holomorphic subvariety. These results
will be used at various points in the discussion in the body of this book.
Appendix B

Vector Bundles

B.1 Definitions

A complex vector bundle of rank $r$ over a topological space $M$ is a topological space $\lambda$ with a continuous mapping $\pi : \lambda \rightarrow M$ such that (i) in an open neighborhood $U$ of each point $p \in M$ there is a commutative diagram

\[
\begin{array}{ccc}
\lambda \cap \pi^{-1}(U) & \xrightarrow{\lambda_U} & U \times \mathbb{C}^r \\
\pi \downarrow & & \pi_1 \downarrow \\
M \cap U & \xrightarrow{} & U
\end{array}
\]

where $\lambda_U$ is a homeomorphism from $\pi^{-1}(U)$ to the product $U \times \mathbb{C}^r$ and $\pi_1$ is the projection of the product to its first factor; and (ii) in an intersection $U \cap V$ of two such neighborhoods of $p$ there is a continuous mapping

\[
\lambda_{VU} : U \cap V \rightarrow \text{Gl}(r, \mathbb{C})
\]

such that the composite mapping $\lambda_V \circ \lambda_{VU}^{-1} : (U \cap V) \times \mathbb{C}^r \rightarrow (U \cap V) \times \mathbb{C}^r$ has the form

\[
(\lambda_V \circ \lambda_{VU}^{-1})(p, t) = (p, \lambda_{VU}(p)t)
\]

for any point $(p, t) \in (U \cap V) \times \mathbb{C}^r$. The space $M$ is called the base space of the vector bundle $\lambda$, the mapping $\pi$ is called the projection, the mappings $\lambda_U$ are called the coordinate mappings or local coordinates, the linear transformations $\lambda_{VU}(p)$ are called the coordinate transition functions, and the inverse image $\lambda_p = \pi^{-1}(p)$ of a point $p \in M$ is called the fibre over the point $p$. The local product structure provided by the homeomorphism $\lambda_U$ describes a point in the open subset $\pi^{-1}(U) \subset \lambda$ by a pair $(p, t_U) \in U \times \mathbb{C}^r$, where the vector $t_U \in \mathbb{C}^r$ is the fibre coordinate of that point in terms of the local product structure over $U$; the vector $t_U$ will be viewed as a column vector of length $r$ when explicit formulas are required, and the coordinate transition functions then will be viewed as $r \times r$
complex matrices. If \( p \in U \cap V \) the fibre coordinates of points in \( \pi^{-1}(p) \) in terms of the local product structures over \( U \) and \( V \) are related by

\[
(B.4) \quad t_V = \lambda_{UV}(p) t_U.
\]

The simplest example of a complex vector bundle of rank \( r \) over a topological space \( M \) is the product bundle or trivial bundle, the product \( \lambda = M \times \mathbb{C}^r \) where \( \pi \) is the natural projection to the first factor; for this bundle all the coordinate transition functions can be taken to be the identity mapping \( \lambda_{UV}(p) = I \) since the coordinate mappings can be taken to be the identity mapping. A complex vector bundle of rank 1 also is called a complex line bundle; for a complex line bundle the coordinate transition functions are merely nowhere vanishing functions in the intersections \( U \cap V \). If the base space \( M \) is a topological manifold of dimension \( n \) and the subsets \( U \subset M \) are coordinate neighborhoods in \( M \) that are identified with subsets of \( \mathbb{R}^n \) then the local coordinate mappings \( \lambda_U \) impose on the space \( \lambda \) the structure of a topological manifold for which \( \dim \lambda = n + r \); in addition, since the homeomorphisms \( \lambda_V \circ \lambda_U^{-1} \) belong to the pseudogroup \( CL \) consisting of local homeomorphisms between products \( \mathbb{R}^n \times \mathbb{C}^r \) that are complex linear mappings on \( \mathbb{C}^r \), the manifold \( \lambda \) is a \( CL \) manifold. If the base space \( M \) is a \( C^\infty \) manifold and the coordinate transition functions \( \lambda_{UV}(p) \) are \( C^\infty \) functions the manifold \( \lambda \) is a \( C^\infty \) manifold and the bundle is said to be a \( C^\infty \) vector bundle. If the base space \( M \) is a complex manifold and the coordinate transition functions \( \lambda_{UV}(p) \) are holomorphic functions the manifold \( \lambda \) is a complex manifold and the bundle is said to be a holomorphic vector bundle. If the coordinate transition functions \( \lambda_{UV}(p) \) are locally constant functions the bundle is called a flat vector bundle. A holomorphic vector bundle also has the weaker structure of a \( C^\infty \) vector bundle, and a flat vector bundle also has the weaker structure of a holomorphic vector bundle.

A cross-section of a complex vector bundle \( \lambda \) over a topological space \( M \) is a continuous mapping \( f : M \longrightarrow \lambda \) such that \( \pi \circ f(p) = p \) for each point \( p \in M \). The composition of a cross-section \( f \) and the coordinate mapping \( \lambda_U \) over an open subset \( U \subset M \) has the form

\[
(B.5) \quad (\lambda_U \circ f)(p) = (p, f_U(p)) \in M \times \mathbb{C}^r
\]

for any point \( p \in U \), where \( f_U : U \longrightarrow \mathbb{C}^r \) is a continuous mapping called the local form of the cross-section over \( U \); thus \( f_U(p) \) is the fibre coordinate of the point \( f(p) \in \lambda \) in terms of the local product structure over \( U \). It is clear that a cross-section \( f \) is described completely by its local form over subsets \( U \subset M \), and that the local forms satisfy

\[
(B.6) \quad f_V(p) = \lambda_{UV}(p) f_U(p) \quad \text{for} \quad p \in U \cap V
\]

since the fibre coordinates satisfy (B.4). A cross-section \( f \) of a \( C^\infty \) complex vector bundle \( \lambda \) is a \( C^\infty \) cross-section if the mapping \( f : M \longrightarrow \lambda \) is a \( C^\infty \) mapping, or equivalently if the local forms \( f_U : U \longrightarrow \mathbb{C}^r \) are \( C^\infty \) mappings; a cross-section \( f \) of a holomorphic vector bundle \( \lambda \) is a holomorphic cross-section.
if the mapping \( f : M \rightarrow \lambda \) is a holomorphic mapping, or equivalently if the
local forms \( f_U : U \rightarrow \mathbb{C}^r \) are holomorphic mappings; and a cross-section \( f \) of
a flat vector bundle \( \lambda \) is a flat cross-section if the local forms \( f_U : U \rightarrow \mathbb{C}^r \)
are locally constant mappings. Cross-sections can be added and multiplied by
complex constants, by using the structure of a complex vector space in the fibre;
thus the set of cross-sections has the natural structure of a complex vector space.
Clearly linear combinations of \( C^\infty \) cross-sections of a \( C^\infty \) vector bundle again are
\( C^\infty \) cross-sections, and correspondingly for holomorphic or flat cross-sections; so
the set of all continuous, \( C^\infty \), holomorphic or flat cross-sections of a complex
vector bundle having the appropriate regularity are also complex vector spaces.
The vector space of continuous cross-sections of a vector bundle \( \lambda \) is denoted by
\( \Gamma (M, C(\lambda)) \), the vector space of \( C^\infty \) cross-sections of a \( C^\infty \) vector bundle \( \lambda \) over
a \( C^\infty \) manifold \( M \) is denoted by \( \Gamma (M, \mathcal{E}(\lambda)) \), the vector space of holomorphic
cross-sections of a holomorphic vector bundle \( \lambda \) over a complex manifold
\( M \) is denoted by \( \Gamma (M, \mathcal{O}(\lambda)) \), and the vector space of flat cross-sections of a flat
vector bundle \( \lambda \) over a topological manifold \( M \) is denoted by \( \Gamma (M, \mathcal{F}(\lambda)) \).

If \( \lambda^i \) for \( 1 \leq i \leq n \) are vector bundles over a topological space \( M \) described by
coordinate transition functions \( \lambda^i_U^V \) for \( 1 \leq i \leq n \) their direct sum \( \lambda^1 \oplus \cdots \oplus \lambda^n \)
is the vector bundle with the coordinate transition functions \( \lambda^1_U^V \oplus \cdots \oplus \lambda^n_U^V \) and their tensor product \( \lambda^1 \otimes \cdots \otimes \lambda^n \) is the vector bundle with the coordinate transition functions \( \lambda^1_U^V \otimes \cdots \otimes \lambda^n_U^V \), where

\[
\lambda^1_U^V \oplus \cdots \oplus \lambda^n_U^V = \begin{pmatrix}
\lambda^1_U^V & 0 & \cdots & 0 \\
0 & \lambda^2_U^V & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda^n_U^V
\end{pmatrix}
\]

and \( \lambda^1_U^V \otimes \cdots \otimes \lambda^n_U^V \) is the linear transformation on tensors \( v_{U, i_1 i_2 \cdots i_n} \) defined by

\[
v_{U, i_1 i_2 \cdots i_n} = \sum_{j_1, \ldots, j_n} \lambda^1_U^V, i_1 j_1 \lambda^2_U^V, i_2 j_2 \cdots \lambda^n_U^V, i_n j_n v_{V, j_1 j_2 \cdots j_n}.
\]

It is evident that

\[
\text{rank}(\lambda^1 \oplus \cdots \oplus \lambda^n) = \text{rank} \lambda^1 + \cdots + \text{rank} \lambda^n \quad \text{and} \\
\text{rank}(\lambda^1 \otimes \cdots \otimes \lambda^n) = (\text{rank} \lambda^1) \cdots (\text{rank} \lambda^n).
\]

If \( \lambda^1 = \cdots = \lambda^n = \lambda \) the tensor product is denoted by \( \lambda \otimes^n \). The tensor product
\( \lambda^1 \otimes \lambda^2 \) of two vector bundles can be viewed alternatively as a vector bundle in
which the fibres as well as the coordinate transition functions are matrices; for
in this case (B.8) becomes

\[
v_{U, i_1 i_2} = \sum_{j_1, j_2} \lambda^1_U^V, i_1 j_1 v_{V, j_1 j_2} \lambda^2_U^V, i_2 j_2,
\]

and when the values \( v_{U, i_1 i_2} \) are interpreted as entries in a matrix \( v_U \) this is the
matrix identity

\[
v_U = \lambda_U^V v_V^t \lambda_U^V
\]
where \( \lambda^T \) is the transpose of the matrix \( \lambda \). A tensor product of line bundles \( \lambda \) and \( \lambda \) is a line bundle, and the notation usually is simplified by setting \( \lambda \otimes \lambda = \lambda \lambda \); correspondingly the notation for the tensor product of a line bundle \( \sigma \) and a vector bundle \( \lambda \) usually also is simplified by setting \( \sigma \otimes = \sigma \lambda \). The tensor product of \( n \) copies of a line bundle \( \lambda \) with itself usually is denoted by \( \lambda^n \).

If \( \lambda = \{ \lambda_U \} \) is a vector bundle of rank \( r \) over a topological space \( M \) then for any group homomorphism \( \theta : \text{Gl}(r, \mathbb{C}) \rightarrow \text{Gl}(s, \mathbb{C}) \) the mappings \( \theta(\lambda_U) \) can be taken as the coordinate transition functions describing a vector bundle \( \theta(\lambda) \) of rank \( s \) over \( M \). For example, associated to any vector bundle \( \lambda \) of rank \( r \) over \( M \) is its determinant bundle \( \det \lambda \), the line bundle over \( M \) described by the coordinate transition functions \( \det \lambda_U \), and its dual bundle \( \lambda^* = \lambda^{-1} \), the vector bundle of rank \( r \) described by the coordinate transition functions \( \lambda^{*}_U = \lambda^{-1}_U \). Similarly to any vector bundle \( \lambda \) of rank \( r \) over \( M \) can be associated its adjoint bundle \( \text{Ad} \lambda \), the vector bundle of rank \( r^2 \) described by the coordinate transition functions \( \text{Ad}(\lambda_U) \) where \( \text{Ad} : \text{Gl}(r, \mathbb{C}) \rightarrow \text{Gl}(r^2, \mathbb{C}) \) is the adjoint representation, the mapping that associates to a matrix \( A \in \text{Gl}(r, \mathbb{C}) \) the linear transformation on the vector space \( \mathbb{C}^{r \times r} \) of \( r \times r \) complex matrices defined by \( \text{Ad}(A)Z = AZA^{-1} \). The linear subspace \( \mathbb{C}_0^{r \times r} \subset \mathbb{C}^{r \times r} \) consisting of matrices of trace zero is preserved under the adjoint representation, and the restriction of the adjoint representation to this subspace \( \mathbb{C}_0^{r \times r} \) is another group homomorphism \( \text{Ad}_0 : \text{Gl}(r, \mathbb{C}) \rightarrow \text{Gl}(r^2 - 1, \mathbb{C}) \) that can be used to associate to the vector bundle \( \lambda \) its restricted adjoint bundle \( \text{Ad}_0 \lambda \) of rank \( r^2 - 1 \), defined by the coordinate transition functions \( \text{Ad}_0(\lambda_U) \). The changes of coordinates in the fibres of the bundle \( \text{Ad} \lambda \) are given by \( Z_U = \text{Ad}(\lambda_U)Z_V = \lambda_U Z_V \lambda_U \) where \( Z_U, Z_V \in \mathbb{C}^{r \times r} \), so in view of (B.11) there is the natural identification

\[
\text{Ad} \lambda = \lambda \otimes \lambda^*
\]

that is quite commonly used.

If \( \lambda \) and \( \sigma \) are vector bundles of ranks \( r \) and \( s \) over the same space \( M \), with projections \( \pi_\lambda : \lambda \rightarrow M \) and \( \pi_\sigma : \sigma \rightarrow M \), a bundle homomorphism \( \phi : \sigma \rightarrow \lambda \) is a continuous mapping between the topological spaces \( \sigma \) and \( \lambda \) such that (i) the diagram

\[
\begin{array}{ccc}
\sigma & \xrightarrow{\phi} & \lambda \\
\pi_\sigma & \downarrow & \pi_\lambda \\
U & \xrightarrow{=} & U
\end{array}
\]

is commutative, so that \( \phi(\sigma_p) \subset \lambda_p \) for the fibres over any point \( p \in M \); and (ii) the restriction \( \phi|_{\sigma_p} : \sigma_p \rightarrow \lambda_p \) of the mapping \( \phi \) to the fibre \( \sigma_p \) is a linear mapping for each point \( p \in M \). In terms of the fibre coordinates \( t_U \) for the bundle \( \lambda \) and \( s_U \) for the bundle \( \sigma \) over an open subset \( U \subset M \) the composite mapping \( \lambda_U \circ \phi \circ \sigma_U^{-1} : U \times \mathbb{C}^s \rightarrow U \times \mathbb{C}^r \) has the form

\[
(\lambda_U \circ \phi \circ \sigma_U^{-1})(p, s_U) = (p, t_U) = (p, \phi_U(p) s_U)
\]
where \( \phi_U(p) : \mathbb{C}^n \rightarrow \mathbb{C}^r \) is a linear mapping for each point \( p \in U \) that is a continuous function of the point \( p \in U \), called the local form of the homomorphism \( \phi \). If \( p \in U \cap V \)

\[
(p, \phi_V(p) s_V) = (\lambda_V \circ \phi \circ \sigma_V^{-1})(p, s_V) \\
= (\lambda_V \circ \lambda_U^{-1}) \circ (\lambda_U \circ \phi \circ \sigma_U^{-1}) \circ (\sigma_U \circ \sigma_V^{-1})(p, s_V) \\
= (\lambda_V \circ \lambda_U^{-1}) \circ (\lambda_U \circ \phi \circ \sigma_U^{-1})(p, \sigma_{UV}(p) s_V) \\
= (\lambda_V \circ \lambda_U^{-1})(p, \phi_V(p) \cdot \sigma_{UV}(p) s_V) \\
= (p, \lambda_{V'}(p) \cdot \phi_U(p) \cdot \sigma_{UV}(p) s_V);
\]

consequently

\[
\phi_V(p) = \lambda_{V'}(p) \cdot \phi_U(p) \cdot \sigma_{UV}(p) \quad \text{if} \quad p \in U \cap V.
\]

The homomorphism \( \phi \) is a \( C^\infty \) homomorphism if \( \lambda \) and \( \sigma \) are \( C^\infty \) bundles and the mapping \( \phi : \sigma \rightarrow \lambda \) is a \( C^\infty \) mapping, or equivalently if the local forms \( \phi_{U}(p) \) are \( C^\infty \) functions; the homomorphism is a holomorphic homomorphism if \( \lambda \) and \( \sigma \) are holomorphic bundles and the mapping \( \phi : \sigma \rightarrow \lambda \) is a holomorphic mapping, or equivalently if the local forms \( \phi_{U}(p) \) are holomorphic functions; and the homomorphism \( \phi \) is a flat homomorphism if \( \sigma \) and \( \lambda \) are flat bundles and the local forms \( \phi_{U}(p) \) are locally constant functions. It is evident from (B.15) that if \( \phi = \{\phi_U\} \) and \( \psi = \{\psi_U\} \) are two homomorphisms from a vector bundle \( \sigma \) to a vector bundle \( \lambda \) over \( M \) then \( a\phi + b\psi = \{a\phi_U + b\psi_U\} \) is also a homomorphism from \( \sigma \) to \( \lambda \) for any complex constants \( a, b \in \mathbb{C} \); the set of homomorphisms from \( \sigma \) to \( \lambda \) thus naturally form a complex vector space, denoted by \( \text{Hom}(\sigma, \lambda) \). If the bundles \( \lambda \) and \( \sigma \) are \( C^\infty \) the set of \( C^\infty \) homomorphisms form a vector subspace \( \text{Hom}_C(\sigma, \lambda) \subset \text{Hom}(\sigma, \lambda) \), as do the further subspaces \( \text{Hom}_C(\sigma, \lambda) \) of holomorphic homomorphisms between holomorphic vector bundles and \( \text{Hom}_\mathcal{O}(\sigma, \lambda) \) of flat homomorphisms between flat vector bundles. When \( \sigma = \lambda \) vector bundle homomorphisms also are called endomorphisms of that bundle. Since the composition of two endomorphisms is again an endomorphism it is evident that the set of endomorphisms \( \text{End}(\lambda) = \text{Hom}(\lambda, \lambda) \) has the natural structure of a complex algebra; of course the same is true for special classes of endomorphisms such as \( \text{End}_\mathcal{O}(\lambda), \text{End}_\mathcal{O}(\lambda) \) and \( \text{End}_\mathcal{O}(\lambda) \).

If \( \phi : \sigma \rightarrow \lambda \) is a bundle homomorphism, the rank of the linear mapping \( \phi|_{\sigma_p} : \sigma_p \rightarrow \lambda_p \) is called the rank of the homomorphism \( \phi \) at the point \( p \in M \) and is denoted by \( \text{rank}_p(\phi) \); of course \( \text{rank}_p(\phi) = \text{rank}_{\phi_U}(p) \) in terms of the local form \( \phi_U \) of the homomorphism \( \phi \) for any coordinate neighborhood \( U \) containing the point \( p \). The maximal rank of a homomorphism \( \phi \) at all the points of \( M \) is called simply the rank of the homomorphism \( \phi \) and is denoted by \( \text{rank}(\phi) \); thus \( \text{rank}(\phi) = \sup_{p \in M} \text{rank}_p(\phi) \). The rank of a homomorphism \( \phi \) can vary from point to point on the set \( M \), except in the case of a flat homomorphism of flat vector bundles over a connected topological space. The condition that \( \text{rank}_p(\phi) \leq n \) amounts to the vanishing of all \((n + 1) \times (n + 1)\) subdeterminants of the matrix \( \phi_U(p) \) at the point \( p \in U \); so for a holomorphic homomorphism \( \phi \) between two
holomorphic vector bundles over a complex manifold $M$ the set of points $p \in M$ at which $\text{rank}_p \phi \leq t$ is either the entire complex manifold $M$ or a holomorphic subvariety of $M$, and the set of points $p \in M$ at which $\text{rank}_p \phi < \text{rank} \phi$ is a proper holomorphic subvariety of $M$. A homomorphism $\phi$ is said to be of constant rank if $\text{rank}_p \phi = \text{rank} \phi$ at all points $p \in M$. A bundle homomorphism $\phi : \sigma \to \lambda$ is injective (surjective) if its restriction $\phi|_p : \sigma_p \to \lambda_p$ is an injective linear mapping (a surjective linear mapping) over each point $p \in M$; it is an isomorphism if it is both injective and surjective, or equivalently if it has an inverse vector bundle homomorphism $\psi : \lambda \to \sigma$. Isomorphic bundles of course have the same rank; and a homomorphism between two vector bundles $\lambda$ and $\sigma$ for which $\text{rank} \lambda = \text{rank} \sigma = r$ is an isomorphism if and only if the homomorphism is of constant rank $r$.

For many purposes it is not necessary to consider the local product structures of a vector bundle over a space $M$ for all open subsets of $M$, but suffices to consider only those for a single open covering of $M$. If $\lambda$ is a vector bundle of rank $r$ over a topological space $M$ then for any sufficiently fine open covering $\mathcal{U} = \{U_\alpha\}$ of $M$ there will be coordinate mappings

$$\lambda_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$$

for the bundle $\lambda$; and in an intersection $U_\alpha \cap U_\beta$ as in (B.3) there are the coordinate transition functions $\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(r, \mathbb{C})$ for which

$$(\lambda_\alpha \circ \lambda_{\beta}^{-1})(p, t) = (p, \lambda_{\alpha\beta}(p)t).$$

The collection $\{U_\alpha, \lambda_{\alpha\beta}\}$ of the open subsets $U_\alpha \subset M$ and coordinate transition functions $\lambda_{\alpha\beta}$ is called a coordinate bundle describing the vector bundle $\lambda$. The fibre coordinates $t_\alpha$ and $t_\beta$ of a point of $\lambda$ lying over a point $p \in U_\alpha \cap U_\beta \subset M$ are related by

$$t_\alpha = \lambda_{\alpha\beta}(p)t_\beta \quad \text{for} \quad p \in U_\alpha \cap U_\beta$$

as in (B.4). It is clear that the coordinate transition functions $\lambda_{\alpha\beta}$ satisfy the compatibility conditions

$$\lambda_{\alpha\alpha}(p) = I \quad \text{if} \quad p \in U_\alpha,$$

$$\lambda_{\alpha\beta}(p) \cdot \lambda_{\beta\alpha}(p) = I \quad \text{if} \quad p \in U_\alpha \cap U_\beta,$$

$$\lambda_{\alpha\beta}(p) \cdot \lambda_{\beta\gamma}(p) \cdot \lambda_{\gamma\alpha}(p) = I \quad \text{if} \quad p \in U_\alpha \cap U_\beta \cap U_\gamma.$$

The sets $U_\alpha$, $U_\beta$, $U_\gamma$ in (B.17) are not necessarily distinct; the second condition follows from the first and third upon setting $\gamma = \alpha$, but is included separately in (B.17) for emphasis. Any collection of open subsets $U_\alpha \subset M$ covering $M$ and of continuous mappings

$$\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{GL}(r, \mathbb{C})$$
satisfying the compatibility conditions \((B.17)\) form a coordinate bundle describing a vector bundle over \(M\). Indeed it is a straightforward matter to verify that, in terms of the equivalence relation on pairs \((p_\alpha, t_\alpha) \in U_\alpha \times \mathbb{C}^r\) defined by \((p_\alpha, t_\alpha) \sim (p_\beta, t_\beta)\) whenever \(p_\alpha = p_\beta = p \in M\) and \(t_\alpha = \lambda_{\alpha\beta}(p)t_\beta\), which is an equivalence relation in the usual sense as an immediate consequence of \((B.17)\), the quotient of the disjoint union of the products \(U_\alpha \times \mathbb{C}^r\) by this equivalence relation is a complex vector bundle described by the coordinate bundle \(\{U_\alpha, \lambda_{\alpha\beta}\}\). This is a particularly common and useful way of describing complex vector bundles; \(\mathcal{C}^\infty\), holomorphic or flat complex vector bundles can be described in this way for mappings \((B.18)\) that are \(\mathcal{C}^\infty\), holomorphic, or locally constant. It should be noted that for there to be a description of a given vector bundle \(C\) described in this way for mappings \((B.18)\) that are \(\mathcal{C}^\infty\), holomorphic, or flat vector bundles.

Two coordinate bundles \(\{U_\alpha, \lambda_{\alpha\beta}\}\) and \(\{V_k, \sigma_{kl}\}\) that describe the same complex vector bundle over \(M\) are called equivalent coordinate bundles. If the fibre coordinates are \(t_\alpha\) over \(U_\alpha\) and \(t_k\) over \(V_k\) equivalence means that in addition to the relations \((B.16)\) between the fibre coordinates \(t_\alpha\) and \(t_\beta\) over intersections \(U_\alpha \cap U_\beta\) and the corresponding relations \(t_k = \sigma_{kl}t_l\) between the fibre coordinates \(t_k\) and \(t_l\) over \(V_k \cap V_l\) there are further relations of the form

\[
(B.19) \quad t_\alpha = \mu_{\alpha k}(p)t_k \quad \text{and} \quad t_k = \mu_{k\alpha}(p)t_\alpha \quad \text{for} \quad p \in U_\alpha \cap V_k
\]

between the fibre coordinates \(t_\alpha\) over \(U_\alpha\) and \(t_k\) over \(V_k\) for some continuous mappings

\[
(B.20) \quad \mu_{\alpha k}, \ \mu_{k\alpha} : U_\alpha \cap V_k \to \text{GL}(r, \mathbb{C}).
\]

Consequently in addition to the compatibility conditions \((B.17)\) for the coordinate bundle \(\{U_\alpha, \lambda_{\alpha\beta}\}\) and the corresponding conditions for the coordinate bundle \(\{V_k, \sigma_{kl}\}\) there are the further compatibility conditions

\[
(B.21) \quad \lambda_{\alpha\beta}(p)\mu_{\beta m}(p)\mu_{ma}(p) = I \quad \text{for} \quad p \in U_\alpha \cap U_\beta \cap V_m,
\]

\[
\mu_{\alpha l}(p)\sigma_{lm}(p)\mu_{mo}(p) = I \quad \text{for} \quad p \in U_\alpha \cap V_l \cap V_m.
\]

The corresponding conditions for other orders of the products of the coordinate transition functions follow automatically from these relations; and for the special cases in which \(\beta = \alpha\) or \(l = k\) it follows that

\[
(B.22) \quad \mu_{\alpha k}(p)\mu_{k\alpha}(p) = I.
\]

Conversely two coordinate bundles \(\{U_\alpha, \lambda_{\alpha\beta}\}\) and \(\{V_k, \sigma_{kl}\}\) of the same rank over \(M\) are equivalent coordinate bundles if there are mappings \((B.20)\) satisfying \((B.21)\), since in that case the collection of all the sets \(U_\alpha\) and \(V_k\) and of
all the mappings $\lambda_{\alpha\beta}, \sigma_{kl}, \mu_{ak}\mu_{k\alpha}$ form a coordinate bundle over $M$ describing
a vector bundle over $M$ that is also described by the two separate coordinate
bundles $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$. When it is useful or necessary to specify an
additional regularity condition for the vector bundle the coordinate bundles are
said to be $C^\infty$ equivalent or holomorphic equivalent or flat equivalent coordinate
bundles; for the equivalence of bundles with these further regularity conditions
the mappings $\lambda_{\alpha\beta}, \sigma_{kl}, \mu_{ak}$ also must satisfy the appropriate regularity conditions.
A somewhat simpler and more useful condition for the equivalence of two
coordinate bundles arises from the observation that if $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$
are equivalent coordinate bundles and if $p \in U_\alpha \cap U_\beta \cap V_k \cap V_l$ then it follows from (B.21) that
$\mu_{ka}(p)\lambda_{\alpha\beta}(p)\mu_{\beta l}(p) = \mu_{ka}(p)\mu_{ak}(p)\mu_{k\beta}(p) : \mu_{\beta l}(p) = \mu_{k\beta}(p)\mu_{\beta l}(p) = \sigma_{kl}(p)$ and consequently

\begin{equation}
\sigma_{kl}(p) = \mu_{ka}(p)\lambda_{\alpha\beta}(p)\mu_{\beta l}(p) \quad \text{for} \quad p \in U_\alpha \cap U_\beta \cap V_k \cap V_l.
\end{equation}

Conversely if $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$ are two coordinate bundles of the same
rank over $M$ and there are mappings (B.20) satisfying (B.22) and (B.23) then
these two coordinate bundles are equivalent; indeed when $k = l$ the equations
(B.23) reduce to the first equations in (B.21) while when $\alpha = \beta$ they reduce to the second equations in (B.21), and consequently the two coordinate bundles are equivalent. In particular a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ describes a trivial
bundle if and only if it is equivalent to the coordinate bundle described by a
single coordinate neighborhood $V_k = M$; and in that case condition (B.23) takes the form

\begin{equation}
\lambda_{\alpha\beta}(p) = \mu_{\alpha}(p)\mu_{\beta}(p)^{-1}.
\end{equation}

On the other hand for two coordinate bundles defined in terms of the same
covering of $M$, so for two coordinate bundles $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{U_\alpha, \sigma_{\alpha\beta}\}$, condition
(B.23) for the case that $V_k = U_\alpha$ and $V_l = U_\beta$ takes the form

\begin{equation}
\sigma_{\alpha\beta}(p) = \mu_{\alpha}(p)\lambda_{\alpha\beta}(p)\mu_{\beta}(p)^{-1} \quad \text{for} \quad p \in U_\alpha \cap U_\beta
\end{equation}

where

\begin{equation}
\mu_{\alpha} = \mu_{\alpha\alpha} : U_\alpha \longrightarrow \text{Gl}(r, \mathbb{C});
\end{equation}

thus this condition must be satisfied if $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{U_\alpha, \sigma_{\alpha\beta}\}$ are equivalent
coordinate bundles. Conversely if this condition is satisfied then for any point
$p \in U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

\begin{align*}
\sigma_{\alpha\beta}(p) &= \mu_{\alpha}(p)\lambda_{\alpha\beta}(p)\mu_{\beta}(p)^{-1} \\
&= \mu_{\alpha}(p)\lambda_{\alpha\gamma}(p)\cdot \lambda_{\gamma\delta}(p)\cdot \lambda_{\delta\beta}(p)\mu_{\beta}(p)^{-1} \\
&= \mu_{\alpha\gamma}(p)\lambda_{\gamma\delta}(p)\mu_{\delta\beta}(p)
\end{align*}

where $\mu_{\alpha\gamma}(p) = \mu_{\alpha}(p)\lambda_{\alpha\gamma}(p)$ and $\mu_{\delta\beta}(p) = \lambda_{\delta\beta}(p)\mu_{\beta}(p)^{-1}$, and since this is just
(B.23) it follows that the two coordinate bundles are equivalent. Consequently
(B.25) is a necessary and sufficient condition for the equivalence of the two coordinate bundles.

If a vector bundle $\lambda$ over $M$ is described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ and if $f \in \Gamma(M, C^r(\lambda))$ then the mappings $f_\alpha = f_{U_\alpha}$ of the local form of the cross-section $f$, as defined by (B.5) for the open subsets $U_\alpha$, describe that cross-section completely. It follows from (B.16) that these mappings satisfy

$$f_\alpha(p) = \lambda_{\alpha\beta}(p) f_\beta(p)$$

for any point $p \in U_\alpha \cap U_\beta$; conversely any collection of mappings $f_\alpha : U_\alpha \rightarrow C^r$ satisfying (B.27) describe a cross-section $f \in \Gamma(M, C(L))$. If vector bundles $\sigma$ and $\lambda$ are described by coordinate bundles $\{U_\alpha, \sigma_{\alpha\beta}\}$ and $\{U_\alpha, \lambda_{\alpha\beta}\}$ in terms of a the same covering $\{U_\alpha\}$ of $M$, the local form of a homomorphism $\phi : \sigma \rightarrow \lambda$ between these two vector bundles for the subsets $U_\alpha$ consist of linear mappings $\phi_\alpha(p) = \phi_{U_\alpha}$ defined for points $p \in U_\alpha$; and as in (B.15) these linear mappings satisfy

$$\phi_\alpha(p) = \lambda_{\alpha\beta}(p) \phi_\beta(p) \sigma_{\alpha\beta}(p)$$

for $p \in U_\alpha \cap U_\beta$.

Conversely any collection of linear mappings $\phi_\alpha$ satisfying these conditions describes a vector bundle homomorphism $\phi : \sigma \rightarrow \lambda$.

### B.2 Basic Properties

Vector bundles of rank $r > 1$ are more complicated than line bundles in many ways, so it may be useful to discuss their basic properties in a bit more detail here. A cross-section $\phi = \{\phi_\alpha\}$ of a vector bundle $\lambda$ described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ over a topological space $M$ satisfies $\phi_\alpha = \lambda_{\alpha\beta} \phi_\beta$ over any intersection $U_\alpha \cap U_\beta$, as in (B.27); and that can be viewed as the special case of (B.28) in which $\sigma_{\alpha\beta}(p) = 1$ for all points $p \in U_\alpha \cap U_\beta$, so $\phi$ can be identified with a bundle homomorphism $\phi : 1 \rightarrow \lambda$ from the trivial line bundle 1 to the bundle $\lambda$. Conversely any such homomorphism can be viewed as a cross-section of the bundle $\lambda$, so for instance there is the natural identification

$$\text{Hom}_\mathcal{O}(1, \lambda) = \Gamma(M, \mathcal{O}(\lambda))$$

and correspondingly for the other regularity classes of bundles. Similarly a collection of $s$ cross-sections of the bundle $\lambda$ can be viewed as a homomorphism $\phi : I_s \rightarrow \lambda$ from the trivial vector bundle of rank $s$ to the bundle $\lambda$ and conversely, so that there is the further natural identification

$$\text{Hom}_\mathcal{O}(I_s, \lambda) = \Gamma(M, \mathcal{O}(\lambda))^s.$$
\begin{equation}
\lambda_\alpha(p) = \begin{pmatrix}
\alpha_\alpha(p) \\ \alpha_\beta(p) \\ \alpha_\gamma(p)
\end{pmatrix}
\end{equation}

\[\text{(B.37)}\]

In any intersection \(U_i \cap U_j\), so these local functions describe a global function on \(M\) that is called the determinant of the endomorphism \(\phi\) and is denoted by 

\[\det \phi = \det \phi(p) \in \text{End}(\Lambda)\] 

so from (B.33) it follows that

\[\text{End}(\Lambda) = \text{Hom}(\Lambda^*) \cong \text{Hom}(\Lambda^* \otimes \Lambda) = \text{Hom}(\Lambda^* \otimes \Lambda^*),\]

End(e(\Lambda)) = \text{Hom}(\Lambda^* \otimes \Lambda^*) = \text{Hom}(\Lambda^* \otimes \Lambda^*).

\[\text{(B.34)}\]

In particular if \(e = \Lambda\) it follows from (B.33) and correspondingly for the other regularity classes. In particular if \(e = \Lambda\), then a subbundle \(A\) of rank \(r\) is an endomorphism

\[\text{End}(\Lambda) = \text{Hom}(\Lambda^* \otimes \Lambda^*) = \text{Hom}(\Lambda^* \otimes \Lambda^*) = \text{Hom}(\Lambda^* \otimes \Lambda^*).

\[\text{(B.35)}\]

Hence that \(\phi \in \text{Hom}(\Lambda^* \otimes \Lambda^*)\); thus taking the transpose of the coordinate functions of a bundle homomorphism yields the natural isomorphism 

\[\text{Hom}(\Lambda^*) \cong \text{Hom}(\Lambda^* \otimes \Lambda),\]

\[\text{(B.36)}\]

\[\text{(B.37)}\]

Incidentally it follows from (B.31) that

\[\text{det}(\phi) = \text{det}(\phi(p)) \in \text{End}(\Lambda)\]

\[\text{Hom}(\Lambda^*) \cong \text{Hom}(\Lambda^* \otimes \Lambda^*),\]

\[\text{(B.33)}\]

\[\text{(B.32)}\]

\[\text{(B.31)}\]

\[\text{(B.30)}\]

\[\text{(B.29)}\]

\[\text{(B.28)}\]

\[\text{(B.27)}\]

\[\text{(B.26)}\]

\[\text{(B.25)}\]

\[\text{(B.24)}\]

\[\text{(B.23)}\]

\[\text{(B.22)}\]

\[\text{(B.21)}\]

\[\text{(B.20)}\]

\[\text{(B.19)}\]

\[\text{(B.18)}\]

\[\text{(B.17)}\]

\[\text{(B.16)}\]

\[\text{(B.15)}\]

\[\text{(B.14)}\]

\[\text{(B.13)}\]

\[\text{(B.12)}\]

\[\text{(B.11)}\]

\[\text{(B.10)}\]

\[\text{(B.9)}\]

\[\text{(B.8)}\]

\[\text{(B.7)}\]

\[\text{(B.6)}\]

\[\text{(B.5)}\]

\[\text{(B.4)}\]

\[\text{(B.3)}\]

\[\text{(B.2)}\]

\[\text{(B.1)}\]
where $\sigma_{\alpha\beta}(p) \in \text{Gl}(s, \mathbb{C})$ are the coordinate transition functions describing the subbundle $\sigma \subset \lambda$ and $\tau_{\alpha\beta}(p) \in \text{Gl}(r-s, \mathbb{C})$ are the coordinate transition functions describing a vector bundle of rank $r-s$ over $M$ that is called the quotient bundle and is denoted by $\tau = \lambda/\sigma$. The remaining entries of the matrix $\lambda_{\alpha\beta}(p)$ of course can be written $\sigma_{\alpha\beta}(p)x_{\alpha\beta}(p)$ for some $s \times (r-s)$ matrices $x_{\alpha\beta}(p)$, since the matrices $\sigma_{\alpha\beta}(p)$ are nonsingular. Conversely whenever the coordinate transition functions for a coordinate bundle $\lambda$ can be put into the form (B.37) where $0 < \text{rank } \sigma_{\alpha\beta} = s < r$ then the subset of $\lambda$ consisting of the first $s$ elements of the column vectors comprising the fibre $\mathbb{C}^r$ form a subbundle $\sigma \subset \lambda$. Clearly $\text{rank } \lambda = \text{rank } \sigma + \text{rank } \tau$ when $\sigma \subset \lambda$ and $\tau = \lambda/\sigma$.

A vector bundle $\lambda$ is said to be reducible if it contains a nontrivial subbundle, and otherwise is said to be irreducible; thus $\lambda$ is reducible precisely when its coordinate transition functions can be put into the form (B.37) nontrivially. On the other hand a vector bundle $\lambda$ is said to be decomposable if it is a nontrivial direct sum $\lambda = \sigma \oplus \tau$ of two other vector bundles, and otherwise is said to be indecomposable; thus $\lambda$ is decomposable precisely when its coordinate transition functions can be put into the form (B.37) in a nontrivial way and $x_{\alpha\beta} = 0$. For example the tensor product $\lambda \otimes \lambda$ of a vector bundle with itself is the direct sum of the subbundle of symmetric tensors and the subbundle of skew-symmetric tensors, hence $\lambda \otimes \lambda$ is decomposable. Reducibility and decomposability depend of course upon the regularity category being considered; for instance a reducible holomorphic vector bundle also is reducible when viewed as a $C^\infty$ vector bundle, but the converse is not necessarily true since there may be a $C^\infty$ equivalence of coordinate bundles exhibiting the reducibility of the vector bundle but not a holomorphic equivalence. Of course any decomposable bundle is reducible, or equivalently any irreducible bundle is indecomposable; but the converse is not always true.

A collection of vector bundles and bundle homomorphisms

\[
0 \rightarrow \sigma \xrightarrow{\phi} \lambda \xrightarrow{\psi} \tau \rightarrow 0
\]

is called a short exact sequence of vector bundles if its restriction to the fibres over any point is a short exact sequence of vector spaces and linear mappings. For example if the coordinate bundle of $\lambda$ has the form (B.37), so that $\sigma$ is a subbundle and $\tau = \lambda/\sigma$ is the quotient bundle, there is a short exact sequence of vector bundles (B.38) in which $\phi$ and $\psi$ are the bundle homomorphisms described by the local forms

\[
\phi_\alpha(p) = \begin{pmatrix} I_s \\ 0 \end{pmatrix}, \quad \psi_\alpha(p) = \begin{pmatrix} 0 \\ I_t \end{pmatrix}
\]

where $I_s$ is the $s \times s$ identity matrix, $I_t$ is the $t \times t$ identity matrix, $r = \text{rank } \lambda$, $s = \text{rank } \sigma$, $t = \text{rank } \tau$ and $r = s + t$. The homomorphism $\phi$ is the inclusion mapping of the subbundle $\sigma \subset \lambda$, and the homomorphism $\psi$ is the projection mapping to the quotient bundle $\tau$. On the other hand for any short exact sequence of vector bundles (B.38) the homomorphisms $\phi$ and $\psi$ both must be of maximal rank at each point, so by passing to equivalent coordinate bundles
they can be represented by coordinate functions of the form (B.39); and in that case the bundle $\lambda$ is described by a coordinate bundle of the form (B.37), so $\sigma$ is a subbundle of $\lambda$ and $\tau = \lambda/\sigma$ is the quotient bundle $\tau$. Thus the existence of a short exact sequence (B.38) is equivalent to the condition that $\sigma$ is a subbundle of $\lambda$ with quotient bundle $\tau = \lambda/\sigma$. Note that the dual of the exact sequence (B.38) is the exact sequence

$$0 \to \tau^* \xrightarrow{i^\psi} \lambda^* \xrightarrow{i^\phi} \sigma^* \to 0$$

exhibiting $\tau^*$ as a subbundle of $\lambda^*$ with $\sigma^* = \lambda^*/\tau^*$ as quotient bundle, as follows from the observation that

$$\lambda^*_{\alpha \beta} = \begin{pmatrix} \sigma^*_{\alpha \beta} & 0 \\ -\tau^*_{\alpha \beta} & \tau^*_{\alpha \beta} \end{pmatrix}.$$ 

This is an alternate form of the description (B.37) of a subbundle and quotient bundle.

The short exact sequence of vector bundles (B.38) can be viewed not just as expressing the reducibility of the vector bundle $\lambda$ but also as describing $\lambda$ as an extension of the subbundle $\sigma$ by the bundle $\tau$, thus as a new vector bundle formed by combining the bundles $\sigma$ and $\tau$. Two extensions $\lambda_1, \lambda_2$ of the bundle $\sigma$ by the bundle $\tau$ are called equivalent if there is a bundle homomorphism $\phi : \lambda_1 \to \lambda_2$ such that

$$0 \to \sigma \xrightarrow{1} \lambda_1 \xrightarrow{\phi} \tau \xrightarrow{1} 0$$

is a commutative diagram of short exact sequences, where $1$ denotes the identity homomorphism. It is easy to see from this diagram that $\phi$ is an isomorphism, and that equivalence in this sense is an equivalence relation in the usual sense. In particular when $\lambda = \sigma \oplus \tau$ the extension is said to be the trivial extension of vector bundles. The set of equivalence classes of extensions of a continuous vector bundle $\sigma$ by a continuous vector bundle $\tau$ is denoted by $\text{Ext}_C(\sigma, \tau)$; correspondingly the sets of equivalence classes of extensions of $C^\infty$, holomorphic, or flat vector bundles are denoted by $\text{Ext}_E(\sigma, \tau)$, $\text{Ext}_C(\sigma, \tau)$ or $\text{Ext}_F(\sigma, \tau)$. These sets have the natural structures of complex vector spaces arising from the following explicit descriptions.

