[chapter] [chapter]

SOME TOPICS IN THE FUNCTION THEORY OF COMPACT RIEMANN SURFACES

PRELIMINARY VERSION

Robert C. Gunning
Fine Hall, Princeton University
Princeton, NJ 08544

January 24, 2018

Contents

Part I

Differentials and Integrals

Chapter 1

Divisors and Line Bundles

1.1 The Divisor Sequence

A *Riemann surface* M is a one-dimensional connected complex manifold[1]. The local properties of a meromorphic function at a point $p \in M$ are described almost entirely by the *order* of the function at that point, denoted by $\mathrm{ord}_p(f)$, which is a positive integer n if the function has a zero of order n at the point p, is a negative integer $-n$ if the function has a pole of order n at the point p, and is zero otherwise. The global properties of a meromorphic function on a compact Riemann surface are determined almost completely when the order of the function is given at each point of the surface; and that is done most conveniently in terms of divisors. In general a *divisor* on a Riemann surface M is a mapping $\mathfrak{d} : M \to \mathbb{Z}$ such that $\mathfrak{d}(p) \neq 0$ only at a discrete set of points $p \in M$; that set of points is called the *support* of the divisor, and is denoted by $|\mathfrak{d}|$. The set of all divisors on M clearly form an abelian group under pointwise addition of functions. The zero element of the group, called the *zero divisor* or the *trivial divisor*, is the divisor that is identically zero on M. The group of divisors is partially ordered by setting $\mathfrak{d}_1 \geq \mathfrak{d}_2$ whenever $\mathfrak{d}_1(p) \geq \mathfrak{d}_2(p)$ for all points $p \in M$; in particular a divisor $\mathfrak{d}$ is a *positive divisor*, traditionally also called an *effective divisor*, if $\mathfrak{d}(p) \geq 0$ for all points $p \in M$. Any divisor $\mathfrak{d}$ on M can be written uniquely as the difference $\mathfrak{d} = \mathfrak{d}_+ - \mathfrak{d}_-$ of two effective divisors with disjoint supports, the divisors defined by

$$(1.1) \qquad \mathfrak{d}_+(p) = \max\big(\mathfrak{d}(p), 0\big), \quad \mathfrak{d}_-(p) = \max\big(-\mathfrak{d}(p), 0\big).$$

On a compact Riemann surface M a divisor $\mathfrak{d}$ is nonzero only at finitely many points of the surface; the *degree* of the divisor is the integer $\deg \mathfrak{d} = \sum_{p \in M} \mathfrak{d}(p)$. A customary and useful notation is to write a divisor in the form $\mathfrak{d} = \sum_i \nu_i \cdot p_i$ where $\{p_i\}$ is a discrete set of points of M and $\mathfrak{d}(p_i) = \nu_i$ while $\mathfrak{d}(p) = 0$ at all points other than those in the set $\{p_i\}$; it is clear that for a divisor written

[1] For the definitions and basic properties of complex manifolds and of holomorphic and meromorphic functions on complex manifolds see Appendix A.2.

this way $\deg \mathfrak{d} = \sum_i \nu_i$ and $|\mathfrak{d}| \subset \bigcup_i p_i$, where this inclusion is an equality of sets if $\nu_i \neq 0$ for all indices i. The groups of divisors in the open subsets of a Riemann surface M, with the obvious restriction homomorphisms, form a complete presheaf[2] of abelian groups over M; the associated sheaf is the *sheaf of divisors* on M, denoted by $\mathcal{D}$, and it is clear that $\mathcal{D}$ is a fine sheaf over M. The group of divisors on any open subset $U \subset M$ can be identified with the group $\Gamma(U, \mathcal{D})$ of sections of the sheaf $\mathcal{D}$ over that subset.

To any meromorphic function f that is not identically zero on a Riemann surface M there can be associated its divisor $\mathfrak{d}(f)$, defined by $\mathfrak{d}(f)(p) = \mathrm{ord}_p(f)$ for all points $p \in M$. When this divisor is written as the difference $\mathfrak{d}(f) = \mathfrak{d}_+(f) - \mathfrak{d}_-(f)$ of two effective divisors with disjoint supports, $\mathfrak{d}_+(f)$ is the *divisor of zeros* of the function f and $\mathfrak{d}_-(f)$ is the *divisor of poles* of the function; $\mathfrak{d}_+(f)$ lists all zeros of the function f with their orders, while $\mathfrak{d}_-(f)$ lists the poles of f with their orders. The mapping that associates to any nontrivial meromorphic function on M its divisor is a homomorphism

$$\mathfrak{d} : \Gamma(M, \mathcal{M}^*) \longrightarrow \Gamma(M, \mathcal{D}) \tag{1.2}$$

from the multiplicative group of nontrivial meromorphic functions on M to the additive group of divisors on M. The kernel of this homomorphism consists of those meromorphic functions having order zero at all points of M, so is the multiplicative group $\Gamma(M, \mathcal{O}^*)$ of nowhere vanishing holomorphic functions on M; consequently (1.2) can be extended to the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}^*) \xrightarrow{\iota} \Gamma(M, \mathcal{M}^*) \xrightarrow{\mathfrak{d}} \Gamma(M, \mathcal{D}) \tag{1.3}$$

in which ι is the inclusion mapping. The image of the homomorphism $\mathfrak{d}$ consists of those divisors on M that are the divisors of global meromorphic functions on M, customarily called *principal divisors* on M. Two divisors $\mathfrak{d}_1$ and $\mathfrak{d}_2$ that differ by a principal divisor are called *linearly equivalent* divisors, and the linear equivalence of these two divisors is denoted by $\mathfrak{d}_1 \sim \mathfrak{d}_2$. It is clear that this is an equivalence relation in the usual sense, and that the principal divisors form one equivalence class. It is actually a nontrivial equivalence relation on compact Riemann surfaces; an initial necessary condition that divisors on a compact Riemann surface must satisfy in order to be principal divisors is the following.

Theorem 1.1 *If f is a nontrivial meromorphic function on a compact Riemann surface M then* $\deg \mathfrak{d}(f) = 0$.

Proof: Select a finite triangulation[3] of the compact Riemann surface M by 2-dimensional simplices σ_j such that the support of the divisor $\mathfrak{d}(f)$ of the meromorphic function f is disjoint from the boundaries $\partial\sigma_j$ of all of these simplices. By the residue theorem the degree of the divisor $\mathfrak{d}(f)$ is given by

$$\deg \mathfrak{d}(f) = \sum_j \frac{1}{2\pi i} \int_{\partial\sigma_j} d\log f(z) = \frac{1}{2\pi i} \int_{\sum_j \partial\sigma_j} d\log f(z);$$

[2]For the definition and basic properties of sheaves and presheaves see Appendix C.1.
[3]For the topological properties of surfaces see Appendix D.

and since $\sum_j \partial\sigma_j = 0$ it follows that $\deg \mathfrak{d}(f) = 0$, which suffices for the proof.

For any nontrivial meromorphic function f on a compact Riemann surface M it follows from the preceding theorem that $0 = \deg \mathfrak{d}(f) = \deg \mathfrak{d}_+(f) - \deg \mathfrak{d}_-(f)$ and hence that $\deg \mathfrak{d}_+(f) = \deg \mathfrak{d}_-(f)$; this common value is called the *degree* of the meromorphic function f and is denoted by $\deg f$. Clearly $\deg f \geq 0$ for any nontrivial meromorphic function on a compact Riemann surface, and $\deg f = 0$ if and only if f is everywhere holomorphic and nowhere zero on M, hence is a nonzero complex constant as an immediate consequence of the maximum modulus theorem. Not all divisors of degree 0 on a compact Riemann surface M are the divisors of meromorphic functions on M though; but it is obvious that any divisor is locally the divisor of a nontrivial meromorphic function, or equivalently that any germ of a divisor is the divisor of a germ of a nontrivial meromorphic function. Thus there is the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}^* \xrightarrow{\iota} \mathcal{M}^* \xrightarrow{\mathfrak{d}} \mathcal{D} \longrightarrow 0 \tag{1.4}$$

on any Riemann surface M, where ι is the natural inclusion homomorphism and $\mathfrak{d}$ is the homomorphism that associates to any germ of a nontrivial meromorphic function the germ of its divisor, the local form of the homomorphism (1.2). The problem of determining which divisors are the divisors of meromorphic functions on M then can be approached by using the exact cohomology sequence[4] associated to the exact sequence of sheaves (1.4), which begins

$$\begin{aligned} (1.5)\quad 0 \longrightarrow \Gamma(M,\mathcal{O}^*) \xrightarrow{\iota} \Gamma(M,\mathcal{M}^*) \xrightarrow{\mathfrak{d}} \Gamma(M,\mathcal{D}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^1(M,\mathcal{O}^*) \xrightarrow{\iota} H^1(M,\mathcal{M}^*) \xrightarrow{\mathfrak{d}} H^1(M,\mathcal{D}) = 0 \end{aligned}$$

where δ is the coboundary homomorphism and $H^1(M,\mathcal{D}) = 0$ since $\mathcal{D}$ is a fine sheaf. The first line of this exact sequence is just the exact sequence (1.3); sheaf cohomology theory yields the coboundary homomorphism δ connecting the two lines and the homomorphisms between the higher dimensional cohomology groups. The exactness of the sequence (1.5) at the group $\Gamma(M,\mathcal{D})$ is just the assertion that a divisor $\mathfrak{d} \in \Gamma(M,\mathcal{D})$ is a principal divisor if and only if $\delta(\mathfrak{d}) = 0 \in H^1(M,\mathcal{O}^*)$; and it follows that $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\delta(\mathfrak{d}_1) = \delta(\mathfrak{d}_2)$, so linear equivalence of divisors is just the equivalence relation defined by the group homomorphism δ.

The preceding can be put into a more concrete form by tracing through the coboundary homomorphism in the exact cohomology sequence (1.5). For any divisor $\mathfrak{d} \in \Gamma(M,\mathcal{D})$ on a Riemann surface M there is a covering $\mathfrak{U} = \{U_\alpha\}$ of M by open subsets $U_\alpha \subset M$ such that the restriction of the divisor $\mathfrak{d}$ to each set U_α is the divisor of a nontrivial meromorphic function f_α in U_α; the set of these functions can be viewed as a cochain $f \in C^0(\mathfrak{U},\mathcal{M})$, and the coboundary of this cochain is the 1-cocycle $\delta f = \lambda \in Z^1(\mathfrak{U},\mathcal{O}^*)$ that represents the image $\delta(\mathfrak{d}) \in H^1(M,\mathcal{O}^*)$. Explicitly as the multiplicative analogue of (C.7)

$$\lambda_{\alpha\beta} = \frac{f_\beta}{f_\alpha}; \tag{1.6}$$

[4] For the cohomology of sheaves see Appendix C.2.

the functions $\lambda_{\alpha\beta}$ are holomorphic and nowhere vanishing in the intersections $U_\alpha \cap U_\beta$, since f_α and f_β have the same divisors there, and they clearly satisfy the skew-symmetry condition $\lambda_{\alpha\beta} = \lambda_{\beta\alpha}^{-1}$ and the cocycle condition

$$\lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha} = 1 \quad \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma. \tag{1.7}$$

The cocycle λ depends on the choices of the local meromorphic functions f_α; but any other meromorphic function $\tilde{f}_\alpha$ in U_α having the same divisor as f_α must be of the form $\tilde{f}_\alpha = f_\alpha h_\alpha$ for a nowhere vanishing holomorphic function h_α in the set U_α, and the cocycle associated to the functions $\tilde{f}_\alpha$ is $\tilde{\lambda}_{\alpha\beta} = h_\alpha^{-1}\lambda_{\alpha\beta}h_\beta$ which is a cohomologous cocycle and represents the same cohomology class in $H^1(M, \mathcal{O}^*)$. The image of the coboundary homomorphism δ consists of those cohomology classes in $H^1(M, \mathcal{O}^*)$ that are trivial in $H^1(M, \mathcal{M}^*)$, hence that are represented for a sufficiently fine open covering $\mathfrak{U} = \{U_\alpha\}$ of M by cocycles $\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ that have the form (1.6) for some meromorphic functions f_α in the sets U_α.

Elements of the cohomology group $H^1(M, \mathcal{O}^*)$ of a Riemann surface M can be viewed as holomorphic line bundles[5] over M. To any divisor $\mathfrak{d} \in \Gamma(M, \mathcal{D})$ there thus can be associated the holomorphic line bundle $\delta(\mathfrak{d}) \in H^1(M, \mathcal{O}^*)$ in the exact sequence (1.5); but it is more convenient and more customary to associate to a divisor $\mathfrak{d}$ the holomorphic line bundle $\zeta_{\mathfrak{d}} = \delta(-\mathfrak{d})$, called the *line bundle of the divisor* $\mathfrak{d}$. If $\mathfrak{d}$ is locally the divisor of meromorphic functions f_α in the open sets U_α of a covering $\mathfrak{U}$ then $-\mathfrak{d}$ is locally the divisor of the meromorphic functions f_α^{-1}, so as in (1.6) the line bundle $\zeta_{\mathfrak{d}}$ is described by the cocycle

$$\zeta_{\mathfrak{d}\alpha\beta} = \frac{f_\beta^{-1}}{f_\alpha^{-1}} = \frac{f_\alpha}{f_\beta}. \tag{1.8}$$

Since $\mathfrak{d}(f_1 f_2) = \mathfrak{d}(f_1) + \mathfrak{d}(f_2)$ it is evident from the preceding equation that

$$\zeta_{\mathfrak{d}_1+\mathfrak{d}_2} = \zeta_{\mathfrak{d}_1} \cdot \zeta_{\mathfrak{d}_2} \tag{1.9}$$

for any divisors $\mathfrak{d}_1, \mathfrak{d}_2$, so the mapping that associates to any divisor $\mathfrak{d}$ the line bundle $\zeta_{\mathfrak{d}}$ of that divisor is a homomorphism from the additive group of divisors to the multiplicative group $H^1(M, \mathcal{O}^*)$ of holomorhic line bundles over M. From the exact sequence (1.5) it follows that $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\delta(\mathfrak{d}_1) = \delta(\mathfrak{d}_2)$; and since $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $-\mathfrak{d}_1 \sim -\mathfrak{d}_2$ it follows that

$$\mathfrak{d}_1 \sim \mathfrak{d}_2 \quad \text{if and only if} \quad \zeta_{\mathfrak{d}_1} = \zeta_{\mathfrak{d}_2}, \tag{1.10}$$

so the linear equivalence of divisors corresponds to the equality of their associated line bundles.

A **cross-section** of a holomorphic line bundle $\zeta \in H^1(M, \mathcal{O}^*)$ described by a cocycle $\zeta_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ is a collection of complex-valued functions f_α defined in the open subsets U_α and satisying

$$f_\alpha = \zeta_{\alpha\beta} f_\beta \quad \text{in the intersections} \quad U_\alpha \cap U_\beta. \tag{1.11}$$

[5]For the definition and basic properties of holomorphic line bundles see Appendix B, and for the description of holomorphic line bundles in terms of sheaf cohomology see Appendix C.2.

A **holomorphic cross-section** is one for which the functions f_α are holomorphic; the set of all holomorphic cross-sections of the line bundle ζ is denoted by $\Gamma\big(M, \mathcal{O}(\zeta)\big)$, and clearly this is a complex vector space under the addition and scalar multiplication of the functions f_α. The vector spaces $\Gamma\big(M, \mathcal{M}(\zeta)\big)$ of meromorphic cross-sections, $\Gamma\big(M, \mathcal{E}(\zeta)\big)$ of $\mathcal{C}^\infty$ complex-valued cross-sections and $\Gamma\big(M, \mathcal{C}(\zeta)\big)$ of continuous complex-valued cross-sections are defined correspondingly. Any cross-section $f \in \Gamma(M, \mathcal{M}(\lambda))$ is described by meromorphic functions f_α in coordinate neighborhoods U_α; the *order* of the cross-section f at a point $p \in M$ can be defined by $\mathrm{ord}_p(f) = \mathrm{ord}_p(f_\alpha)$ whenever $p \in U_\alpha$, for $\mathrm{ord}_p(f_\alpha) = \mathrm{ord}_p(\lambda_{\alpha\beta} f_\beta) = \mathrm{ord}_p(f_\beta)$ if $p \in U_\alpha \cap U_\beta$ since $\lambda_{\alpha\beta}$ is holomorphic and nowhere vanishing there; the *divisor* $\mathfrak{d}(f)$ of the cross-section f then is defined by $\mathfrak{d}(f)(p) = \mathrm{ord}_p(f)$ for all points $p \in M$, just as for meromorphic functions on M.

Since (1.8) and (1.11) are really the same equation it is evident that for any holomorphic line bundle ζ

$$\zeta = \zeta_{\mathfrak{d}(f)} \quad \text{for any} \quad f \in \Gamma\big(M, \mathcal{M}(\zeta)\big). \tag{1.12}$$

For many purposes positive or effective divisors play a special role. A *complete linear system* $|\mathfrak{d}_0|$ is defined traditionally as the set of all effective divisors that are linearly equivalent to the divisor $\mathfrak{d}_0$. By (1.10) this consists of all effective divisors $\mathfrak{d}$ such that $\zeta_\mathfrak{d} = \zeta_{\mathfrak{d}_0}$; equivalently by (1.12) this consists of all divisiors $\mathfrak{d}(f)$ of holomorphic cross-sections $f \in \Gamma\big(M, \mathcal{O}(\zeta_{\mathfrak{d}_0})\big)$. Two holomorphic cross-sections f_1, f_2 of a holomorphic line bundle have the same divisor if and only if their quotient f_1/f_2 is a holomorphic and nowhere vanishing function on M; and since M is compact that quotient must therefore be a nonzero complex constant. It is traditional for any vector space V to let $\mathbb{P}(V)$ denote the set of equivalence classes of nonzero vectors $v_1, v_2 \in V$ where v_1 and v_2 are equivalent if and only if $v_1 = cv_2$ for some complex constant c. For a finite dimensional complex vector space this set $\mathbb{P}\mathbb{C}^{n+1}$ is just the standard projective space $\mathbb{P}^n$ of dimension n. With this notation therefore there is the natural identification

$$|\mathfrak{d}| = \mathbb{P}\Gamma\big(M, \mathcal{O}(\zeta_0)\big). \tag{1.13}$$

Thus another of the standard concepts of classical algebraic geometry has a natural interpretation in terms of holomorphic line bundles.

1.2 The Characteristic Class of a Line Bundle

The line bundle of the divisor $\mathfrak{d} = 1 \cdot p$ for a single point $p \in M$ is denoted by ζ_p and is called a *point bundle* over the Riemann surface M; the bundle ζ_p thus is characterized as that holomorphic line bundle over M having a nontrivial holomorphic cross-section with a simple zero at the point $p \in M$ and no other zeros on M. For any divisor $\mathfrak{d} = \sum_i \nu_i \cdot p_i$ in M it is obvious that $\zeta_\mathfrak{d} = \prod_i \zeta_{p_i}^{\nu_i}$, so all line bundles of divisors on M can be built up from point bundles over

M. If $f_1, f_2 \in \Gamma(M, \mathcal{M}(\lambda))$ are two nontrivial meromorphic cross-sections of a holomorphic line bundle λ over a compact Riemann surface M their quotient f_1/f_2 is a meromorphic function on M, so from Theorem 1.1 it follows that $0 = \deg \mathfrak{d}(f_1/f_2) = \deg \mathfrak{d}(f_1) - \deg \mathfrak{d}(f_2)$; consequently *the degrees of the divisors of all nontrivial meromorphic cross-sections of a holomorphic line bundle λ over a compact Riemann surface are the same.* This observation can be used to define the *characteristic class* or *Chern class* of a holomorphic line bundle λ over a compact Riemann surface M by

$$c(\lambda) = \deg \mathfrak{d}(f) \text{ for any } f \in \Gamma(M, \mathcal{M}(\lambda)),\ f \neq 0; \tag{1.14}$$

in particular

$$c(\zeta_{\mathfrak{d}}) = \deg \mathfrak{d} \text{ for any divisor } \mathfrak{d}. \tag{1.15}$$

The characteristic class is defined in this way though only for those holomorphic line bundles that have nontrivial meromorphic cross-sections. It will be demonstrated later in this chapter that all holomorphic line bundles over a compact Riemann surface do have nontrivial meromorphic cross-sections; thus the characteristic class actually is well defined for any holomorphic line bundle over M. However the characteristic class of a line bundle really is a purely topological invariant of the line bundle and can be described in various other ways, which provide definitions of the characteristic class for any holomorphic line bundle without recourse to the basic existence theorem. One alternative definition is through a curvature integral expressed in terms of the differential operators[6] ∂ and $\overline{\partial}$.

Theorem 1.2 *Let λ be a holomorphic line bundle over a compact Riemann surface M described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ in terms of a covering of M by open sets U_α. If $r_\alpha > 0$ are $\mathcal{C}^\infty$ functions in the sets U_α such that $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in $U_\alpha \cap U_\beta$ then $\overline{\partial}\partial \log r_\alpha = \overline{\partial}\partial \log r_\beta$ in $U_\alpha \cap U_\beta$, so the local differential forms $\overline{\partial}\partial \log r_\alpha$ describe a global differential form of degree 2 on the surface M. The integral*

$$\frac{1}{2\pi i}\int_M \overline{\partial}\partial \log r_\alpha \tag{1.16}$$

is independent of the choice of the functions r_α; and if λ has a nontrivial meromorphic cross-section the value of this integral is the characteristic class $c(\lambda)$ of the line bundle λ.

Proof: If $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in $U_\alpha \cap U_\beta$ then $\log r_\alpha = \log \lambda_{\alpha\beta} + \log \overline{\lambda_{\alpha\beta}} + \log r_\beta$ in that intersection; and since $\log \lambda_{\alpha\beta}$ is holomorphic $\overline{\partial} \log \lambda_{\alpha\beta} = \partial \log \overline{\lambda_{\alpha\beta}} = 0$ and $\partial \log \lambda_{\alpha\beta} = d \log \lambda_{\alpha\beta}$, so

$$\partial \log r_\alpha = d \log \lambda_{\alpha\beta} + \partial \log r_\beta \tag{1.17}$$

[6] The differential operators ∂ and $\overline{\partial}$ are defined in Appendix A.1.

and $\overline{\partial}\partial \log r_\alpha = \overline{\partial}\partial \log r_\beta$ as asserted. For any other $\mathcal{C}^\infty$ functions $s_\alpha > 0$ in the sets U_α satisfying $s_\alpha = |\lambda_{\alpha\beta}|^2 s_\beta$ in $U_\alpha \cap U_\beta$ it is evident that $s_\alpha = h\, r_\alpha$ where $h > 0$ is a $\mathcal{C}^\infty$ function defined on the entire surface M; then $\partial \log h$ is a $\mathcal{C}^\infty$ differential form on the compact manifold M so $\int_M \overline{\partial}\partial \log h = \int_M d(\partial \log h) = 0$ by Stokes's Theorem and consequently

$$\int_M \overline{\partial}\partial \log s_\alpha = \int_M \left(\overline{\partial}\partial \log h + \overline{\partial}\partial \log r_\alpha\right) = \int_M \overline{\partial}\partial \log r_\alpha,$$

showing that the value of the integral (1.16) is independent of the choice of the functions r_α. If $f \in \Gamma(M, \mathcal{M}(\lambda))$ is a nontrivial meromorphic cross-section of the line bundle λ and $\mathfrak{d}(f) = \sum_j \nu_j \cdot p_j$ then by definition $c(\lambda) = \deg \mathfrak{d}(f) = \sum_j \nu_j$. Choose coordinate neighborhoods U_α covering M such that each point p_j is contained in an open disc $D_j \subset U_{\alpha_j}$ for a coordinate neighborhood U_{α_j} and $\overline{D_j} \subset U_{\alpha_j}$ while $\overline{D_j} \cap \overline{U_\beta} = \emptyset$ whenever $\beta \neq \alpha_j$. Set $r_\alpha = |f_\alpha|^2$ if $\alpha \neq \alpha_j$ for any j; and let r_{α_j} be a modification of the function $|f_{\alpha_j}|^2$ within the disc D_j so that r_{α_j} is a $\mathcal{C}^\infty$ and strictly positive function in U_{α_j}, as is clearly possible. The functions r_α so defined then are $\mathcal{C}^\infty$ in the coordinate neighborhoods U_α and satisfy $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in any intersection $U_\alpha \cap U_\beta$; and $\overline{\partial}\partial \log r_\alpha = \overline{\partial}\partial(\log f_\alpha + \log \overline{f}_\alpha) = 0$ in the complement of the union $\bigcup_j D_j$ since the functions f_α are holomorphic there. Then from the residue calculus and Stokes's theorem it follows that

$$\begin{aligned}
\frac{1}{2\pi i} \int_M \overline{\partial}\partial \log r_\alpha &= \frac{1}{2\pi i} \sum_j \int_{D_j} \overline{\partial}\partial \log r_{\alpha_j} = \frac{1}{2\pi i} \sum_j \int_{D_j} d\partial \log r_{\alpha_j} \\
&= \frac{1}{2\pi i} \sum_j \int_{\partial D_j} \partial \log r_{\alpha_j} = \frac{1}{2\pi i} \sum_j \int_{\partial D_j} d \log f_{\alpha_j} \\
&= \sum_j \nu_j = c(\lambda)
\end{aligned}$$

since $\partial \log r_{\alpha_j} = d \log f_{\alpha_j}$ on ∂D_j, and that suffices to conclude the proof.

It is easy to see that for any holomorphic line bundle λ over a compact Riemann surface M there exist functions r_α satisfying the conditions of the preceding theorem for any finite open covering $\{U_\alpha\}$ of M. Indeed choose open subsets V_α such that $\overline{V_\alpha} \subset U_\alpha$ and that the sets V_α also cover M. For any open set U_α of the initial covering there exists as usual a $\mathcal{C}^\infty$ function r^α_α with support in U_α such that $r^\alpha_\alpha(p) \geq 0$ in U_α and $r^\alpha_\alpha > 0$ in V_α; and in terms of this function set $r^\alpha_\beta(p) = \lambda_{\beta\alpha}(p) r^\alpha_\alpha(p)$ for $p \in U_\alpha \cap U_\beta$ and $r^\alpha_\beta(p) = 0$ for $p \in U_\beta \sim U_\alpha$, from which it is clear that $r^\alpha_\beta(p) = \lambda_{\beta\gamma}(p) r^\alpha_\gamma(p)$ for $p \in U_\beta \cap U_\gamma$. The sums $r_\alpha = \sum_\delta r^\delta_\alpha$ then satisfy the conditions of the theorem. The integral (1.16) thus is a well defined invariant associated to any line bundle λ, and by the preceding theorem this invariant is equal to the characteristic class of λ if that bundle has a nontrivial meromorphic cross-section; this thus provides a definition of the characteristic class of an arbitrary holomorphic line bundle over a compact Riemann surface, which reduces to the preceding definition (1.14) for those holomorphic line bundles that have nontrivial meromorphic cross-sections.

Corollary 1.3 *If λ is a holomorphic line bundle over a compact Riemann surface M and $c(\lambda) < 0$ then the bundle λ has no nontrivial holomorphic cross-sections.*

Proof: If λ has a nontrivial holomorphic cross-section f then $c(\lambda)$ is defined by (1.14) so that $c(\lambda) = \deg \mathfrak{d}(f) \ge 0$, and that suffices for the proof.

Corollary 1.4 *If λ is a holomorphic line bundle over a compact Riemann surface M and $c(\lambda) = 0$ then*

$$\dim \Gamma(M, \mathcal{O}(\lambda)) = \begin{cases} 1 & \textit{if } \lambda \textit{ is analytically equivalent to the trivial bundle,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof: If the line bundle λ has characteristic class $c(\lambda) = 0$ and has a nontrivial holomorphic cross-section f then $c(\lambda)$ is defined by (1.14) so that $0 = c(\lambda) = \deg \mathfrak{d}(f)$, and consequently the cross-section f is holomorphic and nowhere vanishing on M. Thus when the line bundle is defined by coordinate transition functions $\lambda_{\alpha\beta}$ in intersections $U_\alpha \cap U_\beta$ of coordinate neighborhoods on M the cross-section f is described by holomorphic and nowhere vanishing functions f_α in the sets U_α such that $\lambda_{\alpha\beta} = f_\alpha / f_\beta$ in $U_\alpha \cap U_\beta$; and that is just the condition that the line bundle λ is analytically equivalent to the trivial line bundle. On the other hand any holomorphic cross-section of the trivial line bundle $\lambda = M \times \mathbb{C}$ over M is just a holomorphic function on the compact Riemann surface M so by the maximum modulus theorem is constant; thus $\dim \Gamma(M, \mathcal{O}(\lambda)) = 1$ for the trivial line bundle λ, and that suffices for the proof.

An alternative approach to the characteristic class of a holomorphic line bundle uses a different exact sequence of sheaves. Over an arbitrary Riemann surface M there is the exact sequence of sheaves

$$(1.18) \qquad 0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{C} \xrightarrow{e} \mathcal{C}^* \longrightarrow 0$$

in which ι is the natural inclusion mapping of the sheaf of locally constant integer-valued functions, the trivial sheaf with stalk $\mathbb{Z}$, into the sheaf $\mathcal{C}$ of germs of complex-valued continuous functions and e is the homomorphism that associates to a germ $f \in \mathcal{C}$ the germ $e(f) = \exp 2\pi i\, f \in \mathcal{C}^*$ of a nowhere vanishing complex-valued continuous function. The associated exact cohomology sequence contains the segment

$$H^1(M, \mathcal{C}) \xrightarrow{e} H^1(M, \mathcal{C}^*) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \xrightarrow{\iota} H^2(M, \mathcal{C}),$$

in which $H^1(M, \mathcal{C}^*)$ can be viewed as the set of topological line bundles over M. Since $\mathcal{C}$ is a fine sheaf $H^1(M, \mathcal{C}) = H^2(M, \mathcal{C}) = 0$, so this segment of an exact sequence reduces to the isomorphism

$$(1.19) \qquad \delta : H^1(M, \mathcal{C}^*) \xrightarrow{\simeq} H^2(M, \mathbb{Z}).$$

The image $\delta(\lambda) \in H^2(M, \mathbb{Z})$ of a topological line bundle $\lambda \in H^1(M, \mathcal{C}^*)$ under this isomorphism thus characterizes the topological equivalence class of λ completely, and every cohomology class in $H^2(M, \mathbb{Z})$ is the image of some topological line bundle over M. There is a similar exact sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{E} \xrightarrow{e} \mathcal{E}^* \longrightarrow 0 \tag{1.20}$$

involving $\mathcal{C}^\infty$ functions rather than merely continuous functions; and since $\mathcal{E}$ also is a fine sheaf the associated exact cohomology sequence leads in the same way to an isomorphism

$$\delta : H^1(M, \mathcal{E}^*) \xrightarrow{\simeq} H^2(M, \mathbb{Z}) \tag{1.21}$$

analogous to (1.19) but in which $H^1(M, \mathcal{E}^*)$ can be viewed as the set of $\mathcal{C}^\infty$ line bundles over M. These isomorphisms can be described more concretely by tracing through the coboundary homomorphism in the exact sequence of sheaves, just as was done for the coboundary homomorphism in the exact sequence (1.5). If the line bundle λ is described by coordinate transition functions $\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{C}^*)$ or $Z^1(\mathfrak{U}, \mathcal{E}^*)$ there are single-valued branches of the logarithms

$$f_{\alpha\beta} = \frac{1}{2\pi i} \log \lambda_{\alpha\beta} \tag{1.22}$$

in the intersections $U_\alpha \cap U_\beta$ after passing to a suitable refinement of the covering if necessary. The image $\delta(\lambda) \in H^2(M, \mathbb{Z})$ is the cohomology class represented by the integral cocycle $n \in Z^1(\mathfrak{U}, \mathbb{Z})$ for which

$$n_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta}. \tag{1.23}$$

Since this is the same construction for either of the homomorphisms (1.19) or (1.21) it is evident that the image $\delta(\lambda) \in H^2(M, \mathbb{Z})$ of a $\mathcal{C}^\infty$ line bundle λ is the same cohomology class whether calculated through (1.19) or (1.21); consequently *a topological line bundle over an arbitrary Riemann surface M is topologically equivalent to a $\mathcal{C}^\infty$ line bundle, and two $\mathcal{C}^\infty$ line bundles are equivalent if and only if they are topologically equivalent.* Of course a holomorphic line bundle λ can be viewed either as a $\mathcal{C}^\infty$ line bundle or as a topological line bundle, and either of these structures is described completely by the cohomology class $\delta(\lambda) \in H^2(M, \mathbb{Z})$.

Since $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ when M is a compact connected two-dimensional orientable manifold the cohomology class $\delta\lambda \in H^2(M, \mathbb{Z})$ associated to a line bundle λ over a compact Riemann surface M can be identified with an integer. It is useful in the present discussion to view the integral cohomology group as a subgroup $H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{C}) \cong \mathbb{C}$ of the complex cohomology group, since the latter group can be handled analytically through the deRham isomorphism. There is of course a choice in the identification of a cohomology class in $H^2(M, \mathbb{C})$ with a complex number, since the identification $H^2(M, \mathbb{C}) \cong \mathbb{C}$ is determined only up to a linear mapping of $\mathbb{C}$; the choice made here is convenient for present purposes. A cohomology class $c \in H^2(M, \mathbb{C})$ is described by a cocycle $c_{\alpha\beta\gamma} \in Z^2(\mathfrak{U}, \mathbb{C})$ in terms of a covering $\mathfrak{U} = \{U_\alpha\}$ of the surface M. When viewed as a cocycle in $Z^2(\mathfrak{U}, \mathcal{E})$ the cocycle $c_{\alpha\beta\gamma}$ is cohomologous to zero after a suitable refinement of the covering $\mathfrak{U}$ since $\mathcal{E}$ is a fine sheaf; so

$$c_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} \tag{1.24}$$

for some $\mathcal{C}^\infty$ functions $f_{\alpha\beta}$ in the intersections $U_\alpha \cap U_\beta$. Since $c_{\alpha\beta\gamma}$ are constants

$$0 = d\, c_{\alpha\beta\gamma} = d\, f_{\beta\gamma} - d\, f_{\alpha\gamma} + d\, f_{\alpha\beta},$$

and consequently the differential forms $d\, f_{\alpha\beta}$ form a cocycle in $Z^1(\mathfrak{U}, \mathcal{E}^1)$. The sheaf $\mathcal{E}^1$ of $\mathcal{C}^\infty$ differential forms of degree 1 also is a fine sheaf, so after a further refinement of the covering $\mathfrak{U}$ if necessary

$$d\, f_{\alpha\beta} = \phi_\beta - \phi_\alpha \tag{1.25}$$

for some $\mathcal{C}^\infty$ differential 1-forms ϕ_α in the neighborhoods U_α. Then

$$0 = d\, d\, f_{\alpha\beta} = d\, \phi_\beta - d\, \phi_\alpha$$

so the differential 2-forms $d\, \phi_\alpha$ and $d\, \phi_\beta$ agree in the intersection $U_\alpha \cap U_\beta$; consequently these local differential forms describe a differential 2-form on the entire compact Riemann surface M. The deRham isomorphism[7]

$$I : H^2(M, \mathbb{C}) \longrightarrow \mathbb{C} \tag{1.26}$$

associates to the cohomology class $c \in H^2(M, \mathbb{C})$ represented by the cocycle $c_{\alpha\beta\gamma}$ the complex number

$$I(c) = \int_M d\, \phi_\alpha. \tag{1.27}$$

In terms of this isomorphism the characteristic class of a holomorphic line bundle can be described as follows.

Theorem 1.5 *If $\delta\lambda \in H^2(M, \mathbb{C})$ is the cohomology class associated to a holomorphic line bundle λ over a compact Riemann surface M through the isomorphism* (1.20) *or* (1.21) *then $c(\lambda) = -I(\delta\lambda)$ is the characteristic class of that line bundle.*

Proof: The cohomology class $\delta\lambda \in H^2(M, \mathbb{Z}) \subset H^2(M, \mathbb{C})$ is represented by the cocycle $n_{\alpha\beta\gamma}$ defined in (1.19) in terms of the functions $f_{\alpha\beta}$ of (1.22); so $I(\delta\lambda) \in \mathbb{C}$ is the complex number defined through equations (1.25) and (1.27) in terms of the functions $f_{\alpha\beta}$. It follows from (1.17) that

$$d\, f_{\alpha\beta} = \frac{1}{2\pi i} \partial \log r_\alpha - \frac{1}{2\pi i} \partial \log r_\beta$$

[7] For details see the discussion in Appendix D.2.

in terms of the functions r_α of Theorem 1.2, so in (1.25) it is possible to take $\phi_\alpha = -\frac{1}{2\pi i}\partial \log r_\alpha$; then $d\phi_\alpha = -\frac{1}{2\pi i}\overline{\partial}\partial \log r_\alpha$ and consequently in (1.27)

$$I(\delta\lambda) = \int_M d\phi_\alpha = -\frac{1}{2\pi i}\int_M \overline{\partial}\partial \log r_\alpha = -c(\lambda)$$

by Theorem 1.2, which suffices for the proof.

That a negative sign appears in the formula for the characteristic class $c(\lambda)$ in the preceding theorem reflects the definition of the line bundle of a divisor $\mathfrak{d}$ as the line bundle $\zeta_{\mathfrak{d}} = \delta(-\mathfrak{d})$ in order that $c(\zeta_{\mathfrak{d}}) = \deg \mathfrak{d}$.

1.3 The Dolbeault Calculus

Further properties of holomorphic line bundles follow from a further examination of the differential operator $\overline{\partial}$, beginning with a variant of the Cauchy Integral Formula that provides a partial inverse to that differential operator. In the statement of this result the *support* of a complex valued function in the plane is the closure of the set of points at which the function is nonzero.

Theorem 1.6 *If g is a $\mathcal{C}^\infty$ function with compact support in the complex plane $\mathbb{C}$ there is a $\mathcal{C}^\infty$ function f in $\mathbb{C}$ such that $\partial f/\partial \overline{z} = g$.*

Proof: When the complex variable ζ is expressed in polar coordinates as $\zeta = re^{i\theta}$ then

$$\frac{d\overline{\zeta} \wedge d\zeta}{\zeta} = 2ie^{-i\theta} dr \wedge d\theta,$$

so this differential form remains bounded near the origin although not actually defined at the origin. Since g is $\mathcal{C}^\infty$ and has compact support the integral

$$f(z) = \frac{i}{2\pi}\int_{\mathbb{C}} g(z+\zeta)\frac{d\overline{\zeta} \wedge d\zeta}{\zeta} \tag{1.28}$$

is a $\mathcal{C}^\infty$ function of the variable z in the entire plane and can be differentiated by differentiating under the integral sign. For any fixed point $z \in \mathbb{C}$ let D be a disc centered at the origin in the plane of the variable ζ and having sufficiently large radius that $g(z+\zeta)$ vanishes for all points $\zeta \notin D$; and let D_ϵ be another disc centered at the origin in the plane of the variable ζ contained in D and having radius ϵ. Then

$$\begin{aligned}\frac{\partial f(z)}{\partial \overline{z}} &= \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial g(z+\zeta)}{\partial \overline{z}}\frac{d\overline{\zeta} \wedge d\zeta}{\zeta} \;=\; \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial g(z+\zeta)}{\partial \overline{\zeta}}\frac{d\overline{\zeta} \wedge d\zeta}{\zeta} \\ &= \lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{D\sim\overline{D}_\epsilon} d\left(\frac{g(z+\zeta)}{\zeta}d\zeta\right) \;=\; \lim_{\epsilon\to 0}\frac{-i}{2\pi}\int_{\partial D_\epsilon}\frac{g(z+\zeta)}{\zeta}d\zeta\end{aligned}$$

by Stokes's Theorem, since the differential form $g(z+\zeta)\,\zeta^{-1}d\zeta$ vanishes on ∂D. If the circle D_ϵ is parametrized by $\zeta = \epsilon e^{i\theta}$ then

$$\frac{\partial f(z)}{\partial \overline{z}} = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\theta=0}^{2\pi} g(z+\epsilon e^{i\theta})d\theta = g(z)$$

as desired, to conclude the proof.

Generally the function f of the preceding theorem does not have compact support, although of course it must be holomorphic outside the support of g. For many purposes only the following local version of this theorem is of interest.

Corollary 1.7 *If g is the germ of a $\mathcal{C}^\infty$ function at a point in the complex plane there is a germ f of a $\mathcal{C}^\infty$ function at that point such that $\partial f/\partial\overline{z} = g$.*

Proof: Any germ of a $\mathcal{C}^\infty$ function can be represented by a $\mathcal{C}^\infty$ function of compact support in the complex plane, so this assertion is an immediate consequence of the preceding theorem.

A modification of the proof of the preceding Theorem 1.6 is useful at a later point in the discussion; but it is perhaps natural to include it here as another form of solution of the $\overline{\partial}$ equation.

Theorem 1.8 *For any $\delta > 0$ there is a $\mathcal{C}^\infty$ function s in the complex plane with support in the disc $|z| \le \delta/2$ such that for any bounded open subsets $U \subset V \subset \mathbb{C}$ for which the distance from $\overline{U}$ to $\mathbb{C} \sim V$ is greater than δ and for any $\mathcal{C}^\infty$ function f in the complex plane with support contained in U there is a $\mathcal{C}^\infty$ function g in the complex plane with support contained in V for which*

$$f(z) = \frac{\partial g(z)}{\partial\overline{z}} + \frac{i}{2}\int_U f(\zeta)s(\zeta - z)d\zeta\wedge d\overline{\zeta}. \tag{1.29}$$

Proof: Choose a $\mathcal{C}^\infty$ function $r(z)$ in the complex plane such that

$$r(z) = \begin{cases} 1 & \text{if} \quad |z| \le \frac{\delta}{4} \\ \\ 0 & \text{if} \quad |z| \ge \frac{\delta}{2} \end{cases}$$

and set

$$s(z) = \begin{cases} 0 & \text{if} \qquad |z| < \frac{\delta}{4} \\ \\ -\dfrac{1}{\pi}\dfrac{\partial}{\partial\overline{z}}\left(\dfrac{r(z)}{z}\right) & \text{if} \quad 0 < |z| < \delta \\ \\ 0 & \text{if} \qquad |z| > \frac{\delta}{2}; \end{cases}$$

clearly the function $s(z)$ is $\mathcal{C}^\infty$ in the entire complex plane and its support is contained in the disc $|z| \le \delta/2$. For any $\mathcal{C}^\infty$ function f in $\mathbb{C}$ with support in U set

$$g(z) = \frac{i}{2\pi}\int_{\mathbb{C}} f(z+\zeta)r(\zeta)\frac{d\overline{\zeta}\wedge d\zeta}{\zeta}, \tag{1.30}$$

reminiscent of the integral (1.28) used in the proof of Theorem 1.6. The function g thus defined is a $\mathcal{C}^\infty$ function in the plane of the complex variable ζ and can be differentiated by differentiating under the integral sign. Furthermore if the distance from z to U exceeds $\delta/2$ then $g(z) = 0$ since $f(z+\zeta) = 0$ whenever $|\zeta| \le \delta/2$ while $r(\zeta) = 0$ whenever $\zeta \ge \delta/2$; thus the support of the function g is contained in V. For any fixed point $z \in U$ let D be a disc centered at the origin in the plane of the complex variable ζ with radius sufficiently large that $f(z+\zeta) = 0$ whenever $\zeta \notin D$, and let D_ϵ be another disc centered at the origin in the plane of the complex variable ζ with radius ϵ. Then

$$\begin{aligned}
\frac{\partial g(z)}{\partial \overline{z}} &= \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial f(z+\zeta)}{\partial \overline{z}} r(\zeta)\frac{d\overline{\zeta}\wedge d\zeta}{\zeta} = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial f(z+\zeta)}{\partial \overline{\zeta}} r(\zeta)\frac{d\overline{\zeta}\wedge d\zeta}{\zeta} \\
&= \lim_{\epsilon\to 0}\frac{i}{2\pi}\int_{D\sim D_\epsilon} \frac{\partial f(z+\zeta)}{\partial \overline{\zeta}}\frac{r(\zeta)}{\zeta} d\overline{\zeta}\wedge d\zeta \\
&= \lim_{\epsilon\to 0}\frac{i}{2\pi}\left(\int_{D\sim D_\epsilon} d\Big(f(z+\zeta)\frac{r(\zeta)}{\zeta}d\zeta\Big) - \int_{D\sim D_\epsilon} f(z+\zeta)\frac{\partial}{\partial\overline{\zeta}}\Big(\frac{r(\zeta)}{\zeta}\Big)d\overline{\zeta}\wedge d\zeta\right) \\
&= \lim_{\epsilon\to 0}\frac{-i}{2\pi}\int_{\partial D_\epsilon}\frac{f(z+\zeta)}{\zeta}d\zeta - \frac{i}{2}\int_{\mathbb{C}} f(z+\zeta)s(\zeta)d\zeta\wedge d\overline{\zeta},
\end{aligned}$$

$\blacksquare$

where Stokes's Theorem is used to replace the integral over $D \sim D_\epsilon$ with the integral over the boundary of this region and the integrand vanishes on the boundary of D while $r(\zeta) = 1$ for $\zeta \in \partial D_\epsilon$ for sufficiently small ϵ. The integral over ∂D_ϵ is just $f(z)$, as in the proof of Theorem 1.6, and the desired result then follows from a change of variable in the second integral.

For any Riemann surface M and any integers $0 \le p, q \le 1$ the vector spaces $\Gamma(U, \mathcal{E}^{(p,q)})$ of $\mathcal{C}^\infty$ differential forms of type (p,q) over open subsets $U \subset M$ clearly form a complete presheaf over M; the associated sheaf is denoted by $\mathcal{E}^{(p,q)}$, and the spaces $\Gamma(U, \mathcal{E}^{(p,q)})$ can be identified with the spaces of sections of this sheaf. The differential operator $\overline{\partial}$ is a homomorphism

$$\overline{\partial} : \Gamma(U, \mathcal{E}^{(p,0)}) \longrightarrow \Gamma(U, \mathcal{E}^{(p,1)}) \tag{1.31}$$

between these presheaves of complex vector spaces so induces a sheaf homomorphism

$$\overline{\partial} : \mathcal{E}^{(p,0)} \longrightarrow \mathcal{E}^{(p,1)} \tag{1.32}$$

between the associated sheaves. If $p = 0$ so $\mathcal{E}^{(0,0)} = \mathcal{E}$ it follows from the Cauchy-Riemann equations that the kernel of the sheaf homomorphism (1.32) is the sheaf $\mathcal{O}$ of germs of holomorphic functions on M, and it follows from Corollary 1.7 that for any germ $\phi = g\,d\overline{z} \in \mathcal{E}^{(0,1)}$ there is a germ $f \in \mathcal{E}$ such that $\partial f/\partial\overline{z} = g$ and hence that $\overline{\partial} f = \phi$; thus there is the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O} \xrightarrow{\iota} \mathcal{E} \xrightarrow{\overline{\partial}} \mathcal{E}^{(0,1)} \longrightarrow 0 \tag{1.33}$$

over M, in which ι is the natural inclusion homomorphism. If $p = 1$ the kernel of the sheaf homomorphism (1.32) is the sheaf of germs of differential forms $\phi = f\,dz$ of type $(1,0)$ for which $\partial f/\partial\overline{z} = 0$, so for which the function f is holomorphic; these are called germs of *holomorphic differential forms*, often also called germs of *holomorphic abelian differentials* or of *abelian differentials of the first kind*, and the sheaf of such germs is denoted by $\mathcal{O}^{(1,0)}$. It follows again from Corollary 1.7 that for any germ $\phi = g\,dz\wedge d\overline{z} \in \mathcal{E}^{(1,1)}$ there is a germ $f \in \mathcal{E}$ such that $-(\partial f/\partial\overline{z}) = g$ and hence $\overline{\partial}(-f\,dz) = -g\,d\overline{z}\wedge dz = g\,dz\wedge d\overline{z} = \phi$; thus there is also the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}^{(1,0)} \xrightarrow{\ \iota\ } \mathcal{E}^{(1,0)} \xrightarrow{\ \overline{\partial}\ } \mathcal{E}^{(1,1)} \longrightarrow 0 \tag{1.34}$$

over M, in which ι is the natural inclusion homomorphism.

More generally suppose that λ is a holomorphic line bundle over the Riemann M and that λ is described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ in terms of a covering of M by open coordinate neighborhoods U_α. The vector spaces $\Gamma(U,\mathcal{E}(\lambda))$, $\Gamma(U,\mathcal{O}(\lambda))$ and $\Gamma(U,\mathcal{M}(\lambda))$ of $\mathcal{C}^\infty$, holomorphic and meromorphic cross-sections of λ in the open subsets $U \subset M$ clearly form complete presheaves of abelian groups over M; the associated sheaves are denoted by $\mathcal{E}(\lambda)$, $\mathcal{O}(\lambda)$ and $\mathcal{M}(\lambda)$ respectively, and the spaces $\Gamma(U,\mathcal{E}(\lambda))$, $\Gamma(U,\mathcal{O}(\lambda))$ and $\Gamma(U,\mathcal{M}(\lambda))$ of cross-sections of λ over U can be identified with the spaces of sections of the corresponding sheaves over U. The vector spaces $\Gamma(U,\mathcal{E}^{(p,q)}(\lambda))$ consisting of $\mathcal{C}^\infty$ differential forms ϕ_α of type (p,q) in the intersections $U \cap U_\alpha$ such that $\phi_\alpha = \lambda_{\alpha\beta}\,\phi_\beta$ in the $U \cap U_\alpha \cap U_\beta$ also form a complete presheaf of abelian groups over M; the associated sheaf is denoted by $\mathcal{E}^{(p,q)}(\lambda)$, and the vector space $\Gamma(U,\mathcal{E}^{(p,q)}(\lambda))$ can be identified with the space of sections of the sheaf $\mathcal{E}^{(p,q)}(\lambda)$ over U. Of course $\mathcal{E}(\lambda) = \mathcal{E}^{(0,0)}(\lambda)$ so the sheaf $\mathcal{E}(\lambda)$ also can be considered as a sheaf of germs of differential forms that are cross-sections of the line bundle λ. If $\phi = \{\phi_\alpha\} \in \Gamma(U,\mathcal{E}^{(p,0)}(\lambda))$ then $\overline{\partial}\phi_\alpha = \lambda_{\alpha\beta}\,\overline{\partial}\phi_\beta$ in each intersection $U_\alpha \cap U_\beta$ since the coordinate transition functions $\lambda_{\alpha\beta}$ are holomorphic; thus the differential operator $\overline{\partial}$ describes homomorphisms

$$\overline{\partial} : \Gamma(U,\mathcal{E}^{(p,0)}(\lambda)) \longrightarrow \Gamma(U,\mathcal{E}^{(p,1)}(\lambda)) \tag{1.35}$$

of presheaves, which induce homomorphsms

$$\overline{\partial} : \mathcal{E}^{(p,0)}(\lambda) \longrightarrow \mathcal{E}^{(p,1)}(\lambda) \tag{1.36}$$

of the associated sheaves. The kernel of this homomorphism for $p = 0$ is the sheaf $\mathcal{O}(\lambda)$ of germs of holomorphic cross-sections of the line bundle λ and for $p = 1$ is the sheaf $\mathcal{O}^{(1,0)}(\lambda)$ of germs of holomorphic differential forms that are cross-sections of the line bundle λ. The homomorphisms (1.35) reduce to the homomorphisms 1.31) in sufficiently small open neighborhoods of any points of M, so the exact sequences of sheaves (1.34) naturally induce the exact sequences of sheaves

$$0 \longrightarrow \mathcal{O}^{(p,0)}(\lambda) \xrightarrow{\ \iota\ } \mathcal{E}^{(p,0)}(\lambda) \xrightarrow{\ \overline{\partial}\ } \mathcal{E}^{(p,1)}(\lambda) \longrightarrow 0 \tag{1.37}$$

for $p = 0$ or 1, in which ι is the inclusion mapping.

The same result clearly holds for differential forms that are cross-sections of a holomorphic vector bundle of any rank over a Riemann surface M, since such cross-sections are described by finite sets of exact sequences (1.33) or (1.34). From the exact sequence of sheaves (1.37) there follows as usual an associated exact sequence of cohomology groups, leading to the following result.

Theorem 1.9 (Theorem of Dolbeault) *If λ is a holomorphic line bundle over an arbitrary Riemann surface M then*

$$\begin{aligned} H^1(M, \mathcal{O}(\lambda)) &\cong \frac{\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))}{\overline{\partial}\Gamma(M, \mathcal{E}^{(0,0)}(\lambda))}, \\ H^q(M, \mathcal{O}(\lambda)) &= 0 \quad \textit{for} \quad q \geq 2. \end{aligned}$$

Proof: The long exact cohomology sequence associated to the exact sequence of sheaves (1.37) for $p = 0$ includes the segment

$$\Gamma(M, \mathcal{E}(\lambda)) \xrightarrow{\overline{\partial}} \Gamma(M, \mathcal{E}^{(0,1)}(\lambda)) \xrightarrow{\delta} H^1(M, \mathcal{O}(\lambda)) \xrightarrow{\iota} H^1(M, \mathcal{E}(\lambda));$$

and since $\mathcal{E}(\lambda)$ is a fine sheaf $H^1(M, \mathcal{E}(\lambda)) = 0$ so the homomorphism δ yields the first isomorphism of the theorem. The same exact cohomology sequence also includes the segments

$$H^{q-1}(M, \mathcal{E}^{(0,1)}(\lambda)) \xrightarrow{\delta} H^q(M, \mathcal{O}(\lambda)) \xrightarrow{\iota} H^q(M, \mathcal{E}(\lambda))$$

for all $q \geq 1$; since $\mathcal{E}^{(0,1)}(\lambda)$ is a fine sheaf $H^{q-1}(M, \mathcal{E}^{(0,1)})) = H^q(M, \mathcal{E}(\lambda)) = 0$ for all $q \geq 2$, from which it follows that $H^q(M, \mathcal{O}(\lambda)) = 0$ for all $q \geq 2$. That suffices to conclude the proof.

The Theorem of Dolbeault also holds for holomorphic vector bundles as well as for holomorphic line bundles, with essentially the same proof. Further information about the first cohomology groups $H^1(M, \mathcal{O}(\lambda))$ can be obtained by strengthening Theorem 1.6 as follows.

Theorem 1.10 *If g is a $\mathcal{C}^\infty$ function in an open subset $U \subseteq \mathbb{C}$ there is a $\mathcal{C}^\infty$ function f in U such that $\partial f / \partial \overline{z} = g$.*

Proof: It is sufficient to prove the theorem for a connected set U, so that will be assumed in the proof. Select a sequence of connected open subsets $U_n \subset U$ such that (i) $\overline{U}_n$ is compact, (ii) $\overline{U}_n \subset U_{n+1}$, (iii) $\bigcup_{n=1}^{\infty} U_n = U$, (iv) any function holomorphic in U_{n-1} can be approximated uniformly in $\overline{U}_{n-2}$ by functions holomorphic in U_n. The existence of such a sequence of subsets is a standard result in complex analysis, rather like the Runge theorem.[8] It will then be demonstrated by induction on n that there is a sequence of functions f_n such

[8] See for instance the discussion in E. Hille, *Analytic Function Theory*, vol. II, pages 299 ff., (Ginn, 1962), or in W. Rudin, *Real and Complex Analysis*, pages 255 ff., (McGraw Hill, 1966).

that (v) f_n is a $\mathcal{C}^\infty$ function in U_n, (vi) $\partial f_n/\partial\overline{z} = g$ in U_n, (vii) $f_n(z) - f_{n-1}(z)$ is holomorphic in U_{n-1}, (viii) $|f_n(z) - f_{n-1}(z)| < 2^{-n}$ for all $z \in \overline{U}_{n-2}$. For this purpose note from Theorem 1.6 that for any index n there is a $\mathcal{C}^\infty$ function h_n on the set U such that $\partial h_n/\partial\overline{z} = g$ in U_n; for it is possible to modify the function g outside U_n so that it has compact support in U. If $n = 1$ take $f_1 = h_1$, and there is nothing further to show. If $n \geq 2$ suppose that the functions $f_1, \ldots, f_{n-1}$ have been determined so that they satisfy (v), (vi), (vii) and (viii). Both h_n and f_{n-1} are $\mathcal{C}^\infty$ functions in U_{n-1} and $\partial(h_n - f_{n-1})/\partial\overline{z} = g - g = 0$ in U_{n-1} so $h_n - f_{n-1}$ actually is holomorphic in U_{n-1}. By (iv) there exists a holomorphic function g_n in U_n such that $|h_n - f_{n-1} - g_n| < 2^{-n}$ in $\overline{U}_{n-2}$. The functions $f_1, \ldots, f_{n-1}, f_n = h_n - g_n$ then also satisfy (v), (vi), (vii) and (viii), which completes the induction. The next step is to show that the sequence f_n converges to a $\mathcal{C}^\infty$ function f in U and that this limit has the desired properties. If $k \geq n+2$ it follows from (viii) that $|f_k(z) - f_{k-1}(z)| < 2^{-k}$ for all $z \in U_n$ and from (vii) that the functions $f_k(z) - f_{k-1}(z)$ are holomorphic in U_n; so the series $\sum_{k=n+2}^{\infty}(f_k(z) - f_{k-1}(z))$ is a uniformly convergent series of holomorphic functions in U_n. If $m \geq n+2$ and $z \in U_n$

$$f_m(z) = f_{n+1}(z) + \sum_{k=n+2}^{m} (f_k(z) - f_{k-1}(z))\,;$$

but then sequence $f_m(z)$ converges uniformly in U_n to the function

$$f(z) = f_{n+1}(z) + \sum_{k=n+2}^{\infty} (f_k(z) - f_{k-1}(z))$$

that differs from $f_{n+1}(z)$ by a holomorphic function, and consequently $f(z)$ is a $\mathcal{C}^\infty$ function in U_n and $\partial f/\partial\overline{z} = \partial f_{n+1}/\partial\overline{z} = g$ in U_n. That is true for all sets U_n, and that suffices to conclude the proof.

Corollary 1.11 *If $U \subset \mathbb{C}$ is an open subset of the complex plane and λ is a holomorphic line bundle over U then λ is analytically trivial and $H^p(U, \mathcal{O}(\lambda)) = 0$ for all $p > 0$.*

Proof: Consider the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \stackrel{\iota}{\longrightarrow} \mathcal{O} \stackrel{e}{\longrightarrow} \mathcal{O}^* \longrightarrow 0 \tag{1.38}$$

over the subset $U \subset \mathbb{C}$ in which ι is the natural inclusion mapping of the sheaf $\mathbb{Z}$ of locally constant integer-valued functions to the sheaf $\mathcal{O}$ of germs of holomorphic functions and e is the homomorphism that associates to a germ $f \in \mathcal{O}$ the germ $e(f) = \exp 2\pi i f \in \mathcal{O}^*$ of a nowhere vanishing holomorphic function; this is just the holomorphic version of the exact sheaf sequences (1.18) and (1.20). The associated exact cohomology sequence contains the segment

$$H^1(U, \mathcal{O}) \stackrel{e}{\longrightarrow} H^1(U, \mathcal{O}^*) \stackrel{\delta}{\longrightarrow} H^2(U, \mathbb{Z}). \tag{1.39}$$

It follows from the preceding theorem that for any differential form $\phi = g\,d\overline{z} \in \Gamma(U, \mathcal{E}^{(0,1)})$ there is a function $f \in \Gamma(U, \mathcal{E})$ for which $\partial f/\partial\overline{z} = g$ and hence $\overline{\partial} f = g\,d\overline{z} = \phi$; consequently $H^1(U, \mathcal{O}) = 0$ by the Theorem of Dolbault, Theorem 1.9, for the special case of the trivial line bundle. Furthermore $H^2(U, \mathbb{Z}) = 0$ for an arbitrary open subset of the complex plane. It therefore follows from the exact sequence (1.39) that $H^1(U, \mathcal{O}^*) = 0$, which is just the condition that any holomorphic line bundle over U is analytically trivial. Thus if λ is a holomorphic line bundle over U then $\lambda \cong 1$, so by what has just been proved $H^1(U, \mathcal{O}(\lambda)) = H^1(U, \mathcal{O}) = 0$. Of course $H^p(U, \mathcal{O}(\lambda)) = 0$ for all $p > 1$ by the Dolbeault Theorem again, and that suffices to conclude the proof.

An application of the preceding corollary yields a method for calculating the cohomology groups of Riemann surfaces with coefficients in the sheaf $\mathcal{O}(\lambda)$ of germs of holomorphic cross-sections of a holomorphic line bundle.

Theorem 1.12 (Theorem of Leray) *If $\mathfrak{U}$ is a covering of the Riemann surface M by open coordinate neighborhoods then for any holomorphic line bundle λ over M and any integer $p \geq 0$ the natural homomorphism*

$$\iota^*_{\mathfrak{U}} : H^p(\mathfrak{U}, \mathcal{O}(\lambda)) \longrightarrow H^p(M, \mathcal{O}(\lambda))$$

is an isomorphism.

Proof: Since any intersection of coordinate neighborhoods in the covering $\mathfrak{U}$ is again a coordinate neighborhood, so can be viewed as an open subset of $\mathbb{C}$, it follows from Corollary1.11 that $H^p(U_{\alpha_1} \cap \cdots \cap U_{\alpha_q}, \mathcal{O}(\lambda)) = 0$ for all $p > 0$. Thus the covering $\mathfrak{U}$ is a Leray covering of the Riemann surface M for the sheaf $\mathcal{O}(\lambda)$, so by the general Theorem of Leray as discussed in Appendix C.2 the natural homomorphisms $\iota^*_{\mathfrak{U}}$ are isomorphisms for all indices $p \geq 0$. That suffices for the proof.

The identification $H^1(M, \mathcal{O}(\lambda)) \cong H^1(\mathfrak{U}, \mathcal{O}(\lambda))$ is very convenient for explicit calculations in these cohomology groups; it is worth examining this in more detail, since such calculations will be used repeatedly in the subsequent discussion. Suppose that the line bundle λ is described by a coordinate bundle $\{U_\alpha,\, \lambda_{\alpha\beta}\}$ in terms of a covering of the Riemann surface M by open coordinate neighborhoods U_α. The local coordinates in the bundle λ describe any point in λ over the coordinate neighborhood U_α as a pair (p, f_α) where $p \in U_\alpha$ is the projection of the point to the base space M and $f_\alpha \in \mathbb{C}$ is the fibre coordinate of the point. Thus if $s \in \Gamma(M, \mathcal{O}(\lambda))$ is a holomorphic cross-section of the bundle λ then for any point $p \in U_\alpha$ the value $s(p)$ of the cross-section s at the point p is described by the pair $\big(p, f_\alpha(p)\big)$ in terms of the fibre coordinate $f_\alpha(p) \in \mathbb{C}$; and the value $f_\alpha(p)$ is a holomorphic function of the point $p \in U_\alpha$, so the cross-section s over the neighborhood U_α can be identified with the holomorphic function f_α in the coordinate neighborhood $U_\alpha \subset M$. A cochain $s \in C^p(\mathfrak{U}, \mathcal{O}(\lambda))$ consists of sections $s_{\alpha_0 \cdots \alpha_p} \in \Gamma(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{O}(\lambda))$ for each ordered set of $p+1$ open subsets $U_{\alpha_0}, \ldots, U_{\alpha_p}$ of the covering $\mathfrak{U}$, where these sections are skew-symmetric in the indices $\alpha_0,\ldots,\alpha_p$; and the section $s_{\alpha_0\cdots\alpha_p}$ can be identified with a holomorphic function $f_{\alpha_0\cdots\alpha_p}$ in the intersection $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}$ in terms of the fibre coordinate over U_{α_p} when the intersection is viewed as a subset of the last coordinate neighborhood U_{α_p}. *This identification will be used consistently in the subsequent discussion; thus cochains in $C^p(\mathfrak{U},\mathcal{O}(\lambda))$ will be identified without further comment as collections of holomorphic functions in the intersections $U_{\alpha_0}\cap\cdots\cap U_{\alpha_p}\subset M$ in terms of the fibre coordinates in λ over U_{α_p}.* Some care must be taken when using this identification though. For example the skew-symmetry of the sections $s_{\alpha_0\cdots\alpha_p}$ in the indices $\alpha_0,\ldots,\alpha_p$ does not mean that the holomorphic functions $f_{\alpha_0\cdots\alpha_p}$ that represent these sections are skew symmetric in these indices. The section $s_{\alpha_0\cdots\alpha_p}$ is identified with a holomorphic function $f_{\alpha_0\cdots\alpha_p}$ in terms of the fibre coordinates of λ over U_{α_p}, while for any permutation $\pi\in\mathfrak{S}_{p+1}$ of the integers $0,1,\ldots,p$ the section $s_{\alpha_{\pi 0}\cdots\alpha_{\pi p}}$ is identified with a holomorphic function $f_{\alpha_{\pi 0}\cdots\alpha_{\pi p}}$ in terms of the fibre coordinates of λ over the subset $U_{\alpha_{\pi p}}$; when the identity $s_{\alpha_{\pi 0}\cdots\alpha_{\pi p}}=(\operatorname{sign}\pi)\cdot s_{\alpha_0\cdots\alpha_p}$ is expressed in terms of the fibre coordinate over the coordinate neighborhood U_{α_p} it takes the form

$$f_{\alpha_0\cdots\alpha_p}=(\operatorname{sign}\pi)\,\lambda_{\alpha_p\,\alpha_{\pi p}}\;f_{\alpha_{\pi_0}\cdots\alpha_{\pi p}} \tag{1.40}$$

since the fibre coordinates over a point in $U_{\alpha_p}\cap U_{\alpha_{\pi p}}$ are related by $f_{\alpha_p}=\lambda_{\alpha_p\,\alpha_{\pi p}}f_{\alpha_{\pi p}}$. Thus a 1-cochain $s\in C^1(\mathfrak{U},\mathcal{O}(\lambda))$ consists of sections $s_{\alpha\beta}\in\Gamma(U_\alpha\cap U_\beta,\mathcal{O}(\lambda))$ that are skew-symmetric in the indices α, β and is identified with a collection of holomorphic functions $f_{\alpha\beta}\in\Gamma(U_\alpha\cap U_\beta,\mathcal{O})$ satisfying the skew-symmetry condition

$$f_{\alpha\beta}=-\lambda_{\beta\alpha}\;f_{\beta\alpha}\quad\text{in}\quad U_\alpha\cap U_\beta. \tag{1.41}$$

A 0-cochain $s\in C^0(\mathfrak{U},\mathcal{O}(\lambda))$ consists just of sections $s_\alpha\in\Gamma(U_\alpha,\mathcal{O}(\lambda))$ and is identified with a collection of holomorphic functions $f_\alpha\in\Gamma(U_\alpha,\mathcal{O})$; but there is of course no skew-symmetry involved in this case.

The coboundary of a 0-cochain $s\in C^0(\mathfrak{U},\mathcal{O}(\lambda))$ is the 1-cochain $(\delta s)_{\alpha\beta}=s_\beta-s_\alpha$; and if the 0-cochain is identified with a collection of holomorphic functions $f_\alpha\in\Gamma(U_\alpha,\mathcal{O})$ and its coboundary δs is identified with a collection of holomorphic functions $(\delta f)_{\alpha\beta}\in\Gamma(U_\alpha\cap U_\beta,\mathcal{O})$ then in terms of the fibre coordinates over the coordinate neighborhood U_β

$$(\delta f)_{\alpha\beta}=f_\beta-\lambda_{\beta\alpha}f_\alpha\quad\text{in}\quad U_\alpha\cap U_\beta. \tag{1.42}$$

In this case $(\delta f)_{\alpha\beta}=-\lambda_{\beta\alpha}(f_\alpha-\lambda_{\alpha\beta}f_\beta)=-\lambda_{\beta\alpha}(\delta f)_{\beta\alpha}$, so the skew-symmetry condition (1.41) holds automatically. The 0-cochain s thus is a 0-cocycle if and only if the functions f_α satisfy

$$f_\alpha=\lambda_{\alpha\beta}f_\beta\quad\text{in}\quad U_\alpha\cap U_\beta; \tag{1.43}$$

that is just the condition that the functions f_α are a cross-section of the line bundle λ, yielding the usual identification $Z^0(\mathfrak{U},\mathcal{O}(\lambda))\cong\Gamma(M,\mathcal{O}(\lambda))$. Correspondingly the coboundary of a 1-cochain $s\in C^1(\mathfrak{U},\mathcal{O}(\lambda))$ is the 2-cochain

$(\delta s)_{\alpha\beta\gamma} = s_{\beta\gamma} - s_{\alpha\gamma} + s_{\alpha\beta}$; and if the 1-cochain s is identified with a collection of holomorphic functions $f_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O})$ and its coboundary δs is identified with a collection of holomorphic functions $(\delta f)_{\alpha\beta\gamma} \in \Gamma(U_\alpha \cap U_\beta \cap U_\gamma, \mathcal{O})$ then in terms of the fibre coordinates over the coordinate neighborhood U_γ

$$(\delta f)_{\alpha\beta\gamma} = f_{\beta\gamma} - f_{\alpha\gamma} + \lambda_{\gamma\beta} f_{\alpha\beta} \quad \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma. \tag{1.44}$$

The 1-cochain s thus is a 1-cocycle if and only if the functions $f_{\alpha\beta}$ satisfy

$$f_{\alpha\gamma} = \lambda_{\gamma\beta} f_{\alpha\beta} + f_{\beta\gamma} \quad \text{in} \quad U_\alpha \cap U_\beta \cap U_\gamma. \tag{1.45}$$

1.4 Finite Dimensionality

A fundamental result in the theory of compact Riemann surfaces is that the cohomology groups $H^p\big(M, \mathcal{O}(\lambda)\big)$ of a compact Riemann surface M are finite-dimensional complex vector spaces for any holomorphic line bundle λ over M. For $p \ge 2$ these cohomology groups are trivial by Dolbeault's theorem, Theorem 1.9; and if $c(\lambda) < 0$ these groups are trivial by Corollary 1.3. For the case $p = 0$ it is even possible to establish quite simply the following upper bound for these groups[9].

Theorem 1.13 *If λ is a holomorphic line bundle over a compact Riemann surface M and if $c(\lambda) > 0$ then*

$$\dim \Gamma\big(M, \mathcal{O}(\lambda)\big) \le c(\lambda) + 1. \tag{1.46}$$

Proof: Suppose to the contrary that λ is a holomorphic line bundle of characterstic class $c(\lambda) = r \ge 0$ for which $\dim \Gamma(M, \big(\mathcal{O}(\lambda)\big) > r + 1$; and consider the divisor $\mathfrak{d} = r \cdot a$ for a point $a \in M$. Let

$$\phi : \Gamma(M, \big(\mathcal{O}(\lambda)\big) \longrightarrow \mathbb{C}^{r+1} \tag{1.47}$$

be the linear mapping that associates to an arbitrary holomorphic cross-section $f \in \Gamma(M, \big(\mathcal{O}(\lambda)\big)$ the vector $\phi(f) = \Big(f(a), f'(a), \ldots, f^{(r)}(a)\Big) \in \mathbb{C}^{r+1}$, where the derivatives are taken with respect to a local coordinate z at the point $a \in M$. If $\phi(f) = 0$ for a nontrivial cross-section $f \in \Gamma(M, \big(\mathcal{O}(\lambda)\big)$ then f vanishes at least to the order $r + 1$ at the point a, and consequently $c(\lambda) \ge r + 1$ in view of the definition (1.14) of the characteristic class of the bundle λ. That is a contradiction, so the kernel of the linear mapping (1.47) is trivial; that means that the linear mapping f is injective, hence that $\dim \Gamma(M, \big(\mathcal{O}(\lambda)\big) \le r+1$, and that suffices for the proof.

The proof that the vector spaces $H^1(M, \mathcal{O}(\lambda))$ are finite-dimensional when the Riemann surface M is compact proceeds by introducing an appropriate topology on the space of cochains for a fixed coordinate covering $\mathfrak{U}$. There

[9] See the book *An Introduction to Riemann Surfaces* by Terence Napier and Mohan Ramachandran.

are various ways of doing this; perhaps the simplest is to use a Hilbert space topology, even though it is not intrinsically defined. It is a standard result in complex analysis[10] that the subspace

$$\Gamma_2(U,\mathcal{O}) = \Big\{ f \in \Gamma(U,\mathcal{O}) \ \Big| \ \int_U |f(z)|^2 dx \wedge dy < \infty \Big\}$$

of square-integrable holomorphic functions on an open subset $U \subset \mathbb{C}$ is a Hilbert space with the inner product

$$(f,g) = \int_U f(z)\overline{g(z)}dx \wedge dy$$

and the corresponding norm $\|f\|^2 = (f,f)$. Furthermore if U and V are open sets and $U \subset V$ then the restriction mapping

$$\rho_{UV} : \Gamma_2(V,\mathcal{O}) \longrightarrow \Gamma_2(U,\mathcal{O})$$

is a bounded linear mapping between these two Hilbert spaces; if $\overline{U} \subset V$ is compact this restriction mapping is even a compact mapping by Vitali's Theorem, so the image of any bounded subset of $\Gamma_2(V,\mathcal{O})$ has compact closure in the space $\Gamma_2(U,\mathcal{O})$.

Theorem 1.14 (Finite Dimensionality Theorem) *If λ is a holomorphic line bundle over a compact Riemann surface M then the cohomology groups $H^0(M,\mathcal{O}(\lambda))$ and $H^1(M,\mathcal{O}(\lambda))$ are finite-dimensional complex vector spaces.*

Proof: Since M is compact there is a finite covering $\mathfrak{W}$ of M by open coordinate neighborhoods W_α. Choose open subsets $U_\alpha \subset W_\alpha$ and $V_\alpha \subset W_\alpha$ that form coverings $\mathfrak{U}$ and $\mathfrak{V}$ of M where $\overline{U}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset W_\alpha$, so that $\overline{U}_\alpha$ and $\overline{V}_\alpha$ are compact. If μ_1^* and μ_2^* are the group homomorphisms (C.15) induced by the refining mappings corresponding to the inclusions $V_\alpha \subset W_\alpha$ and $U_\alpha \subset V_\alpha$ respectively and $\iota_{\mathfrak{U}}^*$ is the natural homomorphism (C.16) from the cohomology group of the covering $\mathfrak{U}$ to the cohomology of the space M then by the Theorem of Leray, Theorem (1.12), the homomorphisms $\iota_{\mathfrak{U}}^*$, $\iota_{\mathfrak{V}}^* = \iota_{\mathfrak{U}}^* \circ \mu_2^*$ and $\iota_{\mathfrak{W}}^* = \iota_{\mathfrak{U}}^* \circ \mu_2^* \circ \mu_1^*$ from the sequence

$$H^p(\mathfrak{W},\mathcal{O}(\lambda)) \xrightarrow{\mu_1^*} H^p(\mathfrak{V},\mathcal{O}(\lambda)) \xrightarrow{\mu_2^*} H^p(\mathfrak{U},\mathcal{O}(\lambda)) \xrightarrow{\iota_{\mathfrak{U}}^*} H^p(M,\mathcal{O}(\lambda)) \tag{1.48}$$

are isomorphisms; consequently the homomorphisms μ_1^* and μ_2^* also are isomorphisms. In the identification

$$C^p(\mathfrak{V},\mathcal{O}(\lambda)) \cong \bigoplus_\alpha \Gamma(V_{\alpha_0} \cap \cdots \cap V_{\alpha_p},\mathcal{O})$$

of p-cochains for the covering $\mathfrak{V}$ with collections of holomorphic functions in the finitely many intersections of $p+1$-tuples of sets from the covering $\mathfrak{V}$ let

[10] See for instance E. Hille, *Analytic Function Theory*, vol.II, pages 325 ff, (Ginn, 1962).

$C_2^p(\mathfrak{V}, \mathcal{O}(\lambda)) \subset C^p(\mathfrak{V}, \mathcal{O}(\lambda))$ be the subgroup consisting of cochains that are identified with square integrable holomorphic functions, so that

$$C_2^p(\mathfrak{V}, \mathcal{O}(\lambda)) \cong \bigoplus_{\alpha} \Gamma_2(V_{\alpha_0} \cap \cdots \cap V_{\alpha_p}, \mathcal{O});$$

this exhibits $C_2^p(\mathfrak{V}, \mathcal{O}(\lambda))$ as a finite direct sum of Hilbert spaces and hence as a Hilbert space itself. The coordinate transition functions $\lambda_{\alpha\beta}$ of the line bundle λ are holomorphic in $W_\alpha \cap W_\beta$ so are bounded in $V_\alpha \cap V_\beta$; hence the coboundary mappings (1.42) take square integrable cochains into square integrable cochains and are bounded linear mappings $\delta : C_2^p(\mathfrak{V}, \mathcal{O}(\lambda)) \longrightarrow C_2^{p+1}(\mathfrak{V}, \mathcal{O}(\lambda))$ between Hilbert spaces. The kernels $Z_2^p(\mathfrak{V}, \mathcal{O}(\lambda))$ of these mappings consequently are closed subspaces of a Hilbert space so also are Hilbert spaces. The same considerations of course also apply to the covering $\mathfrak{U}$.

The square integrable first cohomology group is defined by

$$H_2^1(\mathfrak{V}, \mathcal{O}(\lambda)) = \frac{Z_2^1(\mathfrak{V}, \mathcal{O}(\lambda))}{\delta C_2^0(\mathfrak{V}, \mathcal{O}(\lambda))};$$

this is a well defined complex vector space but cannot be viewed as a Hilbert space since the subspace $\delta C_2^0(\mathfrak{V}, \mathcal{O}(\lambda)) \subset Z_2^1(\mathfrak{V}, \mathcal{O}(\lambda))$ has not been shown to be a closed linear subspace, although that will follow from the conclusion of the finite dimensionality theorem. The natural inclusion of square integrable cochains into all cochains induces a linear mapping

$$\rho_{\mathfrak{V}}^* : H_2^1(\mathfrak{V}, \mathcal{O}(\lambda)) \longrightarrow H^1(\mathfrak{V}, \mathcal{O}(\lambda)) \tag{1.49}$$

between these two complex vector spaces, which will be shown to be an isomorphism. For this purpose first suppose that $f_{\alpha\beta} \in Z_2^1(\mathfrak{V}, \mathcal{O}(\lambda))$ is a cocycle that is cohomologous to zero in $Z^1(\mathfrak{V}, \mathcal{O}(\lambda))$; thus $f_{\alpha\beta}$ is the coboundary of a cochain $f_\alpha \in C^0(\mathfrak{V}, \mathcal{O}(\lambda))$ satisfying (1.42). Any point $p \in \partial V_\alpha$ is contained in some set V_β. The function $f_{\alpha\beta}$ is square-integrable in $V_\alpha \cap V_\beta$ by assumption, the function f_β is continuous in a full open neighborhood of the point p since $p \in V_\beta$, and the function $\lambda_{\alpha\beta}$ is holomorphic in $W_\alpha \cap W_\beta$ and hence is bounded in a full open neighborhood of the point p; consequently the function f_α is also square integrable in an open neighborhood of the point p in V_α. The closure $\overline{V}_\alpha$ is compact, so finitely many of these neighborhoods cover the boundary of that set, and it follows that f_α is square integrable on the full set V_α; consequently $f_\alpha \in C_2^0(\mathfrak{V}, \mathcal{O}(\lambda))$, which shows that $\rho_{\mathfrak{V}}^*$ is injective. Since $\overline{V}_\alpha \subset W_\alpha$ and $\overline{V}_\alpha$ is compact the isomorphism μ_1^* in (1.48) factors through square-integrable cohomology so can be written as the composition $\mu_1^* = \rho_{\mathfrak{V}}^* \circ \nu_1^*$ of the linear mappings ν_1^* and $\rho_{\mathfrak{V}}^*$ in the sequence

$$H^1(\mathfrak{W}, \mathcal{O}(\lambda)) \xrightarrow{\nu_1^*} H_2^1(\mathfrak{V}, \mathcal{O}(\lambda)) \xrightarrow{\rho_{\mathfrak{V}}^*} H^1(\mathfrak{V}, \mathcal{O}(\lambda));$$

and since $\rho_{\mathfrak{V}}^*$ has just been shown to be an injection and the composition $\rho_{\mathfrak{V}}^* \circ \nu_1^* = \mu_1^*$ is an isomorphism it follows that $\rho_{\mathfrak{V}}^*$ is also an isomorphism as asserted.

The arguments just applied to the covering $\mathfrak{V}$ also can be applied to the covering $\mathfrak{U}$; hence the linear mapping

$$\rho_{\mathfrak{U}}^* : H_2^1(\mathfrak{U}, \mathcal{O}(\lambda)) \longrightarrow H^1(\mathfrak{U}, \mathcal{O}(\lambda)) \tag{1.50}$$

analogous to (1.49) is also an isomorphism. To conclude the proof of the theorem it then suffices to show that $H_2^1(\mathfrak{U}, \mathcal{O}(\lambda))$ is a finite dimensional complex vector space. In view of the isomorphisms (1.49) and (1.50) the isomorphism μ_2^* in (1.48) induces an isomorphism

$$\mu_{2H}^* : H_2^1(\mathfrak{V}, \mathcal{O}(\lambda)) \longrightarrow H_2^1(\mathfrak{U}, \mathcal{O}(\lambda)) \tag{1.51}$$

between these two complex vector spaces. Moreover since $\overline{U}_\alpha \subset V_\alpha$ the homomorphism

$$\mu_{2Z}^* : Z_2^1(\mathfrak{V}, \mathcal{O}(\lambda)) \longrightarrow Z_2^1(\mathfrak{U}, \mathcal{O}(\lambda)) \tag{1.52}$$

corresponding to this inclusion is a compact linear mapping between these two Hilbert spaces. Consider then the Hilbert space

$$A = C_2^0(\mathfrak{U}, \mathcal{O}(\lambda)) \oplus Z_2^1(\mathfrak{V}, \mathcal{O}(\lambda))$$

and the bounded linear mapping

$$(\delta, \mu_{2Z}^*) : A \longrightarrow Z_2^1(\mathfrak{U}, \mathcal{O}(\lambda))$$

defined by $(\delta, \mu_{2Z}^*)(f, g) = \delta f + \mu_{2Z}^*(g)$. The mapping (δ, μ_{2Z}^*) is surjective; for since (1.51) is an isomorphism of vector spaces any square-integrable cocycle of the covering $\mathfrak{U}$ must be cohomologous to a square-integrable cocycle coming from the covering $\mathfrak{V}$. The difference $(\delta, \mu_{2Z}^*) - (0, \mu_{2Z}^*)$ is just the coboundary mapping

$$\delta : C_2^0(\mathfrak{U}, \mathcal{O}(\lambda)) \longrightarrow Z_2^1(\mathfrak{U}, \mathcal{O}(\lambda)),$$

so the theorem is a consequence of the following general lemma[11].

Lemma 1.15 *If X and Y are Hilbert spaces and if both $\phi : X \longrightarrow Y$ and $\psi : X \longrightarrow Y$ are bounded linear mappings where ϕ is surjective and ψ is compact then $Y/(\phi - \psi)(X)$ is finite dimensional.*

Proof: Let $\phi^* : Y \longrightarrow X$ and $\psi^* : Y \longrightarrow X$ be the adjoint mappings to ϕ and ψ respectively; then ϕ^* is an injective mapping with closed range and ψ^* is a compact mapping. The first step is to show that the kernel K of the mapping $\phi^* - \psi^*$ is a finite-dimensional subspace of Y. For this purpose suppose that $\{y_n\}$ is any bounded sequence of elements of K. Since ψ^* is compact then after passing to a subsequence if necessary the sequence $\psi^*(y_n)$ converges; consequently the sequence $\phi^*(y_n) = \psi^*(y_n)$ also converges. Since ϕ^* is injective and has closed range it is a homeomorphism between Y and its range; hence

[11]For the properties of Hilbert space used in the proof of this lemma see for instance W. Rudin *Functional Analysis*, (McGraw-Hill, 1991).

the sequence $\{y_n\}$ converges, which shows that K is locally compact hence finite dimensional. The next step is to show that $\phi^* - \psi^*$ has closed range. Indeed after factoring out by K it can be assumed that $\phi^* - \psi^*$ is injective. Consider a sequence of elements $y_n \in Y$ such that $(\phi^* - \psi^*)(y_n) \longrightarrow x$. If $\{y_n\}$ has a bounded subsequence then as before it is possible to assume that $\psi^*(y_n)$ converges; but then $\phi^*(y_n) = (\phi^* - \psi^*)(y_n) + \psi^*(y_n)$ converges, so again y_n converges to an element y and $(\phi^* - \psi^*)(y) = x$. On the other hand if $\|y_n\| \longrightarrow \infty$ the elements $y'_n = y_n/\|y_n\|$ have norm 1 and

$$(\phi^* - \psi^*)(y'_n) = \frac{1}{\|y_n\|}(\phi^* - \psi^*)(y_n) \longrightarrow 0;$$

again it can be assumed that $\psi^*(y'_n)$ converges, hence that $\phi^*(y_n)$ and y'_n converge, and if $y' = \lim y_n$ then $\|y'\| = 1$ and $(\phi^* - \psi^*)(y') = 0$, which contradicts the assumption that $\phi^* - \psi^*$ is one-to one so this case cannot occur. To conclude the proof of the lemma using the results of the preceding two steps note that since $\phi^* - \psi^*$ has closed range the same is true of $\phi - \psi$; so the quotient space $Y' = Y/(\phi - \psi)(X)$ is a Hilbert space. The surjective mapping ϕ induces a surjective mapping $\phi' : X \longrightarrow Y'$, and the compact mapping ψ induces a compact mapping $\psi' : X \longrightarrow Y'$; and since $\phi' = \psi'$ the space Y' is locally compact hence finite dimensional as desired, which suffices to conclude the proof.

The finite dimensionality theorem also holds for cohomology groups with coefficients in the sheaf of germs of holomorphic cross-sections of a complex vector bundle; for the essential part of the proof really uses only the compactness of the operation of restriction of holomorphic functions to compact subsets of their domain of definition, and that is true either for single functions or for vectors of functions with the supremum norm.

1.5 The Serre Duality Theorem

The traditional approach to the further examination of the cohomology group $H^1(M, \mathcal{O}(\lambda))$ for a compact Riemann surface is through potential theory[12]. The discussion here however will follow an alternative approach introduced by J-P. Serre[13] which proceeds by considering the space of linear functionals

$$T : H^1(M, \mathcal{O}(\lambda)) \longrightarrow \mathbb{C} \tag{1.53}$$

[12]The use of potential theory in Riemann surfaces goes back to Riemann's inaugural dissertation in Göttingen in 1851, "Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse", *Collected Works*, pp. 1 - 48, and was a crucial tool in the great classical work on Riemann surfaces by H. Weyl, *Die Idee der Riemannschen Fläche* (Teubner, 1923); [English translation *The Concept of a Riemann Surface* (Addison-Wesley, 1955)].

[13]See the paper by J-P. Serre "Un Théorème de dualité", Comment. Math. Helv., vol. 29 (1955), pp. 9 - 26.

on the finite dimensional complex vector space $H^1(M, \mathcal{O}(\lambda))$. By the Theorem of Dolbeault, Theorem 1.9,

$$H^1(M, \mathcal{O}(\lambda)) \cong \frac{\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))}{\overline{\partial}\Gamma(M, \mathcal{E}^{(0,0)}(\lambda))}, \tag{1.54}$$

so the linear functionals on $H^1(M, \mathcal{O}(\lambda))$ can be identified with the linear functionals on $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ that vanish on the subspace $\overline{\partial}\Gamma(M, \mathcal{E}(^{(0,0)}(\lambda))$. It suffices just to consider continuous linear functionals when the infinite dimensional complex vector space $\Gamma(M, \mathcal{E}(^{(0,0)}(\lambda))$ has a suitable structure as a topological vector space[14]. To introduce the appropriate topology suppose that λ is a holomorphic line bundle over a compact Riemann surface M and is described by a coordinate bundle$\{V_\alpha, \lambda_{\alpha\beta}\}$ in terms of a finite covering of M by open coordinate neighborhoods V_α; and let $z_\alpha = x_\alpha + i\, y_\alpha$ be the local coordinates in V_α. Introduce on $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ for $q = 0$ or 1 the norms

$$||\phi||_N = \sup_\alpha \sup_{\nu_1+\nu_2 \le N} \sup_{z_\alpha \in V_\alpha} \left| \frac{\partial^{\nu_1+\nu_2} f_\alpha(x_\alpha, y_\alpha)}{\partial x_\alpha^{\nu_1} \partial y_\alpha^{\nu_2}} \right| \tag{1.55}$$

for a cross-section $\phi \in \Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ where $\phi_\alpha = f_\alpha$ for $q = 0$ and $\phi_\alpha = f_\alpha d\overline{z}_\alpha$ for $q = 1$. It is evident that these are norms in the customary sense that (i) $||\phi||_N \ge 0$ and this is an equality if and only if $\phi = 0$, (ii) $||\phi_1 + \phi_2||_N \le ||\phi_1||_N + ||\phi_2||_N$, (iii) $||c\,\phi||_N = |c| \cdot ||\phi||_N$ for any complex constant c; and it is also evident that $||\phi||_N \le ||\phi||_{N+1}$. Such a collection of norms determines on the vector space $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ the structure of a locally convex topological vector space by taking as a basis for the open neighborhoods of the origin the sets $V_{N,k}$ consisting of those cross-sections $\phi \in \Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ such that $||\phi||_N < 1/k$ for positive integers N, k; the norms (1.55) are continuous functions in this topology, and the topology can be defined equivalently by the translation invariant metric

$$\rho(\phi, \psi) = \sup_N \frac{c_N\, ||\phi - \psi||_N}{1 + ||\phi - \psi||_N} \tag{1.56}$$

where c_N are positive numbers such that $\lim_{N\to\infty} c_N = 0$. It is evident from the definition (1.55) of these norms that any Cauchy sequence in $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ converges in this topology, so the topological vector space $\Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$ is complete in this norm, hence is a Fréchet space. A linear functional

$$T : \Gamma(M, \mathcal{E}^{(0,q)}(\lambda)) \longrightarrow \mathbb{C} \tag{1.57}$$

is continuous in this topology if and only if there are positive integers M, N such that

$$|T(\phi)| \le M||\phi||_N \tag{1.58}$$

for all $\phi \in \Gamma(M, \mathcal{E}^{(0,q)}(\lambda))$.

[14] For the properties of topological vector spaces used here see for instance the book by Walter Rudin, *Functional Analysis*, McGraw-Hill, 1991, or that by Casper Goffman and George Pedrick, *First Course in Functional Analysis*, Prentice-Hall, 1965.

Lemma 1.16 *The linear mapping*

$$\overline{\partial} : \Gamma(M, \mathcal{E}^{(0,0)}(\lambda)) \longrightarrow \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$$

is a continuous linear mapping between these two Fréchet spaces, and its image is a closed linear subspace of $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$*; consequently linear functionals on the quotient space* $H^1(M, \mathcal{O}(\lambda))$ *can be identified with continuous linear functionals on the Fréchet space* $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ *that vanish on the subspace* $\overline{\partial}\Gamma(M, \mathcal{E}^{(0,0)}(\lambda)) \subset \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$.

Proof: To simplify the notation for this proof let $A = \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ and $B = \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$. The linear mapping $\overline{\partial} : A \longrightarrow B$ is continuous as an immediate consequence of the obvious inequality $||\overline{\partial}\phi||_N \le ||\phi||_{N+1}$. The kernel K of this linear mapping is a closed subspace of A, so the quotient A/K also is a Fréchet space and the induced mapping $\overline{\partial} : A/K \longrightarrow B$ is a continuous linear mapping with a trivial kernel. If $L \subset B$ is a finite dimensional subspace of B complementary to $\overline{\partial}(A/K)$ the product $(A/K) \times L$ is a Fréchet space and the mapping $(\overline{\partial} + \iota) : (A/K) \times L \longrightarrow B$ defined by $(\overline{\partial} + \iota)(a, l) = \overline{\partial}(a) + l$ for $a \in A$, $l \in L$ is a surjective continuous linear mapping with a trivial kernel; hence by the open mapping theorem it is an isomorphism of Fréchet spaces. Since $(A/K) \times 0 \subset (A/K) \times L$ is a closed subspace it follows that its isomorphic image $(\overline{\partial} + \iota)\big((A/K) \times 0\big) = \overline{\partial}A$ is a closed subspace of B. In that case the quotient space $B/\overline{\partial}A$ with the topology it inherits from B also is a Fréchet space; and any linear functional on the finite dimensional quotient space $B/\overline{\partial}A$ is necessarily continuous hence amounts to a continuous linear functional on B that vanishes on $\overline{\partial}A$, which suffices to conclude the proof.

For an example of such a continuous linear functional suppose that λ is a holomorphic line bundle over a compact Riemann surface M and is described by a coordinate bundle $\{V_\alpha, \lambda_{\alpha\beta}\}$ in terms of a finite covering $\mathfrak{V}$ of M by coordinate neighborhoods V_α. A cross-section $\tau \in \Gamma(M, \mathcal{E}^{(1,0)}(\lambda^{-1}))$ is described by $\mathcal{C}^\infty$ differential forms τ_α of type $(1,0)$ in the coordinate neighborhoods V_α such that $\tau_\alpha = \lambda_{\alpha\beta}^{-1}\tau_\beta$ in intersections $V_\alpha \cap V_\beta$. If $\phi \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ is described by $\mathcal{C}^\infty$ differential forms ϕ_α of type $(0,1)$ in the coordinate neighborhoods V_α such that $\phi_\alpha = \lambda_{\alpha\beta}\phi_\beta$ in intersections $V_\alpha \cap V_\beta$ then $\tau_\alpha \wedge \phi_\alpha = \lambda_{\alpha\beta}^{-1}\tau_\beta \wedge \lambda_{\alpha\beta}\phi_\beta = \tau_\beta \wedge \phi_\beta$ in $V_\alpha \cap V_\beta$; so the product $\tau \wedge \phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ is a differential form of type $(1,1)$ defined on the entire compact Riemann surface M. The integral

$$T_\tau(\phi) = \int_M \tau_\alpha \wedge \phi_\alpha \tag{1.59}$$

is clearly a continuous linear functional (1.57) for $q = 1$, and is a nontrivial linear functional so long as $\tau \neq 0$. If $g \in \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ is a cross-section described by $\mathcal{C}^\infty$ functions g_α in the coordinate neighborhoods V_α such that $g_\alpha = \lambda_{\alpha\beta}\, g_\beta$ in $V_\alpha \cap V_\beta$ then $g_\alpha\tau_\alpha = \lambda_{\alpha\beta}g_\beta \cdot \lambda_{\alpha\beta}^{-1}\tau_\beta = g_\beta\,\tau_\beta$ in $V_\alpha \cap V_\beta$, so these local products describe a global differential form of type $(1,0)$ on M. By Stokes's Theorem $\int_M d(g_\alpha\tau_\alpha) = 0$, hence $0 = \int_M d(g_\alpha\tau_\alpha) = \int_M \overline{\partial}(g_\alpha\tau_\alpha) = \int_M \overline{\partial}g_\alpha \wedge \tau_\alpha + \int_M g_\alpha \cdot \overline{\partial}\tau_\alpha$

and consequently

$$T_\tau(\overline{\partial} g_\alpha) = \int_M g_\alpha \cdot \overline{\partial}\tau_\alpha. \tag{1.60}$$

Therefore $T_\tau(\overline{\partial} g) = 0$ for all cross-sections $g \in \Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ if and only if $\int_M g_\alpha \cdot \overline{\partial}\tau_\alpha = 0$ for all cross-sections g, so that $\overline{\partial}\tau_\alpha = 0$ and τ_α are holomorphic differential forms; thus the linear functional T_τ vanishes on the linear subspace $\overline{\partial}\Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ if and only if $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$, and in that case determines a linear functional (1.53). The deeper result is the converse assertion that all linear functionals (1.53) are of this form.