**Theorem B.1** For any vector bundles $\sigma$ and $\tau$ on a topological space $M$ there is a canonical identification

$$\text{Ext}_C(\sigma, \tau) = H^1(M, \mathcal{C}(\sigma \otimes \tau^*)) .$$

If $M$ is a $C^\infty$ manifold and the bundles are $C^\infty$ bundles there is in addition the canonical identification

$$\text{Ext}_E(\sigma, \tau) = H^1(M, \mathcal{E}(\sigma \otimes \tau^*)) ;$$
if the manifold \( M \) and the bundles are holomorphic there is the further canonical identification
\[
\text{Ext}_\mathcal{O}(\sigma, \tau) = H^1(M, \mathcal{O}(\sigma \otimes \tau^*));
\]
and if the bundles are flat there is the canonical identification
\[
\text{Ext}_F(\sigma, \tau) = H^1(M, F(\sigma \otimes \tau^*)).
\]

**Proof:** The short exact sequence of vector bundles (B.38) expresses the condition that the vector bundle \( \lambda \) can be described by a coordinate bundle of the form (B.37), in which \( \sigma_{\alpha\beta} \) and \( \tau_{\alpha\beta} \) are coordinate bundles describing the vector bundles \( \sigma \) and \( \tau \) and the extension itself is described by the matrices \( x_{\alpha\beta} \). As before let \( r = \text{rank} \lambda, s = \text{rank} \sigma, \) and \( t = \text{rank} \tau, \) where \( r = s + t.\) In order that the matrices (B.37) satisfy the consistency conditions (B.17) to be a coordinate bundle the matrices \( x_{\alpha\beta} \) must be such that \( x_{\alpha\alpha} = 0 \) and
\[
\begin{pmatrix}
\sigma_{\alpha\beta} & \sigma_{\alpha\beta} x_{\alpha\beta} \\
0 & \tau_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
\sigma_{\beta\gamma} & \sigma_{\beta\gamma} x_{\beta\gamma} \\
0 & \tau_{\beta\gamma}
\end{pmatrix}
= \begin{pmatrix}
\sigma_{\alpha\gamma} & \sigma_{\alpha\gamma} x_{\alpha\gamma} \\
0 & \tau_{\alpha\gamma}
\end{pmatrix}
\]
in any intersection \( U_\alpha \cap U_\beta \cap U_\gamma, \) which is easily seen to be just the condition that \( \sigma_{\alpha\beta} \sigma_{\beta\gamma} x_{\beta\gamma} + \sigma_{\alpha\beta} x_{\alpha\beta} \tau_{\beta\gamma} = \sigma_{\alpha\gamma} x_{\alpha\gamma}, \) or alternatively that
\[
(B.43)
\]
\[x_{\alpha\gamma} = \sigma_{\alpha\beta} x_{\alpha\beta} \tau_{\beta\gamma} + x_{\beta\gamma};\]
and that is just the condition that the matrices \( x_{\alpha\beta} \) describe a one-cocycle
\[
(B.44)
\]
of the covering \( \mathcal{U} = \{U_\alpha\} \) with coefficients in the sheaf of germs of continuous cross-sections of the vector bundle \( \sigma \otimes \tau^* \), the condition for the vector bundle \( \sigma \otimes \tau^* \) paralleling the corresponding condition for line bundles as in (1.36). The same considerations of course apply to extensions of more restrictive regularity classes of vector bundles; for instance extensions of a holomorphic vector bundle \( \sigma \) by a holomorphic vector bundle \( \tau \) are holomorphic vector bundles \( \lambda \) described by cocycles in the group \( Z^1(\mathcal{U}, \mathcal{C}(\sigma \otimes \tau^*)) \). The extensions \( \lambda_1, \lambda_2 \) described by two cocycles \( x_{1\alpha\beta}, x_{2\alpha\beta} \in Z^1(\mathcal{U}, \mathcal{C}(\sigma \otimes \tau^*)) \) are equivalent if and only if there is a bundle homomorphism \( \phi : \lambda_1 \rightarrow \lambda_2 \) leading to a commutative diagram of exact sequences of the form (B.42). The homomorphism \( \phi \) can be described in a suitable refinement of the covering of \( M \) by coordinate functions \( \phi_\alpha, \) which must be of the form
\[
(B.45)
\]
for some matrices \( f_\alpha, \) since \( \phi \) induces the identity mapping on the subbundle \( \sigma \) and the quotient bundle \( \tau. \) The condition that these coordinate functions describe a bundle homomorphism \( \phi : \lambda_1 \rightarrow \lambda_2 \) is (B.28), which is equivalent to
\[
\begin{pmatrix}
I_s & f_\alpha \\
0 & I_t
\end{pmatrix}
\begin{pmatrix}
\sigma_{\alpha\beta} & \sigma_{\alpha\beta} x_{1\alpha\beta} \\
0 & \tau_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
I_s & f_\beta \\
0 & I_t
\end{pmatrix}
\]
\( \begin{pmatrix}
\sigma_{\alpha\beta} & \sigma_{\alpha\beta} x_{2\alpha\beta} \\
0 & \tau_{\alpha\beta}
\end{pmatrix} \)
in any intersection $U_\alpha \cap U_\beta$; and that amounts to the condition that the matrices $x_{\alpha \beta}$ satisfy

$$\sigma_{\alpha \beta} x_{1\alpha \beta} + f_{\alpha} \tau_{\alpha \beta} = \sigma_{\alpha \beta} f_{\beta} + \sigma_{\alpha \beta} x_{2\alpha \beta}$$

or equivalently

$$(B.46) \quad x_{1\alpha \beta} - x_{2\alpha \beta} = f_{\beta} - \sigma_{\beta \alpha} f_{\alpha} \tau_{\alpha \beta},$$

which is the condition that the cocycle $x_{2\alpha \beta} - x_{1\alpha \beta}$ is the coboundary of the cochain $f_\alpha \in C^0(\Omega, C(\sigma \otimes \tau^*))$, again the condition for the vector bundle $\sigma \otimes \tau^*$ paralleling the corresponding condition for line bundles as in (1.34). Thus the set of equivalence classes of extensions Ext($\sigma, \tau$) is in one-to-one correspondence with the cohomology classes in $H^1(M, C(\sigma \otimes \tau^*))$ represented by the cocycles $x_{\alpha \beta} \in Z^1(\Omega, C(\sigma \otimes \tau^*))$, and similarly for extensions of the more restrictive regularity classes.

That suffices to conclude the proof.

**Corollary B.2** For any vector bundles $\sigma, \tau$ on a topological manifold $M$

$$\text{Ext}_C(\sigma, \tau) = 0;$$

and if the bundles and the manifold are $C^\infty$ then

$$\text{Ext}_E(\sigma, \tau) = 0.$$

**Proof:** Since the sheaves $C(\sigma \otimes \tau^*)$ and $E(\sigma \otimes \tau^*)$ are fine sheaves

$$H^1(M, C(\sigma \otimes \tau^*)) = H^1(M, E(\sigma \otimes \tau^*)) = 0,$$

as in the discussion of the cohomology groups of fine sheaves on page 442 of Appendix C.2; the corollary is an immediate consequence of this observation and the preceding theorem.

For emphasis, and for convenience of reference, the preceding corollary can be restated equivalently as follows.

**Corollary B.3** Reducibility and decomposability are equivalent properties for continuous or $C^\infty$ vector bundles.

**Proof:** The preceding corollary shows that any reducible continuous or $C^\infty$ vector bundle is decomposable, while as noted earlier the converse always holds; that suffices for the proof.

It should be noted particularly though that distinct extension classes in Ext$_O(\sigma, \tau)$ can lead to analytically equivalent vector bundles. The simplest instance of this, which arises sufficiently often to merit a separate statement for purposes of reference, is the following.

**Lemma B.4** Two nontrivial extension classes $x, y \in \text{Ext}_O(\sigma, \tau)$ describe analytically equivalent holomorphic vector bundles whenever $y = cx$ for some nonzero complex constant $c \in \mathbb{C}$. 
Proof: This is an immediate consequence of the identity
\[
\begin{pmatrix} cI & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \sigma_{\alpha \beta} & \sigma_{\alpha \beta} x_{\alpha \beta} \\ 0 & \tau_{\alpha \beta} \end{pmatrix} \begin{pmatrix} c^{-1}I & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha \beta} & \sigma_{\alpha \beta} c x_{\alpha \beta} \\ 0 & \tau_{\alpha \beta} \end{pmatrix},
\]
which suffices for a proof.

It is not the case that a reducible holomorphic or flat vector bundle is necessarily decomposable, as will become evident as the discussion continues. It is particularly useful to have available some simple tests to see whether a particular extension of holomorphic vector bundles is trivial or not. Note that if the short exact sequence (B.38) is the trivial extension, so that \( \lambda = \sigma \oplus \tau \), there is also the bundle homomorphism \( \theta : \tau \rightarrow \lambda \) described by the coordinate functions
\[
(B.47) \quad \theta_\alpha(p) = \begin{pmatrix} 0 \\ I_t \end{pmatrix};
\]
and the composition \( \psi \theta : \tau \rightarrow \tau \) is the identity homomorphism. A short exact sequence (B.38) is said to \textit{split} if there is a homomorphism \( \theta : \tau \rightarrow \lambda \) such that \( \psi \theta = I \) is the identity homomorphism; thus if (B.38) is a trivial extension then the short exact sequence splits.

**Theorem B.5** A short exact sequence of holomorphic vector bundles
\[
(B.48) \quad 0 \rightarrow \sigma \stackrel{\phi}{\rightarrow} \lambda \stackrel{\psi}{\rightarrow} \tau \rightarrow 0
\]
splits if and only if \( \lambda = \sigma \oplus \tau \).

Proof: It has been noted already that for the trivial extension \( \lambda = \sigma \oplus \tau \) the short exact sequence (B.48) splits. Conversely suppose that the short exact sequence (B.48) splits, so that there is a bundle homomorphism \( \theta : \tau \rightarrow \lambda \) for which the composition \( \psi \theta : \tau \rightarrow \tau \) is the identity homomorphism, and let \( r = \text{rank } \lambda, \ s = \text{rank } \sigma, \) and \( t = \text{rank } \tau \) so that \( r = s + t \). When the vector bundle \( \lambda \) is described by a coordinate bundle \( \lambda_{\alpha \beta} \) of the form (B.37) the coordinate functions of the bundle homomorphisms \( \phi \) and \( \psi \) have the form (B.39); and since \( \psi \theta = I_t \) the coordinate functions of the homomorphism \( \theta \) must have the form
\[
\theta_\alpha = \begin{pmatrix} \theta'_\alpha \\ I_t \end{pmatrix}
\]
where \( \theta'_\alpha \) is an \( s \times t \) matrix. These matrices satisfy
\[
\begin{pmatrix} \theta'_\alpha \\ I_t \end{pmatrix} \tau_{\alpha \beta} = \begin{pmatrix} \sigma_{\alpha \beta} & \sigma_{\alpha \beta} x_{\alpha \beta} \\ 0 & \tau_{\alpha \beta} \end{pmatrix} \begin{pmatrix} \theta'_\beta \\ I_t \end{pmatrix},
\]
so \( \theta'_\alpha \tau_{\alpha \beta} = \sigma_{\alpha \beta} \theta'_\beta + \sigma_{\alpha \beta} x_{\alpha \beta} \) or equivalently
\[
(B.49) \quad x_{\alpha \beta} = \sigma_{\beta \alpha} \theta'_\alpha \tau_{\alpha \beta} - \theta'_\beta.
\]
That is just the condition that the cocycle $x_{\alpha\beta} \in Z^2(M, \mathcal{O}(\sigma \otimes \tau^*))$ is the coboundary of the cochain $\theta'_\alpha \in C^1(M, \mathcal{O}(\sigma \otimes \tau^*))$ as in (B.46), hence that this cocycle represents the trivial cohomology class in $H^2(M, \mathcal{O}(\sigma \otimes \tau^*))$; and by Theorem B.1 that is just the condition that the extension is trivial, hence that $\lambda = \sigma \oplus \tau$. That concludes the proof.

There is another sometimes useful interpretation of the cohomology class $x \in H^1(M, \mathcal{O}(\sigma \otimes \tau^*))$ describing an extension of holomorphic vector bundles. Associated to the short exact sequence of holomorphic vector bundles (B.38) describing this extension is the exact sequence
\[(B.50) \quad 0 \longrightarrow \mathcal{O}(\sigma) \xrightarrow{\phi} \mathcal{O}(\lambda) \xrightarrow{\psi} \mathcal{O}(\tau) \longrightarrow 0\]
of sheaves of germs of holomorphic cross-sections of these bundles, since over a sufficiently small coordinate neighborhood the bundle $\lambda$ is the direct sum of the bundles $\sigma$ and $\tau$. The exact cohomology sequence associated to this exact sequence of sheaves includes the segment
\[(B.51) \quad 0 \longrightarrow \Gamma(M, \mathcal{O}(\sigma)) \xrightarrow{\phi} \Gamma(M, \mathcal{O}(\lambda)) \xrightarrow{\psi} \Gamma(M, \mathcal{O}(\tau)) \xrightarrow{\delta} H^1(M, \mathcal{O}(\sigma)).\]

**Theorem B.6** If $\lambda$ is a holomorphic vector bundle over a complex manifold $M$ and $\lambda$ is the extension of a vector bundle $\sigma$ by a vector bundle $\tau$ described by a cohomology class $x \in \text{Ext}(\sigma, \tau) = H^1(M, \mathcal{O}(\sigma \otimes \tau^*))$ then multiplication by this cohomology class $x$ yields a homomorphism
\[(B.52) \quad x : \Gamma(M, \mathcal{O}(\tau)) \longrightarrow H^1(M, \mathcal{O}(\sigma))\]
that is precisely the coboundary mapping $\delta$ in the exact cohomology sequence (B.51); so if $K \subset \Gamma(M, \mathcal{O}(\tau))$ is the kernel of the homomorphism (B.52) then
\[(B.53) \quad \gamma(\lambda) = \gamma(\sigma) + \dim K.\]

**Proof:** Suppose that the vector bundle $\lambda$ is described by a coordinate bundle of the form (B.37) for a covering $\mathcal{U}$ of the surface $M$. For any holomorphic cross-section $f_{2\alpha} \in \Gamma(M, \mathcal{O}(\tau))$ of the vector bundle $\tau$ it follows from the cocycle condition (B.43) that
\[x_{\alpha\gamma} f_{2\gamma} = \sigma_{\alpha\beta} x_{\alpha\beta} f_{2\beta} + x_{\beta\gamma} f_{2\beta},\]
which is just the condition that the products $x_{\alpha\beta} f_{2\beta}$ describe a cocycle in $Z^1(\mathcal{U}, \mathcal{O}(\sigma))$; thus multiplication by the matrices $x_{\alpha\beta}$ determines a homomorphism (B.52). A cross-section $f_{2\alpha} \in \Gamma(M, \mathcal{O}(\tau))$ is in the kernel $K$ of this homomorphism if and only if the cocycle $x_{\alpha\beta} f_{2\beta}$ is a coboundary, so if and only if after a refinement of the covering if necessary there will be holomorphic functions $f_{1\alpha}$ in the open sets of the covering $\mathcal{U}$ such that
\[\sigma_{\beta\alpha} f_{1\alpha} - f_{1\beta} = x_{\alpha\beta} f_{2\beta}\]
as in (B.46). This condition, together with the condition that the functions \( f_{2\alpha} \) are a cross-section of the bundle \( \tau \), are easily seen to amount to the condition that
\[
\begin{pmatrix}
  f_{1\alpha} \\
  f_{2\alpha}
\end{pmatrix} =
\begin{pmatrix}
  \sigma_{\alpha\beta} & \sigma_{\alpha\beta} x_{\alpha\beta} \\
  0 & \tau_{\alpha\beta}
\end{pmatrix}
\begin{pmatrix}
  f_{1\beta} \\
  f_{2\beta}
\end{pmatrix},
\]
which in turn is just the condition that
\[
f_{\alpha} = \begin{pmatrix} f_{1\alpha} \\ f_{2\alpha} \end{pmatrix} \in \Gamma(U_{\alpha}, \mathcal{O}(\lambda)).
\]
Thus a cross-section \( f_{2\alpha} \in \Gamma(M, \mathcal{O}(\tau)) \) is in the kernel \( K \) if and only if it is the image of a cross-section \( f_{\alpha} \in \Gamma(U_{\alpha}, \mathcal{O}(\lambda)) \) under the inclusion mapping \( \phi \); that is precisely the condition satisfied by the coboundary mapping in the exact cohomology sequence (B.51), which identifies the homomorphism (B.52) with that coboundary mapping. The sequence (B.51) yields the exact sequence
\[
0 \rightarrow \Gamma(M, \mathcal{O}(\sigma)) \xrightarrow{\phi} \Gamma(M, \mathcal{O}(\lambda)) \xrightarrow{\psi} K \rightarrow 0,
\]
from which it follows immediately that \( \gamma(\lambda) = \gamma(\sigma) + \dim K \). That suffices to conclude the proof.

For some purposes the particular extensions involved in building up a vector bundle from bundles of smaller ranks are not relevant; and for these purposes it is convenient to introduce another construction. For any complex manifold \( M \) let \( V(M) \) be the free abelian group generated by all holomorphic vector bundles over \( M \), and let \( V_0(M) \subset V(M) \) be the subgroup generated by the expressions \( \lambda - \sigma - \tau \) whenever \( \lambda, \sigma, \tau \) are holomorphic vector bundles for which there is a short exact sequence
\[
0 \rightarrow \sigma \rightarrow \lambda \rightarrow \tau \rightarrow 0. \tag{B.54}
\]
The quotient group \( V(M)/V_0(M) = K(M) \) is called the Grothendieck group of holomorphic vector bundles of the manifold \( M \). For any exact sequence of holomorphic vector bundles (B.54) it is evident that \( \det \lambda = (\det \sigma)(\det \tau) \); thus if the operation of taking the determinant line bundle of a holomorphic vector bundle is extended to a homomorphism \( \det : V(M) \rightarrow H^1(M, \mathcal{O}^*) \) by setting
\[
\det(\lambda_1 + \cdots + \lambda_n) = (\det \lambda_1) \cdots (\det \lambda_n)
\]
then this homomorphism is trivial on the subgroup \( V_0(M) \subset V(M) \) and consequently induces a homomorphism
\[
\det : K(M) \rightarrow H^1(M, \mathcal{O}^*).
\]
It is thus possible to define the determinant line bundle of an arbitrary element in the Grothendieck group \( K(M) \). At least some other constructions for vector bundles also can be extended to the Grothendieck group; but the further discussion discussion of this topic will be deferred.
If \( \phi : \sigma \rightarrow \lambda \) is a homomorphism of vector bundles over a manifold \( M \) the kernel of \( \phi \) is the union of the kernels of the linear mappings \( \phi_p : \sigma_p \rightarrow \lambda_p \) on the fibres of these bundles over all the points \( p \in M \). The kernel of \( \phi \) is a well defined subset of the vector bundle \( \sigma \) and is a linear subspace of each fibre of \( \sigma \); but if the rank of the homomorphism \( \phi \) is not constant the dimensions of these linear subspaces may vary with the point \( p \in M \), so the kernel cannot be a subbundle of \( \sigma \). The image of \( \phi \) correspondingly is the subset \( \phi(\sigma) \subset \lambda \), which is a well defined subset of the vector bundle \( \lambda \) and is a linear subspace of each fibre of \( \lambda \); but again if the rank of the homomorphism \( \phi \) is not constant then this subset too may not be a subbundle of \( \lambda \).

**Lemma B.7** If \( F : U \rightarrow \mathbb{C}^{r \times s} \) is a continuous, \( C^\infty \), holomorphic, or locally constant mapping from an open neighborhood \( U \subset \mathbb{C}^n \) of the origin in the space \( \mathbb{C}^n \) to the space of \( r \times s \) complex matrices, and if \( \text{rank } F(z) = t \) at all points \( z \in U \), then in an open subneighborhood \( V \subset U \) of the origin there are mappings \( A : V \rightarrow \text{Gl}(r, \mathbb{C}) \) and \( B : V \rightarrow \text{Gl}(s, \mathbb{C}) \) such that

\[
A(z)F(z)B(z) = \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix}
\]

at all points \( z \in V \), where \( I_t \) is the \( t \times t \) identity matrix; and these mappings have the same regularity properties as the mapping \( F \).

**Proof:** Multiplying the matrix \( F(z) \) on the right by a matrix \( B(z) \) has the effect of replacing the columns of the matrix \( F(z) \) by linear combinations of those columns with coefficients from the matrix \( B \). By multiplying on the right by a nonsingular constant matrix it can be arranged that the first \( t \) columns of the matrix \( F(z) \) are of rank \( t \) at the origin; and they remain of rank \( t \) at all points of a sufficiently small open neighborhood \( V \) of the origin. By then multiplying on the right by another nonsingular matrix, which has the effect of subtracting the appropriate linear combinations of the first \( t \) columns from the last \( s - t \) columns, it can be arranged that the last \( s - t \) columns of the matrix \( F(z) \) vanish; the coefficients of these linear combinations are determined explicitly by Cramer’s rule, so are continuous, \( C^\infty \), holomorphic or locally constant according to the regularity of the entries of the matrix \( F(z) \). Multiplying the matrix \( F(z) \) on the left by a matrix \( A(z) \) has the corresponding effect on the rows of \( F(z) \), so it can be arranged similarly that the last \( r - t \) rows of the matrix \( F(z) \) also vanish. The leading \( t \times t \) block of the resulting matrix, consisting of the only nonzero terms in this matrix, then is of rank \( t \) throughout \( V \); so by multiplying on the left or right by another nonsingular matrix that block can be reduced to the identity matrix of rank \( t \) as asserted, which suffices to conclude the proof.

**Theorem B.8** If \( \phi : \sigma \rightarrow \lambda \) is a homomorphism of constant rank \( t \) between two vector bundles over a manifold \( M \) there is a commutative diagram of short
exact sequences of vector bundles and bundle homomorphisms of the form

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \lambda_1 & \xrightarrow{\rho_1} & \lambda & \xrightarrow{\rho_2} & \lambda_2 & \longrightarrow & 0 \\
\end{array}
\]

(B.55)

in which \( \text{rank} \lambda_1 = \text{rank} \sigma_1 = t \) and the bundle homomorphism \( \theta \) is an isomorphism. The bundles and bundle homomorphisms in this diagram are \( \mathcal{C}^\infty \), holomorphic, or flat if the initial bundles have those regularity properties.

**Proof:** If the bundles \( \sigma \) and \( \lambda \) are described by coordinate bundles \( \{ U_\alpha, \sigma_{\alpha\beta} \} \) and \( \{ U_\alpha, \lambda_{\alpha\beta} \} \) the local form of the homomorphism \( \phi \) is described by the matrix functions \( \phi_\alpha : U_\alpha \rightarrow \mathbb{C}^{r \times s} \) such that \( \phi_\alpha(p)\sigma_{\alpha\beta}(p) = \lambda_{\alpha\beta}(p)\phi_\beta(p) \) at all points \( p \in U_\alpha \cap U_\beta \), as in (B.28), and the matrices \( \phi_\alpha \) are all of constant rank \( t \).

It follows from the preceding lemma that after passing to a refinement of the covering if necessary there are mappings \( A_\alpha : U_\alpha \rightarrow \text{Gl}(r, \mathbb{C}) \) and \( B_\alpha : U_\alpha \rightarrow \text{Gl}(s, \mathbb{C}) \) such that \( A_\alpha(p)\phi_\alpha(p)B_\alpha(p) = \psi_\alpha(p) \) for all points \( p \in U_\alpha \) where

\[
\psi_\alpha(p) = \begin{pmatrix} I_t & 0 \\ 0 & 0 \end{pmatrix},
\]

in which \( I_t \) is the identity matrix of rank \( t \); and the mappings \( A_\alpha \) and \( B_\alpha \) have the same regularity as the mapping \( \phi_\alpha \) and the bundles \( \lambda \) and \( \sigma \). Then

\[
A_\alpha(p)\phi_\alpha(p)B_\alpha(p)^{-1}\sigma_{\alpha\beta}(p)B_\beta(p) = A_\alpha(p)\lambda_{\alpha\beta}(p)A_\beta(p)^{-1}A_\beta(p)\phi_\beta(p)B_\beta(p)
\]

or equivalently

\[
\psi_\alpha(p)\tilde{\sigma}_{\alpha\beta}(p) = \tilde{\lambda}_{\alpha\beta}(p)\psi_\beta(p)
\]

for all points \( p \in U_\alpha \cap U_\beta \), where \( \tilde{\sigma}_{\alpha\beta}(p) = B_\alpha(p)^{-1}\sigma_{\alpha\beta}(p)B_\beta(p) \) and \( \tilde{\lambda}_{\alpha\beta}(p) = A_\alpha(p)\lambda_{\alpha\beta}(p)A_\beta(p)^{-1} \); thus the vector bundles \( \sigma \) and \( \lambda \) can be described by the coordinate bundles \( \tilde{\sigma}_{\alpha\beta} \) and \( \tilde{\lambda}_{\alpha\beta} \), and the homomorphism \( \phi \) by the coordinate functions \( \psi_\alpha \). When the coordinate bundles \( \tilde{\sigma}_{\alpha\beta} \) and \( \tilde{\lambda}_{\alpha\beta} \) are decomposed into matrix blocks corresponding to the decomposition of the matrices \( \psi_\alpha \) then

\[
\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{11\alpha\beta} & \sigma_{12\alpha\beta} \\ \sigma_{21\alpha\beta} & \sigma_{22\alpha\beta} \end{pmatrix} = \begin{pmatrix} \lambda_{11\alpha\beta} & \lambda_{12\alpha\beta} \\ \lambda_{21\alpha\beta} & \lambda_{22\alpha\beta} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},
\]

and consequently

(B.56)

\[
\sigma_{11\alpha\beta} = \lambda_{11\alpha\beta} \quad \text{and} \quad \sigma_{12\alpha\beta} = \lambda_{21\alpha\beta};
\]

so the coordinate transition functions for the two bundles have the form

\[
\tilde{\sigma}_{\alpha\beta} = \begin{pmatrix} \sigma_{11\alpha\beta} & 0 \\ \sigma_{21\alpha\beta} & \sigma_{22\alpha\beta} \end{pmatrix}, \quad \tilde{\lambda}_{\alpha\beta} = \begin{pmatrix} \lambda_{11\alpha\beta} & \lambda_{12\alpha\beta} \\ 0 & \lambda_{22\alpha\beta} \end{pmatrix}.
\]

Thus the vector bundles \( \sigma \) and \( \lambda \) are reducible, and there are short exact sequences

\[
0 \longrightarrow \sigma_2 \longrightarrow \sigma \longrightarrow \sigma_1 \longrightarrow 0
\]
in which the vector bundles $\sigma_i$ are described by the coordinate bundles $\sigma_{i\alpha\beta}$ and the vector bundles $\lambda_i$ are described by the coordinate bundles $\lambda_{i\alpha\beta}$. Moreover the bundle homomorphism $\theta : \sigma_1 \rightarrow \lambda_1$ defined by the identity mapping is an isomorphism and is just the homomorphism $\phi$ applied to the quotient bundle $\sigma_1 \subset \sigma$ with image contained in the subbundle $\lambda_1 \subset \lambda$; that yields the commutative diagram of the theorem, and suffices for the proof.

The mapping $\theta : \sigma_1 \rightarrow \lambda_1$ in the preceding theorem is a homomorphism between two vector bundles of the same rank, and consequently its determinant is a well defined cross-section $\det \theta \in \Gamma(M, \mathcal{O}(\det \lambda_1)(\det \sigma_1)^{-1})$: this cross-section is called the determinant of the initial bundle homomorphism $\phi : \sigma \rightarrow \lambda$, and as such is denoted by $\det \phi$. It is worth pointing out explicitly that if rank $\sigma = \text{rank} \phi$ in the commutative diagram (B.55) then $\sigma_2 = 0$ and the second line reduces to the assertion that $\sigma_1 \cong \sigma$; correspondingly if rank $\lambda = \text{rank} \phi$ then $\lambda_2 = 0$ and the first line reduces to the assertion that $\lambda_1 \cong \lambda$. Of course if rank $\sigma = \text{rank} \lambda = \text{rank} \phi$ the theorem is rather vacuous. The theorem is most useful in the following form.

**Corollary B.9** If $\phi : \sigma \rightarrow \lambda$ is a homomorphism of constant rank $t$ between vector bundles $\sigma, \lambda$ over a manifold $M$, where rank $\sigma = s$ and rank $\lambda = r$, there is an exact sequence of vector bundles

\[
0 \rightarrow \sigma_2 \xrightarrow{\rho_2^3} \sigma \xrightarrow{\phi} \lambda \xrightarrow{\rho_2^2} \lambda_2 \rightarrow 0
\]

over $M$, where rank $\sigma_2 = s - t$ and rank $\lambda_2 = r - t$. The bundles and bundle homomorphisms are $C^\infty$, holomorphic, or flat if the initial bundles and bundle homomorphisms have those regularity properties.

**Proof:** This follows from the preceding theorem by a chase through the diagram (B.55). From the top short exact sequence it follows that rank $\lambda_2 = \text{rank} \lambda - \text{rank} \lambda_1 = r - t$, and from the bottom exact sequence it follows that rank $\sigma_2 = \text{rank} \sigma - \text{rank} \sigma_1 = s - t$. From these two exact sequences it also follows that $\rho_2^3$ is injective and $\rho_2$ is surjective. From the commutativity of (B.55) it follows that $\rho_2 \cdot \rho_2^3 = \rho_1 \cdot \theta \cdot \rho_1^3 \cdot \rho_2^2 = 0$ since $\rho_1^3 \cdot \rho_2^2 = 0$. If $s \in \sigma$ and $\phi(s) = 0$ then from the commutativity of (B.55) again $0 = \phi(s) = \rho_1 \cdot \theta \cdot \rho_1^3(s)$; since $\theta$ and $\rho_1$ are injective necessarily $\rho_1^3(s) = 0$ and hence $s = \rho_2^3(s_2)$ for some $s_2 \in \sigma_2$. So (B.57) is exact at the bundle $\sigma$. From the commutativity of (B.55) yet again $\rho_2 \cdot \phi = \rho_2 \cdot \rho_1 \cdot \theta \cdot \rho_1^3 = 0$ since $\rho_2 \cdot \rho_1 = 0$. Finally if $t \in \lambda$ and $\rho_2(t) = 0$ then $t = \rho_1(t_1)$ for some $t_1 \in \lambda_1$; since $\theta$ and $\rho_1^3$ are surjective necessarily $t_1 = \theta \cdot \rho_1^3(s)$ for some $s \in \sigma$ and $t = \rho_1 \cdot \theta \cdot \rho_1^3(s) = \phi(s)$, so (B.57) is exact at the bundle $\lambda$, and that concludes the proof.

**Corollary B.10** The kernel and image of a continuous, $C^\infty$, holomorphic or flat vector bundle homomorphism of constant rank are both subbundles of the same regularity class.
Proof: In the exact sequence (B.57) of the preceding corollary the kernel of the homomorphism $\phi : \sigma \rightarrow \lambda$ is the vector bundle $\sigma_2$, so the kernel of a bundle homomorphism of constant rank is a subbundle; and the image of $\phi$ is the kernel of $\rho_2$, so it is a subbundle by the first part of the proof of the present corollary, and that concludes the proof.
Appendix C

Sheaves

C.1 General Properties

Sheaves\(^1\) were introduced into complex analysis in the early 1950’s, in part to provide a tool for passing systematically from local to global results and in part to handle more readily some of the rather complicated semi-local properties of holomorphic functions of several variables. A \textit{sheaf} of abelian groups over a topological space \(M\) is a topological space \(S\) with a mapping \(\pi : S \rightarrow M\) such that: (i) \(\pi\) is a surjective local homeomorphism; (ii) for each point \(p \in M\) the inverse image \(\pi^{-1}(p) \subset S\) has the structure of an abelian group; and (iii) the group operations are continuous in the topology of \(S\). To clarify condition (iii), the product \(S \times S\) of a sheaf \(S\) with itself can be given the product topology, and the subset \(S \times_{\pi} S\) consisting of those points \((s_1, s_2)\) such that \(\pi(s_1) = \pi(s_2)\) inherits a topology as a subset of \(S \times S\); the mapping that takes a point \((s_1, s_2) \in S \times_{\pi} S\) to the point \(s_1 - s_2 \in S\) is a well defined mapping \(S \times_{\pi} S \rightarrow S\) between two topological spaces, and (iii) is just the condition that this mapping is continuous. There are of course analogous definitions for sheaves of rings or of other algebraic structures. It is convenient to speak of a sheaf without specifying the algebraic structure when the particular structure is not relevant, and to speak of a sheaf of abelian groups or of another special algebraic structure when that structure is of particular significance. The space \(M\) is called the \textit{base space} of the sheaf, the mapping \(\pi\) is called the \textit{projection}, and the subset \(\pi^{-1}(p) = S_p\) is called the \textit{stalk} over the point \(p \in M\). The simplest example of a sheaf of groups is a \textit{product sheaf} or \textit{trivial sheaf} over \(M\), the Cartesian product \(S = M \times G\) of the space \(M\) and a discrete group \(G\) with the product topology.

and the projection mapping $\pi : M \times G \to M$ to the first factor; the product sheaf with the group $G$ usually is denoted just by $G$, and when $G$ is the zero group it is called the zero sheaf and is denoted by 0. If $N \subseteq M$ the restriction $S|N$ of a sheaf $S$ of groups over $M$ to the subset $N$ is clearly a sheaf of groups over $N$; in particular the restriction of $S$ to a point $p \in M$ is just the stalk $S|p = S_p$ of the sheaf $S$ over $p$. A section of a sheaf $S$ over a subset $U \subseteq M$ of its base $M$ is a continuous mapping $s : U \to S$ such that the composition $\pi \circ s : U \to U$ is the identity mapping; the set of all sections of $S$ over $U$ is denoted by $\Gamma(U, S)$. By condition (i) in the definition of a sheaf it follows that for any point $s \in S$ there is an open neighborhood $V$ of $s$ in $S$ such that the restriction $\pi|V$ of the projection to that set is a homeomorphism between $V \subseteq S$ and an open subset $U \subseteq M$; the inverse of the restriction $\pi|V$ then is a section of the sheaf $S$ over $U$, so there is a section of the sheaf $S$ through any point $s \in S$ and the images of sections over the open subsets of $M$ form a basis for the topology of $S$. Any two sections through a point $s \in S$ coincide locally with the inverse of the projection mapping $\pi$, so any two sections of the sheaf $S$ that agree at a point $p \in M$ necessarily agree in a full open neighborhood of $p$ in $M$. By condition (iii) in the definition of a sheaf it follows that for any sections $s_1, s_2 \in \Gamma(U, S)$ the mapping that associates to a point $p \in U$ the difference $s_1(p) - s_2(p) \in S_p$ also is a section; thus the set $\Gamma(U, S)$ of sections of $S$ over any subset $U \subseteq M$ has the natural structure of an abelian group, and the corresponding result holds for sheaves of other algebraic structures.

A sheaf over a topological space $M$ is described fully by the collection of its sections over the open subsets of $M$; indeed that is one of the standard ways in which to construct a sheaf. To make this more precise, a presheaf $\{S_U, \rho_{V,U}\}$ of abelian groups over a topological space $M$ is a collection (i) of abelian groups $S_U$ indexed by the open subsets $U \subseteq M$, with $S_\emptyset = 0$, and (ii) of group homomorphisms $\rho_{V,U} : S_U \to S_V$ indexed by pairs $V \subseteq U$ of open subsets of $M$ such that $\rho_{V,V}$ is the identity mapping and $\rho_{W,U} \rho_{V,U} = \rho_{W,V}$ whenever $W \subseteq V \subseteq U$. There are analogous definitions for presheaves of other algebraic structures; and as in the case of sheaves it is convenient to speak of a presheaf without specifying the algebraic structure when the particular structure is not relevant, and to speak of a presheaf of abelian groups or of another special algebraic structure when that structure is of particular significance. The set of sections $S_p = \Gamma(U, S)$ of a sheaf $S$ over the open subsets $U \subseteq M$ with the natural restrictions $\rho_{V,U}$ of sections over a set $U$ to a subset $V \subseteq U$ clearly form a presheaf, which is called the associated presheaf of the sheaf. Conversely to any presheaf $\{S_U, \rho_{V,U}\}$ over $M$ there is an associated sheaf constructed as follows. For any point $p \in M$ let $U_p$ be the collection of all open subsets $U \subseteq M$ that contain $p$ and let $S^*_p$ be the disjoint union of the sets $S_U$ for all $U \in U_p$. Two elements $s_U \in S_U$ and $s_V \in S_V$ are considered to be equivalent if there is a subset $W \subseteq M$ such that $p \in W \subseteq U \cap V$ and $\rho_{W,V}(s_V) = \rho_{W,U}(s_U)$; it is easy to see that this is an equivalence relation in the usual sense. The set $S^*_p$ of equivalence classes of elements in $S^*_p$ is a well defined group, known as the direct limit of the partially ordered collection of groups $S_U$. Let $S = \bigcup_{p \in M} S_p$ be the union of these groups and $\pi : S \to M$ be the mapping for which $\pi(S_p) = p$;
and introduce on $\mathcal{S}$ the topology for which the images in $\mathcal{S}$ of the elements $s_u \in \mathcal{S}_u$ for all the open subsets $U \subset M$ are a basis for the open subsets of $\mathcal{S}$. It is straightforward to verify that $\mathcal{S}$ is a sheaf over $M$ with the projection $\pi$ and with the same algebraic structure as that of the presheaf. It may be the case that the sheaf associated to a nontrivial presheaf is the zero sheaf, as for instance when all the homomorphisms $\rho_{V,U}$ are the zero mappings; so some conditions must be imposed on presheaves to ensure that they determine interesting sheaves. A presheaf $\{\mathcal{S}_U, \rho_{V,U}\}$ over $M$ is called a complete presheaf provided that whenever an open subset $U \subset M$ is covered by open subsets $U_\alpha \subset M$ (i) if $\rho_{U_\alpha,U}(s) = 0$ for an element $s \in \mathcal{S}_U$ and all subsets $U_\alpha$ then $s = 0$; and (ii) if $\rho_{U_\alpha \cap U_\beta, U_\alpha}(s_\alpha) = \rho_{U_\alpha \cap U_\beta, U_\beta}(s_\beta)$ for some elements $s_\alpha \in \mathcal{S}_{U_\alpha}$ and all pairs of subsets $U_\alpha$, $U_\beta$ then there is an element $s \in \mathcal{S}_U$ such that $s_\alpha = \rho_{U_\alpha,U}(s)$ for all $U_\alpha$. It is evident that the presheaf of sections of a sheaf is a complete presheaf; and it is straightforward to verify that a complete presheaf can be identified naturally with the presheaf of sections of its associated sheaf.

As examples of particular interest here, the collection of rings $\mathcal{O}_U$ of meromorphic functions in the open subsets $U \subset \mathbb{C}^n$ clearly form a complete presheaf of rings over $\mathbb{C}^n$; the associated sheaf, denoted by $\mathcal{O}$, is called the sheaf of germs of holomorphic functions over $\mathbb{C}^n$, and there is the natural identification $\mathcal{O}_U \cong \Gamma(U, \mathcal{O})$ for any open subset $U \subset \mathbb{C}^n$. Similarly the collection of fields $\mathcal{M}_U$ of meromorphic functions is a complete presheaf of fields over $\mathbb{C}^n$; the associated sheaf, denoted by $\mathcal{M}$, is called the sheaf of germs of meromorphic functions over $\mathbb{C}^n$, and there is the natural identification $\mathcal{M}_U \cong \Gamma(U, \mathcal{M})$ for any open subset $U \subset \mathbb{C}^n$. The sheaf $\mathcal{E}$ of germs of $\mathcal{C}^\infty$ functions are sheaves of rings defined correspondingly, and the sheaves $\mathcal{E}^{(p,q)}$ of germs of $\mathcal{C}^\infty$ complex valued differential forms of type $(p,q)$ are sheaves of abelian groups. All of these sheaves are defined purely locally, so can be considered as sheaves over arbitrary complex manifolds as well as over subsets of $\mathbb{C}^n$.