Theorem 1.17 (Serre Duality Thorem) *If λ is a holomorphic line bundle over a compact Riemann surface M the continuous linear functionals on the topological vector space $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ that vanish on the closed linear subspace $\overline{\partial}\Gamma(M, \mathcal{E}^{(0,0)}(\lambda))$ are precisely the linear functionals T_τ for cross-sections $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$.*

Proof: Suppose that λ is defined by a coordinate bundle $\{W_\alpha, \lambda_{\alpha\beta}\}$ for a finite covering $\mathfrak{W}$ of the surface M by bounded coordinate neighborhoods $W_\alpha \subset M$ with local coordinates z_α. Choose open subsets U_α and V_α that form coverings $\mathfrak{U}$ and $\mathfrak{V}$ of M with $\overline{U}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset W_\alpha$, so that $\overline{U}_\alpha$ and $\overline{V}_\alpha$ are compact; and choose a positive constant $\delta > 0$ such that when the coordinate neighborhood W_α is viewed as a bounded open subset of the complex plane of the variable z_α both the distance from U_α to the complement of V_α and the distance from V_α to the complement of W_α are greater than δ for all α. Introduce on the vector space $\Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ the topology defined by the norms (1.55) in terms of the covering of M by the coordinate neighborhoods V_α. For any index α let $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda)) \subset \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ be the subspace consisting of those cross-sections $\phi \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ with support contained in V_α and let $\Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ be the vector space consisting of ordinary $\mathcal{C}^\infty$ differential forms of type $(0,1)$ with support in the coordinate neighborhood V_α. To any differential form $\phi_\alpha = f_\alpha d\overline{z}_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ there can be associated the cross-section $\iota_\alpha(\phi_\alpha) \in \Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$ for which $\phi_\beta = \lambda_{\beta\alpha}\phi_\alpha$ in any intersection $V_\alpha \cap V_\beta$ and $\phi_\beta = 0$ otherwise; all elements of $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$ arise in this way, so the mapping

$$\iota_\alpha : \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)}) \longrightarrow \Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda)) \tag{1.61}$$

thus defined is an isomorphism between these two complex vector spaces. The vector space $\Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ can be made into a topological vector space by the norms

$$||\phi_\alpha||_{N,\alpha} = \sup_{\nu_1+\nu_2 \le N} \sup_{z_\alpha \in V_\alpha} \left| \frac{\partial^{\nu_1+\nu_2} f_\alpha(x_\alpha, y_\alpha)}{\partial x_\alpha^{\nu_1} \partial y_\alpha^{\nu_2}} \right| \tag{1.62}$$

for any differential form $\phi_\alpha = f_\alpha d\overline{z}_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$; and with this topology

$\Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ also is a Fréchet space. For any $\phi_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ clearly

$$\begin{aligned}\|\iota_\alpha(\phi_\alpha)\|_n &= \sup_\beta \sup_{\nu_1+\nu_2\le N} \sup_{z_\beta \in V_\alpha \cap V_\beta} \left| \frac{\partial^{\nu_1+\nu_2}\big(\lambda_{\beta\alpha}(z_\beta) f_\alpha(z_\beta)\big)}{\partial x_\beta^{\nu_1}\partial y_\beta^{\nu_2}} \right| \\ &= \|\phi_\alpha\|_{N,\alpha} + \sup_{\beta\ne\alpha} \sup_{\nu_1+\nu_2\le N} \sup_{z_\beta \in V_\alpha \cap V_\beta} \left| \frac{\partial^{\nu_1+\nu_2}\big(\lambda_{\beta\alpha}(z_\beta) f_\alpha(z_\beta)\big)}{\partial x_\beta^{\nu_1}\partial y_\beta^{\nu_2}} \right|.\end{aligned}$$

It follows from this identity first that $\|\iota_\alpha(\phi_\alpha)\|_N \ge \|\phi_\alpha\|_{N,\alpha}$ and second that $\|\iota_\alpha(\phi_\alpha)\|_N \le C_N\|\phi_\alpha\|_{N,\alpha}$ for some constant $C_N > 0$, since the functions $\lambda_{\beta\alpha}(z_\beta)$ and all their partial derivatives are defined in $W_\alpha \cap W_\beta \supset \overline{V}_\alpha \cap \overline{V}_\beta$ and hence are uniformly bounded in $\overline{V}_\alpha \cap \overline{V}_\beta$ as are all the partial derivatives $\partial^{k_1+k_2}x_\alpha/\partial^{k_1}x_\beta\partial^{k_2}y_\beta$ and $\partial^{k_1+k_2}y_\alpha/\partial^{k_1}x_\beta\partial^{k_2}y_\beta$; therefore the linear mapping (1.61) is an isomorphism of Fréchet spaces. If the support of the differential form $\phi_\alpha = f_\alpha d\overline{z}_\alpha \in \Gamma_0(V_\alpha, \mathcal{E}^{(0,1)})$ is contained in the subset $U_\alpha \subset V_\alpha$ and the coordinate neighborhood W_α is viewed as an open subset of the complex plane of the variable ζ_α it follows from Theorem 1.8 that $f_\alpha = \partial g_\alpha/\partial\overline{z}_\alpha + h_\alpha$ for $\mathcal{C}^\infty$ functions g_α and h_α with supports contained in V_α. The function g_α can be extended to a section $\iota_\alpha(g_\alpha) \in \Gamma(M, \mathcal{E}(\lambda))$ by the obvious analogue of the construction of the isomorphism (1.61); and when the differential forms $\phi_\alpha = f_\alpha d\overline{z}_\alpha$ and $\psi_\alpha = h_\alpha d\overline{z}_\alpha$ are extended by the isomorphism (1.61) then $\phi_\alpha = \overline{\partial} g_\alpha + \psi_\alpha$ so

$$\iota_\alpha(\phi_\alpha) = \overline{\partial}\iota_\alpha(g_\alpha) + \iota_\alpha(\psi_\alpha)$$

since $\iota_\alpha(\overline{\partial} g_\alpha) = \overline{\partial}\iota_\alpha(g_\alpha)$. If $T : \Gamma(M, \mathcal{E}^{(0,1)}(\lambda)) \longrightarrow \mathbb{C}$ is a continuous linear functional that vanishes on the subspace $\overline{\partial}\Gamma(M, \mathcal{E}(\lambda))$ it then follows that

$$T\big(\iota_\alpha(\phi_\alpha)\big) = T\big(\iota_\alpha(\psi_\alpha)\big). \tag{1.63}$$

As in (1.29) the function $h_\alpha(z_\alpha)$ is given explicitly by the integral

$$h_\alpha(z_\alpha) = \frac{i}{2}\int_{U_\alpha} f_\alpha(\zeta_\alpha) s(\zeta_\alpha - z_\alpha) d\zeta_\alpha \wedge d\overline{\zeta}_\alpha,$$

which can be written as a limit of Riemann sums so that

$$\psi_\alpha(z_\alpha) = h_\alpha(z_\alpha) d\overline{z}_\alpha = \lim \sum_j f_\alpha(\zeta_j) s(\zeta_j - z_\alpha) dz_\alpha \cdot \Delta_j$$

for local elements of area Δ_j. For any fixed point $\zeta_j \in U_\alpha$ the expression $s(\zeta_j - z_\alpha)dz_\alpha$ is a $\mathcal{C}^\infty$ differential form with support contained in V_α, since the support of the function $s(z)$ is contained in a disc of radius $\delta/2$ about the origin; the extensions $\iota_\alpha\big(s(\zeta_j - z_\alpha)dz_\alpha\big)$ thus are well defined elements of $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$. Since $s(z)$ is a $\mathcal{C}^\infty$ function it follows from the continuity of the functional T that the images

$$T\Big(\iota_\alpha\big(s(\zeta_j - z_\alpha)dz_\alpha\big)\Big) = t_\alpha(\zeta_j)$$

are $\mathcal{C}^\infty$ functions of the variable $\zeta_j \in U_\alpha$. The Riemann sums and their partial derivatives converge uniformly to the integral over U_α, so it follows further from the continuity of the functional T that

$$\begin{aligned} T\big(\iota_\alpha(\psi_\alpha)\big) &= T\Big(\lim \sum_j f_\alpha(\zeta_j)s(\zeta_j - z_\alpha)dz_\alpha \cdot \Delta_j\Big) \\ &= \lim \sum_j f_\alpha(\zeta_j)T\Big(s(\zeta_j - z_\alpha)dz_\alpha\Big) \cdot \Delta_j \\ &= \lim \sum_j f_\alpha(\zeta_j)t_\alpha(\zeta_i)\Delta_j = \frac{i}{2}\int_{U_\alpha} f_\alpha(\zeta)t_\alpha(\zeta)d\zeta \wedge d\overline{\zeta}; \end{aligned}$$

consequently in view of (1.63)

$$T\big(\iota_\alpha(\phi_\alpha)\big) = \frac{i}{2}\int_{U_\alpha} \tau_\alpha \wedge \phi_\alpha \tag{1.64}$$

where $\tau_\alpha = t_\alpha(\zeta)d\zeta$. If the support of $\phi \in \Gamma(M, \mathcal{E}^{(0,1)}(\lambda))$ is contained in the intersection $U_\alpha \cap U_\beta$ then ϕ can be considered as being an element of either $\Gamma_\alpha(M, \mathcal{E}^{(0,1)}(\lambda))$ or $\Gamma_\beta(M, \mathcal{E}^{(0,1)}(\lambda))$, so

$$\begin{aligned} T\big(\iota_\alpha(\phi)\big) &= \frac{i}{2}\int_{U_\alpha \cap U_\beta} \tau_\alpha \wedge \phi_\alpha \\ &= \frac{i}{2}\int_{U_\alpha \cap U_\beta} \tau_\beta \wedge \phi_\beta = \frac{i}{2}\int_{U_\alpha \cap U_\beta} \tau_\beta \wedge \lambda_{\beta\alpha}\phi_\alpha; \end{aligned}$$

and if that holds for any such form ϕ then necessarily $\tau_\alpha = \tau_\beta\, \lambda_{\beta\alpha}$ in $U_\alpha \cap U_\beta$ so that the differential forms τ_α in the coordinate neighborhoods U_α are a cross-section $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\lambda^{-1}))$. If ρ_α is a $\mathcal{C}^\infty$ partition of unity subordinate to the covering $\{U_\alpha\}$ then any cross-section $\phi \in \Gamma(M, \mathcal{O}^{(0,1)}(\lambda))$ can be written as the sum $\phi = \sum_\alpha \rho_\alpha \phi$ of cross-sections $\rho_\alpha\phi \in \Gamma(M, \mathcal{O}^{(0,1)}(\lambda))$; if ϕ is described by differential forms ϕ_β in the subsets U_β then $\rho_\alpha\phi$ is described by the differential forms $\rho_\alpha\phi_\beta$ in the subsets U_β. In particular the product $\rho_\alpha\phi_\alpha$ can be viewed as a differential form $\rho_\alpha\phi_\alpha \in \Gamma_0(U_\alpha, \mathcal{E}^{(0,1)})$, and then $\iota_\alpha(\rho_\alpha\phi_\alpha) = \rho_\alpha\phi$ so that by (1.64)

$$\begin{aligned} T(\phi) = T\left(\sum_\alpha \rho_\alpha\phi\right) &= \sum_\alpha T\big(\rho_\alpha\phi\big) = \sum_\alpha T\big(\iota_\alpha(\rho_\alpha\phi_\alpha)\big) \\ &= \sum_\alpha \frac{i}{2}\int_{U_\alpha} \tau_\alpha \wedge \rho_\alpha\phi_\alpha = \frac{i}{2}\int_M \sum_\alpha \rho_\alpha \cdot (\tau_\alpha \wedge \phi_\alpha). \end{aligned}$$

However in an intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods $\tau_\alpha \wedge \phi_\alpha = \tau_\beta \wedge \phi_\beta$ as noted before, so these local differential forms describe a global differential form $\tau \wedge \phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ on the entire Riemann surface M and consequently

$$T(\phi) = \frac{i}{2}\int_M \sum_\alpha \rho_\alpha \cdot (\tau \wedge \phi) = \int_M \left(\frac{i}{2}\tau\right) \wedge \phi = T_{i\tau/2}(\phi);$$

and since T vanishes on the subspace $\overline{\partial}\Gamma(M,\mathcal{E}^{(0,0)}(\lambda))$ it follows as a consequence of (1.60) as before that $i\,\tau/2 \in \Gamma(M,\mathcal{O}^{(1,0)}(\lambda^{-1}))$, which suffices to conclude the proof.

Corollary 1.18 *If λ is a holomorphic line bundle over a compact Riemann surface M*

$$\dim H^1(M,\mathcal{O}(\lambda)) = \dim \Gamma(M,\mathcal{O}^{(1,0)}(\lambda^{-1})). \tag{1.65}$$

Proof: Since $H^1(M,\mathcal{O}(\lambda))$ is a finite dimensional complex vector space its dimension is equal to the dimension of its dual space, which in view of (1.54) and the preceding theorem is isomorphic to the vector space $\Gamma(M,\mathcal{O}^{(1,0)}(\lambda^{-1}))$, and that suffices for the proof.

Theorem 1.17 also holds for holomorphic vector bundles just as for holomorphic line bundles, with the proper interpretation. Thus for cross-sections $\phi = \{\phi_\alpha\} \in \Gamma(M,\mathcal{E}^{(0,1)}(\lambda))$ and $\tau = \{\tau_\alpha\} \in \Gamma(M,\mathcal{O}^{(1,0)}(\lambda^*))$ viewed as column vectors of differential forms, where the dual vector bundle λ^* is defined by the dual coordinate transition functions $\lambda^*_{\alpha\beta} = {}^t\lambda^{-1}_{\alpha\beta}$, it follows that ${}^t\tau_\alpha \wedge \phi_\alpha = {}^t\tau_\beta \wedge \phi_\beta$ in $U_\alpha \cap U_\beta$ and consequently that

$$T_\tau(\phi) = \int_M {}^t\tau_\alpha \wedge \phi_\alpha \tag{1.66}$$

is a well defined linear functional on the vector space $\Gamma(M,\mathcal{O}^{(1,0)}(\lambda^*))$. The rest of the proof of the Serre Duality Theorem carries through unchanged, although in the final statement for vector bundles the dual vector bundle λ^* replaces the inverse line bundle λ^{-1}.

Chapter 2

The Lüroth Semigroup

2.1 An Example: The Riemann Sphere

The basic topological invariant of a holomorphic line bundle λ over a compact Riemann surface M is its characteristic class $c(\lambda) \in \mathbb{Z}$, which characterizes the underlying topological line bundle completely. The basic analytic invariant is the dimension of the finite dimensional vector space $\Gamma(M, \mathcal{O}(\lambda))$, denoted by

$$\gamma(\lambda) = \dim \Gamma(M, \mathcal{O}(\lambda)). \tag{2.1}$$

This section will begin the investigation of the relations between these two invariants. For the simplest Riemann surface, the *Riemann sphere* or equivalently the one-dimensional complex projective space $\mathbb{P}^1$, this relation is somewhat anomalous but can be described completely quite easily. The Riemann sphere $\mathbb{P}^1$ is no doubt quite familiar, since it arises naturally in a number of contexts in almost all discussions of basic complex analysis. It is constructed from two copies U_0 and U_1 of the complex plane, with the complex coordinates z_0 and z_1 respectively, by identifying nonzero values z_0 and z_1 whenever $z_0 = 1/z_1$; thus the two sets U_0 and U_1 form a coordinate covering of the resulting Riemann surface $\mathbb{P}^1$, and the intersection $U_0 \cap U_1$ consists of all points $z_0 \neq 0$ in the coordinate neighborhood U_0 and all points $z_1 \neq 0$ in the coordinate neighborhood U_1. The surface $\mathbb{P}^1$ is topologically a two-sphere, the one-point compactification of the complex plane U_0 that arises by the addition of the point $z_1 = 0$ to the complex plane U_0. A customary notation is to denote points in U_0 by the complex variable $z = z_0$ and to denote the point $z_1 = 0$ by $z = \infty$, viewed as the point added to compactify the plane U_0.

Theorem 2.1 *For divisors $\mathfrak{d}$ on the Riemann sphere $\mathbb{P}^1$*
(i) *$\mathfrak{d}$ is a principal divisor if and only if $\deg \mathfrak{d} = 0$;*
(ii) *$\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\deg \mathfrak{d}_1 = \deg \mathfrak{d}_2$; and*
(iii) *if $\zeta_{\mathfrak{d}}$ is the line bundle of a divisor $\mathfrak{d}$ of degree $n = \deg \mathfrak{d} \geq 0$ then*

$$\gamma(\zeta_{\mathfrak{d}}) = \deg \mathfrak{d} + 1 = c(\zeta_{\mathfrak{d}}) + 1 \tag{2.2}$$

and the cross-sections in $\Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}}))$ *can be identified with polynomials of degree* n *in the variable* z_0.

Proof: (i) By definition the principal divisors on $\mathbb{P}^1$ are those divisors that are the divisors of meromorphic functions on $\mathbb{P}^1$. A rational function $f(z_0)$ of the complex variable z_0 can be viewed as a meromorphic function on the Riemann surface $\mathbb{P}^1$; $f(z_0)$ is of course a meromorphic function of the variable z_0 in the coordinate neighborhood U_0, and $f(1/z_1)$ is a meromorphic function of the variable z_1 in the coordinate neighborhood U_1. If

$$f(z_0) = \prod_i (z_0 - a_i)^{n_i} = \prod_i \left(\frac{1 - a_i z_1}{z_1}\right)^{n_i}$$

where n_i are positive or negative integers then $\mathfrak{d}(f) = \sum_i n_i \cdot p_i - (\sum_i n_i) \cdot \infty$ where $p_i \in U_0$ are the points with coordinates $z_0 = a_i$, $z_1 = 1/a_i$, so $\mathfrak{d}(f)$ is a divisor of degree zero; any divisor on $\mathbb{P}^1$ of degree zero is of this form, so any divisor of degree zero on $\mathbb{P}^1$ is the divisor of a meromorphic function on $\mathbb{P}^1$. Conversely the divisor of an arbitrary meromorphic function on $\mathbb{P}^1$ is of degree zero by Theorem 1.1, and that suffices to demonstrate (i).
(ii) By definition $\mathfrak{d}_1 \sim \mathfrak{d}_2$ if and only if $\mathfrak{d}_1 - \mathfrak{d}_2$ is a principal divisor; so since $\deg(\mathfrak{d}_1 - \mathfrak{d}_2) = \deg \mathfrak{d}_1 - \deg \mathfrak{d}_2$ it is evident that (ii) follows from (i).
(iii) It follows from (ii) that if $\mathfrak{d}$ is a divisor of degree n and $p \in \mathbb{P}^1$ is any point in the Riemann sphere then $\mathfrak{d} \sim n \cdot p$. In particular if the point p has local coordinates $z_0 = a$, $z_1 = 1/a$ for a nonzero complex number $a \in \mathbb{C}$ then the function $h_0(z_0) = z_0 - a$ is holomorphic in U_0 with a simple zero at p and at no other point of U_0 while the function $h_1(z_1) = 1 - a\, z_1$ is holomorphic in U_1 with a simple zero at p and at no other point of U_1; then as in (1.8) the point bundle ζ_a is described by the coordinate transition functions

$$\zeta_{a,01} = \frac{h_0(z_0)}{h_1(z_1)} = \frac{z_0 - a}{1 - a z_1} = z_0 \tag{2.3}$$

in $U_0 \cap U_1$, so the line bundle $\zeta_{\mathfrak{d}}$ of the divisor $\mathfrak{d}$ is described by the coordinate transition functions $\zeta_{\mathfrak{d},01} = z_0^n$ in $U_0 \cap U_1$. A general holomorphic cross-section $f \in \Gamma(\mathbb{P}^1, \mathcal{O}(\zeta_{\mathfrak{d}}))$ is described by an entire function $f_0(z_0)$ of the variable z_0 and an entire function $f_1(z_1)$ of the variable z_1 such that $f_0(z_0) = z_0^n f_1(1/z_0)$ for $z_0 \neq 0$. The function $f_1(1/z_0)$ is bounded for large values of z_0, so the entire function $f_0(z_0)$ is bounded by $C|z_0|^n$ for some constant $C > 0$ for large values of z_0; if $n \geq 0$ the function $f_0(z_0)$ consequently must be a polynomial of degree n in the variable z. Conversely if $f_0(z_0)$ is a polynomial of degree n in the variable z_0 then $f_1(z_1) = z_1^n f_0(1/z_1)$ is a polynomial of degree n in the variable z_1, and the entire functions $f_0(z_0)$ and $f_1(z_1)$ describe a holomorphic cross-section of the bundle $\zeta_{\mathfrak{d}}$. Since the space of polynomials $f_0(z_0)$ of degree n has dimension $n + 1$ it follows that $\gamma(\zeta_{\mathfrak{d}}) = n + 1$, and that suffices to conclude the proof.

Theorem 2.2 *Any holomorphic line bundle over the Riemann sphere* $\mathbb{P}^1$ *is the line bundle of a divisor on* $\mathbb{P}^1$ *so has nontrivial meromorphic cross-sections.*

Proof: The Riemann sphere is covered by the two coordinate neighborhoods U_0 and U_1, so alternatively can be viewed as covered by the two discs

$$D_0 = \left\{ z_0 \in U_0 \;\middle|\; |z_0| < 2 \right\} \quad \text{and} \quad D_1 = \left\{ z_1 \in U_1 \;\middle|\; |z_1| < 2 \right\}$$

for which

$$D_0 \cap D_1 = \left\{ z_0 \in U_0 \;\middle|\; \frac{1}{2} < |z_0| < 2 \right\}.$$

Since any holomorphic line bundle is analytically trivial over each disc D_i by Corollary 1.11 it follows that any holomorphic line bundle λ over $\mathbb{P}^1$ can be described by a coordinate line bundle $(D_\alpha, \lambda_{\alpha\beta})$ in terms of this covering, where the coordinate transition function λ_{01} in the intersection $D_0 \cap D_1$ is a holomorphic and nowhere vanishing function of the complex variable z_0 in the annulus $1/2 < |z_0| < 2$. A local branch of the holomorphic function $\log \lambda_{01}(z_0)$ near the point $z_0 = 1$ can be continued analytically once around the origin in this annulus, and upon this continuation its value will increase by $2\pi i\, n$ for some integer n; hence $f(z_0) = \log\left(z_0^{-n}\lambda_{01}(z_0)\right)$ is a single-valued holomorphic function in the annulus, and $\lambda_{01}(z_0) = z_0^n \exp f(z_0)$. By using the Cauchy integral formula the function $f(z_0)$ can be represented as usual as the difference $f(z_0) = f_0(z_0) - f_1(z_1)$ of a holomorphic function $f_0(z_0)$ in the disc D_0 and a holomorphic function in the exterior of the circle $|z_0| = 1/2$, where the latter function can be viewed equivalently as a holomorphic function $f_1(z_1)$ in the disc D_1. The exponentials $h_j(z_j) = \exp f_j(z_j)$ are holomorphic and nowhere vanishing functions in the discs D_j, and $\lambda_{01}(z_0) = z_0^n h_0(z_0)/h_1(z_1)$ in the intersection $D_1 \cap D_2 \subset \mathbb{P}^1$; hence the holomorphic line bundle λ is analytically equivalent to the holomorphic line bundle defined by the coordinate transition function z_0^n, and it is evident from (2.3) that this is the line bundle of a divisor of degree n. That suffices to conclude the proof.

The two preceding theorems provide a complete characterization of holomorphic line bundles over the Riemann sphere $\mathbb{P}^1$ and a description of their properties.

Corollary 2.3 (i) *There is a unique holomorphic line bundle ζ of characteristic class $c(\zeta) = 1$ on $\mathbb{P}^1$, and $\zeta = \zeta_p$ for any point $p \in M$.*
(ii) *For any integer n the line bundle ζ^n is the unique holomorphic line bundle of characteristic class $c(\zeta^n) = n$ on $\mathbb{P}^1$, and $\gamma(\zeta^n) = \max(n+1, 0)$.*

Proof: (i) A holomorphic line bundle ζ of characterstic class $c(\zeta) = 1$ is the line bundle of a divisor $\mathfrak{d}$ by Theorem 2.2, and $\deg \mathfrak{d} = 1$ by (1.15). Since any two divisors of the same degree are linearly equivalent by Theorem 2.1 (ii) the divisor $\mathfrak{d}$ is linearly equivalent to the divisor $1 \cdot p$ for any chosen base point $p \in M$ so $\zeta = \zeta_p$, which demonstrates (i).
(ii) If ζ is a line bundle for which $c(\zeta) = n$ then ζ is the line bundle of a divisor $\mathfrak{d}$ for which $\deg(\mathfrak{d}) = n$; and that divisor is linearly equivalent to the divisor $n \cdot p$. for any chosen point $p \in M$ so $\zeta = \zeta_{n \cdot p} = \zeta_p^n$. If $n < 0$ then $\gamma(\lambda) = 0$ by Corollary 1.3 while if $n \geq 0$ then $\gamma(\lambda) = n+1$ by Theorem 2.1 (iii), which altogether is the desired result. That suffices for the proof.

2.2 The Role of Point Bundles

Although there is not an equally simple description of all holomorphic line bundles over more general Riemann surfaces in terms of point bundles, nonetheless point bundles play a significant role in the study of holomorphic line bundles over arbitrary compact Riemann surfaces.

Theorem 2.4 (i) *The point bundles over an arbitrary compact Riemann surface M can be characterized as those holomorphic line bundles λ over M such that*

$$c(\lambda) = 1 \quad \textit{and} \quad \gamma(\lambda) > 0. \tag{2.4}$$

(ii) *If ζ_p is a point bundle over M then*

$$\gamma(\zeta_p) = \begin{cases} 2 & \textit{if} \quad M = \mathbb{P}^1, \\ 1 & \textit{if} \quad M \neq \mathbb{P}^1. \end{cases} \tag{2.5}$$

Proof: (i) If ζ_p is a point bundle over M then $c(\zeta_p) = 1$ by (1.15); and since there is a nontrivial holomorphic cross-section of ζ_p necessarily $\gamma(\zeta_p) > 0$. Conversely if λ is a holomorphic line bundle over M for which $\gamma(\lambda) > 0$ then $\lambda = \zeta_{\mathfrak{d}}$ is the line bundle of some positive divisor $\mathfrak{d}$ on M; and if $c(\zeta_{\mathfrak{d}}) = 1$ then $\deg \mathfrak{d} = 1$ by (1.15) so $\mathfrak{d} = 1 \cdot p$ for some point $p \in M$.
(ii) If $\gamma(\zeta_p) > 1$ for a point bundle ζ_p over M choose two linearly independent holomorphic cross-sections $f_1, f_2 \in \Gamma(M, \mathcal{O}(\zeta_p))$ and let their divisors be $\mathfrak{d}(f_1) = 1 \cdot p_1$ and $\mathfrak{d}(f_2) = 1 \cdot p_2$. If $p_1 = p_2$ the quotient $f = f_1/f_2$ is a function that is holomorphic everywhere on the compact Riemann surface M, so by the maximum modulus theorem it must be a constant; but that contradicts the assumption that the two functions are linearly independent. Thus $p_1 \neq p_2$, and the quotient $f = f_1/f_2$ is a nonconstant meromorphic function on M with the divisor $\mathfrak{d}(f) = 1 \cdot p_1 - 1 \cdot p_2$. This function can be viewed as a holomorphic mapping from the Riemann surface M to the Riemann sphere $\mathbb{P}^1$ in the obvious manner: near any regular point the function f takes finite values in the coordinate neighborhood $U_0 \subset \mathbb{P}^1$, while near its pole the function $1/f$ takes finite values in the coordinate neighborhood $U_1 \subset \mathbb{P}^1$. This mapping takes the single point p_1 to the origin in the coordinate neighborhood U_0, and the single point p_2 to the point ∞ in the coordinate neighborhood U_0, or equivalently to the origin in the coordinate neighborhood U_1. For any complex number c the function $f(z_0) - c$ also has a single simple pole, so it must have a single simple zero; thus the function f itself takes a single point of M to the complex value c. Altogether the function f describes a one-to-one holomorphic mapping from M to $\mathbb{P}^1$, so M is analytically equivalent to $\mathbb{P}^1$. Then $\gamma(\zeta_p) = 2$ by Corollary 2.3, and that suffices to conclude the proof of the theorem.

The preceding theorem provides a characterization of the Riemann sphere among all compact Riemann surfaces and also yields the following properties of point bundles over compact Riemann surfaces other than the Riemann sphere.

Corollary 2.5 *If M is a compact Riemann surface other than the Riemann sphere $\mathbb{P}^1$ the point bundles ζ_p for distinct points $p \in M$ are analytically inequivalent.*

Proof: If $\zeta_p = \zeta_q$ for two distinct points $p, q \in M$ then this bundle has one cross-section with a simple zero at p and no other points and another cross-section with a simple zero at q and no other points; but then $\gamma(\zeta) > 1$ so by the preceding theorem M is the Riemann sphere, which suffices for the proof.

Thus if M is a compact Riemann surface other than the Riemann sphere $\mathbb{P}^1$ the mapping that associates to each point $p \in M$ the point bundle ζ_p is a one-to-one correspondence

$$\zeta : M \longrightarrow \Big\{ \zeta \in H^1(M, \mathcal{O}^*) \Big| c(\zeta) = 1, \ \gamma(\zeta) > 0 \Big\}; \tag{2.6}$$

this provides a concrete representation of Riemann surfaces that will be examined in some detail in the discussion of the Abel-Jacobi mapping (3.4). Point bundles often are used in the study of other holomorphic line bundles over compact Riemann surfaces through an application of the following observation.

Lemma 2.6 (i) *If ζ_p is a point bundle and λ is any other holomorphic line bundle on a compact Riemann surface M*

$$\gamma(\lambda) \le \gamma(\lambda\zeta_p) \le \gamma(\lambda) + 1. \tag{2.7}$$

(ii) *Further $\gamma(\lambda\zeta_p) = \gamma(\lambda)$ if and only if all holomorphic cross-sections of the bundle $\lambda\zeta_p$ vanish at the point p.*

Proof: If $h \in \Gamma(M, \mathcal{O}(\zeta_p))$ is a nontrivial holomorphic cross-section of the point bundle ζ_p multiplication by h clearly yields an injective homomorphism

$$\times h : \Gamma(M, \mathcal{O}(\lambda)) \longrightarrow \Gamma(M, \mathcal{O}(\lambda\zeta_p)),$$

the image of which consists precisely of all holomorphic cross-sections of the bundle $\lambda\zeta_p$ that vanish at the point p since $h(p) = 0$; in particular therefore $\gamma(\lambda) \le \gamma(\lambda\zeta_p)$. If $\gamma(\lambda\zeta_p) = 0$ then $\gamma(\lambda) = 0$ and the asserted result holds trivially. Otherwise choose a basis $f_1, \ldots, f_n \in \Gamma(M, \mathcal{O}(\lambda\zeta_p))$ where $n = \gamma(\lambda\zeta_p)$. If $f_i(p) = 0$ for all of these cross-sections then the mapping $\times h$ is surjective and $\gamma(\lambda) = \gamma(\lambda\zeta_p)$. If for instance $f_1(p) \ne 0$ then the mapping $\times h$ is not surjective, so $\gamma(\lambda) \le \gamma(\lambda\zeta_p) - 1$; the differences $g_i(z) = f_i(z) - \big(f_i(p)/f_1(p)\big)f_1(z)$ for $2 \le i \le n$ are $n - 1$ linearly independent holomorphic cross-sections of the bundle $\lambda\zeta_p$ that vanish at the point p so are the images under the injective homomorphism $\times h$ of $n - 1$ linearly independent holomorphic cross-sections of λ, and consequently $\gamma(\lambda) \ge n - 1 = \gamma(\lambda\zeta_p) - 1$. That suffices to conclude the proof.

An application of this auxiliary result yields a minor refinement of the upper bound for the dimension $\gamma(\lambda)$ in Theorem 1.46, but stilll generally not an effective upper bound.

Theorem 2.7 *If λ is a holomorphic line bundle over a compact Riemann surface M and if $c(\lambda) > 0$ then $\gamma(\lambda) \leq c(\lambda) + 1$. If the equality holds for any holomorphic line bundle over M then $M = \mathbb{P}^1$.*

Proof: The result will be demonstrated by induction on the integer $n = c(\lambda)$. The case $n = 1$ follows immediately from Theorem 2.4. Assume therefore that the result has been demonstrated for all integers strictly less than n, and consider a holomorphic line bundle λ for which $c(\lambda) = n > 1$. If $\gamma(\lambda) \geq n + 1$ then from the preceding lemma it follows that $\gamma(\lambda\zeta_p^{-1}) \geq \gamma(\lambda) - 1 \geq n$ for a point bundle ζ_p; but $c(\lambda\zeta_p^{-1}) = n-1$ so from the induction hypothesis it follows that $M = \mathbb{P}^1$, and in that case $\gamma(\lambda) = n+1$ by Corollary 2.3. Thus $\gamma(\lambda) \leq n+1$ and equality holds only when $M = \mathbb{P}^1$, which concludes the induction step and the proof.

2.3 The Base Decomposition of Line Bundles

Over a compact Riemann surface M holomorphic line bundles λ for which $\gamma(\lambda) = 1$, such as point bundles, are the line bundles of unique divisors on M; however if $\gamma(\lambda) > 1$ the bundle λ is the line bundle of the divisors of some linearly independent cross-sections, hence of a number of distinct divisors. Nonetheless there are some unique divisors that can be associated to such line bundles. If λ is a holomorphic line bundle over a compact Riemann surface M the *divisor of common zeros* of a finite or infinite collection of nontrivial holomorphic cross-sections $f_i \in \Gamma(M, \mathcal{O}(\lambda))$ is the positive divisor on M defined by

$$\mathfrak{d}(f_1, f_2, \ldots)(a) = \inf_i \operatorname{ord}_a(f_i). \tag{2.8}$$

The degree of this divisor is the *number of common zeros* of the collection of cross-sections; the cross-sections have no common zeros precisely when this divisor is the trivial divisor. The *base divisor* of a holomorphic line bundle λ for which $\gamma(\lambda) > 0$ is the positive divisor $\mathfrak{b}(\lambda)$ defined by

$$\mathfrak{b}(\lambda)(a) = \inf \Big\{ \operatorname{ord}_a(f) \Bigm| f \in \Gamma(M, \mathcal{O}(\lambda)),\ f \neq 0 \Big\}, \tag{2.9}$$

or equivalently it is the divisor of common zeros of the set of all nontrivial holomorphic cross-sections of the line bundle λ; of course if $\gamma(\lambda) = 0$ the only holomorphic cross-section of λ is that which vanishes identically, so the base divisor $\mathfrak{b}(\lambda)$ is undefined. If $\mathfrak{b}(\lambda) = \sum_i \nu_i \cdot a_i$ then all holomorphic cross-sections of the bundle λ vanish at the point a_i to order at least ν_i, and there are cross-sections that vanish at the point a_i to order exactly ν_i. The points that appear in the base divisor $\mathfrak{b}(\lambda)$ with strictly positive coefficients are called the *base points* of the line bundle λ; thus a point $a \in M$ is a base point of a line bundle λ if and only if $\gamma(\lambda) > 0$ and all holomorphic cross-sections of λ vanish at the point a. If $\mathfrak{d} \geq 0$ is a positive divisor and $\gamma(\zeta_{\mathfrak{d}}) > 0$ clearly $\mathfrak{d} = \mathfrak{b}(\zeta_{\mathfrak{d}}) + \mathfrak{d}'$ for another positive divisor $\mathfrak{d}' \geq 0$, since there is a holomorphic cross-section of the line bundle $\zeta_{\mathfrak{d}}$ that vanishes at the divisor $\mathfrak{d}$; in particular if a is a base point of the line bundle $\zeta_{\mathfrak{d}}$ of a divisor $\mathfrak{d} \geq 0$ for which $\gamma(\zeta_{\mathfrak{d}}) > 0$ then the point a must appear in the divisor $\mathfrak{d}$. A holomorphic line bundle λ is *base-point-free* if $\gamma(\lambda) > 0$ and $\mathfrak{b}(\lambda) = 0$, or equivalently if $\gamma(\lambda) > 0$ and for any point of M there exists a holomorphic cross-section of λ that is nonzero at that point. In particular the trivial line bundle $\lambda = 1$ is base-point-free, since its holomorphic cross-sections are complex constants; indeed it is the only line bundle λ with $c(\lambda) = 0$ that is base-point-free, since $\gamma(\lambda) = 0$ for any other line bundle $\lambda \neq 1$ with $c(\lambda) = 0$. The set of base-point-free holomorphic line bundles on a compact Riemann surface M is denoted by $\mathcal{B}(M)$, or simply by $\mathcal{B}$ if it is either clear from context or irrelevant just which Riemann surface M is being considered.

Theorem 2.8 *If λ and σ are base-point-free holomorphic line bundles on a compact Riemann surface M their product $\lambda\sigma$ also is base-point-free.*

Proof: If λ, σ are base-point-free line bundles then for any point $a \in M$ there are cross-sections $f \in \Gamma(M, \mathcal{O}(\lambda))$ and $g \in \Gamma(M, \mathcal{O}(\sigma))$ such that $f(a) \neq 0$ and $g(a) \neq 0$; and then the cross-section $fg \in \Gamma(M, \mathcal{O}(\lambda\sigma))$ has the property that $f(a)g(a) \neq 0$, so the product bundle also is base-point-free. That suffices for the proof.

The set $\mathcal{B}(M)$ of base-point-free holomorphic line bundles on a compact Riemann surface thus is closed under multiplication of line bundles, so sometimes is called the *semigroup of base-point-free holomorphic line bundles* on M. As a consequence the characteristic classes $c(\lambda)$ of the base-point-free holomorphic line bundles $\lambda \in \mathcal{B}(M)$ form a semigroup of nonnegative integers, called the *Lüroth semigroup* of the surface M and denoted by $\mathcal{L}(M)$. As an example, on the Riemann sphere $\mathbb{P}^1$ by Corollary 2.3 there is a unique holomorphic line bundle ζ for which $c(\zeta) = 1$ and $\gamma(\zeta) = 2$; and it is equal to the point bundle ζ_a for any point $a \in M$, so if $f_1, f_2 \in \Gamma\big(\mathbb{P}^1, \mathcal{O}(\zeta)\big)$ is a basis for the holomorphic cross-sections then some linear combination of these cross-sections vanshes at any point of M so the two sections have no common zeros. It follows that the line bundle ζ is base-point-free, hence the Lüroth semigroup $\mathcal{L}(\mathbb{P}^1)$ begins with 1 and consequently it consists of all integers $n \geq 1$.

For some purposes it is convenient to use one or the other of the following two alternative characterizations of base-point-free holomorphic line bundles.

Lemma 2.9 *On a compact Riemann surface M a holomorphic line bundle λ is base-point-free if and only if it has two holomorphic cross-sections with no common zeros on M.*

Proof: If λ has two holomorphic cross-sections with no common zeros then of course it is base-point-free. Conversely suppose that λ is base-point-free and let $f_i \in \Gamma(M, \mathcal{O}(\lambda))$ be a basis for the space of holomorphic cross-sections for $0 \leq i \leq n$. If $n = 0$ the cross-section f_0 must have no zeros, so it and the zero cross-section have no common zeros. If $n = 1$ the two cross-sections f_0 and f_1 must have no common zeros. If $n \geq 2$ consider the divisor $\mathfrak{d}(f_0) = \sum_j \nu_j \cdot a_j$

of the first of these cross-sections. A cross-section $f = \sum_{i=1}^{n} x_i f_i$ vanishes at the point a_1 if and only if the coefficients x_i satisfy the linear equation $\sum_{i=1}^{n} x_i f_i(a_1) = 0$. Since $f_i(a_1) \ne 0$ for at least one index $i \ge 1$ and the equation involves at least two variables the set of solutions is a proper linear subspace of the space $\mathbb{C}^n = \{(x_1, \ldots, x_n)\}$ of all the coefficients x_i; so its complement, the set of coefficients for which $f(a_1) \ne 0$, is a dense open subset of $\mathbb{C}^n$. The same is true for each of the points a_j, and since the intersection of finitely many dense open subsets of $\mathbb{C}^n$ is again a dense open subset there must exist coefficients $(x_1, \ldots, x_n)$ describing a holomorphic cross-section f that does not vanish at any of the points a_j; this cross-section and the cross-section f_0 thus have no common zeros, and that suffices for the proof.

It is evident from the proof of the preceding lemma that if $\gamma(\lambda) > 2$ then in general any pair of holomorphic cross-sections will have no common zeros. More precisely, all pairs h_1, h_2 of holomorphic cross-section of λ can be expressed in terms of a basis $f_1, f_2, \ldots f_n$ of these cross-sections as $h_i = \sum_{j=1}^{n} a_{ij} f_i$ for a $2 \times n$ matrix $A = \{a_{ij}\}$; the set of those matrices describing pairs of cross-sections with no common zeros is a dense open subset of the space $\mathbb{C}^{2n}$ of such matrices.

Lemma 2.10 (i) *A holomorphic line bundle λ on a compact Riemann surface M is base-point-free if and only if $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) - 1$ for all points $a \in M$.*
(ii) *A point $a \in M$ on a compact Riemann surface M is a base point of a holomorphic bundle λ over M for which $\gamma(M) > 0$ if and only $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda)$.*

Proof: If $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) - 1$ then $\gamma(\lambda) > 0$. Lemma 2.6 asserts that $\gamma(\lambda\zeta_a^{-1}) \le \gamma(\lambda) \le \gamma(\lambda\zeta_a^{-1}) + 1$ for any line bundle λ and any point $a \in M$, and that $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda)$ if and only if all holomorphic cross-sections of the bundle λ vanish at the point a. Both (i) and (ii) are obvious consequences, and that suffices for the proof.

Although not all holomorphic line bundles are base-point-free, any line bundle λ for which $\gamma(\lambda) > 0$ can be described by its base divisor and an associated base-point-free holomorphic line bundle as follows.

Theorem 2.11 (Base Decomposition Theorem) *On a compact Riemann surface M a holomorphic line bundle λ with $\gamma(\lambda) > 0$ is uniquely expressible as the product $\lambda = \lambda_0 \zeta_{\mathfrak{b}(\lambda)}$ of a base-point-free line bundle λ_0 and the line bundle $\zeta_{\mathfrak{b}(\lambda)}$ of the base divisor $\mathfrak{b}(\lambda)$ of λ, and $\gamma(\zeta_{\mathfrak{b}(\lambda)}) = 1$. For any nontrivial holomorphic cross-section $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{b}(\lambda)}))$ multiplication by h is an isomorphism*

$$\times h : \Gamma(M, \mathcal{O}(\lambda_0)) \xrightarrow{\cong} \Gamma(M, \mathcal{O}(\lambda)), \tag{2.10}$$

hence $\gamma(\lambda_0) = \gamma(\lambda)$.

Proof: If λ is a holomorphic line bundle with $\gamma(\lambda) > 0$ set $\lambda_0 = \lambda\zeta_{\mathfrak{b}(\lambda)}^{-1}$ where $\mathfrak{b}(\lambda)$ is the base divisor of λ; the base divisor $\mathfrak{b}(\lambda)$ of course is determined uniquely by the line bundle λ, hence so are the line bundles $\zeta_{\mathfrak{b}(\lambda)}$ and λ_0. Multiplication by any nontrivial holomorphic cross-section $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{b}(\lambda)}))$ is an injective linear homomorphism of the form (2.10). If $\mathfrak{d}(h) = \mathfrak{b}(\lambda)$ then it is clear from the definition of the base divisor that for any cross-section $f \in \Gamma(M, \mathcal{O}(\lambda))$ the quotient f/h is everywhere holomorphic, hence is a cross-section $f_0 = f/h \in \Gamma(M, \mathcal{O}(\lambda_0))$ for which $(\times h)(f_0) = f$; consequently the homomorphism $\times h$ also is surjective, hence is an isomorphism. Therefore if $f_i \in \Gamma(M, \mathcal{O}(\lambda_0))$ is a basis for the space of holomorphic cross-sections of the line bundle λ_0 then $hf_i \in \Gamma(M, \mathcal{O}(\lambda))$ is a basis for the space of holomorphic cross-sections of the line bundle $\lambda = \lambda_0 \zeta_{\mathfrak{b}(\lambda)}$; hence $\mathfrak{b}(\lambda_0)$ is the divisor of common zeros of the cross-sections f_i while $\mathfrak{b}(\lambda)$ is the divisor of common zeros of the cross-sections hf_i so $\mathfrak{b}(\lambda) = \mathfrak{b}(\lambda) + \mathfrak{b}(\lambda_0)$ and consequently $\mathfrak{b}(\lambda_0) = \emptyset$ so λ_0 is base-point-free. Finally if $g \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{b}(\lambda)}))$ is an arbitrary nontrivial holomorphic cross-section, not necessarily vanishing at the base divisor $\mathfrak{b}(\lambda)$, nonetheless the corresponding homomorphism $\times g$ also is an isomorphism, since it is injective and the two vector spaces of cross-sections have the same dimension; but then $\mathfrak{b}(\lambda) = \mathfrak{d}(g) = \mathfrak{d}(h)$ so g is necessarily a constant multiple of h, hence $\gamma(\zeta_{\mathfrak{b}(\lambda)}) = 1$. That suffices to conclude the proof.

This expression of a holomorphic line bundle λ for which $\gamma(\lambda) > 0$ as the product of a base-point-free line bundle λ_0 and the line bundle $\zeta_{\mathfrak{b}(\lambda)}$ associated to the base divisor $\mathfrak{b}(\lambda)$ is called the *base decomposition* of the line bundle λ. As an illustrative example of the base decomposition of a line bundle, if $\gamma(\lambda) = 1$ then λ is the line bundle of a unique positive divisor $\mathfrak{d}$ and λ has the base decomposition as the product $\lambda = 1 \cdot \zeta_{\mathfrak{d}}$ of the base-point-free identity line bundle 1 and the line bundle $\zeta_{\mathfrak{d}}$ of the base divisor $\mathfrak{d} = \mathfrak{b}(\lambda)$. To each base-point-free line bundle λ_0 there can be associated the set of holomorphic line bundles with base decomposition $\lambda = \lambda_0 \zeta_{\mathfrak{b}}$, parametrized by the appropriate positive divisors $\mathfrak{b}$; alternatively to each positive divisor $\mathfrak{b}$ for which $\gamma(\zeta_{\mathfrak{b}}) = 1$ there can be associated the set of holomorphic line bundles λ with the base decomposition $\lambda = \lambda_0 \zeta_{\mathfrak{b}}$, parametrized by the appropriate set of base-point-free line bundles λ_0. These are convenient descriptions of the set of holomorphic line bundles over a given compact Riemann surface; but some care must be taken since not every product of a base-point-free line bundle and a line bundle $\zeta_{\mathfrak{b}}$ for which $\gamma(\zeta_{\mathfrak{b}}) = 1$ is the base decomposition of the product.

Corollary 2.12 *If λ_0 is a base-point-free holomorphic line bundle on a compact Riemann surface M and $\mathfrak{b} \geq 0$ is a positive divisor on M, the product $\lambda = \lambda_0 \zeta_{\mathfrak{b}}$ is the base decomposition of the line bundle λ if and only if $\gamma(\lambda) = \gamma(\lambda_0)$.*

Proof: If $\lambda = \lambda_0 \zeta_{\mathfrak{b}}$ is the base decomposition of λ then $\gamma(\lambda) = \gamma(\lambda_0)$ by the preceding theorem. Conversely if $\gamma(\lambda_0 \zeta_{\mathfrak{b}}) = \gamma(\lambda_0)$ for a base-point-free line bundle λ_0 and a divisor $\mathfrak{b} \geq 0$, and if $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{b}}))$ is a nontrivial holomorphic cross-section for which $\mathfrak{d}(h) = \mathfrak{b}$, multiplication by h is an injective homomorphism

$$\times h : \Gamma(M, \mathcal{O}(\lambda_0)) \longrightarrow \Gamma(M, \mathcal{O}(\lambda_0 \zeta_{\mathfrak{b}}))$$

which also must be surjective since the two spaces of holomorphic cross-sections have the same dimension; consequently all holomorphic cross-sections of the line bundle $\lambda_0\zeta_{\mathfrak{b}}$ are multiples of h so must vanish at the divisor $\mathfrak{b}$. If all of these cross-sections vanish at a divisor $\mathfrak{b}+\mathfrak{a}$ where $\mathfrak{a} \geq 0$ is a nontrivial positive divisor then all of the holomorphic cross-sections of the line bundle λ_0 also vanish at $\mathfrak{a}$, which is impossible since the line bundle λ_0 is base-point-free. Thus $\mathfrak{b}$ is the base divisor of the line bundle $\lambda_0\zeta_{\mathfrak{b}}$, and hence this is the base decomposition of the bundle $\lambda = \lambda_0\zeta_{\mathfrak{b}}$ and $\gamma(\zeta_{\mathfrak{d}}) = 1$.

Theorem 2.13 *If a holomorphic line bundle λ over a Riemann surface M has the base decomposition $\lambda = \lambda_0\zeta_{\mathfrak{b}}$ then for any point $a \in M$ either $\gamma(\lambda\zeta_a) = \gamma(\lambda)$ and the product bundle $\lambda\zeta_a$ has the base decomposition $\lambda\zeta_a = \lambda_0\zeta_{\mathfrak{b}+a}$ or $\gamma(\lambda\zeta_a) > \gamma(\lambda)$ and the product bundle $\lambda\zeta_a$ has the base decomposition*

$$\lambda\zeta_a = \lambda_0'\zeta_{\mathfrak{b}'} \text{ for the base-point free bundle } \lambda_0' = \lambda_0\lambda_a\zeta_{\mathfrak{b}''} \tag{2.11}$$

where $\mathfrak{b} = \mathfrak{b}' + \mathfrak{b}''$ for some divisors $\mathfrak{b}' \geq 0$ and $\mathfrak{b}'' \geq 0$.

Proof: Since $\lambda = \lambda_0\zeta_{\mathfrak{b}}$ is a base decomposition then $\gamma(\lambda) = \gamma(\lambda_0)$ by the preceding Corolllary 2.12; so if $\gamma(\lambda\zeta_a) = \gamma(\lambda) = \gamma(\lambda_0)$ then by the preceding Corollary 2.12 again $\lambda\zeta_a = \lambda_0\zeta_{\mathfrak{b}+a}$ is a base decomposition. On the other hand if $\gamma(\lambda\zeta_a) > \gamma(\lambda)$ then $\gamma(\lambda\zeta_a) = \gamma(\lambda)+1$ by Lemma 2.6; so if $f_i \in \Gamma(M, \mathcal{O}(\lambda_0))$ is a basis for the space of holomorphic cross-sections of the line bundle λ_0 and $h_{\mathfrak{b}} \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{b}}))$ and $h_a \in \Gamma(M, \mathcal{O}(\zeta_a))$ are nontrivial holomorphic cross-sections with the divisors $\mathfrak{d}(h_{\mathfrak{b}}) = \mathfrak{b}$ and $\mathfrak{d}(h_a) = a$ the products $f_i h_{\mathfrak{b}} h_a$ together with one other cross-section g are a basis for $\Gamma(M, \mathcal{O}(\lambda\zeta_a))$. Not all of these cross-sections vanish at the point a, since otherwise it would followfrom Corollary 2.12 as in the first part of the proof that $\gamma(\lambda\zeta_a) = \gamma(\lambda)$. If all these cross-sections vanish at a point $b \neq a$ then since the cross-sections f_i have no common zeros and $h_a(b) \neq 0$ it must be the case that $h_{\mathfrak{b}}(b) = 0$ hence that b is a point in the divisor $\mathfrak{b}$; therefore if $\mathfrak{b}'$ is the base divisor of the line bundle $\lambda\zeta_a$ then $\mathfrak{b}'$ must be part of the divisor $\mathfrak{b}$, so $\mathfrak{b} = \mathfrak{b}' + \mathfrak{b}''$ for another divisor $\mathfrak{b}'' \geq 0$. Thus the base decomposition of the line bundle $\lambda\zeta_a = \lambda_0\zeta_{\mathfrak{b}'}\zeta_{\mathfrak{b}''}\zeta_a$ has the form $\lambda\zeta_a = \lambda_0'\zeta_{\mathfrak{b}'}$ for a base-point-free line bundle λ_0' which must be the bundle $\lambda_0' = \lambda_0\zeta_a\zeta_{\mathfrak{b}''}$. That suffices to conclude the proof.

Corollary 2.14 *If $r_0 \in \mathcal{L}(M)$ is an integer in the Lüroth semigroup $\mathcal{L}(M)$ of a Riemann surface M, if a line bundle λ over M has the base decomposition $\lambda = \lambda_0\zeta_{\mathfrak{b}}$ for a base-point-free line bundle λ_0 for which $c(\lambda_0) = r_0$, and if $\gamma(\lambda\zeta_a) > \gamma(\lambda)$ for some point $a \in M$ then there is an integer $r \in \mathcal{L}(M)$ in the range $r_0 + 1 \leq r \leq r_0 + 1 + \deg \mathfrak{b}$.*

Proof: If $\gamma(\lambda\zeta_a) > \gamma(\lambda)$ it follows from the preceding theorem that $\mathfrak{b} = \mathfrak{b}' + \mathfrak{b}''$ for some divisors $\mathfrak{b}' \geq 0, \mathfrak{b}'' \geq 0$ for which $\lambda_0\zeta_a\zeta_{\mathfrak{b}''}$ is base-point-free, and consequently $c(\lambda_0\zeta_a\zeta_{\mathfrak{b}''}) = r_0 + 1 + \deg \mathfrak{b}'' \in \mathcal{L}(M)$; since $0 \leq \deg \mathfrak{b}'' \leq \deg \mathfrak{b}$ that suffices for the proof.

2.4 Mappings to Projective Space

The Riemann sphere $\mathbb{P}^1$ discussed in Section 2.1 is the simplest complex projective space. General complex projective spaces also play an important role in the study of compact Riemann surfaces, so a brief survey is included here to establish the notation. Two nonzero vectors v_1, v_2 in a complex vector space V of dimension $r+1$ are considered as equivalent if $v_1 = cv_2$ for some nonzero complex constant $c \in \mathbb{C}$. This is clearly an equivalence relation $\sim$ on the subset $V^\times \subset V$ consisting of nonzero vectors, the complement of the origin in V. The quotient of $V^\times$ under this equivalence relation is by definition the complex vector space $\mathbb{P}(V) = V^\times/\sim$ of dimension r. For the standard vector space $\mathbb{C}^{r+1}$ the notation is simplified by setting $\mathbb{P}^r = \mathbb{P}(\mathbb{C}^{r+1})$. The equivalence class of the nonzero vector $v = (z_0, z_1, \ldots, z_r)$ in $\mathbb{P}^r$ will be denoted by $[z_0, z_1, \ldots, z_r]$, and that will be called the description of the point $v/\sim \in \mathbb{P}^r$ by *homogeneous coordinates* in $\mathbb{P}^r$. If $U_j \subset \mathbb{P}^r$ denotes the subset of $\mathbb{P}^r$ consisting of points $[z_0, z_1, \ldots, z_r] \in \mathbb{P}^r$ for which $z_j \neq 0$ and $z^j_k = z_k/z_j \in \mathbb{C}$ then points in $U_j \subset \mathbb{P}^r$ can be described uniquely by the vectors

$$z^j = \left(z^j_0, z^j_1, \ldots z^j_{j-1}, z^j_{j+1} \ldots, z^j_r\right) \in \mathbb{C}^r.$$

This provides local coordinates in the subset $U_j \subset \mathbb{P}^r$, identifying that subset with the subspace $\mathbb{C}^r$; this is the description of a point by *inhomogeneous coordinates*. In the intersection $U^i \cap U^j$ of two coordinate neighborhoods the local coordinates are related by $z^i = \rho_{ij} z^j$ where $\rho_{ij} = z_j/z_i$. This describes the structure of a compact complex manifold of dimension r on the projective space $\mathbb{P}^r$.

If λ is a base-point-free holomorphic line bundle over M and is defined by coordinate transition functions $\lambda_{\alpha\beta}$ in terms of a coordinate covering $\{U_\alpha\}$ of the Riemann surface M, and if $f_{\alpha i} \in \Gamma(M, \mathcal{O}(\lambda))$ for $0 \le i \le r$ are any $r+1$ holomorphic cross-sections having no common zeros on M, then for any point $z \in U_\alpha$ the vector

$$F_\alpha(z) = \left(f_{\alpha 0}(z), f_{\alpha 1}(z), \ldots, f_{\alpha r}(z)\right) \in \mathbb{C}^{r+1} \tag{2.12}$$

is nonzero so it can be viewed as describing a point $[F_\alpha(z)]$ in the projective space $\mathbb{P}^r$ in homogeneous coordinates; and since $F_\alpha(z) = \lambda_{\alpha,\beta}(z) F_\beta(z)$ for any point $z \in U_\alpha \cap U_\beta$ it follows that $[F_\alpha(z)] = [F_\beta(z)]$ for $z \in U_\alpha \cap U_\beta$ so the local mappings $[F_\alpha(z)]$ can be viewed as describing a holomorphic mapping $F : M \longrightarrow \mathbb{P}^r$ by associating to any point $z \in M$ the point

$$F(z) = [F_\alpha(z)] = [f_{\alpha 0}(z), f_{\alpha 1}(z), \ldots, f_{\alpha r}(z)] \in \mathbb{P}^r. \tag{2.13}$$

If the cross-sections $f_{\alpha i}$ have some common zeros the preceding construction can be modified so as still to provide a holomorphic mapping into $\mathbb{P}^r$.

Theorem 2.15 (i). *Any $r+1$ nontrivial cross-sections $f_{\alpha i} \in \Gamma(M, \mathcal{O}(\lambda))$ of a holomorphic line bundle λ over a compact Riemann surface M describe a*

holomorphic mapping $F : (M \sim |\mathfrak{d}_0|) \longrightarrow \mathbb{P}^r$ *from the complement of the suport* $|\mathfrak{d}_0| \subset M$ *of the divisor* $\mathfrak{d}_0 = \mathfrak{d}(f_{\alpha 0}, \ldots, f_{\alpha r})$ *of common zeros of these cross-sections to the complex projective space* $\mathbb{P}^r$*.*
(ii) *If* $h \in \Gamma\big(M, \mathcal{O}(\zeta_{\mathfrak{d}_0})\big)$ *is a holomorphic cross-section for which* $\mathfrak{d}(h) = \mathfrak{d}_0$ *the mapping in* (i) *extends to a holomorphic mapping* $\tilde{F} : M \longrightarrow \mathbb{P}^r$ *described by the* $r+1$ *holomorphic cross-sections* $\tilde{f}_{\alpha i} = f_{\alpha i}/h_\alpha$ *of the base-point-free holomorphic line bundle* $\lambda\zeta_{\mathfrak{d}_0}^{-1}$*.*

Proof: (i) Since not all the cross-sections $f_{\alpha i}$ vanish identically the image $F(z) = [f_{\alpha 0}(z), \ldots, f_{\alpha r}(z)] \in \mathbb{P}^r$ is well defined for all points $z \in M$ except for the finitely many points z that are the common zeros of the cross-sections $f_{\alpha i}$, the finitely many points $z \in |\mathfrak{d}|_0$ of the divisor $\mathfrak{d}_0$; and that defines a holomorphic mapping $F : (M \sim |\mathfrak{d}|) \longrightarrow \mathbb{P}^r$.
(ii) The cross-section $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}}))$ has the divisor $\mathfrak{d}(h) = \mathfrak{d}_0$ so the quotients $f_{\alpha i}/h \in \Gamma(M, \mathcal{O}(\lambda\zeta_{\mathfrak{d}}^{-1}))$ are holomorphic cross-sections with no common zeros on the surface M; and consequently the line bundle $\lambda\zeta_{\mathfrak{d}_0}^{-1}$ is base-point-free. These cross-sections describe a holomorphic mapping $\tilde{F} : M \longrightarrow \mathbb{P}^n$ for which $\tilde{F}|(M - |\mathfrak{d}|) = F$, and that suffices for the proof.

Actually all holomorphic mappings from a compact Riemann surface M into complex projective spaces can be described in this way.

Theorem 2.16 *Any holomorphic mapping* $F : M \longrightarrow \mathbb{P}^r$ *from a compact Riemann surface* M *into an* r*-dimensional complex projective space for* $r \geq 1$ *can be described by a collection of* $r+1$ *holomorphic cross-sections* $f_i \in \Gamma(M, \mathcal{O}(\lambda_F))$ *with no common zeros of a uniquely determined base-point-free holomorphic line bundle* λ_F *over* M*.*

Proof: In terms of homogeneous coordinates $[w_0, \ldots, w_r]$ in the projective space $\mathbb{P}^r$, let $V_k \subset \mathbb{P}^r$ be the open subset in $\mathbb{P}^r$ consisting of points for which $w_k \neq 0$. The quotients $w_i^k = w_i/w_k$ are local inhomogeneous coordinates in V_k for indices $0 \leq i \leq r$, $i \neq k$, identifying the subset $V_k \subset \mathbb{P}^r$ with the space $\mathbb{C}^r$ of these coordinates. The inverse image $U_k = F^{-1}(V_k) \subset M$ is an open subset of the Riemann surface M and the image $F(z) \in V_k$ of any point $z \in U_k$ has uniquely determined inhomogeneous coordinates $w_i^k(z)$, which are holomorphic functions in the open subset $U_k \subset M$. The image $F(z)$ of a point $z \in U_k \cap U_l$ then is described by the two sets of inhomogeneous coordinates $w_i^k(z)$ and $w_i^l(z)\}$. Since these coordinates describe the same point of $\mathbb{P}^r$ necessarily $w_i^k(z) = \lambda_{kl}(z) w_i^l(z)$ for a nonzero complex number $\lambda_{kl}(z)$ at each point $z \in U_k \cap U_l$. It is evident from this definition that the function $\lambda_{kl}(z) = w_i^k(z)/w_i^l(z)$ is a holomorphic function in $U_k \cap U_l$ and that $\lambda_{kl}(z)\lambda_{lm}(z)\lambda_{mk}(z) = 1$ whenever $z \in U_k \cap U_l \cap U_m$; thus these functions are the coordinate transition functions for a holomorphic line bundle λ_F over the Riemann surface M, and the functions $w_i^k(z)$ for any fixed index i describe a holomorphic cross-section $w_i \in \Gamma\big(M, \mathcal{O}(\lambda_F)\big)$ of the line bundle λ_F. Since not all the coordinates w_i^k of any point in $\mathbb{P}^r$ are zero it follows that not all the holomorphic cross-sections $w_i(z)$ vanish at any point $z \in M$; consequently the line bundle λ_F is base-point-free. Any holomorphic line bundle is the line bundle $\zeta_{\mathfrak{d}(f)}$ of the divisor of any meromorphic cross-section of that bundle, so the line bundle λ_F is determined uniquely by the mapping F; and the mapping $F : M \longrightarrow \mathbb{P}^r$ is just the holomorphic mapping described by these cross-sections, which suffices for the proof.

The base-point-free holomorphic line bundle λ_F in the description of a holomorphic mapping $F : M \longrightarrow \mathbb{P}^r$ in the preceding theorem is called the *line bundle determined by the mapping* F. Conversely to any base-point-free holomophic line bundle λ over M there is associated uniquely the holomorphic mapping $F_\lambda : M \longrightarrow \mathbb{P}^r$ defined by a basis of the holomorphic cross-sections of λ, where $r = \gamma(\lambda) - 1$; this is called the *projective mapping defined by the line bundle* λ. By Remmert's Proper Mapping Theorem[1] the image $F(M) \subset \mathbb{P}^r$ of any holomorphic mapping $F : M \longrightarrow \mathbb{P}^r$ from the compact Riemann surface is a one-dimensional irreducible holomorphic subvariety of the projective space $\mathbb{P}^r$, and as such is an algebraic curve in $\mathbb{P}^r$ by Chow's Theorem. The image of course may have some singularities, so it is not necessarily a holomorphic submanifold of $\mathbb{P}^r$.

Theorem 2.17 (Projective Mapping Theorem) *Let $F_\lambda : M \longrightarrow \mathbb{P}^r$ be the holomorphic mapping defined by the space of holomorphic cross-sections of a base-point-free holomorphic line bundle λ over the Riemann surface M.*
(i) *For any point $a \in M$ the inverse image $F_\lambda^{-1}(a) \subset M$ consists of the point $a \in M$ and any points in the base divisor $\mathfrak{d}(\lambda\zeta_a^{-1})$ of the line bundle $\lambda\zeta_a^{-1}$.*
(ii) *The mapping F_λ is a bijective mapping if and only if the line bundle $\lambda\zeta_a^{-1}$ is base-point-free for all points $a \in M$.*
(iii) *The mapping F_λ is a nonsingular holomorphic mapping at the point a if and only if the line bundle $\lambda\zeta_a^{-1}$ is base-point-free.*
(iv) *If $\lambda\zeta_a^{-1}$ is base-point-free for all points $a \in M$ then the mapping F_λ is a biholomorphic mapping from M to a connected one-dimensional submanifold $F_\lambda(M) \subset \mathbb{P}^r$.*

Proof: (i) For any point $a \in M$ it is possible to choose a basis $\{f_i\}$ for the holomorphic cross-sections of the line bundle λ in a coordinate neighborhood of the point $a \in M$ so that $F(a) = [1, 0, \ldots, 0] \in \mathbb{P}^r$, hence so that $f_0(a) \neq 0$ but $f_1(a) = \cdots = f_r(a) = 0$. The cross-sections $f_1, \ldots, f_r$ then are a basis for the space of those cross-sections of the bundle λ that vanish at the point $a \in M$; and if $h \in \Gamma(M, \mathcal{O}(\zeta_a))$ is a nontrivial holomorphic cross-section of the point bundle ζ_a that vanishes at the poin a then the quotients $f_1/h, \ldots, f_n/h$ are a basis for the space of holomorphic cross-sections $\Gamma(M, \mathcal{O}(\lambda\zeta_a^{-1}))$, since it follows from Lemma 2.6 that $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) - 1$ for the base-point-free line bundle λ. Then $F(b) = F(a)$ for another point $b \in M$ if and only if $f_0(b) \neq 0$ but $f_1(b) = \cdots = f_n(b) = 0$, hence if and only if $f_1(b)/h(b) = \cdots = f_r(b)/h(b) = 0$, which is just the condition that $b \in \mathfrak{b}(\lambda\zeta_a^{-1})$, and that demonstrates part (i).
(ii) It follows from (i) that the mapping F_λ is bijective if and only if $n(a) = 0$ for all points $a \in M$, which is just the condition that $\mathfrak{b}(\lambda\zeta_a^{-1}) = \emptyset$ for all $a \in M$ or

[1] That result is discussed on page423 in Appendix A.3.

equivalently that $\lambda\zeta_a^{-1}$ is base-point-free for all $a \in M$, and that demonstrates part (ii).
(iii) Choose coordinates in $\mathbb{P}^r$ such that $F(a) = [1, 0, \ldots, 0] \in \mathbb{P}^r$. The quotients z_i/z_0 of the homogeneous coordinates $[z_0, \ldots z_r]$ in $\mathbb{P}^r$ are local inhomogeneous coordinates in an open neighborhood of the point $[1, 0, \ldots, 0] \in \mathbb{P}^r$ for $1 \le i \le r$; and in terms of these coordinates the holomorphic mapping $F : M \longrightarrow \mathbb{P}^r$ is described by the quotient functions $g_i(z) = f_i(z)/f_0(z)$ for $1 \le i \le r$. The derivative of the mapping F at the point a is the vector $F'(a) = \big\{g_i'(a)\big\} = \big\{f_i'(a)/f_0(a)\big\}$ since $f_i(a) = 0$ for $1 \le i \le r$; hence the mapping F is nonsingular at the point a if and only if $f_i'(a) \ne 0$ for some index i in the range $1 \le i \le r$. Since $f_i(a) = h(a) = 0$, but $h'(a) \ne 0$, where h has a simple zero at the point $a \in M$, it follows that the value of the cross-section $f_i(z)/h(z) \in \Gamma(M, \mathcal{O}(\lambda\zeta_a^{-1}))$ at the point $a \in M$ actually is given by $f_i'(a)/h'(a)$; thus the mapping F is nonsingular at the point $a \in M$ precisely when not all the holomorphic cross-sections of the bundle $\lambda\zeta_a^{-1}$ vanish at the point $a \in M$, which is just the condition that the line bundle $\lambda\zeta_a^{-1}$ is base-point-free, and that demonstrates (iii).
(iv) If $\lambda\zeta_a^{-1}$ is base-point-free for all points $a \in M$ then by (ii) the mapping F_λ is bijective and by (iii) it is nonsingular at all points of M, so the image $F_\lambda(M)$ is a submanifold of $\mathbb{P}^r$ and the mapping F_λ is a biholomorhic mapping, which suffices for the proof.

That there do exist nonsingular imbeddings of a compact Riemann surface M into various projective spaces will follow from a characterization of the base-point-free holomorphic line bundles over M. The special case of mappings $F : M \longrightarrow \mathbb{P}^1$ also is of considerable interest. For any two holomorphic cross-sections $f_0, f_1 \in \Gamma\big(M, \mathcal{O}(\lambda)\big)$ with no common zeros, where λ is a base-point-free holomorphic line bundle over M, the quotient $f = f_1/f_0$ is a meromorphic function of degree $r = c(\lambda)$ which describes a mapping $F : M \longrightarrow \mathbb{P}^1$ of degree r, meaning that for all but finitely many points of $\mathbb{P}^1$ the inverse image $f^{-1}(a)$ consists of r distinct points of M. These and the more general mappings between Riemann surfaces will be discussed in more detail later; the point here is that the degrees of holomorphic mappings $F : M \longrightarrow \mathbb{P}^1$ are just the integers in the Lüroth semigroup of M.

2.5 The Analytic Euler Characteristic

To examine further some relations between the dimension $\gamma(\lambda)$ of the space of holomorphic cross-sections of a general holomorphic line bundle λ over a compact Riemann surface M and the characteristic class $c(\lambda)$ of that line bundle it is convenient to introduce the expression

$$\chi(\lambda) = \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)), \tag{2.14}$$

a finite integer as a consequence of the Finite Dimensionality Theorem, Theorem 1.14; this is called the *Euler characteristic* of the line bundle λ.

Lemma 2.18 *If λ is a holomorphic line bundle over a compact Riemann surface M then*

$$\chi(\lambda\zeta_p) - c(\lambda\zeta_p) = \chi(\lambda) - c(\lambda) \tag{2.15}$$

for any point bundle ζ_p.

Proof: If $h \in \Gamma(M, \mathcal{O}(\zeta_p))$ is a nontrivial holomorphic cross-section multiplication by h is an injective sheaf homomorphism $\times h : \mathcal{O}(\lambda) \longrightarrow \mathcal{O}(\lambda\zeta_p)$; this leads to the exact exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}(\lambda) \xrightarrow{\times h} \mathcal{O}(\lambda\zeta_p) \longrightarrow S \longrightarrow 0, \tag{2.16}$$

in which $\mathcal{S}$ is the quotient sheaf. Since the cross-section h has a simple zero at the point p and is otherwise nonvanishing it follows that at any point $a \in M$ other than p the homomorphism $\times h$ is an isomorphism, and consequently $\mathcal{S}_a = 0$. On the other hand if z is a local coordinate centered at the point p the elements in the stalk $\mathcal{O}_p(\lambda\zeta_p)$ can be identified with germs of holomorphic functions of the variable z at the origin, and the functions that are in the image of the homomorphism $\times h$ are those functions that vanish at the point p. Therefore associating to the germ f of a holomorphic function representing an element in the stalk $\mathcal{O}_p(\lambda\zeta_p)$ the value $f(p)$ is a mapping $\mathcal{O}_p(\lambda\zeta_p) \longrightarrow \mathbb{C}$ with kernel the image of multiplication by h; and that in turn yields an identification $\mathcal{S}_p \cong \mathbb{C}$. The precise identification depends of course on the choice of a local coordinate z in terms of which the bundle λ is trivialized, but is really not needed at all; more important are first that $H^0(M, \mathcal{S}) = \Gamma(M, \mathcal{S}) \cong \mathbb{C}$ and second that the sheaf $\mathcal{S}$ is a fine sheaf, since it is nontrivial at just a single point of M, so $H^1(M, \mathcal{S}) = 0$. Therefore the exact cohomology sequence associated to the exact sequence of sheaves (2.16) begins

$$\begin{aligned} 0 \longrightarrow H^0(M, \mathcal{O}(\lambda)) \xrightarrow{\times h} H^0(M, \mathcal{O}(\lambda\zeta_p)) \longrightarrow \mathbb{C} \longrightarrow \\ \longrightarrow H^1(M, \mathcal{O}(\lambda)) \xrightarrow{\times h} H^1(M, \mathcal{O}(\lambda\zeta_p)) \longrightarrow 0. \end{aligned}$$

The alternating sum of the dimensions of the spaces in an exact sequence of vector spaces such as this is zero, as can be seen most simply by decomposing the exact sequence into a collection of short exact sequences; consequently

$$\begin{aligned} 0 &= \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^0(M, \mathcal{O}(\lambda\zeta_p)) + 1 \\ &\quad - \dim H^1(M, \mathcal{O}(\lambda)) + \dim H^1(M, \mathcal{O}(\lambda\zeta_p)) \\ &= \chi(\lambda) - \chi(\lambda\zeta_p) + 1. \end{aligned}$$

Since $c(\lambda\zeta_p) = c(\lambda) + 1$ this yields the desired result, thereby concluding the proof.

An immediate consequence of this lemma is the fundamental existence theorem for compact Riemann surfaces.

Theorem 2.19 (Existence Theorem) *A holomorphic line bundle on a compact Riemann surface M has nontrivial meromorphic cross-sections; indeed for any choice of a base point $p \in M$ any holomorphic line bundle λ over M has nontrivial meromorphic cross-sections with poles at most at the point $p \in M$.*

Proof: Iterating the preceding lemma yields the result that

$$\chi(\lambda\zeta_p^n) - c(\lambda\zeta_p^n) = \chi(\lambda) - c(\lambda) \tag{2.17}$$

for any integer n, or explicitly

$$\begin{aligned}\dim H^0(M, \mathcal{O}(\lambda\zeta_p^n)) - \dim H^1(M, \mathcal{O}(\lambda\zeta_p^n)) - c(\lambda) - n \\ = \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)) - c(\lambda)\end{aligned}$$

since $c(\lambda\zeta_p^n) = c(\lambda) + n$; and hence

$$\begin{aligned}\dim H^0(M, \mathcal{O}(\lambda\zeta_p^n)) &= n + \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)) \\ &\qquad + \dim H^1(M, \mathcal{O}(\lambda\zeta_p^n)) \\ &\geq n + \dim H^0(M, \mathcal{O}(\lambda)) - \dim H^1(M, \mathcal{O}(\lambda)).\end{aligned}$$

It is evident from this that $\dim \Gamma(M, \mathcal{O}(\lambda\zeta_p^n)) = \dim H^0(M, \mathcal{O}(\lambda\zeta_p^n)) > 0$ for sufficiently large $n > 0$, so there is a nontrivial holomorphic cross-section f of the bundle $\lambda\zeta_p^n$ for some $n > 0$; and if $h \in \Gamma(M, \mathcal{O}(\zeta_p))$ is a nontrivial holomorphic section of the point bundle ζ_p the quotient f/h^n is a nontrivial meromorphic cross-section of the bundle λ with poles at most at the point $p \in M$, which concludes the proof.

Corollary 2.20 *Any holomorphic line bundle λ over a compact Riemann surface M is the line bundle of a divisor on M, so there is the exact sequence*

$$0 \longrightarrow \Gamma(M, \mathcal{O}^*) \xrightarrow{\iota} \Gamma(M, \mathcal{M}^*) \xrightarrow{\mathfrak{d}} \Gamma(M, \mathcal{D}) \xrightarrow{\delta} H^1(M, \mathcal{O}^*) \longrightarrow 0.$$

Proof: By the Existence Theorem any line bundle λ has a nontrivial meromorphic cross-section, and then λ is the line bundle of the divisor of this cross-section. That means that the coboundary mapping in the exact cohomology sequence (1.5) is surjective, so that sequence reduces to the exact sequence of the present corollary, which suffices for the proof.

The exactness of the cohomology sequence of the preceding corollary was demonstrated by showing that the coboundary homomorphism in the exact sequence (1.5) is surjective; and since $\mathcal{D}$ is a fine sheaf $H^1(M, \mathcal{D}) = 0$, so it follows further from (1.5) that

$$H^1(M, \mathcal{M}^*) = 0 \quad \text{on any compact Riemann surface } M. \tag{2.18}$$

By applying the Serre Duality Theorem the preceding result can be extended as follows.

Theorem 2.21 *If λ is a holomorphic line bundle over a compact Riemann surface M then $H^1(M, \mathcal{M}(\lambda)) = 0$.*

Proof: If $f_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{M}(\lambda))$ is a cocycle in a coordinate covering $\mathfrak{U}$ of the Riemann surface M choose a divisor $\mathfrak{d}$ on M such that $\deg \mathfrak{d} > 2g - 2 - c(\lambda)$ and $\mathfrak{d} + \mathfrak{d}(f_{\alpha\beta}) \geq 0$ and let $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}})$ be a holomorphic cross-section with $\mathfrak{d}(h) = \mathfrak{d}$. Since $f_{\alpha\gamma} = \lambda_{\gamma\beta} f_{\alpha\beta} + f_{\beta\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$ by (1.45) while $h_\gamma = \zeta_{\mathfrak{d},\gamma\beta} h_\beta$ in $U_\beta \cap U_\gamma$ it follows that $f_{\alpha\gamma} h_\gamma = \zeta_{\mathfrak{d},\gamma\beta} \lambda_{\gamma\beta} f_{\alpha\beta} h_\beta + f_{\beta\gamma} h_\gamma$, which is just the condition that the holomorphic functions $f_{\alpha\beta} h_\beta$ describe a cocycle in $Z^1(\mathfrak{U}, \mathcal{O}(\zeta_{\mathfrak{d}}\lambda))$. By Corollary 1.18 to the Serre Duality Theorem $\dim H^1(M, \mathcal{O}(\zeta_{\mathfrak{d}}\lambda)) = \dim \Gamma(M, \mathcal{O}^{(1,0)}(\zeta_{\mathfrak{d}}^{-1}\lambda^{-1})$ and $\dim \Gamma(M, \mathcal{O}^{(1,0)}(\zeta_{\mathfrak{d}}\lambda^{-1})) = \dim \Gamma(M, \mathcal{O}(\kappa\zeta_{\mathfrak{d}}\lambda^{-1})) = 0$ since $c(\kappa\zeta_{\mathfrak{d}}^{-1}\lambda^{-1}) = 2g - 2 - \deg \mathfrak{d} - c(\lambda) < 0$; consequently after passing to a refinement of the covering if necessary there will be holomorphic functions g_α in the sets U_α such that $f_{\alpha\beta} h_\beta = g_\beta - \lambda_{\beta\alpha}\zeta_{\mathfrak{d},\beta\alpha}$ in the intersections $U_\alpha \cap U_\beta$. The quotients $f_\alpha = g_\alpha / h_\alpha$ are then meromorphic functions in the sets U_α such that $f_{\alpha\beta} = f_\beta - \lambda_{\beta\alpha} f_\alpha$ in the intersections $U_\alpha \cap U_\beta$, which is the condition that the meromorphic cocycle $f_{\alpha\beta}$ is cohomologous to zero, and that concludes the proof.

Another consequence of the Existence Theorem is an explicit formula for the Euler characteristic of any holomorphic line bundle over a compact Riemann surface in terms of the characterisic class of that bundle. In this formula the *arithmetic genus* of a compact Riemann surface M is defined to be the integer

$$g_a = \dim H^1(M, \mathcal{O}). \tag{2.19}$$

Theorem 2.22 (Euler Characteristic Theorem) *If λ is a holomorphic line bundle over a compact Riemann surface M of arithmetic genus g_a then*

$$\chi(\lambda) = c(\lambda) + 1 - g_a. \tag{2.20}$$

Proof: Any holomorphic line bundle λ over M is the line bundle of a divisor on M by the Existence Theorem, so λ can be written as a product $\lambda = \prod_i \zeta_{p_i}^{n_i}$; it then follows by iterating Lemma 2.18 that $\chi(\lambda) - c(\lambda) = \chi(1) - c(1)$ where 1 is the trivial holomorphic line bundle. The characteristic class of the trivial bundle is $c(1) = 0$; and since a holomorphic cross-section of the trivial line bundle is a holomorphic function on the compact Riemann surface M, so is a constant by the maximum modulus theorem, it follows that $\dim H^0(M, \mathcal{O}) = \dim \Gamma(M, \mathcal{O}) = 1$. The Euler class of the trivial bundle therefore is $\chi(1) = \dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O})) = 1 - g_a$, and that suffices for the proof.

2.6 The Riemann-Roch Theorem

Reference to the first cohomology group can be removed by applying the Serre Duality Theorem, Theorem 1.17; as in Corollary 1.18, the Serre Duality Theorem implies that

$$\dim H^1\big(M, \mathcal{O}(\lambda)\big) = \dim \Gamma\big(M, \mathcal{O}^{(1,0)}(\lambda^{-1})\big) \tag{2.21}$$

for any holomophic line bundle λ, where these are finite-dimensional vector spaces. To reinterpret this result, holomorphic differential forms $\phi \in \Gamma(M, \mathcal{O}^{(1,0)})$ on an arbitrary Riemann surface M can be written explicitly in a local coordinate neighborhood U_α with local coordinate z_α in the form $\phi = f_\alpha dz_\alpha$ for a holomorphic function f_α in U_α. In the intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods $\phi = f_\alpha\, dz_\alpha = f_\beta dz_\beta$ and consequently

$$f_\alpha = \frac{dz_\beta}{dz_\alpha} f_\beta = \left(\frac{dz_\alpha}{dz_\beta}\right)^{-1} f_\beta. \tag{2.22}$$

The derivatives

$$\kappa_{\alpha\beta} = \frac{dz_\beta}{dz_\alpha} = \left(\frac{dz_\alpha}{dz_\beta}\right)^{-1} \tag{2.23}$$

are holomorphic and nowhere vanishing functions in the intersections $U_\alpha \cap U_\beta$ of pairs of sets, and it follows immediately from the chain rule for differentiation that $\kappa_{\alpha\beta} \cdot \kappa_{\beta\gamma} \cdot \kappa_{\gamma\alpha} = 1$ in any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$; thus $\{U_\alpha, \kappa_{\alpha\beta}\}$ is a holomorphic coordinate line bundle over M, describing a holomorphic line bundle. The same construction can be applied to the union of any two coordinate coverings of the surface, from which it is evident that the line bundle described by these coordinate line bundles is independent of the choice of a coo/rdinate covering of the surface; that line bundle is called the *canonical bundle* of the Riemann surface M, and is denoted by κ. By (2.22) the coefficients f_α of a holomorphic differential form on M are a holomorphic cross-section of the canonical line bundle κ, so there results the natural identification $\mathcal{O}^{(1,0)} \cong \mathcal{O}(\kappa)$ or more generally

$$\mathcal{O}^{(1,0)}(\lambda) \cong \mathcal{O}(\kappa\lambda) \quad \text{for any line bundle } \lambda; \tag{2.24}$$

Serre duality in the form (2.21) then can be rewritten

$$\dim H^1\big(M, \mathcal{O}(\lambda)\big) = \dim \Gamma\big(M, \mathcal{O}(\kappa\lambda^{-1})\big) \tag{2.25}$$

for any holomorphic line bundle.

Further results arise from a holomorphic form of the deRham exact sequence of differential forms. The germ of a holomorphic differential form on a Riemann surface is the germ ϕ of a differential form of type $(1,0)$ such that $\overline{\partial}\phi = 0$, as on page 15. Clearly the exterior derivative of the germ of a holomorphic function is the germ of a holomorphic differential form. Conversely a germ ϕ of a holomorphic differential form on a Riemann surface is the germ of a closed differential 1-form since $d\phi = \overline{\partial}\phi = 0$, so by the Poincaré lemma ϕ is the exterior derivative of the germ of a function f; but if $\phi = df = \partial f + \overline{\partial} f$ then $\overline{\partial} f = 0$ since ϕ is of type $(1,0)$, so f is the germ of a holomorphic function. Thus there is the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \xrightarrow{\iota} \mathcal{O} \xrightarrow{d} \mathcal{O}^{(1,0)} \longrightarrow 0 \tag{2.26}$$

on an arbitrary Riemann surface, the holomorphic version of the deRham exact sequence of sheaves. From this exact sequence of sheaves there follows the exact cohomology sequence

$$\begin{aligned} 0 \longrightarrow \Gamma(M,\mathbb{C}) \xrightarrow{\iota} \Gamma(M,\mathcal{O}) \xrightarrow{d} \Gamma(M,\mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^1(M,\mathbb{C}) \xrightarrow{\iota} \\ \xrightarrow{\iota} H^1(M,\mathcal{O}) \xrightarrow{d} H^1(M,\mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^2(M,\mathbb{C}) \longrightarrow 0 \end{aligned}$$

since $H^2(M,\mathcal{O}) = 0$ by the Dolbeault Theorem, Theorem 1.9. If M is compact every holomorphic function is constant, so $\Gamma(M,\mathbb{C}) = \Gamma(M,\mathcal{O}) = \mathbb{C}$. Furthermore it follows from (2.24) and the Serre Duality Theorem (2.25 that

$$\begin{aligned} \dim H^1(M,\mathcal{O}^{(1,0)}) &= \dim H^1(M,\mathcal{O}(\kappa)) = \dim \Gamma(M,\mathcal{O}^{(1,0)}(\kappa^{-1})) \\ &= \dim \Gamma(M,\mathcal{O}) = 1 \end{aligned}$$

while $\dim H^2(M,\mathbb{C}) = 1$ as well, a standard topological result. Thus for a compact Riemann surface this exact cohomology sequence reduces to the short exact sequence

$$0 \longrightarrow \Gamma(M,\mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^1(M,\mathbb{C}) \xrightarrow{\iota} H^1(M,\mathcal{O}) \longrightarrow 0. \tag{2.27}$$

In this sequence $\dim H^1(M,\mathbb{C}) = 2g$ where g is the topological genus[2] of the compact surface M. On the other hand by the Serre Duality Theorem (2.25 again it also follows that $g_a = \dim H^1(M,\mathcal{O}) = \dim \Gamma(M,\mathcal{O}^{(1,0)})$, so (2.27) implies that

$$g_a = \dim H^1(M,\mathcal{O}) = \dim \Gamma(M,\mathcal{O}^{(1,0)}) = \frac{1}{2}\dim H^1(M,\mathbb{C}) = g. \tag{2.28}$$

Thus the arithmetic genus g_a of a compact Riemann surface is equal to its topological genus g; this common value subsequently will be called simply the *genus* of the surface. In these terms the Euler Characteristic Theorem can be rewritten in the following form.

Theorem 2.23 (Riemann-Roch Theorem) *For any holomorphic line bundle λ over a compact Riemann surface M of genus g*

$$\gamma(\lambda) - \gamma(\kappa\lambda^{-1}) = c(\lambda) + 1 - g. \tag{2.29}$$

Proof: From the Serre Duality Theorem (2.25 it follows that

$$\dim H^1(M,\mathcal{O}(\lambda)) = \dim \Gamma(M,\mathcal{O}^{(1,0)}(\lambda^{-1})) = \gamma(\kappa\lambda^{-1});$$

and with these observations the Euler Characteristic Theorem, Theorem 2.22, takes the form (2.29), with the genus g in place of the arithmetic genus g_a in view of (2.28). That suffices for the proof.

[2]See the discussion of the topology of surfaces in Appendix D.

Corollary 2.24 (Canonical Bundle Theorem) *The canonical bundle κ of a compact Riemann surface M of genus g is characterized by*

$$c(\kappa) = 2g - 2 \quad \textit{and} \quad \gamma(\kappa) = g. \tag{2.30}$$

Proof: Since $\gamma(\kappa) = \dim \Gamma(M, \mathcal{O}^{(1,0)}) = g$ by (2.28) and since $\gamma(\kappa\kappa^{-1}) = \gamma(1) = 1$ it follows immediately from (2.23) for the special case in which $\lambda = \kappa$ that $c(\kappa) = 2g - 2$. On the other hand if λ is a holomorphic line bundle with $c(\lambda) = 2g - 2$ and $\gamma(\lambda) = g$ then it follows from (2.29) that $g - \gamma(\kappa\lambda^{-1}) = 2g - 2 + 1 - g = g - 1$, hence that $\gamma(\kappa\lambda^{-1}) = 1$; but since $c(\kappa\lambda^{-1}) = 0$ it follows from Corollary 1.4 that $\kappa\lambda^{-1}$ is the identity bundle, hence that $\lambda = \kappa$. That suffices to conclude the proof.

The Riemann-Roch Theorem in the form given in Theorem 2.24 can be rephrased in a more symmetric way in terms of an auxiliary expression that is useful in various other contexts as well. The *Clifford index* of a holomorphic line bundle λ on a compact Riemann surface is defined by

$$C(\lambda) = c(\lambda) - 2\big(\gamma(\lambda) - 1\big), \tag{2.31}$$

so is another integral invariant associated to a holomorphic line bundle.

Corollary 2.25 (Brill-Noether Formula) *The Clifford index of a holomorphic line bundle λ over a compact Riemann surface satisfies the symmetry condition*

$$C(\lambda) = C(\kappa\lambda^{-1}), \tag{2.32}$$

where κ is the canonical bundle of the surface.

Proof: From the definition of the Clifford index and the Riemann-Roch Theorem (2.29) it follows that

$$\begin{aligned} C(\lambda) &= c(\lambda) - 2\big(\gamma(\lambda) - 1\big) \\ &= c(\lambda) - 2\big(\gamma(\kappa\lambda^{-1}) + c(\lambda) + 1 - g - 1\big) \\ &= c(\kappa\lambda^{-1}) - 2\big(\gamma(\kappa\lambda^{-1}) - 1\big) = C(\kappa\lambda^{-1}), \end{aligned}$$

which concludes the proof.

There is yet another formulation of the Riemann-Roch Theorem that is quite commonly used, one that is expressed entirely in terms of meromorphic functions and differential forms and avoids any mention of line bundles. It is customarily expressed in terms of the complex vector spaces $\mathcal{L}(\mathfrak{d})$ associated to divisors $\mathfrak{d}$ on the surface M, defined by

$$\mathcal{L}(\mathfrak{d}) = \Big\{ f \in \Gamma(M, \mathcal{M}) \ \Big| \ \mathfrak{d}(f) + \mathfrak{d} \ge 0 \Big\}. \tag{2.33}$$

Note that vector spaces $\mathcal{L}(\mathfrak{d}_1)$ and $\mathcal{L}(\mathfrak{d}_2)$ are isomorphic whenever the divisors $\mathfrak{d}_1$ and $\mathfrak{d}_2$ are linearly equivalent; indeed if $\mathfrak{d}_1 \sim \mathfrak{d}_2$ there is a meromorphic function g on M with $\mathfrak{d}(g) = \mathfrak{d}_1 - \mathfrak{d}_2$ and multiplication by g defines an isomorphism

$$\times g : \mathcal{L}(\mathfrak{d}_1) \xrightarrow{\cong} \mathcal{L}(\mathfrak{d}_2). \tag{2.34}$$

Thus the dimension of the vector space $\mathcal{L}(\mathfrak{d})$ depends only on the linear equivalence class of the divisor $\mathfrak{d}$. If $\mathfrak{d} = \sum_i \nu_i \cdot p_i$ for distinct points $p_i \in M$ then $\mathcal{L}(\mathfrak{d})$ consists of those meromorphic functions f on M having a zero at p_i of order at least $-\nu_i$ if $\nu_i \le 0$ and having a pole at p_i of order at most ν_i if $\nu_i \ge 0$. These vector spaces of meromorphic functions have played a major role in the study of function theory on compact Riemann surfaces from the earliest period. If $\zeta_{\mathfrak{d}}$ is the line bundle associated to the divisor $\mathfrak{d}$ and $h \in \Gamma(M, \mathcal{M}(\zeta_{\mathfrak{d}}))$ is a meromorphic cross-section with $\mathfrak{d}(h) = \mathfrak{d}$ then multiplication by h defines an isomorphism

$$\times h : \mathcal{L}(\mathfrak{d}) \xrightarrow{\cong} \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}})), \tag{2.35}$$

so that

$$\dim \mathcal{L}(\mathfrak{d}) = \dim \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}})). \tag{2.36}$$

The divisors of the holomorphic cross-sections $f \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}}))$ are precisely the effective divisors that are linearly equivalent to the divisor $\mathfrak{d}$, a collection of divisors called a *complete linear system* and traditionally denoted by $|\mathfrak{d}|$. This system of divisors can be identified with the projective space $\mathbb{P}\mathcal{L}(\mathfrak{d})$, since two cross-sections $f_1, f_2 \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}}))$ have the same divisor if and only if $f_1 = cf_2$ for some nonzero complex constant c; thus $|\mathfrak{d}| = P\mathcal{L}(\mathfrak{d})$ has the natural structure of a complex projective space of dimension $\gamma(\zeta_{\mathfrak{d}}) - 1$.

There is a corresponding definition for differential forms, expressed in terms of meromorphic differential forms. A *germ of a meromorphic differential form* is an expression of the form $f\,dz$ for the germ of a meromorphic function $f \in \mathcal{M}$; the sheaf of germs of meromorphic differential forms is denoted by $\mathcal{M}^{(1,0)}$, and the global sections in $\Gamma(M, \mathcal{M}^{(1,0)})$ are called *meromorphic differential forms* on M. These are not quite a subset of the space of $\mathcal{C}^\infty$ differential forms of type $(1,0)$ on M, since the meromorphic differential forms are not differentiable at their singularities; but the space of holomorphic differential forms can be identified in the obvious way with a subspace of the space of meromorphic differential forms. Of course there is again the natural identification $\Gamma(M, \mathcal{M}^{(1,0)}) \cong \Gamma(M, \mathcal{M}(\kappa))$, so that meromorphic differential forms can be identified with meromorphic cross-sections of the canonical bundle in the same way that holomorphic differential forms can be identified with holomorphic cross-sections of the canonical bundle; and the notion of the divisor of a meromorphic differential form is consequently well defined. In these terms, let

$$\mathcal{L}^{(1,0)}(\mathfrak{d}) = \left\{ f\,dz \in \Gamma(M, \mathcal{M}^{(1,0)}) \;\middle|\; \mathfrak{d}(f) + \mathfrak{d} \ge 0 \right\}. \tag{2.37}$$

Again if $\mathfrak{d} = \sum_i \nu_i \cdot p_i$ for distinct points $p_i \in M$ then $\mathcal{L}^{(1,0)}(\mathfrak{d})$ consists of those meromorphic differential forms $f\,dz$ on M having a zero at p_i of order at least $-\nu_i$ if $\nu_i \le 0$ and having a pole at p_i of order at most ν_i if $\nu_i \ge 0$. If $h \in \Gamma(M, \mathcal{M}(\zeta_{\mathfrak{d}}))$ is a meromorphic cross-section with $\mathfrak{d}(h) = \mathfrak{d}$ then multiplication by h defines an isomorphism

$$\times h : \mathcal{L}^{(1,0)}(\mathfrak{d}) \xrightarrow{\cong} \Gamma(M, \mathcal{O}^{(1,0)}(\zeta_{\mathfrak{d}})) = \Gamma(M, \mathcal{O}(\kappa\zeta_{\mathfrak{d}})), \tag{2.38}$$

and since $\dim \Gamma(M, \mathcal{O}^{(1,0)}(\zeta_{\mathfrak{d}})) = \dim \Gamma(M, \mathcal{O}(\kappa\zeta_{\mathfrak{d}}))$ it follows that

$$\dim \mathcal{L}^{(1,0)}(\mathfrak{d}) = \gamma(\kappa\zeta_{\mathfrak{d}}). \tag{2.39}$$

In these terms, the Riemann-Roch Theorem can be rephrased as follows.

Corollary 2.26 (Riemann-Roch Theorem) *If $\mathfrak{d}$ is a divisor on a compact Riemann surface M of genus g then*

$$\dim \mathcal{L}(\mathfrak{d}) - \dim \mathcal{L}^{(1,0)}(-\mathfrak{d}) = \deg \mathfrak{d} + 1 - g. \tag{2.40}$$

Proof: This follows immediately from the Riemann-Roch Theorem (2.29) in Corollary 2.24, applied to the line bundle $\lambda = \zeta_{\mathfrak{d}}$, in view of the identifications (2.36) and (2.39), and that suffices for a proof.

A slight variant of this version of the Riemann-Roch Theorem replaces the meromorphic differential forms by their divisors. The divisors on M that are associated to the canonical bundle are called *canonical divisors* on M, and are customarily denoted by $\mathfrak{k}$; it is important to keep in mind that $\mathfrak{k}$ does not represent a single divisor, but rather any of a large class of linearly equivalent divisors. The divisor of any holomorphic differential form on M is a positive canonical divisor, for instance, and the divisor of any meromorphic differential form on M is a not necessarily positive canonical divisor. Alternatively, a canonical divisor is any divisor $\mathfrak{k}$ with the property that $\zeta_{\mathfrak{k}} = \kappa$. A divisor $\mathfrak{d}'$ is *residual* to a divisor $\mathfrak{d}$ if the sum of these divisors is a canonical divisor, that is, if $\mathfrak{d}' + \mathfrak{d} = \mathfrak{k}$; the divisor $\mathfrak{d}$ of course is then residual to the divisor $\mathfrak{d}'$, so that this is a dual relationship between divisors. In these terms, the Riemann-Roch Theorem can be rephrased yet again as follows.

Corollary 2.27 (Riemann-Roch Theorem) *If $\mathfrak{d}'$ is the residual divisor to a divisor $\mathfrak{d}$ on a compact Riemann surface M of genus g then*

$$\dim \mathcal{L}(\mathfrak{d}) - \dim \mathcal{L}(\mathfrak{d}') = \deg \mathfrak{d} + 1 - g. \tag{2.41}$$

Proof: This follows immediately from the preceding Corollary 2.26 upon noting that $\dim \mathcal{L}(\mathfrak{d}') = \dim \mathcal{L}(\mathfrak{k} - \mathfrak{d}) = \gamma(\zeta_{\mathfrak{k}-\mathfrak{d}}) = \gamma(\kappa\zeta_{-\mathfrak{d}}) = \dim \mathcal{L}^{(1,0)}(-\mathfrak{d})$ as a consequence of (2.39), and that suffices for the proof.

The Riemann-Roch Theorem has useful applications to the examination of base-point free holomorphic line bndles.