A subsheaf of a sheaf $\mathcal{S}$ of abelian groups over a topological space $M$ is an open subset $\mathcal{R} \subset \mathcal{S}$ such that $\mathcal{R}_p = \mathcal{R} \cap \mathcal{S}_p$ is a subgroup of $\mathcal{S}_p$ for each point $p \in M$; a subsheaf of a sheaf of abelian groups over $M$ clearly is itself a sheaf of abelian groups over $M$, and subsheaves of sheaves of other algebraic structures are defined correspondingly. If $\mathcal{R}$ is a subsheaf of a sheaf $\mathcal{S}$ of abelian groups over $M$ the quotient groups $\mathcal{S}_p/\mathcal{R}_p$ are well defined for each point $p \in M$ and the union $\mathcal{T} = \bigcup_{p \in M} \mathcal{S}_p/\mathcal{R}_p$ with the natural quotient topology is another sheaf of abelian groups over $M$ called the quotient sheaf and denoted by $\mathcal{S}/\mathcal{R}$. A homomorphism between two sheaves $\mathcal{R}$ and $\mathcal{S}$ of abelian groups over the same base space $M$ is a continuous mapping $\phi : \mathcal{R} \rightarrow \mathcal{S}$ that commutes with the projections of the two sheaves, so that $\phi(\mathcal{R}_p) \subset \mathcal{S}_p$ for each point $p \in M$, and that restricts to group homomorphisms $\phi|_{\mathcal{R}_p} : \mathcal{R}_p \rightarrow \mathcal{S}_p$ between the stalks of the two sheaves at all points $p \in M$. It is a straightforward matter to verify that a homomorphism between the two sheaves is always an open mapping. The kernel of a homomorphism $\phi : \mathcal{R} \rightarrow \mathcal{S}$ between two sheaves of abelian groups is the union of the kernels of the homomorphisms $\phi|_{\mathcal{R}_p} : \mathcal{R}_p \rightarrow \mathcal{S}_p$ between the
stalks of the sheaves, and easily is seen to be a subsheaf of the sheaf $\mathcal{R}$. The image of the homomorphism $\phi$ is the union of the images of the homomorphisms $\phi: \mathcal{R}_p \rightarrow \mathcal{S}_p$ between the stalks of the sheaves, and also easily is seen to be a subsheaf of the sheaf $\mathcal{S}$ since a sheaf homomorphism is an open mapping. An isomorphism between two sheaves of abelian groups is a homomorphism with an inverse that is also a homomorphism; a homomorphism $\phi: \mathcal{R} \rightarrow \mathcal{S}$ is an isomorphism if and only if it is injective, has trivial kernel, and is surjective, has the full sheaf $\mathcal{S}$ as its image. The inclusion mapping $\iota: \mathcal{R} \rightarrow \mathcal{S}$ of a subsheaf $\mathcal{R} \subset \mathcal{S}$ of abelian groups into the sheaf $\mathcal{S}$ is injective, and the natural homomorphism from the sheaf $\mathcal{S}$ to the quotient sheaf $\mathcal{S}/\mathcal{R}$ is surjective. A sequence

$$\phi_{n-2}: \mathcal{S}_{n-1} \xrightarrow{\phi_{n-1}} \mathcal{S}_n \xrightarrow{\phi_n} \mathcal{S}_{n+1}$$

of sheaves of abelian groups and homomorphisms is an exact sequence if for each $n$ the image of $\phi_{n-1}$ is precisely the kernel of $\phi_n$; a short exact sequence of sheaves of abelian groups is an exact sequence of the form

$$(C.1) \quad 0 \rightarrow \mathcal{R} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathcal{T} \rightarrow 0$$

in which 0 are zero sheaves. That (C.1) is an exact sequence means that $\phi$ is injective, that its image $\phi(\mathcal{R}) \subset \mathcal{S}$ is the kernel of the homomorphism $\psi$, and that $\psi$ is surjective; or equivalently it just means that $\phi$ is an imbedding of $\mathcal{R}$ as a subsheaf of $\mathcal{S}$ and $\psi$ identifies the quotient sheaf $\mathcal{S}/\mathcal{R}$ with the image sheaf $\mathcal{T}$. It is easy to see that if (C.1) is a short exact sequence of sheaves of abelian groups over a topological space $M$ then the induced sequence of sections

$$(C.2) \quad 0 \rightarrow \Gamma(M, \mathcal{R}) \xrightarrow{\phi} \Gamma(M, \mathcal{S}) \xrightarrow{\psi} \Gamma(M, \mathcal{T})$$

is exact; but the homomorphism $\psi$ on sections is not necessarily surjective. For example if $e$ is the mapping that sends the germ that sends the germ of a holomorphic function $f(z)$ to the germ of the nowhere vanishing homomorphic function $\exp 2\pi i f(z)$ there is the exact sequence of sheaves

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \rightarrow 0$$

over $\mathbb{C}^n$; for any open subset $M \subset \mathbb{C}^n$ the induced sequence of sections

$$0 \rightarrow \Gamma(M, \mathbb{Z}) \rightarrow \Gamma(M, \mathcal{O}) \xrightarrow{e} \Gamma(M, \mathcal{O}^*)$$

is exact, but the mapping $e$ on sections is not necessarily surjective when $M$ is not simply connected. A measure of the extent to which such a sequence of sections fails to be exact is provided by the cohomology theory of sheaves.
C.2 Sheaf Cohomology

Although there are more general approaches to the cohomology theory of sheaves, for present purposes it is most convenient to consider skew-symmetric Čech cohomology. For any covering $\mathcal{U}$ of a topological space $M$ by open subsets $U_\alpha \subset M$ let $\mathcal{U}^p$ be the collection of all ordered $(p+1)$-tuples $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p})$ of sets of $\mathcal{U}$ with nonempty intersection $|\sigma| = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \emptyset$. A (skew-symmetric) $p$-cochain $s$ of the covering $\mathcal{U}$ with coefficients in a sheaf of abelian groups $\mathcal{S}$ over $M$ is a mapping that associates to each $(p+1)$-tuple $\sigma = (U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_p}) \in \mathcal{U}^p$ a section

$$s_{\alpha_0, \alpha_1, \ldots, \alpha_p} \in \Gamma(|\sigma|, \mathcal{S})$$

over the intersection $|\sigma|$ such that for any permutation $\pi \in \mathfrak{S}_{p+1}$ of the indices $(0,1,\ldots,p)$

$$s_{\alpha_{\pi_0}, \alpha_{\pi_1}, \ldots, \alpha_{\pi_p}} = (\text{sgn } \pi) \cdot s_{\alpha_0, \alpha_1, \ldots, \alpha_p}$$

where $\text{sgn } \pi$ is the sign of the permutation $\pi$. For example a 0-cochain $s$ associates to each set $U_{\alpha_0}$ a section $s_{\alpha_0} \in \Gamma(U_{\alpha_0}, \mathcal{S})$; and a 1-cochain $s$ associates to each ordered pair of sets $(U_{\alpha_0}, U_{\alpha_1})$ with a nonempty intersection $U_{\alpha_0} \cap U_{\alpha_1} \neq \emptyset$ a section $s_{\alpha_0, \alpha_1} \in \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathcal{S})$ such that $s_{\alpha_1, \alpha_0} = -s_{\alpha_0, \alpha_1}$, so that in particular $s_{\alpha_0, \alpha_0} = 0$. The set of all $p$-cochains is denoted by $C^p(\mathcal{U}, \mathcal{S})$ and clearly is an abelian group under addition. The coboundary homomorphism $\delta$ is the group homomorphism

$$\delta : C^p(\mathcal{U}, \mathcal{S}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{S})$$

for any $p \geq 0$ taking a cochain $s \in C^p(\mathcal{U}, \mathcal{S})$ to the cochain $\delta s \in C^{p+1}(\mathcal{U}, \mathcal{S})$ that associates to each $(p+2)$-tuple $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_{p+1}}) \in \mathcal{U}^{p+1}$ the section

$$\delta s_{\alpha_0, \ldots, \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho_{|\sigma|} (s_{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_{p+1}})$$

where $\rho_{|\sigma|}(s)$ is the restriction of the section $s$ to the intersection $|\sigma|$. A straightforward calculation\(^2\) shows that the coboundary of a skew-symmetric cochain satisfies the skew-symmetry condition (C.4) and that $\delta \delta = 0$. The kernel of the coboundary homomorphism is called the subgroup of $p$-cocycles and is denoted by $Z^p(\mathcal{U}, \mathcal{S}) \subset C^p(\mathcal{U}, \mathcal{S})$; the image $\delta C^{p-1}(\mathcal{U}, \mathcal{S}) \subset C^p(\mathcal{U}, \mathcal{S})$ is called the subgroup of $p$-coboundaries. Since $\delta \delta = 0$ it follows that $\delta C^{p-1}(\mathcal{U}, \mathcal{S}) \subset Z^p(\mathcal{U}, \mathcal{S})$ for $p > 0$; the group

$$H^p(\mathcal{U}, \mathcal{S}) = \begin{cases} Z^p(\mathcal{U}, \mathcal{S}) & \text{for } p > 0, \\ \delta C^{p-1}(\mathcal{U}, \mathcal{S}) & \text{for } p = 0 \end{cases}$$

\(^2\)For details see G-IIIE.
is called the \( p \)-th (skew-symmetric) \textit{Čech cohomology group} of the covering \( \mathcal{U} \) with coefficients in the sheaf \( \mathcal{S} \). For example if \( s \in C^0(\mathcal{U}, \mathcal{S}) \) then

\[(\delta s)_{\alpha_0, \alpha_1}(a) = s_{\alpha_1}(a) - s_{\alpha_0}(a) \quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1},\]

and clearly \( \delta s_{\alpha_1, \alpha_0} = -\delta s_{\alpha_0, \alpha_1} \). The cochain \( s \) is a cocycle if and only if

\[s_{\alpha_0}(a) = s_{\alpha_1}(a) \quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1}\]

so that the local sections \( s_{\alpha_0} \) are the restrictions to the various sets \( U_\alpha \) of a section \( s \in \Gamma(M, \mathcal{S}) \) over all of \( M \); thus there is the natural identification

\[H^0(\mathcal{U}, \mathcal{S}) = \Gamma(M, \mathcal{S}).\]

If \( s \in C^1(\mathcal{U}, \mathcal{S}) \) then

\[(\delta s)_{\alpha_0, \alpha_1, \alpha_2}(a) = s_{\alpha_1, \alpha_2}(a) - s_{\alpha_0, \alpha_2}(a) + s_{\alpha_0, \alpha_1}(a)\]

\[\text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2},\]

which is easily seen to satisfy the skew-symmetry condition (C.4); the cochain is a cocycle \( s \in Z^1(\mathcal{U}, \mathcal{S}) \) if and only if

\[s_{\alpha_0, \alpha_1}(a) + s_{\alpha_1, \alpha_2}(a) + s_{\alpha_2, \alpha_0}(a) = 0 \quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}.\]

A special case of particular interest in this book is that of the sheaf \( \mathcal{S} = \mathcal{O}^* \) of germs of nowhere vanishing holomorphic functions on a complex manifold \( M \), a sheaf of multiplicative abelian groups. A cocycle \( s \in Z^1(\mathcal{U}, \mathcal{O}^*) \) is a collection of nowhere vanishing holomorphic functions \( s_{\alpha_0, \alpha_1}(z) \) in the intersections \( U_{\alpha_0} \cap U_{\alpha_1} \) such that

\[s_{\alpha_0, \alpha_0}(z) = 1 \quad \text{for} \quad z \in U_{\alpha_0}\]

\[s_{\alpha_0, \alpha_1}(z) = s_{\alpha_1, \alpha_0}(z)^{-1} \quad \text{for} \quad z \in U_{\alpha_0} \cap U_{\alpha_1}\]

\[s_{\alpha_0, \alpha_1}(z)s_{\alpha_1, \alpha_2}(z)s_{\alpha_2, \alpha_0}(z) = 1 \quad \text{for} \quad z \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2};\]

thus if \( s \in Z^1(\mathcal{U}, \mathcal{O}^*) \) then the cross-sections \( s_{\alpha_0, \alpha_1} \) satisfy (B.17) so that \( \{U_\alpha, s_{\alpha_0, \alpha_1}\} \) is a coordinate line bundle describing a holomorphic line bundle over \( M \). The cocycle \( s \in Z^1(\mathcal{U}, \mathcal{O}^*) \) is a coboundary if and only if

\[s_{\alpha_0, \alpha_1}(z) = t_{\alpha_1}(z)t_{\alpha_0}(z)^{-1} \quad \text{for} \quad z \in U_{\alpha_0} \cap U_{\alpha_1}\]

where \( t_{\alpha_0}(z) \) are nowhere vanishing holomorphic functions in the open subsets \( U_\alpha \), hence if and only if the cross-sections \( s_{\alpha_0, \alpha_1} \) satisfy (B.24) so that the coordinate line bundle \( \{U_\alpha, s_{\alpha_0, \alpha_1}\} \) describes the trivial holomorphic line bundle over \( M \). Thus the cohomology group \( H^1(\mathcal{U}, \mathcal{O}^*) \) can be identified with the set of those holomorphic line bundles over \( M \) that can be described by coordinate line bundles in terms of the covering \( \mathcal{U} \).
C.2. SHEAF COHOMOLOGY

There remains the question of the relations between the cohomology groups of different coverings of the space $M$. A covering $\mathcal{U}$ of $M$ is called a refinement of the covering $\mathcal{V}$ if there is a mapping $\mu : \mathcal{V} \rightarrow \mathcal{U}$ which associates to each set $V_\alpha \in \mathcal{V}$ a set $\mu(V_\alpha) = U_{\mu(\alpha)} \in \mathcal{U}$ such that $V_\alpha \subset U_{\mu(\alpha)}$. The mapping $\mu$, called a refining mapping, induces group homomorphisms $\mu : C^p(\mathcal{U}, S) \rightarrow C^p(\mathcal{V}, S)$ for any sheaf $S$ of abelian groups over $M$; for a cochain $s \in C^p(\mathcal{U}, S)$ the image $\mu s \in C^p(\mathcal{V}, S)$ is the cochain that associates to each $\sigma = (V_{\alpha_0}, \ldots, V_{\alpha_p}) \in \mathcal{V}^p$ the section

$$(C.14) \quad (\mu s)_{\alpha_0,\ldots,\alpha_p} = \rho_{|\sigma|}(s_{\mu(\alpha_0),\ldots,\mu(\alpha_p)})$$

where $\rho_{|\sigma|}(s)$ is the restriction of the section $s$ to the intersection $|\sigma|$. This homomorphism clearly commutes with the coboundary homomorphism $\delta$ and consequently induces group homomorphisms

$$(C.15) \quad \mu^* : H^p(\mathcal{U}, S) \rightarrow H^p(\mathcal{V}, S).$$

Of course if $\mathcal{W}$ is a refinement of the covering $\mathcal{U}$ there may be a number of different refining mappings; but a straightforward calculation\(^3\) shows that the induced homomorphisms $\mu^*$ of the cohomology groups are independent of the choice of a refining mapping. In the disjoint union of the cohomology groups $H^p(\mathcal{U}, S)$ for all coverings $\mathcal{U}$ two cohomology classes $s \in H^p(\mathcal{U}, S)$ and $t \in H^p(\mathcal{W}, S)$ are considered to be equivalent if there is a common refinement $\mathcal{W}$ of the coverings $\mathcal{U}$ and $\mathcal{W}$, with refining mappings $\mu_\mathcal{W} : \mathcal{W} \rightarrow \mathcal{U}$ and $\mu_\mathcal{W} : \mathcal{W} \rightarrow \mathcal{U}$, such that $\mu_\mathcal{W}^*(s) = \mu_\mathcal{W}^*(t)$; this easily is seen to be an equivalence relation in the usual sense. The set of equivalence classes is a well defined abelian group, the direct limit of the directed set of groups indexed by coverings $\mathcal{U}$ of $M$, called the (skew-symmetric) Čech cohomology group of the space $M$ with coefficients in the sheaf $S$ and denoted by $H^p(M, S)$. For any covering $\mathcal{U}$ there is then the natural homomorphism

$$(C.16) \quad i_\mathcal{U}^* : H^p(\mathcal{U}, S) \rightarrow H^p(M, S)$$

that takes a cohomology class in $H^p(\mathcal{U}, S)$ to its equivalence class in $H^p(M, S)$; and for any refining mapping $\mu : \mathcal{W} \rightarrow \mathcal{U}$ these homomorphisms commute in the sense that $i_\mathcal{U}^* = i_\mathcal{W}^* \circ \mu^*$. For example, since $H^0(\mathcal{U}, S) = \Gamma(M, S)$ for any covering $\mathcal{U}$ it follows that

$$(C.17) \quad H^0(M, S) = \Gamma(M, S),$$

so $H^0(\mathcal{U}, S) \cong H^0(M, S)$ for any covering $\mathcal{U}$ of $M$. The cohomology groups $H^p(\mathcal{U}, S)$ and $H^p(M, S)$ generally are not isomorphic for $p > 0$, although they are for some special covers of suitably regular topological spaces.

To any short exact sequence (C.1) of sheaves of abelian groups over a topological space $M$ there corresponds the exact sequence of sections (C.2). Since

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\(^3\)See Theorem G-IIIE8
the cochain groups $C^p(\mathcal{U}, S)$ are just the direct sums of groups of sections over various subsets of $M$ there are corresponding exact sequences

\begin{equation}
0 \longrightarrow C^p(\mathcal{U}, \mathcal{R}) \xrightarrow{\phi} C^p(\mathcal{U}, S) \xrightarrow{\psi} C^p(\mathcal{U}, T)
\end{equation}

of cochain groups. The homomorphisms $\phi$ and $\psi$ commute with the coboundary operators and consequently induce homomorphisms

\begin{align*}
\phi^* : H^p(\mathcal{U}, \mathcal{R}) &\longrightarrow H^p(\mathcal{U}, S), \\
\psi^* : H^p(\mathcal{U}, S) &\longrightarrow H^p(\mathcal{U}, T).
\end{align*}

If $M$ is a paracompact Hausdorff space, a Hausdorff space such that every open covering has a locally finite refinement, these homomorphisms can be combined and lead to the exact cohomology sequence\(^4\)

\begin{equation}
0 \longrightarrow \Gamma(M, \mathcal{R}) \xrightarrow{\phi^*} \Gamma(M, S) \xrightarrow{\psi^*} \Gamma(M, T) \xrightarrow{\delta^*} H^1(M, \mathcal{R}) \xrightarrow{\phi^*} H^1(M, S) \xrightarrow{\psi^*} H^1(M, T) \xrightarrow{\delta^*} \cdots
\end{equation}

\begin{equation}
\cdots \xrightarrow{\delta^*} H^p(M, \mathcal{R}) \xrightarrow{\phi^*} H^p(M, S) \xrightarrow{\psi^*} H^p(M, T) \xrightarrow{\delta^*} H^{p+1}(M, \mathcal{R}) \xrightarrow{\phi^*} H^{p+1}(M, S) \xrightarrow{\psi^*} H^{p+1}(M, T) \xrightarrow{\delta^*} \cdots
\end{equation}

for suitable connecting homomorphisms $\delta^*$. To define these connecting homomorphisms and demonstrate the exactness of the sequence (C.19) extend the exact sequences (C.18) to the short exact sequences

\begin{equation}
0 \longrightarrow C^p(\mathcal{U}, \mathcal{R}) \xrightarrow{\phi} C^p(\mathcal{U}, S) \xrightarrow{\psi} C^p(\mathcal{U}, T) \longrightarrow 0
\end{equation}

in which $\overline{C}^p(\mathcal{U}, T) \subset C^p(\mathcal{U}, T)$ is the image of the homomorphism $\psi$. These short exact sequences are mapped to one another by the coboundary homomorphism $\delta$, leading to the following commutative diagram of abelian groups and

\(^4\)See Theorem G-IIID2.
Each row is exact by (C.20) while the sheaf cohomology groups measure the inexactness of the columns, in the sense that $H^p(U, R) = \ker \delta_p / \text{im} \delta_{p-1}$ and similarly for the cohomology groups $H^p(U, S)$ and $H^p(U, T)$, where the latter are defined in terms of the cochain groups $C^p(U, T)$. A simple diagram chase shows that under the induced homomorphisms on the cohomology groups the sequences $H^p(U, R) \xrightarrow{\phi^*} H^p(U, S) \xrightarrow{\psi^*} H^p(U, T)$ are exact sequences. For any $t \in C^p(U, T)$ for which $\delta t = 0$ select an element $s \in C^p(U, S)$ for which $t = \psi(s)$. Since $\psi(\delta s) = \delta \psi(s) = \delta t = 0$ it follows from the exactness of the next row that there is an element $r \in C^{p+1}(U, R)$ such that $\phi(r) = \delta s$; and $\phi(\delta r) = \delta \phi(r) = \delta \delta s = 0$, so since $\phi$ is an inclusion it follows that $\delta r = 0$. The homomorphism $\delta^* : \mathbb{Z}^p(U, T) \rightarrow \mathbb{Z}^{p+1}(U, R)$ is defined by $\delta^*(t) = r$. Further diagram chases show first that the cohomology class of $r$ is independent of the choice of $s$, next that cohomologous elements $t$ lead to cohomologous elements $\delta^* t$, and finally that there results a long exact cohomology sequence of the form

$$
\cdots \rightarrow H^p(U, R) \xrightarrow{\delta^*} H^p(U, S) \xrightarrow{\psi^*} \mathbb{H}^p(U, T) \xrightarrow{\delta^*} H^{p+1}(U, S) \rightarrow \cdots.
$$

There is a corresponding exact cohomology sequence for any refinement $\mathcal{V}$ of the covering $\mathcal{U}$, and it is easy to see that the homomorphisms induced by the refining mapping commute with the homomorphisms in these exact sequences; it follows readily from this that there results the exact sequence

$$
\cdots \rightarrow H^p(M, R) \xrightarrow{\delta^*} H^p(M, S) \xrightarrow{\psi^*} \mathbb{H}^p(M, T) \xrightarrow{\delta^*} H^{p+1}(M, S) \cdots.
$$

Finally it is a straightforward matter to show that if $M$ is a paracompact Hausdorff space then $\mathbb{H}^p(M, T) \cong H^p(M, T)$, since for any locally finite covering $\mathcal{U}$
it is possible to choose a locally finite refinement \( \mathfrak{W} \) in which the sets \( V_\alpha \) are sufficiently small that sections of the sheaf \( T \) over intersections of these sets are the images of sections of the sheaf \( S \). That demonstrates the exactness of the sequence (C.19).

Various auxiliary sheaves often are used to calculate cohomology groups explicitly. A sheaf \( S \) of abelian groups over a topological space \( M \) is a fine sheaf if for any locally finite open covering \( \mathfrak{U} = \{ U_\alpha \} \) of \( M \) there are sheaf homomorphisms \( \epsilon_\alpha : S \to S \) such that (i) \( \epsilon_\alpha(s) = 0 \) if \( s \in S_\alpha \) for a point \( a \in M \sim U_\alpha \), and (ii) \( \sum_\alpha \epsilon_\alpha(s) = s \) for any \( s \in S \); the latter sum is finite since the covering \( \mathfrak{U} \) is locally finite so by (i) only finitely many entries in the sum are nonzero. The collection of homomorphisms \( \{ \epsilon_\alpha \} \) is called a partition of unity for the sheaf \( S \) subordinate to the covering \( \mathfrak{U} \). For example the sheaves \( \mathcal{C}(\lambda) \) and \( \mathcal{E}(\lambda) \) of continuous or \( \mathcal{C}^\infty \) cross-sections of a holomorphic line bundle \( \lambda \) over a Riemann surface \( M \) are fine sheaves since it is a standard result of analysis that for any locally finite open covering \( \mathfrak{U} = \{ U_\alpha \} \) of \( M \) there are \( \mathcal{C}^\infty \) real-valued functions \( \epsilon_\alpha \) on \( M \) such that \( \epsilon_\alpha(a) \geq 0 \) at each point \( a \in M \), that the support of the function \( \epsilon_\alpha \) is contained in \( U_\alpha \), and that \( \sum_\alpha \epsilon_\alpha(a) = 1 \) at each point \( a \in M \); and multiplication of the sheaves \( \mathcal{C}(\lambda) \) or \( \mathcal{E}(\lambda) \) by such functions \( \epsilon_\alpha \) is a partition of unity for these sheaves. The basic property of fine sheaves is that \( H^p(M, S) = 0 \) for all \( p > 0 \) if \( S \) is a fine sheaf over a paracompact Hausdorff space. To see this it suffices to show that \( H^p(\mathfrak{U}, S) = 0 \) for a locally finite covering \( \mathfrak{U} \) of \( M \). The first step in doing so is to demonstrate that if \( s \in Z^p(\mathfrak{U}, S) \) is a cocycle for \( p > 0 \) and if \( s(a) = 0 \) whenever \( a \in M \sim U_\beta \) for some set \( U_\beta \) of the covering \( \mathfrak{U} \) then the cocycle \( s \) is cohomologous to zero. Indeed for such a cocycle \( s \) consider the \( (p - 1) \)-cochain \( s^2 \in C^{p-1}(\mathfrak{U}, S) \) that associates to a \( p \)-tuple \( \sigma = (U_\alpha_0, \ldots, U_{\alpha_{p-1}}) \in \mathfrak{U}^{p-1} \) the cross-section over \( |\sigma| \) defined by

\[
s^2_{\alpha_0, \ldots, \alpha_{p-1}}(a) = \begin{cases} s_{\beta, \alpha_0, \ldots, \alpha_{p-1}}(a) & \text{if } a \in |\sigma| \cap U_\beta, \\ 0 & \text{if } a \in |\sigma| \sim |\sigma| \cap U_\beta, \end{cases}
\]

noting that this is a well defined cross-section since the cocycle \( s \) vanishes outside \( U_\beta \). Since \( s \) is a cocycle it follows that for any \( (p + 2) \)-tuple \( \tau = (U_\beta, U_{\alpha_0}, \ldots, U_{\alpha_p}) \in \mathfrak{U}^{p+1} \) and any point \( a \in |\tau| \)

\[
0 = (\delta s)_{\beta, \alpha_0, \ldots, \alpha_p}(a) = s_{\alpha_0, \ldots, \alpha_p}(a) - \sum_{j=0}^p (-1)^j s_{\beta, \alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_p}(a) = s_{\alpha_0, \ldots, \alpha_p}(a) - \sum_{j=0}^p (-1)^j s^2_{\alpha_0, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_p}(a);
\]

this identity holds trivially if \( \alpha \notin U_\beta \), since the cocycle \( s \) vanishes outside \( U_\beta \), so it actually holds for all points \( a \in |\sigma| \) where \( \sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p}) \) and consequently \( s = \delta s^2 \), showing that the cocycle \( s \) is cohomologous to zero. Next choose a
partition of unity $\epsilon_\beta$ for the sheaf $S$ subordinate to the covering $\mathcal{U}$. The sheaf mappings $\epsilon_\beta : S \rightarrow S$ then induce homomorphisms $\epsilon_\beta : \Gamma(U, S) \rightarrow \Gamma(U, S)$ between the sections of the sheaf $S$ over any open subset $U \subset M$; and since the cochain groups $\mathcal{C}^p(\mathcal{U}, S)$ consist of sections over subsets of $M$ these sheaf mappings also induce homomorphisms $\epsilon_\beta : \mathcal{C}^p(U, S) \rightarrow \mathcal{C}^p(U, S)$. Then for any cocycle $s \in \mathcal{Z}^p(U, S)$ the image $\epsilon_\beta(s) \in \mathcal{Z}^p(U, S)$ vanishes outside the set $U_\beta$ so there are cochains $t^\beta \in \mathcal{C}^{p-1}(U, S)$ such that $\delta t^\beta = \epsilon_\beta(s)$; and then $\delta \sum_\beta t^\beta = \sum_\beta \epsilon_\beta(s) = 0$, and consequently the cocycle $s$ is cohomologous to zero as desired.

If for a sheaf $S$ of abelian groups over a paracompact Hausdorff space $M$ there is an exact sequence of the form

\[(C.21) \quad 0 \rightarrow S \rightarrow S_0 \xrightarrow{d_0} S_1 \xrightarrow{d_1} S_2 \xrightarrow{d_2} \cdots \]

in which all the sheaves $S_j$ are fine sheaves, a sequence called a fine resolution of the sheaf $S$, the cohomology groups of $S$ can be calculated in terms of the groups of cross-sections of these auxiliary fine sheaves. Explicitly the cohomology groups of $S$ are isomorphic to the cohomology groups of the not necessarily exact sequence

\[(C.22) \quad 0 \rightarrow \Gamma(M, S) \rightarrow \Gamma(M, S_0) \xrightarrow{d_0^*} \Gamma(M, S_1) \xrightarrow{d_1^*} \Gamma(M, S_2) \xrightarrow{d_2^*} \cdots \]

in the sense that

\[(C.23) \quad H^q(M, S) \cong \frac{\ker d_q^*}{\text{im } d_{q-1}^*} \]

If $K_j \subset S_j$ is the kernel of the homomorphism $d_j$ the initial segment of the long exact sequence (C.21) is equivalent to the short exact sequence

\[0 \rightarrow S \rightarrow S_0 \xrightarrow{d_0} K_1 \rightarrow 0;\]

and since $S_0$ has trivial cohomology groups in strictly positive dimensions it follows from the exact cohomology sequence associated to this short exact sequence of sheaves that

\[H^1(M, S) \cong \frac{\Gamma(M, K_1)}{d_0^* \Gamma(M, S_0)} \]

and

\[H^q(M, S) \cong H^{q-1}(M, K_1) \quad \text{for } q > 1.\]

The remainder of the long exact sequence (C.21) is equivalent to the collection of short exact sequences

\[0 \rightarrow K_j \rightarrow S_j \xrightarrow{d_j} K_{j+1} \rightarrow 0\]

for $j > 0$; and since the sheaves $S_j$ also have trivial cohomology groups in strictly positive dimensions it follows from the exact cohomology sequence associated to these short exact sequences of sheaves that

\[H^1(M, K_j) \cong \frac{\Gamma(M, K_{j+1})}{d_j^* \Gamma(M, S_j)}\]
and

\[ H^q(M, K_{j+1}) \cong H^{q+1}(M, K_j) \quad \text{for} \quad j, q > 0. \]

From these sets of isomorphisms it follows that

\[ H^q(M, S) \cong H^{q-1}(M, K_1) \cong H^{q-2}(M, K_2) \cong \cdots \cong H^1(M, K_{q-1}) \]

and hence that

\[ H^q(M, S) \cong \frac{\Gamma(M, K_q)}{d_{q-1}^* \Gamma(M, S_{q-1})} \quad \text{for} \quad q > 0, \]

where of course \( \Gamma(M, K_q) \) is just the kernel of the homomorphism

\[ d_q^* : \Gamma(M, S_q) \longrightarrow \Gamma(M, S_{q+1}) \]

and consequently this suffices to demonstrate (C.23).

This result can be used to calculate the cohomology groups of sheaves in another way. A covering \( \mathcal{U} \) of a topological space \( M \) is called a Leray covering for a sheaf of abelian groups \( S \) if \( H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, S) = 0 \) for all indices \( p \geq 0, q \geq 1 \) and all \( U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \in \mathcal{U}^p \). In these terms the Theorem of Leray\(^5\) asserts that if \( \mathcal{U} \) is a Leray covering of a paracompact Hausdorff space \( M \) for a sheaf of abelian groups \( S \) then the natural homomorphisms (C.16) are isomorphisms; thus for paracompact Hausdorff spaces the cohomology groups \( H^p(M, S) \) can be calculated in terms of any Leray covering of the space \( M \) for the sheaf \( S \). To demonstrate this result construct a fine resolution of the sheaf \( S \) over \( M \) as follows. For any open subset \( U \subset M \) let \( \Gamma^*(U, S) \) be the group of not necessarily continuous cross-sections of the sheaf \( S \) over \( U \), the group of quite arbitrary mappings \( f : U \longrightarrow S \) such that \( \pi f(p) = p \) for all points \( p \in U \). The set of such groups form a complete presheaf over \( M \), and the associated sheaf \( S^* \) is a fine sheaf since for any locally finite covering \( \mathcal{U} = \{ U_\alpha \} \) of \( M \) and any subsets \( K_\alpha \subset U_\alpha \) that are pairwise disjoint and also cover \( M \) the mappings \( \rho_\alpha : S^* \longrightarrow S^* \) for which \( \rho_\alpha(s) = s \) if \( s \in K_\alpha \) and \( \rho_\alpha(s) = 0 \) otherwise form a partition of unity for the sheaf \( S^* \) for the covering \( \mathcal{U} \). The same argument shows that the restriction of the sheaf \( S^* \) to any open subset of \( M \) is a fine sheaf over that subset. The inclusion \( \Gamma(U, S) \subset \Gamma^*(U, S) \) of continuous cross-sections into the group of not necessarily continuous cross-sections is a homomorphism of presheaves which leads to an imbedding \( t : S \longrightarrow S^* \). For the fine resolution of \( S \) take \( S_0 = S^*, S_1 = (S_0/S)^*, \) and so on. This is a fine resolution over the entire space \( M \) or over any open subset of \( M \); so the cohomology of \( M \) or of any open subset of \( M \) with coefficients in the sheaf \( S \) can be calculated from this fine resolution. In particular since \( \mathcal{U} \) is assumed to be a Leray covering \( H^1(|\sigma|, S) = 0 \) for any intersection \( |\sigma| = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \) of sets \( U_\alpha \) of \( \mathcal{U} \), so the sequence of sections

\[ 0 \longrightarrow \Gamma(|\sigma|, S) \longrightarrow \Gamma(|\sigma|, S_0) \xrightarrow{d_0^*} \Gamma(|\sigma|, S_1) \xrightarrow{d_1^*} \cdots \]

\(^5\)See Theorem G-IIE5.
over $|\sigma|$ is exact. The cochain groups are finite direct sums of these sequences of sections, so there is also the exact sequence

$$0 \longrightarrow C^q(|\sigma|, S) \longrightarrow C^q(|\sigma|, S_0) \overset{d_0^*}{\longrightarrow} C^q(|\sigma|, S_1) \overset{d_1^*}{\longrightarrow} \cdots.$$ 

The coboundary homomorphisms commute with the homomorphisms of these exact sequences, so there results the following commutative diagram of abelian groups and group homomorphisms.

$$
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \Gamma(M, S) & \Gamma(M, S_0) & \Gamma(M, S_1) & \Gamma(M, S_2) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & C^0(\mathcal{U}, S) & C^0(\mathcal{U}, S_0) & C^0(\mathcal{U}, S_1) & C^0(\mathcal{U}, S_2) \\
\delta & \delta & \delta & \delta \\
0 & C^1(\mathcal{U}, S) & C^1(\mathcal{U}, S_0) & C^1(\mathcal{U}, S_1) & C^1(\mathcal{U}, S_2) \\
\delta & \delta & \delta & \delta \\
0 & C^2(\mathcal{U}, S) & C^2(\mathcal{U}, S_0) & C^2(\mathcal{U}, S_1) & C^2(\mathcal{U}, S_2) \\
\delta & \delta & \delta & \delta \\
0 & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

All rows except the first are exact, while the cohomology of $M$ with coefficients in the sheaf $S$ measures the extent to which the first row fails to be exact as in (C.23). All columns except the first are exact, since the sheaves $S_j$ are all fine sheaves, while the cohomology of the covering $\mathcal{U}$ with coefficients in the sheaf $S$ measures the extent to which the first column fails to be exact. A straightforward diagram chase then yields the desired result.
Appendix D

Topology of Surfaces

D.1 Homotopy

This appendix contains a brief survey of some of the basic topological properties of surfaces, an acquaintance with which is presupposed in the discussion in this book. The choice of topics and the order of presentation are not those of a standard introduction to the topology of surfaces, in particular are not those most appropriate for a rigorous development of the subject with complete proofs. Instead those results that are of primary interest in the study of compact Riemann surfaces will be discussed from a rather intuitive and geometric point of view; proofs can be found in the references noted.

A surface is a connected two-dimensional topological manifold; only orientable surfaces will be considered here. A path on a surface $M$ is a continuous image of the closed unit interval $[0,1]$ oriented in the direction of increasing parameter values; the beginning point is the image of 0 and the end point is the image of 1. The path is closed if its beginning and end points coincide, and is simple if distinct real numbers in $[0,1]$ have distinct images, except possibly for the beginning and end points; if the beginning and end point do coincide the path is a simple closed path. Two closed paths $\sigma : [0,1] \rightarrow M$ and $\tau : [0,1] \rightarrow M$ beginning and ending at $p$ are homotopic if there is a continuous mapping $F : [0,1]^2 \rightarrow M$ of the unit square $[0,1]^2 = \{(t_1, t_2) \mid 0 \leq t_i \leq 1\}$ into $M$ such that $F(t_1, 0) = \sigma(t_1)$, $F(t_1, 1) = \tau(t_1)$, $F(0, t_2) = F(1, t_2) = p$; this is readily seen to be an equivalence relation in the usual sense. The fundamental group $\pi_1(M, p_0)$ of a surface $M$ at a point $p_0 \in M$ is the set of homotopy classes of closed paths in $M$ beginning and ending at the point $p_0$. The product $\sigma \cdot \tau$ of two paths $\sigma$ and $\tau$ beginning and ending at $p_0$ is the path that arises by traversing first $\sigma$ and then $\tau$. If $\sigma$ is homotopic to $\sigma'$ and $\tau$ is homotopic to $\tau'$ then $\sigma \tau$ is homotopic to $\sigma' \tau'$, so this defines a group structure on $\pi_1(M, p_0)$. The

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identity element in the group \( \pi_1(M, p_0) \) is the homotopy class of the constant mapping of the unit interval to the point \( p \), while the inverse of an element is the homotopy class of that path traversed in the reverse direction. To any choice of a path \( \sigma \) from a point \( p_0 \in M \) to another point \( q_0 \in M \) there corresponds the natural isomorphism \( \sigma^* : \pi_1(M, p_0) \longrightarrow \pi_1(M, q_0) \) that associates to the homotopy class of a closed path \( \tau \) beginning and ending at the point \( p_0 \) the homotopy class of the closed path \( \sigma^{-1} \tau \sigma \) beginning and ending at the point \( q_0 \), since if \( \tau' \sim \tau \) then \( \sigma^{-1} \tau' \sigma \sim \sigma^{-1} \tau \sigma \). The isomorphism \( \sigma^* \) depends only on the homotopy class of the path \( \sigma \), any two such isomorphisms differ by an inner automorphism of the fundamental group, and any inner automorphism of the fundamental group can be realized in this way. In the subsequent discussion the homotopy classes represented by a path \( \sigma \) also will be denoted by \( \sigma \) to avoid complicating the notation; it should be quite clear from context what is meant in any particular case. A topological space \( M \) is simply connected if its fundamental group is trivial.

The fundamental group is closely related to properties of covering spaces, which play an important role in the analytic study of Riemann surfaces. A covering space over a surface \( M \) is a surface \( N \) together with a continuous mapping \( \pi : N \longrightarrow M \), called the covering projection, such that each point of \( M \) has an open neighborhood \( U \) for which the inverse image \( \pi^{-1}(U) \) consists of a collection of disjoint open subsets of \( N \) each of which is homeomorphic to \( U \) under the covering projection. The universal covering space \( \tilde{M} \) over \( M \) is the unique simply connected covering space over \( M \). There is a properly discontinuous group \( \Gamma \) of homeomorphisms acting without fixed points on the universal covering space \( \tilde{M} \), the covering translation group, such that the quotient space \( \tilde{M}/\Gamma \) is homeomorphic to the surface \( M \) and the natural quotient mapping \( \tilde{\pi} : \tilde{M} \longrightarrow \tilde{M}/\Gamma = M \) is the covering projection mapping; that the group \( \Gamma \) is properly discontinuous means that for each point \( z \in \tilde{M} \) there is an open neighborhood \( U \) of \( z \) in \( \tilde{M} \) such that \( S(U) \cap T(U) = \emptyset \) whenever \( S, T \) are distinct elements of \( \Gamma \). For any contractible open subset \( U \subset M \) the elements of the group \( \Gamma \) permute the connected components of \( \pi^{-1}(U) \) transitively. For any subgroup \( \Gamma_N \subset \Gamma \) the quotient space \( N = \tilde{M}/\Gamma_N \) is a surface and the natural mappings \( \tilde{\pi} \) and \( \pi \) in the diagram

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\pi} & \tilde{M}/\Gamma_N \\
\| & & \| \\
N & \xrightarrow{\pi} & M
\end{array}
\]

are covering projections; and conversely any for any covering projection \( \pi : N \longrightarrow M \) there corresponds a subgroup \( \Gamma_N \subset \Gamma \) such that the covering projection \( \pi \) is that arising from the action of the group \( \Gamma_N \) on the universal covering space \( \tilde{M} \). The covering projection \( \pi : N \longrightarrow M \) is a regular covering if the subgroup \( \Gamma_N \) is a normal subgroup of \( \Gamma \); in that case the quotient group \( \Gamma/\Gamma_N \) acts as a group of covering transformations of the space \( N \) with quotient \( M \).

The covering translation group \( \Gamma \) is isomorphic to the fundamental group of the surface \( M \), and the isomorphism can be made canonical by the choice of a
D.1. HOMOTOPY

Figure D.1: A marking of a compact oriented surface $M$.

A compact oriented surface $M$ together with a base point $z_0 \in \tilde{M}$. A surface $M$ together with a base point in its universal covering space $\tilde{M}$ is called a pointed surface. For a pointed surface $M$ with base point $z_0 \in \tilde{M}$ let $\pi : \tilde{M} \rightarrow M$ be the universal covering projection and $\pi_{z_0} : \Gamma \rightarrow \pi_1(M, p(z_0))$ be the mapping that associates to a covering translation $T \in \Gamma$ the homotopy class in $\pi_1(M, \pi(z_0))$ of the image $\tau = \pi(\tilde{\tau})$ in $M$ of any path $\tilde{\tau} \subset \tilde{M}$ from $z_0$ to $Tz_0$; since $\tilde{M}$ is simply connected the homotopy class of the path $\tau$ is independent of the choice of the path $\tilde{\tau}$. If $\tilde{\sigma} \subset \tilde{M}$ is a path from the base point $z_0$ to the point $Sz_0$ for another covering translation $S \in \Gamma$ then the path $\tilde{\sigma} \cdot S\tilde{\tau}$ extends from $z_0$ to the point $STz_0$ so $\pi_{z_0}(ST) = \pi(\tilde{\sigma} \cdot (S\tilde{\tau})) = \pi(\tilde{\sigma}) \cdot \pi(\tilde{\tau}) = \pi_{z_0}(S) \cdot \pi_{z_0}(T)$, showing that $\pi_{z_0}$ is a group homomorphism; that it is an isomorphism is a simple consequence of the simple connectivity of $\tilde{M}$. Changing the base point $z_0 \in \tilde{M}$ to $Az_0$ for a covering translation $A \in \Gamma$ has the effect of changing the isomorphism $\pi_{z_0}$ by an inner automorphism of the group $\Gamma$, and any inner automorphism of the group $\Gamma$ can be realized in this way.

A compact orientable surface can be represented as a sphere with $g$ handles, where the integer $g$ is called the genus of the surface; thus a surface of genus $g = 0$ is just a sphere, a surface of genus $g = 1$ is a torus, a surface of genus $g = 2$ is a sphere with two handles, and so on. A surface of genus $g > 0$ can be represented as a sphere with $g$ handles in a number of different ways though. A marking of a compact oriented surface $M$ of genus $g > 0$ with universal covering space $\tilde{M}$ and covering projection $\pi : \tilde{M} \rightarrow M$ is the choice of a base point $z_0 \in \tilde{M}$. A surface $M$ together with a base point in its universal covering space $\tilde{M}$ is called a pointed surface. Of course the choice of a base point $z_0 \in \tilde{M}$ yields automatically the choice of the base point $\pi(z_0) \in M$ where $\pi : \tilde{M} \rightarrow M$ is the covering projection; but for a canonical isomorphism between the fundamental group and the covering translation group it is necessary to choose a base point in the universal covering space.

\textsuperscript{2}It is more customary to define a pointed surface as a surface $M$ together with the choice of a base point in $M$ itself. Of course the choice of a base point $z_0 \in \tilde{M}$ yields automatically the choice of the base point $\pi(z_0) \in M$ where $\pi : \tilde{M} \rightarrow M$ is the covering projection; but for a canonical isomorphism between the fundamental group and the covering translation group it is necessary to choose a base point in the universal covering space.
point \( z_0 \in \tilde{M} \), of a representation of \( M \) as a sphere with \( g \) handles, ordered as the first handle, the second handle, and so on, and of \( 2g \) simple closed paths \( \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \) in \( \tilde{M} \) beginning and ending at the point \( p_0 = \pi(z_0) \in M \), disjoint except for the common point \( p_0 \) and such that the paths \( \alpha_i \) and \( \beta_i \) encircle the \( i \)-th handle as sketched in Figure D.1; the surface \( M \) together with a marking is a marked surface. Any orientation-preserving homeomorphism of the surface \( M \) that preserves the base point \( p_0 \in M \) transforms a marking of the surface to another marking; two markings related in this way are called equivalent markings of the surface, and it is really the equivalence classes of markings of a surface that are of primary interest. When the \( 2g \) paths \( \alpha_i \) and \( \beta_i \) are removed from the surface \( M \) the result is a contractible open subset \( D \subset M \). The boundary of \( D \) can be traversed from the interior of \( D \) in the positive sense of the orientation it inherits from the orientation of the surface, beginning at the point 1 in Figure D.1 and proceeding first along the path \( \alpha_1 \) in the direction of its orientation back to the point 2, then along the path \( \beta_1 \) in the direction of its orientation back to the point 3, then along the path \( \alpha_1 \) but in the reverse direction to its orientation back to the point 4, then along the path \( \beta_1 \) but again in the reverse direction to its orientation back to the point 5, then along the path \( \alpha_2 \) in the direction of its orientation back to the point 6, and so on; the traverse ends by proceeding along the path \( \beta_g \) in the reverse direction to its orientation back to the initial point 1.

If \( \tilde{\alpha}_i \subset \tilde{M} \) is the lifting of the path \( \alpha_i \subset M \) to a path in the universal covering space \( \tilde{M} \) beginning at the base point \( \tilde{z}_0 \in \tilde{M} \) the end point of the path \( \tilde{\alpha}_i \) is the point \( \tilde{A}_i z_0 \in \tilde{M} \) for a uniquely determined covering translation \( A_i \in \Gamma \), that element of the group \( \Gamma \) for which \( \pi_{\tilde{z}_0}(A_i) \in \pi_1(M,p_0) \) is the homotopy class of the path \( \alpha_i \) under the canonical isomorphism from the covering translation group \( \Gamma \) to the fundamental group of the surface \( M \). Correspondingly if \( \tilde{\beta}_i \subset \tilde{M} \) is the lifting of the path \( \beta_i \subset M \) to a path beginning at \( \tilde{z}_0 \) it will end at the point \( \tilde{B}_i z_0 \) for a covering translation \( B_i \in \Gamma \) for which \( \pi_{\tilde{z}_0}(B_i) \in \pi_1(M,p_0) \) is the homotopy class of the path \( \beta_i \). To simplify the formulas in the subsequent discussion it is convenient to introduce the commutators

\[
C_i = [A_i, B_i] = A_i B_i A_i^{-1} B_i^{-1} \in \Gamma
\]

of these covering translations. The inverse image \( \pi^{-1}(D) \subset \tilde{M} \) of the open subset \( D \subset M \) consists of a collection of disjoint open subsets of the universal covering space \( \tilde{M} \) that are homeomorphic to \( D \) under the covering projection \( \pi : \tilde{M} \rightarrow M \) and that are permuted by the action of the covering translation group \( \Gamma \); the boundaries of the disjoint connected components of \( \pi^{-1}(D) \) consist of the images under various covering translations in \( \Gamma \) of the paths \( \tilde{\alpha}_i \) and \( \tilde{\beta}_i \).

The set \( \Delta \) sketched in Figure D.2 is that connected component of \( \pi^{-1}(D) \) with the base point \( \tilde{z}_0 \) and the paths \( \tilde{\alpha}_1 \) and \( \tilde{\beta}_g \) on its boundary; it is called the fundamental domain for the action of the covering translation group \( \Gamma \). The translates \( T \Delta \) for all covering translations \( T \in \Gamma \) are disjoint subsets of \( \tilde{M} \) that cover the entire space \( \tilde{M} \) except for points over the removed paths \( \alpha_i \) and \( \beta_i \). The boundary of \( \Delta \) can be traversed from the interior of \( \Delta \) by lifting the traverse...
of the boundary of $D$, beginning at the point 1 and proceeding first along the path $\hat{\alpha}_1$ covering $\alpha_1$ to the point 2, then along the path $A_1\hat{\beta}_1$ covering $\beta_1$ to the point 3, then in the reverse direction along the path $C_1B_1\hat{\alpha}_1$ covering $\alpha_1$ to the point 4, then in the reverse direction along the path $C_1\hat{\beta}_1$ covering $\beta_1$ to the point 5, then along the path $c_1\hat{\alpha}_1$ covering $\alpha_1$ to the point 6, and so on, ending back at the point 1. The vertices of $\Delta$ are the images of the base point $z_0$ under the indicated elements of the group $\Gamma$, and the particular lifts of the paths $\hat{\alpha}_i$ and $\hat{\beta}_i$ that form the boundary of $\Delta$ are determined by their beginning points as in Figure D.2. A neighborhood of the base point $z_0 \in \tilde{M}$ is mapped homeomorphically to a neighborhood of the point $p_0 \in M$ by the projection $\pi$; consequently $2g$ translates of the fundamental domain $\Delta$ meet at the vertex $z_0$ in a manner reflecting the configuration of paths emerging from the point $p_0 \in M$ as sketched in Figure D.1, and similarly of course at all the points $\Gamma z_0$. The surface $M$ itself can be constructed from the fundamental domain $\Delta$ by identifying the sides $\hat{\alpha}_1$ and $C_1B_1\hat{\alpha}_1$ and the other pairs of sides correspondingly; this is the traditional “scissors and paste” description of a compact surface, probably most familiar for surfaces of genus $g = 1$ described by identifying the opposite sides of a parallelogram in the traditional treatment of elliptic functions.