Theorem 2.28 *Let M be a compact Riemann surface of genus g.*
(i) *Any holomorphic line bundle λ on M with $c(\lambda) \geq 2g$ is base-point-free.*
(ii) *If $g > 0$ all holomorphic line bundles λ on M for which $c(\lambda) = 2g - 1$ are base-point-free except for those bundles of the form $\lambda = \kappa\zeta_a$ for some point*

$a \in M$, and none of the latter bundles is base-point-free.
(iii) *If $g > 0$ the canonical bundle κ is base-point-free; all other holomorphic line bundles λ on M for which $c(\lambda) = 2g-2$ also are base-point-free except for those bundles of the form $\lambda = \kappa\zeta_a\zeta_b^{-1}$ for two distinct points $a, b \in M$, and none of the latter bundles is base-point-free.*

Proof: (i) If λ is a holomorphic line bundle over M for which $c(\lambda) \geq 2g$ then $c(\kappa\lambda^{-1}) < c(\kappa\lambda^{-1}\zeta_a) < 0$ for any point $a \in M$, so $\gamma(\kappa\lambda^{-1}) = \gamma(\kappa\lambda^{-1}\zeta_a) = 0$ by Corollary 1.3 and it follows from the Riemann-Roch Theorem (2.29) that $\gamma(\lambda) = c(\lambda) + 1 - g \geq g + 1 > 0$ and $\gamma(\lambda\zeta_a^{-1}) = c(\lambda\zeta_a^{-1}) + 1 - g = \gamma(\lambda) - 1$; so by Lemma 2.10 the bundle λ is base-point-free.
(ii) If $c(\lambda) = 2g - 1$ then $c(\kappa\lambda^{-1}) < 0$ so $\gamma(\kappa\lambda^{-1}) = 0$ by Corollary 1.3 and it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = c(\lambda) + 1 - g = g > 0$. If λ is not base-point-free then by Lemma 2.10 there is a point $a \in M$ such that $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) = g$; and since $c(\lambda\zeta_a^{-1}) = 2g - 2$ the Canonical Bundle Theorem, Theorem 2.24, shows that $\lambda\zeta_a^{-1} = \kappa$. On the other hand if $\lambda = \kappa\zeta_a$ for a point $a \in M$ then $\gamma(\lambda\zeta_a^{-1}) = \gamma(\kappa) = g = \gamma(\lambda)$ so by Lemma 2.10 the bundle $\kappa\zeta_a$ is not base-point-free.
(iii) Since $g > 0$ it follows from Theorem 2.4 that $\gamma(\zeta_a) = 1$ for any point $a \in M$; then by the Riemann-Roch Theorem $\gamma(\kappa\zeta_a^{-1}) = \gamma(\zeta_a) + g - 2 = g - 1 = \gamma(\kappa) - 1$, since $\gamma(\kappa) = g$ by the Canonical Bundle Theorem, Theorem 2.24, so by Lemma 2.10 the bundle κ is base-point-free. If λ is a line bundle for which $c(\lambda) = 2g-2$ then $c(\kappa\lambda^{-1}) = 0$, so if $\lambda \neq \kappa$ then $\kappa\lambda^{-1} \neq 1$ and $\gamma(\kappa\lambda^{-1}) = 0$ by Corollary 1.4; it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = g - 1$. The case $g = 1$ is slightly special, for $\gamma(\lambda) = g-1 = 0$ as just demonstrated so λ is not base-point-free; however $c(\lambda) = 0$ so $c(\lambda\,\zeta_b) = 1$ for any point $b \in M$, and from the Riemann-Roch Theorem it follows that $\gamma(\lambda\zeta_b) = \gamma(\kappa\lambda^{-1}\zeta_b^{-1}) + 1 = 1$, since $c(\kappa\lambda^{-1}\zeta_b^{-1}) < 0$, so $\lambda\zeta_b = \zeta_a$ for some point $a \in M$ by Theorem 2.4 and consequently $\lambda = \zeta_a\zeta_b^{-1}$. In the more general case if $g > 1$ and the bundle λ is not base-point-free then by Lemma 2.10 there is a point $a \in M$ for which $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) = g - 1$. It then follows from the Riemann-Roch Theorem that $\gamma(\kappa\lambda^{-1}\zeta_a) = \gamma(\lambda\zeta_a^{-1}) + 2 - g = 1$, and since $c(\kappa\lambda^{-1}\zeta_a) = 1$ then $\kappa\lambda^{-1}\zeta_a = \zeta_b$ is a point bundle by Theorem 2.4; consequently $\lambda = \kappa\zeta_a\zeta_b^{-1}$, where $a \neq b$ since $\lambda \neq \kappa$. Conversely if $\lambda = \kappa\zeta_a\zeta_b^{-1}$ for points $a \neq b$ on the Riemann surface M then $\lambda \neq \kappa$ so $\gamma(\lambda) = g - 1$, and $\gamma(\lambda\zeta_a^{-1}) = \gamma(\kappa\zeta_b^{-1}) = \gamma(\zeta_b) + g - 2 = g - 1$ as well; hence by Lemma 2.10 the bundle λ is not base-point-free, which suffices to conclude the proof.

Since the canonical bundle of a compact Riemann surface of genus $g > 0$ is base-point-free by part (iii) of the preceding theorem, it follows that in part (ii) the product $\lambda = \kappa\zeta_a$ is the base decomposition of λ for any point $a \in M$.

2.7 Dual Base Divisors

The Riemann-Roch theorem closely relates the holomorphic lime bundles λ and $\kappa\lambda^{-1}$, so it is convenient to view them as *dual line bundles* and to set

$\lambda^* = \kappa\lambda^{-1}$. If the dual bundle λ^* satisfies $\gamma(\lambda^*) > 0$ then its base divisor $\mathfrak{b}(\lambda^*)$ is well defined; it is called the *dual base divisor* of the line bundle λ and it is denoted by $\mathfrak{b}^*(\lambda) = \mathfrak{b}(\lambda^*)$. The points appearing in the dual base divisor are called the *dual base points* of the line bundle λ. In terms of the dual line bundle the Riemann-Roch Theorem can be written

$$\gamma(\lambda) = \gamma(\lambda^*) + c(\lambda) + 1 - g \tag{2.42}$$

and has several immediate implications for properties of the line bundle λ itself.

Theorem 2.29 *Let λ be a holomorphic line bundle over a compact Riemann surface M of genus $g > 0$ and let λ^* be its dual bundle.*
(i) *If $\gamma(\lambda^*) = 0$ then*

$$\gamma(\lambda\zeta_{\mathfrak{d}}) = \gamma(\lambda) + \deg \mathfrak{d} \quad \textit{for any positive divisor } \mathfrak{d} \tag{2.43}$$

(ii) *If λ^* is base-point-free then*

$$\gamma(\lambda\zeta_a) = \gamma(\lambda) \quad \textit{for any point } a \in M. \tag{2.44}$$

(iii) *If $\mathfrak{b}^*$ is the dual base divisor of λ then*

$$\gamma(\lambda\zeta_{\mathfrak{b}^*}) = \gamma(\lambda) + \deg \mathfrak{b}^*, \quad \textit{and} \tag{2.45}$$

$$\gamma(\lambda\zeta_{\mathfrak{b}^*}\zeta_a) = \gamma(\lambda) + \deg \mathfrak{b}^* \quad \textit{for any point } a \in M, \textit{ while} \tag{2.46}$$

$$\gamma(\lambda\zeta_a) = \gamma(\lambda) \quad \textit{for any point } a \notin \mathfrak{b}^*. \tag{2.47}$$

Proof: (i) It follows from the Riemann-Roch Theorem (2.29) that for any divisor $\mathfrak{d}$ on M

$$\gamma(\lambda\zeta_{\mathfrak{d}}) - \gamma(\lambda) = \gamma(\kappa\lambda^{-1}\zeta_{\mathfrak{d}}^{-1}) - \gamma(\kappa\lambda^{-1}) + \deg \mathfrak{d}. \tag{2.48}$$

If $\gamma(\kappa\lambda^{-1}) = 0$ then $\gamma(\kappa\lambda^{-1}\zeta_{\mathfrak{d}}^{-1}) = 0$ for any positive divisor $\mathfrak{d}$, since multiplication by a nontrivial holomorphic cross-section $f \in \Gamma(M, \zeta_{\mathfrak{d}})$ is an injective linear mapping $\times f : \Gamma(M, \kappa\lambda^{-1}\zeta_{\mathfrak{d}}^{-1}) \longrightarrow \Gamma(M, \kappa\lambda^{-1})$; and in that case (2.43) is an immediate consequence of (2.48).
(ii) If $\kappa\lambda^{-1}$ is base-point-free then $\gamma(\kappa\lambda^{-1}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}) - 1$ for any point $a \in M$, and in that case (2.44) is an immediate consequence of (2.48) for the divisor $\mathfrak{d} = 1 \cdot a$.
(iii) If $\kappa\lambda^{-1} = \lambda_0\zeta_{\mathfrak{b}^*}$ is the base decomposition of this line bundle then $\gamma(\lambda_0\zeta_{\mathfrak{b}^*}) = \gamma(\lambda_0)$ by Theorem 2.12 (i), or since $\lambda_0 = \kappa\lambda^{-1}\zeta_{\mathfrak{b}^*}^{-1}$ equivalently $\gamma(\kappa\lambda^{-1}) = \gamma(\kappa\lambda^{-1}\zeta_{\mathfrak{b}^*}^{-1})$; and in that case (2.45) is an immediate consequence of (2.48) for the divisor $\mathfrak{d} = \mathfrak{b}^*$.
Since λ_0 is base-point-free $\gamma(\lambda_0\zeta_a^{-1}) = \gamma(\lambda_0) - 1$ for any point $a \in M$ by Lemma 2.10, or equivalently $(\kappa\lambda^{-1}\zeta_{\mathfrak{b}^*}^{-1}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}\zeta_{\mathfrak{b}^*}^{-1}) - 1$, and it then follows from (2.48) for the line bundle $\lambda\zeta_{\mathfrak{b}^*}$ in place of λ and the divisor $\mathfrak{d} = 1 \cdot a$

that $\gamma(\lambda\zeta_{\mathfrak{b}^*}\zeta_a) = \gamma(\lambda\zeta_{\mathfrak{b}^*})$; this together with (2.45) yields (2.46).
Finally since $\mathfrak{b}^*$ is the common divisor of all the holomorphic cross-sections of the bundle $\lambda_0\zeta_{\mathfrak{b}^*}$, not all of these cross-sections vanish at a point $a \notin \mathfrak{b}^*$, and it then follows from Lemma 2.6 that $\gamma(\lambda_0\zeta_{\mathfrak{b}^*}\zeta_a^{-1}) = \gamma(\lambda_0\zeta_{\mathfrak{b}^*}) - 1$ or equivalently $\gamma(\kappa\lambda^{-1}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1}) - 1$; and in that case (2.47) is an immediate consequence of (2.48) for the divisor $\mathfrak{d} = 1 \cdot a$. That suffices for the proof.

Corollary 2.30 (i) *If M is a compact Riemann surface of genus $g > 0$ and λ is a holomorphic line bundle over M with the dual base divisor $\mathfrak{b}^* = b_1 + \cdots + b_n$ then*

$$\gamma(\lambda\zeta_a) = \begin{cases} \gamma(\lambda) + 1 & \textit{if} \quad a \in \mathfrak{b}^*, \\ \gamma(\lambda) & \textit{if} \quad a \notin \mathfrak{b}^*. \end{cases} \tag{2.49}$$

(ii) *If in addition the bundle λ is base-point-free then whenever $\mathfrak{b}^* = \mathfrak{b}' + \mathfrak{b}''$ for some positive divisors $\mathfrak{b}'$ and $\mathfrak{b}''$ the line bundle $\lambda\zeta_{\mathfrak{b}'}$ is also base-point-free, and $\gamma(\lambda\zeta_{\mathfrak{b}'}\zeta_a) = \gamma(\lambda\zeta_{\mathfrak{b}'})$ for any point $a \notin \mathfrak{b}^*$.*

Proof: (i) If $\mathfrak{b}^* = \emptyset$ the line bundle $\kappa\lambda^{-1}$ is base-point-free and (2.49) is just (2.44) of the preceding theorem. Otherwise the line bundle $\kappa\lambda^{-1}$ has the base decomposition $\kappa\lambda^{-1} = \lambda_0\zeta_{\mathfrak{b}^*}$ for a base-point-free holomorphic line bundle λ_0 and the positive divisor $\mathfrak{b}^* = b_1 + \cdots + b_n$. It then follows from (2.47) in the preceding theorem that $\gamma(\lambda\zeta_a) = \gamma(\lambda)$ for any point $a \notin \mathfrak{b}^*$. On the other hand it follows from (2.45) in the preceding theorem that

$$\gamma(\lambda\zeta_{b_1 + \cdots b_n}) = \gamma(\lambda) + n,$$

while by Lemma 2.6

$$\gamma(\lambda\zeta_{b_1+\cdots+b_{i-1}}) \le \gamma(\lambda\zeta_{b_1+\cdots+b_{i-1}+b_i}) \le \gamma(\lambda\zeta_{b_1+\cdots+b_{i-1}}) + 1$$

for $1 \le i \le n$; it is then evident from the two preceding equations that

$$\gamma(\lambda\zeta_{b_1+\cdots+b_i}) = \gamma(\lambda\zeta_{b_1+\cdots+b_{i-1}}) + 1 \tag{2.50}$$

for $1 \le i \le n$. In particular $\gamma(\lambda\zeta_{b_1}) = \gamma(\lambda) + 1$, hence $\gamma(\lambda\zeta_a) = \gamma(\lambda) + 1$ for any point $a \in \mathfrak{b}^*$.
(ii) If λ is base-point-free then since $\gamma(\lambda\zeta_{b_1}) = \gamma(\lambda) + 1$ by (2.50) it follows from Theorem 2.12 (ii) that $\lambda\zeta_{b_1}$ is base-point-free; then since $\gamma(\lambda\zeta_{b_1+b_2}) = \gamma(\lambda\zeta_{b_1}) + 1$ by (2.50) it also follows from Theorem 2.12 (ii) that $\lambda\zeta_{b_1+b_2}$ is base-point-free; and by repeating this argument it follows that all the line bundles $\zeta_{b_1+\cdots+b_i}$ for $1 \le i \le n$ are base-point-free. For any point $a \notin \mathfrak{b}^*$ it follows from (2.46) and (2.47) of Theorem 2.29 (iii) that $\gamma(\lambda\zeta_a\zeta_{\mathfrak{b}}^*) = \gamma(\lambda\zeta_a) + n$; it is then possible to apply to the line bundle $\lambda\zeta_a$ the argument leading to (2.50) in part (i) of the proof to show that

$$\gamma(\lambda\zeta_a\zeta_{b_1+\cdots+b_i}) = \gamma(\lambda\zeta_a\zeta_{b_1+\cdots+b_{i-1}}) + 1$$

for $1 \le i \le n$, and consequently that

$$\gamma(\lambda\zeta_a\zeta_{b_1+\cdots+b_i}) = \gamma(\lambda\zeta_a) + i$$

for $1 \le i \le n$. On the other hand it follows from (2.50) that

$$\gamma(\lambda\zeta_{b_1+\cdots+b_i}) = \gamma(\lambda) + i,$$

and since $\gamma(\lambda\zeta_a) = \gamma(\lambda)$ by (2.47) it follows that

$$\gamma(\lambda\zeta_a\zeta_{b_1+\cdots+b_i}) = \gamma(\lambda\zeta_{b_1+\cdots+b_i}),$$

which suffices to conclude the proof.

For most base-point-free holomorphic line bundles λ on a compact Riemann surface M of genus $g > 0$ the bundle $\kappa\lambda^{-1}$ is not base-point-free; indeed all bundles λ for which $\gamma(\lambda) \ge 2g$ are base-point-free by Theorem 2.28, but $\gamma(\kappa\lambda^{-1}) = 0$ for all of these bundles. On the other hand there are base-point-free holomorphic line bundles λ on M such that $\kappa\lambda^{-1}$ is base-point-free; for instance the identity bundle 1 is base-point-free, as observed on page 39, and the canonical bundle $\kappa = \kappa \cdot 1^{-1}$ is base-point-free by Theorem 2.28 (iii). A pair of base-point-free holomorphic line bundles (λ_1, λ_2) is called a *dual pair of base-point-free holomorphic line bundles* over M if $\lambda_1\lambda_2 = \kappa$ is the canonical bundle of M; the pair of holomorphic line bundles $(1, \kappa)$ thus is an example of a dual pair of base-point-free holomorphic line bundles. Of course it may be the case that $\lambda_1 = \lambda_2$ for a particular dual pair of base-point-free holomorphic line bundles, as for instance the pair $(1, 1)$ on a surface of genus $g = 1$ since in that case $\kappa = 1$ by Corollary **??**. Although the line bundles appearing in dual pairs of base-point-free holomorphic line bundles are somewhat special base-point-free line bundles, nonetheless many base-point-free holomorphic line bundles on M can be expressed in terms of dual pairs of base-point-free line bundles over M.

Corollary 2.31 *To any base-point-free holomorphic line bundle λ for which $\gamma(\kappa\lambda^{-1}) > 0$ over a compact Riemann surface M of genus $g > 0$ there correspond a unique dual pair of base-point-free holomorphic line bundles (λ_1, λ_2) over M and a unique positive divisor $\mathfrak{b}^{**}$ such that $\lambda_1 = \lambda\zeta_{\mathfrak{b}^{**}}$ and $\lambda_2 = \kappa\lambda^{-1}\zeta_{\mathfrak{b}^{**}}^{-1}$.*

Proof: If λ is a base-point-free holomorphic line bundle over M for which $\gamma(\kappa\lambda^{-1}) > 0$ either the line bundle $\kappa\lambda^{-1}$ is base-point-free, in which case $(\lambda, \kappa\lambda^{-1})$ already is a dual pair of base-point-free holomorphic line bundles; or alternatively the line bundle $\kappa\lambda^{-1}$ has a base decomposition $\kappa\lambda^{-1} = \lambda_2\zeta_{\mathfrak{b}^{**}}$ for a base-point-free holomorphic line bundle λ_2 and a positive divisor $\mathfrak{b}^{**}$, where the divisor $\mathfrak{b}^{**}$ is uniquely determined as the base divisor of the line bundle $\kappa\lambda^{-1}$ and hence the line bundle λ_2 also is uniquely determined. In the latter case it follows from Corollary 2.30 (ii) that the line bundle $\lambda_1 = \lambda\zeta_{\mathfrak{b}^{**}}$ is base-point-free; and since $\kappa\lambda^{-1} = \lambda_2\zeta_{\mathfrak{b}^{**}}$ then $\kappa = \lambda\zeta_{\mathfrak{b}^{**}} \cdot \lambda_2 = \lambda_1 \cdot \lambda_2$, so (λ_1, λ_2) is a dual pair of base-point-free line bundles over M. That suffices to conclude the proof.

Corollary 2.32 *If M is a compact Riemann surface of genus $g > 0$ and r is an integer in Lüroth semigroup $\mathcal{L}(M)$ of M such that $0 \le r \le g-2$ and that $r+1$ is not in the Lüroth semigroup of M, then any base-point-free holomorphic line bundle λ such that $c(\lambda) = r$ is part of a dual pair (λ, λ_0) of base-point-free holomorphic line bundles over M.*

Proof: If λ is base-point-free and $c(\lambda) = r$ where $0 \le r \le g$ then by the Riemann-Roch theorem in the form of Theorem 2.24 it follows that $\gamma(\kappa\lambda^{-1}) = \gamma(\lambda) + g - 1 - r \ge g - 1 - r > 0$, and consequently there is a base decomposition $\kappa\lambda^{-1} = \lambda_0\zeta_{\mathfrak{b}}$ for a base-point-free holomorphic line bundle λ_0 and a positive divisor $\mathfrak{b}^{**}$. If $\mathfrak{b}^{**} \ne \emptyset$ then for any point $a \in \mathfrak{b}^{**}$ it follows from Corollary 2.30 (ii) that the line bundle $\lambda\zeta_a$ also is base-point-free, and therefore that $r+1 \in \mathcal{L}(M)$. By assumption that is not the case, and therefore $\mathfrak{b}^{**} = \emptyset$ so $\kappa\lambda^{-1} = \lambda_0$ is base-point-free and consequently (λ, λ_0) is a dual pair of base-point-free holomorphic line budles over M, which concludes the proof.

Chapter 3

Jacobi and Picard Varieties

3.1 Period Matrices

Holomorphic differential forms on a compact Riemann surface often are called *holomorphic abelian differentials* or *abelian differentials of the first kind.* They are of interest only on surfaces of genus $g > 0$, since as noted earlier there are no nontrivial holomorphic differential forms on a compact Riemann surface of genus $g = 0$. When a compact Riemann surface M of genus $g > 0$ is identified with the quotient $M = \widetilde{M}/\Gamma$ of its universal covering space $\widetilde{M}$ by the group Γ of covering translations[1] a holomorphic function on M can be viewed alternatively as a Γ-invariant holomorphic function on $\widetilde{M}$. It is convenient to be able to pass back and forth between these two perspectives quite freely, and often will be done without explicit comment; in particular no attempt will be made to use a notation that distinguishes between these perspectives. That should not cause any confusion, since in most cases the relevant interpretation will be apparent from the context and generally it does not matter anyway. A holomorphic abelian differential on M can be viewed as a Γ-invariant holomorphic differential form on $\widetilde{M}$ in the same way, and will be viewed as such without further comment whenever it is convenient to do so. On the other hand the integral

$$w(z, a) = \int_a^z \omega \tag{3.1}$$

of a holomorphic abelian differential ω on the Riemann surface M is a well defined holomorphic function of the variables $(z, a) \in M \times M$ locally, since ω is a closed differential form, but is inevitably a multiple-valued function of these variables in the large; however the monodromy theorem ensures that this integral is a single-valued holomorphic function on the simply connected complex manifold $\widetilde{M} \times \widetilde{M}$, independent of the choice of the path of integration on $\widetilde{M}$.

[1] A survey of the topological properties of surfaces prerequisite to the discussion here can be found in Appendix D.

This function is called a *holomorphic abelian integral* on the Riemann surface M, although of course really it is defined as a holomorphic function on the universal covering space $\widetilde{M}$ in both variables. A holomorphic abelian integral clearly satisfies the symmetry condition $w(z,a) = -w(a,z)$, and $w(z,z) = 0$ for all points $z \in \widetilde{M}$. It is more convenient in many circumstances to view a holomorphic abelian integral as a function of the first variable only, and to allow it to be modified by an arbitrary additive constant; in such cases the function is denoted by $w(z)$ rather than $w(z,a)$, but also is called a holomorphic abelian integral. It must be kept in mind though that the abelian integral $w(z)$ is determined by the abelian differential ω only up to an arbitrary additive constant. For any choice of the abelian integral $w(z)$ the integral (3.1) is given by $w(z,a) = w(z) - w(a)$.

Lemma 3.1 *The holomorphic abelian integrals $w(z)$ on a compact Riemann surface M of genus $g > 0$ can be characterized as those holomorphic functions on the universal covering space $\widetilde{M}$ of the surface that satisfy*

$$w(Tz) = w(z) + \omega(T) \quad \textit{for all} \quad T \in \Gamma \tag{3.2}$$

for a group homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$, where Γ is the covering translation group of M.

Proof: If $w(z)$ is a holomorphic abelian integral on M then $dw(Tz) = dw(z)$ for any covering translation $T \in \Gamma$, since $dw = \omega$ is invariant under Γ; therefore $w(Tz) = w(z) + \omega(T)$ for some complex constant $\omega(T)$. For any two covering translations $S, T \in \Gamma$

$$\begin{aligned} &w(STz) = w(z) + \omega(ST) \quad \text{and} \\ &w(STz) = w(S \cdot Tz) = w(Tz) + \omega(S) = w(z) + \omega(T) + \omega(S); \end{aligned}$$

consequently $\omega(ST) = \omega(S) + \omega(T)$ so the mapping $T \longrightarrow \omega(T)$ is a group homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$. Conversely if $w(z)$ is a holomorphic function on $\widetilde{M}$ that satisfies (3.2) for some homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$ then $\omega = dw$ is invariant under Γ so is a holomorphic abelian differential on M, and the function $w(z)$ is the integral of ω and hence is a holomorphic abelian integral. That suffices for the proof.

Clearly the homomorphism ω in (3.2) is unchanged when the abelian integral $w(z)$ is replaced by $w(z)+c$ for a complex constant c, so it is determined uniquely by the abelian differential $\omega = dw$; it is called the *period class* of the holomorphic abelian differential ω. The mapping that associates to a holomorphic abelian differential its period class is a homomorphism $\Gamma(M, \mathcal{O}^{(1,0)}) \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C})$ between these two additive abelian groups.

Lemma 3.2 *A holomorphic abelian differential on a compact Riemann surface of genus $g > 0$ is determined uniquely by its period class.*

Proof: If the period class $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$ of a holomorphic abelian differential ω is identically zero then in view of (3.2) any associated holomorphic abelian integral is a Γ-invariant holomorphic function w on $\widetilde{M}$, or equivalently is a holomorphic function on the compact Riemann surface M; so by the maximum modulus theorem it must be constant and therefore $\omega = dw = 0$, which suffices for the proof.

A homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$ is necessarily trivial on the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$, so induces a homomorphism from the abelianized group $\Gamma/[\Gamma, \Gamma] \cong H_1(M)$ to the complex numbers and therefore can be viewed as an element in the dual group $\mathrm{Hom}(H_1(M), \mathbb{C}) = H^1(M, \mathbb{C})$; and conversely any cohomology class $\omega \in H^1(M, \mathbb{C}) = \mathrm{Hom}(H_1(M), \mathbb{C})$ is induced by a unique homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$. A homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C})$ consequently is determined uniquely either by its values $\omega(\tau_j)$ on a basis $\tau_j \in H_1(M)$ for the homology of M or by its values $\omega(T_j)$ on covering transformations $T_j \in \Gamma$ generating the covering translation group Γ of M. Under the canonical isomorphism $\pi_{z_0} : \Gamma \longrightarrow \pi_1(M, \pi(z_0))$ between the covering translation group and the fundamental group of the surface determined by the choice of a base point $z_0 \in \widetilde{M}$, as discussed in Appendix D.1, an element $T \in \Gamma$ is associated to the homotopy class of the image $\pi(\tilde{\tau}) \subset M$ under the covering projection $\pi : \widetilde{M} \longrightarrow M$ of any path $\tilde{\tau} \subset \widetilde{M}$ from the base point $z_0 \in \widetilde{M}$ to the point $Tz_0 \in \widetilde{M}$. If ω is a holomorphic abelian differential on M and $w(z)$ is its integral then $\int_{\pi(\tilde{\tau})} \omega = \int_{\tilde{\tau}} \omega = w(Tz_0) - w(z_0) = \omega(T)$, so the period class represents the periods of the holomorphic differential form in the customary sense. The integral of course depends only on the homology class represented by the path $\pi(\tilde{\tau}) \subset M$.

Holomorphic abelian differentials on a compact Riemann surface M of genus $g > 0$ can be identified with holomorphic cross-sections of the canonical bundle κ of M, extending the local identification (2.24); consequently by the Canonical Bundle Theorem, Corollary 2.24, the set of holomorphic abelian differentials form a complex vector space of dimension g. If $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for $1 \le i \le g$ is a basis for the holomorphic abelian differentials on M and $\tau_j \in H_1(M)$ for $1 \le j \le 2g$ is a basis for the homology of M the values $\omega_{ij} = \int_{\tau_j} \omega_i$ can be viewed as forming a $g \times 2g$ complex matrix Ω; this is called the *period matrix* of the Riemann surface in terms of the bases ω_i and τ_j. An arbitrary holomorphic abelian differential ω on M can be expressed as the sum $\omega = \sum_{i=1}^{g} c_i \omega_i$ for some complex constants c_i, and an arbitrary homology class τ on M can be expressed as the sum $\tau = \sum_{j=1}^{2g} n_j \tau_j$ for some integers n_j; consequently $\omega(\tau) = \sum_{i=1}^{g} \sum_{j=1}^{2g} c_i \omega_i(n_j \tau_j) = \sum_{i=1}^{g} \sum_{j=1}^{2g} c_i \omega_{ij} n_j$, or in matrix notation $\omega(\tau) = {}^t c\, \Omega\, n$ for the column vectors $c = \{c_i\} \in \mathbb{C}^g$ and $n = \{n_j\} \in \mathbb{Z}^{2g}$. Thus all the periods of the holomorphic abelian differentials on M can be expressed in this way in terms of the period matrix Ω for any choice of bases ω_i and τ_j.

Theorem 3.3 *The period matrix Ω of a compact Riemann surface of genus*

$g > 0$ *is a nonsingular period matrix*[2].

Proof: By definition the period matrix Ω is a nonsingular period matrix if and only if its columns are linearly independent over the real numbers; and by Lemma F.1 that is equivalent to the condition that the associated full period matrix $\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}$ is a nonsingular $2g \times 2g$ matrix in the usual sense. If $\Omega = A + iB$ for some $g \times 2g$ real matrices A and B then

$$\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix} = \begin{pmatrix}\mathrm{I} & i\,\mathrm{I}\\ \mathrm{I} & -i\,\mathrm{I}\end{pmatrix}\begin{pmatrix}A\\ B\end{pmatrix}$$

where I is the $g \times g$ identity matrix; hence the full period matrix is nonsingular if and only if the $2g \times 2g$ real matrix $C = \begin{pmatrix}A\\ B\end{pmatrix}$ is nonsingular. If the matrix C is singular there is a nontrivial real row vector $y \in \mathbb{R}^{2g}$ such that $y\,C = 0$; and if $y = (y_1, y_2)$ for some real row vectors $y_i \in \mathbb{R}^g$ then $y_1A + y_2B = 0$. The vector $c = y_1 - iy_2 \in \mathbb{C}^g$ consequently is a nontrivial complex row vector such that

$$\Re(c\,\Omega) = \Re\Big((y_1 - iy_2)(A + iB)\Big) = y_1 + y_2 = 0,$$

where $\Re(z)$ denotes the real part of the complex vector z. The periods of the nontrivial holomorphic abelian differential $\omega = \sum_i c_i\omega_i$ on a basis for the homology group $H_1(M)$ are the entries of the vector $c\,\Omega$ so are purely imaginary; consequently the period $\omega(T)$ is purely imaginary for any covering translation $T \in \Gamma$. If $w(z)$ is the integral of the nontrivial holomorphic abelian differential ω then $|\exp w(Tz)| = |\exp\big(w(z) + \omega(T)\big)| = |\exp w(z)|$ for every covering translation $T \in \Gamma$ since $|\exp \omega(T)| = 1$; hence $|\exp w(z)|$ is a well defined continuous function on the compact Riemann surface M so by the maximum modulus theorem the holomorphic function $\exp w(z)$ must be constant. The abelian integral $w(z)$ itself then is also constant and $\omega = dw = 0$, a contradiction since $\omega \neq 0$. That suffices to conclude the proof.

The essence of the proof of the preceding theorem is the observation that *a nontrivial holomorphic abelian differential cannot have purely imaginary periods*, or of course purely real periods either; that is one restriction on the possible period matrices of Riemann surfaces, but there are deeper restrictions that will be discussed later. The following simple consequences of the preceding theorem are useful in various circumstances.

Corollary 3.4 *If $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for $1 \le i \le g$ is a basis for the holomorphic abelian differentials on a compact Riemann surface M of genus $g > 0$ then the closed differential 1-forms ω_i and $\overline{\omega}_i$ for $1 \le i \le g$ are a basis for the deRham group $\mathfrak{H}^1(M)$ of closed differential forms of degree 1 on M modulo exact differential forms.*

[2] The definition and properties of nonsingular period matrices are discussed in Appendix F.1.

Proof: The period class of any closed differential form ϕ of degree 1 on M is determined by the periods $\phi(\tau_j)$ on a basis $\tau_j \in H_1(M)$; and a collection of $2g$ closed differential forms ϕ_i form a basis for the deRham group precisely when the $2g \times 2g$ complex matrix $\{\phi_i(\tau_j)\}$ is nonsingular. In particular for the differential forms $\phi_i = \omega_i$ for $1 \le i \le g$ and $\phi_i = \overline{\omega}_{i-g}$ for $g+1 \le i \le 2g$ the period matrix is $\{\phi_i(\tau_j)\} = \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}$; and since this is a nonsingular matrix as a consequence of the preceding theorem it follows that these differential forms are a basis for the deRham group, thereby concluding the proof.

Corollary 3.5 *An element $T \in \Gamma$ in the covering translation group of a compact Riemann surface M of genus $g > 0$ is contained in the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ if and only if $\omega(T) = 0$ for the period classes ω of all holomorphic abelian differentials on M.*

Proof: It was already noted that the period class ω of a holomorphic abelian differential vanishes on any element $T \in [\Gamma, \Gamma]$. Conversely if the period classes of all holomorphic abelian differentials vanish on an element $T \in \Gamma$ then the preceding corollary shows that the period classes of all differential one-forms also vanish on T; that means that homotopy class represented by the covering translation T determines the trivial homology class in the natural homomorphism $\Gamma \longrightarrow H_1(M) = \Gamma/[\Gamma, \Gamma]$, which suffices for the proof.

Theorem 3.6 *The period matrices of a compact Riemann surface M of genus $g > 0$ for all choices of bases for the holomorphic abelian differentials on M and for the homology of M are a full class of equivalent*[3] *period matrices.*

Proof: Two bases $\tilde{\omega}_i$ and ω_i for the holomorphic abelian differentials on M are related by $\tilde{\omega}_i = \sum_{k=1}^{g} a_{ik}\omega_k$ for an arbitrary nonsingular complex matrix $A = \{a_{ik}\} \in \mathrm{Gl}(g, \mathbb{C})$, and two bases $\tilde{\tau}_j$ and τ_j for the homology of M are related by $\tilde{\tau}_j = \sum_{l=1}^{2g} \tau_l q_{lj}$ for an arbitrary invertible integral matrix $Q = \{q_{lj}\} \in \mathrm{Gl}(2g, \mathbb{Z})$. The associated period matrices $\tilde{\Omega} = \{\tilde{\omega}_i(\tilde{\tau}_j)\}$ and $\Omega = \{\omega_k(\tau_l)\}$ then are related by $\tilde{\omega}_i(\tilde{\tau}_j) = \sum_{k=1}^{g}\sum_{l=1}^{2g} a_{ik}\omega_k(\tau_l)q_{lj}$, or in matrix terms $\tilde{\Omega} = A\Omega Q$. That is just the condition (F.1) that these two period matrices are equivalent period matrices, which suffices for the proof.

A period matrix Ω for the Riemann surface M of genus $g > 0$ describes a lattice subgroup $\mathcal{L}(\Omega) = \Omega\,\mathbb{Z}^{2g} \subset \mathbb{C}^g$, which in turn describes a complex torus $J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$. Since the period matrices of the surface M are equivalent period matrices, by the preceding theorem, it follows from Corollary F.9 that the complex tori described by these period matrices are biholomorphic complex manifolds; thus there is really a unique such complex manifold, called the *Jacobi variety* of the Riemann surface M and denoted by $J(M)$. The Jacobi varieties of compact Riemann surfaces play a major role in almost any discussion of Riemann surfaces.

[3]The equivalence of period matrices is defined and discussed in appendix F.1.

If ω_i is a basis for the holomorphic abelian differentials on a compact Riemann surface M of genus $g > 0$ the associated integrals $w_i(z)$ can be taken as the components of a column vector $\tilde{w}(z) = \{w_i(z)\} \in \mathbb{C}^g$; the mapping that associates to any point $z \in \widetilde{M}$ the vector $\tilde{w}(z)$ is a holomorphic mapping $\tilde{w} : \widetilde{M} \longrightarrow \mathbb{C}^g$. For any covering translation $T \in \Gamma$ the values $\omega_i(T)$ of the period classes of these holomorphic abelian differentials can be viewed correspondingly as the components of a column vector $\omega(T) = \{\omega_i(T)\} \in \mathbb{C}^g$; and the mapping that associates to a covering translation $T \in \Gamma$ the vector $\omega(T)$ is a group homomorphism $\omega \in \mathrm{Hom}(\Gamma, \mathbb{C}^g)$, which of course can be viewed alternatively as a group homomorphism $\omega \in \mathrm{Hom}(H_1(M), \mathbb{C}^g)$. The image $\omega(\tau_j) \in \mathbb{C}^g$ of one of the homology classes τ_j forming a basis for the homology $H_1(M)$ of the surface M is just column j of the period matrix Ω of the surface M for the bases ω_i and τ_j, or equivalently is one of the generators of the lattice subgroup $\mathcal{L}(\Omega)$; consequently the image subgroup $\omega(\Gamma) = \omega(H_1(M)) \subset \mathbb{C}^g$ is precisely the lattice subgroup $\mathcal{L}(\Omega) \subset \mathbb{C}^g$. Each of the integrals $w_i(z)$ satisfies (3.2), so altogether

$$\tilde{w}(Tz) = \tilde{w}(z) + \omega(T) \quad \text{for all } T \in \Gamma. \tag{3.3}$$

This shows that points of $\widetilde{M}$ that are mapped to one another under the covering translation group Γ, and hence have the same image under the covering projection $\pi : \widetilde{M} \longrightarrow M$, have as their images under the mapping $\tilde{w} : \widetilde{M} \longrightarrow \mathbb{C}^g$ points of $\mathbb{C}^g$ that are mapped to one another by translation by vectors in the lattice subgroup $\mathcal{L}(\Omega)$, and hence have the same image under the covering projection $\pi : \mathbb{C}^g \longrightarrow J(\Omega)$; consequently the mapping $\tilde{w} : \widetilde{M} \longrightarrow \mathbb{C}^g$ induces a holomorphic mapping $w : M \longrightarrow J(\Omega)$, and these mappings together with the covering projection mappings π form the commutative diagram of holomorphic mappings

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\tilde{w}"] \arrow[d, "\pi"'] & \mathbb{C}^g \arrow[d, "\pi"'] \\ M = \widetilde{M}/\Gamma \arrow[r, "w"] & J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega). \end{tikzcd} \tag{3.4}$$

The abelian integrals $w_i(z)$ are determined only up to arbitrary complex constants, so the mapping w is determined only up to an arbitrary translation in the complex torus $J(\Omega)$. The choice of another basis for the holomorphic abelian differentials has the effect of replacing the period matrix Ω by $A\Omega$ for a nonsingular matrix $A \in \mathrm{Gl}(g, \mathbb{C})$ and correspondingly replacing the mapping w by $A\,w$; this can be viewed as the result of composing the initial holomorphic mapping $w : M \longrightarrow J(\Omega)$ with the composition of that mapping and the biholomorphic mapping $A : J(\Omega) \longrightarrow J(A\Omega)$ between these complex tori, so it really amounts to the same mapping from M to the Jacobi variety $J(M)$ of the Riemann surface M. The mapping $w : M \longrightarrow J(M)$ so defined is called the *Abel-Jacobi mapping* of the Riemann surface into its Jacobi variety. Again though this mapping is determined only up to arbitrary translations in the complex torus $J(M)$.

Theorem 3.7 *A holomorphic mapping $f : M \longrightarrow T$ from a compact Riemann surface M of genus $g > 0$ to a complex torus T can be factored uniquely as the composition $f = h \circ w$ of the Abel-Jacobi mapping $w : M \longrightarrow J(M)$ from M to its Jacobi variety $J(M)$ and a holomorphic mapping $h : J(M) \longrightarrow T$ from the Jacobi variety to the complex torus T, and this property characterizes the Abel-Jacobi mapping.*

Proof: Suppose that the torus T is represented as the quotient $T = \mathbb{C}^h/\mathcal{L}$ for a lattice subgroup $\mathcal{L} \subset \mathbb{C}^h$. A holomorphic mapping $f : M \longrightarrow T$ lifts to a holomorphic mapping $\tilde{f} : \widetilde{M} \longrightarrow C^h$ from the universal covering space $\widetilde{M}$ of M to the universal covering space $\mathbb{C}^h$ of the torus T; and a mapping $\tilde{f} : \widetilde{M} \longrightarrow C^h$ induces a mapping $f : M \longrightarrow T$ between the quotient spaces if and only if for any covering translation $T \in \Gamma$ and any point $z \in \widetilde{M}$ there is a lattice vector $\lambda \in \mathcal{L}$ such that

$$\tilde{f}(Tz) = \tilde{f}(z) + \lambda. \tag{3.5}$$

Since the lattice subgroup $\mathcal{L} \subset \mathbb{C}^h$ is discrete the lattice vector λ in (3.5) must be independent of the point $z \in \widetilde{M}$ so can be viewed as a function $\lambda(T)$ of the covering translation $T \in \Gamma$; and it is evident from (3.5) that $\lambda(ST) = \lambda(S)+\lambda(T)$ for any two covering translations $S, T \in \Gamma$, so the function $\lambda(T)$ is a group homomorphism $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C})$. By Lemma 3.1 the component functions $f_i(z)$ of the mapping $\tilde{f}$ must be holomorphic integrals on M; so if $w_j(z)$ is a basis for the holomorphic abelian integrals on M then $f_i(z) = \sum_{j=1}^{g} a_{ij} w_j(z) + a_i$ for some uniquely determined complex constants a_{ij}, a_i, or in matrix notation

$$\tilde{f}(z) = A\tilde{w}(z) + a \tag{3.6}$$

for the matrix $A = \{a_{ij}\} \in \mathbb{C}^{h\times g}$, the vector $a = \{a_i\} \in \mathbb{C}^h$ and the mapping $\tilde{w} : \widetilde{M} \longrightarrow \mathbb{C}^g$ described by the component functions $w_j(z)$. It follows from (3.2) and (3.5) that $\lambda(T) = \tilde{f}(Tz) - \tilde{f}(z) = A\big(w(Tz) - w(z)\big) = A\,\omega(T)$ for any covering translation $T \in \Gamma$; the vectors $\omega(T)$ generate the lattice subgroup $\mathcal{L}(\Omega)$ described by the period matrix Ω of the surface M in terms of the chosen basis for the holomorphic abelian integrals while $\lambda(T) \in \mathcal{L}$, so $A\mathcal{L}(\Omega) \subset \mathcal{L}$ and as in Theorem F.6 the affine mapping $\tilde{h}(t) = At + a$ from $\mathbb{C}^g$ to $\mathbb{C}^h$ induces a holomorphic mapping $h : J(M) \longrightarrow T$ from the Jacobi variety $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$ of M to the complex torus $T = \mathbb{C}^h/\mathcal{L}$. The holomorphic mapping $\tilde{w}$ induces the Abel-Jacobi mapping $w : M \longrightarrow J(\Omega)$ as in the commutative diagram (3.4), so since $\tilde{f} = \tilde{h} \circ \tilde{w}$ by (3.6) it follows that $f = h \circ w$. To show that this property characterizes the Abel-Jacobi mapping suppose that $w_0 : M \longrightarrow J_0(M)$ is a holomorphic mapping from M to another complex torus $J_0(M)$ such that any holomorphic mapping $f : M \longrightarrow T$ from M to a complex torus T can be factored uniquely as the composition $f = h_0 \circ w_0$ of the mapping $w_0 : M \longrightarrow J_0$ and a holomorphic mapping $h_0 : J_0 \longrightarrow T$. Then in particular for this mapping w_0 and for the Abel-Jacobi mapping w there are unique holomorphic mappings $h_0 : J_0(M) \longrightarrow J(M)$ and $h : J(M) \longrightarrow J_0(M)$ such that $w = h_0 \circ w_0$ and $w_0 = h \circ w$. The uniqueness implies that $h \circ h_0$ and $h_0 \circ h$ are both identity mappings and consequently that $h : J(M) \longrightarrow J_0(M)$ is a biholomorphic mapping, thus identifying the complex torus $J_0(M)$ with the Jacobi variety $J(M)$. That suffices for the proof.

3.2 Flat Line Bundles

Paralleling the exact sequence of sheaves (2.26) over M is the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C}^* \xrightarrow{\iota} \mathcal{O}^* \xrightarrow{\mathrm{dl}} \mathcal{O}^{(1,0)} \longrightarrow 0 \tag{3.7}$$

in which ι is the natural inclusion homomorphism and $\mathrm{dl}(f) = df/f = d\ \log f$ for any germ f of a nowhere vanishing holomorphic function. The exactness of (3.7) follows immediately from the exactness of (2.26), since $df/f = \mathrm{dl}(f) = 0$ precisely when f is constant and a germ $\phi \in \mathcal{O}^{(1,0)}$ can be written $\phi = dg$ for some germ $g \in \mathcal{O}$ so $\phi = \mathrm{dl}(f)$ where $f = \exp g$.

Theorem 3.8 *On a compact Riemann surface M of genus $g > 0$ there is the exact sequence*

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^1(M, \mathbb{C}^*) \xrightarrow{\iota} H^1(M, \mathcal{O}^*) \xrightarrow{c} \mathbb{Z} \longrightarrow 0, \tag{3.8}$$

where δ is the coboundary mapping induced by the exact sequence of sheaves (3.7), ι is the homomorphism induced by the natural inclusion $\mathbb{C}^* \subset \mathcal{O}^*$ and $c(\lambda) \in \mathbb{Z}$ is the characteristic class of a holomorphic line bundle $\lambda \in H^1(M, \mathcal{O}^*)$.

Proof: The exact cohomology sequence arising from the exact sequence of sheaves (3.7) begins

$$\begin{aligned} 0 \longrightarrow \Gamma(M, \mathbb{C}^*) \xrightarrow{\iota} \Gamma(M, \mathcal{O}^*) \xrightarrow{\mathrm{dl}} \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta} \\ \xrightarrow{\delta} H^1(M, \mathbb{C}^*) \xrightarrow{\iota} H^1(M, \mathcal{O}^*) \xrightarrow{\mathrm{dl}} H^1(M, \mathcal{O}^{(1,0)}) \longrightarrow \cdots . \end{aligned} \tag{3.9}$$

Since M is compact every holomorphic function on M is constant, by the maximum modulus theorem, so the homomorphism $\iota : \Gamma(M, \mathbb{C}^*) \longrightarrow \Gamma(M, \mathcal{O}^*)$ is an isomorphism and consequently the homomorphism $\mathrm{dl} : \Gamma(M, \mathcal{O}^*) \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)})$ is the zero homomorphism; thus (3.9) reduces to the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^1(M, \mathbb{C}^*) \xrightarrow{\iota} H^1(M, \mathcal{O}^*) \xrightarrow{\mathrm{dl}} H^1(M, \mathcal{O}^{(1,0)}). \tag{3.10}$$

In terms of an open coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of M, a cohomology class $\lambda \in H^1(M, \mathcal{O}^*)$ can be represented by a cocycle $\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ and its image $\mathrm{dl}(\lambda) \in H^1(M, \mathcal{O}^{(1,0)})$ then is represented by the cocycle $d \log \lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^{(1,0)})$. By the Theorem of Dolbeault, Theorem 1.9, for the special case that the line bundle λ is the canonical bundle $\lambda = \kappa$ so that $\mathcal{O}^{(1,0)} \cong \mathcal{O}(\kappa)$, there is the isomorphism

$$H^1(M, \mathcal{O}^{(1,0)}) \cong \frac{\Gamma(M, \mathcal{E}^{(1,1)})}{\overline{\partial}\Gamma(M, \mathcal{E}^{(1,0)})}. \tag{3.11}$$

Explicitly this isomorphism is induced by the mapping that associates to a cross-section $\phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ the cohomology class of the cocycle $\phi_{\alpha\beta} = \phi_\beta - \phi_\alpha \in Z^1(\mathfrak{U}, \mathcal{O}^{(1,0)})$ for any differential forms $\phi_\alpha \in \Gamma(U_\alpha, \mathcal{E}^{(1,0)})$ such that $\overline{\partial}\phi_\alpha = \phi$. As noted following the proof of Theorem 1.2, there are $\mathcal{C}^\infty$ functions $r_\alpha > 0$ in the coordinate neighborhoods U_α such that $r_\alpha = |\lambda_{\alpha\beta}|^2 r_\beta$ in the intersections $U_\alpha \cap U_\beta$, hence such that $d \log \lambda_{\alpha\beta} = \partial \log r_\alpha - \partial \log r_\beta$; therefore for the cocycle $\phi_{\alpha\beta} = d \log \lambda_{\alpha\beta}$ it is possible to take $\phi_\alpha = -\partial \log r_\alpha$, so the cohomology class $\mathrm{dl}(\lambda) \in H^1(M, \mathcal{O}^{(1,0)})$ can be represented in the isomorphism (3.11) by the differential form $\phi = \overline{\partial}\phi_\alpha = -\overline{\partial}\partial \log r_\alpha$. By the Serre Duality Theorem, Theorem 1.17, again for the special case of the line bundle $\lambda = \kappa$, the dual space to the vector space (3.11) consists of the linear functionals (1.59) associated to the holomorphic cross-sections $\tau \in \Gamma(M, \mathcal{O}^{(1,0)}(\kappa^{-1})) = \Gamma(M, \mathcal{O})$; since M is compact these cross-sections are merely complex constants, so the vector space (3.11) can be identified with the complex numbers $\mathbb{C}$ by associating to the global differential form $\phi \in \Gamma(M, \mathcal{E}^{(1,1)})$ the integral $\int_M c\phi$ for any choice of a normalizing complex constant $c \in \mathbb{C}$. For present purposes take $c = -1/(2\pi i)$, so the differential form $\phi = -\overline{\partial}\partial \log r_\alpha$ corresponds to the complex constant $\frac{1}{2\pi i}\int_M \overline{\partial}\partial \log r_\alpha$, which by Theorem 1.2 is just the characteristic class $c(\lambda)$ of the line bundle λ. Thus the homomorphism $\mathrm{dl} : H^1(M, \mathcal{O}^*) \longrightarrow H^1(M, \mathcal{O}^{(1,0)})$ in (3.10) can be identified with the mapping that associates to a line bundle $\lambda \in H^1(M, \mathcal{O}^*)$ its characteristic class $c(\lambda) \in \mathbb{Z}$, and that suffices to conclude the proof.

The group $H^1(M, \mathbb{C}^*)$ is the set of cohomology classes defined by cocycles $\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathbb{C}^*)$, where a cocycle is described by complex constants $\lambda_{\alpha\beta}$ associated to intersections $U_\alpha \cap U_\beta$ and satisfying $\lambda_{\alpha\beta}\lambda_{\beta\gamma}\lambda_{\gamma\alpha} = 1$ in intersections $U_\alpha \cap U_\beta \cap U_\gamma$ and a cocycle $\lambda_{\alpha\beta}$ is a coboundary if and only if $\lambda_{\alpha\beta} = c_\alpha/c_\beta$ for some nonzero complex constants c_α. Thus the group $H^1(M, \mathbb{C}^*)$ is the analogue of the group $H^1(M, \mathcal{O}^*)$ of holomorphic line bundles in which holomorphic functions are replaced by complex constants; hence the group $H^1(M, \mathbb{C}^*)$ is called the group of *flat line bundles* over the Riemann surface M. Any flat line bundle can be viewed as a holomorphic line bundle, which is the natural homomorphism $\iota : H^1(M, \mathbb{C}^*) \longrightarrow H^1(M, \mathcal{O}^*)$ in the exact sequence (3.8).

Corollary 3.9 *A holomorphic line bundle λ over a compact Riemann surface of genus $g > 0$ is holomorphically equivalent to a flat line bundle if and only if $c(\lambda) = 0$.*

Proof: This is an immediate consequence of the exact sequence (3.8) of the preceding theorem, and is included explicitly here only as a convenience for later reference.

It is useful to keep in mind the explicit form of the mappings in the exact sequence (3.8). For this purpose consider a covering $\mathfrak{U}$ of M by contractible open coordinate neighborhoods U_α such that any nonempty intersection $U_\alpha \cap U_\beta$ is connected. The image of the coboundary mapping δ is the set of all flat line bundles that are trivial as holomorphic line bundles. Any holomorphic abelian differential $\omega(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ can be written locally as the exterior derivative $\omega(z) = dw_\alpha(z)$ of a holomorphic function $w_\alpha(z)$ in each coordinate neighborhood U_α, and the image $\delta\,\omega \in H^1(M, \mathbb{C}^*)$ is the cohomology class of the cocycle

$$\lambda_{\alpha\beta} = \exp(w_\beta(z) - w_\alpha(z)) \quad \text{for} \quad z \in U_\alpha \cap U_\beta, \tag{3.12}$$

where $w_\beta(z) - w_\alpha(z)$ is a complex constant in the connected set $U_\alpha \cap U_\beta$ since $d\big(w_\beta(z) - w_\alpha(z)\big) = \omega(z) - \omega(z) = 0$. The inclusion mapping ι associates to any flat line bundle the holomorphic line bundle it represents; a cocycle $\lambda_{\alpha\beta}$ consisting of complex constants can be viewed as well as a cocycle consisting of holomorphic functions. The kernel of the mapping dl consists of those holomorphic line bundles that can be represented by flat line bundles. If λ is a holomorphic line bundle described by a cocycle $\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ then $dl(\lambda)$ is described by the cocycle $\omega_{\alpha\beta} = d\log\lambda_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{O}^{1,0})$. If $c(\lambda) = 0$ then by the exactness of the sequence (3.8) the cocycle $\omega_{\alpha\beta}$ must be a coboundary, so $\omega_{\alpha\beta} = \phi_\beta(z) - \phi_\alpha(z)$ for some holomorphic differentials $\phi_\alpha(z)$ in the sets U_α; hence $d\log\lambda_{\alpha\beta} = d(\omega_\beta(z) - \omega_\alpha(z))$ so $\log\lambda_{\alpha\beta} = w_\beta(z) - w_\alpha(z) + c_{\alpha\beta}$ for some complex constants $c_{\alpha\beta}$, where $dw_\alpha(z) = \omega_\alpha(z)$, and

$$\lambda_{\alpha\beta} = \exp\big(c_{\alpha\beta} + w_\beta(z) - w_\alpha(z)\big). \tag{3.13}$$

That is the condition that the cocycle $\lambda_{\alpha\beta}$ is cohomologous to the flat cocycle $\exp c_{\alpha\beta}$ in $H^1(M, \mathcal{O}^*)$.

As in (1.19) the condition that $c(\lambda) = 0$ is equivalent to the condition that the line bundle λ is topologically trivial; thus the holomorphic line bundles that can be represented by flat line bundles are precisely those that are topologically trivial. The subgroup of topologically trivial holomorphic line bundles on a compact Riemann surface M is called the *Picard group* of M, and is denoted by $P(M)$; and in these terms the preceding theorem can be rephrased as follows.

Corollary 3.10 *On a compact Riemann surface M of genus $g > 0$ there is the exact sequence*

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta} H^1(M, \mathbb{C}^*) \xrightarrow{p} P(M) \longrightarrow 0 \tag{3.14}$$

where $P(M)$ is the Picard group of the surface M, p is the homomorphism that associates to a flat line bundle the holomorphic line bundle it represents, and δ is the coboundary homomorphism defined by (3.12).

Proof: This also follows immediately from the exact sequence (3.8) of the preceding theorem, since the kernel of the mapping c in the exact sequence (3.8) is precisely the Picard group $P(M)$ of the surface M; so no further proof is necessary.

3.3 Factors of Automorphy

There is a usefully explicit alternative description of the group of line bundles on a compact Riemann surface M of genus $g > 0$. A continuous *factor of automorphy* for the action of the covering translation group Γ on the universal covering space $\widetilde{M}$ of M is a mapping $\lambda : \Gamma \times \widetilde{M} \longrightarrow \mathbb{C}^*$ that is continuous on $\widetilde{M}$ and satisfies

$$\lambda(ST, z) = \lambda(S, Tz)\lambda(T, z) \quad \text{for all } S,\, T \in \Gamma, z \in \widetilde{M}. \tag{3.15}$$

The factor of automorphy is *holomorphic* if the functions $\lambda(T, z)$ are holomorphic functions of the variable $z \in \widetilde{M}$, and is *flat* if the functions $\lambda(T, z)$ are constant in the variable $z \in \widetilde{M}$; clearly a flat factor of automorphy is just a group homomorphism $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$, and any flat factor of automorphy is also a holomorphic factor of automorphy. The set of factors of automorphy form an abelian group under multiplication $\lambda_1(T, z) \cdot \lambda_2(T, z)$ of the functions $\lambda(T, z)$; the holomorphic factors of automorphy are a subgroup of the group of all factors of automorphy, and the flat factors of automorphy are a subgroup of the group of holomorphic factors of automorphy. Two factors of automorphy $\lambda_1(T, z)$ and $\lambda_2(T, z)$ are *equivalent* if there is a continuous mapping $f : \widetilde{M} \longrightarrow \mathbb{C}^*$ such that

$$\lambda_1(T, z) = f(Tz)\lambda_2(T, z)f(z)^{-1} \quad \text{for all } T \in \Gamma, z \in \widetilde{M}; \tag{3.16}$$

in that case $\lambda_2(T, z) = f(Tz)^{-1}\lambda_1(T, z)f(z)$, so this relation is symmetric. Two holomorphic factors of automorphy are *holomorphically equivalent* if they are equivalent and the function $f(z)$ in (3.16) is holomorphic. Analogously two flat factors of automorphy could be considered flatly equivalent if they are equivalent and the function $f(z)$ in (3.16) is constant; but that is the case only when $\lambda_1(T) = \lambda_2(T)$, so flat equivalence just amounts to coincidence and is not worth introducing as a separate notion. It is clear that equivalence of factors of automorphy and holomorphic equivalence of holomorphic factors of automorphy are equivalence relations in the usual sense. A *relatively automorphic function* for a factor of automorphy $\lambda(T, z)$ is a continuous function $f(z)$ on the universal covering space $\widetilde{M}$ such that

$$f(Tz) = \lambda(T, z)f(z) \quad \text{for all } T \in \Gamma,\, z \in \widetilde{M}. \tag{3.17}$$

A *holomorphic relatively automorphic function* for a holomorphic factor of automorphy is a relatively automorphic function that is holomorphic in the variable $z \in \widetilde{M}$. Analogously a flat relatively automorphic function for a flat factor of automorphy could be defined as a relatively automorphic function that is constant in the variable $z \in \widetilde{M}$; but clearly there is such a function only when the flat factor of automorphy is the trivial factor $\lambda(T) = 1$ for all $T \in \Gamma$, so this also is not a useful auxiliary notion. A comparison of equations (3.16) and (3.17) shows that two factors of automorphy $\lambda_1(T, z)$ and $\lambda_2(T, z)$ are equivalent if and only if there is a nowhere vanishing relatively automorphic function for the factor of automorphy $\lambda_1(T, z) \cdot \lambda_2(T, z)^{-1}$, or of course equivalently for the factor of automorphy $\lambda_2(T,z)\cdot\lambda_1(T,z)^{-1}$; and if these factors of automorphy are holomorphic they are holomorphically equivalent if and only if there is a holomorphic nowhere vanishing relatively automorphic function for the factor of automorphy $\lambda_1(T,z)\cdot\lambda_2(T,z)^{-1}$, or of course equivalently for the factor of automorphy $\lambda_2(T,z)\cdot\lambda_1(T,z)^{-1}$.

If $\lambda(T,z)$ is a factor of automorphy for the action of the covering translation group Γ of the Riemann surface M then to each covering translation $T\in\Gamma$ there can be associated the mapping $T_\lambda:\widetilde{M}\times\mathbb{C}\longrightarrow\widetilde{M}\times\mathbb{C}$ defined by

$$T_\lambda(z,\,t)=(Tz,\,\lambda(T,z)t)\in\widetilde{M}\times\mathbb{C}\quad\text{for}\quad(z,\,t)\in\widetilde{M}\times\mathbb{C}.$$

It follows from the defining condition (3.15) for a factor of automorphy that for any two covering translations $S,T\in\Gamma$ the associated mappings satisfy

$$\begin{aligned}(ST)_\lambda(z,\,t)&=\big(STz,\,\lambda(ST,z)t\big)=\big(S\cdot Tz,\,\lambda(S,Tz)\cdot\lambda(T,z)\big)\\&=S_\lambda\big(Tz,\,\lambda(T,z)t\big)=S_\lambda\big(T_\lambda(z,\,t)\big);\end{aligned}$$

so this exhibits the covering translation group as a group of continuous mappings of the space $\widetilde{M}\times\mathbb{C}$ to itself, or of holomorphic mappings if the factor of automorphy is holomorphic. The group action on the product $\widetilde{M}\times\mathbb{C}$ commutes with the action of the covering translation group on the universal covering space $\widetilde{M}$ itself under the natural projection $\pi:\widetilde{M}\times\mathbb{C}\longrightarrow\widetilde{M}$, yielding the commutative diagram

$$\begin{array}{ccc}\widetilde{M}\times\mathbb{C} & \xrightarrow{T_\lambda} & \widetilde{M}\times\mathbb{C}\\ \pi\big\downarrow & & \pi\big\downarrow\\ \widetilde{M} & \xrightarrow{T} & \widetilde{M};\end{array}\tag{3.18}$$

it follows that the projection $\pi:\widetilde{M}\times\mathbb{C}\longrightarrow\widetilde{M}$ induces a mapping $\pi_\lambda:\lambda\longrightarrow M$ between the quotient spaces $\lambda=(\widetilde{M}\times\mathbb{C})/\Gamma$ and $M=\widetilde{M}/\Gamma$.

Theorem 3.11 (i) *If $\lambda(T,z)$ is a factor of automorphy for the action of the covering translation group Γ of a compact Riemann surface M of genus $g>0$ the quotient space $\lambda=(\widetilde{M}\times\mathbb{C})/\Gamma$ has the natural structure of a line bundle over M with the projection mapping $\pi_\lambda:\lambda\longrightarrow M$. Relatively automorphic functions for the factor of automorphy $\lambda(T,z)$ correspond to cross-sections of λ; and the line bundles corresponding to two factors of automorphy are equivalent if and only if the factors of automorphy are equivalent.*
(ii) *A holomorphic factor of automorphy $\lambda(T,z)$ describes in this way a holomorphic line bundle λ, and holomorphic relatively automorphic functions for $\lambda(T,z)$ correspond to holomorphic cross-sections of λ; the line bundles described by two holomorphic factors of automorphy are holomorphically equivalent if and only if the factors of automorphy are holomorphically equivalent.*
(iii) *A flat factor of automorphy $\lambda(T,z)$ describes in this way a flat line bundle λ over M.*

Proof: The quotient space λ can be described most conveniently by using a coordinate covering $\mathfrak{U}$ of the Riemann surface M by connected and simply connected coordinate neighborhoods U_α such that the intersections $U_\alpha \cap U_\beta$ of pairs of these coordinate neighborhoods also are connected; there exist such coordinate coverings on any Riemann surface. The inverse image of a coordinate neighborhood $U_\alpha \subset M$ under the covering projection $\pi : \widetilde{M} \longrightarrow M$ is a disjoint collection of open coordinate neighborhoods $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$, each of which is biholomorphic to U_α under the covering projection $\pi : \tilde{U}_{\alpha\, i} \longrightarrow U_\alpha$; the sets $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$ form a coordinate covering $\tilde{\mathfrak{U}}$ of the universal covering space $\widetilde{M}$. The images $U_{\alpha\, i} = \pi(\tilde{U}_{\alpha\, i}) \subset M$ can be taken as another coordinate covering $\pi(\tilde{\mathfrak{U}})$ of the Riemann surface M itself, although it must be kept in mind that the point sets $U_{\alpha\, i}$ for all indices i actually coincide with the point set U_α although they are considered as being different sets of the covering. If $U_\alpha \cap U_\beta \neq \emptyset$ then since this intersection is connected by assumption it follows that for any two components $\tilde{U}_{\alpha\, i}$ and $\tilde{U}_{\beta\, j}$ there is a uniquely determined covering translation $T_{\alpha\, i,\ \beta\, j} \in \Gamma$ such that

$$\tilde{U}_{\alpha\, i} \cap \pi^{-1}(U_\alpha \cap U_\beta) = T_{\alpha\, i, \beta\, j}\Big(\tilde{U}_{\beta\, j} \cap \pi^{-1}(U_\alpha \cap U_\beta)\Big).$$

Of course the point sets $U_{\alpha\, i} \cap U_{\beta\, j}$ for any indices i, j coincide with the point set $U_\alpha \cap U_\beta$; but $\tilde{U}_{\alpha\, i} \cap \tilde{U}_{\beta\, j} \neq \emptyset$ in $\widetilde{M}$ if and only if $T_{\alpha\, i, \beta\, j} = I$. Since $T\tilde{U}_{\alpha\, i} \cap \tilde{U}_{\alpha\, i} = \emptyset$ for any covering translation $T \in \Gamma$ other than the identity it follows that

$$T_\lambda(\tilde{U}_{\alpha\, i} \times \mathbb{C}) \cap (\tilde{U}_{\alpha\, i} \times \mathbb{C}) = \emptyset \quad \text{if} \quad T \neq I;$$

therefore the set $\tilde{U}_{\alpha\, i} \times \mathbb{C}$ can be identified with a subset of the quotient space $(\widetilde{M} \times \mathbb{C})/\Gamma = \lambda$, and a local coordinate $z_{\alpha\, i}$ in $\tilde{U}_{\alpha\, i}$ and the variable $t_{\alpha\, i} \in \mathbb{C}$ can be used as local coordinates $(z_{\alpha\, i}, t_{\alpha\, i})$ in this subset of the quotient space λ. In terms of these coordinates the mapping $\pi_\lambda : \lambda \longrightarrow M$ is just the natural projection $\tilde{U}_{\alpha\, i} \times \mathbb{C} \longrightarrow U_\alpha$. Points $(z_{\alpha\, i},\ t_{\alpha\, i}) \in \tilde{U}_{\alpha\, i} \times \mathbb{C}$ and $(z_{\beta\, j},\ t_{\beta\, j}) \in \tilde{U}_{\beta j} \times \mathbb{C}$ represent the same point in the quotient space λ if and only if

$$z_{\alpha\, i} = T_{\alpha\, i,\ \beta\, j}\ z_{\beta\, j} \quad \text{and} \quad t_{\alpha\, i} = \lambda(T_{\alpha\, i,\ \beta\, j},\ z_{\beta\, j}) t_{\beta\, j}; \tag{3.19}$$

this is a linear relation between the fibre coordinates that is continuous in the local coordinates on M, so determines on λ the structure of a complex line bundle over M. Of course if the factor of automorphy is holomorphic the line bundle is holomorphic, and if the factor of automorphy is flat the line bundle is flat. If $f(z)$ is a relatively automorphic function for the factor of automorphy $\lambda(T, z)$ then the restrictions of this function to the coordinate neighborhoods $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$ satisfy $f(z_{\alpha\, i}) = \lambda(T_{\alpha\, i, \beta\, j},\ z_{\beta\, j}) f(z_{\beta\, j})$ so they describe a cross-section of the line bundle λ. Conversely a cross-section of λ is described by functions $f_{\alpha\, i}(z)$ in the coordinate neighborhoods $\tilde{U}_{\alpha\, i}$ such that

$$f_{\alpha\, i}(z_{\alpha\, i}) = \lambda(T_{\alpha\, i,\ \beta\, j},\ z_{\beta\, j}) f_{\beta\, j}(z_{\beta\, j}) \tag{3.20}$$

whenever $z_{\alpha\, i} \in \tilde{U}_{\alpha\, i}$, $z_{\beta\, j} \in \tilde{U}_{\beta\, j}$, $\pi(z_{\alpha\, i}) = \pi(z_{\beta\, j})$. Since $\lambda(T_{\alpha\, i, \beta\, j}, z) = 1$ whenever $\tilde{U}_{\alpha\, i} \cap \tilde{U}_{\beta\, j} \neq \emptyset$ it follows that $f_{\alpha\, i}(z) = f_{\beta\, j}(z)$ for any point $z \in$

$\tilde{U}_{\alpha\, i} \cap \tilde{U}_{\beta\, j}$, so the local functions $f_{\alpha\, i}(z_{\alpha\, i})$ combine to form a single global function on the entire covering space $\widetilde{M}$; and it follows from (3.20) that this function is a relatively automorphic function for the factor of automorphy λ. Two factors of automorphy $\lambda_1(T,z)$ and $\lambda_2(T,z)$ are equivalent if and only if the factor of automorphy $\lambda_1(T,z)\lambda_2(T,z)^{-1}$ has a nowhere vanishing relatively automorphic function, and two line bundles λ_1 and λ_2 are equivalent if and only if the line bundle $\lambda_1\lambda_2^{-1}$ has a nowhere vanishing cross-section; since relatively automorphic functions correspond to cross-sections of the line bundle they describe it follows that two factors of automorphy are equivalent if and only if the line bundles they describe are equivalent. The same result of course holds for holomorphic line bundles, and that suffices to conclude the proof.

The preceding theorem shows that a factor of automorphy for the covering translation group of a compact Riemann surface M describes a line bundle over M, and that two factors of automorphy are equivalent if and only if they describe equivalent line bundles over M; thus the mapping that associates to a factor of automorphy the line bundle it describes is an injective mapping from the multiplicative group of equivalence classes of factors of automorphy into the group $H^1(M,\mathcal{O}^*)$ of line bundles over M. The corresponding statement holds for holomorphic factors of automorphy and flat factors of automorphy, so that the appropriate equivalence classes of these special factors of automorphy are mapped injectively into the groups $H^1(M,\mathcal{O}^*)$ and $H^1(M,\mathbb{C}^*)$ respectively. In all three cases the mapping will be shown to be surjective as well, so line bundles over M can be identified with equivalence classes of factors of automorphy for the covering translation group of M and correspondingly for holomorphic and flat line bundles. The discussion here though will be limited to the identification of flat line bundles with flat factors of automorphy; the proof is simplest for that case, an almost immediate consequence of the observation that any flat line bundle over a simply connected surface such as the universal covering space $\widetilde{M}$ is a trivial flat bundle, and will be given in the next corollary. The corresponding result for the other cases will be demonstrated in Chapter **??** by constructing an explicit holomorphic factor of automorphy with any prescribed characteristic class.

Corollary 3.12 *The mapping that associates to a flat factor of automorphy in* $\mathrm{Hom}(\Gamma,\mathbb{C}^*)$ *the flat line bundle that it describes is an isomorphism*

$$\phi : \mathrm{Hom}(\Gamma,\mathbb{C}^*) \xrightarrow{\cong} H^1(M,\mathbb{C}^*).$$

Proof: If a flat factor of automorphy describes a trivial flat line bundle that line bundle has a flat nowhere vanishing cross-section; this cross-section corresponds to a nowhere vanishing flat relatively automorphic function for the flat factor of automorphy, which as noted means that the flat factor of automorphy is trivial. Thus the homomorphism $\phi : \mathrm{Hom}(\Gamma,\mathbb{C}^*) \longrightarrow H^1(M,\mathbb{C}^*)$ is injective, and to conclude the proof it is only necessary to show that it is also surjective. For this purpose let $\mathfrak{U} = \{U_\alpha\}$ be a coordinate covering of M with the properties as in the proof of the preceding theorem. A flat line bundle λ over M is described by constant coordinate transition functions in each nonempty intersection $U_\alpha \cap U_\beta$; it is convenience to denote these coordinate transition functions by $\lambda(U_\alpha, U_\beta)$ for this proof. The same line bundle λ can be described in terms of the covering $\pi(\tilde{\mathfrak{U}})$ by the coordinate transition functions $\lambda(U_{\alpha\, i}, U_{\beta\, j}) = \lambda(U_\alpha, U_\beta)$ in each nonempty intersection $U_{\alpha\, i} \cap U_{\beta\, j}$. The bundle λ induces a flat line bundle $\tilde{\lambda}$ over the universal covering space $\widetilde{M}$ described in terms of the covering $\tilde{\mathfrak{U}}$ by the coordinate transition functions $\tilde{\lambda}(\tilde{U}_{\alpha\, i}, \tilde{U}_{\beta\, j}) = \lambda(U_\alpha, U_\beta)$ in each nonempty intersection $\tilde{U}_{\alpha\, i} \cap \tilde{U}_{\beta\, j}$; it is evident from its definition that this coordinate bundle is invariant under the covering translation group Γ. Since the coordinate transition functions are constants a constant cross-section of the bundle $\tilde{\lambda}$ over a coordinate neighborhood $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$ can be extended to a constant cross-section over any coordinate neighborhood $\tilde{U}_{\beta\, j} \subset \widetilde{M}$ that meets $\tilde{U}_{\alpha\, i}$; this cross-section can be extended further in the same way, and since $\widetilde{M}$ is simply connected the usual monodromy argument shows that there results a well defined cross-section of the induced line bundle $\tilde{\lambda}$ over $\widetilde{M}$. This cross-section is described by complex constants $\phi(\tilde{U}_{\alpha\, i})$ in the coordinate neighborhoods $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$ such that

$$\phi(\tilde{U}_{\alpha\, i}) = \tilde{\lambda}(\tilde{U}_{\alpha\, i}, \tilde{U}_{\beta\, j})\ \phi(\tilde{U}_{\beta\, j}) \quad \text{if} \quad \tilde{U}_{\alpha\, i} \cap \tilde{U}_{\beta\, j} \neq \emptyset. \tag{3.21}$$

Set $\phi(U_{\alpha\, i}) = \phi(\tilde{U}_{\alpha\, i})$ and note that the coordinate transition functions

$$\sigma(U_{\alpha\, i}, U_{\beta\, j}) = \phi(U_{\alpha\, i})^{-1} \lambda(U_{\alpha\, i}, U_{\beta\, j}) \phi(U_{\beta\, j}) \quad \text{for} \quad U_{\alpha\, i} \cap U_{\beta\, j} \neq \emptyset \tag{3.22}$$

describe a flat coordinate bundle σ over M in terms of the covering $\pi(\tilde{\mathfrak{U}})$ and that this bundle is flatly equivalent to the initial coordinate bundle λ. For any covering translation $T \in \Gamma$ and any coordinate neighborhood $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$ set $\lambda(\tilde{U}_{\alpha\, i}, T) = \sigma(TU_{\alpha\, i}, U_{\alpha\, i})$, which is well defined since $TU_{\alpha\, i} \cap U_{\alpha\, i} \neq \emptyset$. If $\tilde{U}_{\alpha\, i} \cap \tilde{U}_{\beta\, j} \neq \emptyset$ then of course $T\tilde{U}_{\beta\, j} \cap T\tilde{U}_{\alpha\, i} \neq \emptyset$ as well and it is evident upon comparing (3.21) and (3.22) that $\sigma(U_{\alpha\, i}, U_{\beta\, j}) = \sigma(TU_{\beta\, j}, TU_{\alpha\, i}) = 1$; so from the compatibility conditions for the coordinate transition functions $\sigma(U_{\alpha\, i}, U_{\beta\, j})$ it follows that whenever $U_{\alpha\, i} \cap U_{\beta\, j} \neq \emptyset$

$$\begin{aligned}
\lambda(\tilde{U}_{\alpha\, i}, T) &= \sigma(TU_{\alpha\, i}, U_{\alpha\, i}) \\
&= \sigma(TU_{\beta\, j}, TU_{\alpha\, i}) \sigma(TU_{\alpha\, i}, U_{\alpha\, i}) \sigma(U_{\alpha\, i}, U_{\beta\, j}) \\
&= \sigma(TU_{\beta\, j}, U_{\beta\, j}) = \lambda(\tilde{U}_{\beta\, j}, T).
\end{aligned}$$

Thus the constants $\lambda(\tilde{U}_{\alpha\, i}, T)$ are independent of the coordinate neighborhood $\tilde{U}_{\alpha\, i} \subset \widetilde{M}$ so can be labeled just $\lambda(T)$. Then $\lambda(ST) = \lambda(\tilde{U}_{\alpha\, i}, ST) = \sigma(STU_{\alpha\, i}, U_{\alpha\, i}) = \sigma(STU_{\alpha\, i}, TU_{\alpha\, i}) \sigma(TU_{\alpha\, i}, U_{\alpha\, i}) = \lambda(T\tilde{U}_{\alpha\, i}, S) \lambda(\tilde{U}_{\alpha\, i}, T) = \lambda(S)\lambda(T)$ for any two covering translations $S, T \in \Gamma$, so the constants $\lambda(T)$ describe a homomorphism $\lambda \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$. As in the proof of the preceding theorem this factor of automorphy describes a flat line bundle over M for which the coordinate transition functions in an intersection $U_{\alpha\, i} \cap U_{\beta\, j}$ are the constants $\lambda(T_{\alpha\, i, \beta\, j}) = \lambda(\tilde{U}_{\beta\, j},\, T_{\alpha\, i, \beta\, j}) = \sigma(T_{\alpha\, i, \beta\, j} U_{\beta\, j},\, U_{\beta\, j}) = \sigma(U_{\alpha\, i}, U_{\beta\, j})$ which are the coordinate transition functions of the flat line bundle σ flatly equivalent to λ. That suffices to conclude the proof.

Theorem 3.13 *On a compact Riemann surface M of genus $g > 0$ with universal covering space $\widetilde{M}$ and covering transformation group Γ there is the exact sequence of abelian groups*

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}) \xrightarrow{\delta_0} \operatorname{Hom}(\Gamma, \mathbb{C}^*) \xrightarrow{p_0} P(M) \longrightarrow 0, \tag{3.23}$$

where $P(M)$ is the Picard group of the surface, p_0 is the homomorphism that associates to a flat factor of automorphy in $\operatorname{Hom}(\Gamma, \mathbb{C}^*)$ the holomorphic line bundle it describes, and δ_0 is the homomorphism that associates to a holomorphic abelian differential $\omega \in \Gamma(M, \mathcal{O}^{(1,0)})$ the flat factor of automorphy

$$\lambda_\omega(T) = \exp -2\pi i\, \omega(T) \tag{3.24}$$

where $\omega(T)$ is the period class of the differential ω.

Proof: The isomorphism of Corollary 3.12 can be combined with the exact sequence (3.14) of Corollary 3.10 to yield the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(M, \mathcal{O}^{(1,0)}) & \xrightarrow{\delta} & H^1(M, \mathbb{C}^*) & \xrightarrow{p} & P(M) & \longrightarrow & 0 \\ & & \Big\| & & \phi \Big\uparrow \cong & & \Big\| & & \\ 0 & \longrightarrow & \Gamma(M, \mathcal{O}^{(1,0)}) & \xrightarrow{\delta_0} & \operatorname{Hom}(\Gamma, \mathbb{C}^*) & \xrightarrow{p_0} & P(M) & \longrightarrow & 0 \end{array} \tag{3.25}$$

in which $\delta_0 = \phi^{-1} \cdot \delta$ and $p_0 = p \cdot \phi$; and it merely remains to describe the homomorphisms δ_0 and p_0 more explicitly. Clearly p_0 associates to a flat factor of automorphy the holomorphic line bundle represented by the flat line bundle described by that factor of automorphy, or more briefly the holomorphic line bundle described by that factor of automorphy. A holomorphic abelian differential $\omega \in \Gamma(M, \mathcal{O}^{(1,0)})$ is the exterior derivative $\omega = d\,w$ of its integral $w(z)$ on $\widetilde{M}$; the period class of ω then is defined by $\omega(T) = w(Tz) - w(z)$, and to this period class can be associated the flat factor of automorphy $\tilde{\delta}_0(\omega) \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ defined by $\tilde{\delta}_0(\omega)(T) = \exp -2\pi i\, \omega(T)$. This flat factor of automorphy describes a flat line bundle $\phi\big(\tilde{\delta}_0(\omega)\big)$ over M as in Theorem 3.11. Explicitly if $\mathfrak{U} = \{U_\alpha\}$ is a covering of M by contractible coordinate neighborhoods U_α with connected intersections, the connected components $\tilde{U}_{\alpha\, j}$ of the inverse image $\pi^{-1}(U_\alpha)$ of a coordinate neighborhood U_α under the covering projection $\pi : \widetilde{M} \longrightarrow M$ form a coordinate covering of $\widetilde{M}$; and the images $U_{\alpha\, j} = \pi(\tilde{U}_{\alpha\, j})$ under the covering projection π can be viewed as a coordinate covering of the surface M itself. As in (3.19) the coordinate transition functions of the line bundle $\phi\big(\tilde{\delta}_0(\omega)\big)$ in an intersection $U_{\alpha\, j} \cap U_{\beta\, k}$ are the constants

$$\phi\big(\tilde{\delta}_0(\omega)\big)_{\alpha j,\ \beta k} = \tilde{\delta}_0(\omega)(T_{\alpha j,\ \beta k}) = \exp -2\pi i \omega(T_{\alpha j,\ \beta k}) \tag{3.26}$$

for the covering translation $T_{\alpha j,\ \beta k} \in \Gamma$ that takes a point $z_{\beta\, k} \in U_{\beta\, k}$ to the point $z_{\alpha\, j} = T_{\alpha j,\ \beta k} z_{\beta\, k} \in U_{\alpha\, j}$. On the other hand in each set $\tilde{U}_{\alpha\, j}$ the abelian differential ω is the exterior derivative of the restriction $w_{\alpha\, j} = w\big|\tilde{U}_{\alpha\, j}$ of the holomorphic abelian integral w; so in each set $U_{\alpha\, j}$ the abelian differential ω is the exterior derivative $\omega = dw_{\alpha j}$ of the function $w_{\alpha\, j}$ when the latter is viewed as a function in the set $U_{\alpha\, j}$. As in (3.12) the coordinate transition functions of the flat line bundle $\delta(\omega)$ at a point $z \in U_{\alpha\, i} \cap U_{\beta\, j}$, where $z = \pi(z_{\alpha\, j}) = \pi(z_{\beta\, k})$ for points $z_{\alpha\, j} \in \tilde{U}_{\alpha\, j}$ and $z_{\beta\, k} \in \tilde{U}_{\beta\, k}$, are the constants

$$\begin{aligned}
\delta(\omega)_{\alpha\, j,\ \beta\, k} &= \exp 2\pi i\big(w_{\beta\, k}(z) - w_{\alpha\, j}(z)\big)\\
&= \exp 2\pi i\big(w(z_{\beta\, k}) - w(z_{\alpha\, j})\big)\\
&= \exp -2\pi i\big(w(T_{\alpha\, j,\beta\, k} z_{\beta\, k}) - w(z_{\beta\, k})\big)\\
&= \exp -2\pi i \omega(T_{\alpha\, j,\beta\, k}).
\end{aligned}$$

Comparing this with (3.26) shows that $\phi\big(\tilde{\delta}_0(\omega)\big)_{\alpha j,\ \beta k} = \delta(\omega)_{\alpha\, j,\ \beta\, k}$; consequently the homomorphism δ_0 in the exact sequence (3.23) is just this homomorphism $\tilde{\delta}_0$ described by (3.24), and that suffices to conclude the proof.

3.4 The Canonical Parametrization of the Picard Group

The description of the Picard group in the exact sequence (3.23) can be made more explicit by using a convenient parametrization of the group of flat factors of automorphy. In terms of a basis $\tau_j \in H_1(M)$ for the homology group of the surface M associate to any complex vector $t = \{t_j\} \in \mathbb{C}^{2g}$ the homomorphism $\rho_t \in \operatorname{Hom}(\Gamma, \mathbb{C}^*) = \operatorname{Hom}(H_1(M), \mathbb{C}^*)$ for which

$$\rho_t(\tau_j) = \exp 2\pi i\, t_j. \tag{3.27}$$

Any homomorphism in $\operatorname{Hom}(\Gamma, \mathbb{C}^*)$ is of this form for some vector $t \in \mathbb{C}^{2g}$, two vectors $t_1,\, t_2 \in \mathbb{C}^{2g}$ determine the same homomorphism if and only if they differ by an integral vector, and the mapping ρ that associates to a vector $t \in \mathbb{C}^{2g}$ the homomorphism $\rho_t \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ is a homomorphism from the additive group $\mathbb{C}^{2g}$ to the multiplicative group $\operatorname{Hom}(\Gamma, \mathbb{C}^*)$ of flat factors of automorphy, yielding the exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{C}^{2g} \xrightarrow{\ \rho\ } \operatorname{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow 0 \tag{3.28}$$

called the *canonical parametrization* of flat factors of automorphy for the surface M associated to the basis τ_j for the homology group $H_1(M)$. For another basis $\tilde{\tau}_j \in H_1(M)$ there is a corresponding exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{C}^{2g} \xrightarrow{\ \tilde{\rho}\ } \operatorname{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow 0. \tag{3.29}$$

The two homomorphisms ρ and $\tilde{\rho}$ are surjective, so for any vector $t \in \mathbb{C}^{2g}$ there will be at least one vector $\tilde{t} \in \mathbb{C}^{2g}$ such that $\tilde{\rho}_{\tilde{t}} = \rho_t$. If $\tilde{\tau}_j = \sum_{k=1}^{2g} q_{jk}\tau_k$ for a nonsingular matrix $Q = \{q_{jk}\} \in \mathrm{Gl}(2g, \mathbb{Z})$ it follows from (3.27) that

$$\begin{aligned}\exp 2\pi i \tilde{t}_j &= \tilde{\rho}_{\tilde{t}}(\tilde{\tau}_j) = \rho_t(\tilde{\tau}_j) = \rho_t\left(\sum_{k=1}^{2g} q_{jk}\tau_k\right) \\ &= \prod_{k=1}^{2g} \rho_t(\tau_k)^{q_{jk}} = \exp 2\pi i \sum_{k=1}^{2g} q_{jk} t_k,\end{aligned} \tag{3.30}$$

and consequently $\tilde{t}_j = n_j + \sum_{k=1}^{2g} q_{jk} t_k$ for some integers $n_j \in \mathbb{Z}$, or in matrix terms $\tilde{t} = n + Qt$ for the vector $n = \{n_j\} \in \mathbb{Z}^{2g}$. Since $\tilde{\rho}_n = 1$ for any integral vector $n \in \mathbb{Z}^{2g}$ by (3.28), the two canonical parametrizations of flat factors of automorphy are related by $\tilde{\rho}_{Qt} = \rho_t$; and since linear transformation defined by the matrix $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ maps the subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ to itself, there is the commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^{2g} \arrow[r, "\iota"] \arrow[d, "Q", "\cong"'] & \mathbb{C}^{2g} \arrow[r, "\rho"] \arrow[d, "Q", "\cong"'] & \mathrm{Hom}(\Gamma, \mathbb{C}^*) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z}^{2g} \arrow[r, "\iota"] & \mathbb{C}^{2g} \arrow[r, "\tilde{\rho}"] & \mathrm{Hom}(\Gamma, \mathbb{C}^*) \arrow[r] & 0, \end{tikzcd} \tag{3.31}$$

showing the effect of a change of basis for the homology group $H_1(M)$ on the exact sequence (3.28). It follows that the complex Lie group structure on the group $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ arising from the identification $\mathrm{Hom}(\Gamma, \mathbb{C}^*) \cong \mathbb{C}^{2g}/\mathbb{Z}^{2g}$ in the exact sequence (3.28) is independent of the choice of a basis for the homology group $H_1(M)$; thus the group $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ has a uniquely determined structure as a noncompact complex manifold.

When the parametrization (3.28) of the group $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ of flat line bundles is composed with the surjective homomorphism p_0 in the exact sequence (3.23) describing the Picard group $P(M)$ of the Riemann surface M, there results a surjective homomorphism

$$P = p_0 \cdot \rho : \mathbb{C}^{2g} \longrightarrow P(M) \tag{3.32}$$

called the *canonical parametrization* of the Picard group associated to the basis for the homology group $H_1(M)$; this parametrization can be described more explicitly as follows.

Theorem 3.14 *If M is a compact Riemann surface of genus $g > 0$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ there is the exact sequence*

$$0 \longrightarrow \mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{P} P(M) \longrightarrow 0, \tag{3.33}$$

where ι is the natural inclusion homomorphism, Ω is the period matrix of M and P is the canonical parametrization of the Picard group $P(M)$ in terms of these bases.

Proof: In addition to the canonical parametrization (3.28) of flat factors of automorphy introduce the parametrization

$$\sigma : \mathbb{C}^g \xrightarrow{\cong} \Gamma(M, \mathcal{O}^{(1,0)}) \tag{3.34}$$

that associates to any vector $s = (s_1, \ldots, s_g) \in \mathbb{C}^g$ the holomorphic abelian differential $\sigma(s) = \sum_{k=1}^{g} s_k \omega_k$ in terms of the basis $\{\omega_k\}$; and introduce as well the linear mapping ${}^t\Omega : \mathbb{C}^g \longrightarrow \mathbb{C}^{2g}$ defined by the negative of the transpose of the period matrix Ω of the surface M in terms of the bases $\{\omega_k\}$ and $\{\tau_j\}$. These homomorphisms can be combined in the following diagram

$$\begin{tikzcd}
 & & & 0 & & \\
0 \arrow[r] & \Gamma(M, \mathcal{O}^{(1,0)}) \arrow[r, "\delta_0"] & & \mathrm{Hom}(\Gamma, \mathbb{C}^*) \arrow[u] \arrow[r, "p_0"] & P(M) \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{C}^g \arrow[u, "\sigma", "\cong"'] \arrow[rr, "-{}^t\Omega"] & & \mathbb{C}^{2g} \arrow[u, "\rho"] & & \\
 & & & \mathbb{Z}^{2g} \arrow[u, "\iota"] & &
\end{tikzcd} \tag{3.35}$$

The first row is the exact sequence(3.23) of Theorem 3.13 and the long column is the exact sequence (3.28) in the canonical parametrization of flat factors of automorphy. That this is a commutative diagram will be demonstrated by showing that $\rho \cdot (-{}^t\Omega) = \delta_0 \cdot \sigma$. If $s \in \mathbb{C}^g$ the image $\omega = \sigma(s)$ is the holomorphic abelian differential $\omega = \sum_{k=1}^{g} s_k \omega_k$, and by (3.24) the image $\delta_0 \cdot \sigma(s)$ is the homomorphism $\lambda_1 \in \mathrm{Hom}(\Gamma, \mathbb{C}^*) = \mathrm{Hom}(H_1, \mathbb{C}^*)$ for which

$$\begin{aligned}
\lambda_1(\tau_j) = \exp -2\pi i \omega(T_j) \quad &= \quad \exp -2\pi i \sum_{k=1}^{g} s_k \omega_k(T_j) \\
&= \exp -2\pi i \sum_{k=1}^{g} s_k \omega_{kj};
\end{aligned}$$

by (3.27) the image $\lambda_2 = \rho \cdot \big(-{}^t\Omega(s) \big) \in \mathrm{Hom}(\Gamma, \mathbb{C}^*) = \mathrm{Hom}(H_1, \mathbb{C}^*)$ is the homomorphism for which

$$\lambda_2(\tau_j) = \rho_{-{}^t\Omega s}(\tau_j) = \exp -2\pi i \sum_{k=1}^{g} s_k \omega_{kj}.$$

Comparing these two equations shows that $\rho \cdot (-{}^t\Omega) = \delta_0 \cdot \sigma$, hence that the diagram (3.35) is commutative. By definition the canonical parametrization $P : \mathbb{C}^{2g} \longrightarrow P(M)$ of the Picard group is the composition $P = p_0 \cdot \rho$. For the exactness of the sequence (3.33), it is clear from the diagram (3.35) that $(p_0 \cdot \rho)(n - {}^t\Omega s) = (p_0 \cdot \rho \cdot \iota)(n) + (p_0 \cdot \delta_0 \cdot \sigma)(s) = 0$ for any $n \in \mathbb{Z}^{2g}$ and $s \in \mathbb{C}^g$;

conversely if $t \in \mathbb{C}^{2g}$ and $(p_0 \cdot \rho)(t) = 0$ then $\rho(t) = (\delta_0 \cdot \sigma)(s) = \big(\rho \cdot (-\,{}^t\Omega)\big)(s)$ for some $s \in \mathbb{C}^g$, hence $\rho(t + {}^t\Omega s) = 0$ so $t + {}^t\Omega s = \iota(n)$ for some $n \in \mathbb{Z}^{2g}$ and therefore $t \in \mathbb{Z}^{2g} - {}^t\Omega\mathbb{C}^g = \mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g$. That suffices to conclude the proof.

The exact sequence (3.33) depends on the bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M and $\tau_j \in H_1(M)$ for the homology of M. There is a corresponding exact sequence

$$0 \longrightarrow \mathbb{Z}^{2g} + {}^t\tilde{\Omega}\mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{\tilde{P}} P(M) \longrightarrow 0$$

for the period matrix $\tilde{\Omega}$ and the canonical parametrization $\tilde{P}$ of the Picard group associated to other bases $\tilde{\omega}_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tilde{\tau}_j \in H_1(M)$. If $\tilde{\omega}_i = \sum_{k=1}^{g} a_{ik}\omega_k$ and $\tilde{\tau}_j = \sum_{l=1}^{2g} q_{jl}\tau_l$ for nonsingular matrices $A = \{a_{ik}\} \in \mathrm{Gl}(g, \mathbb{C})$ and $Q = \{q_{jl}\} \in \mathrm{Gl}(2g, \mathbb{Z})$ the two period matrices are related by

$$\begin{aligned}(3.36)\qquad \tilde{\Omega} = \{\tilde{\omega}_i(\tilde{\tau}_j)\} &= \Big\{\sum_{k=1}^{g}\sum_{l=1}^{2g} a_{ik}\omega_k(q_{jl}\tau_l)\Big\} = \Big\{\sum_{k=1}^{g}\sum_{l=1}^{2g} a_{ik}\omega_{kl}q_{jl}\Big\} \\ &= A\Omega\,{}^tQ.\end{aligned}$$

The canonical parametrizations ρ and $\tilde{\rho}$ of flat factors of automorphy are related as in (3.31), so that $\tilde{\rho}_{Qt} = \rho_t$ and consequently $\tilde{P}(Qt) = p_0(\tilde{\rho}_{Qt}) = p_0(\rho_t) = P(t)$ for all $t \in \mathbb{C}^{2g}$; and the kernel of the homomorphism $\tilde{P}$ is the subgroup $\mathbb{Z}^{2g} + {}^t\tilde{\Omega}\mathbb{C}^g = \mathbb{Z}^{2g} + Q\,{}^t\Omega\,{}^tA\mathbb{C}^g = Q(\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g)$ in view of (3.36). Thus there is the commutative diagram of exact sequences

$$(3.37)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g & \xrightarrow{\iota} & \mathbb{C}^{2g} & \xrightarrow{P} & P(M) & \longrightarrow & 0 \\ & & {\scriptstyle Q}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle Q}\big\downarrow{\scriptstyle\cong} & & \big\| & & \\ 0 & \longrightarrow & \mathbb{Z}^{2g} + {}^t\tilde{\Omega}\mathbb{C}^g & \xrightarrow{\iota} & \mathbb{C}^{2g} & \xrightarrow{\tilde{P}} & P(M) & \longrightarrow & 0 \end{array}$$

in which the vertical homomorphisms are isomorphisms; and this shows that the canonical parametrization P of the Picard group $P(M)$ transforms canonically under a change in the basis for the homology group $H_1(M)$ of the surface M.

Corollary 3.15 *The Picard group $P(M)$ of a compact Riemann surface M of genus $g > 0$ has a uniquely defined structure as the complex torus $J(\Pi)$ described by the inverse*[4] *period matrix Π to a period matrix Ω of M.*

Proof: For any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M and $\tau_j \in H_1(M)$ for the homology of M the exact sequence (3.33) of Theorem 3.14 yields the isomorphism

$$(3.38)\qquad P(M) \cong \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g};$$

[4] The inverse period matrix is defined in Appendix F.1, and its basic properties are discussed there.

and by Corollary F.15 in Appendix F.1 this quotient group is the complex torus $J(\Pi)$ described by the inverse period matrix Π to the period matrix Ω. Since the period matrices of a compact Riemann surface M for any choices of bases for the holomorphic abelian differentials and the homology of M are equivalent period matrices by Theorem 3.6, their inverse period matrices are also equivalent period matrices as noted on page 531; hence the structure of $P(M)$ as a complex torus is intrinsically defined, and is independent of the choices of these bases. That suffices for the proof.

It should be kept in mind that the Picard group $P(M)$ has a natural group structure, so when viewed as a complex torus it is not just a complex manifold but is an abelian compact Lie group with a specified identity element, the trivial complex line bundle represented by the origin $0 \in \mathbb{C}^{2g}$ in the exact sequence (3.33). When only the structure of a complex manifold is relevant, the Picard group often is called the *Picard variety* of the Riemann surface M, but still is denoted by $P(M)$. The Picard variety $P(M)$ like the Jacobi variety $J(M)$ of a compact Riemann surface is a complex torus of dimension g canonically associated to the Riemann surface M. The two complex tori $J(\Omega)$ and $P(M)$ clearly are closely related, indeed are defined by period matrices that are inverse to one another; their relationship will be discussed further in Section 4.6.

3.5 Alternative Descriptions of the Picard Group

There is a convenient and more explicit description of the holomorphic line bundles forming the Picard group $P(M)$ of a compact Riemann surface M.

Theorem 3.16 (i) *If M is a compact Riemann surface of genus $g > 0$ then for any choice of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M and $\tau_j \in H_1(M)$ for the homology of M there is the exact sequence*

$$0 \longrightarrow {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \xrightarrow{\iota} {}^t\overline{\Omega}\mathbb{C}^g \xrightarrow{P_0} P(M) \longrightarrow 0 \tag{3.39}$$

where Ω is the period matrix of M in terms of these bases, Π is the inverse period matrix to Ω, ι is the natural inclusion homomorphism and P_0 is the restriction of the canonical parametrization $P : \mathbb{C}^g \longrightarrow P(M)$ of the Picard group to the linear subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$.
(ii) *Any line bundle in $P(M)$ is holomorphically equivalent to the flat line bundle $\rho_{{}^t\overline{\Omega}t}$ for some vector $t \in \mathbb{C}^g$; and flat line bundles $\rho_{{}^t\overline{\Omega}t}$ and $\rho_{{}^t\overline{\Omega}s}$ are holomorphically equivalent if and only if $s - t \in \Pi\mathbb{Z}^{2g}$.*

Proof: Since by Theorem 3.14 the canonical parametrization of the Picard group is a homomorphism $P : \mathbb{C}^{2g} \longrightarrow P(M)$ with the subgroup ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ in its kernel, it follows from the direct sum decomposition

$$\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g \tag{3.40}$$

of (F.9) in Appendix F.1 that the restriction of the homomorphism P to the subgroup ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$ is a surjective group homomorphism

$$P_0 : {}^t\overline{\Omega}\mathbb{C}^g \longrightarrow P(M); \tag{3.41}$$

and it follows further from Theorem 3.14 that the kernel of the restriction P_0 is the intersection $(\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g) \cap {}^t\overline{\Omega}\mathbb{C}^g$. By using the natural projection operators (F.11) for the direct sum decomposition (3.40) as given in Appendix F.1, any $n \in \mathbb{Z}^{2g}$ can be decomposed as $n = {}^t\overline{\Omega}\Pi n \oplus {}^t\Omega\overline{\Pi} n$, and consequently

$$(\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g) \cap {}^t\overline{\Omega}\mathbb{C}^g = {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}.$$

That suffices to demonstrate the exactness of the sequence (3.39); the remaining statement of the corollary as an immediate consequence of this exactness, and that concludes the proof.