Any closed path $\tau$ on the marked surface $M$ beginning at the point $p_0$ can be deformed homotopically to a closed path lying entirely on the boundary of $D$, so is homotopic to the product of the paths $\alpha_i$ and $\beta_i$ and their inverses in some order; thus the fundamental group $\pi_1(M,p_0)$ is generated by the group
elements $\alpha_i$ and $\beta_i$. It is evident from Figure D.2 that the boundary of the fundamental domain $\Delta$ can be described as the product

\[(D.2) \quad \partial \Delta = \prod_{i=1}^{g} \left( (C_1 \cdots C_{i-1} \tilde{\alpha}_i) \cdot (C_1 \cdots C_{i-1} A_i \tilde{\beta}_i) \right) \left( (C_1 \cdots C_i B_i \tilde{\alpha}_i)^{-1} \cdot (C_1 \cdots C_i \tilde{\beta}_i)^{-1} \right),\]

where the product is taken in increasing order of the index $i$ and a product $C_1 \cdots C_{i-1}$ is interpreted as being the identity element when $i = 1$; and that this product is homotopic to the identity element, so that the homotopy classes $\alpha_i$ and $\beta_i$ in $\pi_1(M,p_0)$ are subject to the relation

\[(D.3) \quad I = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \cdots \beta_{g-1}^{-1} \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1} \cdot \]

All the relations between these generators are consequences of this single relation, so the fundamental group $\pi_1(M,p_0)$ has the presentation as the quotient of the free group $F$ on the symbols $\alpha_i$ and $\beta_i$ by the normal subgroup $K$ generated by the relation (D.3). The fundamental group is isomorphic to the covering translation group; so the covering translation group $\Gamma$ can be described correspondingly as the quotient of the free group $F$ on the symbols $A_i$ and $B_i$ by the normal subgroup $K$ generated by the single word

\[(D.4) \quad C = C_1 \cdots C_g = A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1}\]

in the commutator subgroup $[F,F] \subset F$. Conversely whenever $\alpha_i$, $\beta_i$ are generators of the fundamental group $\pi_1(M,p_0)$ of a surface $M$ of genus $g$, subject only to the relation (D.3), these generators can be taken to arise from the paths of a geometric marking of the surface; for many purposes this associated presentation of the fundamental group or covering translation group of the surface is the most significant aspect of a marking of the surface.

There are situations in which it is necessary to consider the noncompact surfaces that arise by removing from a surface $M$ a set of $n$ points $q_1, \ldots, q_n$. By expanding the holes made by removing the points $q_1, \ldots, q_n$ to small discs about these points and expanding the whole made by removing the point $q_n$ any closed path in the complement $D \sim (q_1 \cup \cdots \cup q_n)$ can be deformed homotopically to a path on the boundary of the set $D$ together with paths $\gamma_1, \ldots, \gamma_{n-1}$ from the point $p_0$ out to small circles around the points $q_1, \ldots, q_{n-1}$ and then back to the point $p_0$ that can be visualized by considering the domain $D$ and its pairs of boundary paths as sketched in Figure D.3. The collection of the paths

\[\text{[Footnote 3]} \quad \text{That any presentation of the fundamental group with the single relation (D.3) can be realized by a geometric marking of the surface is a consequence of the result of J. Nielsen that an automorphism of the covering translation group can be realized by a homeomorphism of the surface. This result can be found in the paper by J. Nielsen, "Untersuchungen zur Topologie der geschlossenen zweiseitige Flächen" I, Acta Math. 50 (1927), pp. 189-358; an English translation is in J. Nielsen, Collected Papers I, Birkhäuser, 1986, pp. 223-341. See also the discussion in the book by W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory, Interscience, 1966, page 176.}\]
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The set resulting from the removal of the points $q_i$ from $D$ can be shrunk to the union of the paths $\alpha_i, \beta_i, \gamma_i$.

$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{n-1}$ amounts to the set consisting of $2g + n - 1$ circles with a single point in common, a set called a bouquet of circles; its fundamental group, and therefore the fundamental group of the complement $D \sim (q_1 \cup \cdots \cup q_n)$, is the free group generated by the paths $\alpha_i, \beta_i, \gamma_j$, for $1 \leq i \leq g$ and $1 \leq j \leq n - 1$, a free group on $2g + n - 1$ generators.

D.2 Homology

To examine the homology groups of a compact Riemann surface $M$ of genus $g > 0$ it is convenient to assume that the surface is triangulated\footnote{That every compact surface can be triangulated is a classical result; the definition and general properties of triangulations are discussed in the general references cited on page 447.}. The group $C_i$ of $i$-dimensional chains of the triangulated surface $M$ is the free abelian group generated by the $i$-dimensional oriented simplices of the triangulation. The boundary of a 2-dimensional simplex is the 1-dimensional chain consisting of the sum of the three 1-dimensional simplices forming the boundary of the triangle, oriented so that the boundary is traversed in the natural orientation; and the boundary of a 1-dimensional simplex is the 0-dimensional chain consisting of the end point of the simplex minus the beginning point of the simplex, as sketched in Figure D.4. The mapping that associates to any chain the sum of the boundaries of the simplices comprising that chain is a group homomorphism $\partial_i : C_i \rightarrow C_{i-1}$ for $i = 1, 2$; it is apparent from Figure D.4 that $\partial_1 \partial_2 = 0$. The kernel of the homomorphism $\partial_i$ for $i = 1, 2$ is the subgroup $Z_i \subset C_i$ of $i$-
dimensional cycles, and the image of the homomorphism $\partial_{i+1}$ for $i = 0, 1$ is the subgroup $B_i \subset C_i$ of $i$-dimensional boundaries. Clearly $B_1 \subset Z_1$ since $\partial_1 \partial_2 = 0$; the 1-dimensional homology group of the surface $M$ is defined to be the quotient group $H_1(M) = Z_1 / B_1$. There are no boundaries $B_2 \subset C_2$ so the 2-dimensional homology group is defined as the quotient group $H_2(M) = Z_2(M)$; and every cochain in $C_0$ can be viewed as a cocycle so the 0-dimensional homology group is defined as the quotient group $H_0(M) = C_0 / B_0$. A basic result is that the homology groups are independent of the choice of the triangulation. Customarily two cycles that differ from one another by a boundary are called homologous, so that alternatively the homology group can be viewed as the group of homology classes of cycles on $M$. The rank of the group $H_i(M)$ is called the $i$-th Betti number of the surface $M$ and is denoted by $b_i$.

All 2-cycles of the surface $M$ are integral multiples of the fundamental cycle, the sum of all of the 2-dimensional simplices of the triangulation; the fundamental cycle usually is denoted simply by $M$; consequently

(D.5) \[ H_2(M) \cong \mathbb{Z}, \]

or equivalently $b_2 = 1$. On a marked surface, with the marking described by a base point $z_0 \in M$ and paths $\alpha_i$ and $\beta_i$ as in Figure D.2, any 1-cycle is homologous to a sum of the cycles $\alpha_i$ and $\beta_i$, and no nontrivial combination of these cycles is homologous to zero; consequently

(D.6) \[ H_1(M) \cong \mathbb{Z}^{2g}, \]

or equivalently $b_1 = 2g$. A basic result is that the homology group $H_1(M)$ is the abelianization of the fundamental group $\pi_1(M, p_0)$; the homology group is a simpler invariant than the fundamental group since it ignores the information carried by the commutator subgroup of the fundamental group. The homology classes represented by the paths $\alpha_i, \beta_i$ also are denoted by $\alpha_i, \beta_i$, to avoid complicating the notation; it should be quite clear from context what is meant in any particular case. If $p_0 = \pi(z_0) \in M$, the composition of the homomorphism $\pi_{z_0} : \Gamma \rightarrow \pi_1(M, p_0)$ and the mapping from the fundamental group to the first homology group yields the natural identification

(D.7) \[ H_1(M) \cong \frac{\Gamma}{[\Gamma, \Gamma]}, \]
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describing the homology of $M$ in terms of the covering translation group. Since the mapping $\pi_{z_0}$ changes by an inner automorphism when the base point is changed, it follows that this identification is canonical, independent of the choice of the base point. Finally any 0-dimensional chain is a cycle, and two such cycles are homologous precisely when the sums of the multiplicities of the points involved coincide, so that

$$(D.8)\quad H_0(M) \cong \mathbb{Z},$$

or equivalently $b_0 = 1$.

The alternating sum of the ranks of the homology groups of the surface $M$, the expression

$$(D.9)\quad \chi(M) = \sum_{i=0}^{3} (-1)^i \text{rank } H_i(M) = b_0 - b_1 + b_2,$$

is called the Euler characteristic of the surface $M$; in view of the formulas for the Betti numbers just derived, it follows that

$$(D.10)\quad \chi(M) = 2 - 2g$$

where $g$ is the genus of the surface. Since the homology groups arise from the exact sequences

$$0 \rightarrow Z_2 \rightarrow C_2 \xrightarrow{\partial} B_1 \rightarrow 0,$$
$$0 \rightarrow Z_1 \rightarrow C_1 \xrightarrow{\partial} B_0 \rightarrow 0,$$
$$0 \rightarrow Z_0 \rightarrow C_0 \xrightarrow{\partial} 0,$$

it follows that the Euler characteristic can be expressed alternatively as

$$\chi(M) = \text{rank } H_2(M) - \text{rank } H_1(M) + \text{rank } H_0(M)$$
$$= \text{rank } Z_2 - (\text{rank } Z_1 - \text{rank } B_1) + (\text{rank } Z_0 - \text{rank } B_0)$$
$$= (\text{rank } Z_2 + \text{rank } B_1) - (\text{rank } Z_1 + \text{rank } B_0) + \text{rank } Z_0$$
$$= \text{rank } C_2 - \text{rank } C_1 + \text{rank } C_0.$$

Here $\text{rank } C_i = n_i$ is just the total number of $i$-dimensional simplices in the triangulation, so there results the Euler formula

$$(D.11)\quad \chi(M) = n_0 - n_1 + n_2;$$

this expresses the Euler characteristic directly in terms of the number of simplices in any triangulation of the surface.

For some purposes it is more convenient to consider the singular homology groups rather than the homology groups associated to a triangulation of the surface. A singular simplex of a surface $M$ is a continuous mapping of a standard simplex, either a point, a line segment, or a triangle, into the surface $M$; and the singular chain complex of $M$ is the free abelian group generated by the
singular simplices of \( M \). The boundary of a singular simplex is the element of the singular chain complex consisting of the singular simplices that are formed by restricting the mapping of a standard simplex into \( M \) to the boundary of that simplex. Again the boundary of a boundary is zero, so it is possible to define the singular homology groups of a surface \( M \) as the homology groups of this chain complex. It is a standard result that the homology groups formed from the singular complex of a surface are isomorphic to the homology groups formed from a triangulation of the surface. The singular homology groups are clearly invariantly defined, so this shows that the homology groups defined in terms of a triangulation really are independent of the choice of the triangulation.

Just as important as the homology groups, and in some ways even more convenient, are the dual cohomology groups. If \( C_i \) is the group or \( \mathbb{Z} \)-module of \( i \)-dimensional chains in a triangulation of the surface \( M \) and \( R \) is any \( \mathbb{Z} \)-module then \( C^i(R) = \text{Hom}(C_i, R) \) is the group of \( i \)-dimensional cochains of that triangulation with coefficients in the \( \mathbb{Z} \)-module \( R \). Of primary interest here are the cases in which \( R = \mathbb{Z}, \mathbb{R}, \) or \( \mathbb{C} \), and it will be assumed henceforth that \( R \) is one of these modules. The boundary homomorphisms \( \partial : C_{i+1} \rightarrow C_i \) naturally lead to dual coboundary homomorphisms \( \delta : C^i(R) \rightarrow C^{i+1}(R) \), where \( \delta(\phi)(c) = \phi(\partial c) \) for any cochain \( \phi \in C^i(R) = \text{Hom}(C_i, R) \) and any chain \( c \in C_{i+1} \); and \( \delta \delta = 0 \) since \( \partial \partial = 0 \). The kernel of the homomorphism \( \delta \) is the subgroup \( Z^i(R) \subset C^i(R) \) of \( i \)-dimensional cocycles with coefficients in \( R \), and the image of that homomorphism is the subgroup \( B^i(R) = \delta C^{i-1}(R) \) of \( i \)-dimensional coboundaries with coefficients in \( R \); clearly \( B^i(R) \subset Z^i(R) \) since \( \delta \delta = 0 \). The quotient group \( H^i(M, R) = Z^i(R)/B^i(R) \) is the \( i \)-th cohomology group of the surface \( M \) with coefficients \( R \). In the case of surfaces the situation is particularly simple, for \( H^i(M, \mathbb{R}) \cong \text{Hom}(H_i(M), \mathbb{R}) \) since the homology groups are free abelian groups; in particular

\[
H^i(M, \mathbb{Z}) \cong \text{Hom}(H_i(M), \mathbb{Z}),
\]

\[
H^i(M, \mathbb{R}) = H^i(M, \mathbb{Z}) \otimes \mathbb{R}, \quad H^i(M, \mathbb{C}) = H^i(M, \mathbb{Z}) \otimes \mathbb{C}
\]

for \( i = 0, 1, 2 \). Furthermore there is the canonical identification

\[
H^1(M, \mathbb{Z}) = \text{Hom}(\Gamma, \mathbb{Z}),
\]

since homomorphisms from \( \Gamma \) to the abelian group \( \mathbb{Z} \) are necessarily trivial on the commutator subgroup \( [\Gamma, \Gamma] \subset \Gamma \) and \( \Gamma/[\Gamma, \Gamma] \cong H_1(M) \).

For any \( \mathbb{Z} \)-module \( R \) the cohomology groups \( H^i(M, R) \) can be identified with the sheaf cohomology groups of \( M \) with coefficients in the constant sheaf \( R \). Any finite open covering of a compact Riemann surface \( M \) has a refinement \( \mathcal{U} \) consisting of nonempty open sets \( U_i \) such that any 4 of the sets \( U_i \) have an empty intersection\(^5\). Associated to this covering of \( M \) is the two-dimensional simplicial complex in which the vertices are the sets \( U_i \), the one-simplices are pairs of distinct intersecting sets \( U_i \cap U_j \neq \emptyset \), and the two-simplices are triples

of distinct intersecting sets $U_i \cap U_j \cap U_k \neq \emptyset$; this simplicial complex can be viewed as a simplicial approximation to the topological space $M$, associating to each set $U_i$ a point in that set, associating to each intersection $U_i \cap U_j$ a segment connecting the points associated to these two separate sets, and associating to the intersection $U_i \cap U_j \cap U_k$ the triangle formed by the segments associated to the separate pairs of intersecting sets. The sheaf cochain groups of the covering $\mathcal{U}$ with coefficients in the constant sheaf $\mathcal{R}$ can be identified with the ordinary cochains $C_i$ of this simplicial complex, and the coboundary operators then clearly coincide so the two sets of cochain groups lead to the same cohomology groups.

The cohomology groups of $M$ with real coefficients can be expressed in terms of differential forms by deRham’s Theorem. For an arbitrary $C^\infty$ manifold $M$ let $\mathcal{E}^p$ be the sheaf of germs of $C^\infty$ complex-valued differential forms of degree $p$ on $M$ and let $d : \mathcal{E}^p \to \mathcal{E}^{p+1}$ be the operator of exterior differentiation, which satisfies $dd = 0$. The kernel of the operator $d$ is the subsheaf $\mathcal{E}^p_c \subset \mathcal{E}^p$ of closed differential forms of degree $p$, and the local form of deRham’s Theorem is the assertion that the sequence

(D.14) \[ 0 \to \mathcal{E}^{p-1}_c \to \mathcal{E}^p \xrightarrow{d} \mathcal{E}^p_c \to 0 \]

is an exact sequence of sheaves for any degree $p > 0$. Of course $\mathcal{E}^0_c = \mathcal{C}$, the subsheaf of $\mathcal{E}^0$ consisting of germs of constant functions; and $\mathcal{E}^n_c = \mathcal{E}^n$ and $\mathcal{E}^p = 0$ for $p > n$ for a manifold $M$ of dimension $n$. For surfaces $\mathcal{E}^2_c = \mathcal{E}^2$ and $\mathcal{E}^p = 0$ for $p > 2$, so that there are just the two exact sequences

(D.15) \[ 0 \to \mathcal{C} \to \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1_c \to 0 \]

and

(D.16) \[ 0 \to \mathcal{E}^1_c \to \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \to 0. \]

The sequences of cross-sections of these two exact sequence of sheaves are not necessarily exact at the right end; the extent to which it fails to be exact is measured by the deRham groups of the manifold $M$, the quotient groups

(D.17) \[ \mathcal{H}^1(M) = \frac{\Gamma(M, \mathcal{E}^1)}{d\Gamma(M, \mathcal{E}^0)} \quad \text{and} \quad \mathcal{H}^2(M) = \frac{\Gamma(M, \mathcal{E}^2)}{d\Gamma(M, \mathcal{E}^1)}. \]

where as usual $\Gamma(M, \mathcal{E}^p)$ denotes the space of sections of the sheaf $\mathcal{E}^p$. The sheaves $\mathcal{E}^p$ are fine sheaves, so that $H^q(M, \mathcal{E}^{p-1}) = 0$ whenever $p,q > 0$, and from the exact cohomology sequences associated to the exact sequences of sheaves (D.15) (D.16) there result the global form of deRham’s Theorem, the isomorphisms

(D.18) \[ \mathcal{H}^p(M) \cong H^p(M, \mathcal{C}) \quad \text{for} \quad p = 1, 2. \]

A general discussion of differential forms on differentiable manifolds, and proofs of the basic properties that will be used here, can be found in M. Spivak, Calculus on Manifolds, (Benjamin, 1965) as well as in W. Fulton, Algebraic Topology (Springer, 1995), among other places. The general properties of sheaves that arise in this discussion are reviewed in Appendix C.
The cohomology class in $H^p(M, \mathbb{C})$ that is associated to the element in the deRham group $\mathcal{H}^p(M)$ represented by a closed differential form $\phi \in \Gamma(M, \mathcal{E}_p)$ under the deRham isomorphism (D.18) is the period class of the differential form $\phi$. This abstract cohomological interpretation of the period class has a more geometric form; the classical statement of deRham’s Theorem is the assertion that the mapping that associates to a differential form $\phi \in \Gamma(M, \mathcal{E}_p)$ the linear functional on the homology group $H_p(M)$ defined by integration of the differential form $\phi$ along representative cycles is an isomorphism between the deRham group $\mathcal{H}^p(M)$ and the cohomology group $H^p(M, \mathbb{C})$ in each dimension $p$. In particular the exact differential forms, those in $d\Gamma(M, \mathcal{E}^{p-1})$, are precisely the differential forms having zero integrals along all the cycles of $M$; and any linear functional on the cycles can be represented as the integral of a suitable closed differential form. Two closed differential forms $\phi, \psi$ that differ by an exact differential form are said to be cohomologous, and that is indicated by writing $\phi \sim \psi$; so the deRham isomorphism can be rephrased as the assertion that the space of cohomology classes of closed differential forms of degree $p$ is a vector space that is naturally dual to the homology group $H_p(M)$ by integration. The deRham isomorphism for real cohomology is described correspondingly. The subgroup of cohomology classes of closed differential forms having integral periods on all the cycles of $M$ form a lattice subgroup of the deRham group that is naturally isomorphic to the integral cohomology group $H^p(M, \mathbb{Z})$.

The exterior product of any two closed differential forms $\phi, \psi \in \Gamma(M, \mathcal{E}_1)$ is a closed differential form $\phi \wedge \psi \in \Gamma(M, \mathcal{E}_2)$; and if $\phi \sim \phi'$ and $\psi \sim \psi'$ then clearly $\phi \wedge \psi \sim \phi' \wedge \psi'$, so this yields a well defined skew-symmetric bilinear mapping

$$\mathcal{H}^1(M) \times \mathcal{H}^1(M) \longrightarrow \mathcal{H}^2(M),$$

the cup product mapping. Under the deRham isomorphism through the period classes of these differential forms this induces the skew-symmetric bilinear mapping

$$H^1(M, \mathbb{C}) \times H^1(M, \mathbb{C}) \longrightarrow H^2(M, \mathbb{C}),$$

that associates to cohomology classes $\phi, \psi \in H^1(M, \mathbb{C})$ a cohomology class $\phi \cup \psi$ called the cup product of these cohomology classes. Since the manifold $M$ is two-dimensional the composition of the exterior product mapping in the deRham group and the isomorphism $\mathcal{H}^2(M) \cong \mathbb{C}$ that associates to the deRham class represented by a differential form $\phi \in \Gamma(M, \mathcal{E}_2)$ the value $\int_M \phi$ is the skew-symmetric bilinear mapping

$$\mathcal{H}^1(M) \times \mathcal{H}^1(M) \longrightarrow \mathbb{C}$$

that associates to the deRham classes represented by any two differential forms $\phi, \psi \in \Gamma(M, \mathcal{E}_1)$ the complex number

$$(\phi, \psi) = \int_M \phi \wedge \psi;$$

this is called the intersection form on the surface $M$. In terms of a basis $\tau_j \in H_1(M)$ for the homology of $M$ and the dual basis $\phi_i \in \Gamma(M, \mathcal{E}_1^*)$ for the deRham
group $H^1(M)$, characterized by the period conditions $\int_{\gamma_i} \phi_i = \delta^i_j$ for $1 \leq i, j \leq 2g$, the intersection form is described by the intersection matrix $P = \{ p_{ij} \}$, the $2g \times 2g$ skew-symmetric integral matrix with entries

$$
p_{ij} = (\phi_i, \phi_j) = \int_M \phi_i \wedge \phi_j.
$$

For the basis associated to a marking of the surface the intersection matrix has the following normal form.

**Theorem D.1** If $M$ is a compact oriented surface of genus $g > 0$ with a marking described by covering translations $A_j, B_j \in \Gamma$ and if $\phi_i \in \Gamma(M, \mathcal{E}_1)$ is the dual basis for the first deRham group of $M$ then in terms of this basis the intersection matrix is the $2g \times 2g$ basic skew-symmetric matrix

$$
J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
$$

where $I$ is the $g \times g$ identity matrix and $0$ is the $g \times g$ zero matrix.

**Proof:** When the differential forms $\phi_i$ are viewed as $\Gamma$-invariant differential forms on the universal covering space $\tilde{M}$ their integrals $f_i(z) = \int_{\gamma_i} \phi_i(z)$ are functions on the universal covering space $\tilde{M}$ such that $f_i(A_j z) = f_i(z) + \delta^i_j$, $f_i(B_j z) = f_i(z) + \delta^i_{i+g}$, and $f_i(C_j z) = f_i(z)$ for the commutator $C_j = [A_j, B_j]$. By Stokes’s Theorem

$$
p_{jk} = \int_M \phi_j \wedge \phi_k = \int_\Delta d(f_j \phi_k) = \int_{\partial \Delta} f_j \phi_k
$$
in terms of the fundamental polygon $\Delta$. The boundary $\partial \Delta$ is described explicitly in equation (D.2), and it follows that

$$
p_{jk} = \sum_{i=1}^g \int_{C_1 \cdots C_{i-1} \tilde{\alpha}_i - C_1 \cdots C_i B_i \tilde{\alpha}_i} f_j(z) \phi_k(z) + \sum_{i=1}^g \int_{C_1 \cdots C_{i-1} A_i \tilde{\beta}_i - C_1 \cdots C_i \tilde{\beta}_i} f_j(z) \phi_k(z)
$$

$$
= \sum_{i=1}^g \int_{\tilde{\alpha}_i} \left( f_j(z) \phi_k(z) - (f_j(z) + \delta^i_{i+g}) \phi_k(z) \right)
$$

$$
+ \sum_{i=1}^g \int_{\tilde{\beta}_i} \left( (f_j(z) + \delta^i_{i+g}) \phi_k(z) - f_j(z) \phi_k(z) \right)
$$

$$
= -\sum_{i=1}^g \delta^i_{i+g} \int_{\tilde{\alpha}_i} \phi_k(z) + \sum_{i=1}^g \delta^i_{i+g} \int_{\tilde{\beta}_i} \phi_k(z)
$$

$$
= \sum_{i=1}^g \left( -\delta^i_{i+g} \delta^k + \delta^i_{i+g} \delta^k \right)
$$

$$
= \delta^{i+g}_{k} - \delta^i_{k+g}.
$$
which is the result asserted and thereby concludes the proof.

**Corollary D.2** Any two intersection matrices $P$ and $\tilde{P}$ for a compact oriented surface of genus $g > 0$ are related by $\tilde{P} = QP^tQ$ for some matrix $Q \in \text{Gl}(2g, \mathbb{Z})$, and consequently $\det P = 1$ for any intersection matrix $P$.

**Proof:** If $P$ is the intersection matrix of the surface $M$ in terms of a basis $\phi_i$ of the deRham group the intersection matrix $\tilde{P}$ in terms of another basis $\tilde{\phi}_i = \sum_{j=1}^{2g} q_{ij} \phi_j$ has the form

$$\tilde{p}_{ij} = \int_M \tilde{\phi}_i \wedge \tilde{\phi}_j = \sum_{k,l=1}^{2g} q_{ik} \phi_k \wedge q_{jl} \phi_l = \sum_{k,l=1}^{2g} \int_M q_{ik} p_{kl} q_{jl},$$

or in matrix terms $\tilde{P} = QP^tQ$. Since one intersection matrix is the basic skew-symmetric matrix $J$ by the preceding theorem it follows that any other intersection matrix is of the form $PJ^tQ$ for some invertible matrix $Q$ and consequently $\det P = \det J = 1$. That suffices for the proof.
Appendix E

Cohomology of Groups

E.1 Definitions and Basic Properties

Various analytical and geometrical constructions that arise in the study of compact Riemann surfaces involve the action of the covering translation group on the universal covering space of the surface. The universal covering space is both topologically and analytically trivial, in natural senses, so structures on the quotient space to a considerable extent are determined by the structure of the covering translation group; in particular the cohomology of the covering translation group reflects significant properties of the geometry of the quotient space. Since the cohomology of groups possibly is not so familiar and the notation that will be adopted here is not altogether standard, among other things in that groups will be viewed as acting on the right rather than on the left, a brief survey of the notation and of some of the basic properties of the cohomology of groups will be included in this appendix.¹

A multiplicative group Γ acts as a group of operators on the right on an additive abelian group V if there is a mapping $V \times \Gamma \rightarrow V$ that associates to any elements $v \in V$ and $T \in \Gamma$ an element $v|T \in V$ such that:

(i) for each $T \in \Gamma$ the mapping $v \mapsto v|T$ is an automorphism of the group V;
(ii) if $I \in \Gamma$ is the identity then $v|I = v$ for all $v \in V$;
(iii) $v|(T_1T_2) = (v|T_1)|T_2$ for all $v \in V$ and $T_1, T_2 \in \Gamma$.

If Γ acts as a group of operators on the right on two additive abelian groups $V_1, V_2$, a Γ-homomorphism $\phi : V_1 \rightarrow V_2$ is a homomorphism of abelian groups such that $\phi(v|T) = \phi(v)|T$ for all $v \in V_1$ and all $T \in \Gamma$.

For any multiplicative group Γ and for any integer $n \geq 0$ let $X_n(\Gamma)$ be the additive free abelian group generated by the symbols $(T_0, T_1, \ldots, T_n)$ for arbitrary $T_i \in \Gamma$, but where $(T_0, T_1, \ldots, T_n) = 0$ if $T_i = T_{i-1}$ for any index $i$.

¹A more detailed treatment of this material can be found in S. MacLane, Homology, (Springer, 1994), to which reference is made for the proofs that are not included here.
The group $\Gamma$ acts as a group of operators on the right on the abelian group $X_n(\Gamma)$ by setting
\begin{equation}
(T_0, T_1, \ldots, T_n)|T = (T_0 T, T_1 T, \ldots, T_n T)
\end{equation}
for the free generators of $X_n(\Gamma)$. For any index $n > 0$ introduce the group homomorphism
\begin{equation}
\partial : X_n(\Gamma) \longrightarrow X_{n-1}(\Gamma)
\end{equation}
defined on the free generators of $X_n$ by
\begin{equation}
\partial(T_0, T_1, \ldots, T_n) = \sum_{i=0}^{n} (-1)^i (T_0, \ldots, T_{i-1}, T_i+1, \ldots, T_n);
\end{equation}
it is a straightforward exercise to verify that this is compatible with the condition that $(T_0, T_1, \ldots, T_n) = 0$ if $T_i = T_{i-1}$. These homomorphisms clearly commute with the operation of $\Gamma$ on $X_n(\Gamma)$, so are also $\Gamma$-homomorphisms; and it is another straightforward exercise to verify that $\partial \partial = 0$. On the other hand it is somewhat less straightforward to see that
\begin{equation}
X_0(\Gamma) \xrightarrow{\partial} X_1(\Gamma) \xleftarrow{\partial} X_2(\Gamma) \xleftarrow{\partial} \cdots
\end{equation}
is an exact sequence of $\Gamma$-homomorphisms. To demonstrate that, introduce the group homomorphisms $\sigma : X_n(\Gamma) \longrightarrow X_{n+1}(\Gamma)$ defined on the free generators of $X_n$ by $\sigma(T_0, T_1, \ldots, T_n) = (I, T_0, T_1, \ldots, T_n)$ where $I \in \Gamma$ is the identity element. One more straightforward calculation shows that $\partial \sigma + \sigma \partial = I$ is the identity homomorphism on $X_n(\Gamma)$ for $n > 0$; hence if $f \in X_n(\Gamma)$ for $n > 0$ and if $\partial f = 0$ then $f = (\partial \sigma + \sigma \partial) f = \partial(\sigma f)$, so the sequence (E.3) is exact. The cohomology groups of $\Gamma$ with coefficients in an abelian group $V$ on which $\Gamma$ acts on the right are defined to be the cohomology groups of the sequence of $\Gamma$-homomorphisms
\begin{equation}
\text{Hom}_\Gamma(X_0(\Gamma), V) \xrightarrow{\delta} \text{Hom}_\Gamma(X_1(\Gamma), V) \xrightarrow{\delta} \text{Hom}_\Gamma(X_2(\Gamma), V) \xrightarrow{\delta} \cdots,
\end{equation}
where $\text{Hom}_\Gamma$ denotes the group of $\Gamma$-homomorphisms and $\delta(f) = f \circ \partial$. In more detail, the group $C^n_0(\Gamma, V) = \text{Hom}_\Gamma(X_n(\Gamma), V)$, called the group of homogeneous $n$-cochains of $\Gamma$ with coefficients in $V$, can be described alternatively as
\begin{equation}
C^n_0(\Gamma, V) = \left\{ f : \Gamma^{n+1} \longrightarrow V \mid \begin{array}{l}
f(T_0 T, T_1 T, \ldots, T_n T) = \\
f(T_0, T_1, \ldots, T_n)|T, \text{ and} \\
f(T_0, T_1, \ldots, T_n) = 0 \\
\text{if } T_i = T_{i-1} \text{ for any } i,
\end{array} \right\}
\end{equation}
since a homomorphism $f \in \text{Hom}(X_n(\Gamma), V)$ is determined by its values on the free generators of $X_n(\Gamma)$. The coboundary homomorphism
\begin{equation}
\delta : C^n_0(\Gamma, V) \longrightarrow C^{n+1}_0(\Gamma, V)
\end{equation}
E.1. DEFINITIONS AND BASIC PROPERTIES

takes an \( n \)-cochain \( f \in C_0^n(\Gamma, V) \) to the \((n+1)\)-cochain \( \delta f \in C_0^{n+1}(\Gamma, V) \) defined by

\[
\delta f(T_0, T_1, \ldots, T_{n+1}) = f\partial(T_0, T_1, \ldots, T_{n+1})
= \sum_{i=0}^{n+1} (-1)^i f(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_{n+1}).
\]

A cochain \( f \) is a \textit{cocycle} if \( \delta f = 0 \) and is a \textit{coboundary} if \( f = \delta g \) for some cochain \( g \); the cocycles are the kernels of the coboundary homomorphisms and form subgroups \( Z_0^n(\Gamma, V) \subset C_0^n(\Gamma, V) \) for all \( n \geq 0 \), while the coboundaries are the images of the coboundary homomorphisms and form subgroups \( B_0^n(\Gamma, V) \subset C_0^n(\Gamma, V) \) for all \( n > 0 \), where the latter definition is extended by setting \( B_0^0(\Gamma, V) = 0 \). Since \( \partial\partial = 0 \) every coboundary is a cocycle, or equivalently \( B_0^n(\Gamma, V) \subset Z_0^n(\Gamma, V) \); the quotient groups are the \textit{cohomology groups}

\[
H^n(\Gamma, V) = \frac{Z_0^n(\Gamma, V)}{B_0^n(\Gamma, V)}
\]

of the group \( \Gamma \) with coefficients in \( V \) for all indices \( n \geq 0 \).

The cohomology groups can be shown to satisfy the expected naturality properties, although the details will not be included here; in particular any \( \Gamma \)-homomorphism \( \phi : V_1 \rightarrow V_2 \) naturally induces homomorphisms

\[
\phi^* : H^n(\Gamma, V_1) \rightarrow H^n(\Gamma, V_2),
\]

and the compositions of \( \Gamma \)-homomorphisms induce the corresponding compositions of homomorphisms of the cohomology groups. As for any cohomology theory, a critical property is that to any short exact sequence of \( \Gamma \)-homomorphisms

\[
0 \rightarrow V_1 \xrightarrow{\phi} V_2 \xrightarrow{\psi} V_3 \rightarrow 0
\]

there is associated a long exact sequence of cohomology groups

\[
\cdots \rightarrow H^n(\Gamma, V_1) \xrightarrow{\phi} H^n(\Gamma, V_2) \xrightarrow{\psi} H^n(\Gamma, V_3) \xrightarrow{\delta} H^{n+1}(\Gamma, V_1) \rightarrow \cdots
\]

The proof of the exactness of the cohomology sequence in general will not be included, since it parallels quite closely the proof of the exactness of the corresponding cohomology sequence in sheaf cohomology; but the proof of the exactness in those special cases in which explicit forms of the connecting homomorphism \( \delta \) are needed will be included, and the proof of the general case can be constructed by following the pattern of the proof of the exactness in those special cases. The naturality properties, the exactness of the cohomology sequence, and the identification of the cohomology groups in a few standard cases can be shown to characterize the cohomology groups intrinsically.

In calculations it is often more convenient to use the group \( C^n(\Gamma, V) \) of \textit{inhomogeneous cochains} defined by

\[
C^n(\Gamma, V) = \left\{ v : \Gamma^n \rightarrow V \mid \begin{array}{l} v(T_1, \ldots, T_n) = 0 \quad \text{if} \quad T_i = I \quad \text{for any} \quad i \end{array} \right\}.
\]
To a homogeneous cochain \( f \in C^n_0(\Gamma, V) \) there can be associated its inhomogeneous form \( v \) defined by

\[
(E.10) \quad v(T_1, \ldots, T_n) = f(I, T_n, T_{n-1}T_n, T_{n-2}T_{n-1}T_n, \ldots, T_1T_2 \cdots T_n)
\]

for any \( T_i \in \Gamma \). If \( S_i = T_{n-i+1}T_{n-i+2} \cdots T_n \) for \( 1 \leq i \leq n \), so that \( S_1 = T_n, \ S_2 = T_{n-1}T_n, \ S_3 = T_{n-2}T_{n-1}T_n, \ldots, S_n = T_1T_2 \cdots T_n \), then conversely \( T_n = S_1 \) and \( T_{n-i+1} = S_iS_{i-1}^{-1} \) for \( 2 \leq i \leq n \); and the homogeneous cochain can be recaptured from its inhomogeneous form by

\[
(E.11) \quad f(I, S_1, S_2, \ldots, S_n) = v(S_nS_{n-1}^{-1}, S_{n-1}S_{n-2}^{-1}, \ldots, S_2S_1^{-1}, S_1)
\]

since the homogeneous cochain \( f \) clearly is determined fully just by the values \( f(I, S_1, S_2, \ldots, S_n) \). The condition that \( f(I, S_1, S_2, \ldots, S_n) = 0 \) if \( S_i = I \) or \( S_i = S_{i-1} \) for any index \( i \) in the range \( 2 \leq i \leq n \) corresponds to the condition that \( v(T_1, \ldots, T_n) = 0 \) if \( T_i = I \) for any index \( i \) in the range \( 1 \leq i \leq n \). The mapping that associates to a homogeneous cochain its inhomogeneous form thus is an isomorphism from the group \( C^n_0(\Gamma, V) \) of homogeneous cochains to the group \( C^n(\Gamma, V) \) of inhomogeneous cochains. In particular to a homogeneous cochain \( f \in C^n_0(\Gamma, V) \) there is associated the inhomogeneous form \( v = f(I) \), so that \( C^n(\Gamma, V) = V \); and the homogeneous form is determined by its inhomogeneous form since \( f(T) = f(I)|T = v|T \). If \( v \in C^{n-1}(\Gamma, V) \) is the inhomogeneous form of a cochain \( f \in C^n_0(\Gamma, V) \) and if \( w \in C^n(\Gamma, V) \) is the inhomogeneous form of the coboundary \( \delta f \in C^n_0(\Gamma, V) \) then for any elements \( S_i, T_i \in \Gamma \)

\[
w(T_1, \ldots, T_n) = (\delta f)(I, T_n, T_{n-1}T_n, \ldots, T_1T_2 \cdots T_n)
\]

\[
= (\delta f)(I, S_1, S_2, \ldots, S_n)
\]

\[
= f(S_1, S_2, \ldots, S_n) + \sum_{i=1}^{n} (-1)^i f(I, S_1, \ldots, S_{i-1}, S_{i+1} \ldots S_n)
\]

\[
= f(I, S_2S_1^{-1}, \ldots, S_nS_{n-1}^{-1})|S_1 + \sum_{i=1}^{n} (-1)^i f(I, S_1, \ldots, S_{i-1}, S_{i+1} \ldots S_n)
\]

\[
= v(S_nS_{n-1}^{-1}, S_{n-1}S_{n-2}^{-1}, \ldots, S_2S_1^{-1})|S_1
\]

\[
- v(S_nS_{n-1}^{-1}, S_{n-1}S_{n-2}^{-1}, \ldots, S_2S_1^{-1})S_2
\]

\[
+ \sum_{i=2}^{n-1} (-1)^i v(S_nS_{n-1}^{-1}, S_{i+2}S_{i+1}^{-1}, S_{i+1}S_{i-1}^{-1}, S_{i-1}S_{i-2}^{-1}, \ldots, S_1)
\]

\[
+ (-1)^n v(S_nS_{n-1}^{-1}, S_{n-2}S_{n-3}^{-1}, \ldots, S_2S_1^{-1}, S_1)
\]

\[
= v(T_1, T_2, \ldots, T_{n-1})|T_n
\]

\[
- v(T_1, T_2, \ldots, T_{n-2}, T_{n-1}T_n)
\]

\[
+ \sum_{i=2}^{n-1} (-1)^i v(T_1, \ldots, T_{n-i}, T_{n-i}T_{n-i+1}, T_{n-i+2}, \ldots, T_n)
\]

\[
+ (-1)^n v(T_2, T_3, \ldots, T_n).
\]
This expresses the coboundary operator in terms of the inhomogeneous cocycles; but it is perhaps clearer to change the index of summation and to rewrite the formula as

$$
(\delta v)(T_1, \ldots, T_n) = v(T_1, T_2, \ldots, T_{n-1})|T_n + (-1)^n v(T_2, T_3, \ldots, T_n) \\
+ \sum_{i=1}^{n-1} (-1)^{n+i} v(T_1, \ldots, T_{i-1}, T_i T_{i+1}, T_{i+2}, \ldots, T_n)
$$

for any inhomogeneous cochain $v \in C^{n-1}(\Gamma, V)$. For the initial cases

$$
\begin{align*}
\forall v \in C^0(\Gamma, V) : \quad (\delta v)(T_1) &= v|T_1 - v, \\
\forall v \in C^1(\Gamma, V) : \quad (\delta v)(T_1, T_2) &= v(T_1)|T_2 + v(T_2) - v(T_1 T_2) \\
\forall v \in C^2(\Gamma, V) : \quad (\delta v)(T_1, T_2, T_3) &= v(T_1, T_2)|T_3 - v(T_2, T_3) \\
&\quad + v(T_1 T_2, T_3) - v(T_1, T_2 T_3).
\end{align*}
$$

The \textit{inhomogeneous cocycles} are the cochains $v \in C^n(\Gamma, V)$ such that $\delta v = 0$ and form subgroups $Z^n(\Gamma, V) \subset C^n(\Gamma, V)$, while the \textit{inhomogeneous coboundaries} are the cochains $v \in \delta C^{n-1}(\Gamma, V)$ and form subgroups $B^n(\Gamma, V) \subset C^n(\Gamma, V)$ where $B^0(\Gamma, V) = 0$. The cohomology groups are isomorphic to the quotients

$$
H^n(\Gamma, V) \cong \frac{Z^n(\Gamma, V)}{B^n(\Gamma, V)} \quad \text{for } n > 0
$$

while

$$
H^0(\Gamma, V) \cong Z^0(\Gamma, V) = V^\Gamma
$$

where

$$
V^\Gamma = \left\{ v \in V \mid v|T = v \quad \text{for all } T \in \Gamma \right\}
$$

is the subgroup of $\Gamma$-invariant elements of $V$. Then for $n = 1$ the group of inhomogeneous 1-cocycles is

$$
Z^1(\Gamma, V) = \left\{ v : \Gamma \to V \mid v(T_1)|T_2 = v(T_1 T_2) - v(T_2) \quad v(I) = 0 \right\}
$$

while the subgroup of inhomogeneous 1-coboundaries is

$$
B^1(\Gamma, V) = \left\{ v : \Gamma \to V \mid v(T) = w|T - w \quad \text{for some } w \in V \right\},
$$
and for \( n = 2 \)
\[(E.19)\]
\[
Z^2(\Gamma, V) = \left\{ v: \Gamma \times \Gamma \to V \mid \begin{aligned}
v(T_1, T_2)T_3 &= v(T_1, T_2T_3) - v(T_1T_2, T_3) + v(T_2, T_3) \\
v(I, T) &= v(T, I) = 0
\end{aligned} \right\}
\]
while
\[(E.20)\]
\[
B^2(\Gamma, V) = \left\{ v: \Gamma \times \Gamma \to V \mid v(T_1, T_2) = w(T_1T_2) - w(T_2) - w(T_1)T_2) \right\}
\]
where \( w: \Gamma \to V \) and \( w(I) = 0 \).

**E.2 Example: Trivial Group Action**

A particularly simple case is that in which a multiplicative group \( \Gamma \) acts trivially on an additive abelian group \( V \), so that \( vT = v \) for all \( T \in \Gamma \) and all \( v \in V \), or equivalently \( V^\Gamma = V \); by (E.15) then
\[(E.21)\]
\[
H^0(\Gamma, V) \cong V \quad \text{if } \Gamma \text{ acts trivially on } V.
\]
Next by (E.17) the inhomogeneous 1-cocycles are mappings \( v: \Gamma \to V \) such that \( v(I) = 0 \) and \( v(T_1T_2) = v(T_1) + v(T_2) \), so \( Z^1(\Gamma, V) = \text{Hom}(\Gamma, V) \); by (E.18) the inhomogeneous 1-coboundaries are trivial, and consequently
\[(E.22)\]
\[
H^1(\Gamma, V) \cong \text{Hom}(\Gamma, V) \quad \text{if } \Gamma \text{ acts trivially on } V.
\]
The second cohomology group is equally interesting and possibly less familiar. By (E.19) the group of inhomogeneous two-cocycles consists of those mappings \( v: \Gamma \times \Gamma \to V \) such that \( v(I, T) = v(T, I) = 0 \) and
\[(E.23)\]
\[
v(R, S) - v(R, ST) + v(RS, T) - v(S, T) = 0
\]
for any \( R, S, T \in \Gamma \); and by (E.20) the subgroup \( B^2(\Gamma, V) \) of inhomogeneous two-coboundaries consists of those two-cocycles of the form
\[(E.24)\]
\[
v(S, T) = w(S) + w(T) - w(ST)
\]
for a mapping \( w: \Gamma \to V \) such that \( w(I) = 0 \). While there are natural direct interpretations of the quotient cohomology group, what is quite useful for present purposes is a rather more indirect interpretation of the second cohomology group in terms of a presentation of the group \( \Gamma \), following H. Hopf\(^2\) and beginning with the following preliminary observation.

\(^2\)H. Hopf, "Fundamentalgruppe und zweite Bettische Gruppe," *Commentarii Mathematici Helvetici* 14(1941), pp. 257-309. See the historical discussion in MacLane’s *Homology*, p.137.
Lemma E.1 If $\Gamma$ is a finitely generated free group acting trivially on the right on an abelian group $V$ then $H^2(\Gamma, V) = 0$.

Proof: By (E.24) it is only necessary to show that for any inhomogeneous cocycle $v \in Z^2(\Gamma, V)$ there is a mapping $w : \Gamma \rightarrow V$ such that $w(1) = 0$ and

$$w(ST) = w(S) + w(T) - v(S,T)$$

for all $S,T \in \Gamma$. If $v \in Z^2(\Gamma, V)$ it follows from (E.23) for $R = S^{-1} = T$ that $v(T,T^{-1}) = v(T^{-1},T)$. Now choose arbitrary values $w(T_1) \in V$ for a set of free generators $T_i$ of the group $\Gamma$, and set $w(T_i^{-1}) = v(T_i, T_i^{-1}) - w(T_i) = v(T_i^{-1}, T_i) - w(T_i)$. Since any element of the free group $\Gamma$ can be written uniquely as a product of the symbols $T_i$ and $T_i^{-1}$, equation (E.25) can be used to define $w(S)$ for any element $S \in \Gamma$ if it is demonstrated that $w(T_iT_i^{-1}) = w(T_i^{-1}T_i) = 0$ for each free generator $T_i$ and that the value assigned to $w(RST)$ for any elements $R,S,T \in \Gamma$ is independent of the way in which this triple product is associated. The first follows readily from the way in which $w(T_i^{-1})$ is defined, while the second is a consequence of the cocycle condition (E.23) and can be verified by a straightforward calculation. That suffices for the proof.