A different explicit description of the Picard group $P(M)$ arises by restricting the exact sequence (3.33) of Theorem 3.14 to the real linear subspace $\mathbb{R}^{2g} \subset \mathbb{C}^{2g}$ rather than to the complex linear subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$. It is apparent from the definition (3.27) that $|\rho_t(\tau_j)| = 1$ for all the vectors $\tau_j \in H_1(M)$ precisely when $t \in \mathbb{R}^{2g}$; so under the canonical parametrization (3.28) of flat factors of automorphy, real vectors $t \in \mathbb{R}^{2g}$ parametrize precisely those flat factors of automorphy for which $|\rho_t(T)| = 1$ for all $T \in \Gamma$, called *unitary flat factors of automorphy.* The flat line bundles represented by these factors of automorphy under the isomorphism of Corollary 3.12 are those that can be described by flat coordinate bundles $f_{\alpha\beta}$ for some coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of M such that $|f_{\alpha\beta}| = 1$; they are called *unitary flat line bundles.* Since any holomorphic function $f_{\alpha\beta}$ for which $|f_{\alpha\beta}| = 1$ is necessarily constant, unitary flat line bundles can be characterized alternatively as those holomorphic line bundles that can be represented in terms of some coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the surface M by holomorphic coordinate line bundles $\rho_{\alpha\beta}$ such that $|\rho_{\alpha\beta}| = 1$ for all intersections $U_\alpha \cap U_\beta$.

Corollary 3.17 *If M is a compact Riemann surface M of genus $g > 0$ there is the exact sequence*

$$0 \longrightarrow \mathbb{Z}^{2g} \xrightarrow{\iota} \mathbb{R}^{2g} \xrightarrow{P_r} P(M) \longrightarrow 0 \tag{3.42}$$

where ι is the natural inclusion homomorphism and P_r is the restriction of the canonical parametrization $P : \mathbb{C}^{2g} \longrightarrow P(M)$ of the Picard group in terms of any basis for the homology $H_1(M)$ of M to the real subspace $\mathbb{R}^{2g} \subset \mathbb{C}^{2g}$; thus any line bundle in $P(M)$ is holomorphically equivalent to a unique unitary line bundle.

Proof: Let Ω be the period matrix of the surface M in terms of some bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and set $\Omega = \Omega' + i\,\Omega''$ for real $g \times 2g$ matrices Ω' and Ω''. For any complex vectors $t \in \mathbb{C}^{2g}$ and $z = x + i\,y \in \mathbb{C}^g$ clearly $\Im(t - {}^t\Omega z) = \Im(t) - {}^t\Omega' y - {}^t\Omega'' x$, where $\Im(t)$ denotes the imaginary part of the complex number t. Since the full period matrix $\binom{\Omega}{\overline{\Omega}}$ is nonsingular so is the $2g \times 2g$ real matrix $({}^t\Omega' \ {}^t\Omega'')$, so for any vector $t \in \mathbb{C}^{2g}$ there is a vector $z \in \mathbb{C}^g$ such that $\Im(t - {}^t\Omega z) = 0$; then from the exact sequence (3.33) of Theorem 3.14 it follows that $P(t) = P(t - {}^t\Omega z)$ where $t - {}^t\Omega z \in \mathbb{R}^{2g}$. Therefore the restriction P_r of the homomorphism P in the exact sequence (3.33) to the subspace $\mathbb{R}^{2g} \subset \mathbb{C}^{2g}$ has as its image the full Picard group $P(M)$. As for the kernel of the restriction P_r, if $t \in \mathbb{R}^{2g}$ and $P_r(t) = 0$ then $t - n \in {}^t\Omega\mathbb{C}^g$ for some $n \in \mathbb{Z}^{2g}$ by the exact sequence (3.33), and since $t-n$ is real it follows by conjugation that $t-n \in {}^t\overline{\Omega}\mathbb{C}^g$ as well; but in view of the direct sum decomposition (3.40) that can be the case only if $t - n = 0$, so $t \in \mathbb{Z}^{2g}$. That demonstrates the exactness of the sequence (3.42). From this it follows that any holomorphic line bundle in $P(M)$ is the image $P_0(t)$ of a real vector $t \in \mathbb{C}^{2g}$, so is represented by a unitary line bundle; and since all the points in $\mathbb{R}^{2g}$ that have the same image $P_0(t) \in P(M)$ are just the points $t + \mathbb{Z}^{2g}$ that represent the same unitary flat factor of automorphy, that suffices to conclude the proof.

The preceding corollary is very useful in that it selects a unique flat line bundle representing any holomorphic line bundle in the Picard group $P(M)$. It has the disadvantage that it does not directly describe the structure of the group $P(M)$ as a complex manifold; when the complex structure is of principal interest it is usually more convenient to use the identification of the Picard group with the complex torus $J(\Pi)$ as in Corollary 3.16. The groups $H^1(M, \mathbb{C}^*)$ and $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ of flat line bundles and flat factors of automorphy also have natural complex structures provided by the canonical parametrization (3.29) of flat factors of automorphy and the isomorphism of Corollary 3.12; and the mappings p of Corollary 3.10 and p_0 of Theorem 3.13 when viewed as mappings between complex manifolds have the following structure.

Theorem 3.18 *In terms of the natural complex structures on the groups of flat line bundles and of flat factors of automorphy over a compact Riemann surface M of genus $g > 0$, the mappings*

$$p : H^1(M, \mathbb{C}^*) \longrightarrow P(M), \quad p_0 : \mathrm{Hom}(\Gamma, \mathbb{C}^*) \longrightarrow P(M) \tag{3.43}$$

are holomorphic mappings exhibiting the complex manifolds $H^1(M, \mathbb{C}^*)$ and $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ as holomorphic fibre bundles over the complex torus $P(M)$ with fibre the complex vector space $\mathbb{C}^g$ and group the lattice subgroup $\overline{\Pi}\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ acting on the vector space $\mathbb{C}^g$ by translation, where Π is the inverse period matrix to a period matrix of M.

Proof: Choose bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, and let Ω be the period matrix of the surface M in terms of these bases and Π be the inverse period matrix to Ω. The group $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ has the complex structure arising from its identification with the quotient of $\mathbb{C}^{2g}$ by the action of the subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ by translation, as in the canonical parametrization (3.29) of flat factors of automorphy for the surface M. The group $P(M)$ has the structure of the complex torus arising as the quotient of the complex manifold ${}^t\overline{\Omega}\mathbb{C}^g$ by the action of the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \subset {}^t\overline{\Omega}\mathbb{C}^g$ by translation, as in the exact sequence (3.39) of Corollary 3.16. The holomorphic mapping P_0 in the latter exact sequence can be viewed alternatively as the result of factoring the canonical parametrization $P = p_0 \cdot \rho : \mathbb{C}^{2g} \longrightarrow P(M)$ of the Picard group through the natural projection

$$(3.44) \qquad {}^t\overline{\Omega}\Pi : \mathbb{C}^{2g} \longrightarrow {}^t\overline{\Omega}\mathbb{C}^g$$

in the direct sum decomposition (3.40), yielding the commutative diagram

$$(3.45) \qquad \begin{array}{ccc} \mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g & \xrightarrow{\ \rho\ } & \mathrm{Hom}(\Gamma, \mathbb{C}^*) \\ {\scriptstyle {}^t\overline{\Omega}\Pi}\big\downarrow & & {\scriptstyle p_0}\big\downarrow \\ {}^t\overline{\Omega}\mathbb{C}^{2g} & \xrightarrow{\ P_0\ } & P(M). \end{array}$$

When $n \in \mathbb{Z}^{2g}$ is decomposed as $n = {}^t\Omega\overline{\Pi}n \oplus {}^t\overline{\Omega}\Pi n$ in the direct sum decomposition (3.40), the action of n on $\mathbb{C}^{2g}$ by translation decomposes correspondingly as the mapping of $\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^{2g}$ to itself given by

$$(3.46) \qquad ({}^t\Omega s,\ {}^t\overline{\Omega}t) \longrightarrow \big({}^t\Omega(s + \overline{\Pi}n),\ {}^t\overline{\Omega}(t + \Pi n)\big);$$

hence the action of n on $\mathbb{C}^{2g}$ by translation commutes with the action of ${}^t\overline{\Omega}\Pi n$ on the complex manifold ${}^t\overline{\Omega}\mathbb{C}^g$ by translation, in the commutative diagram (3.45), so the mapping (3.44) induces the holomorphic mapping

$$(3.47) \qquad p_0 : \mathrm{Hom}(\Gamma, \mathbb{C}^*) = \frac{{}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g}{\mathbb{Z}^{2g}} \longrightarrow \frac{{}^t\overline{\Omega}\mathbb{C}^g}{{}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}} = P(M)$$

between the quotient spaces. If $U_\alpha \subset {}^t\overline{\Omega}\mathbb{C}^g$ is an open subset that is disjoint from any of its translates under the action of the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$ then the subset

$$(3.48) \qquad {}^t\Omega\mathbb{C}^g \oplus U_\alpha \subset {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g$$

is an open subset of $\mathbb{C}^{2g}$ that is disjoint from any of its translates under the action (3.46) of the group $\mathbb{Z}^{2g}$, so this subset can be identified with the inverse image $p_0^{-1}(U_\alpha) \subset \mathrm{Hom}(\Gamma, \mathbb{C}^*)$; thus if the points of the open subset U_α are parametrized as ${}^t\overline{\Omega}t_\alpha \in U_\alpha$ for local coordinates $t_\alpha \in \mathbb{C}^g$ and if the points in the linear subspace ${}^t\Omega\mathbb{C}^g$ are parametrized as ${}^t\Omega s_\alpha \in {}^t\Omega\mathbb{C}^g$ for local coordinates $s_\alpha \in \mathbb{C}^g$ then the pairs $(s_\alpha, t_\alpha) \in \mathbb{C}^g \times \mathbb{C}^g$ can be taken as local coordinates in $p_0^{-1}(U_\alpha) = {}^t\Omega\mathbb{C}^g \oplus U_\alpha$. In terms of these local coordinates the holomorphic mapping (3.47) is just the restriction of the natural projection $\mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{C}^g$ to the second factor, so it exhibits the mapping p_0 as a local fibration with fibre $\mathbb{C}^g$. Over an intersection $U_\alpha \cap U_\beta \subset {}^t\overline{\Omega}\mathbb{C}^g$ it is possible to take the same fibre coordinates $s_\alpha = s_\beta$ in the fibres ${}^t\Omega\mathbb{C}^g$; thus when viewed as a fibre bundle over the entire complex manifold ${}^t\Omega\mathbb{C}^g$ rather than over the quotient $P(M)$, the fibration (3.47) is a trivial fibre bundle. However for another local coordinate system in $P(M)$ corresponding to the translate

$$U_\beta = U_\alpha + {}^t\overline{\Omega}\Pi n_{\beta\alpha} \quad \text{for} \quad n_{\beta\alpha} \in \mathbb{Z}^{2g} \tag{3.49}$$

it follows from (3.46) that the fibre coordinates are related by

$$s_\beta = s_\alpha + \overline{\Pi} n_{\beta\alpha}; \tag{3.50}$$

of course this also holds over the intersection $U_\alpha \cap U_\beta$, for which $n_{\beta\alpha} = 0$. Altogether then the holomorphic mapping (3.47) exhibits the complex manifold $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ as a holomorphic fibre bundle over $P(M)$ with fibres the complex vector space $\mathbb{C}^g$ and coordinate transition functions in the fibres given by (3.50). The complex manifold $H^1(M, \mathbb{C}^*)$ is biholomorphic to the complex manifold $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ through the isomorphism ϕ of Corollary 3.12, and the two mappings p and p_0 are related as in the commutative diagram (3.25) in the proof of Theorem 3.13; through that diagram the results just derived for the mapping p_0 extend immediately to the corresponding results for the mapping p, and that suffices to conclude the proof.

The fibre bundles (3.43) are described quite explicitly by the coordinate transition functions (3.50) in terms of a covering of the manifold $P(M)$ by coordinate neighborhoods represented by subsets $U_\alpha \subset {}^t\overline{\Omega}\mathbb{C}^g$. These bundles also can be be described globally, paralleling the use of factors of automorphy in describing line bundles over a Riemann surface; and the global description is useful in examining these fibre bundles a bit more closely.

Corollary 3.19 *Let M be a compact Riemann surface of genus $g > 0$ and Π be the inverse period matrix to a period matrix of M.*
(i) *Cross-sections of the fibre bundles* (3.43) *can be identified with mappings $\tilde{f} : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ from the universal covering space of the manifold $P(M)$ to the fibres $\mathbb{C}^g$ such that*

$$\tilde{f}(t + \Pi n) = \tilde{f}(t) + \overline{\Pi} n \quad \textit{for all} \quad n \in \mathbb{Z}^{2g}; \tag{3.51}$$

and these mappings in turn are in one-to-one correspondence with mappings

$$f : J(\Pi) \longrightarrow J(\overline{\Pi})$$

between the complex tori described by the period matrices Π and $\overline{\Pi}$.
(ii) *The fibre bundles* (3.43) *are topologically trivial.*
(iii) *Holomorphic cross-sections of the fibre bundles* (3.43) *are in one-to-one correspondence with with triples (A, Q, a_0) where (A, Q) is a Hurwitz relation from the period matrix Π to the period matrix $\overline{\Pi}$ and $a_0 \in J(\overline{\Pi})$.*
(iv) *The fibre bundles* (3.43) *are holomorphically trivial if and only if the complex tori $J(\Pi)$ and $J(\overline{\Pi})$ are isogenous.*

Proof: (i) With the notation as in the proof of the preceding theorem, cross-sections of the bundle (3.47) are described by mappings $f_\alpha : U_\alpha \longrightarrow \mathbb{C}^g$ that agree in intersections $U_\alpha \cap U_\beta$ and that satisfy

$$f_\alpha({}^t\overline{\Omega} t + {}^t\overline{\Omega}\Pi n) = f_\alpha({}^t\overline{\Omega} t) + {}^t\overline{\Pi} n \tag{3.52}$$

for all points ${}^t\overline{\Omega}t \in U_\alpha$ and all $n \in \mathbb{Z}^{2g}$. Since these local mappings agree in intersections $U_\alpha \cap U_\beta$ they can be combined to yield a mapping $\tilde{f} : {}^t\overline{\Omega}\mathbb{C}^g \longrightarrow \mathbb{C}^g$; and as a consequence of (3.52) this global mapping satisfies $\tilde{f}({}^t\overline{\Omega}t + {}^t\overline{\Omega}\Pi n) = \tilde{f}({}^t\overline{\Omega}t) + {}^t\overline{\Pi}n$, which is just (3.51) in terms of the parametrization of the linear subspace ${}^t\overline{\Omega}\mathbb{C}^g$ by parameter values $t \in \mathbb{C}^g$. Conversely any mapping $\tilde{f}$ satisfying (3.51) when viewed as a mapping defined on the parametrized subspace ${}^t\overline{\Omega}\mathbb{C}^g$ restricts to coordinate neighborhoods $U_\alpha \subset {}^t\overline{\Omega}\mathbb{C}^g$ to yield local mappings satisfying (3.52), and these local mappings describe a cross-section of the fibre bundle. The condition that a mapping $\tilde{f} : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ satisfies (3.51) is equivalent to the condition that this mapping commutes with the natural projections p_i to the quotient groups in the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^g & \xrightarrow{\tilde{f}} & \mathbb{C}^g \\ {\scriptstyle p_1}\big\downarrow & & {\scriptstyle p_2}\big\downarrow \\ \mathbb{C}^g/\Pi Z^{2g} & \xrightarrow{f} & \mathbb{C}^g/\overline{\Pi}\mathbb{Z}^{2g} \end{array}$$

and consequently that the mapping $\tilde{f}$ induces a mapping f between the quotient groups.
(ii) There is a real linear topological homeomorphism $\phi : J(\Pi) \longrightarrow J(\overline{\Pi})$, since any two complex tori of the same dimension are homeomorphic. When the component functions of this homeomorphism ϕ are identified with real linear mappings $\tilde{f}_i : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ satisfying (3.51) the linear functions $\tilde{f}_i$ are linearly independent, so any continuous cross-section $\tilde{f}$ can be written uniquely as a linear combination $\tilde{f} = \sum_{i=1}^g c_i \tilde{f}_i$ for some constants c_i; consequently the fibre bundles (3.43) are topologically trivial.
(iii) As in (i) holomorphic cross-sections of the fibre bundle (3.43) are in one-to-one correspondence with holomorphic mappings $f : J(\Pi) \longrightarrow J(\overline{\Pi})$; and by Theorem F.9 in Appendix F.1 these holomorphic mappings are in one-to-one correspondence with with triples (A, Q, a_0) where (A, Q) is a Hurwitz relation from the period matrix Π to the period matrix $\overline{\Pi}$ and $a_0 \in J(\overline{\Pi})$.
(iv) Since by (iii) holomorphic cross-sections of the fibre bundle (3.43) are constant complex linear functions, the fibre bundle (3.43) is holomorphically trivial if and only if there are g holomorphic cross-sections f_i that are linearly independent complex linear functions; when viewed as the coordinate functions of a mapping $f\colon J(\Pi) \longrightarrow J(\overline{\Pi})$ as in (i), the condition that these coordinate functions are linearly independent complex linear functions is equivalent to the condition that the holomorphic mapping $f\colon J(\Pi) \longrightarrow J(\overline{\Pi})$ is locally biholomorphic, hence that this mapping is an isogeny between the two complex tori. That suffices to conclude the proof.

3.6 The Riemann Matrix Theorem

The Picard and Jacobi varieties of a compact Riemann surface M of genus $g > 0$ have additional properties arising from the multiplicative structure in the cohomology group $H^1(M)$. If $\tau_j \in H_1(M)$ is a basis for the homology of M and $\phi_j \in \mathfrak{H}^1(M)$ is a dual basis for the first deRham group of the surface, so that ϕ_j are closed differential forms of degree 1 on M such that $\int_{\tau_k} \phi_j = \delta^j_k$ for $1 \le k \le 2g$ in terms of the Kronecker delta, the intersection matrix of the surface M in terms of the basis $\{\tau_j\}$ is the $2g \times 2g$ skew-symmetric integral matrix $P = \{p_{jk}\}$ where $p_{jk} = \int_M \phi_j \wedge \phi_k$. If $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ is a basis for the holomorphic abelian differentials then $\omega_i \sim \sum_{j=1}^{2g} \omega_{ij}\phi_j$ in terms of the basis ϕ_j, where $\sim$ denotes cohomologous differential forms and $\{\omega_{ij}\} = \Omega$ is the period matrix of the surface in terms of these bases.

Theorem 3.20 (Riemann Matrix Theorem) *If M is a compact Riemann surface of genus $g > 0$, and if Ω is the period matrix and P is the intersection matrix of that surface in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, then*
(i) $\Omega P\, {}^t\Omega = 0$ *and*
(ii) $i\,\Omega P\, {}^t\overline{\Omega}$ *is a positive definite Hermitian matrix.*

Proof: First $\omega_i \wedge \omega_j = 0$ for any two holomorphic abelian differentials ω_i, ω_j, since the exterior product is a differential form of type $(2,0)$ so must vanish identically; consequently

$$0 = \int_M \omega_i \wedge \omega_j = \sum_{kl} \omega_{ik}\omega_{jl} \int_M \phi_k \wedge \phi_l = \sum_{kl} \omega_{ik}\omega_{jl} p_{kl},$$

which in matrix terms is (i). Next in a coordinate neighborhood U_α with local coordinate $z_\alpha = x_\alpha + iy_\alpha$ a holomorphic abelian differential ω can be written $\omega = f_\alpha\, dz_\alpha$ for some holomorphic function f_α; consequently

$$i\,\omega \wedge \overline{\omega} = i|f_\alpha|^2 dz_\alpha \wedge d\overline{z}_\alpha = 2|f_\alpha|^2 dx_\alpha \wedge dy_\alpha.$$

Since $dx_\alpha \wedge dy_\alpha$ is the local element of area in the canonical orientation of the Riemann surface M the integral of this differential form is non-negative, indeed is strictly positive so long as $\omega \ne 0$. Therefore if $\omega = \sum_i c_i\omega_i$ for some complex constants c_i and if $h_{ij} = i\int_M \omega_i \wedge \overline{\omega}_j$ then

$$0 \le i\int_M \omega \wedge \overline{\omega} = i\sum_{ij} c_i\overline{c}_j \int_M \omega_i \wedge \overline{\omega}_j = \sum_{ij} c_i\overline{c}_j h_{ij},$$

and equality occurs only when $c_i = 0$ for all i; this is just the condition that the matrix $H = \{h_{ij}\}$, which is readily seen to be Hermitian, is positive definite. Furthermore

$$h_{ij} = i\int_M \sum_{kl} \omega_{ik}\phi_k \wedge \overline{\omega}_{jl}\phi_l = i\sum_{kl} \omega_{ik}\overline{\omega}_{jl} p_{kl},$$

or in matrix terms $H = i\,\Omega P\, {}^t\overline{\Omega}$; consequently (ii) follows, and that concludes the proof.

Traditionally condition (i) of the preceding theorem is called *Riemann's equality* and condition (ii) is called *Riemann's inequality.* These two conditions taken together amount to the condition that Ω is a Riemann matrix[5] with principal matrix P. Since $\det P = 1$ by Corollary D.2 in Appendix D.2 the entries of the matrix P are relatively prime, so P is a primitive principal matrix for the Riemann matrix Ω; the pair (Ω, P) thus describes a polarized Riemann matrix called the *polarized period matrix* of the surface M in terms of these bases. It was demonstrated in Theorem 3.6 that the period matrices of a Riemann surface for various choices of bases are equivalent period matrices; the corresponding result holds for the polarized period matrices as well.

Corollary 3.21 *The polarized period matrices (Ω, P) of a Riemann surface M of genus $g > 0$ for arbitrary choices of bases for the holomorphic abelian differentials and the homology of M are a full equivalence class of polarized Riemann matrices.*

Proof: Two bases ω_i and $\tilde{\omega}_i$ for the holomorphic abelian differentials on the Riemann surface M are related by $\tilde{\omega}_i = \sum_{k=1}^{g} a_{ik}\omega_k$ for an arbitrary nonsingular complex matrix $A = \{a_{ik}\} \in \mathrm{Gl}(g, \mathbb{C})$, and two bases τ_j and $\tilde{\tau}_j$ for the homology of the surface M are related by $\tilde{\tau}_j = \sum_{l=1}^{2g} \tau_l q_{lj}$ for an arbitrary invertible integral matrix $Q = \{q_{lj}\} \in \mathrm{Gl}(2g, \mathbb{Z})$. The two period matrices for these two bases then are related by $\tilde{\Omega} = A\,\Omega\,Q$, as in Theorem 3.6. If ϕ_j and $\tilde{\phi}_j$ are bases for the deRham group of M dual to the bases τ_j and $\tilde{\tau}_j$ for the homology of M then $\tilde{\phi}_m = \sum_{n=1}^{2g} \phi_n \tilde{q}_{nm}$ for an invertible integral matrix $\tilde{Q} = \{\tilde{q}_{lj}\} \in \mathrm{Gl}(2g, \mathbb{Z})$; and $\delta_j^m = \int_{\tilde{\tau}_j} \tilde{\phi}_m = \sum_{l,n=1}^{2g} q_{lj} \int_{\tau_l} \phi_n \tilde{q}_{nm} = \sum_{l,n=1}^{2g} q_{lj} \delta_l^n \tilde{q}_{nm} = \sum_{l=1}^{2g} q_{lj} \tilde{q}_{lm}$ so ${}^tQ\,\tilde{Q} = \mathrm{I}$ and consequently $\tilde{Q} = {}^tQ^{-1}$. Therefore the intersection matrices P and $\tilde{P}$ for the two homology bases are related by $\tilde{p}_{jk} = \int_M \tilde{\phi}_j \wedge \tilde{\phi}_k = \sum_{l,m=1}^{2g} \tilde{q}_{lj}\tilde{q}_{mk} \int_M \phi_l \wedge \phi_m = \sum_{l,m=1}^{2g} \tilde{q}_{lj}\tilde{q}_{mk} p_{lm}$, or in matrix terms $\tilde{P} = {}^t\tilde{Q}\,P\,\tilde{Q} = Q^{-1}\,P\,{}^tQ^{-1}$. Altogether then the two polarized Riemann matrices (Ω, P) and $(\tilde{\Omega}, \tilde{P})$ are related by $\tilde{\Omega} = A\,\Omega\,Q$ and $\tilde{P} = Q^{-1}\,P\,{}^tQ^{-1}$ for arbitrary matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$; and by (F.33) in Appendix F.3 that is precisely the description of a complete equivalence class of polarized Riemann matrices, so that suffices for the proof.

It follows from Theorem D.1 in Appendix D.2 and its corollary that the intersection matrices for a compact Riemann surface of genus $g > 0$ are precisely the $2g \times 2g$ integral matrices $Q\mathrm{J}Q^{-1}$ where $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ is an arbitrary invertible integral matrix and J is the basic $2g \times 2g$ integral skew-symmetric matrix

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}; \tag{3.53}$$

thus the polarized period matrix of a compact Riemann surface of genus $g > 0$ is always equivalent to a polarized Riemann matrix of the form (Ω, J), a princi-

[5]The definition of a Riemann matrix and of various related notions, and a survey of some of the basic properties of Riemann matrices, are given in Appendix F.3.

pally polarized Riemann matrix. By Lemma F.22 in Appendix F.3 any principally polarized Riemann matrix is in turn equivalent to a normalized principally polarized Riemann matrix, one of the form $\big((\mathrm{I} \quad Z), \mathrm{J}\big)$ where I is the identity matrix of rank g and $Z \in \mathfrak{H}_g$ is a matrix in the Siegel upper half-space of rank g, although not to a unique normalized principally polarized Riemann matrix; but by Lemma F.23 in Appendix F.3 two normalized principally polarized Riemann matrices $\big((\mathrm{I} \quad Z),\, \mathrm{J}\big)$ and $\big((\mathrm{I} \quad \tilde{Z}),\, \mathrm{J}\big)$ are equivalent if and only if

$$\tilde{Z} = (A + ZC)^{-1}(B + ZD) \tag{3.54}$$

for a symplectic modular matrix

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}).$$

Thus if $\mathcal{A}_g = \mathfrak{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$ is the quotient of the complex manifold $\mathfrak{H}_g$ by the action (3.54) of the symplectic modular group on the Siegel upper half-space, the moduli space of equivalence classes of principally polarized Riemann matrices, there is associated to any compact Riemann surface M of genus $g > 0$ a unique point $Z(M) \in \mathcal{A}_g$, the point in the quotient space $\mathcal{A}_g$ represented by the matrix Z for any normalized principally polarized period matrix for the surface M. The image point is called the *Riemann modulus* of the Riemann surface M. That of course raises the questions (i) whether the mapping that associates to a compact Riemann surface M of genus $g > 0$ its Riemann modulus $Z(M) \in \mathcal{A}_g$ is an injective mapping, so that the Riemann moduli describe the biholomorphic equivalence classes of Riemann surfaces; and (ii) how to describe or characterize the image in $\mathcal{A}_g$ of the set of Riemann moduli for all compact Riemann surfaces of genus $g > 0$. These are basic questions in the further study of compact Riemann surfaces, to be considered in somewhat more detail later. For the present, though, the normalized principally polarized period matrices of a Riemann surface will be used just to determine canonical bases for the spaces of holomorphic abelian differentials.

If M is a marked Riemann surface[6] of genus $g > 0$ the marking determines a basis for the homology $H_1(M)$ consisting of homology classes $\tau_j = \alpha_j,\ \ \tau_{g+j} = \beta_j$ for $1 \le j \le g$. In terms of this basis the intersection matrix of the surface M is the basic skew-symmetric matrix J, by Theorem D.1 of Appendix D.2, so the period matrix Ω of the surface is a principally polarized Riemann matrix; and this matrix can be reduced to a normalized principally polarized Riemann matrix $\Omega = (\mathrm{I} \quad Z)$, where $Z \in \mathfrak{H}_g$ represents the Riemann modulus of the surface M.

Theorem 3.22 *If M is a marked Riemann surface of genus $g > 0$, with the marking representing homology classes $\alpha_j, \beta_j \in H_1(M)$, there is a uniquely determined basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M such that*

$$\int_{\alpha_j} \omega_i = \delta^i_j \quad \textit{for} \quad 1 \le i \le g; \tag{3.55}$$

[6]The notion and properties of marked surfaces are discussed in Appendix D.2.

the remaining periods are the entries

$$\int_{\beta_j} \omega_i = z_{ij} \tag{3.56}$$

in a matrix $Z \in \mathfrak{H}_g$ in the Siegel upper half-space of rank g that represents the Riemann modulus for the Riemann surface M.

Proof: If $\tilde{\omega}_j$ is a basis for the holomorphic abelian differentials on M and $\tau_j \in H_1(M)$ is the basis for the homology of M arising from the marking of M then the period matrix of the surface in terms of these bases is a principally polarized Riemann matrix Ω. By Lemma F.22 there is a uniquely determined nonsingular matrix $A = \{a_{ij}\} \in \mathrm{Gl}(g, \mathbb{C})$ such that $A^{-1}\Omega$ is a normalized principally polarized Riemann matrix, for which $A^{-1}\Omega = (\mathrm{I} \quad Z)$ where $Z \in \mathfrak{H}_g$. Then $(\mathrm{I} \quad Z)$ is the period matrix of the surface M in terms of the basis $\omega_i = \sum_{j=1}^{g} a_{ij}\tilde{\omega}_j$ for the holomorphic abelian differentials on M in terms of the given basis $\tau_j \in H_1(M)$, so these abelian differentials have the periods (3.55) and (3.56) and that suffices for the proof.

The basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on a marked Riemann surface satisfying the conditions of the preceding theorem is called the *canonical basis* for the holomorphic abelian differentials on that marked surface. Such bases are particularly simple for many purposes and are quite commonly used in the study of compact Riemann surfaces; but they do depend on the choice of a marking, so are not intrinsic to the surface itself.

Another important consequence of the Riemann Matrix Theorem is that *the Jacobi and Picard varieties of a compact Riemann surface are biholomorphic complex manifolds.* Indeed by Theorem F.20 in Appendix F.3, if $(\Omega,\ P)$ is a polarized Riemann matrix for which $\det P = 1$ and Π is the inverse period matrix to Ω then $(\Pi,\ {}^tP^{-1})$ is a polarized Riemann matrix equivalent to $(\Omega,\ P)$, where the equivalence is exhibited by the Hurwitz relation $(\Omega P\,{}^t\overline{\Omega},\ P)$ from the period matrix Π to the period matrix Ω; and it follows from this that the complex tori $J(\Omega)$ and $J(\Pi)$ are biholomorphic complex manifolds. The explicit form of this biholomorphic mapping is often quite useful in the examination of Riemann surfaces.

Theorem 3.23 *If M is a compact Riemann surface of genus $g > 0$ and (Ω, P) is the polarized period matrix of that surface in terms of bases for the holomorphic abelian differentials and the homology of the surface then the linear mapping $\Omega P : \mathbb{C}^{2g} \longrightarrow \mathbb{C}^g$ defined by the $g \times 2g$ complex matrix ΩP defines a biholomorphic mapping*

$$(\Omega P)^* : \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g} \longrightarrow \frac{\mathbb{C}^g}{\Omega\mathbb{Z}^{2g}} \tag{3.57}$$

from the Picard variety

$$P(M) = \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g}$$

of the surface M onto its Jacobi variety

$$J(M) = \frac{\mathbb{C}^g}{\Omega\mathbb{Z}^{2g}}.$$

Proof: Since $\Omega\, P\, {}^t\Omega = 0$ by Riemann's equality, the linear subspace ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ is in the kernel of the linear mapping described by the matrix ΩP; and since rank $\Omega P = g$ the subspace ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ is precisely the kernel of this linear mapping. Further $\det P = 1$ by Corollary D.2 in Appendix D.2, so $P \in \mathrm{Gl}(2g, \mathbb{Z})$ and hence $P\mathbb{Z}^{2g} = \mathbb{Z}^{2g}$; consequently $\Omega P\mathbb{Z}^{2g} = \Omega\mathbb{Z}^{2g}$, which suffices to conclude the proof.

Chapter 4

Meromorphic Differentials of the Second Kind

4.1 Principal Parts of Meromorphic Functions and Differentials

Meromorphic differential forms on a compact Riemann surface M often are called *meromorphic abelian differentials* on M. The sheaf $\mathcal{M}^{(1,0)}$ of germs of meromorphic abelian differentials was defined on page 53; and paralleling the identification (2.24) on page 50 there is the natural identification $\mathcal{M}^{(1,0)}(\lambda) \cong \mathcal{M}(\kappa\lambda)$ of the sheaf of germs of meromorphic abelian differentials that are cross-sections of a holomorphic line bundle λ with the sheaf of germs of meromorphic cross-sections of the product $\kappa\lambda$ of the line bundle λ with the canonical bundle κ of the Riemann surface M. The divisors of meromorphic abelian differentials are the not necessarily positive canonical divisors $\mathfrak{k}$; as noted in the discussion of divisors of holomorphic abelian differentials, $\mathfrak{k}$ does not denote a single divisor but rather a linear equivalence class of divisors on M.

If $a \in U_\alpha$ is a pole of the meromorphic abelian differential μ in the coordinate neighborhood $U_\alpha \subset M$ and if γ is a simple closed path in U_α that encircles the point a once in the positive orientation and contains no other singularities of the differential μ on the path itself or in its interior the integral

$$\operatorname{res}_a(\mu) = \frac{1}{2\pi i}\int_\gamma \mu \tag{4.1}$$

is called the *residue* of the abelian differential μ at the point a; since the differential μ is a $\mathcal{C}^\infty$ closed differential form except at its poles it is clear from Stokes's Theorem that the value of this integral is independent of the choice of the path γ, subject to the stated conditions. If z_α is a local coordinate centered at the point $a \in M$ and the differential is written $\mu = f_\alpha dz_\alpha$ then the residue $\operatorname{res}_a(\mu)$ is the coefficient of z_α^{-1} in the Laurent expansion of the function f_α in terms of this coordinate; that coefficient consequently is independent of the choice of the local coordinate. The corresponding coefficient in the Laurent expansion of a meromorphic function, on the other hand, depends on the choice of the local coordinate; a direct calculation for the case of a simple pole is quite convincing. Thus the residue of a meromorphic function really is not well defined independently of the choice of a local coordinate system, while that of a meromorphic abelian differential is; that should be kept in mind to avoid possible confusion.

Theorem 4.1 *The sum of the residues of a meromorphic abelian differential at all its poles on a compact Riemann surface is zero.*

Proof: Let a_i be the finitely many poles of a meromorphic abelian differential μ on the surface M. For each pole a_i let γ_i be a path encircling that pole once in the positive orientation and containing no other poles of the differential μ on the path itself or in its interior, let Δ_i be the interior of the path γ_i, and assume further that these paths are chosen so that the closed sets $\overline{\Delta}_i = \Delta_i \cup \gamma_i$ are disjoint. Note that the path $\cup_i \gamma_i$ can be viewed either as the boundary of the set $\cup_i \Delta_i$ or as the negative of the boundary of the complementary set $M \sim \cup_i \Delta_i$, that is, as the boundary of the latter set with the reversed orientation. The differential form μ is a closed differential form except at the points a_i, hence in particular μ is a closed differential form on the set $M \sim \cup_i \Delta_i$ so $d\mu = 0$ there; consequently

$$2\pi i \sum_i \operatorname{res}_{a_i}(\mu) = \int_{\cup_i \gamma_i} \mu = -\int_{\partial(M \sim \cup_i \Delta_i)} \mu = -\int_{M \sim \cup_i \Delta_i} d\mu = 0$$

by Stokes's Theorem, and that suffices to complete the proof.

A meromorphic function on a compact Riemann surface can be described uniquely up to an additive constant by specifying its singularities; and a meromorphic abelian differential can be described uniquely up to an additive holomorphic abelian differential by specifying its singularities. To make this more precise, the sheaf of germs of holomorphic functions $\mathcal{O}$ is a subsheaf of the sheaf of germs of meromorphic functions $\mathcal{M}$, so there is a well defined quotient sheaf $\mathcal{P} = \mathcal{M}/\mathcal{O}$ called the *sheaf of principal parts* on M. Similarly the sheaf of germs of holomorphic differential forms $\mathcal{O}^{(1,0)}$ is a subsheaf of the sheaf of germs of meromorphic differential forms $\mathcal{M}^{(1,0)}$, so there is a well defined quotient sheaf $\mathcal{P}^{(1,0)} = \mathcal{M}^{(1,0)}/\mathcal{O}^{(1,0)}$ called the *sheaf of differential principal parts* on M. There are thus the two exact sequences of sheaves

$$0 \longrightarrow \mathcal{O} \xrightarrow{\iota} \mathcal{M} \xrightarrow{p} \mathcal{P} \longrightarrow 0, \tag{4.2}$$

$$0 \longrightarrow \mathcal{O}^{(1,0)} \xrightarrow{\iota} \mathcal{M}^{(1,0)} \xrightarrow{p} \mathcal{P}^{(1,0)} \longrightarrow 0, \tag{4.3}$$

in each of which ι is the natural inclusion homomorphism and p is the homomorphism that associates to the germ of a meromorphic function $f_a \in \mathcal{M}_a$ or differential $\mu_a \in \mathcal{M}_a^{(1,0)}$ at a point $a \in M$ its *principal part* $p(f_a) = \mathfrak{p}_a \in \mathcal{P}_a$ or

$p(\mu_a) = \mathfrak{p}_a^{(1,0)} \in \mathcal{P}_a^{(1,0)}$ at the point a, the element in the quotient sheaf that it represents.

The sheaves $\mathcal{P}$ and $\mathcal{P}^{(1,0)}$ and the sheaf homomorphisms p in these exact sequences can be described more concretely in quite familiar terms. A germ of a meromorphic function $f_a \in \mathcal{M}_a$ at any point $a \in M$ has a Laurent expansion in any local coordinate z_α centered at the point a, and the negative terms in that Laurent expansion determine the given germ uniquely up to the germ of a holomorphic function. Thus the image $\mathfrak{p}_a = p(f_a) \in \mathcal{P}_a$, the principal part of the germ f_a at the point a, is described completely by the negative terms in the Laurent expansion of f_a in terms of the local coordinate z_α, and of course any Laurent series with finitely many negative terms describes the germ of some meromorphic function; hence $\mathcal{P}_a$ *can be identified with the set of finite negative Laurent expansions* in a local coordinate centered at the point a. The negative terms in the Laurent expansion in one local coordinate centered at the point a completely determine the negative terms in the Laurent expansion in any other local coordinate centered at that point; but the explicit formula for the relation between these two Laurent expansions depends on the particular local coordinates. The germ of a meromorphic abelian differential μ_a at a point $a \in M$ can be expressed in terms of any local coordinate z_α centered at that point as $\mu_a = f_\alpha dz_\alpha$ where $f_\alpha \in \mathcal{M}_a$, and the principal part of the differential μ_a is determined completely by the principal part of the coefficient function f_α. Thus $\mathcal{P}_a^{(1,0)}$ *can be identified with* $\mathcal{P}_a \cdot dz_\alpha$ in terms of any local coordinate z_α centered at the point a. Again the negative terms in the Laurent expansion in one local coordinate centered at the point a completely determine the negative terms in the Laurent expansion in any other local coordinate centered at that point; but the explicit formula for the relation between these two Laurent expansions depends on the particular local coordinates. The one exceptional case is that of a meromorphic abelian differential having a simple pole with residue r; in that case the single negative term in the Laurent expansion for any local coordinate z_α centered at that point is always $(r/z_\alpha)\, dz_\alpha$. That this is the only case in which the coefficients of the negative terms in the Laurent expansion are independent of the choice of local coordinate is quite evident upon considering just simple changes of the local coordinate of the form $\tilde{z} = cz$. A section $\mathfrak{p} \in \Gamma(M, \mathcal{P})$, called a *principal part on* M, is described by listing principal parts $\mathfrak{p}_{a_\nu} \in \mathcal{P}_{a_\nu}$ of meromorphic functions at a discrete set of points $\{a_\nu\}$ of the Riemann surface M; correspondingly a section $\mathfrak{p}^{(1,0)} \in \Gamma(M, \mathcal{P}^{(1,0)})$, called a *differential principal part on* M, is described by listing the principal parts $\mathfrak{p}_{a_\nu}^{(1,0)} \in \mathcal{P}_{a_\nu}^{(1,0)}$ of meromorphic abelian differentials at a discrete set of points $\{a_\nu\}$ of the Riemann surface M. If a principal part or differential principal part $\mathfrak{p}$ consists of Laurent expansions with poles of order n_ν at distinct points $a_\nu \in M$ the *divisor* of that principal part is the divisor $\mathfrak{d}(\mathfrak{p}) = \sum_\nu n_\nu \cdot a_\nu$; it is locally just the polar divisor $\mathfrak{d}_-(f_\alpha)$ for any local meromorphic function or differential form f_α that has the principal part $\mathfrak{p}$.

Theorem 4.2 *On a compact Riemann surface M there are isomorphisms*

$$\frac{\Gamma(M,\mathcal{P})}{p\big(\Gamma(M,\mathcal{M})\big)} \cong H^1(M,\mathcal{O}) \quad \textit{and} \quad \frac{\Gamma(M,\mathcal{P}^{(1,0)})}{p\big(\Gamma(M,\mathcal{M}^{(1,0)})\big)} \cong H^1(M,\mathcal{O}^{(1,0)}),$$

where p is the linear mapping that associates to a meromorphic function or differential form on M its principal part.

Proof: The exact cohomology sequence associated to the exact sequence of sheaves (4.2) contains the segment

$$\Gamma(M,\mathcal{M}) \xrightarrow{p} \Gamma(M,\mathcal{P}) \xrightarrow{\delta_1} H^1(M,\mathcal{O}) \longrightarrow H^1(M,\mathcal{M});$$

and since $H^1(M,\mathcal{M}) = 0$ by Corollary 2.21 that yields the first isomorphism. Similarly the exact cohomology sequence associated to the exact sequence of sheaves (4.3) contains the segment

$$\Gamma(M,\mathcal{M}^{(1,0)}) \xrightarrow{p} \Gamma(M,\mathcal{P}^{(1,0)}) \xrightarrow{\delta_2} H^1(M,\mathcal{O}^{(1,0)}) \longrightarrow H^1(M,\mathcal{M}^{(1,0)});$$

and since $H^1(M,\mathcal{M}^{(1,0)}) = H^1(M,\mathcal{M}(\kappa)) = 0$ by Corollary 2.21 that yields the second isomorphism and concludes the proof.

This provides an interesting and useful alternative interpretation of the cohomology groups $H^1(M,\mathcal{O})$ and $H^1(M,\mathcal{O}^{(1,0)})$ as measures of the obstruction to a principal part or differential principal part being the principal part of a global meromorphic function or differential form. The cohomology groups can be eliminated from the statement of the preceding theorem by using the Serre Duality Theorem.

Theorem 4.3 *On a compact Riemann surface M there are isomorphisms*

$$\delta_1 : \frac{\Gamma(M,\mathcal{P})}{p\big(\Gamma(M,\mathcal{M})\big)} \xrightarrow{\cong} \Gamma(M,\mathcal{O}^{(1,0)})^* \quad \textit{and} \quad \delta_2 : \frac{\Gamma(M,\mathcal{P}^{(1,0)})}{p\big(\Gamma(M,\mathcal{M}^{(1,0)})\big)} \xrightarrow{\cong} \mathbb{C},$$

where for any principal part $\mathfrak{p} \in \Gamma(M,\mathcal{P})$ the image $\delta_1(\mathfrak{p})$ is the element in the dual space to $\Gamma(M,\mathcal{O}^{(1,0)})$ that takes the value

$$\delta_1(\mathfrak{p})(\omega) = \sum_{a\in M} \operatorname{res}_a(\mathfrak{p}\omega)$$

on a holomorphic abelian differential $\omega \in \Gamma(M,\mathcal{O}^{(1,0)})$, and for any differential principal part $\mathfrak{p}^{(1,0)} \in \Gamma(M,\mathcal{P}^{(1,0)})$

$$\delta_2(\mathfrak{p}^{(1,0)}) = \sum_{a\in M} \operatorname{res}_a(\mathfrak{p}^{(1,0)}).$$

Proof: The first isomorphism in Theorem 4.2 arises from the exact cohomology sequence associated to the exact sequence of sheaves (4.2). Explicitly for a principal part $\mathfrak{p}$ on the Riemann surface M choose a covering $\mathfrak{U}$ of the Riemann surface M by open sets U_α such that each pole of the principal part $\mathfrak{p}$ is contained within a single coordinate neighborhood U_α, and such that there are meromorphic functions f_α in the sets U_α with the principal parts $p(f_\alpha) = \mathfrak{p}|U_\alpha$; the differences $f_{\alpha\beta} = f_\beta - f_\alpha$ are holomorphic in the intersections $U_\alpha \cap U_\beta$ and form a cocycle in $Z^1(\mathfrak{U}, \mathcal{O})$ representing the image of the principal part $\mathfrak{p}$ in the cohomology group $H^1(M, \mathcal{O})$. The functions f_α can be modified so that they become $\mathcal{C}^\infty$ within the sets U_α without changing their values in any intersections $U_\alpha \cap U_\beta$; indeed merely multiply each function f_α by a $\mathcal{C}^\infty$ function that is identically 0 near the pole in U_α and is identically 1 in all other subsets U_β of the covering $\mathfrak{U}$. The modified functions $\tilde{f}_\alpha$ then form a $\mathcal{C}^\infty$ cochain with coboundary $f_{\alpha\beta}$, and the global differential form $\overline{\partial}\tilde{f}_\alpha \in \Gamma(M, \mathcal{E}^{(0,1)})$ represents the cohomology class of the cocycle $f_{\alpha\beta}$ under the isomorphism $H^1(M, \mathcal{O}) \cong \Gamma(M, \mathcal{E}^{(0,1)})/\overline{\partial}\Gamma(M, \mathcal{E})$ of the Theorem of Dolbeault, Theorem 1.9. By the Serre Duality Theorem, Theorem 1.17, the dual space to the quotient vector space $\Gamma(M, \mathcal{E}^{(0,1)})/\overline{\partial}\Gamma(M, \mathcal{E})$ is the vector space $\Gamma(M, \mathcal{O}^{(1,0)})$, where a cross-section $\tau \in \Gamma(M, \mathcal{O}^{(1,0)})$ associates to the cross-section $\omega \in \overline{\partial}\tilde{f}_\alpha$ the value $T_\omega(\overline{\partial}\tilde{f}_\alpha) = \int_M \omega \wedge \overline{\partial}\tilde{f}_\alpha$; of course this can be intrinsic for convenience by multiplying by an arbitrary complex constant. The integrand in this formula is identically zero except near the poles of $\mathfrak{p}$ where the meromorphic functions f_α have been modified, since otherwise $\tilde{f}_\alpha = f_\alpha$ is holomorphic. Therefore if Δ_i are disjoint open neighborhoods of the distinct poles a_i of the principal part $\mathfrak{p}$ within which the functions f_α have been modified then

$$\begin{aligned} T_\omega(\overline{\partial}\tilde{f}_\alpha) &= \frac{1}{2\pi i}\sum_i \int_{\Delta_i} \overline{\partial}\tilde{f}_\alpha \wedge \omega = \frac{1}{2\pi i}\sum_i \int_{\Delta_i} d\big(\tilde{f}_\alpha \omega\big) \\ &= \frac{1}{2\pi i}\sum_i \int_{\partial\Delta_i} \tilde{f}_\alpha \omega = \frac{1}{2\pi i}\sum_i \int_{\partial\Delta_i} f_\alpha \omega \\ &= \sum_i \operatorname{res}_{a_i}(f_\alpha \omega) \end{aligned}$$

by Stokes's Theorem, since the functions $\tilde{f}_\alpha$ and f_α coincide on the boundaries of the sets Δ_i. That yields the first isomorphism. The same argument carries through for a differential principal part $\mathfrak{p}^{(1,0)}$, simply replacing the functions f_α by differential forms in applying the Serre Duality Theorem; the integration does not involve integrating against a holomorphic abelian differential but just against a constant. That yields the second isomorphism and concludes the proof of the theorem.

If $\mathfrak{p}_a \in \mathcal{P}_a$ is a principal part at the point a and f is a meromorphic function in an open neighborhood of a with that principal part, and if ω is a holomorphic differential form in an open neighborhood of the point a, then the product $f\,\omega$ is a meromorphic abelian differential near the point a that is determined by the principal part $\mathfrak{p}_a$ only up to the product of the holomorphic abelian differential ω and an arbitrary holomorphic function near that point; but the residue $\operatorname{res}_a(f\,\omega)$ appearing in the statement of the preceding theorem is independent of the choice of this additive term, so is well defined. The preceding theorem provides criteria for determining which principal parts or differential principal parts on a compact Riemann surface are the principal parts of meromorphic functions or meromorphic abelian differentials on the surface.

Corollary 4.4 *On any compact Riemann surface M*
(i) *there exists a meromorphic function with the principal part $\mathfrak{p} \in \Gamma(M, \mathcal{P})$ if and only if*

$$\sum_{a \in M} \mathrm{res}_a(\mathfrak{p}\omega) = 0$$

for all holomorphic abelian differentials $\omega \in \Gamma(M, \mathcal{O}^{(1,0)})$;
(ii) *there exists a meromorphic abelian differential with the differential principal part $\mathfrak{p}^{(1,0)} \in \Gamma(M, \mathcal{P}^{(1,0)})$ if and only if*

$$\sum_{a \in M} \mathrm{res}_a(\mathfrak{p}^{(1,0)}) = 0.$$

Proof: The first isomorphism of the preceding theorem shows that a principal part $\mathfrak{p} \in \Gamma(M, \mathcal{P})$ is the principal part of a meromorphic function, that is, is contained in the subgroup $p(\Gamma(M, \mathcal{M}))$, if and only if $\delta_1(\mathfrak{p})$ is the trivial linear functional on the space of holomorphic abelian differentials, hence if and only if $\sum_{a \in M} \mathrm{res}_a(\mathfrak{p}\omega) = 0$ for all holomorphic abelian differentials ω. Correspondingly the second isomorphism of the preceding theorem shows that a differential principal part $\mathfrak{p}^{(1,0)} \in \Gamma(M, \mathcal{P}^{(1,0)})$ is the principal part of a meromorphic differential, that is, is contained in the subgroup $p(\Gamma(M, \mathcal{M}^{(1,0)}))$, if and only if $0 = \delta_2(\mathfrak{p}^{(1,0)}) = \sum_{a \in M} \mathrm{res}_a(\mathfrak{p}^{(1,0)})$. That suffices for the proof.