The preceding lemma is also true for a finitely generated free group $\Gamma$ acting trivially on a multiplicative abelian group, such as the group $\mathbb{C}^*$ or any finite subgroup of $\mathbb{C}^*$, rather than on an additive abelian group $V$; for the argument used only the commutativity of the coefficient group $V$. For present purposes the principal application of the preceding lemma is to the following general result.

Theorem E.2 (Hopf’s Theorem) If a group $\Gamma$ acts trivially on the right on an abelian group $V$ and if $\Gamma$ has a presentation $\Gamma = F/K$, where $F$ is a finitely generated free group and $K \subset F$ is a normal subgroup, then

$$H^2(\Gamma, V) \cong \frac{\text{Hom}(K/[K,F], V)}{\text{im}(\text{Hom}(F,V))}$$

where $[K,F] \subset K$ is the normal subgroup of $F$ generated by commutators $[S,T]$ for $S \in K$ and $T \in F$ and

$$i : \text{Hom}(F,V) \rightarrow \text{Hom}(K/[K,F], V)$$

is the restriction of a homomorphism in $\text{Hom}(F,V)$ to the subgroup $K$.

Proof: If $v \in Z^2(\Gamma, V)$ and $p : F \rightarrow \Gamma$ is the natural quotient mapping then $v_p(S,T) = v(p(S), p(T)) \in Z^2(F, V)$. Since $H^2(F, V) = 0$ by the preceding lemma there is an inhomogeneous 1-cocycle $w \in \text{C}^1(F, V)$ such that $v_p = \delta w$, hence such that $v_p(S,T) = w(S) + w(T) - w(ST)$ for all $S,T \in F$. If $S \in K$ then $v_p(S,T) = v(p(S), p(T)) = v(1, p(T)) = 0$ and hence $w(ST) = w(S) + w(T)$, and the same of course is true if $T \in K$. One consequence of this observation is
that \( w|K \in \text{Hom}(K, V) \). Another consequence is that if \( S \in K \) and \( T \in F \) then \( T^{-1}ST \in K \) and \( w(S) + w(T) = w(ST) = w(T \cdot T^{-1}ST) = w(T) + w(T^{-1}ST) \); therefore \( w(T^{-1}ST) = w(S) \), and consequently \( w|[K,F] = 0 \) so the cochain \( w \) restricts to a homomorphism \( w|K \in \text{Hom}(K/[K,F], V) \). Since any two cochains that have the same coboundary \( v \in Z^2(\Gamma, V) \) differ by an element of \( Z^1(F, V) = \text{Hom}(F, V) \), there results a well defined homomorphism

\[
(E.28) \quad p^* : Z^2(\Gamma, V) \longrightarrow \frac{\text{Hom}(K/[K,F], V)}{i(\text{Hom}(F,V))}.
\]

The kernel of this homomorphism consists of those cocycles \( v \in Z^2(\Gamma, V) \) such that \( v_p = \delta w \) for a cochain \( w \in C^1(F, V) \) which, after modification by the addition of a cocycle in \( Z^1(F, V) = \text{Hom}(F, V) \), can be supposed to satisfy \( w|K = 0 \); but then \( w \in C^1(\Gamma, V) \) so \( v = \delta w \in B^2(\Gamma, V) \) and hence the kernel of the homomorphism \( p^* \) is the subgroup \( B^2(\Gamma, V) \subset Z^2(\Gamma, V) \). To conclude the proof it remains only to show that the homomorphism \( p^* \) is surjective. Any element \( v \in \text{Hom}(K, V) \) can be extended to a mapping \( w : F \longrightarrow V \) by choosing a coset decomposition \( F = \cup_i K T_i \), choosing arbitrary values \( w(T_i) \in V \), and setting \( w(ST_i) = w(S) + w(T_i) \) for all \( S \in K \). If \( R \in K \) and \( T \in F \) then \( T = ST_i \) for some \( S \in K \) and \( w(RT) = w(RST_i) = w(RS) + w(T_i) = w(R) + w(S) + w(T_i) = w(R) + w(T) \). If \( w|[K,F] = 0 \) as well then for any \( S \in K \) necessarily \( w(T_iS) = w(S\cdot S^{-1}T_iST_i^{-1}T_i) = w(S[S^{-1},T_i]T_i) = w(S[S^{-1},T_i]) + w(T_i) = w(S) + w(T_i) \); and as in the preceding argument it is also the case that \( w(RT) = w(R) + w(T) \) whenever \( R \in F \) and \( T \in K \). The expression \( v(S,T) = w(S) + w(T) - w(ST) \) is a cocycle \( v \in Z^2(F, V) \). If \( R \in K \) then \( v(RS,T) = w(RS) + w(T) - w(RST) = w(R) + w(S) + w(T) - w(R) - w(ST) = v(S,T) \), so that \( v(S,T) \) depends only on the coset of \( T \) modulo \( K \); and the same argument shows that \( v(S,T) \) also depends only on the coset of \( S \) modulo \( K \). That shows that actually \( v \in Z^2(\Gamma, V) \) as desired, hence concludes the proof.

In the special cases in which the group \( \Gamma \) has a presentation \( \Gamma = F/K \) where \( F \) is a free group and \( K \subset F \) is actually a subgroup \( K \subset [F,F] \) of the commutator subgroup of \( F \), Hopf’s Theorem can be restated in a simpler and more explicit form: this is the special case that is of interest for surface groups.

**Corollary E.3** If a group \( \Gamma \) acts trivially on the right on an abelian group \( V \) and if \( \Gamma \) has a presentation \( \Gamma = F/K \), where \( F \) is a finitely generated free group and \( K \subset F \) is a normal subgroup such that \( K \subset [F,F] \), the natural quotient mapping \( p : F \longrightarrow \Gamma \) induces an isomorphism

\[
(E.29) \quad p^* : H^2(\Gamma, V) \longrightarrow \frac{\text{Hom}(K/[K,F], V)}{i(\text{Hom}(F,V))}.
\]

This isomorphism takes the cohomology class represented by an inhomogeneous cocycle \( v \in Z^2(\Gamma, V) \) to the homomorphism in \( \text{Hom}(K/[K,F], V) \) that is the restriction to \( K \subset [F,F] \) of the mapping \( w : [F,F] \longrightarrow V \) for which
\[ w([S,T]) = v([p(S), p(T), p(T)p(S)]) - v(p(S), p(T)) + v(p(T), p(S)) \]

for any \( S, T \in F \) and

\[ w(C_1C_2) = w(C_1) + w(C_2) - v(p(C_1), p(C_2)) \]

for any commutators \( C_1, C_2 \in [F, F] \).

\textbf{Proof:} If \( K \subset [F, F] \) then \( w|K = 0 \) for any homomorphism \( w \in \text{Hom}(F, V) \), so \( i(\text{Hom}(F, V)) = 0 \) and the isomorphism (E.26) of the preceding theorem takes the simpler form (E.29). In the proof of the preceding theorem the homomorphism \( w \in \text{Hom}(K/[K, F], V) \) associated to an inhomogeneous cocycle \( v \in Z^2(\Gamma, V) \) is the restriction to \( K \subset [F, F] \) of any inhomogeneous cochain \( w \in C^1(F, V) \) such that \( \delta w = v_p \) for the cocycle \( v_p \in Z^2(\Gamma, V) \) defined by

\[ v_p(S, T) = w(S) + w(T) - w(ST), \]

which is (E.31) in the special cases in which \( S, T \in [F, F] \). Since \( w(1) = 0 \) it follows from this for \( S = T^{-1} \) that

\[ w(T^{-1}) = -w(T) + v_p(T^{-1}, T) = -w(T) + v_p(T, T^{-1}) \]

for any \( T \in F \). It also follows that

\[
\begin{align*}
  w([S,T]) &= w(ST(TS)^{-1}) \\
  &= w(ST) + w((TS)^{-1}) - v_p(ST, (TS)^{-1}) \\
  &= w(ST) - w(TS) + v_p(TS, (TS)^{-1}) - v_p(ST, (TS)^{-1}) \\
  &= -v_p(S, T) + v_p(T, S) + v_p(TS, (TS)^{-1}) - v_p(ST, (TS)^{-1});
\end{align*}
\]

but upon replacing \( R \) by \([S, T], S \) by \( TS \), and \( T \) by \((TS)^{-1}\) the cocycle condition (E.23) takes the form

\[ 0 = v_p([S,T], TS) - v_p(TS, (TS)^{-1}) + v_p(ST, (TS)^{-1}) - v_p([S,T], I), \]

so since \( v_p([S,T], I) = v(p([S,T]), I) = 0 \) then

\[ w([S,T]) = -v_p(S, T) + v_p(T, S) + v_p([S,T], TS), \]

which is (E.30). That suffices to conclude the proof.
E.3 Example: Surface Groups

When a Riemann surface is represented as the quotient of its universal covering space \( \tilde{M} \) by the covering translation group \( \Gamma \), the group \( \Gamma \) acts on the right on the complex vector space \( V^p = \Gamma(\tilde{M}, \mathcal{E}^p) \) of \( C^\infty \) differential forms of degree \( p \) on \( \tilde{M} \) by \( (\phi(T))(z) = \phi(Tz) \). The differential forms on \( \tilde{M} \) that are invariant under this action of the group \( \Gamma \) are precisely the differential forms on the quotient space \( M \), so in view of (E.15)

\[
H^0(\Gamma, V^p) = \Gamma(M, \mathcal{E}^p) \quad \text{for} \quad p = 0, 1, 2.
\]

To see that

\[
H^q(\Gamma, V^p) = 0 \quad \text{for} \quad p = 0, 1, 2 \quad \text{and} \quad q > 0,
\]
a homogeneous \( q \)-cocycle \( w(T_0, T_1, \ldots, T_q) \in Z^q_0(\Gamma, V^p) \) can be viewed as a \( C^\infty \) differential form \( w(T_0, T_1, \ldots, T_q, z) \) of degree \( p \) on the manifold \( \tilde{M} \) indexed by the elements \( T_0, \ldots, T_q \in \Gamma \). For any simply-connected open subset \( U \subset M \) the complete inverse image \( \pi^{-1}(U) \subset \tilde{M} \) is the set \( \Gamma \tilde{U} = \{ T\tilde{U} \mid T \in \Gamma \} \), where \( \tilde{U} \) is a connected component of \( \pi^{-1}(U) \) and \( T_1 \tilde{U} \cap T_2 \tilde{U} = \emptyset \) whenever \( T_1 \neq T_2 \). For any \( C^\infty \) function \( r(z) \) on \( M \) with support contained in \( U \), viewed as a \( \Gamma \)-invariant \( C^\infty \) function on \( \tilde{M} \) with support contained in \( \pi^{-1}(U) = \Gamma \tilde{U} \), the product \( r(z)w(T_0, T_1, \ldots, T_q; z) \) also is a \( C^\infty \) differential form on \( \tilde{M} \), so is a homogeneous \( q \)-cocycle, and its support is contained in \( \Gamma \tilde{U} \). The homogeneous \((q-1)\)-cochain in \( \Gamma \tilde{U} \) defined by

\[
v(T_0, T_1, \ldots, T_q-1; z) = r(z)w(I, T_0, T_1, \ldots, T_q-1; z) \quad \text{for} \quad z \in U,
\]

\[
v(T_0, T_1, \ldots, T_q-1; Tz) = v(T_0 T, T_1 T, \ldots, T_q-1 T; z) \quad \text{for} \quad z \in U, \ T \neq 1,
\]
can be extended to all of \( \tilde{M} \) by setting it equal to zero outside \( \Gamma \tilde{U} \). It is a straightforward calculation to verify that the coboundary of the cochain \( v(T_0, T_1, \ldots, T_q-1; z) \) is the cocycle \( r(z)w(T_0, T_1, \ldots, T_q; z) \), so this cocycle is cohomologous to zero. Since any cocycle can be written as a sum of such cocycles for functions \( r_i(z) \) forming a \( C^\infty \) partition of unity on \( M \) it follows that any cocycle in \( Z^q(\Gamma, V^p) \) is cohomologous to zero as asserted.

From the exact sequence of sheaves

\[
0 \longrightarrow \mathcal{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1_c \longrightarrow 0
\]
on the universal covering space \( \tilde{M} \), where \( \mathcal{E}^1_c \) is the sheaf of closed \( C^\infty \) differential forms of degree 1 and \( d \) is exterior differentiation, there follows the exact sheaf cohomology sequence beginning

\[
0 \longrightarrow \mathcal{C} \longrightarrow \Gamma(\tilde{M}, \mathcal{E}^0) \xrightarrow{d} \Gamma(\tilde{M}, \mathcal{E}^1_c) \xrightarrow{\delta} H^1(\tilde{M}, \mathcal{C});
\]
E.3. EXAMPLE: SURFACE GROUPS

since \( M \) is contractible \( H^1(\tilde{M}, \mathbb{C}) = 0 \), so this reduces to the exact sequence

\[
(E.35) \quad 0 \longrightarrow \mathbb{C} \longrightarrow V^0 \overset{d}{\longrightarrow} V_c^1 \longrightarrow 0,
\]

where \( V^0 = \Gamma(\tilde{M}, \mathcal{E}^0) \) as before and \( V_c^1 = \Gamma(\tilde{M}, \mathcal{E}_c^1) \). This is an exact sequence of right \( \Gamma \)-modules, so there results the exact cohomology sequence beginning

\[
(E.36) \quad 0 \longrightarrow H^0(\Gamma, \mathbb{C}) \longrightarrow H^0(\Gamma, V^0) \overset{d}{\longrightarrow} H^0(\Gamma, V_c^1) \overset{\delta}{\longrightarrow} H^1(\Gamma, \mathbb{C}) \longrightarrow H^1(\Gamma, V^0);
\]

and \( H^1(\Gamma, V^0) = 0 \) by (E.33). The coboundary mapping \( \delta \) in this exact sequence can be described explicitly by a diagram chase through the cochain complex associated to the exact sequence of \( \Gamma \) homomorphisms (E.35), the commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C^0(\Gamma, \mathbb{C}) & \longrightarrow & C^0(\Gamma, V^0) & \longrightarrow & C^0(\Gamma, V_c^1) & \longrightarrow & 0 \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
0 & \longrightarrow & C^1(\Gamma, \mathbb{C}) & \longrightarrow & C^1(\Gamma, V^0) & \longrightarrow & C^1(\Gamma, V_c^1) & \longrightarrow & 0 \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
0 & \longrightarrow & C^2(\Gamma, \mathbb{C}) & \longrightarrow & C^2(\Gamma, V^0) & \longrightarrow & C^2(\Gamma, V_c^1) & \longrightarrow & 0 \\
\delta & \downarrow & \delta & \downarrow & \delta & \downarrow & \\
\end{array}
\]

An inhomogeneous cocycle \( \phi \in C^0(\Gamma, V_c^1) \) representing a cohomology class in \( H^0(\Gamma, V_c^1) \) is just a closed differential form on the universal covering space \( \tilde{M} \) that is invariant under the covering translation group as in (E.15). This differential form can be written as the exterior derivative \( \phi = df \) of a \( C^\infty \) function \( f \) on \( M \), and this function in turn is a cochain \( f \in C^0(\Gamma, V^0) \) that maps to \( \phi \) under the \( \Gamma \)-homomorphism \( d \) in the first line of the commutative diagram (E.37). The coboundary of this cochain is a 1-cochain \( \delta f \in C^1(\Gamma, V^0) \), which actually is a cocycle contained in the cochain group \( C^1(\Gamma, \mathbb{C}) \); so by (E.22) it can be viewed as a homomorphism \( p_1(\phi) \in \text{Hom}(\Gamma, \mathbb{C}) \). By (E.13) this homomorphism is given explicitly by

\[
(E.38) \quad p_1(\phi)(T) = \delta f(T) = f(Tz) - f(z) = \int_z^{Tz} \phi
\]

so it is just the usual period class of the closed differential form \( \phi \); thus the exact cohomology sequence (E.36) reduces to the deRham isomorphism

\[
(E.39) \quad p_1 : \frac{\Gamma(M, \mathcal{E}_c^1)}{d\Gamma(M, \mathcal{E}^0)} \cong H^1(\Gamma, \mathbb{C}) = \text{Hom}(\Gamma, \mathbb{C})
\]

where \( p_1 \) is the usual period mapping (E.38).
For a possibly more interesting and less familiar result, the next segment of the exact cohomology sequence associated to the exact sequence (E.35) is the exact sequence

\[ H^1(\Gamma, V^0) \longrightarrow H^1(\Gamma, V^1_c) \xrightarrow{\delta} H^2(\Gamma, \mathbb{C}) \longrightarrow H^2(\Gamma, V^0); \]

and since \( H^1(\Gamma, V^0) = H^2(\Gamma, V^0) = 0 \) by (E.33) this reduces to the isomorphism

\[ (E.40) \quad \delta : H^1(\Gamma, V^1_c) \xrightarrow{\cong} H^2(\Gamma, \mathbb{C}). \]

The coboundary mapping giving this isomorphism can be described explicitly by another diagram chase through the commutative diagram (E.37). A cocycle in \( C^1(\Gamma, V^1_c) \) representing a cohomology class \( \theta \in H^1(M, V^1_c) \) is a collection of \( \mathcal{C}^\infty \) closed differential 1-forms \( \theta(T, z) \) on the universal covering space \( \tilde{M} \) such that \( \theta(I, z) = 0 \) and that \( \theta(T_1T_2, z) = \theta(T_1, T_2z) + \theta(T_2, z) \), the cocycle condition (E.13). Each of these differential forms can be written as the exterior derivative \( \theta(T, z) = df(T, z) \) of a \( \mathcal{C}^\infty \) function \( f(T, z) \) on \( \tilde{M} \), and this collection of functions is a cochain \( f \in C^1(\Gamma, V^0) \) that maps to the cochain \( \theta \) under the \( \Gamma \)-homomorphism \( d \) in the second line of the commutative diagram (E.37). The coboundary of this cocycle is a cochain \( \delta f \in C^2(\Gamma, V^0) \), which actually is a cocycle contained in the cochain group \( C^2(\Gamma, \mathbb{C}) \). By (E.13) this cocycle is given explicitly by

\[ (E.41) \quad \delta f(S, T) = f(S, Tz) + f(T, z) - f(ST, z). \]

The next segment of the exact cohomology sequence arising from the exact sequence of sheaves (E.34) is

\[ H^1(\tilde{M}, \mathcal{E}^0) \longrightarrow H^1(\tilde{M}, E^1_c) \xrightarrow{\delta} H^2(\tilde{M}, \mathbb{C}) \longrightarrow H^2(\tilde{M}, \mathcal{E}^0); \]

and \( H^1(\tilde{M}, \mathcal{E}^0) = H^2(\tilde{M}, \mathcal{E}^0) = 0 \) since \( \mathcal{E}^0 \) is a fine sheaf while \( H^2(\tilde{M}, \mathbb{C}) = 0 \) since the universal covering space \( \tilde{M} \) is contractible, so this exact sequence reduces to the identity

\[ (E.42) \quad H^1(\tilde{M}, E^1_c) = 0. \]

The exact cohomology sequence arising from the exact sequence of sheaves

\[ (E.43) \quad 0 \longrightarrow \mathcal{E}^1_c \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \longrightarrow 0 \]

on the universal covering space \( \tilde{M} \) includes the segment

\[ 0 \longrightarrow \Gamma(\tilde{M}, \mathcal{E}^1_c) \longrightarrow \Gamma(\tilde{M}, \mathcal{E}^1) \xrightarrow{d} \Gamma(\tilde{M}, \mathcal{E}^2) \xrightarrow{\delta} H^1(\tilde{M}, \mathcal{E}^1_c), \]

in which \( H^1(\tilde{M}, \mathcal{E}^1_c) = 0 \) by (E.42); so this amounts to the exact sequence of \( \Gamma \)-homomorphisms

\[ (E.44) \quad 0 \longrightarrow V^1_c \longrightarrow V^1 \xrightarrow{d} V^2 \longrightarrow 0, \]
from which there follows the exact cohomology sequence containing the segment

\[ 0 \to H^0(\Gamma, V^1_\mathcal{C}) \to H^0(\Gamma, V^1) \xrightarrow{d} H^0(\Gamma, V^2) \xrightarrow{\delta} H^1(\Gamma, V^1_\mathcal{C}) \to H^1(\Gamma, V^1). \]

Since \( H^1(\Gamma, V^1) = 0 \) by (E.33) while \( H^0(\Gamma, V^q) = \Gamma(M, \mathcal{E}^q) \) by (E.32) this reduces to the isomorphism

\[ \delta : \frac{\Gamma(M, \mathcal{E}^2)}{d\Gamma(M, \mathcal{E}^1)} \cong H^1(\Gamma, V^1_\mathcal{C}). \]

The coboundary mapping giving this isomorphism can be described explicitly by a diagram chase through the cochain complex associated to the exact sequence of \( \Gamma \)-homomorphisms (E.44), the commutative diagram

\[
\begin{array}{cccccc}
0 & \to & C^0(\Gamma, V^1_\mathcal{C}) & \to & C^0(\Gamma, V^1) & \xrightarrow{d} & C^0(\Gamma, V^2) & \to & 0 \\
0 & \to & C^1(\Gamma, V^1_\mathcal{C}) & \to & C^1(\Gamma, V^1) & \xrightarrow{d} & C^1(\Gamma, V^2) & \to & 0.
\end{array}
\]

An inhomogeneous cocycle \( \phi \in C^0(\Gamma, V^2) \) representing a cohomology class in \( H^0(\Gamma, V^2) \) is a \( C^\infty \) differential 2-form on the universal covering space \( \widetilde{M} \) that is invariant under the covering translation group, as in (E.15). Since \( \phi \) is automatically closed it can be written as the exterior derivative \( \phi = d\psi \) of a \( C^\infty \) differential 1-form \( \psi \) on \( \widetilde{M} \), and this differential form in turn is a cochain \( \psi \in C^0(\Gamma, V^1) \) that maps to \( \phi \) under the \( \Gamma \)-homomorphism \( d \) in the first line of the commutative diagram (E.46). The coboundary of this cochain is a 1-cocycle \( \delta \psi \in C^1(\Gamma, V^1) \), which actually is a cocycle contained in the cochain group \( C^1(\Gamma, V^1_\mathcal{C}) \) and represents the cohomology class that is the image of the class \( \phi \) under the isomorphism (E.45); by (E.13) this cocycle is explicitly

\[ \delta(\phi)(T) = \delta \psi(T) = \psi(Tz) - \psi(z). \]

Combining the isomorphisms (E.40) and (E.45) yields the isomorphism

\[ p_2 : \frac{\Gamma(M, \mathcal{E}^2)}{d\Gamma(M, \mathcal{E}^1)} \cong H^2(\Gamma, \mathbb{C}). \]

Combining the explicit descriptions of the isomorphisms (E.40) and (E.45) given in (E.41) and (E.47) shows that the isomorphism \( p_2 \) associates to the differential 2-form \( \phi \) on the surface \( M \) the cohomology class in \( H^2(\Gamma, \mathbb{C}) \) represented by the cocycle \( p_2(\phi) \in Z^2(\Gamma, \mathbb{C}) \) for which

\[ p_2(\phi)(S, T) = f(S, Tz) + f(T, z) - f(ST, z) \]

\[
\text{where } \phi = d\psi \text{ and } \psi(Tz) - \psi(z) = df(T, z).
\]

This too can be viewed as a period isomorphism, extending (E.39) to the next higher dimension.
The more familiar period mapping of course is the isomorphism

$$(E.50) \quad p : \frac{\Gamma(M, E^2)}{d\Gamma(M, E)} \xrightarrow{\sim} \mathbb{C}$$

that associates to any $C^\infty$ differential form $\phi \in \Gamma(M, E^2)$ its period $p(\phi) = \int_M \phi$; that this mapping is an isomorphism is just the classical deRham theorem. Combining the two isomorphisms (E.48) and (E.50) leads to an isomorphism

$$(E.51) \quad p \cdot p_2^{-1} : H^2(\Gamma, \mathbb{C}) \xrightarrow{\sim} \mathbb{C},$$

which can be described quite explicitly in terms of a marking of the surface $M$. As discussed in Appendix D.1, a marking of a compact Riemann surface $M$ of genus $g > 0$ is a representation of $M$ as a sphere with $g$ handles together with the choice of a base point $z_0 \in \hat{M}$ in the universal covering space of $M$ and a collection of $2g$ simple closed paths $\alpha_i, \beta_i \subset M$ as in Figure D.1. When the paths $\alpha_i$ and $\beta_i$ are lifted to simple paths $\tilde{\alpha}_i, \tilde{\beta}_i \subset \hat{M}$ beginning at the base point $z_0 \in \hat{M}$ their end points are $A_i z_0$ and $B_i z_0$, where $A_i, B_i \in \Gamma$ are covering translations corresponding to the homotopy classes of the paths $\alpha_i, \beta_i$; and the surface $M$ can be recaptured from the fundamental domain $\Delta \subset \hat{M}$ bounded by pairs of translates of the paths $\tilde{\alpha}_i, \tilde{\beta}_i$ by identifying the boundary paths as in Figure D.2. The covering translations $A_i, B_i$ are generators of the group $\Gamma$ and are subject to the single relation $C_1 \cdot C_2 \cdot \cdots C_g = 1$ for the commutators $C_i = [A_i, B_i]$. Alternatively the group $\Gamma$ has a presentation as the quotient $\Gamma = F/K$ of the free group $F$ on $2g$ generators $\hat{A}_i, \hat{B}_i$, representing the generators $A_i, B_i$ of $\Gamma$, modulo the normal subgroup $K \subset F$ generated by the element $\hat{C} = \hat{C}_1 \cdot \hat{C}_2 \cdot \cdots \hat{C}_g$ for the commutators $C_i = [\hat{A}_i, \hat{B}_i]$.

**Theorem E.4** In terms of the presentation of the covering translation group $\Gamma$ of a compact Riemann surface $M$ of genus $g > 0$ derived from a marking of $M$, the image under the isomorphism $p \cdot p_2^{-1} : H^2(\Gamma, \mathbb{C}) \longrightarrow \mathbb{C}$ of the cohomology class $v \in H^2(\Gamma, \mathbb{C})$ represented by a cocycle $v(S, T) \in Z^2(\Gamma, \mathbb{C})$ is the complex number

$$p \cdot p_2^{-1}(v) = \sum_{i=1}^{g} \left( v(C_1 \cdots C_{i-1}, C_i) \right)$$

$$-v(C_i, B_i A_i) + v(A_i, B_i) - v(B_i, A_i) \right).$$

**Proof:** It follows from the isomorphism (E.48) in the explicit form (E.49) that, after replacing the cocycle $v(S, T)$ by a cohomologous cocycle if necessary, it can be assumed that $v(v(S, T)) = f(S, T z) + f(T, z) - f(ST, z)$ for some $C^\infty$ functions $f(T, z)$ on $\hat{M}$ indexed by covering translations $T \in \Gamma$, where $df(T, z) = \psi(T z) - \psi(z)$ and $d\psi \phi$ is a differential form $\phi \in \Gamma(M, E^2)$; thus $v = p_2(\phi)$ and consequently $p \cdot p_2^{-1}(v) = p(\phi) = \int_M \phi$. The functions $f(T, z)$ can be modified by
E.3. EXAMPLE: SURFACE GROUPS

a suitable additive constant so that \( f(T, z_0) = 0 \) for each \( T \in \Gamma \), which amounts to replacing the cocycle \( v(S, T) \) by yet another cohomologous cocycle; and then \( v(S, T) = f(S, Tz_0) \). By Stokes’s Theorem for the region \( \Delta \) in Figure D.2 it follows that

\[
\begin{align*}
    p \cdot p_2^{-1}(v) &= \int_M \phi = \int_{\Delta} d\psi = \int_{\partial \Delta} \psi \\
    &= \sum_{i=1}^{g} \int \left( C_{1 \cdots C_{i-1}} \bar{\alpha}_i - C_{1 \cdots C_i} B_i \tilde{\alpha}_i \right) \psi(z) \\
    &\quad + \sum_{i=1}^{g} \int \left( C_{1 \cdots C_{i-1}} A_i \beta_i - C_{1 \cdots C_i} \tilde{\beta}_i \right) \psi(z) \\
    &= \sum_{i=1}^{g} \left( \psi(C_1 \cdots C_{i-1} z) - \psi(C_1 \cdots C_i B_i z) \right) \\
    &\quad + \sum_{i=1}^{g} \left( \psi(C_1 \cdots C_{i-1} A_i z) - \psi(C_1 \cdots C_i z) \right) \\
    &= \sum_{i=1}^{g} \left( df(C_1 \cdots C_{i-1}, z) - df(C_1 \cdots C_i B_i, z) \right) \\
    &\quad + \sum_{i=1}^{g} \left( df(C_1 \cdots C_{i-1} A_i, z) - df(C_1 \cdots C_i, z) \right) \\
    &= \sum_{i=1}^{g} \left( f(C_1 \cdots C_{i-1}, A_i z_0) - f(C_1 \cdots C_i B_i, A_i z_0) \right) \\
    &\quad + \sum_{i=1}^{g} \left( f(C_1 \cdots C_{i-1} A_i, B_i z_0) - f(C_1 \cdots C_i, B_i z_0) \right) \\
    &= \sum_{i=1}^{g} \left( v(C_1 \cdots C_{i-1}, A_i) - v(C_1 \cdots C_i B_i, A_i) \right) \\
    &\quad + \sum_{i=1}^{g} \left( v(C_1 \cdots C_{i-1} A_i, B_i) - v(C_1 \cdots C_i, B_i) \right).
\end{align*}
\]

By using the cocycle condition

\[
v(T_1 T_2, T_3) = v(T_1, T_2 T_3) - v(T_1, T_2) + v(T_2, T_3)
\]

following from (E.13) and noting that

\[
v(C_1 \cdots C_{i-1} \cdot C_i, B_i A_i) \quad = \quad v(C_1 \cdots C_{i-1}, A_i B_i) \\
- v(C_1 \cdots C_{i-1}, C_i) + v(C_i, B_i A_i)
\]
this can be rewritten
\[
p \cdot p^{-1}(v) = \sum_{i=1}^{g} \left( v(C_1 \cdots C_{i-1}, A_i B_i) - v(C_1 \cdots C_i, B_i A_i) + v(A_i, B_i) - v(B_i, A_i) \right) \]
\[
= \sum_{i=1}^{g} \left( v(C_1 \cdots C_{i-1}, C_i) - v(C_i, B_i A_i) + v(A_i, B_i) - v(B_i, A_i) \right),
\]
and that suffices to conclude the proof.

The explicit form (E.52) of the isomorphism \( p \cdot p^{-1} \) can be interpreted alternatively in terms of Hopf's Theorem in the simplified form given in Corollary E.3. A cohomology class \( v \in H^2(\Gamma, \mathbb{C}) \) is determined uniquely by its image \( p^* (v) \) under the isomorphism
\[
(E.53) \quad p^* : H^2(\Gamma, \mathbb{C}) \longrightarrow \text{Hom}(K/[K,F], \mathbb{C})
\]
of (E.29); and since the group \( K \) is generated by the single element \( \tilde{C} \in K \) the image homomorphism \( p^* (v) \) in turn is determined uniquely by its value \( p^* (v)(\tilde{C}) \in \mathbb{C} \) on this generator.

**Corollary E.5** In terms of the presentation of the covering translation group \( \Gamma \) of a compact Riemann surface \( M \) of genus \( g > 0 \) derived from a marking of \( M \), for which \( \Gamma \cong F/K \) where \( K \subset F \) is the normal subgroup of the free group \( F \) generated by a single commutator \( \tilde{C} \in F \), the image under the isomorphism \( p^* \) of the cohomology class \( v \in H^2(\Gamma, \mathbb{C}) \) represented by a cocycle \( v(S,T) \in Z^2(\Gamma, \mathbb{C}) \) is the homomorphism \( p^* (v) \in \text{Hom}(K/[K,F], \mathbb{C}) \) characterized by
\[
(E.54) \quad p^* (v)(\tilde{C}) = -p \cdot p^{-1}(v)
\]
where \( p \cdot p^{-1}(v) \) has the explicit form as in the preceding theorem.

**Proof:** If \( v(S,T) \in Z^2(\Gamma, \mathbb{C}) \) is a cocycle representing the cohomology class \( v \in H^2(\Gamma, \mathbb{C}) \) then by Corollary E.3 the image homomorphism \( p^* (v) \) is the restriction to \( K \subset [F,F] \) of the mapping \( w : [F,F] \longrightarrow \mathbb{C} \) determined by the cocycle \( v(S,T) \) through the two conditions (E.30) and (E.31). From (E.30) it follows that
\[
w(\tilde{C}_i) = w([\tilde{A}_i, \tilde{B}_i]) = v(C_i, B_i A_i) - v(A_i, B_i) + v(B_i, A_i).
\]
and from (E.31) it follows by induction on \( g \) that
\[
w(\tilde{C}_1 \cdots \tilde{C}_g) = \sum_{i=1}^{g} \left( w(\tilde{C}_i) - v(C_1 \cdots C_{i-1}, C_i) \right)
\]
with the understanding that \( v(C_1 \cdots C_{i-1}, C_i) = 0 \) if \( i = 1 \). Combining these two observations shows that

\[
p^*(v)(\tilde{C}) = w(\tilde{C}) = w(\tilde{C}_1 \tilde{C}_2 \cdots \tilde{C}_g) = \sum_{i=1}^{g} \left( v(C_i, B_i A_i) - v(A_i, B_i) + v(B_i, A_i) - v(C_1 \cdots C_{i-1}, C_i) \right),
\]

so in view of (E.52) it follows that \( p^* (\tilde{C}) = -p \cdot p_2^{-1} \), which suffices for the proof.

The negative sign in (E.54) is yet another instance of conventional choices made in interpreting abstract cohomology groups in concrete terms, such as the convention that associates to a divisor \( \mathfrak{d} \) the holomorphic line bundle \( \zeta_\mathfrak{d} = \delta(-\mathfrak{d}) \) as discussed on page 6. Some useful properties of the cohomology of surface groups follow from these various results about the period classes of closed differential forms on a compact Riemann surface.

**Theorem E.6** If \( M \) is a compact Riemann surface of genus \( g > 0 \) with the covering translation group \( \Gamma \), and if \( v \in H^2(\Gamma, \mathbb{C}) \) is a cohomology class such that \( p \cdot p_2^{-1}(v) \in \mathbb{Z} \), then the cohomology class \( v \) can be represented by an integral cocycle \( v(S,T) \in \mathbb{Z}^2(\Gamma, \mathbb{Z}) \).

**Proof:** Choose a marking of the surface \( M \), in terms of which the covering translation group \( \Gamma \) can be presented as the quotient \( \Gamma = F/K \) of a free group \( F \) modulo the normal subgroup \( K \subset F \) generated by a single commutator \( \tilde{C} \in K < [F,F] \) as before. For any cohomology class \( v \in H^2(\Gamma, \mathbb{C}) \) the image \( p^*(v) \in \text{Hom}(K/[K,F], \mathbb{C}) \) under the isomorphism (E.53) is the homomorphism that is characterized by \( p^*(v)(\tilde{C}) = -p \cdot p_2^{-1}(v) \) as in Theorem E.5; therefore if \( p \cdot p_2^{-1}(v) \in \mathbb{Z} \) then \( p^*(v)(\tilde{C}) \in \mathbb{Z} \), and since \( K \) is generated by the single commutator \( \tilde{C} \) it follows further that \( p^*(v) \in \text{Hom}(K/[K,F], \mathbb{Z}) \). Corollary E.3 for the case that \( V = \mathbb{Z} \) is the isomorphism \( p^* : H^2(\Gamma, \mathbb{Z}) \rightarrow \text{Hom}(K/[K,F], \mathbb{Z}) \), and consequently \( p^*(v) \) is the image of an integral cohomology class so the cohomology class \( v \) can be represented by an integral cocycle, which suffices to conclude the proof.

A special case of a general construction in the cohomology of groups plays a role in the study of surface groups. To any two inhomogeneous 1-cocycles \( v_i \in Z^1(\Gamma, \mathbb{C}) = \text{Hom}(\Gamma, \mathbb{C}) \) of the group \( \Gamma \) acting trivially on the complex numbers \( \mathbb{C} \) there can be associated the 2-cocycle \( v_1 \cup v_2 \in C^2(\Gamma, \mathbb{C}) \) defined by

\[
(E.55) \quad (v_1 \cup v_2)(T_1, T_2) = v_1(T_1) \cdot v_2(T_2) \quad \text{for all} \quad T_1, T_2 \in \Gamma;
\]
that it is a cocycle can be demonstrated by noting by (E.13) that
\[ \delta v(T_1, T_2, T_3) = v(T_1, T_2) - v(T_2, T_3) + v(T_1 T_2, T_3) - v(T_1, T_2 T_3) \]
\[ = v_1(T_1)v_2(T_2) - v_1(T_2)v_2(T_3) + \left( v_1(T_1) + v_1(T_2) \right) v_2(T_3) \]
\[ - v_1(T_1) \left( v_2(T_2) + v_2(T_3) \right) \]
\[ = 0. \]

The cohomology class of this cocycle is called the \textit{cup product} of the cohomology classes \( v_1 \in H^1(\Gamma, \mathbb{C}) \) and also is denoted by \( v_1 \cup v_2 \). This operation is a reflection in the cohomology of groups of the exterior product of differential forms, in the following sense.

**Theorem E.7** If \( \phi_i \in \Gamma(M, \mathcal{E}^1) \) are closed differential 1-forms on a compact Riemann surface \( M \) of genus \( g > 0 \) the period period class \( p_2(\phi_1 \wedge \phi_2) \in H^2(\Gamma, \mathbb{C}) \) of their exterior product can be expressed in terms of the period classes \( p_1(\phi_i) \in H^1(\Gamma, \mathbb{C}) \) of these 1-forms by

\[ p_2(\phi_1 \wedge \phi_2) = p_1(\phi_1) \cup p_1(\phi_2). \]

**Proof:** If \( \phi_i(z) = df_i(z) \) for some functions \( f_i \in \Gamma(\tilde{M}, \mathcal{E}^0) \) then as in (E.38) the period classes of these differential forms are represented by the cocycles \( v_i(T) = f_i(Tz) - f_i(z) \) for any covering translation \( T \in \Gamma \). The product form \( \phi(z) = \phi_1(z) \wedge \phi_2(z) \) can be written as the derivative \( \phi(z) = d\psi(z) \) of the differential form \( \psi(z) = f_1(z)\phi_2(z) \) on \( \tilde{M} \); and \( \psi(Tz) - \psi(z) = v_1(T) \cdot \phi_2(z) = df(T, z) \) for the function \( f(T, z) = v_1(T) \cdot v_2(z) \). It then follows from (E.49) that the period class of the differential form \( \phi \) is represented by the cocycle

\[ v(S, T) = v_1(S)f_2(Tz) + v_1(T) \cdot f_2(z) - v_1(ST) \cdot f_2(z) \]
\[ = v_1(S) \cdot \left( v_1(T) + f_2(z) \right) + v_1(T) \cdot f_2(z) - v_1(ST) \cdot f_2(z) \]
\[ = v_1(S) \cdot v_1(T), \]

and that suffices to conclude the proof.

The factors of automorphy describing holomorphic line bundles over compact Riemann surfaces can be interpreted in terms of the cohomology of groups. The exact sequence of sheaves

\[ 0 \longrightarrow \mathbb{Z} \overset{i}{\longrightarrow} \mathcal{O} \overset{\epsilon}{\longrightarrow} \mathcal{O}^* \longrightarrow 0 \]

over the universal covering space \( \tilde{M} \) of a compact Riemann surface \( M \) of genus \( g \) as in (1.29), in which \( \epsilon(f) = \exp 2\pi i f \) for any \( f \in \mathcal{O} \), leads to an exact cohomology sequence beginning

\[ 0 \longrightarrow \Gamma(\tilde{M}, \mathbb{Z}) \overset{i}{\longrightarrow} \Gamma(\tilde{M}, \mathcal{O}) \overset{\epsilon}{\longrightarrow} \Gamma(\tilde{M}, \mathcal{O}^*) \longrightarrow 0, \]

since \( H^1(\tilde{M}, \mathbb{Z}) = 0 \) for the simply connected surface \( \tilde{M} \); and there is the natural identification \( \Gamma(\tilde{M}, \mathbb{Z}) \cong \mathbb{Z} \). When the covering translation group \( \Gamma \) of
the surface $M$ acts on the right on these groups of cross-sections by setting $(f|T)(z) = f(Tz)$ the exact sequence (E.58) can be viewed as an exact sequence of $\Gamma$-homomorphisms; and it leads to an exact group cohomology sequence containing the segment

\[(E.59)\]

$$H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{O})) \xrightarrow{\delta} H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{O}^*)) \xrightarrow{\delta} H^2(\Gamma, \mathbb{Z}) \xrightarrow{\delta} H^2(\Gamma, \Gamma(\widehat{M}, \mathcal{O})).$$

An inhomogeneous 1-cocycle $\lambda \in Z^1(\Gamma, \Gamma(\widehat{M}, \mathcal{O}^*))$ is a collection of holomorphic and nowhere vanishing functions $\lambda(T, z)$ on $\widehat{M}$ such that $\lambda(I, z) = 1$ and $\lambda(ST, z) = \lambda(S, Tz)\lambda(T, z)$, the multiplicative form of the cocycle condition (E.17); hence it is a holomorphic factor of automorphy for the action of the covering translation group $\Gamma$. A 1-coboundary is a 1-cocycle $\lambda(T, z)$ of the form $\lambda(T, z) = h(Tz)/h(z)$ for a holomorphic nowhere vanishing function $h(z)$ on $\widehat{M}$, the multiplicative form of (E.18); hence it is a holomorphically trivial holomorphic factor of automorphy. Therefore the cohomology group $H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{O}^*))$ can be identified with the group of holomorphic equivalence classes of holomorphic factors of automorphy for the covering translation group of the surface $M$, which by Theorem 3.11 in turn can be identified with the group of holomorphic equivalence classes of holomorphic line bundles over $M$. From the usual chase through the diagram of cochain groups associated to the exact sequence of $\Gamma$-homomorphisms (E.58), the diagram analogous to (E.37), it follows that the coboundary mapping

\[(E.60)\]

$$\delta : H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{O}^*)) \longrightarrow H^2(\Gamma, \mathbb{Z})$$

associates to the cohomology class represented by a factor of automorphy $\lambda(T, z)$ the cohomology class $\delta(\lambda) \in H^2(\Gamma, \mathbb{Z})$ represented by the cocycle $\delta(\lambda)(S, T) \in Z^2(\Gamma, \mathbb{Z})$ given explicitly by

\[(E.61)\]

$$\delta(\lambda)(S, T) = f(S, Tz) + f(T, z) - f(ST, z)$$

where $\lambda(T, z) = \exp 2\pi i f(T, z)$; this cohomology class is called the characteristic class of the factor of automorphy $\lambda(T, z)$. Parallel constructions can be carried out for the sheaf $\mathcal{C}$ of germs of continuous functions and the sheaf $\mathcal{E}$ of germs of $C^\infty$ functions on $\widehat{M}$; so the cohomology group $H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{C}^*))$ can be identified and the group of equivalence classes of continuous factors of automorphy while the group $H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{E}^*))$ can be identified with the group of equivalence classes of $C^\infty$ factors of automorphy. In these cases the analogues of the exact sequence (E.59) reduce to the isomorphisms

$$\delta : H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{C}^*)) \xrightarrow{\cong} H^2(\Gamma, \mathbb{Z}),$$

$$\delta : H^1(\Gamma, \Gamma(\widehat{M}, \mathcal{E}^*)) \xrightarrow{\cong} H^2(\Gamma, \mathbb{Z}),$$

since $H^i(\Gamma, \Gamma(\widehat{M}, \mathcal{C})) = 0$ for $i > 0$ by (E.33) and $H^i(\Gamma, \Gamma(\widehat{M}, \mathcal{C})) = 0$ for $i > 0$ by the corresponding argument. Thus the characteristic class of a holomorphic
factor of automorphy provides a complete description of the continuous or $C^\infty$ equivalence class of that factor of automorphy.

**Theorem E.8** If $\lambda(T, z)$ is a holomorphic factor of automorphy for the covering translation group $\Gamma$ of a compact Riemann surface $M$ of genus $g > 0$ then the image $p \cdot p_2^{-1}(\delta(\lambda)) \in \mathbb{Z}$ of the characteristic class $\delta(\lambda) \in H^2(\Gamma, \mathbb{Z})$ of the factor of automorphy $\lambda(T, z)$ is equal to the characteristic class of the holomorphic line bundle represented by that factor of automorphy.

**Proof:** A meromorphic relatively automorphic function $f(z)$ for the factor of automorphy $\lambda(T, z) \in H^1(\Gamma, \Gamma(\tilde{M}, \mathcal{O}^*))$ corresponds to a meromorphic cross-section of the holomorphic line bundle $\lambda$ represented by that factor of automorphy, as in Theorem 3.11; so by definition (1.9) the characteristic class of the line bundle $\lambda$ is the integer $\deg d(\lambda)$, where $d(\lambda)$ is the divisor of the function $f(z)$ on the Riemann surface $M$. If $M$ is identified with the quotient $M = \tilde{M}/\Gamma$ of its universal covering space $\tilde{M}$ by the group $\Gamma$ of covering translations and if $\Delta \subset \tilde{M}$ is a fundamental domain for the action of $\Gamma$ on $\tilde{M}$ as in the discussion of marked surfaces in Appendix D.1, where $\Delta$ is chosen so that there are no zeros or poles of the function $f(z)$ on its boundary $\partial \Delta$, then

$$\tag{E.62} \deg d(\lambda) = \frac{1}{2\pi i} \int_{\partial \Delta} d \log f(z)$$

by the residue theorem. Now the factor of automorphy can be written $\lambda(T, z) = \exp 2\pi i f(T, z)$ for some holomorphic functions $f(T, z)$ on $\tilde{M}$; and its characteristic is the cohomology class represented by the 2-cocycle

$$\delta(\lambda)(S, T) = f(S, Tz) + f(T, z) - f(ST, z) \in \mathbb{Z}^2(\Gamma, \mathbb{Z}).$$

If $\psi(z)$ is any $C^\infty$ differential form on $\tilde{M}$ such that $\psi(Tz) - \psi(z) = d f(T, z)$ for all $T \in \Gamma$ it follows from (E.49) that the cocycle $\delta(\lambda)(S, T)$ represents the period class $p_2(\phi)$ of the differential form $\phi = d \psi$, and consequently that

$$\tag{E.63} p \cdot p_2^{-1}(\delta(\lambda)) = p(\phi) = \int_{\tilde{M}} \phi.$$
since $\partial \log \lambda(T, z) = d \log \lambda(T, z)$ and $\partial \log \lambda(T, z) = 0$; consequently if $\phi(z) = d\psi(z)$ it follows from (E.62) and (E.63) that

\[
p \cdot p_2^{-1}(\delta(\lambda)) = \int_M \phi = \int_\Delta d\psi = \int_{\partial\Delta} \psi = \frac{1}{2\pi i} \int_{\partial\Delta} \partial \log r(z) = \frac{1}{2\pi i} \int_{\partial\Delta} d\log |f(z)|^2
\]

\[
= \frac{1}{2\pi i} \int_{\partial\Delta} d\log f(z) = \deg \mathcal{O}(f),
\]

since $r(z) = |f(z)|^2$ on $\partial\Delta$, and that suffices to conclude the proof.
Appendix F

Complex Tori

F.1 Period Matrices

A lattice subgroup \( \mathcal{L} \subset \mathbb{C}^g \) in the space of \( g \) complex variables is an additive subgroup generated by \( 2g \) vectors in \( \mathbb{C}^g \) that are linearly independent over the real numbers. These \( 2g \) vectors viewed as column vectors of length \( g \) can be taken as the columns of a \( g \times 2g \) complex matrix \( \Omega \), and \( \mathcal{L} = \Omega \mathbb{Z}^{2g} \subset \mathbb{C}^g \) also is called the lattice subgroup described by the period matrix \( \Omega \) and is denoted by \( \mathcal{L} = \mathcal{L}(\Omega) \). A complex \( g \times 2g \) matrix is called a period matrix, and a period matrix with columns that are linearly independent over the real numbers is called a nonsingular period matrix. To a \( g \times 2g \) period matrix \( \Omega \) there can be associated the \( 2g \times 2g \) matrix \( (\Omega \Omega) \), called the full period matrix associated to the period matrix \( \Omega \).

**Lemma F.1** A \( g \times 2g \) period matrix \( \Omega \) is a nonsingular period matrix if and only if its associated \( 2g \times 2g \) full period matrix is an invertible square matrix.

**Proof:** If the column vectors of the period matrix \( \Omega \) are linearly dependent over the real numbers there is a nontrivial real column vector \( x \in \mathbb{R}^{2g} \) such that \( \Omega \cdot x = 0 \). Since the vector \( x \) is real \( \Omega \cdot x = 0 \) as well, so the \( 2g \times 2g \) complex matrix \( (\Omega \Omega) \) is singular. Conversely if the square matrix \( (\Omega \Omega) \) is singular there is a nontrivial complex column vector \( z = x + iy \in \mathbb{C}^{2g} \) such that \( \Omega \cdot z = \Omega \cdot x + i \Omega \cdot y = 0 \); then \( \Omega \cdot \overline{z} = \overline{\Omega} \cdot z = 0 \) as well, so \( \Omega \cdot x = \Omega \cdot y = 0 \), and since not both \( x = 0 \) and \( y = 0 \) the columns of \( \Omega \) must be linearly dependent over the real numbers. That suffices to conclude the proof.

**Theorem F.2** Two nonsingular \( g \times 2g \) period matrices \( \Omega_1 \) and \( \Omega_2 \) describe the same lattice subgroup \( \mathcal{L}(\Omega_1) = \mathcal{L}(\Omega_2) \) if and only if \( \Omega_1 = \Omega_2 Q^{-1} \) for some matrix \( Q \in \text{Gl}(2g, \mathbb{Z}) \).

**Proof:** The group \( \text{Gl}(2g, \mathbb{Z}) \) of \( 2g \times 2g \) integral matrices with integral inverses can be characterized as the set of \( 2g \times 2g \) complex matrices \( Q \) such that \( Q \mathbb{Z}^{2g} = \mathbb{Z}^{2g} \).
In these terms the preceding corollary can be restated as follows.

(F.2) \( \Omega \sim \) the weak equivalence of period matrices is defined by

\[
\begin{pmatrix}
\Omega_2 \\
\Omega_1
\end{pmatrix} Z^{2g} = \begin{pmatrix}
\Omega_1 \\
\Omega_2
\end{pmatrix} Z^{2g}.
\]

The full period matrices are nonsingular by Lemma F.1 so the matrix

\[
Q = \begin{pmatrix}
\Omega_1 \\
\Omega_2
\end{pmatrix}^{-1} \begin{pmatrix}
\Omega_1 \\
\Omega_2
\end{pmatrix}
\]

is well defined; this matrix satisfies \( QZ^{2g} = Z^{2g} \) so \( Q \in \text{Gl}(2g, \mathbb{Z}) \), and since \( \Omega_2 = \Omega_1 Q \) that suffices to conclude the proof.

The linear mapping \( A : \mathbb{C}^g \rightarrow \mathbb{C}^g \) described by a nonsingular complex matrix \( A \in \text{Gl}(g, \mathbb{C}) \) takes a lattice subgroup \( \mathcal{L} \subset \mathbb{C}^g \) to the lattice subgroup \( A\mathcal{L} \subset \mathbb{C}^g \); two lattice subgroups related in this way are called \textit{linearly equivalent} lattice subgroups.

**Corollary F.3** Lattice subgroups \( \mathcal{L}(\Omega_1) \) and \( \mathcal{L}(\Omega_2) \) in \( \mathbb{C}^g \) are linearly equivalent if and only if \( \Omega_1 = A\Omega_2 Q^{-1} \) for matrices \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \).

**Proof:** If \( \mathcal{L}(\Omega_1) = A\mathcal{L}(\Omega_2) = \mathcal{L}(A\Omega_2) \) for some matrix \( A \in \text{Gl}(g, \mathbb{C}) \) then \( \Omega_1 = A\Omega_2 Q^{-1} \) for some matrix \( Q \in \text{Gl}(2g, \mathbb{Z}) \) by the preceding theorem. Conversely if \( \Omega_1 = A\Omega_2 Q^{-1} \) for some matrices \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \) then \( \mathcal{L}(\Omega_1) = \Omega_1 Z^{2g} = A\Omega_2 Q^{-1} Z^{2g} = A\Omega_2 Z^{2g} = A\mathcal{L}(\Omega_2) \). That suffices for the proof.

Two \( g \times 2g \) period matrices \( \Omega_1 \) and \( \Omega_2 \) are called \textit{equivalent period matrices} if \( \Omega_1 = A\Omega_2 Q^{-1} \) for matrices \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \), and the equivalence of these two period matrices is denoted by \( \Omega_1 \sim \Omega_2 \). If it is only the case that \( Q \in \text{Gl}(2g, \mathbb{Q}) \), that \( Q \) is a nonsingular rational matrix, the two matrices are called \textit{weakly equivalent period matrices}, and the weak equivalence of these two period matrices is denoted by \( \Omega_1 \sim \Omega_2 \). It is quite evident that both are equivalence relations in the customary sense, and that equivalent period matrices are weakly equivalent period matrices. To summarize, \textit{the equivalence of period matrices is defined by}

\[
\Omega \sim A\Omega Q^{-1} \quad \text{for any} \quad A \in \text{Gl}(g, \mathbb{C}), \; Q \in \text{Gl}(2g, \mathbb{Z}),
\]

and \textit{the weak equivalence of period matrices is defined by}

\[
\Omega \sim A\Omega Q^{-1} \quad \text{for any} \quad A \in \text{Gl}(g, \mathbb{C}), \; Q \in \text{Gl}(2g, \mathbb{Q}).
\]

In these terms the preceding corollary can be restated as follows.
F.1. PERIOD MATRICES

Corollary F.4 Two lattice subgroups \( \mathcal{L}(\Omega_1) \) and \( \mathcal{L}(\Omega_2) \) in \( \mathbb{C}^g \) are linearly equivalent if and only if the period matrices \( \Omega_1 \) and \( \Omega_2 \) are equivalent period matrices.

Proof: This is equivalent to the preceding Corollary in view of the definition of equivalent period matrices, so no further proof is necessary.

Perhaps it should be repeated for emphasis that equivalence and weak equivalence of period matrices are defined for arbitrary period matrices, not necessarily just for nonsingular period matrices; but these equivalences preserve nonsingularity.

Corollary F.5 A period matrix weakly equivalent (or equivalent) to a nonsingular period matrix is itself a nonsingular period matrix.

Proof: It is course sufficient to demonstrate this corollary just for weakly equivalent period matrices. If \( \Omega_1 \) and \( \Omega_2 \) are weakly equivalent period matrices then by definition \( \Omega_2 = A \Omega_1 Q^{-1} \) for some nonsingular square matrices \( A \) and \( Q \), so the associated full period matrices satisfy
\[
\left( \begin{array}{c} \Omega_2 \\ \Omega_1 \end{array} \right) = \left( \begin{array}{cc} A & 0 \\ 0 & A \end{array} \right) \left( \begin{array}{c} \Omega_1 \\ \Omega_1 \end{array} \right) Q^{-1}.
\]
Since \( \left( \begin{array}{cc} A & 0 \\ 0 & A \end{array} \right) \) and \( Q \) are nonsingular matrices it follows that if one of the two full period matrices is nonsingular so is the other; the desired result is then a consequence of Lemma F.1, and that suffices for the proof.

A complex torus of dimension \( g \) is the quotient \( \mathbb{C}^g/\mathcal{L} \) of the additive group \( \mathbb{C}^g \) by a lattice subgroup \( \mathcal{L} \subset \mathbb{C}^g \). As a quotient group a complex torus has the natural structure of an abelian group. The natural quotient mapping \( \pi : \mathbb{C}^g \rightarrow \mathbb{C}^g/\mathcal{L} \) is the universal covering projection, and the complex torus \( \mathbb{C}^g/\mathcal{L} \) inherits from its universal covering space \( \mathbb{C}^g \) a natural complex structure; with this complex structure the complex torus is a compact complex abelian Lie group. For many purposes though the primary interest is in just the complex manifold structure of a complex torus rather than its full complex Lie group structure.

Theorem F.6 A holomorphic mapping \( f : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2 \) between two complex tori is induced by an affine mapping \( \tilde{f}(z) = Az + a \) between their universal covering spaces, where \( A \in \mathbb{C}^{g_2 \times g_1} \) and \( a \in \mathbb{C}^{g_2} \). An affine mapping \( f(z) = Az + a \) induces a holomorphic mapping \( f : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2 \) between the complex tori for lattice subgroups \( \mathcal{L}_1 \subset \mathbb{C}^{g_1} \) and \( \mathcal{L}_2 \subset \mathbb{C}^{g_2} \) if and only if \( A \mathcal{L}_1 \subset \mathcal{L}_2 \); and this mapping is a group homomorphism if and only if \( a \in \mathcal{L}_2 \).

Proof: A holomorphic mapping \( f : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2 \) lifts to a holomorphic mapping \( \tilde{f} : \mathbb{C}^{g_1} \rightarrow \mathbb{C}^{g_2} \) between the universal covering spaces of these two complex manifolds. A holomorphic mapping \( \tilde{f} : \mathbb{C}^{g_1} \rightarrow \mathbb{C}^{g_2} \) induces a holomorphic mapping \( f : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2 \) between the two quotient groups if
and only if it takes points in $\mathbb{C}^{g_1}$ that differ by a lattice vector in $\mathcal{L}_1$ to points in $\mathbb{C}^{g_2}$ that differ by a lattice vector in $\mathcal{L}_2$, hence if and only if for any point $z \in \mathbb{C}^{g_1}$ and any lattice vector $\lambda_1 \in \mathcal{L}_1$ there is a lattice vector $\lambda_2 \in \mathcal{L}_2$ such that

\begin{equation}
(f.3) \quad \tilde{f}(z + \lambda_1) = \tilde{f}(z) + \lambda_2.
\end{equation}

Since lattice subgroups are discrete the lattice vector $\lambda_2$ must be independent of the point $z$; so for any lattice vector $\lambda_1$ there must be a lattice vector $\lambda_2$ such that (F.3) holds as an identity in the variable $z \in \mathbb{C}^{g_1}$. The partial derivative $\partial\tilde{f}/\partial z_j$ then is a holomorphic mapping from $\mathbb{C}^{g_1}$ to $\mathbb{C}^{g_2}$ that is invariant under the lattice subgroup $\mathcal{L}_1$, so it is bounded in $\mathbb{C}^{g_1}$ and hence constant by the maximum modulus theorem for vector-valued holomorphic mappings; therefore the mapping $\tilde{f}$ must be of the form $\tilde{f}(z) = Az + a$ for some complex matrix $A$ and complex vector $a$. For such a mapping (F.3) reduces to the condition that for any lattice vector $\lambda_1 \in \mathcal{L}_1$ there is a lattice vector $\lambda_2 \in \mathcal{L}_2$ such that $A\lambda_1 = \lambda_2$, hence to the condition that $A\mathcal{L}_1 \subset \mathcal{L}_2$. A holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ is a group homomorphism if and only if for any points $z_1, z_2 \in \mathbb{C}^{g_1}$ the image of their sum is the sum of their images in the torus $\mathbb{C}^{g_2}/\mathcal{L}_2$; for the mapping $f(z) = Az + a$ that is the condition that $\mathcal{A}\mathcal{L}_1 \subset \mathcal{L}_2$, and since the lattice is discrete this must be an identity in the variables $z_i$ so it is just the condition that $a = l_2 \in \mathcal{L}_2$. That suffices to conclude the proof.

**Corollary F.7** A holomorphic mapping between complex tori is the composition of a group homomorphism from one torus to the other and a translation in the image torus.

**Proof:** A holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ between two complex tori is induced by an affine mapping $\tilde{f}(z) = Az + a$ between their universal covering spaces for a matrix $A \in \mathbb{C}^{g_2 \times g_1}$ such that $A\mathcal{L}_1 \subset \mathcal{L}_2$, by the preceding theorem. The mapping $\tilde{f}$ can be written as the composition $\tilde{f} = \tilde{g} \cdot \tilde{h}$ where $\tilde{g}(z) = z + a$ and $\tilde{h}(z) = Az$. The mapping $\tilde{g}(z)$ induces a translation $g$ in the torus $\mathbb{C}^{g_2}/\mathcal{L}_2$, and by the preceding theorem again the mapping $\tilde{h}(z)$ induces a group homomorphism $h : \mathbb{C}^{g_1}/\mathcal{L}_1 \rightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$; since $f = g \cdot h$ that suffices for the proof.

The complex torus for the lattice subgroup $\mathcal{L}(\Omega)$ described by a nonsingular period matrix $\Omega$ is denoted by $J(\Omega)$, so that $J(\Omega) = \mathbb{C}^{g}/\mathcal{L}(\Omega)$. A Hurwitz relation $(A, Q)$ from a period matrix $\Omega_1 \in \mathbb{C}^{g_1 \times 2g_1}$ to a period matrix $\Omega_2 \in \mathbb{C}^{g_2 \times 2g_2}$ is defined to be a pair of matrices $A \in \mathbb{C}^{g_2 \times g_1}$ and $Q \in \mathbb{Z}^{2g_2 \times 2g_1}$ such that

\begin{equation}
(F.4) \quad A\Omega_1 = \Omega_2 Q,
\end{equation}

whether the period matrices are nonsingular period matrices or not. A Hurwitz relation is not a symmetric relation, but rather involves a definite ordering.
of the period matrices \( \Omega_1 \) and \( \Omega_2 \). It is clear from (F.4) that if \((A_1, Q_1)\) and \((A_2, Q_2)\) are Hurwitz relations from \( \Omega_1 \) to \( \Omega_2 \) then so is the linear combination \( n_1(A_1, Q_1) + n_2(A_2, Q_2) = (n_1A_1 + n_2A_2, n_1Q_1 + n_2Q_2) \) for any integers \( n_1, n_2 \in \mathbb{Z} \); thus the set of Hurwitz relations from \( \Omega_1 \) to \( \Omega_2 \) form a \( \mathbb{Z} \)-module. In the special case that \( A \in \text{GL}(g, \mathbb{C}) \) and \( Q \in \text{GL}(2g, \mathbb{Z}) \) the Hurwitz relation (F.4) when written \( \Omega_2 = A\Omega_1Q^{-1} \) amounts to the equivalence (F.1) of the period matrices \( \Omega_1 \) and \( \Omega_2 \).

**Lemma F.8** If \((A, Q)\) is a Hurwitz relation from a nonsingular period matrix \( \Omega_1 \) to a nonsingular period matrix \( \Omega_2 \) then rank \( Q = 2 \) rank \( A \) and either one of the matrices \( A \) or \( Q \) determines the other matrix uniquely.

**Proof:** It is evident that the Hurwitz relation (F.4) is equivalent to the relation

\[
(F.5) \quad \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \bar{\Omega}_1 \end{pmatrix} = \begin{pmatrix} \Omega_2 \\ \bar{\Omega}_2 \end{pmatrix} Q
\]

between the associated full period matrices. If the period matrices \( \Omega_1 \) and \( \Omega_2 \) are nonsingular period matrices the full period matrices are nonsingular square matrices by Lemma F.1; hence it follows from (F.5) that rank \( Q = 2 \) rank \( A \) and that either one of the matrices \( A \) or \( Q \) determines the other uniquely. That suffices for the proof.

**Theorem F.9** (i) Holomorphic mappings \( f : J(\Omega_1) \rightarrow J(\Omega_2) \) from the complex torus described by a nonsingular \( g_1 \times 2g_1 \) period matrix \( \Omega_1 \) to the complex torus described by a nonsingular \( g_2 \times 2g_2 \) period matrix \( \Omega_2 \) are in one-to-one correspondence with triples \((A, Q, a_0)\) where \((A, Q)\) is a Hurwitz relation from \( \Omega_1 \) to \( \Omega_2 \) and \( a_0 \in J(\Omega_2) \); the holomorphic mapping \( f \) corresponding to \((A, Q, a_0)\) is induced by the affine mapping \( f(z) = Az + a \) from \( \mathbb{C}^{g_1} \) to \( \mathbb{C}^{g_2} \) for any point \( a \in \mathbb{C}^{g_2} \) representing the point \( a_0 \in J(\Omega_2) \).

(ii) A holomorphic mapping \( f : J(\Omega_1) \rightarrow J(\Omega_2) \) between two complex tori of the same dimension \( g \) corresponding to a Hurwitz relation \((A, Q)\) from \( \Omega_1 \) to \( \Omega_2 \) is a biholomorphic mapping if and only if \( A \in \text{GL}(g, \mathbb{C}) \) and \( Q \in \text{GL}(2g, \mathbb{Z}) \).

**Proof:** (i) By Theorem F.6 a holomorphic mapping \( f : J(\Omega_1) \rightarrow J(\Omega_2) \) is induced by an affine mapping \( f(z) = Az + a \); and an affine mapping \( f(z) = Az + a \) induces a holomorphic mapping \( f : J(\Omega_1) \rightarrow J(\Omega_2) \) if and only if \( A \Omega_1 = \Omega_2 Q \) for some integral matrix \( Q \in \mathbb{Z}^{2g_2 \times 2g_1} \), which is just the condition that \((A, Q)\) is a Hurwitz relation from \( \Omega_1 \) to \( \Omega_2 \). Two affine mappings \( f_1(z) = A_1z + a_1 \) and \( f_2(z) = A_2z + a_2 \) induce the same mapping \( f : J(\Omega_1) \rightarrow J(\Omega_2) \) precisely when \( (A_1 - A_2)z + (a_1 - a_2) = z + \lambda_2 \) for all points \( z \in \mathbb{C}^{g} \) and some lattice vector \( \lambda_2 \in \mathcal{L}(\Omega_2) \), hence precisely when \( A_1 = A_2 \) and \( a_1 - a_2 = \lambda_2 \); that is the condition that the Hurwitz relations are the same and that \( a_1 \) and \( a_2 \) represent the same point of \( J(\Omega_2) \).

(ii) Let \( f : J(\Omega_1) \rightarrow J(\Omega_2) \) be a holomorphic mapping between two complex tori of dimension \( g \) corresponding to a triple \((A, Q, a)\) where \((A, Q)\) is a Hurwitz relation from \( \Omega_1 \) to \( \Omega_2 \). If \( A \in \text{GL}(g, \mathbb{C}) \) and \( Q \in \text{GL}(2g, \mathbb{Z}) \) then
A^{-1} \in \text{Gl}(g, \mathbb{C}) \text{ and } Q^{-1} \in \text{Gl}(2g, \mathbb{Z}), \text{ and it follows from the Hurwitz relation } A\Omega_1 = \Omega_2 Q \text{ that } A^{-1}\Omega_2 = \Omega_1 Q^{-1}; \text{ thus } (A^{-1}, Q^{-1}) \text{ is a Hurwitz relation from } \Omega_2 \text{ to } \Omega_1, \text{ and hence the affine mapping } h(z) = A^{-1}z - A^{-1}a \text{ describes a holomorphic mapping } h : J(\Omega_2) \rightarrow J(\Omega_1). \text{ Since the affine mapping } fh \text{ is the identity mapping it follows that the induced mapping } fh \text{ also is the identity mapping, and consequently that } f \text{ itself is a biholomorphic mapping. Conversely if } f : J(\Omega_1) \rightarrow J(\Omega_2) \text{ is a biholomorphic mapping the inverse biholomorphic mapping } h = f^{-1} \text{ must be induced by an affine mapping } \hat{h}(z) = Bz + b. \text{ Since the mappings } f \text{ and } h \text{ are inverse to one another, for any point } z \in \mathbb{C}^g \text{ that point and the point } f(\hat{h}(z)) = A(Bz + b) + a \in \mathbb{C}^g \text{ must represent the same point in the torus } J(\Omega_1); \text{ consequently there must be a lattice vector } \lambda_1 \in \mathcal{L}(\Omega_1) \text{ such that } A \mathcal{B}z + Ab + a = z + \lambda_1. \text{ This holds identically in } z \text{ by continuity, hence } AB = I \text{ so } A \in \text{Gl}(g, \mathbb{C}); \text{ and it then follows from Lemma F.8 that } Q \in \text{Gl}(2g, \mathbb{Z}). \text{ That suffices to conclude the proof.}

The image of any holomorphic mapping between two complex tori is a holomorphic subvariety of the image manifold by Remmert’s Proper Mapping Theorem. The holomorphic mapping } f : J(\Omega_1) \rightarrow J(\Omega_2) \text{ corresponding to a Hurwitz relation } (A, Q) \text{ from } \Omega_1 \text{ to } \Omega_2 \text{ is induced by the affine mapping } \hat{f}(z) = Az + a \text{ for some point } a \in J(\Omega_2), \text{ so its image is a connected complex submanifold of } J(\Omega_2) \text{ of dimension equal to the rank of the matrix } A. \text{ If rank } A = \dim J(\Omega_2) \text{ the induced mapping } f \text{ is surjective, with image the full complex torus } J(\Omega_2). \text{ If rank } A = \dim J(\Omega_1) = \dim J(\Omega_2) \text{ the mapping } f \text{ is a surjective and locally biholomorphic mapping between these two complex tori; such a mapping is called an isogeny from the complex torus } J(\Omega_1) \text{ to the complex torus } J(\Omega_2). \text{ When a Hurwitz relation } (A, Q) \text{ determines an isogeny the matrix } A \text{ is nonsingular, and then } Q \text{ also is nonsingular by Lemma F.8; and since } A\Omega_1 = \Omega_2 Q \text{ it follows that } A^{-1}\Omega_2 = \Omega_1 Q^{-1}. \text{ Although } Q^{-1} \text{ is not necessarily an integral matrix it is at least a rational matrix, so } qQ^{-1} \text{ will be integral for some integer } q; \text{ then } (qA^{-1}, qQ^{-1}) \text{ is a Hurwitz relation from the period matrix } \Omega_2 \text{ to the period matrix } \Omega_1. \text{ Thus if there is an isogeny from the complex torus } J(\Omega_1) \text{ to the complex torus } J(\Omega_2) \text{ there also is an isogeny from the complex torus } J(\Omega_2) \text{ to the complex torus } J(\Omega_1). \text{ Two complex tori are isogenous if there is an isogeny from one to another; this clearly is an equivalence relation between complex tori. Of course a biholomorphic mapping is a special case of an isogeny, so biholomorphic complex tori are isogenous.}

**Theorem F.10** An isogeny } f : J(\Omega_1) \rightarrow J(\Omega_2) \text{ that is a group homomorphism induces a group isomorphism } J(\Omega_1)/K \cong J(\Omega_2) \text{ where } K \subset J(\Omega_1) \text{ is a finite subgroup; the mapping } f \text{ exhibits the torus } J(\Omega_1) \text{ as a finite unbranched covering space of the torus } J(\Omega_2).

**Proof:** By Theorem F.6 an isogeny } f : J(\Omega_1) \rightarrow J(\Omega_2) \text{ is induced by an affine mapping } \hat{f}(z) = Az + a \text{ between the universal covering spaces of the complex tori, and } A \text{ must be a nonsingular matrix such that } A\mathcal{L}(\Omega_1) \subset \mathcal{L}(\Omega_2); \text{ for a discussion of Remmert’s Proper Mapping Theorem see page 409 in Appendix A.3.
this isogeny is a group homomorphism if and only if \( a \in \mathcal{L}(\Omega_2) \), in which case the mapping \( f \) also is induced by the linear mapping \( \tilde{f}(z) = Az \). The kernel \( K \subset J(\Omega_1) \) of the homomorphism \( f \) consists of those points of \( J(\Omega_1) \) represented by vectors \( \lambda \in \mathbb{C}^g \) such that \( A\lambda \in \mathcal{L}(\Omega_2) \); consequently \( K = \mathcal{K}/\mathcal{L}(\Omega_1) \) where \( \mathcal{L}(\Omega_1) \subset \mathcal{K} = A^{-1}\mathcal{L}(\Omega_2) \subset \mathbb{C}^g \), and since \( A^{-1} \) is a linear isomorphism \( \mathcal{K} \) is a lattice subgroup of \( \mathbb{C}^g \). As the quotient of two lattice subgroups the group \( \mathcal{K} \) is a finite group. The representation

\[
\mathbb{C}^g \mathcal{L}(\Omega_1) \xrightarrow{\sim} \mathbb{C}^g \mathcal{K} \xrightarrow{\sim} J(\Omega_1)/K \xrightarrow{\sim} J(\Omega_2);
\]

and since the groups are abelian \( \mathcal{L}(\Omega_1) \) is a normal subgroup of \( \mathcal{K} \) so \( J(\Omega_2) = J(\Omega_1)/K \), which suffices to conclude the proof.

**Theorem F.11** The complex tori \( J(\Omega_1) \) and \( J(\Omega_2) \) described by nonsingular period matrices \( \Omega_1 \) and \( \Omega_2 \) are biholomorphic if and only if the period matrices \( \Omega_1 \) and \( \Omega_2 \) are equivalent, and are isogenous if and only if the period matrices \( \Omega_1 \) and \( \Omega_2 \) are weakly equivalent.

**Proof:** By Theorem F.9 the tori \( J(\Omega_1) \) and \( J(\Omega_2) \) are biholomorphic if and only if they are of the same dimension \( g \) and there are matrices \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \) such that \( A\Omega_1 = \Omega_2Q \); and that is precisely the condition (F.1) that the two period matrices are equivalent. The two tori are isogenous if and only they are of the same dimension \( g \) and there are matrices \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \mathbb{Z}^{2g \times 2g} \) such that \( A\Omega_1 = \Omega_2Q \); by Lemma F.8 the matrix \( Q \) has rank \( 2g \) so that \( Q \in \text{Gl}(2g, \mathbb{Q}) \), and that is precisely the condition (F.2) that the two period matrices are weakly equivalent. That suffices for the proof.

A useful alternative description of the complex torus \( J(\Omega) \) involves a matrix \( \Pi \) closely related to the period matrix \( \Omega \).

**Theorem F.12** If \( \Omega \) is a nonsingular period matrix

\[
\left( \frac{\Omega}{\Pi} \right)^{-1} = \left( \frac{\Pi}{\Pi} \right)
\]

where \( \Pi \) also is a nonsingular period matrix.
Proof: The full period matrix associated to the period matrix $\Omega$ is a nonsingular $2g \times 2g$ matrix, so its inverse transpose conjugate exists and can be written in the form \[
abla (\Pi_1)\) for some $g \times 2g$ period matrices $\Pi_1$ and $\Pi_2$. Since
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 \\
\end{pmatrix} = \begin{pmatrix} \Omega \\ \Omega \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} = \begin{pmatrix} \Omega & 0 \\
0 & \Omega \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix} = \begin{pmatrix} \Omega \Pi_1 & \Omega \Pi_2 \\
\Pi_1 & \Pi_2 \end{pmatrix}
\]
it follows that
\[
\Omega \Pi_1 = \overline{\Omega} \Pi_2 = I \quad \text{and} \quad \Omega \Pi_2 = \overline{\Omega} \Pi_1 = 0.
\]
By conjugation $\Omega \Pi_2 = \overline{\Omega} \Pi_2 = I$ and $\overline{\Omega} \Pi_2 = \overline{\Omega} \Pi_2 = 0$ as well, so
\[
\Omega (\Pi_1 - \Pi_2) = \overline{\Omega} (\Pi_1 - \Pi_2) = 0;
\]
and since the full period matrix is nonsingular it follows that $\Pi_2 = \overline{\Pi}_1$. The inverse of the full period matrix of course is also nonsingular, so $\Pi$ itself is a nonsingular period matrix, and that suffices to conclude the proof.

The matrix $\Pi$ of the preceding lemma is called the inverse period matrix to $\Omega$. That $\Pi$ is the inverse period matrix to $\Omega$ when viewed as the identity
\[
I = \begin{pmatrix} \overline{\Omega} \\ \Omega \end{pmatrix} \cdot \begin{pmatrix} \Omega \\ \Omega \end{pmatrix} = \begin{pmatrix} \Omega & 0 \\
0 & \Omega \end{pmatrix} \cdot \begin{pmatrix} \Omega \\ \Omega \end{pmatrix} = \begin{pmatrix} \Omega \Pi & \Omega \Pi \\
\Pi \Omega & \Omega \Pi \end{pmatrix}
\]
is equivalent to the conditions that
\[
(F.6) \quad \Omega \Pi = 0 \quad \text{and} \quad \Omega \Pi = I;
\]
and that $\Pi$ is the inverse period matrix to $\Omega$ when viewed as the identity
\[
I = \begin{pmatrix} \overline{\Omega} \\ \Omega \end{pmatrix} \cdot \begin{pmatrix} \Pi \\ \Pi \end{pmatrix} = \begin{pmatrix} \Pi & \Pi \\
\Pi & \Pi \end{pmatrix} = \overline{\Pi} \Pi + \Omega \Pi
\]
is equivalent to the condition that
\[
(F.7) \quad \overline{\Omega} \Pi + \overline{\Pi} \Pi = I.
\]
Clearly if $\Pi$ is the inverse period matrix to $\Omega$ then $\Omega$ is the inverse period matrix to $\Pi$. Furthermore if $\Omega_1 \simeq \Omega_2$ so that $\Omega_2 = A \Omega_1 Q^{-1}$ where $A \in \text{Gl}(g, \mathbb{C})$ and $Q \in \text{Gl}(2g, \mathbb{Z})$ then
\[
\begin{pmatrix} \Omega_2 \\ \Pi_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\
0 & A \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Pi_1 \end{pmatrix} Q^{-1},
\]
and the complex conjugate of the inverse transpose of this equation is the equation
\[
\begin{pmatrix} \Pi_2 \\ \Pi_2 \end{pmatrix} = \begin{pmatrix} \overline{A}^{-1} & 0 \\
0 & \overline{A}^{-1} \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \Pi_1 \end{pmatrix} tQ,
\]
showing that Π₂ = \( \mathcal{A}^{-1} \Pi_1 \upsilon \) and consequently that Π₁ ≃ Π₂. Similarly of course if Ω₁ ∼ Ω₂ then Π₁ ∼ Π₂ by the same formula. The inverse period matrix to a nonsingular period matrix can be defined more intrinsically in terms of the real bilinear form

\[(F.8) \quad <z, w> = 2 \Re(\upsilon z \overline{w}) = 2 \Re\left(\sum_{i=1}^{g} z_i \overline{w}_i\right)\]

for any column vectors \( z = \{z_i\}, \ w = \{w_i\} \in \mathbb{C}^g \) when \( \mathbb{C}^g \) is viewed as the real linear vector space \( \mathbb{R}^g \); here \( \Re(z) \) denotes the real part of the complex number \( z \). For most purposes it is sufficient to describe the lattice subgroup \( L(\Pi) = \Pi \mathbb{Z}^g \subset \mathbb{C}^g \) rather than the inverse period matrix \( \Pi \) itself.

**Theorem F.13** If \( \Omega \in \mathbb{C}^{g \times 2g} \) is a nonsingular period matrix the lattice subgroup \( L(\Pi) \) described by the inverse period matrix \( \Pi \) is the dual lattice subgroup to \( L(\Omega) \) in terms of the real bilinear form \((F.8)\) in the sense that

\[L(\Pi) = \{ \pi \in \mathbb{C}^g \mid <\omega, \pi> \in \mathbb{Z} \text{ for all } \omega \in L(\Omega) \}\]

**Proof:** The lattice subgroup \( L(\Omega) = \Omega \mathbb{Z}^g \subset \mathbb{C}^g \) is generated over the integers by the column vectors \( \omega_i \in \mathbb{C}^g \) of the matrix \( \Omega \), where \( \omega_i = \{\omega_{ki} \mid 1 \leq k \leq g\} \) in terms of the entries \( \omega_{ki} \) of the matrix \( \Omega \). The dual lattice then is generated over the integers by the \( 2g \) column vectors \( \pi_j \in \mathbb{C}^g \) defined by the conditions that \( <\omega_i, \pi_j> = \delta_{ij} \) for \( 1 \leq i, j \leq 2g \). If \( \pi_j = \{\pi_{kj} \mid 1 \leq k \leq g\} \in \mathbb{C}^g \) and \( \Pi \) is the matrix \( \Pi = \{\pi_{kj}\} \in \mathbb{C}^{g \times 2g} \) this duality condition is just that \( \delta_{ij} = 2 \Re(\omega_i \pi_j) = \upsilon \omega_i \pi_j + \overline{\upsilon} \omega_i \pi_j = \sum_{k=1}^{g} (\omega_{ki} \pi_{kj} + \overline{\omega}_{ki} \pi_{kj}) \), or in matrix terms \( I = \upsilon \Pi + \overline{\upsilon} \Pi \); and by \((F.7)\) that is just the condition that \( \Pi \) is the inverse period matrix to \( \Omega \), which suffices to conclude the proof.

For another use of the inverse period matrix, if \( \Omega \) is a nonsingular period matrix the columns of the \( 2g \times 2g \) matrix \( (\upsilon \Omega \Pi) \) are linearly independent vectors, so there is a direct sum decomposition

\[(F.9) \quad \mathbb{C}^{2g} = \upsilon \Omega \mathbb{C}^g \oplus \overline{\upsilon} \mathbb{C}^g \]

of the complex vector space \( \mathbb{C}^{2g} \) into two complementary linear subspaces, one spanned by the columns of the matrix \( \upsilon \Omega \) and the other spanned by the columns of the matrix \( \overline{\upsilon} \Pi \). It follows from \((F.7)\) that any point \( t \in \mathbb{C}^{2g} \) can be written

\[(F.10) \quad t = \upsilon \Pi t + \overline{\upsilon} \Pi t,\]

which is an explicit formula for splitting a vector \( t \in \mathbb{C}^{2g} \) into its components in the direct sum decomposition \((F.9)\); indeed by \((F.6)\) with the matrices \( \Omega \) and \( \Pi \) interchanged the square matrices \( \upsilon \Pi \overline{\Omega} \) and \( \overline{\upsilon} \Pi \Omega \) are the natural projection operators

\[(F.11) \quad \upsilon \Pi : \upsilon \Omega \mathbb{C}^g \oplus \overline{\upsilon} \mathbb{C}^g \longrightarrow \upsilon \Omega \mathbb{C}^g \]

\[(F.11) \quad \overline{\upsilon} \Pi : \upsilon \Omega \mathbb{C}^g \oplus \overline{\upsilon} \mathbb{C}^g \longrightarrow \overline{\upsilon} \mathbb{C}^g \]
in the direct sum decomposition (F.9), since $\Omega \Omega = \Omega$ while both $\Omega t = \Omega t$ and $\Omega t = 0$ for any $t \in \mathbb{C}^g$, and correspondingly for the complex conjugates. A particularly useful application of the inverse period matrix suggested by these observations is the following.

**Theorem F.14** If $\Omega$ is a nonsingular $g \times 2g$ period matrix there is the exact sequence of abelian groups

$$0 \rightarrow \mathbb{Z}^{2g} + \Omega \mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{\Pi} \frac{\mathbb{C}^g}{\Pi \mathbb{Z}^{2g}} \rightarrow 0$$

where $\iota$ is the natural inclusion homomorphism and $\Pi$ is the linear mapping defined by the inverse period matrix to $\Omega$.

**Proof:** The complex linear mapping $\Pi : \mathbb{C}^{2g} \rightarrow \mathbb{C}^g$ defined by the inverse period matrix $\Pi$ is surjective and has as its kernel the linear subspace $\Omega \mathbb{C}^g$, as is evident from (F.6); and since this linear mapping takes the subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ to the lattice subgroup $\Pi \mathbb{Z}^{2g} \subset \mathbb{C}^g$ that suffices for the proof.

**Corollary F.15** The complex torus $J(\Omega)$ defined by a nonsingular period matrix $\Omega$ can be described alternatively as the quotient group

$$J(\Omega) = \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + \Omega \mathbb{C}^g}$$

where $\Pi$ is the inverse period matrix to $\Omega$.

**Proof:** This follows immediately from the exact sequence of the preceding theorem, when the roles of the period matrices $\Omega$ and $\Pi$ are interchanged, since the complex torus $J(\Omega)$ is the quotient $J(\Omega) = \mathbb{C}^g / \Omega \mathbb{Z}^{2g}$, and that suffices for the proof.

The inverse period matrix also can be used to provide alternative characterizations of Hurwitz relations (F.4) between nonsingular period matrices in terms of either the matrix $A$ or the matrix $Q$.

**Theorem F.16** Let $\Omega_1$ be a $g_1 \times 2g_1$ nonsingular period matrix and $\Omega_2$ be a $g_2 \times 2g_2$ nonsingular period matrix.

(i) A matrix $A \in \mathbb{C}^{g_2 \times g_1}$ is part of a Hurwitz relation $(A, Q)$ from the period matrix $\Omega_1$ to the period matrix $\Omega_2$ if and only if

$$2\Re(\Pi_2 A \Omega_1) \in \mathbb{Z}^{2g_2 \times 2g_1}$$

where $\Pi_2$ is the inverse period matrix to $\Omega_2$ and $\Re(Z)$ denotes the real part of the complex matrix $Z$; and $Q = 2\Re(\Pi_2 A \Omega_1)$.

(ii) A matrix $Q \in \mathbb{Z}^{2g_2 \times 2g_1}$ is part of a Hurwitz relation $(A, Q)$ from the period matrix $\Omega_1$ to the period matrix $\Omega_2$ if and only if

$$\Omega_2 Q^T \Pi_1 = 0 \in \mathbb{C}^{g_2 \times g_1}$$

where $\Pi_1$ is the inverse period matrix to $\Omega_1$; and $A = Q^T \Pi_1$. 
F.1. PERIOD MATRICES

**Proof:** A matrix $A \in \mathbb{C}^{g_2 \times g_1}$ is part of a Hurwitz relation from the period matrix $\Omega_1$ to the period matrix $\Omega_2$ if and only if there is a matrix $Q \in \mathbb{Z}^{2g_2 \times 2g_1}$ satisfying (F.5); and that equation can be rewritten equivalently as

$$Q = \begin{pmatrix} \pi_2 & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A \Omega_1 \\ \pi_1 \end{pmatrix} = \begin{pmatrix} \pi_2 & A \Omega_1 \\ 0 & \pi_1 \end{pmatrix} = 2\Re \left( \pi_2 A \Omega_1 \right).$$

Similarly a matrix $Q \in \mathbb{Z}^{2g_2 \times 2g_1}$ is part of a Hurwitz relation from the period matrix $\Omega_1$ to the period matrix $\Omega_2$ if and only if there is a matrix $A \in \mathbb{C}^{g_2 \times g_1}$ satisfying (F.5); and that equation can be rewritten equivalently as

$$\begin{pmatrix} A \\ 0 \end{pmatrix} = \begin{pmatrix} A \Omega_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_2 & A \Omega_1 \\ 0 & \pi_1 \end{pmatrix} = \begin{pmatrix} \Omega_2 Q & \Omega_1 \\ \pi_2 Q & \pi_1 \end{pmatrix}.$$

That suffices to conclude the proof.

**Corollary F.17** Two nonsingular period matrices $\Omega_1, \Omega_2$ of the same rank $g$ are equivalent if and only if either of the following two equivalent conditions hold:

(i) there is a matrix $A \in \text{Gl}(g, \mathbb{C})$ such that $2\Re(\pi_2 A \Omega_1) \in \text{Gl}(2g, \mathbb{Z})$ where $\pi_2$ is the inverse period matrix to $\Omega_2$;

(ii) there is a matrix $Q \in \text{Gl}(2g, \mathbb{Z})$ such that $\Omega_2 Q \pi_1 = 0$ where $\pi_1$ is the inverse period matrix to $\Omega_1$.

**Proof:** Two nonsingular period matrices $\Omega_1, \Omega_2$ of the same rank $g$ are equivalent if and only if there is a Hurwitz relation $(A, Q)$ from the period matrix $\Omega_1$ to the period matrix $\Omega_2$ where the matrices $A$ and $Q$ are invertible. By Lemma F.8 it is enough just to show that one of the two matrices $A$ or $Q$ is invertible. As in the proof of the preceding theorem, for a given matrix $A$ the matrix $Q$ in the Hurwitz relation is $Q = 2\Re(\pi_2 A \Omega_1)$ while for a given matrix $Q$ there is such a matrix $A$ if and only if $\Omega_2 Q \pi_1 = 0$, and that suffices to conclude the proof.

**Corollary F.18** Two nonsingular period matrices $\Omega_1, \Omega_2$ of the same rank $g$ are weakly equivalent if and only if either of the following two equivalent conditions holds:

(i) there is a matrix $A \in \text{Gl}(g, \mathbb{C})$ such that $2\Re(\pi_2 A \Omega_1) \in \text{Gl}(2g, \mathbb{Q})$ where $\pi_2$ is the inverse period matrix to $\Omega_2$;

(ii) there is a matrix $Q \in \text{Gl}(2g, \mathbb{Q})$ such that $\Omega_2 Q \pi_1 = 0$ where $\pi_1$ is the inverse period matrix to $\Omega_1$. 
APPENDIX F. COMPLEX TORI

Proof: Two nonsingular period matrices $\Omega_1, \Omega_2$ of the same rank $g$ are weakly equivalent if and only if there is the analogue of a Hurwitz relation $(A, Q)$ from the period matrix $\Omega_1$ to the period matrix $\Omega_2$ in which the matrices $A$ and $Q$ are invertible but $Q$ is only rational rather than integral; with this modification the proof is as that of the preceding corollary, and that suffices for the proof.

F.2 Topological Properties

Topologically a complex torus of dimension $g$ is a product of $2g$ circles. To make this more explicit, if $\Omega = (\omega^1, \ldots, \omega^{2g})$ is a nonsingular period matrix its column vectors $\omega^i \in \mathbb{C}^g$ are linearly independent over the real numbers. The real linear mapping that takes a vector $t \in \mathbb{R}^{2g}$ to the vector $z = \Omega t \in \mathbb{C}^g$ is an isomorphism

\[ \Omega : \mathbb{R}^{2g} \longrightarrow \mathbb{C}^g \]  

of real vector spaces that maps the standard basis column vector $\delta^j = \{ \delta^j_k \}$ in $\mathbb{R}^{2g}$ to the column vector $\omega^j \in \mathbb{C}^g$ for $1 \leq j \leq 2g$ and consequently maps the lattice subgroup $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ to the lattice subgroup $L(\Omega) = \Omega \mathbb{Z}^{2g} \subset \mathbb{C}^g$; it therefore determines a one-to-one mapping

\[ \Omega : \mathbb{R}^{2g}/\mathbb{Z}^{2g} \longrightarrow \mathbb{C}^g/\Omega \mathbb{Z}^{2g} \]  

that identifies the real torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ with the complex torus $J(\Omega) = \mathbb{C}^g/\Omega \mathbb{Z}^{2g}$ topologically. If $\Pi$ is the inverse period matrix to $\Omega$ as introduced in Theorem F.12 then the real linear mapping that takes a vector $z \in \mathbb{C}^g$ to the vector

\[ \tilde{\Pi}(z) = \Pi z + \Pi \tau \in \mathbb{R}^{2g} \]  

is the real linear mapping inverse to (F.12) since $\Omega \tilde{\Pi}(z) = \Omega \Pi z + \Omega \Pi \tau = z$ by (F.6) and conversely $\tilde{\Pi}(\Omega t) = \Pi \Omega t + \Pi \Omega \tau = t$ by (F.7); the mapping (F.14) consequently determines the one-to-one mapping

\[ \tilde{\Pi} : \mathbb{C}^g/\Omega \mathbb{Z}^{2g} \longrightarrow \mathbb{R}^{2g}/\mathbb{Z}^{2g} \]  

that is the inverse mapping to (F.13). The real torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g} = (\mathbb{R}/\mathbb{Z})^{2g}$ is the product of $2g$ circles $\mathbb{R}/\mathbb{Z}$, and consequently the complex torus $J(\Omega) = \mathbb{C}^g/\Omega \mathbb{Z}^{2g}$ is topologically the product of $2g$ circles as well.

The first homology group of a circle is the free abelian group $\mathbb{Z}$, so it follows from the Künneth formula\(^2\) for the homology groups of product spaces that the homology groups of a torus $T$ are finitely generated free abelian groups. Consequently the cohomology and homology groups of $T$ are dual to one another, and both can be described fully by considering only the homology and cohomology

groups with real or complex coefficients; and the complex cohomology group can be identified with the tensor product of the real cohomology group with \( \mathbb{C} \). Thus for the purposes at hand it is enough to describe the homology and cohomology groups of \( T \) in terms of the complex deRham groups \( H^p(T) \) of \( T \), the quotients of the groups of closed complex-valued differential forms by the subgroups of exact complex-valued differential forms on the differentiable manifold \( T \), effectively reducing the topological considerations to rather straightforward analytic considerations.

A complex-valued differential \( p \)-form on the torus \( T = \mathbb{R}^{2g}/\mathbb{Z}^{2g} \) can be viewed as a complex-valued differential \( p \)-form on \( \mathbb{R}^{2g} \) that is invariant under translations by vectors in \( \mathbb{Z}^{2g} \); and a complex-valued differential \( p \)-form on \( T \) that is invariant under all translations of the torus \( T \) can be viewed correspondingly as a constant complex-valued differential \( p \)-form on \( \mathbb{R}^{2g} \). Such a differential form can be written in terms of the real coordinates \( t_1, \ldots, t_{2g} \) on \( \mathbb{R}^{2g} \) as

\[
\phi = \sum_{1 \leq i_1 < \cdots < i_p \leq 2g} c_{i_1 \ldots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}
\]

for arbitrary complex constants \( c_{i_1 \ldots i_p} \); alternatively when the coefficients \( c_{i_1 \ldots i_p} \) are extended to all values of the indices \( i_1, \ldots, i_p \) to be skew-symmetric in these indices then

\[
\phi = \frac{1}{p!} \sum_{i_1, \ldots, i_p = 1}^{2g} c_{i_1 \ldots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}.
\]

One of the many applications of the theory of harmonic differential forms\(^3\) is that on a compact Lie group such as a torus any closed differential form is cohomologous to a unique group invariant differential form, and consequently that the deRham group \( \mathcal{H}^p(T) \) of the torus \( T = \mathbb{R}^{2g}/\mathbb{Z}^{2g} \) is isomorphic to the space of constant complex-valued differential \( p \)-forms on \( \mathbb{R}^{2g} \). Since the deRham group is isomorphic to the complex cohomology group it follows that

\[
\dim H^p(T, \mathbb{C}) = \dim \mathcal{H}^p(T) = \binom{2g}{p}.
\]

The exterior product of differential forms determines an exterior product structure on the cohomology group \( H^p(T, \mathbb{C}) \), exhibiting it as the complex exterior algebra generated by the first cohomology group \( H^1(T, \mathbb{C}) \).

The cohomology group \( H^p(T, \mathbb{C}) \) is dual to the homology group \( H_p(T, \mathbb{C}) \), so \( \dim H_p(T, \mathbb{C}) = \binom{2g}{p} \) as a consequence of (F.18). To describe the homology

\(^3\)Invariant integrals on groups were introduced by E. Cartan, and the applications of harmonic differential forms to show that the harmonic differential forms on compact Lie groups are the invariant differentials forms of Cartan and consequently that harmonic differentials can be used to describe the deRham groups was due to W. V. D. Hodge, described in detail in his book *Harmonic Integrals*, Cambridge Univ. Press, 1941. There are many other derivations of the same result; for the case of complex tori a short proof of a quite different sort can be found in the book by C. Birkenhake and H. Lange, *Complex Abelian Varieties*, Springer-Verlag 2004.
group $H_p(T, \mathbb{C})$ of the torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ more concretely, for any $p$ distinct integers $j_1, \ldots, j_p$ in the range $1 \leq j_i \leq 2g$ let $[\delta^{j_1, \ldots, j_p}]$ be the singular $p$-cycle

\[(F.19) \quad [\delta^{j_1, \ldots, j_p}] : [0, 1]^p \rightarrow T\]

that is the composition of the linear mapping from $[0, 1]^p$ to the vector space $\mathbb{R}^{2g}$ defined by

\[(F.20) \quad \delta^{j_1, \ldots, j_p}(s_1, \ldots, s_p) = \sum_{k=1}^{p} s_k \delta^{j_k} \in \mathbb{R}^{2g}\]

for any point $(s_1, \ldots, s_p)$ where $0 \leq s_i \leq 1$, followed by the natural projection

\[(F.21) \quad t \in \mathbb{R}^{2g} \rightarrow [t] \in T = \mathbb{R}^{2g}/\mathbb{Z}^{2g};\]

with the natural orientation provided by the parameter space $\mathbb{R}^p$ these singular cycles are skew-symmetric in the indices $j_1, \ldots, j_p$. For $p = 1$ the singular 1-cycle $[\delta^i]$ can be viewed as being spanned by the column vector $\delta^i$ itself; the singular $p$-cycle $[\delta^{j_1, \ldots, j_p}]$ can be viewed as being spanned by the $p$ column vectors $\delta^{j_1}, \ldots, \delta^{j_p}$ so sometimes it is denoted also by $[\delta^{j_1}] \wedge \ldots \wedge [\delta^{j_p}]$. In terms of the coordinates $(t_1, \ldots, t_{2g})$ on $\mathbb{R}^{2g}$ the image of the mapping (F.20) is described parametrically by $t_l = \sum_{k=1}^{p} \delta^{j_k}_l s_k$; so the differential form $dt_l$ on $\mathbb{R}^{2g}$ induces the differential form $dt_l = \sum_{k=1}^{p} \delta^{j_k} ds_k$ in terms of the parameters $(s_1, \ldots, s_p)$ of the singular $p$-cycle $[\delta^{j_1, \ldots, j_p}]$, and more generally

$$dt_{t_1} \wedge \cdots \wedge dt_{t_p} = \sum_{k_1, \ldots, k_p=1}^{p} \delta^{j_{k_1}}_{t_1} \cdots \delta^{j_{k_p}}_{t_p} ds_{k_1} \wedge \cdots \wedge ds_{k_p}$$

This differential form is clearly 0 unless the set of indices $(l_1, \ldots, l_p)$ is a permutation of the set of indices $(j_1, \ldots, j_p)$; and if $(l_1, \ldots, l_p) = \pi(j_{k_1}, \ldots, j_{k_p})$ for a permutation $\pi \in \mathfrak{S}^p$ then

$$dt_{t_{k_1}} \wedge \cdots \wedge dt_{t_{k_p}} = \text{sgn}(\pi) ds_1 \wedge \cdots \wedge ds_p$$

where $\text{sgn}(\pi)$ is the sign of the permutation $\pi$. It thus follows that

\[(F.22) \quad \int_{[\delta^{j_1, \ldots, j_p}]} dt_{t_1} \wedge \cdots \wedge dt_{t_p} = \delta^{j_1, \ldots, j_p} \]

where $\delta^{j_1, \ldots, j_p}$ is 0 unless $(j_1, \ldots, j_p)$ is a permutation of $(l_1, \ldots, l_p)$ and then is the sign of that permutation. As a consequence it is clear that the differential forms $dt_{t_1} \wedge \cdots \wedge dt_{t_p}$ are dual to the cycles $\delta^{j_1, \ldots, j_p}$, so since these differential forms are a basis for the deRham group $\mathfrak{H}^p(T)$ and hence represent a basis for the cohomology group $H^p(T, \mathbb{C})$ it follows that the homology classes represented
by the $p$-cycles $[\delta^{j_1,\ldots,j_p}]$ for $1 \leq j_1 < \cdots < j_p \leq 2g$ are a basis for the homology group $H_p(T, \mathbb{C})$; this establishes explicitly the duality between the homology and cohomology groups of the torus.

The homeomorphism (F.13) carries the skew-symmetric singular $p$-cycle $[\delta^{j_1,\ldots,j_p}]$ in the torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ to the skew-symmetric singular $p$-cycle

\[(F.23) \quad \Omega[\delta^{j_1,\ldots,j_p}] = [\Omega \delta^{j_1,\ldots,j_p}] = [\omega^{j_1,\ldots,j_p}]\]

in the torus $J(\Omega)$ spanned by the columns $\omega^{j_1,\ldots,j_p}$ of the period matrix $\Omega$, in analogy with (F.19), (F.20), (F.21), and the homology classes represented by the skew-symmetric singular $p$-cycles $[\omega^{j_1,\ldots,j_p}]$ are a basis for the homology $H_p(J(\Omega), \mathbb{C})$ of the complex torus $J(\Omega) = \mathbb{C}g/\Omega \mathbb{Z}^{2g}$. The dual basis for the deRham group $\mathcal{H}^p(J(\Omega))$ then consists of the differential forms

\[(F.24) \quad \phi_j(z) = \sum_{k=1}^{g} (\pi_{kj}) dz + \pi_{kj} \overline{dz}\]

for 1-forms and the exterior products have the corresponding forms.

For another period matrix $\Lambda \in \mathbb{C}^{h \times 2h}$ describing a complex torus $J(\Lambda)$ of dimension $h$, a basis for the homology $H_p(J(\lambda), \mathbb{C})$ is represented by the skew-symmetric singular $p$-cycles $[\lambda^{j_1,\ldots,j_p}]$ spanned by the columns $\lambda^j = \Lambda \delta^j$ of the matrix $\Lambda$, and a dual basis for the deRham group $\mathcal{H}^p(J(\Lambda))$ consists of the differential forms $\psi_{j_1,\ldots,j_p}$ for $1 \leq j_1 < \cdots < j_p \leq 2h$ where in analogy with (F.24)

\[(F.25) \quad \psi_j(w) = \sum_{k=1}^{h} (\sigma_{kj}) dw_k + \sigma_{kj} \overline{dw_k}\]

in terms of the complex coordinates $w_1, \ldots, w_h$ in $\mathbb{C}^h$ and $\Sigma = \{\sigma_{kj}\}$ is the inverse period matrix to the period matrix $\Lambda$; the exterior products have the corresponding forms. Both (F.24) and (F.25) can be rewritten conveniently in matrix notation as

\[(F.26) \quad \phi(z) = \Pi dz + \Pi \overline{dz}\]

and

\[(F.27) \quad \psi(w) = \Sigma dw + \Sigma \overline{dw}\]

where $\phi = \{\phi_j\}$, $\psi = \{\psi_j\}$, $dz = \{dz_j\}$ and $dw = \{dw_j\}$ are viewed as column vectors of differential forms; and in view of (F.6) the inverse of (F.26) is

\[(F.28) \quad dz = \Omega \phi(z) \quad \overline{dz} = \overline{\Pi} \phi(z)\]

If $f : J(\Omega) \rightarrow J(\Lambda)$ is a holomorphic mapping between these complex tori described by a Hurwitz relation $(A, Q)$ from the period matrix $\Omega$ to the period
matrix Λ, so that the mapping f is induced by the linear mapping \( A : \mathbb{C}^h \rightarrow \mathbb{C}^h \) aside from a translation in the vector space \( \mathbb{C}^h \), then under this mapping the differential forms \( dw \) on \( J(Λ) \) induce the differential forms \( f^*(dw) = Adz \) on \( J(Ω) \) in matrix notation; and consequently the differential forms \( ψ \) on \( J(Λ) \) induce the differential forms

\[
f^*(ψ_j(w)) = \sum_{k=1}^{h} q_{jk} φ_k(z) \quad \text{for} \quad 1 \leq j \leq h.
\]

The induced differential \( p \)-forms then are just the wedge products of the induced differential 1-forms. Dually the mapping \( f \) takes the singular 1-cycle \( [ω] \) spanned by column \( l \) of the period matrix \( Ω \) for \( 1 \leq l \leq 2g \) to a linear combination \( f^*([ω]) = \sum_{m=1}^{2g} c_{lm} [λ^m] \) of the singular 1-cycles spanned by the columns of the period matrix \( Λ \), where

\[
q_{jk} = \sum_{l=1}^{2h} q_{jl} δ_k^l = \sum_{l=1}^{2h} \int_{[Λ]} q_{jl} φ_l(z) = \int_{[ω]} f^*(ψ_j(w)) = \int_{f^*([ω])} ψ_j(w) = \sum_{m=1}^{2g} \int_{c_{km}[λ^m]} ψ_j(w) = c_{kj}
\]

and consequently

\[
f^*([ω]) = \sum_{m=1}^{2g} q_{mk} [λ^m];
\]

thus the matrix \( Q \) describes the effect of the mapping \( f \) described by the Hurwitz relation \((A, Q)\) on the first homology groups of the two tori, the group homomorphism \( f_* : H_1(J(Ω_1)) \rightarrow H_1(J(Ω_2)) \) induced by the mapping \( f \), and that extends to the wedge products of the 1-cycles correspondingly. When the mapping \( f \) is the biholomorphic mapping corresponding to a Hurwitz relation \((A, Q)\) that is an equivalence of the period matrices defining the complex tori, the induced homomorphism \( f_* \) is the isomorphism described by the matrix \( Q \in \text{Gl}(2g, \mathbb{Z}) \). When the mapping \( f \) is merely an isogeny corresponding to a Hurwitz relation \((A, Q)\) that is a weak equivalence of the period matrices defining the complex tori, the mapping exhibits the torus \( J(Ω_1) \) as an unbranched covering of the torus \( J(Ω_2) \); the induced homomorphism \( f_* \) described by the matrix \( Q \) determines this covering topologically, since the fundamental groups of complex tori are abelian so coincide with the first homology groups.
F.3 Riemann Matrices

A $g \times 2g$ period matrix $\Omega$ is called a Riemann matrix if there is a skew-
symmetric integral matrix $P$ such that:

(i) $\Omega P^{\dagger} \Omega = 0$, and

(ii) the matrix $H = i \Omega P^{\dagger} \Omega$ is positive definite Hermitian;

the matrix $P$ is called a principal matrix for the Riemann matrix $\Omega$. If $P$ is a
principal matrix for the Riemann matrix $\Omega$ then so is any positive scalar multi-
ple $rP$ that is also an integral matrix; among these multiples there is a unique
one having relatively prime integral entries, called a primitive principal matrix
for the Riemann matrix $\Omega$. There are Riemann matrices admitting principal
matrices not all of which are scalar multiples of one another, hence admitting
a number of distinct primitive principal matrices; these are called singular Rie-
mann matrices. The choice of a principal matrix up to arbitrary positive scalar
multiples, or equivalently the choice of a primitive principal matrix, is called
a polarization of the Riemann matrix $\Omega$; and the pair consisting of a Riemann
matrix $\Omega$ and its polarization is called a polarized Riemann matrix. A polarized
Riemann matrix is denoted either by $(\Omega, P)$, where $P$ is a primitive principal
matrix for the Riemann matrix $\Omega$, or $(\Omega, \{P\})$, where $P$ is any matrix a multiple
of which is a principal matrix for the Riemann matrix $\Omega$. A polarized Riemann
matrix $(\Omega, J)$ where $J$ is the basic skew-symmetric matrix

\[
(F.31) \quad J = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

is called a principally polarized Riemann matrix. Condition (i) in the definition
of a Riemann matrix, often called Riemann’s equality, can be rewritten in terms
of the associated full period matrix as

\[
(F.32) \quad i \begin{pmatrix}
\Omega \\
\Omega
\end{pmatrix} P^{\dagger} \Omega = i \begin{pmatrix}
\Omega P^{\dagger} \Omega & \Omega P^{\dagger} \Omega \\
\Omega P^{\dagger} \Omega & \Omega P^{\dagger} \Omega
\end{pmatrix} = \begin{pmatrix}
H & 0 \\
0 & -H
\end{pmatrix}
\]

where $H = i \Omega P^{\dagger} \Omega$; and condition (ii) in the definition of a Riemann matrix,
often called Riemann’s inequality, is that the Hermitian matrix $H$ is positive
definite. An immediate consequence of this expanded form of the Riemann
matrix conditions is the following auxiliary observation.

**Theorem F.19** A Riemann matrix $\Omega$ is a nonsingular period matrix; and if $P$
is a principal matrix for the Riemann matrix $\Omega$ then $P$ is a nonsingular matrix
and $\det P > 0$.

**Proof:** Since the matrix $H$ in (F.32) is positive definite the right-hand side of
that equation is a nonsingular matrix; consequently both the full period matrix
$(\Omega)$ and the principal matrix $P$ must be nonsingular. Taking the determinant
in the identity (F.32) among $2g \times 2g$ square matrices yields the result that

\[
i^{2g} \left| \det \begin{pmatrix}
\Omega \\
\Omega
\end{pmatrix} \right|^2 \det P = (-1)^g |\det H|^2,
\]
hence that \( \det P > 0 \), and that suffices for the proof.

Two period matrices \( \Omega \) and \( \hat{\Omega} \) were defined to be equivalent in (F.1) if there are matrices \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \) such that \( \hat{\Omega} = A\Omega Q^{-1} \). If \((\Omega, P)\) is a polarized Riemann matrix, so that \( \Omega^t P \Omega = 0 \) and \( i\Omega P \bar{\Omega} \) is positive definite Hermitian, and if \( \hat{P} = Q P^t Q \), so \( \hat{P} \) is a skew-symmetric matrix with relatively prime integral entries, then

\[
\hat{\Omega} \hat{P} \hat{\Omega} = A\Omega Q^{-1} \cdot QP^tQ \cdot \bar{\Omega}^tA = A\Omega^t \Omega P^t \Omega A = 0
\]

and the matrix

\[
i\hat{\Omega} \hat{P} \hat{\Omega} = i A\Omega Q^{-1} \cdot QP^tQ \cdot \bar{\Omega}^tA = iA\Omega^t \Omega \Omega^t \Omega A
\]

is positive definite Hermitian, so \((\hat{\Omega}, \hat{P})\) also is a polarized Riemann matrix. The two polarized Riemann matrices \((\Omega, P)\) and \((\hat{\Omega}, \hat{P})\) are called equivalent polarized Riemann matrices, and the equivalence of these two polarized Riemann matrices is denoted by \((\Omega, P) \simeq (\hat{\Omega}, \hat{P})\). If the period matrices \( \Omega \) and \( \hat{\Omega} \) are just weakly equivalent, so that it is only the case that \( Q \in \text{Gl}(2g, \mathbb{Q}) \), then the matrix \( QP^tQ \) is only a rational matrix; but the other conditions for a polarized Riemann matrix are satisfied, so if \( r \) is a positive rational number such that \( rQP^tQ \) is an integral matrix with relatively prime entries then the pair \((A\Omega Q^{-1}, rQP^tQ)\) is a polarized Riemann matrix. These two polarized Riemann matrices are called weakly equivalent polarized Riemann matrices, and the weak equivalence of these two polarized Riemann matrices is denoted by \((\Omega, P) \sim (\hat{\Omega}, \hat{P})\) or \((\Omega, \{P\}) \sim (\hat{\Omega}, \{\hat{P}\})\). Of course \((A\Omega Q^{-1}, \{rQP^tQ\}) = (A\Omega Q^{-1}, \{QP^tQ\})\), so this is the more convenient notation when considering the weak equivalence of polarized Riemann matrices. Both evidently are equivalence relations in the customary sense. In summary, the equivalence of polarized Riemann matrices is defined by

\[(F.33)\quad (\Omega, P) \simeq (A\Omega Q^{-1}, \{P\}) \quad \text{or} \quad (\Omega, \{P\}) \simeq (A\Omega Q^{-1}, \{P\})
\]

whenever \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \);

and the weak equivalence of polarized Riemann matrices is defined by

\[(F.34)\quad (\Omega, P) \sim (A\Omega Q^{-1}, \{rQP^tQ\}) \quad \text{or} \quad (\Omega, \{P\}) \sim (A\Omega Q^{-1}, \{P\})
\]

whenever \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Q}) \)

where \( r \) is the unique positive rational number such that \( rQP^tQ \) is an integral matrix with relatively prime entries. An important case of this equivalence is the following.
Theorem F.20 If \((\Omega, P)\) is a polarized Riemann matrix, \(r\) is a positive integer such that \(N = r^{-1}P^{-1}\) is an integral matrix with relatively prime entries and \(\Pi\) is the inverse period matrix to \(\Omega\), then \((\Pi, \, ^tN)\) is a polarized Riemann matrix that is weakly equivalent to \((\Omega, P)\); and if in addition \(\det P = 1\) then \(r = 1\) and the weak equivalence is actually an equivalence.

Proof: From Riemann’s equality \(\Omega P \Omega = 0\) and Theorem F.16 (ii) it follows that \(P\) is part of the Hurwitz relation \((A, P)\) from the period matrix \(\Pi\) to the period matrix \(\Omega\) where \(A = \Omega P \, ^t\Omega\). Thus \(A\Pi = \Omega P\), and since the principal matrix \(P\) is nonsingular by Theorem F.19, and consequently the matrix \(A\) also is nonsingular by Lemma F.8, it follows that \(A \in \text{Gl}(g, \mathbb{C})\) and \(P \in \text{Gl}(2g, \mathbb{Q})\) so this Hurwitz relation exhibits the weak equivalence of the period matrices \(\Omega\) and \(\Pi\). In addition \(rP \, ^tN = P\), so the polarized Riemann matrices \((\Pi, \, ^tN)\) and \((\Omega, P)\) are weakly equivalent. Of course if \(\det P = 1\) then \(r = 1\) and \(P \in \text{Gl}(2g, \mathbb{Z})\) so \((\Pi, \, ^tN)\) and \((\Omega, P)\) are equivalent polarized Riemann matrices, and that suffices to conclude the proof.

Corollary F.21 If \((\Omega, P)\) is a polarized Riemann matrix and \(\Pi\) is the inverse period matrix to \(\Omega\) then \((\Omega P \, ^t\Omega, P)\) is a Hurwitz relation from the period matrix \(\Pi\) to the period matrix \(\Omega\) describing an isogeny from the complex torus \(J(\Pi)\) to the complex torus \(J(\Omega)\); and if \(\det P = 1\) this isogeny is a biholomorphic mapping.

Proof: In the proof of the preceding theorem the weak equivalence of the polarized Riemann matrices \((\Pi, \, ^tN)\) and \((\Omega, P)\) was exhibited by the Hurwitz relation \((A, P) = (\Omega P \, ^t\Omega, P)\) from the period matrix \(\Pi\) to the period matrix \(\Omega\), and this Hurwitz relation exhibits an isogeny from the complex torus \(J(\Pi)\) to the complex torus \(J(\Omega)\). If \(\det P = 1\) the weak equivalence is an equivalence and the isogeny is a biholomorphic mapping, and that suffices for the proof.

For some purposes it is convenient to have a more explicit statement of the preceding observations. If \((\Omega, P)\) is a polarized Riemann matrix, \((F.32)\) is an identity between invertible matrices and its inverse transpose is the equation

\[-i \, ^t\left(\frac{\Omega}{\Pi}\right) \, ^{-1}P^* \left(\frac{\Omega}{\Pi}\right)^{-1} = \begin{pmatrix} G & 0 \\ 0 & -\overline{G} \end{pmatrix}\]

or equivalently in terms of the inverse period matrix \(\Pi\)

\[-i \, ^t\left(\frac{\Pi}{\Pi}\right) \, ^{-1}P^* \left(\frac{\Pi}{\Pi}\right)^{-1} = \begin{pmatrix} G & 0 \\ 0 & -\overline{G} \end{pmatrix}\]

where \(P^* = \, ^tP^{-1}\) and \(G = \, ^tH^{-1} = \overline{H^{-1}}\). This equation is equivalent to the identities

\[(F.35) \quad \Pi P^* \, ^t\Pi = 0 \quad i \, \Pi P^* \, \overline{\Pi} = \overline{G},\]
which if $rP^*$ is an integral matrix is the pair of conditions (i) and (ii) showing that $rP^*$ is a principal matrix for the Riemann matrix $\Pi$ since the matrix $G$ is positive definite Hermitian.

Principally polarized Riemann matrices are of particular interest in the study of compact Riemann surfaces, so it is worth examining that special case in somewhat more detail.

**Theorem F.22** (i) For any principally polarized Riemann matrix $(\Omega, J)$ there is a unique nonsingular complex matrix $A$ such that $A\Omega = (I \ Z)$, where $I$ is the identity matrix.

(ii) A period matrix of the form $(I \ Z)$, where $I$ is the identity matrix, is a Riemann matrix with the principal matrix $J$ if and only if the matrix block $Z$ is a complex symmetric matrix with positive definite imaginary part.

**Proof:** If $(\Omega, J)$ is a principally polarized Riemann matrix and the $g \times 2g$ matrix $\Omega$ is decomposed into $g \times g$ square blocks $\Omega = (\Omega_1 \ \Omega_2)$ Riemann’s equality is that

$$0 = (\Omega_1 \ \Omega_2) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \Omega_1 \Omega_2 - \Omega_2 \Omega_1$$

and Riemann’s inequality is that the $g \times g$ matrix

$$H = i(\Omega_1 \ \Omega_2) \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \overline{\Omega_1} \\ \overline{\Omega_2} \end{pmatrix} = i(\Omega_1 \ \Omega_2 - \Omega_2 \ \Omega_1)$$

is positive definite Hermitian. If the square matrix $\Omega_1$ is singular there is a nontrivial row vector $c \in \mathbb{C}^g$ such that $c\Omega_1 = 0$, and then $cHc = ic\Omega_1 \cdot \overline{\Omega}_2 - ic\Omega_2 \cdot \overline{\Omega}_1 = 0$ which contradicts the condition that the matrix $H$ is positive definite Hermitian; therefore the matrix $\Omega_1$ is nonsingular, and if $A = \Omega_1^{-1}$ it follows that $A\Omega = (I \ Z)$ for a $g \times g$ square complex matrix $Z$. The principally polarized Riemann matrix $((I \ Z), J)$ thus is equivalent to $(\Omega, J)$, and it must satisfy the analogues of (F.36) and (F.37). From the analogue to (F.36) it follows that 0 = $\overline{Z} - Z$, so the matrix $Z$ is symmetric; and from the analogue to (F.37) it follows that the matrix $H = i(\overline{Z} - Z) = i(\overline{Z} - Z) = 2\Im(Z)$ is positive definite, where $\Im(Z)$ is the imaginary part of the matrix $Z$. That suffices to conclude the proof.

The set of complex symmetric $g \times g$ matrices $Z = X + iY$ such that the imaginary part $\Im(Z) = Y$ is positive definite is called the Siegel upper half-space of rank $g$ and is denoted by $\mathfrak{H}_g$. In the special case $g = 1$ the space $\mathfrak{H}_1$ is just the ordinary upper half-plane; in general $\mathfrak{H}_g$ is an open convex subspace of the vector space $\mathbb{C}^g$. A principally polarized Riemann matrix $(\Omega, J)$ for which $\Omega = (I \ Z)$ for $Z \in \mathfrak{H}_g$ is called a normalized principally polarized Riemann matrix. In these terms one of the consequences of the preceding theorem is that any principally polarized Riemann matrix is equivalent to a normalized principally polarized Riemann matrix. However distinct normalized principally polarized Riemann matrices still can be equivalent principally polarized Riemann matrices. The
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A description of this situation involves the group $\text{Sp}(2g, \mathbb{Z})$ of integral symplectic matrices of rank $2g$, the group consisting of those $2g \times 2g$ integral matrices $Q$ such that $QJQ = J$ for the basic skew-symmetric matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, as discussed in more detail in Appendix H.

**Theorem F.23** Normalized principally polarized Riemann matrices $(I \ Z, J)$ and $(I \ ˜Z, J)$ are equivalent polarized Riemann matrices if and only if

$$\tilde{Z} = (A + ZC)^{-1}(B + ZD)$$

for a symplectic matrix

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z}).$$

**Proof:** By definition (F.33) two normalized principally polarized Riemann matrices $(Ω, J)$ and $(\tilde{Ω}, J)$ are equivalent polarized Riemann matrices if and only if $\tilde{Ω} = EΩQ$ and $Q^{-1}P'Q^{-1} = J$ for some matrices $E \in \text{Gl}(g, \mathbb{C})$ and $Q \in \text{Gl}(2g, \mathbb{Z})$. It must therefore be the case that $Q \in \text{Sp}(2g, \mathbb{Z})$, and when $Q$ is decomposed into $g \times g$ matrix blocks

$$(I \ ˜Z) = E(I \ Z)Q = E(I \ Z)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = E(A + ZC \ B + ZD);$$

thus $E = (A + ZC)^{-1}$ and $\tilde{Z} = (A + ZC)^{-1}(B + ZD)$, which suffices for the proof.

For $g = 1$ equation (F.38) is just the familiar action of the classical modular group $\text{Sl}(2, \mathbb{Z})$ as a group of biholomorphic mappings of the upper half-plane $\mathfrak{H}_1$ to itself, and the quotient space $\mathcal{A}_1 = \mathfrak{H}_1/\text{Sl}(2, \mathbb{Z})$ is the familiar space of moduli of complex tori, that is, is a space of parameters for biholomorphic equivalence classes of complex tori of dimension 1. By Theorem F.22 normalized principally polarized Riemann matrices are of the form $(I \ Z, J)$ for arbitrary matrices $Z \in \mathfrak{H}_g$, so (F.38) describes an action of the symplectic modular group as a group of biholomorphic mappings of the Siegel upper half-space $\mathfrak{H}_g$ of rank $g$ to itself. The quotient space $\mathcal{A}_g = \mathfrak{H}_g/\text{Sp}(2g, \mathbb{Z})$ then is a well defined topological space with the natural structure of a holomorphic variety of dimension $g(g - 1)/2$, the Siegel moduli space$^4$. The holomorphic variety $\mathcal{A}_g$ can be considered as the space of moduli or of parameters for the set of equivalence classes of principally polarized Riemann matrices; in view of Theorem F.11 the variety $\mathcal{A}_g$ also can be viewed as the space of moduli or of parameters for the set of biholomorphic equivalence classes of complex tori $J(Ω)$ described by period matrices $Ω$ that are Riemann matrices with principal matrix $J$.

F.4 Form Matrices

An alternative characterization and interpretation of polarized Riemann matrices is quite useful for some purposes.

**Theorem F.24** A $g \times 2g$ nonsingular period matrix $\Omega$ is a Riemann matrix if and only if there is a positive definite Hermitian matrix $G$ such that the matrix $N = \mathfrak{N}(\Omega G \Omega)$ is a rational matrix, where $\mathfrak{N}(Z)$ denotes the imaginary part of the complex matrix $Z$.

**Proof:** If $\Omega$ is a Riemann matrix then $\Omega$ is a nonsingular period matrix by the preceding theorem; thus (F.32) is an identity among invertible matrices, and its inverse readily is seen to be the identity

\[
P^{-1} = i \begin{pmatrix} \Omega \quad 0 \\ \Omega \quad -H^{-1} \end{pmatrix} \begin{pmatrix} \Omega \quad 0 \\ 0 \quad -H^{-1} \end{pmatrix} = i \begin{pmatrix} \Omega H^{-1} - \Omega \bar{H}^{-1} \Omega \end{pmatrix} = 2 \mathfrak{N}(\Omega H^{-1} \Omega),
\]

so $P^{-1} = 2 \mathfrak{N}(\Omega H^{-1} \Omega)$ is a rational matrix where $\mathfrak{N}(H^{-1})$ is positive definite Hermitian. Conversely if $\Omega$ is a nonsingular period matrix and $G$ is a positive definite Hermitian matrix such that $N = \mathfrak{N}(\Omega G \Omega)$ is a rational matrix then the inverse of equation (F.39) is (F.32) in which $H = \Omega^{-1}$ and $P = N^{-1}$; the matrix $H$ is positive definite Hermitian and $P$ is rational, so a suitable positive multiple of $P$ is integral and hence is a principal matrix for $\Omega$, and consequently $\Omega$ is a Riemann matrix. That suffices to conclude the proof.

If $\Omega$ is a Riemann matrix a positive definite Hermitian matrix $G$ such that $N = \mathfrak{N}(\Omega G \Omega)$ is an integral matrix is called a form matrix for the Riemann matrix $\Omega$, and the matrix $N$ is called the associated characteristic matrix. If $G$ is a form matrix for the Riemann matrix $\Omega$ then so is any positive scalar multiple $rG$ for which the associated characteristic matrix is an integral matrix; among these multiples there is a unique one for which the associated characteristic matrix is integral with relatively prime entries, called a primitive form matrix for the Riemann matrix $\Omega$.

**Corollary F.25** (i) If $\Omega$ is a Riemann matrix with principal matrix $P$ and $H = i \Omega P \Omega$ then $G = r^{-1} H^{-1}$ is a form matrix for the Riemann matrix $\Omega$ with characteristic matrix $N = r^{-1} P^{-1}$ for any positive number $r$ such that $N$ is integral.

(ii) If $\Omega$ is a Riemann matrix with form matrix $G$ and associated characteristic matrix $N$ then $P = r^{-1} N^{-1}$ is a principal matrix for the Riemann matrix $\Omega$ for any positive number $r$ such that $P$ is integral, and $i \Omega P \Omega = r^{-1} G^{-1}$.

**Proof:** In the proof of the preceding theorem it was demonstrated that (F.32) is equivalent to the condition that $N = 2 \mathfrak{N}(\Omega, G \Omega)$ where $N = P^{-1}$ and $G = \Omega H^{-1}$, from which the corollary follows. That suffices for the proof.
It follows that a polarization of a Riemann matrix $\Omega$ also can be described as the choice of a form matrix for $\Omega$ up to arbitrary positive scalar multiples, or equivalently the choice of a primitive form matrix for $\Omega$; in view of this a polarized Riemann matrix also can be denoted either by $[\Omega, G]$, where $G$ is a primitive form matrix for the Riemann matrix $\Omega$, or by $[\Omega, \{G\}]$, where $G$ is any positive definite Hermitian matrix a multiple of which is a form matrix for the Riemann matrix $\Omega$. Polarizations $(\Omega, P)$ and $[\Omega, G]$ related as in the preceding corollary are considered as describing the same polarization of a Riemann matrix, so

\[
(F.40) \quad ([\Omega, P]) = [\Omega, \{G\}]
\]

where $^*G^{-1} = i \Omega P \overline{\Omega}$ or equivalently $P^{-1} = 2 \Im(\Omega G \overline{\Omega})$.

In view of this the characteristic matrix $N$ of the form matrix $G$ often is called the characteristic matrix of the polarized Riemann matrix $(\Omega, P)$. This alternative description of a polarized Riemann matrix is more convenient for some purposes in that it exhibits the polarization of a Riemann matrix $\Omega$ as a natural property of the complex torus $J(\Omega)$ determined by the period matrix $\Omega$. Indeed a positive definite $g \times g$ Hermitian matrix $G$ can be viewed as describing a constant, or equivalently a translation-invariant, Hermitian metric\(^5\) of the form $\sum_{j,k=1}^{g} g_{j,k} dw_j d\overline{w}_k$ on the torus $J(\Omega)$, the complex form of a Riemannian metric expressed in terms of the complex coordinates $w_j$ on the complex manifold $J(\Omega)$. Associated to this Hermitian metric is the differential form

\[
\phi = \frac{1}{i} \sum_{j,k=1}^{g} g_{j,k} dw_j \wedge d\overline{w}_k
\]

of type $(1,1)$ on the torus $J(\Omega)$. Since the matrix $G$ is Hermitian it follows that $\overline{\phi} = \phi$, hence that $\phi$ is a real differential form; and since the coefficients of this differential form are constant it is a closed differential form. A Hermitian metric with the property that the associated differential form of type $(1,1)$ is closed is called a Kähler metric.

**Theorem F.26** A nonsingular period matrix $\Omega$ is a Riemann matrix if and only if the complex torus $J(\Omega)$ admits a translation-invariant Kähler metric such that the associated differential form has integral periods on all the two-cycles of the torus; the coefficient matrix of the metric is a form matrix for the Riemann matrix $\Omega$, and the associated characteristic matrix describes the periods of this differential form.

**Proof:** The integral of the differential form $\phi$ associated to the Kähler metric described by a positive definite Hermitian matrix $G$ on the 2-cycle $[\omega^{lm}]$ defined

---

\(^5\)The basic properties of Hermitian and Kähler metrics are discussed in most texts on differential geometry that deal with complex as well as real manifolds; see for instance R. O. Wells, *Differential Geometry on Complex Manifolds*, (Prentice-Hall, 1973).
as on page 406 is

\[ n_{lm} = \int_0^{\omega_{lm}} \phi = \frac{1}{i} \sum_{j,k=1}^g \int_{s=0}^1 \int_{t=0}^1 g_{jk} dw_j(s, t) \wedge dw_k(s, t) \]

\[ = \frac{1}{i} \sum_{j,k=1}^g g_{jk} \int_{s=0}^1 \int_{t=0}^1 (\omega_{jl} ds + \omega_{jm} dt) \wedge (\omega_{kl} ds + \omega_{km} dt) \]

\[ = \frac{1}{i} \sum_{j,k=1}^g g_{jk} (\omega_{jl} \omega_{km} - \omega_{jm} \omega_{kl}) \int_{s=0}^1 \int_{t=0}^1 ds \wedge dt \]

\[ = \frac{1}{i} \sum_{j,k=1}^g g_{jk} (\omega_{jl} \omega_{km} - \omega_{jm} \omega_{kl}). \]

When these periods are viewed as forming a \( 2g \times 2g \) matrix \( N = \{ n_{lm} \} \), the preceding equation can be rewritten as the matrix identity

\[ N = \frac{1}{i} \left( t \Omega \bar{G} \Omega - \bar{t} \bar{G} \Omega \right) = \frac{1}{i} \left( t \Omega \bar{G} \Omega - \bar{t} \bar{G} \Omega \right) = 2 \Im \left( \Omega \bar{G} \Omega \right). \]

Consequently the condition that \( \Omega \) is a Riemann matrix, expressed in terms of the form matrix \( G \), is just that the differential form \( \phi \) associated to the matrix \( G \) has integral periods on the basic cycles of the torus \( J(\Omega) \); and these periods form the characteristic matrix associated to the form matrix \( G \), which concludes the proof.

Since the integrated average over the torus \( J(\Omega) \) of any differentiable Kähler metric is a translation-invariant Kähler metric such that the closed differential form \( (1, 1) \) associated to the averaged metric and that associated to the initial metric have the same periods, the period matrix \( \Omega \) of any complex torus \( J(\Omega) \) that admits a differentiable Kähler metric with integral periods is a Riemann matrix; the coefficient matrix of the averaged Kähler metric is a form matrix for \( \Omega \). A Kähler metric with period integrals is called a Hodge metric. Although the topic will not be pursued further here, at least it should be mentioned that the existence of a Hodge metric on a complex torus is equivalent to the condition that the torus is an algebraic variety; that is traditionally approached through the study of theta functions on the torus.\(^6\) Alternatively and more generally, it was demonstrated by K. Kodaira\(^7\) that a compact complex manifold is an algebraic variety if and only if it admits a Hodge metric.

The notions of equivalence and weak equivalence of polarized Riemann matrices can be expressed alternatively in terms of the form matrix as well.\(^{\text{If }} (\Omega, P) \simeq (\Omega', P') \ trold Zahlentheorie und algebraische Ge-omemy, (Springer, 1956); \( D. \) Mumford, Abelian Varieties, (Oxford, 1970); or A. I. Markushevich, Introduction to the Classical Theory of Abelian Functions, Translations of Mathematical Monographs, vol. 96, (American Mathematical Society, 1992).

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\( \tilde{\Omega}, \tilde{P} \) and if the equivalence is described by matrices \( A = \{ a_{ik} \} \in \text{Gl}(g, \mathbb{C}) \) and \( Q = \{ q_{ik} \} \in \text{Gl}(2g, \mathbb{Z}) \) as in (F.33) then the positive definite matrices \( H = i \Omega P \tilde{\Pi} \) and \( \tilde{H} = i A \Omega Q^{-1} P Q^{-1} \cdot Q \tilde{\Pi} = A H \tilde{A} \). The form matrix describing the same polarization of the Riemann matrix \( \Omega \) is \( G = r^{-1} H^{-1} \) as in Corollary F.25, where \( r \) is the unique positive rational number such that the matrix \( N = r^{-1} P^{-1} \) is an integral matrix with relatively prime entries; and since \( \tilde{P}^{-1} = Q^{-1} P^{-1} Q \) where \( Q \in \text{Gl}(2g, \mathbb{Z}) \) it follows that \( \tilde{N} = r^{-1} \tilde{P}^{-1} \) is also an integral matrix with relatively prime entries, so that \( \tilde{G} = r^{-1} \tilde{H}^{-1} \) is the form matrix describing the same polarization of the Riemann matrix \( \tilde{\Omega} \). These two form matrices thus are related by \( \tilde{G} = tA^{-1} G A^{-1} \), and the associated characteristic matrices \( N = r^{-1} P^{-1} \) and \( \tilde{N} = r^{-1} \tilde{P}^{-1} \) are related by \( \tilde{N} = tQ N Q \). In summary then, the equivalence of polarized Riemann matrices described in terms of form matrices is defined by

\[
(F.41) \quad [\Omega, G] \simeq [A \Omega Q, tA^{-1} G A^{-1}] \quad \text{or} \quad [\Omega, \{ G \}] \simeq [A \Omega Q, \{ tA^{-1} G A^{-1} \}]
\]

whenever \( A \in \text{Gl}(g, \mathbb{C}) \) and \( Q \in \text{Gl}(2g, \mathbb{Z}) \),

and the associated characteristic matrices are related by

\[
(F.42) \quad \tilde{N} = tQ N Q.
\]

For weak equivalence the matrix \( Q \) is only a nonsingular rational matrix and the form and characteristic matrices must be multiplied by that positive rational number for which the characteristic matrix is integral with relatively prime entries. This complication can be avoided by considering only the alternative notation for polarized Riemann matrices in terms of families of form matrices, so that the weak equivalence of polarized Riemann matrices described in terms of families of form matrices is defined by

\[
(F.43) \quad [\Omega, \{ G \}] \sim [A \Omega Q, \{ tA^{-1} G A^{-1} \}]
\]

whenever \( A \in \text{Gl}(g, \mathbb{C}) \), and \( Q \in \text{Gl}(2g, \mathbb{Q}) \).

The equivalence of polarized Riemann matrices has an interesting interpretation in terms of the associated complex tori. If \((\tilde{\Omega}, \tilde{P}) = (A \Omega Q, Q^{-1} P Q^{-1})\) are equivalent polarized Riemann matrices with form matrices \( G \) and \( \tilde{G} = tA^{-1} G A^{-1} \) then \( A \Omega = \tilde{\Omega} Q^{-1} \) so \((A, Q^{-1})\) is a Hurwitz relation from the period matrix \( \Omega \) to the period matrix \( \tilde{\Omega} \); and since \( Q \in \text{Gl}(2g, \mathbb{Z}) \) the linear mapping \( A : \mathbb{C}^g \rightarrow \mathbb{C}^g \) induces a biholomorphic mapping \( A : J(\Omega) \rightarrow J(\tilde{\Omega}) \) between the complex tori described by these period matrices, as in Theorem F.9. This biholomorphic mapping, viewed as a nonsingular linear change of coordinates \( \tilde{w}_j = \sum_{k=1}^g a_{jk} w_k \), transforms the translation-invariant Hermitian
APPENDIX F. COMPLEX TORI

metric \[ \sum_{j=1}^{g} \tilde{g}_{j} d\tilde{w}_{j} d\tilde{w}_{k} \] on the torus \( J(\hat{\Omega}) \) described by the form matrix \( \hat{G} \) to the metric

\[(F.44) \quad \sum_{jk=1}^{g} \tilde{g}_{j} d\tilde{w}_{j} d\tilde{w}_{k} = \sum_{jklm=1}^{g} \tilde{g}_{j} a_{jl} \overline{a}_{km} d\tilde{w}_{j} d\tilde{w}_{m} = \sum_{lm=1}^{g} g_{lm} d\tilde{w}_{l} d\tilde{w}_{m},\]

the translation-invariant Hermitian metric on the torus \( J(\Omega) \) described by the form matrix \( G = \hat{A}G\hat{A} \). Thus the equivalence of polarized Riemann matrices amounts to the existence of a biholomorphic mapping between the complex tori described by these period matrices that transforms the translation-invariant metrics describing the polarizations into one another. To phrase this in another way, a polarized complex torus is a complex torus together with a family of translation-invariant Hermitian metrics, all of which are scalar multiples of one another and some of which have integral periods; and in these terms the polarized Riemann matrices in an equivalence class describe the same polarized complex torus, with the various polarized Riemann matrices in the equivalence class merely being descriptions of the same polarized torus in terms of other linear coordinate systems on the torus. This provides an intrinsic interpretation of a polarized Riemann matrix.

If the polarized Riemann matrices \( (\Omega, P) \) and \( (\hat{\Omega}, \hat{P}) = (A\Omega Q, Q^{-1}P'(Q^{-1}) \) are just weakly equivalent the linear mapping \( A : \mathbb{C}^{g} \rightarrow \mathbb{C}^{g} \) is nonsingular and transforms the translation-invariant Hermitian metric on the torus \( J(\hat{\Omega}) \) described by the form matrix \( \hat{G} \) locally to the translation-invariant Hermitian metric on the torus \( J(\Omega) \) described by the form matrix \( G \) as in equation (F.44); but the induced mapping on complex tori is not well defined globally. However for any positive rational number \( r \) for which \( r Q^{-1} \) is an integral matrix the pair \( (r A, r Q^{-1}) \) is a Hurwitz relation from the period matrix \( \Omega \) to the period matrix \( \hat{\Omega} \), so the linear mapping \( r A : \mathbb{C}^{g} \rightarrow \mathbb{C}^{g} \) defines an isogeny \( r A : J(\Omega) \rightarrow J(\hat{\Omega}) \) and this isogeny transforms the family of translation-invariant Hermitian metrics \( \{\hat{G}\} \) defining the polarization of the torus \( J(\hat{\Omega}) \) to the family of translation-invariant Hermitian metrics \( \{G\} \) defining the polarization of the torus \( J(\Omega) \); that the metrics are transformed locally to one another follows from (F.44), and since the metrics are constant the same transformation arises at all points of \( J(\Omega) \) that have the same image in \( J(\hat{\Omega}) \) under this isogeny. Two polarized complex tori are polarized-isogenous if there is an isogeny between the tori that transforms the families of translation-invariant Hermitian metrics defining the polarizations to one another; and in these terms all the polarized Riemann matrices in a weak equivalence class describe polarized-isogenous polarized complex tori, with the various polarized Riemann matrices in the weak equivalence class merely being descriptions of the polarizations of isogenous tori in various linear coordinate systems.
Appendix G

Theta Series

The classical theta function in one variable plays a significant role in the study of elliptic functions, which are just meromorphic functions on compact Riemann surfaces of genus $g = 1$; the theta function in several variables plays an equally significant role in the study of meromorphic functions on complex tori of higher dimensions. Although a detailed discussion of function theory on complex tori in higher dimensions would lead far too far afield\(^1\), at least some of the basic properties of theta functions in several variables required in the discussion in this book will be reviewed here. The classical theta function is defined by the series

\[
\theta(t; z) = \sum_{n \in \mathbb{Z}} \exp \left( \frac{1}{2} \pi i \left( z n^2 + tn \right) \right),
\]

where $t \in \mathbb{C}$ and $z = x + iy \in \mathbb{C}$ with $y > 0$. It is no doubt quite familiar that this is a nontrivial entire function of the complex variable $t$ and a holomorphic function of the complex variable $z$ in the upper half-plane. The theta function in $g$ variables is defined by the analogous series

\[
\Theta(t; Z) = \sum_{n \in \mathbb{Z}^g} \exp \left( \frac{1}{2} \pi i \left( Y n^2 + t n \right) \right)
\]

for a complex vector $t \in \mathbb{C}^g$ and a $g \times g$ complex matrix $Z \in \mathcal{H}_g$, where $\mathcal{H}_g$ is the Siegel upper half-space of rank $g$ consisting of complex symmetric $g \times g$ matrices $Z = X + iY$ with positive definite imaginary part $Y$, as discussed on page 413. Since $Y$ is positive definite, $b_n Y n \geq \epsilon \|n\|^2$ for some $\epsilon > 0$ and all $n \in \mathbb{Z}^g$, so

\[
|b_n(Y n - 2 i t)| \geq \epsilon \|n\|^2 - 2 \|t\| \geq \epsilon \|n\|^2 - 2 \|n\| \cdot \|t\| \geq \frac{\epsilon}{2} \|n\|^2
\]

whenever \( ||n|| \geq \frac{4}{\epsilon} ||t|| \), where as usual \( ||t||^2 = \sum_i |t_i|^2 \); consequently

\[
| \exp 2\pi i \ h(n + t) | = | \exp 2\pi i \ h(Yn + t) |
\]

\[
= | \exp -\pi h(Yn - 2it) | \leq \exp -\frac{1}{2} \pi \epsilon ||n||^2
\]

whenever \( ||n|| \geq \frac{\epsilon}{2} ||t|| \), so the series (G.2) is locally uniformly convergent for \((t, Z) \in \mathbb{C}^g \times \mathcal{S}_g\) and hence it represents a holomorphic function on the product manifold \( \mathbb{C}^g \times \mathcal{S}_g \). For any fixed point \( Z \in \mathcal{S}_g \) this function is nontrivial in the variable \( t \in \mathbb{C}^g \), since (G.2) is a Fourier series with nonzero coefficients.

The parameter of summation \( n \in \mathbb{Z}^g \) in the series (G.2) can be replaced by \( -n \), so that

\[
(G.3) \quad \Theta(t; Z) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} hZn - 'nt \right) = \Theta(-t; Z).
\]

Thus the theta function is an even function of the variable \( t \in \mathbb{C}^g \); and it can be described by either of the series expansions (G.2) or (G.3), so the sign of the term \( 'nt \) in the series expansion can be chosen arbitrarily as convenient. It is clear from (G.2) or (G.3) that

\[
(G.4) \quad \Theta(t + \mu; Z) = \Xi_Z(\mu, t) \cdot \Theta(t; Z) \quad \text{for} \quad \mu \in \mathbb{Z}^g,
\]

since \( \exp 2\pi i \ h\mu = 1 \). On the other for any \( \nu \in \mathbb{Z}^g \) the parameter of summation \( n \in \mathbb{Z}^g \) in (G.2) can be replaced by \( n + \nu \), and since \( Z \) is a symmetric matrix it follows that

\[
\Theta(t; Z) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( n + \nu \right) \left( \frac{1}{2} Z(n + \nu + t) \right)
\]

\[
= \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( h \left( \frac{1}{2} Zn + Z\nu + t \right) + ' \nu \left( \frac{1}{2} Z\nu + t \right) \right)
\]

\[
= \Theta(t + Z\nu; Z) \cdot \exp 2\pi i \left( \frac{1}{2} \nu Z\nu + ' \nu t \right);
\]

consequently

\[
(G.5) \quad \Theta(t + Z\nu; Z) = \Theta(t; Z) \cdot \exp -2\pi i \left( \frac{1}{2} \nu Z\nu + ' \nu t \right) \quad \text{for} \quad \nu \in \mathbb{Z}^g.
\]

Equations (G.4) and (G.5) determine the behaviour of the function \( \Theta(t; Z) \) when the variable \( t \in \mathbb{C}^g \) is translated by any vector \( \lambda \) in the lattice subgroup \( \mathcal{L}(\Omega) = \Omega \mathbb{Z}^{2g} \subset \mathbb{C}^g \) spanned by the columns of the \( g \times 2g \) period matrix \( \Omega = [I \ Z] \). These equations taken together can be written as the condition that

\[
(G.6) \quad \Theta(t + \lambda; Z) = \Xi_Z(\lambda, t) \cdot \Theta(t; Z) \quad \text{for all} \quad \lambda = \mu + Z\nu \in \mathcal{L}(\Omega)
\]

where

\[
(G.7) \quad \Xi_Z(\mu + Z\nu, t) = \exp -2\pi i \left( \frac{1}{2} \nu Z\nu + ' \nu t \right) \quad \text{for all} \quad \mu, \nu \in \mathbb{Z}^g.
\]

That is just the condition that the theta function \( \Theta(t; Z) \) associated to a point \( Z \in \mathcal{S}_g \), viewed as a function of the complex variable \( t \in \mathbb{C}^g \), is a holomorphic
relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda, t)$ for the action of the lattice subgroup $\mathcal{L}(\Omega)$ on the space $\mathbb{C}^g$; this factor of automorphy consequently is called the \textit{theta factor of automorphy}. Of course in view of (G.3) the function $\Theta(t; Z)$ can be viewed as a relatively automorphic function for the larger group that arises by adjoining to the lattice group $\mathcal{L}(\Omega)$ the additional mapping $\iota : t \rightarrow -t$, and extending the factor of automorphy (G.7) to the larger group by setting $\Xi_Z(\iota, t) = 1$; however it is usually more convenient to view the theta function as a symmetric relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda; t)$ for the lattice subgroup $\mathcal{L}(\Omega)$ itself. Slightly more generally, it is also possible to replace the parameter of summation $n \in \mathbb{Z}^g$ in the series (G.2) by $Qn$ for any matrix $Q \in \text{Gl}(g, \mathbb{Z})$; it then follows from a straightforward calculation that

\begin{equation}
\Theta(t; Z) = \Theta('Qt; 'QZQ) \quad \text{for any} \quad Q \in \text{Gl}(g, \mathbb{Z}),
\end{equation}

hence $\Theta('Qt; Z) = \Theta(t; Z)$ for all matrices $Q$ in the subgroup

\begin{equation}
\mathcal{F}(Z) = \left\{ Q \in \text{Gl}(g, \mathbb{Z}) \mid 'QZQ = Z \right\},
\end{equation}

which for some matrices $Z \in \mathcal{H}_g$ can be properly larger than the subgroup consisting merely of $\pm I$. The theta factor of automorphy $\Xi_Z(\lambda, t)$ describes a holomorphic line bundle $\Xi_Z$ over the complex torus $J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega)$, called the \textit{theta bundle} over $J(\Omega)$; and the theta function $\Theta(t; Z)$ describes a holomorphic cross-section of that line bundle. The vector space of holomorphic cross-sections of this line bundle is denoted by $\Gamma(J(\Omega), \mathcal{O}(\Xi_Z))$, paralleling the notation for the space of holomorphic cross-sections of a holomorphic line bundle over a compact Riemann surface; and correspondingly the dimension of this vector space is denoted by $\gamma(\Xi_Z) = \text{dim} \Gamma(J(\Omega), \mathcal{O}(\Xi_Z))$.

\textbf{Theorem G.1} \textit{The space of holomorphic cross-sections of the line bundle $\Xi_Z$ over the complex torus $J(\Omega)$ associated to the period matrix $\Omega = (I \quad Z)$ for any matrix $Z \in \mathcal{H}_g$ is one-dimensional, that is, $\gamma(\Xi_Z) = 1$.}

\textbf{Proof:} The theta function $\Theta(t; Z)$ for a matrix $Z \in \mathcal{H}_g$ describes a nontrivial holomorphic cross-section of the bundle $\Xi_Z$ so $\gamma(\Xi_Z) \geq 1$. On the other hand if $f(t) \in \Gamma(J(\Omega), \mathcal{O}(\Xi_Z))$ then $f(t)$ satisfies (G.4) so that $f(t + \mu) = f(t)$ for all $\mu \in \mathbb{Z}^g$; hence $f(t)$ must have a holomorphic Fourier expansion

\begin{equation}
f(t) = \sum_{n \in \mathbb{Z}^g} a_n \exp 2\pi i n \cdot t
\end{equation}

for some constants $a_n$. Since $f(t)$ also satisfies (G.5), that result for the special
case that \( \nu = \delta_j \), where \( \delta_j \) are the basis vectors in \( \mathbb{R}^g \), shows that

\[
f(t + Z\delta_j) = f(t) \cdot \exp(-2\pi i (t_j + \frac{1}{2}z_{jj}))
= \left( \sum_{n \in \mathbb{Z}^g} a_n \exp(2\pi i \cdot t) \right) \cdot \exp(-2\pi i (t_j + \frac{1}{2}z_{jj}))
= \sum_{n \in \mathbb{Z}^g} a_n \exp(2\pi i \cdot (n - \delta_j) \cdot t - \frac{1}{2}z_{jj})
= \sum_{n \in \mathbb{Z}^g} a_{n+\delta_j} \exp(2\pi i \cdot (n \cdot t - \frac{1}{2}z_{jj})),
\]

where the last equality follows by replacing the parameter of summation \( n \) in the preceding line by \( n + \delta_j \). However, a direct substitution in (G.10) shows that

\[
f(t + Z\delta_j) = \sum_{n \in \mathbb{Z}^g} a_n \exp(2\pi i \cdot (t + Z\delta_j)).
\]

Comparing the Fourier coefficients in the two preceding expansions of \( f(t + Z\delta_j) \) shows that

\[
a_{n+\delta_j} \cdot \exp(-\pi i z_{jj}) = a_n \exp(2\pi i \cdot nZ\delta_j) \quad \text{for all } n;
\]

and since this recurrence relation determines all the coefficients \( a_n \) in terms of \( a_0 \) it follows that the space of holomorphic cross-sections of the bundle \( \Xi_Z \) has dimension at most one, which suffices to conclude the proof.

It follows from the preceding result that the theta function in several variables viewed as a holomorphic function of the variable \( t \in \mathbb{C}^g \) can be described uniquely up to a constant factor as a holomorphic relatively automorphic function for the factor of automorphy \( \Xi_Z \) for the action of the lattice subgroup \( \mathcal{L}(\Omega) \) on the vector space \( \mathbb{C}^g \), where \( \Omega = (I \ Z) \). The explicit formulas for the theta function and its factor of automorphy are for period matrices that are principally polarized Riemann matrices in the normal form \( \Omega = (I \ Z) \) where \( Z \in \mathcal{S}_g \). If \( \Lambda \) is a nonsingular period matrix describing a complex torus \( J(\Lambda) \) of dimension \( h \) for which there is a holomorphic mapping

\[
(G.11) \quad \phi : J(\Lambda) \rightarrow J(\Omega),
\]

the theta function associated to the torus \( J(\Omega) \) induces a holomorphic function associated to the torus \( J(\Lambda) \) that can be viewed as a generalized theta function. Explicitly, if the mapping \( \phi \) is described by a Hurwitz relation \((A, N)\) from the period matrix \( \Lambda \) to the period matrix \( \Omega \) as in Theorem F.9, for a complex matrix \( A \in \mathbb{C}^{g \times h} \) and an integral matrix \( N \in \mathbb{Z}^{2g \times 2h} \), then \( A \cdot \Lambda = \Omega \cdot N \) and the holomorphic function \( f(t) = \Theta(At; Z) \) of the variable \( t \in \mathbb{C}^h \) satisfies

\[
f(t + \Lambda \nu) = \Theta(At + A\Lambda \nu; Z) = \Theta(At + \Omega \cdot N \nu; Z)
= \Xi_Z(\Omega \cdot N \nu, At) \cdot \Theta(At; z) = \Xi_Z(\Omega \nu, At) \cdot f(t)
\]

in the space of \( \Xi_Z \) functions.
for $\nu \in \mathbb{Z}^{2h}$. If $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ where $N_1, N_2 \in \mathbb{Z}^{g \times 2h}$, so that $\Omega \cdot N \nu = N_1 \nu + Z N_2 \nu$, the factor of automorphy $\Xi_Z(\Omega \cdot N \nu, At)$ can be written out explicitly as in (G.7) and the preceding equation takes the form

$$(G.12) \quad f(t + \Lambda \nu) = f(t) \exp -2 \pi i \left( \frac{1}{2} \nu \cdot N_2 Z N_2 \nu + \nu \cdot N_2 A t \right) \quad \text{for } \nu \in \mathbb{Z}^{2h}. $$

Of course the function $f(t)$ may vanish identically if the image of the mapping (G.11) is contained in the zero locus of the theta function $\Theta(t; Z)$. On the other hand if the mapping (G.11) is a biholomorphic mapping then the function $f(t)$ is a nontrivial relatively automorphic function for the induced factor of automorphy, and is determined uniquely up to a constant factor by (G.12).

The zero locus of the theta function $\Theta(t)$ is a proper holomorphic subvariety $\bar{V}_\Theta \subset \mathcal{C}^g$ of dimension $g - 1$ that is invariant under translation by any vector in the lattice subgroup $L(\Omega) \subset \mathcal{C}^g$ spanned by the columns of the period matrix $\Omega$, as a consequence of (G.6). This subvariety thus represents a holomorphic subvariety $V_\Theta = \bar{V}_\Theta / L(\Omega)$ of dimension $g - 1$ in the quotient torus $J(\Omega) = \mathcal{C}^g / L(\Omega)$, sometimes called the theta locus in $J(\Omega)$. Since the theta function is an even function by (G.3) it follows that

$$(G.13) \quad V_\Theta = -V_\Theta \subset J(\Omega); $$

more generally $V_\Theta = tQ V_\Theta$ for any matrix $Q \in F(Z)$ where $F(Z) \subset \text{GL}(g, \mathbb{Z})$ is the subgroup defined by (G.9), as an immediate consequence of (G.8). A particularly interesting finite set of points on the complex torus $J(\Omega)$, actually a finite subgroup of that complex torus, is the set of half-periods, the set of $2^{2g}$ points $\delta_i \in J(\Omega)$ such that $2 \delta_i \in J(M)$ is the point of the torus represented by the origin in $\mathbb{C}^g$; alternatively this is the subset $\frac{1}{2} \Omega \mathbb{Z}^{2g} / \Omega \mathbb{Z}^{2g} \subset J(\Omega)$.

The subset of real half-periods for the period matrix $\Omega = (I \ Z)$ is the subset $\frac{1}{2} \mathbb{Z}^g / \Omega \mathbb{Z}^{2g} \subset J(\Omega)$ consisting of $2^g$ of the half-periods. Frequently points of $\mathbb{C}^{2g}$ representing the half-periods of the torus $\mathbb{C}^{2g} / L(\Omega)$ also are called half-periods; that is essentially an equivalent definition, so the identification of the terms should not cause any confusion.

For any integer $r > 0$ it follows from (G.6) that the function $f(t) = \Theta(t; Z)^r$ is a holomorphic relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda, t)^r$ in terms of the period matrix $\Omega = (I \ Z)$, the $r$-the power of the theta factor of automorphy for that period matrix, so that for any $\mu, \nu \in \mathbb{Z}^g$

$$(G.14) \quad f(t + \mu + Z \nu; Z) = f(t) \cdot \exp -2 \pi i r \left( \frac{1}{2} \nu Z \nu + \nu t \right). $$

The holomorphic relatively automorphic functions for this factor of automorphy are called theta functions of order $r$ for the period matrix $\Omega = (I \ Z)$, and describe holomorphic cross-sections of the $r$-th power $\Xi_Z^r$ of the theta bundle over $M$; as before, the dimension of the vector space of theta functions of order $r$ is denoted by $\gamma(\Xi_Z^r)$.

**Theorem G.2** The vector space of theta functions of order $r > 0$ for the period matrix $\Omega = (I \ Z)$ has dimension $r^g$, that is, $\gamma(\Xi_Z^r) = r^g$. 
Proof: The holomorphic relatively automorphic functions for the theta factor of automorphy $Ξ_Z(\mu + Z\nu)^r$ are holomorphic functions $f(t)$ of the variables $t \in \mathbb{C}^g$ satisfying (G.14). As in the proof of Theorem G.1, for $\nu = 0$ it follows that the function $f(t)$ is invariant under translation through integer vectors, so it has a complex Fourier expansion

\[(G.15) \quad f(t) = \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \langle n, t \rangle\]

for some $c_n \in \mathbb{C}$. Then for $\mu = 0$ it follows from (G.14) that

\[(G.16) \quad f(t + Z\nu) = f(t) \cdot \exp -2\pi ir (\frac{1}{2} \langle \nu, Z\nu \rangle + \langle t, \nu \rangle)\]

\[= \left( \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \langle n, t \rangle \right) \cdot \exp -2\pi ir (\frac{1}{2} \langle \nu, Z\nu \rangle + \langle t, \nu \rangle)\]

\[= \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i (\langle n-r\nu, t \rangle - \frac{1}{2} \langle \nu, Z\nu \rangle)\]

\[= \sum_{n \in \mathbb{Z}^g} c_{n+r\nu} \exp 2\pi i (\langle n, t \rangle - \frac{1}{2} \langle \nu, Z\nu \rangle)\]

where the last equality follows by replacing the parameter of summation $n$ in the preceding line by $n + r\nu$; on the other hand replacing $t$ in (G.15) by $t + Z\nu$ shows that

\[(G.17) \quad f(t + Z\nu) = \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \langle n, (t + Z\nu) \rangle.\]

Upon comparing the Fourier coefficients in the Fourier expansions (G.16) and (G.17) it follows that

\[(G.18) \quad c_{n+r\nu} = c_n \exp 2\pi i (\langle \nu, Z\nu \rangle + \frac{r}{2} \langle \nu, Z\nu \rangle) \quad \text{for all } n \in \mathbb{Z}^g.\]

Since every vector $n \in \mathbb{Z}^g$ can be written uniquely as $n = \delta + r\nu$ for some $\delta = (\delta_1, \ldots, \delta_g) \in \mathbb{Z}^g$ for which $0 \leq \delta_i < r$ and some $\nu \in \mathbb{Z}^g$ it follows that all the coefficients $c_n$ of the Fourier expansion (G.15) are determined by the $r^g$ coefficients $c_\delta$ through the recurrence relation (G.18), where these coefficients $c_\delta$ can be chosen arbitrarily; hence $\gamma(\mathbb{Z}_Z^g) = r^g$, which suffices for the proof.

The special case of second order theta functions is particularly interesting. The preceding theorem shows that the space of second-order theta functions has dimension $2^g$. For notational convenience identify $\mathbb{Z}_{2^g}^g$ with the set of $g$-tuples of integers $\delta = (\delta_1, \ldots, \delta_g)$ where $\delta_i = 0$ or 1, or equivalently, with the set of real half-periods for the period matrix $\Omega$; in these terms, every vector $n \in \mathbb{Z}^g$ can be written uniquely as $n = \delta + r\nu$ for some $\delta \in \mathbb{Z}_{2^g}^g$ and some $\nu \in \mathbb{Z}^g$. Then by using the recurrence relation (G.18) for the Fourier coefficients of second-order theta functions and the symmetry $\langle Z, Z \rangle = Z$, any second-order theta function $f(t)$
can be written

\[(G.19)\]

\[f(t) = \sum_{\delta \in \mathbb{Z}_g^\ast} \sum_{\nu \in \mathbb{Z}_g^\ast} c_{\delta + 2\nu} \exp 2\pi i (\delta + 2\nu)t \]

\[= \sum_{\delta \in \mathbb{Z}_g^\ast} \sum_{\nu \in \mathbb{Z}_g^\ast} c_{\delta} \exp 2\pi i (4\delta \nu + 2\nu \delta Z + 4(\delta + 2\nu)t) \]

\[= \sum_{\delta \in \mathbb{Z}_g^\ast} c_{\delta} \exp 2\pi i 4\delta t \cdot \sum_{\nu \in \mathbb{Z}_g^\ast} \exp 2\pi i (4\nu \delta Z + 4(2t + Z\delta)) \]

\[= \sum_{\delta \in \mathbb{Z}_g^\ast} c_{\delta} \exp 2\pi i 4\delta t \cdot \Theta(2t + Z\delta; 2Z) \]

in view of (G.2), or equivalently

\[(G.20)\]

\[f(t) = \sum_{\delta \in \mathbb{Z}_g^\ast} c_{\delta} \Theta_{2,\delta}(t; Z) \]

where

\[(G.21)\]

\[\Theta_{2,\delta}(t; Z) = \exp 2\pi i 4\delta t \cdot \Theta(2t + Z\delta; 2Z). \]

It follows from (G.6) and (G.7) that for any \(\mu, \nu \in \mathbb{Z}_g^\ast\)

\[\Theta_{2,\delta}(t + \mu + Z\nu; Z) = \exp 2\pi i 4\delta(t + \mu + Z\nu) \cdot \Theta(2t + 2\mu + 2Z\nu + Z\delta; 2Z) \]

\[= \exp 2\pi i 4\delta t + \mu + Z\nu \cdot \Theta(2t + Z\delta; 2Z) \]

\[\cdot \exp -2\pi i (4\nu \delta Z + 4(2t + Z\delta)) \cdot \Theta(2t + Z\delta; 2Z) \]

\[= \exp 2\pi i 4\delta t \cdot \exp -2\pi i (4\nu \delta Z + 2(2t + Z\delta)) \cdot \Theta(2t + Z\delta; 2Z) \]

\[= \exp -2\pi i (4\nu \delta Z + 2(2t + Z\delta)) \Theta_{2,\delta}(t; Z), \]

so in view of (G.14) each of the functions \(\Theta_{2,\delta}(t; Z)\) is a second-order theta function and (G.20) is an expansion of an arbitrary second-order theta function as a linear combination of the 2g functions \(\Theta_{\delta}(t; Z)\), which consequently are a basis for the vector space of second-order theta functions. The Fourier expansions of these basic second-order theta functions also can be read directly from (G.19), since it is clear from the second line of that formula that

\[(G.22)\]

\[\Theta_{2,\delta}(t; Z) = \sum_{\nu \in \mathbb{Z}_g^\ast} \exp 2\pi i 4\nu Z(\nu + \delta) \cdot \exp 2\pi i 4(\delta + 2\nu)t. \]

Thus the basic second-order theta function \(\Theta_{2,\delta}(t; Z)\) has nonzero Fourier coefficients only for indices \(\delta + 2\nu \equiv \delta (\text{mod} 2)\), and all these Fourier coefficients are nonzero; in particular for \(\nu = 0\) the Fourier expansion includes the term \(1 \cdot \exp 2\pi i 4\delta t\), so the functions \(\Theta_{2,\delta}(t; Z)\) are in a natural sense the standard basis for the set of second-order theta functions.
For any integral vectors $\delta, \mu, \nu \in \mathbb{Z}^g$ it follows from (G.6) and (G.7) that

$$\Theta(t + \frac{1}{2} \delta + \mu + Z\nu; Z) = \exp -2\pi i \left(\frac{1}{2} \nu Z \nu + \nu (t + \frac{1}{2} \delta)\right) \cdot \Theta(t + \frac{1}{2} \delta; Z)$$

$$= (-1)^{\nu \cdot \delta} \exp -2\pi i \left(\frac{1}{2} \nu Z \nu + \nu t\right) \cdot \Theta(t + \frac{1}{2} \delta; Z)$$

and consequently that

$$(G.23) \quad \Theta(t + \frac{1}{2} \delta + \mu + Z\nu; Z)^2 = \exp -4\pi i \left(\frac{1}{2} \nu Z \nu + \nu t\right) \cdot \Theta(t + \frac{1}{2} \delta; Z)^2;$$

so in view of (G.14) the squares $\Theta(t + \frac{1}{2} \delta; Z)^2$ for all parameter values $\delta \in \mathbb{Z}_2^g$ are $2^g$ second-order theta functions.

**Theorem G.3** The second-order theta functions functions $\Theta(t + \frac{1}{2} \delta; Z)^2$ for real half-periods $\delta \in \mathbb{Z}_2^g$ can be written in terms of the basic second-order theta functions $\Theta_{2,\delta}(t; Z)$ as

$$(G.24) \quad \Theta(t + \frac{1}{2} \delta; Z)^2 = \sum_{\epsilon \in \mathbb{Z}_2^g} (-1)^{\epsilon \cdot \delta} e^{\pi i \epsilon Z} \Theta(Z\epsilon; 2Z) \cdot \Theta_{2,\epsilon}(t; Z);$$

consequently $\Theta(Z\epsilon; 2Z) \neq 0$ for at least some $\epsilon \in \mathbb{Z}_2^g$, and the dimension of the space of second-order theta functions spanned by the squares $\Theta(t + \frac{1}{2} \delta; Z)^2$ is equal to the number of real half-periods $\epsilon \in \mathbb{Z}_2^g$ such that $\Theta(Z\epsilon; 2Z) \neq 0$.

**Proof:** By the definition (G.2) of the theta function

$$\Theta(t + \frac{1}{2} \delta; Z) = \sum_{n \in \mathbb{Z}^g} (-1)^{h \cdot \delta} \exp 2\pi i \left(\frac{1}{2} \nu Z n + \nu t\right)$$

so

$$\Theta(t + \frac{1}{2} \delta; Z)^2 = \sum_{m,n \in \mathbb{Z}^g} (-1)^{(m+n) \cdot \delta} \exp 2\pi i \left(\frac{1}{2} \nu Z n + \nu t\right)$$

and setting $m = \nu - n$ this can be rewritten

$$\Theta(t + \frac{1}{2} \delta; Z)^2 = \sum_{\nu, n \in \mathbb{Z}^g} (-1)^{\nu \cdot \delta} \exp 2\pi i \left(\frac{1}{2} \nu Z n + \nu t\right) \cdot \sum_{n \in \mathbb{Z}^g} \exp 2\pi i (\nu Z n - \nu Z n)$$

$$= \sum_{\nu \in \mathbb{Z}^g} (-1)^{\nu \cdot \delta} \exp 2\pi i \left(\frac{1}{2} \nu Z n + \nu t\right) \cdot \Theta(Z\nu; 2Z);$$

thus the Fourier series expansion of $\Theta(t + \frac{1}{2} \delta; Z)^2$ is

$$(G.25) \quad \Theta(t + \frac{1}{2} \delta; Z)^2 = \sum_{\nu \in \mathbb{Z}^g} (-1)^{\nu \cdot \delta} a_\nu e^{\pi i \nu t}$$

where

$$(G.26) \quad a_\nu = \exp \pi i (\nu Z \nu) \cdot \Theta(Z\nu; 2Z).$$
This function can be written as a linear combination
\[ \Theta(t + \frac{1}{2} \delta; Z)^2 = \sum_{\epsilon \in \mathbb{Z}_2^g} b_\epsilon \Theta_{2,\epsilon}(t; Z) \]
of the basis \( \Theta_{2,\epsilon}(t; Z) \) for the space of second-order theta functions, and comparing the Fourier coefficients of \( \exp 2\pi i t \) for these second-order theta functions shows that
\[ b_\epsilon = (-1)^\epsilon \delta a_\epsilon, \]
which yields (G.24). The final conclusion of the theorem is an immediate consequence, since the squares \( \Theta(t + \frac{1}{2} \delta; Z)^2 \) are nontrivial second-order theta functions, and that suffices for the proof.

**Corollary G.4** The translates \( \Theta(t + \frac{1}{2} \delta; Z) \) of the theta function satisfy the quadratic equations
\[ \sum_{\delta \in \mathbb{Z}_2^g} (-1)^\gamma \Theta(t + \frac{1}{2} \delta; Z)^2 = 0 \]
for all \( \gamma \in \mathbb{Z}_2^g \) such that \( \Theta(Z\gamma; 2Z) = 0 \).

**Proof:** It is clear that for any \( \epsilon, \gamma \in \mathbb{Z}_2^g \)
\[ \sum_{\delta \in \mathbb{Z}_2^g} (-1)^{(\epsilon-\gamma) \cdot \delta} \Theta(2 + \frac{1}{2} \delta; Z)^2 = \begin{cases} 2g & \text{if } \gamma = \epsilon, \\ 0 & \text{if } \gamma \neq \epsilon \end{cases} \]
From this and (G.24) it follows that
\[ \sum_{\delta \in \mathbb{Z}_2^g} (-1)^{(\epsilon-\gamma) \cdot \delta} \Theta(2 + \frac{1}{2} \delta; Z)^2 = \sum_{\delta, \epsilon \in \mathbb{Z}_2^g} (-1)^{(\epsilon-\gamma) \cdot \delta} e^{\pi i \epsilon Z \epsilon} \Theta(Z\epsilon; 2Z) \cdot \Theta_{2,\epsilon}(t; Z) \]
\[ = 2g e^{\pi i \gamma Z \epsilon} \Theta(Z\gamma; 2Z) \cdot \Theta_{2,\epsilon}(t; Z). \]
In particular then
\[ \sum_{\delta \in \mathbb{Z}_2^g} (-1)^{(\epsilon-\gamma) \cdot \delta} \Theta(2 + \frac{1}{2} \delta; Z)^2 = 0 \]
whenever \( \gamma \in \mathbb{Z}_2^g \) is a real half-period for which \( \Theta(Z\gamma; 2Z) = 0 \), and that suffices for the proof.

More generally the complex tori \( J(\Omega) \) are homogeneous spaces under arbitrary translations, and these translations have a fairly simple effect on theta functions. Indeed if \( \Theta(t; s, Z) = \Theta(s + t; Z) \) is the translate of the theta function \( \Theta(t; z) \) through \( s \) it follows from (G.4), (G.5) and (G.7) that
\[ \Theta(t + \mu + Z\nu; s, Z) = \Theta(s + t + \mu + Z\nu; Z) \]
\[ = \Theta(s + t; Z) \cdot \exp -2\pi i \left( \frac{1}{2} Z \nu (\mu + Z\nu) + (t + s) \right) \]
\[ = \exp -2\pi i \nu s \cdot Z Z (\mu + Z\nu, t) \cdot \Theta(t; s, Z); \]
thus $\Theta(t; s, Z)$ is a relatively automorphic function for the product of the flat factor of automorphy

$$\sigma_s(\mu + Z\nu) = \exp -2\pi i' \nu s$$

and the theta factor of automorphy $\Xi_Z(\mu + Z\nu, t)$. It is clear from its definition that the flat factor of automorphy $\sigma_s(\mu + Z\nu)$ is trivial if and only if $s \in \mathbb{Z}$. Indeed by definition this factor of automorphy is analytically trivial if and only if there is a nowhere vanishing holomorphic function $\phi(t)$ in $\mathbb{C}^g$ such that $\phi(t + \mu + Z\nu) = \sigma_s(\mu + Z\nu)\phi(t)$ for all $\mu, \nu \in \mathbb{Z}^g$. Any such function can be written as $\phi(t) = \exp 2\pi i\lambda(t)$ for some holomorphic function $h(t)$ on $\mathbb{C}^g$, and in terms of this function the flat factor of automorphy $\sigma_s$ is analytically trivial if and only if

$$h(t + \mu + Z\nu) = h(t) - \nu s - m(\mu, \nu) \quad \text{for some } m(\mu, \nu) \in \mathbb{Z}.$$  

If there is such a function $h(t)$ then $\partial_j h(t)$ is invariant under translations through the lattice vectors in $L(\Omega)$ for any index $1 \leq j \leq g$, so $\partial_j h(t)$ is a bounded holomorphic function in $\mathbb{C}^g$ hence it must be constant; the function $h(t)$ consequently must be a linear function $h(t) = \lambda t$ for some $\lambda \in \mathbb{C}^g$. In that case (G.30) reduces to the condition that $\lambda t = \nu s - m(\mu, \nu)$ for all $\mu, \nu \in \mathbb{Z}^g$. In particular for $\nu = 0$ it follows that $\lambda t \in \mathbb{C}$ for all $\mu \in \mathbb{Z}^g$, hence that $\alpha \in \mathbb{Z}^g$, and therefore $\nu s \in L(\Omega)$ for all $\nu \in \mathbb{Z}^g$ so that $s \in L(\Omega)$. Conversely if $s \in L(\Omega)$ so that $s = m + Zn$ for $m, n \in \mathbb{Z}^g$, and if $h(t) = \lambda t$ then $h(t + \mu + Z\nu) - h(t) - \nu(m + Zn) = \lambda t - \nu m \in \mathbb{Z}$ so $\sigma_s(\mu + Z\nu)$ is analytically trivial. In general, if $s \in \frac{1}{2}Z^g$ for an integer $r$ it is clear from (G.29) that $\sigma_s(\mu + Z\nu)^r = 1$ for all $\mu, \nu$, hence that $\Theta(t + \mu + Z\nu; s, Z)^r = \xi_Z(\mu + Z\nu)^r \Theta(t; s, Z)^r$; thus the $r$-th power of the translate $\theta(t; s, Z)$ for any such a value $s$ is also a relatively automorphic function for the $r$-power of the theta factor of automorphy, so

$$\Theta(t; s, Z)^r \in \Gamma(M, \Xi_Z^r) \quad \text{whenever } s \in \frac{1}{2}Z^g.$$  

Classically translates of the theta function were handled by writing a vector $s \in \mathbb{C}^g$ as the sum

$$s = Z\alpha + \beta \quad \text{for some } \alpha, \beta \in \mathbb{C}^g,$$

since the columns of the matrix $\Omega = (I \ Z)$ are a basis for the vector space $\mathbb{C}^g$. The translate $\Theta(t; s, Z) = \Theta(s + t; Z)$ of the theta series through a vector $s$ of the form (G.32) was expressed\(^2\) in terms of a theta series with characteristic $[\alpha|\beta]$, defined by

$$\Theta[\alpha|\beta](t; Z) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} t(n + \alpha) Z(n + \alpha) + \frac{1}{4}(n + \alpha)(t + \beta) \right).$$

\(^2\)The classical notation is $\Theta[\alpha|\beta](t; Z)$, which has been used quite persistently despite its rather ungainly form; in the present book a modified notation is used, for purely aesthetic reasons.
Since
\[
\frac{1}{2} \left( n + \alpha \right) Z(n + \alpha) + t(n + \alpha)(t + \beta) = \frac{1}{2} t n Zn + t(n + Z\alpha + \beta) + \frac{1}{2} t \alpha Z\alpha + t(t + \beta)
\]
the series (G.33) can be written
\[
\text{(G.34)} \quad \Theta[\alpha|\beta](t; Z) = \Theta(t + Z\alpha + \beta; Z) \cdot \exp 2\pi i \left( \frac{1}{2} t \alpha Z\alpha + t(t + \beta) \right),
\]
so the theta function with characteristic \([\alpha, \beta]\) is the product of the exponential of a simple linear function and the translate \(\Theta(t; s, Z)\) of the theta function \(\Theta(t; Z)\) through the vector \(s = Z\alpha + \beta\). It then follows from (G.28) that for any \(\mu, \nu \in \mathbb{Z}^g\)
\[
\text{(G.35)} \quad \Theta[\alpha|\beta](t + \mu + Z\nu; Z) = \Theta(t + \mu + Z\nu + Z\alpha + \beta; Z) \cdot \exp 2\pi i \left( \frac{1}{2} t \alpha Z\alpha + t(t + Z\nu + \beta) \right)
\]
and
\[
= \Theta(t + Z\alpha + \beta; Z) \exp -2\pi i \left( \frac{1}{2} t \nu Z\nu + \nu(t + Z\alpha + \beta) \right) \cdot \exp 2\pi i \left( \frac{1}{2} t \alpha Z\alpha + t(t + Z\nu + \beta) \right)
\]
\[
= \Theta[\alpha|\beta](t; Z) \exp -2\pi i \left( \frac{1}{2} t \nu Z\nu - \nu \alpha + \nu(t + \beta) \right),
\]
which can be rewritten
\[
\text{(G.36)} \quad \Theta[\alpha|\beta](t + \mu + Z\nu; Z) = \sigma[\alpha|\beta](\mu + Z\nu) \cdot \Xi_Z(\mu + Z\nu, t) \cdot \Theta[\alpha|\beta](t; Z)
\]
for the flat factor of automorphy \(\sigma[\alpha|\beta] \in \text{Hom}(\mathcal{L}(\Omega), \mathbb{C}^+\) for which
\[
\text{(G.37)} \quad \sigma[\alpha|\beta](\mu + Z\nu) = \exp 2\pi i \left( \nu \alpha - \nu \beta \right).
\]
The theta function with characteristic \([\alpha, \beta]\) consequently is a relatively automorphic function for the product of the flat factor of automorphy \(\sigma[\alpha|\beta](\mu + Z\nu)\) and the theta factor of automorphy \(\Xi_Z(\mu + Z\nu, t)\).

Theta functions with rational characteristics are particularly interesting. For example, if \(\alpha\) and \(\beta\) are half-integer vectors then \(Z\alpha + \beta\) is a half-period for the period matrix \(\Omega = (1 \ Z)\), while if \(\alpha = 0\) and \(\beta\) is a half-integer vector then \(Z\alpha + \beta\) is a real half-period. In general a theta function \(\Theta[\alpha|\beta](t; Z)\) with half-integral characteristic is either an odd or an even function of \(t\). Of course \(\Theta(t; Z) = \Theta[0|0](t; Z)\) is an even function, as noted in (G.3), and it follows from the definition (G.33) upon replacing the index of summation \(n\) by \(m\) where \(n + \alpha = -(m + \alpha)\), so \(m\) ranges through \(\mathbb{Z}^g\) as \(n\) does, that
\[
\Theta[\alpha|\beta](t; Z) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} t(n + \alpha)Z(n + \alpha) + t(n + \alpha)(-t + \beta) \right)
\]
\[
= \sum_{m \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} t(m + \alpha)Z(m + \alpha) - t(m + \alpha)(-t + \beta) \right)
\]
\[
= \sum_{m \in \mathbb{Z}^g} \exp 2\pi i \left( \frac{1}{2} t(m + \alpha)Z(m + \alpha) + t(m + \alpha)(t + \beta) - 2 \nu \alpha \beta \right)
\]
since $2\beta \in \mathbb{Z}^g$ hence $\exp 4\pi i^t m\beta = 1$; consequently
\[(G.38) \quad \Theta[\alpha|\beta](-t; Z) = \exp -4\pi i^t \alpha \beta \cdot \Theta[\alpha|\beta](t; Z).\]

The half-integral characteristic $[\alpha|\beta]$ is said to be \textit{even} if $4\alpha \beta \in \mathbb{Z}$ is an even integer and \textit{odd} if $4\alpha \beta \in \mathbb{Z}$ is an odd integer, so that
\[(G.39) \quad \exp -4\pi i[\alpha|\beta] = \begin{cases} 1 & \text{if } [\alpha|\beta] \text{ is even}, \\ -1 & \text{if } [\alpha|\beta] \text{ is odd}. \end{cases}\]

Consequently the theta function $\Theta[\alpha|\beta](t; Z)$ with an odd half-integral characteristic is an odd function so has a zero of odd order at the origin; and by (G.34) the zero of the function $\Theta[\alpha|\beta](t; Z)$ at the origin has the same order as that of the zero of the theta function $\Theta(t; Z)$ at the point $Z\alpha + \beta$, so the theta function $\Theta(t; Z)$ has a zero of odd order at any half-integral point $Z\alpha + \beta$ for which $[\alpha|\beta]$ is an odd characteristic. Similarly the theta function $\Theta[\alpha|\beta](t; Z)$ with an even half-integral characteristic is an even function so is either nonzero or has a zero of even order at the origin; and by (G.34) the zero of the function $\Theta[\alpha|\beta](t; Z)$ at the origin has the same order as that of the zero of the theta function $\Theta(t; Z)$ at the point $Z\alpha + \beta$, so the theta function $\Theta(t; Z)$ is either nonzero or has a zero of even order at any half-integral point $Z\alpha + \beta$ for which $[\alpha|\beta]$ is an even characteristic. There are $2^{g-1}(2^g + 1)$ even half-periods and $2^{g-1}(2^g - 1)$ odd half-periods, which is essentially demonstrated in Corollary 6.27 in Chapter 6.