Part (ii) of the preceding corollary complements Theorem 4.1 by showing that the necessary condition given in Theorem 4.1 that a differential principal part on a compact Riemann surface M be the principal part of a meromorphic abelian differential also is sufficient.

For an example, there are no nontrivial holomorphic abelian differentials on the Riemann sphere $M = \mathbb{P}^1$, since $c(\kappa) = 2g - 2 = -2$; so in this case the first assertion of the preceding corollary is just the familiar result that there are meromorphic functions on the Riemann sphere $\mathbb{P}^1$ with arbitrary principal parts. Indeed any principal part

$$\mathfrak{p}_a = \frac{c_n}{(z-a)^n} + \cdots + \frac{c_1}{(z-a)} \tag{4.4}$$

at a finite point $a \in \mathbb{C} \subset \mathbb{P}^1$ is itself a rational function, a meromorphic function on all of $\mathbb{P}^1$, that is holomorphic everywhere except for the pole at the point a; and a polynomial

$$\mathfrak{p}_\infty = c_n z^n + \cdots + c_1 z + c_0 = \frac{c_n}{\zeta^n} + \cdots + \frac{c_1}{\zeta} + c_0 \tag{4.5}$$

where $\zeta = \frac{1}{z}$ also is meromorphic on all of $\mathbb{P}^1$ except for the pole at the point $z = \infty$ or $\zeta = 1/z = 0$. Consequently if $\mathfrak{p}_{a_1}, \ldots, \mathfrak{p}_{a_m}$ are principal parts at points $a_1,\dots,a_m$ the sum $f(z) = \mathfrak{p}_{a_1} + \cdots + \mathfrak{p}_{a_m}$ is a meromorphic function on $\mathbb{P}^1$ with precisely these principal parts, and it is unique up to an additive constant. On the other hand a differential principal part $\mathfrak{p}_a^{(1,0)}$ on $\mathbb{P}^1$ is itself a meromorphic abelian differential on all of $\mathbb{P}^1$ that has poles at the points a and ∞; it has residue c_1 at the point a, and at the point $z = \infty$ it has the expansion in terms of the local coordinate $\zeta = 1/z$

$$\begin{aligned} \mathfrak{p}_a^{(1,0)} &= \left(\frac{c_n\zeta^n}{(1-a\zeta)^n} + \cdots + \frac{c_2\zeta^2}{(1-a\zeta)^2} + \frac{c_1\zeta}{1-a\zeta}\right)\left(-\frac{d\zeta}{\zeta^2}\right) \\ &= \left(-\frac{c_n\zeta^{n-2}}{(1-a\zeta)^n} - \cdots - \frac{c_2}{(1-a\zeta)^2} + \frac{ac_1}{1-a\zeta} - \frac{c_1}{\zeta}\right)d\zeta, \end{aligned}$$

since

$$-\frac{c_1}{(1-a\zeta)\zeta} = \frac{ac_1}{1-a\zeta} - \frac{c_1}{\zeta};$$

thus it has a simple pole at ∞ with residue $-c_1$, so altogether it is a meromorphic abelian differential on $\mathbb{P}^1$ with total residue 0. The differential principal part $\mathfrak{p}_\infty^{(1,0)}$ of (4.5) has the expansion in terms of the local coordinate $\zeta = 1/z$

$$\mathfrak{p}_\infty^{(1,0)} = -\left(\frac{c_n}{\zeta^{n+2}} + \cdots + \frac{c_1}{\zeta^3} + \frac{c_0}{\zeta^2}\right)d\zeta,$$

so it has zero residue at its pole. Any meromorphic abelian differential on $\mathbb{P}^1$ can be written as $f(z)dz = \mathfrak{p}_{a_1}dz + \cdots + \mathfrak{p}_{a_m}dz$ for some differential principal parts as described, so has total residue 0; incidentally any such abelian differential has the form $f(z)dz$ where $f(z)$ is a rational function.

4.2 Meromorphic Abelian Integrals of the Second Kind

The meromorphic abelian differentials that have nontrivial poles but have residue zero at each pole are called *abelian differentials of the second kind*; and the meromorphic abelian differentials having at least one pole with a nonzero residue are called *abelian differentials of the third kind*. Correspondingly a *differential principal part of the second kind* is a differential principal part with zero residues at all its poles, while any other differential principal part is a *differential principal part of the third kind*. By the preceding corollary any differential principal part of the second kind is the principal part of an abelian differential of the second kind; and that differential is determined by its principal part uniquely up to the addition of an arbitrary abelian differential of the first kind. Just as in the case of holomorphic abelian differentials, meromorphic abelian differentials on a compact Riemann surface M of genus $g > 0$ can be viewed as Γ-invariant meromorphic differential forms on the universal covering space $\widetilde{M}$ of the surface M, where Γ is the group of covering translations acting on $\widetilde{M}$. If μ is an abelian differential of the second kind on M viewed as a meromorphic differential form on $\widetilde{M}$ then $\int_\gamma \mu = 0$ for any closed path $\gamma \subset \widetilde{M}$ that avoids the singularities of μ. Indeed since $\widetilde{M}$ is simply connected the path γ is the boundary $\gamma = \partial\Delta$ of a domain $\Delta \subset \widetilde{M}$; and if a_i are the poles of μ in Δ and for each pole γ_i is a closed path in Δ that encircles a_i once in the positive direction, such that the paths γ_i are disjoint and have disjoint interiors, then since μ is a closed differential form in the complement of the points a_i in Δ it follows from Stokes's Theorem and the definition of the residue that $\int_\gamma \mu = \sum_i \int_{\gamma_i} \mu = \sum_i \text{res}_{\,a_i}(\mu) = 0$. Therefore the integral

$$u(z,a) = \int_a^z \mu \tag{4.6}$$

taken along any path in $\widetilde{M}$ that avoids the poles of μ is a well defined meromorphic function of the variables $z, a \in \widetilde{M}$ that is independent of the path of integration; such a function is called a *meromorphic abelian integral* on the Riemann surface M, although of course really it is a meromorphic function on the universal covering space $\widetilde{M}$ in both variables. A meromorphic abelian integral clearly satisfies the conditions

$$u(z,a) = -u(a,z) \quad \text{and} \quad u(z,z) = 0 \tag{4.7}$$

for all points $z \in \widetilde{M}$. It is more convenient in many circumstances to view a meromorphic abelian integral as a function of the first variable only, and to allow it to be modified by an arbitrary additive constant; in such cases the function is denoted by $u(z)$ rather than $u(z,a)$, but also is called a meromorphic abelian integral, just as in the case of holomorphic abelian integrals. It must be kept in mind though that a meromorphic abelian integral $u(z)$ is determined by the meromorphic abelian differential ν only up to an arbitrary additive constant. For any choice of the meromorphic abelian integral $u(z)$ the integral (4.6) is given by $u(z,a) = u(z) - u(a)$ at its regular points.

Lemma 4.5 *The meromorphic abelian integrals $u(z)$ on a compact Riemann surface M of genus $g > 0$ can be characterized as those meromorphic functions on the universal covering space $\widetilde{M}$ of the surface M that satisfy*

$$u(Tz) = u(z) + \mu(T) \quad \textit{for all} \quad T \in \Gamma \tag{4.8}$$

for a group homomorphism $\mu \in \text{Hom}(\Gamma, \mathbb{C})$, where Γ is the covering translation group of M.

Proof: As in the proof of the corresponding result, Lemma 3.1, for holomorphic abelian integrals, a meromorphic abelian integral $u(z)$ on a compact Riemann surface M of genus $g > 0$ clearly satisfies (4.8) since $du(Tz) = du(z)$. The differential $\mu = du$ of any meromorphic function $u(z)$ on $\widetilde{M}$ that satisfies (4.8) is invariant under Γ and has zero residue at each pole so is a meromorphic abelian differential of the second kind; and the function $u(z)$ is an integral of μ so is a meromorphic abelian integral on M. That suffices for the proof.

Clearly the group homomorphism μ in (4.8) is unchanged when the abelian integral $u(z)$ is replaced by $u(z)+c$ for a complex constant c, so it is determined uniquely by the meromorphic abelian differential μ; it is called the *period class* of the abelian differential of the second kind $\mu = du$. The period class can be viewed either as an element of the group $H^1(\Gamma, \mathbb{C}) \cong \mathrm{Hom}(\Gamma, \mathbb{C})$ or as an element of the group $\mathrm{Hom}(H_1(M), \mathbb{C}) = H^1(M, \mathbb{C})$; in the latter case it associates to the homology class of a path $\tau \subset M$ that avoids the singularities of the abelian differential μ the period $\mu(\tau) = \int_\tau \mu$ in the usual sense.

Lemma 4.6 *A meromorphic abelian differential of the second kind on a compact Riemann surface M of genus $g > 0$ is determined by its period class uniquely up to the derivative of a meromorphic function on M.*

Proof: If the period class of a meromorphic abelian differential of the second kind is identically zero its integral is a meromorphic function on the Riemann surface M itself and conversely, and that suffices for the proof.

Theorem 4.7 *On a compact Riemann surface M of genus $g > 0$ let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{jk}\}$ be the intersection matrix of M in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$.*
(i) *The periods $\mu(\tau_j)$ of an abelian differential μ of the second kind on M with the differential principal part $\mathfrak{p}^{(1,0)}$ satisfy*

$$\sum_{j,k=1}^{2g} \omega_{ij}\, p_{jk}\, \mu(\tau_k) = 2\pi i \sum_{a\in M} \mathrm{res}_a(w_i\, \mu) = 2\pi i \sum_{a \in M} \mathrm{res}_a(w_i\, \mathfrak{p}^{(1,0)}) \tag{4.9}$$

for $1 \le i \le g$, where $w_i(z) = \int_a^z \omega_i$ is the integral of the differential form ω.
(ii) *The periods $\mu'(\tau_j)$ and $\mu''(\tau_j)$ of any two abelian differentials μ', μ'' of the second kind on M satisfy*

$$\sum_{j,k=1}^{2g} \mu'(\tau_j)\, p_{jk}\, \mu''(\tau_k) = 2\pi i \sum_{a\in M} \mathrm{res}_a(u'\, \mu'') \tag{4.10}$$

where $u'(z) = \int_a^z \mu'$ is the integral of the meromorphic differential form μ'.

Proof: (i) Let $u(z) = \int_a^z \mu$ be an integral of the differential of the second kind μ, where a is any point other than a pole of μ; the periods of μ thus are given by $\mu(T) = u(Tz) - u(z)$ for all $T \in \Gamma$. Choose disjoint contractible open neighborhoods Δ_j of the poles a_j of μ in M, and for each of these neighborhoods choose a connected component $\tilde{\Delta}_j$ of the inverse image $\pi^{-1}(\Delta_j) \subset \widetilde{M}$ where $\pi : \widetilde{M} \longrightarrow M$ is the covering projection. The set $\tilde{\Delta}_j$ is thus homeomorphic to Δ_j under the covering projection π; and the complete inverse image $\pi^{-1}(\Delta_j) \subset \widetilde{M}$ is the union of the disjoint open sets $T\tilde{\Delta}_j$ for all $T \in \Gamma$. Choose a $\mathcal{C}^\infty$ real-valued function r on M that is identically one on an open neighborhood of $M \sim \cup_j \Delta_j$ and is identically zero in an open neighborhood of each pole a_j; this function also can be viewed as a Γ-invariant function on $\widetilde{M}$. In terms of this auxiliary function introduce the smoothed integral

$$\tilde{u}(z) = \begin{cases} u(z) & \text{for} \quad z \in \widetilde{M} \sim \cup_j \Gamma\tilde{\Delta}_j \\ r(z)u(z) & \text{for} \quad z \in \cup_j \tilde{\Delta}_j \\ \tilde{u}(T^{-1}z) + \mu(T) & \text{for} \quad z \in \cup_j T\tilde{\Delta}_j, \quad T \ne I. \end{cases}$$

Thus $\tilde{u}(z)$ is a $\mathcal{C}^\infty$ function on $\widetilde{M}$, $\tilde{u}(Tz) = \tilde{u}(z) + \mu(T)$ for any covering translation $T \in \Gamma$, and $\tilde{u}(z) = u(z)$ whenever $z \notin \cup_j \Gamma\tilde{\Delta}_j$. The differential form $\tilde{\mu} = d\tilde{u}$ is a $\mathcal{C}^\infty$ closed Γ-invariant differential form on $\widetilde{M}$, or equivalently is a $\mathcal{C}^\infty$ closed differential form on M, that is holomorphic outside the set $\cup_j \Delta_j$ and that has the same periods as the meromorphic abelian differential μ. Let ϕ_j be a basis for the first deRham group of M dual to the chosen basis $\tau_j \in H_1(M)$, so that ϕ_j are $\mathcal{C}^\infty$ closed differential forms of degree 1 on M with the periods $\phi_j(\tau_k) = \delta^j_k$. The $\mathcal{C}^\infty$ differential form $\tilde{\mu}$ and the holomorphic abelian differentials ω_j can be expressed in terms of this basis by

$$\tilde{\mu} \sim \sum_{k=1}^{2g} \mu(\tau_k)\phi_k \quad \text{and} \quad \omega_i \sim \sum_{j=1}^{2g} \omega_{ij}\phi_j,$$

where $\sim$ denotes cohomologous differential forms, those that differ by exact differential forms. Then

$$\int_M \omega_i \wedge \tilde{\mu} = \sum_{j,k=1}^{2g} \int_M \omega_{ij}\phi_j \wedge \mu(\tau_k)\phi_k = \sum_{j,k=1}^{2g} \omega_{ij}\, p_{jk}\, \mu(\tau_k)$$

where $p_{jk} = \int_M \phi_j \wedge \phi_k$ are the entries of the intersection matrix of the surface M in terms of these bases. On the other hand the differential form $\tilde{\mu}$ is holomorphic outside the set $\cup_j \Delta_j$, so that $\omega_i \wedge \tilde{\mu} = 0$ there, and consequently by Stokes's Theorem

$$\begin{aligned} \int_M \omega_i \wedge \tilde{\mu} &= \int_{\cup_j \Delta_j} \omega_i \wedge \tilde{\mu} = \int_{\cup_j \tilde{\Delta}_j} d(w_i \tilde{\mu}) \\ &= \sum_j \int_{\partial\tilde{\Delta}_j} w_i\, \tilde{\mu} = \sum_j \int_{\partial\tilde{\Delta}_j} w_i\, \mu = 2\pi i \sum_j \operatorname{res}_{a_j}(w_i\, \mu) \end{aligned}$$

since $\mu = \tilde{\mu}$ on the boundary of the sets Δ_j. Combining these two observations yields the formula of part (i), since $\operatorname{res}_a(w_i\mu) = \operatorname{res}_a(w_i\mathfrak{p}^{(1,0)})$ for any meromorphic abelian differential μ with the differential principal part $\mathfrak{p}^{(1,0)}$.

(ii) Next for any two abelian differentials μ', μ'' of the second kind let $\{a_j\}$ be the union of the sets of poles of these two differentials, and choose disjoint contractible open neighborhoods Δ_j of these points and a $\mathcal{C}^\infty$ function r as in the preceding part of the argument; and in these terms introduce the smoothed versions $\tilde{u}'(z)$ and $\tilde{u}''(z)$ of the integrals $u'(z) = \int_a^z \mu'$ and $u''(z) = \int_a^z \mu''$, also as in the preceding part of the argument. Then for the $\mathcal{C}^\infty$ closed differential forms $\tilde{\mu}' = d\tilde{u}'$ and $\tilde{\mu}'' = d\tilde{u}''$ it follows that

$$\int_M \tilde{\mu}' \wedge \tilde{\mu}'' = \sum_{j,k=1}^{2g} \int_M \mu'(\tau_j)\phi_j \wedge \mu''(\tau_k)\phi_k = \sum_{j,k=1}^{2g} \mu'(\tau_j)\, p_{jk}\, \mu''(\tau_k).$$

Again $\tilde{\mu}' \wedge \tilde{\mu}'' = 0$ outside the sets Δ_j, since both $\tilde{\mu}'$ and $\tilde{\mu}''$ are holomorphic differential forms there, and consequently by Stokes's theorem

$$\begin{aligned}\int_M \tilde{\mu}' \wedge \tilde{\mu}'' &= \int_{\cup_j \Delta_j} \tilde{\mu}' \wedge \tilde{\mu}'' = \int_{\cup_j \tilde{\Delta}_j} d(\tilde{u}'\tilde{\mu}'') \\ &= \sum_j \int_{\partial\tilde{\Delta}_j} \tilde{u}'\tilde{\mu}'' = \sum_j \int_{\partial\tilde{\Delta}_j} u'\mu'' = 2\pi i \sum_j \operatorname{res}_{a_j}(u'\mu'')\end{aligned}$$

since $\tilde{u}' = u'$ and $\tilde{\mu}'' = \mu''$ on the boundary of the sets Δ_j. Combining these two observations yields the formula of part (ii) and thereby concludes the proof.

The products $w_i\mu$ and $u'\mu''$ in the preceding theorem are meromorphic differential forms on $\widetilde{M}$, so their residues are well defined at least on $\widetilde{M}$; and since $w_i(Tz)\mu(Tz) = w_i(z)\mu(z) + \omega_i(T)\mu(z)$ and $u'(Tz)\mu''(Tz) = u'(z)\mu''(z) + \mu'(T)\mu''(z)$, where the differentials μ and μ'' have zero residue, the residues are the same at any two points of $\widetilde{M}$ that are transforms of one another by covering translations, so these residues actually are well defined on the Riemann surface M itself. The expression (4.10) is not symmetric in the differentials μ' and μ''; indeed since the intersection matrix P is skew-symmetric the left-hand side changes sign when these two differentials are reversed, so the right-hand side also must change signs and thus

$$\operatorname{res}_a(u'\mu'') = -\operatorname{res}_a(u''\mu'). \tag{4.11}$$

Alternatively this is a simple consequence of Stokes's theorem, since

$$\operatorname{res}_a(u'\mu'') + \operatorname{res}_a(u''\mu') = \frac{1}{2\pi i}\int_\gamma (u'\mu'' + u''\mu') = \frac{1}{2\pi i}\int_\gamma d(u'u'') = 0$$

for any simple closed path γ encircling only the singularity a.

4.3 Intrinsic Abelian Differentials of the Second Kind

An abelian differential of the second kind is determined by its principal part only up to the addition of an arbitrary abelian differential of the first kind; but it is possible to normalize the abelian differentials of the second kind in terms of their period classes so that there corresponds to each differential principal part of the second kind a uniquely determined abelian differential of the second kind. To simplify the notation differential principal parts henceforth will be denoted merely by $\mathfrak{p}$ rather than $\mathfrak{p}^{(1,0)}$.

Theorem 4.8 (i) *For any differential principal part of the second kind* $\mathfrak{p}$ *on a compact Riemann surface* M *of genus* $g > 0$ *there are a unique meromorphic abelian differential of the second kind* $\mu_{\mathfrak{p}}$ *and a unique holomorphic abelian differential* $\omega_{\mathfrak{p}}$ *such that* $\mu_{\mathfrak{p}}$ *has the differential principal part* $\mathfrak{p}$ *and has the same period class as the complex conjugate differential* $\overline{\omega}_{\mathfrak{p}}$.
(ii) *The holomorphic abelian differential* $\omega_{\mathfrak{p}}$ *is characterized by the conditions that*

$$\int_M \omega \wedge \overline{\omega}_{\mathfrak{p}} = 2\pi i \sum_{a \in M} \operatorname{res}_a(w\mathfrak{p}) \tag{4.12}$$

for all holomorphic abelian differentials ω, *where* $w(z) = \int_a^z \omega$ *is the integral of the holomorphic differential form* ω.
(iii) *If* $\mathfrak{p}'$ *and* $\mathfrak{p}''$ *are two differential principal parts of the second kind on* M *then the associated abelian differentials of the second kind* $\mu_{\mathfrak{p}'}$ *and* $\mu_{\mathfrak{p}''}$ *satisfy*

$$\sum_{a \in M} \operatorname{res}_a(u_{\mathfrak{p}'}\mu_{\mathfrak{p}''}) = 0, \tag{4.13}$$

where $u_{\mathfrak{p}'}(z) = \int_a^z \mu_{\mathfrak{p}'}$ *is the integral of the meromorphic differential form* $\mu_{\mathfrak{p}'}$.

Proof: (i) Let $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ be a basis for the space of holomorphic abelian differentials on the surface M and $\tau_j \in H_1(M)$ be a basis for the homology of the surface M, and in terms of these bases let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{ij}\}$ be the intersection matrix of M. As in (F.9) in Appendix F.1 there is the direct sum decomposition $\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g$, in which the subspace ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ consists of the period vectors $\{\omega(\tau_j)\}$ of the holomorphic abelian differentials ω on the basis τ_j and the subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$ consists of the period vectors $\{\overline{\omega}(\tau_j)\}$ of the complex conjugates $\overline{\omega}$ of the holomorphic abelian differentials on the basis τ_j. If μ is an abelian differential of the second kind with differential principal part $\mathfrak{p}$ then for any abelian differential of the first kind ω the sum $\mu+\omega$ is an abelian differential of the second kind with the same principal part $\mathfrak{p}$, and all the abelian differentials of the second kind with the differential principal part $\mathfrak{p}$ arise in this way. There is a unique abelian differential of the first kind ω such that the period vector $\{\mu_{\mathfrak{p}}(\tau_j)\} = \{\mu(\tau_j) + \omega(\tau_j)\}$ of the sum $\mu_{\mathfrak{p}} = \mu + \omega$ is contained in the linear subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$, hence such that $\mu_{\mathfrak{p}}(\tau_j) = \overline{\omega}_{\mathfrak{p}}(\tau_j)$ for a uniquely determined holomorphic abelian differential $\omega_{\mathfrak{p}}$, thus demonstrating (i).

(ii) If ϕ_j are closed real differential forms of a basis for the first deRham group of M dual to the basis τ_j, then from the homologies $\omega_i \sim \sum_{j=1}^g \omega_i(\tau_j)\phi_j = \sum_{j=1}^g \omega_{ij}\phi_j$ and $\omega_{\mathfrak{p}} \sim \sum_{j=1}^g \omega_{\mathfrak{p}}(\tau_k)\phi_k$ it follows that

$$\int_M \omega_i \wedge \overline{\omega}_{\mathfrak{p}} = \int_M \sum_{j,k=1}^g \omega_{ij}\overline{\omega}_{\mathfrak{p}}(\tau_k)\phi_j \wedge \phi_k = \sum_{j,k=1}^g \omega_{ij}p_{jk}\overline{\omega}_{\mathfrak{p}}(\tau_k);$$

and since $\overline{\omega}_{\mathfrak{p}}(\tau_k) = \mu_{\mathfrak{p}}(\tau_k)$ as in (i) it follows from (4.9) in Theorem 4.7 that

$$\sum_{j,k=1}^{g} \omega_{ij} p_{jk} \overline{\omega}_{\mathfrak{p}}(\tau_k) = \sum_{j,k=1}^{g} \omega_{ij} p_{jk} \mu_{\mathfrak{p}}(\tau_k) = 2\pi i \sum_{a \in M} \mathrm{res}_a(w_i \mathfrak{p}) \tag{4.14}$$

where $w_i(z) = \int_a^z \omega_i$ is the integral of the holomorphic differential form ω_i. Combining the two preceding equations shows that (ii) holds for the basis ω_i, and consequently it holds for all holomorphic abelian differentials ω.

(iii) Finally if $\mathfrak{p}'$ and $\mathfrak{p}''$ are two differential principal parts of the second kind to which correspond the meromorphic differential forms $\mu_{\mathfrak{p}'}$ and $\mu_{\mathfrak{p}''}$ as in (i), the associated holomorphic abelian differentials $\omega_{\mathfrak{p}'}$ and $\omega_{\mathfrak{p}''}$ satisfy $\omega_{\mathfrak{p}'} \wedge \omega_{\mathfrak{p}''} = 0$, since the product is a differential form of type $(2,0)$ on the Riemann surface M; so from the homologies $\omega_{\mathfrak{p}'} \sim \sum_{j=1}^{g} \omega_{\mathfrak{p}'}(\tau_k)\phi_k$ and $\omega_{\mathfrak{p}''} \sim \sum_{j=1}^{g} \omega_{\mathfrak{p}''}(\tau_k)\phi_k$ it follows that

$$0 = \int_M \overline{\omega}_{\mathfrak{p}'} \wedge \overline{\omega}_{\mathfrak{p}''} = \int_M \sum_{j,k=1}^{g} \overline{\omega}_{\mathfrak{p}'}(\tau_j)\phi_j \wedge \overline{\omega}_{\mathfrak{p}''}(\tau_k)\phi_k = \sum_{j,k=1}^{g} \overline{\omega}_{\mathfrak{p}'}(\tau_j) p_{jk} \overline{\omega}_{\mathfrak{p}''}(\tau_k).$$

Since $\overline{\omega}_{\mathfrak{p}'}(\tau_k) = \mu_{\mathfrak{p}'}(\tau_k)$ and $\overline{\omega}_{\mathfrak{p}''}(\tau_k) = \mu_{\mathfrak{p}''}(\tau_k)$ it follows from (4.10) in Theorem 4.7 that

$$\sum_{j,k=1}^{g} \overline{\omega}_{\mathfrak{p}'}(\tau_j) p_{jk} \overline{\omega}_{\mathfrak{p}''}(\tau_k) = \sum_{j,k=1}^{g} \mu_{\mathfrak{p}'}(\tau_j) p_{jk} \mu_{\mathfrak{p}''}(\tau_k) = 2\pi i \sum_{a \in M} \mathrm{res}\ _a(u_1 \mu_2).$$

Combining the two preceding equations shows that (iii) holds and thereby concludes the proof.

The meromorphic abelian differential $\mu_{\mathfrak{p}}$ of part (i) of the preceding theorem is called the *intrinsic abelian differential of the second kind* with the differential principal part $\mathfrak{p}$, and the holomorphic abelian differential $\omega_{\mathfrak{p}}$ is called its *associated holomorphic abelian differential*; both are determined uniquely by the differential principal part $\mathfrak{p}$. Since linear combinations of the intrinsic meromorphic abelian differentials of the second kind have period classes and principal parts that are the corresponding linear combinations it is evident that

$$\mu_{c_1\mathfrak{p}_1 + c_2\mathfrak{p}_2} = c_1 \mu_{\mathfrak{p}_1} + c_2 \mu_{\mathfrak{p}_2} \tag{4.15}$$

for any differential principal parts $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of the second kind and any complex constants c_1, c_2.

Corollary 4.9 *On a compact Riemann surface M of genus $g > 0$ let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{ij}\}$ be the intersection matrix of M in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$. For any differential principal part $\mathfrak{p}$ of the second kind on M the periods of the intrinsic abelian differential of the second kind $\mu_{\mathfrak{p}}$ are*

$$\mu_{\mathfrak{p}}(T) = -2\pi \sum_{m,n=1}^{g} \sum_{a \in M} g_{mn} \mathrm{res}_a(w_m \mathfrak{p}) \overline{\omega}_n(T) \tag{4.16}$$

for any covering translation $T \in \Gamma$, where $G = \{g_{mn}\} = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\, {}^t\overline{\Omega}$.

Proof: The holomorphic abelian differential $\omega_{\mathfrak{p}}$ associated to the intrinsic abelian differential of the second kind $\mu_{\mathfrak{p}}$ can be written as a sum $\omega_{\mathfrak{p}} = \sum_{l=1}^{g} c_l\omega_l$ for some complex constants c_l, so its periods are $\omega_{\mathfrak{p}}(\tau_k) = \sum_{l=1}^{g} c_l\omega_l(\tau_k) = \sum_{l=1}^{g} c_l\omega_{lk}$. Substituting this into (4.14) yields the identity

$$2\pi i \sum_{a\in M} \mathrm{res}_a(w_m\mathfrak{p}) = \sum_{j,k,l=1}^{g} \omega_{mj}p_{jk}\overline{\omega}_{lk}\overline{c}_l = -i\sum_{l=1}^{g} h_{ml}\overline{c}_l \tag{4.17}$$

where h_{ml} are the entries in the $g \times g$ complex matrix $H = i\Omega P\, {}^t\overline{\Omega}$. This matrix is positive definite Hermitian by Riemann's inequality, Theorem 3.20 (ii), so $G = {}^tH^{-1}$ exists; and if $G = \{g_{mn}\}$ then upon multiplying (4.17) by g_{mn} and summing over m it follows that

$$\overline{c}_n = -2\pi \sum_{m=1}^{g}\sum_{a\in M} g_{mn}\mathrm{res}_a(w_m\mathfrak{p}),$$

hence that

$$\mu_{\mathfrak{p}}(\tau_j) = \overline{\omega}_{\mathfrak{p}}(\tau_j) = \sum_{n=1}^{g} \overline{c}_n\overline{\omega}_{nj} = -2\pi \sum_{m,n=1}^{g} g_{mn}\mathrm{res}_a(w_m\mathfrak{p})\overline{\omega_{nj}}.$$

If the covering translation $T \in \Gamma$ corresponds to a homology class $\tau \in H_1(M)$ and $\tau \sim \sum_{j=1}^{g} n_j\tau_j$ for some integers n_j then $\mu_{\mathfrak{p}}(T) = \mu_{\mathfrak{p}}(\tau) = \sum_{j=1}^{g} n_j\mu_{\mathfrak{p}}(\tau_j)$ and correspondingly $\omega_n(T) = \omega_n(\tau) = \sum_{j=1}^{g} n_j\omega_n(\tau_j) = \sum_{j=1}^{g} n_j\omega_{nj}$; multiplying both sides of the preceding equation by n_j and summing over $j = 1, \ldots, g$ yields (4.17) and thereby concludes the proof.

Since $\omega_{\mathfrak{p}}(T) = \overline{\mu_{\mathfrak{p}}(T)}$ it follows immediately from (4.16) that the associated holomorphic abelian differential is given by

$$\omega_{\mathfrak{p}}(z) = -2\pi \sum_{m,n=1}^{g}\sum_{a\in M} \overline{g_{mn}\mathrm{res}_a(w_m\mathfrak{p})}\omega_n(z). \tag{4.18}$$

It is apparent from this that although the intrinsic abelian differentials of the second kind are complex linear functions of their differential principal parts as in (4.15), the associated holomorphic abelian differentials are complex conjugate functions of their differential principal parts; of course that also is clear from the definitions of these differentials, since the periods of the intrinsic abelian differentials and of their associated holomorphic abelian differentials are complex conjugates of one another. The explicit formulas (4.16) and (4.18) depend on the choice of bases ω_i for the holomorphic abelian differentials and τ_j for the homology of the surface M, but it is clear that $\mu_{\mathfrak{p}}(T)$ and $\omega_{\mathfrak{p}}(z)$ are independent of these choices. It may be comforting to verify that directly, and for that purpose as well as for later use it is convenient to rephrase (4.16) in matrix notation. If $\tilde{w} : \widetilde{M} \longrightarrow \mathbb{C}^g$ is the mapping defined by the column vector $\{w_j(z)\}$ of holomorphic abelian integrals on $\widetilde{M}$ and $\omega \in \operatorname{Hom}(\Gamma, \mathbb{C}^g)$ is the group homomorphism defined by the column vector $\{\omega_j(T)\}$ of period classes of the holomorphic abelian differentials, as in (3.3) and (3.4), then (4.16) can be written equivalently as

$$\mu_{\mathfrak{p}}(T) = -2\pi \sum_{a\in M} \operatorname{res}_a({}^t\tilde{w}\mathfrak{p}) \cdot G \cdot \overline{\omega}(T). \tag{4.19}$$

Of course (4.18) can be rephrased correspondingly. A change of basis for the holomorphic abelian differentials on M has the effect of replacing the vector $\tilde{w}$ by $A\tilde{w}$ and the vector $\overline{\omega}(T)$ by $\overline{A\omega}(T)$, and the form matrix G by ${}^tA^{-1}G\overline{A}^{-1}$ as in equation (F.41) in Appendix F.4; and this change clearly leaves the expression $\operatorname{res}_a({}^t\tilde{w}\mathfrak{p}) \cdot G \cdot \overline{\omega}(T)$ unchanged.

Corollary 4.10 *On a compact Riemann surface M of genus $g > 0$ there is a meromorphic function with the principal part $\mathfrak{p}$ if and only if the intrinsic abelian differential of the second kind $\mu_{d\mathfrak{p}}$ with the differential principal part $d\mathfrak{p}$ has a trivial period class.*

Proof: If there is a meromorphic function f with principal part $\mathfrak{p}$ then df is a meromorphic abelian differential of the second kind with the principal part $d\mathfrak{p}$ and with a trivial period class; since its period class is the same as that of the trivial holomorphic abelian differential $\omega = 0$ it follows from Theorem 4.8 (i) that df is an intrinsic abelian differential of the second kind. Conversely if there is an intrinsic abelian differential of the second kind with principal part $d\mathfrak{p}$ and with trivial period class it is the differential of a meromorphic function with the period class $\mathfrak{p}$. That suffices for the proof.

Theorem 4.11 *Let $a_j \in M$ be n distinct points of a compact Riemann surface M of genus $g > 0$, let $\mathfrak{d} = \sum_{j=1}^n \nu_j \cdot a_j$ be a positive divisor with* $\deg \mathfrak{d} =$ *$r \le g$, and let $\mathfrak{p}_{jk}$ be principal parts of the form $\mathfrak{p}_{jk} = z_j^{-k}$ in terms of local coordinates z_j centered at the points a_j. The period classes of the r intrinsic abelian differentials of the second kind $\mu_{jk} = \mu_{d\mathfrak{p}_{jk}}$ with the differential principal parts $d\mathfrak{p}_{jk}$ for $1 \le j \le n$, $1 \le k \le \nu_j$, are linearly dependent if and only if $\mathfrak{d}$ is a special positive divisor.*

Proof: The period classes of the intrinsic abelian differentials of the second kind μ_{jk} are linearly dependent if and only if there are constants c_{jk} not all of which are zero such that $\sum_{j=1}^n \sum_{k=1}^{\nu_j} c_{jk}\mu_{jk}(T) = 0$ for all covering translations $T \in \Gamma$; and that is just the condition that the nontrivial intrinsic abelian differential of the second kind $\mu = \sum_{j=1}^n \sum_{k=1}^{\nu_j} c_{jk}\mu_{jk}$ with the differential principal part $d\mathfrak{p} = \sum_{j=1}^n \sum_{k=1}^{\nu_j} c_{jk}\ d\mathfrak{p}_{jk}$ has trivial period classes, which by Corollary 4.10 is equivalent to the condition there is a meromorphic function on M with the principal part $\mathfrak{p} = \sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk}\, \mathfrak{p}_{jk}$. By Corollary 4.4 there is a meromorphic function on M with principal part $\mathfrak{p}$ if and only if

$$0 = \sum_{a \in M} \mathrm{res}_a(\mathfrak{p}\omega_i) = \sum_{j=1}^{n} \sum_{k=1}^{\nu_j} c_{jk} \mathrm{res}_{a_j}(\mathfrak{p}_{jk}\omega_i) \tag{4.20}$$

where ω_i is a basis for the holomorphic abelian differentials on M. If $\omega_i = f_i(z_j)dz_j$ is the expression of the differential ω_i in terms of the local coordinate z_j centered at the point a_j then

$$\mathrm{res}_{a_j}(\mathfrak{p}_{jk}\omega_i) = \mathrm{res}_{z_j=0}\Big(z_j^{-k} f_i(z_j)\Big) = \frac{1}{(k-1)!} f_i^{(k-1)}(a_j);$$

so for a fixed index i these residues are just the entries in row i of the $g \times r$ Brill-Noether matrix $\Omega(\mathfrak{d})$ of the divisor $\mathfrak{d}$, as defined in (11.9). In these terms (4.20) takes the form $\Omega(\mathfrak{d}) \cdot c = 0$ for the column vector $c = \{c_{jk}\}$; so since $c \neq 0$ and $r \leq g$ this is equivalent to the condition that rank $\Omega \ < \ r = \deg\ \delta$. By the Riemann-Roch Theorem in the form of Theorem 11.3 this in turn is equivalent to the condition that $\gamma(\zeta_{\mathfrak{d}}) - 1 > 0 = \max(0, \deg \mathfrak{d} - g)$; and as in (11.19) that is just the condition that $\mathfrak{d}$ is a special positive divisor. That suffices for the proof.

Corollary 4.12 *If z is a local coordinate at a point $a \in M$ of a compact Riemann surface M of genus $g > 0$, the period classes of the r intrinsic abelian differentials of the second kind $\mu_k = \mu_{d\mathfrak{p}_k}$ with the differential principal parts $d\mathfrak{p}_k = z^{-k-1}dz$ for $1 \leq k \leq r \leq g$ are linearly dependent if and only if $r \cdot a$ is a special positive divisor.*

Proof: This is just the special case of the preceding theorem in which $n = 1$, $\nu_1 = r$, so nothing further is needed to conclude the proof.

Corollary 4.13 *Let $a_i \in M$ be $r \leq g$ distinct points of a compact Riemann surface M of genus $g > 0$ and $\mathfrak{p}_i$ be the principal parts consisting of a simple pole of residue 1 at the point $a_i \in M$ and no other singularities on M. The period classes of the r intrinsic abelian differentials of the second kind $\mu_i = \mu_{d\mathfrak{p}_i}$ with the differential principal parts $d\mathfrak{p}_i$ for $1 \leq i \leq r$ are linearly dependent if and only if $1 \cdot a_1 + \cdots + 1 \cdot a_g$ is a special positive divisor.*

Proof: This is just the special case of the preceding theorem in which $\nu_j = 1$ for $1 \leq j \leq r$, so nothing further is needed to conclude the proof.

As noted in the proof of Theorem 4.8 (i), if Ω is the period matrix of a compact Riemann surface M of genus $g > 0$ in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ the vectors consisting of the period classes of the holomorphic abelian differentials on the basis τ_j span the linear subspace ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$; and by definition the vectors consisting of the period classes of the intrinsic abelian differentials of the second kind on the basis τ_j lie in the complementary linear subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$ in the direct sum decomposition $\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g$. Consequently when the period classes of all of these abelian differentials are viewed as elements of the complex vector space $\mathrm{Hom}(\Gamma, \mathbb{C})$ of dimension $2g$, the period classes of the differentials of the first kind and the period classes of the intrinsic differentials of the second kind lie in complementary linear subspaces of dimension g. The two preceding corollaries then can be interpreted alternatively as follows in terms of the vector spaces $L^{(1,0)}(\mathfrak{d})$ of meromorphic differential forms as defined in (2.37).

Corollary 4.14 *If a is is a point of a compact Riemann surface M of genus $g > 0$ and $r \ge 0$ then $\dim L^{(1,0)}\big((r+1)\cdot a\big) = g + r$ and the differential forms in $L^{(1,0)}\big((r+1)\cdot a\big)$ are linear combinations of abelian differentials of the first and second kinds. If $0 \le r \le g$ the period classes of these differential forms span a linear subspace of $\mathrm{Hom}(\Gamma, \mathbb{C})$ of dimension $g+r$ if and only if the divisor $r \cdot a$ is a general positive divisor.*

Proof: By definition the vector space $L^{(1,0)}\big((r+1)\cdot a\big)$ consists of those meromorphic differential forms on M that have a pole of order at most $r+1$ at the point a; and from the Riemann-Roch Theorem in the form of Theorem 2.26 it follows that $\dim L^{(1,0)}\big((r+1)\cdot a\big) = g + r$, for by (2.39) $\dim L\big(-(r+1)\cdot a\big) = \gamma(\zeta_a^{-(r+1)}) = 0$ since $c(\zeta_a^{-(r+1)}) < 0$. Since the total residue of any differential form in $L^{(1,0)}\big((r+1)\cdot a\big)$ vanishes by Theorem 4.4 (ii), all of these differential forms must be of the first or second kind. A basis for the intrinsic abelian differentials of the second kind in $L^{(1,0)}\big((r+1)\cdot a\big)$ consists of r differential forms with differential principal parts dz^{-k} for $1 \le k \le r$; by Theorem 4.11 the period classes of these differentials are linearly independent if and only if the divisor $r \cdot a$ is a general positive divisor, so since these period classes and those of the abelian differentials of the first kind lie in complementary linear subspaces of the space $\mathrm{Hom}(\Gamma, \mathbb{C})$ of all period classes that suffices for the proof.

Corollary 4.15 *If $a_1, \dots a_r$ are r distinct points of a compact Riemann surface M of genus $g > 0$ and $\mathfrak{d} = 1\cdot a_1 + \cdots + 1\cdot a_r$ then $\dim L^{(1,0)}\big(2\cdot\mathfrak{d}\big) = g + 2r$ and the subspace of $L^{(1,0)}\big(2\cdot\mathfrak{d}\big)$ spanned by abelian differentials of the first and second kinds has dimension $g+r$. If $1 \le r \le g$ the period classes of the abelian differentials of the first and second kinds in $L^{(1,0)}\big(2\cdot\mathfrak{d}\big)$ span a linear subspace of $\mathrm{Hom}(\Gamma, \mathbb{C})$ of dimension $g+r$ if and only if the divisor $\mathfrak{d}$ is a general positive divisor.*

Proof: As in the proof of the preceding corollary, it follows from the Riemann-Roch Theorem in the form of Theorem 2.26 that $\dim L^{(1,0)}\big(2\cdot\mathfrak{d}\big) = g + 2r$. A basis for the intrinsic abelian differentials of the second kind in $L^{(1,0)}\big(2\cdot\mathfrak{d}\big)$ consists of r differential forms with differential principal parts $z_j^{-2}dz_j$ for local coordinates z_j centered at the points a_j in M, for $1 \le j \le r$. By Theorem 4.11 the period classes of these differentials are linearly independent if and only if the divisor $\mathfrak{d}$ is a general positive divisor; so since these period classes and those of the abelian differentials of the first kind lie in complementary linear subspaces of the space $\operatorname{Hom}(\Gamma, \mathbb{C})$ of all period classes that suffices for the proof.

4.4 Canonical and Green's Abelian Differentials of the Second Kind

The most general abelian differential of the second kind with the differential principal part $\mathfrak{p}$ is of the form $\mu_{\mathfrak{p}} + \omega$ for an arbitrary holomorphic abelian differential ω. Some particular choices of the differential ω yield useful alternative normalizations of the abelian differentials of the second kind, which also are determined uniquely by their differential principal parts. If M is a marked[1] Riemann surface of genus $g > 0$, with the marking described by generators $A_j,\ B_j \in \Gamma$ of the covering translation group Γ of the surface for $1 \le j \le g$, the associated canonical basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials is characterized by the period conditions $\omega_i(A_j) = \delta^i_j$ for $1 \le i,\ j \le g$ as in Theorem 3.22. To any differential principal part of the second kind $\mathfrak{p}$ associate the meromorphic abelian differential

$$\hat{\mu}_{\mathfrak{p}} = \mu_{\mathfrak{p}} - \sum_{j=1}^{g} \mu_{\mathfrak{p}}(A_j)\omega_j, \tag{4.21}$$

called the *canonical abelian differential of the second kind* on the marked surface M with the differential principal part $\mathfrak{p}$.

Theorem 4.16 *Let M be a marked Riemann surface of genus $g > 0$, with the marking described by generators $A_j, B_j \in \Gamma$ of the covering translation group Γ of the surface for $1 \le j \le g$, and let $\mathfrak{p}$ be a differential principal part of the second kind on M.*
(i) *The canonical abelian differential $\hat{\mu}_{\mathfrak{p}}$ is characterized by the conditions that it has the differential principal part $\mathfrak{p}$ and the periods*

$$\hat{\mu}_{\mathfrak{p}}(A_j) = 0 \quad \textit{for} \quad 1 \le j \le g. \tag{4.22}$$

(ii) *The remaining periods of the differential $\hat{\mu}_{\mathfrak{p}}$ are determined by* (i) *and*

$$\hat{\mu}_{\mathfrak{p}}(B_j) = 2\pi i \sum_{a \in M} \operatorname{res}_a(w_j \mathfrak{p}) \quad \textit{for} \quad 1 \le j \le g, \tag{4.23}$$

where $w_j(z) = \int_a^z \omega_j$ is the integral of the abelian differential ω_j.

Proof: (i) The defining equation (4.21) for the differential $\hat{\mu}_{\mathfrak{p}}$ amounts to the conditions that $\hat{\mu}_{\mathfrak{p}}$ has the same differential principal part as the intrinsic abelian differential of the second kind $\mu_{\mathfrak{p}}$ and that

$$\hat{\mu}_{\mathfrak{p}}(A_j) = \mu_{\mathfrak{p}}(A_j) - \sum_{k=1}^{g} \mu_{\mathfrak{p}}(A_k)\omega_k(A_j) = 0$$

[1]The definition and properties of markings of a surface are discussed in Appendix D.1.

since $\omega_k(A_j) = \delta^k_j$.
(ii) The intersection matrix of the surface M in terms of the basis for the homology $H_1(M)$ described by the marking is the basic skew-symmetric matrix J by Theorem D.1 in Appendix D.2; and by Theorem 3.22 the period matrix of M in terms of this basis for the homology $H_1(M)$ and the canonical basis for the holomorphic abelian differentials on M has the form $\Omega = (\mathrm{I} \quad Z)$ for a matrix Z in the Siegel upper half-space $\mathfrak{H}_g$, a $g \times g$ complex symmetric matrix Z with a positive definite imaginary part $Y = \Im(Z)$. It then follows from Corollary 4.9 that

$$\mu_{\mathfrak{p}}(A_j) = -2\pi \sum_{m=1}^{g} \sum_{a \in M} g_{mj} \mathrm{res}_a(w_m \mathfrak{p})$$
$$\mu_{\mathfrak{p}}(B_j) = -2\pi \sum_{m,n=1}^{g} \sum_{a \in M} g_{mn} \overline{z}_{nj} \mathrm{res}_a(w_m \mathfrak{p})$$

where

$$H = i\,(\mathrm{I} \quad Z) \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{I} \\ \overline{Z} \end{pmatrix} = i(\overline{Z} - Z) = 2Y$$

and $G = {}^tH^{-1} = \frac{1}{2} Y^{-1}$. Since $Z - \overline{Z} = 2iY$ and $GY = \frac{1}{2}\mathrm{I}$ it follows from (4.21) that

$$\begin{aligned}
\hat{\mu}_{\mathfrak{p}}(B_j) &= \mu_{\mathfrak{p}}(B_j) - \sum_{k=1}^{g} \mu_{\mathfrak{p}}(A_k)\omega_k(B_j) = \mu_{\mathfrak{p}}(B_j) - \sum_{k=1}^{g} \mu_{\mathfrak{p}}(A_k) z_{kj} \\
&= -2\pi \sum_{m,k=1}^{g} \sum_{a \in M} g_{mk} \overline{z}_{kj} \mathrm{res}_a(w_m \mathfrak{p}) + 2\pi \sum_{m,k=1}^{g} \sum_{a \in M} g_{mk} z_{kj} \mathrm{res}_a(w_m \mathfrak{p}) \\
&= 2\pi \sum_{m,k=1}^{g} \sum_{a \in M} g_{mk}\, 2i y_{kj} \mathrm{res}_a(w_m \mathfrak{p}) = 2\pi i \sum_{a \in M} \mathrm{res}_a(w_j \mathfrak{p}),
\end{aligned}$$

which suffices to conclude the proof.

The advantage of this normalization is the simplicity of the periods in (4.22) and (4.23), a convenience in some calculations; the disadvantage is that this normalization is not intrinsically determined, but depends on the marking of the surface. Another normalization, although one that will play almost no role in the subsequent discussion here, leads to harmonic rather than complex analytic functions. To any differential principal part of the second kind $\mathfrak{p}$ on a compact Riemann surface M of genus $g > 0$ associate the meromorphic abelian differential

$$\breve{\mu}_{\mathfrak{p}} = \mu_{\mathfrak{p}} - \omega_{\mathfrak{p}}. \tag{4.24}$$

This differential has the differential principal part $\mathfrak{p}$ and the periods $\breve{\mu}_{\mathfrak{p}}(T) = \mu_{\mathfrak{p}}(T) - \omega_{\mathfrak{p}}(T) = \overline{\omega}_{\mathfrak{p}}(T) - \omega_{\mathfrak{p}}(T) = -2i\Im\big(\omega_{\mathfrak{p}}(T)\big)$ where $\Im(z)$ denotes the imaginary part of the complex number z; it is readily seen to be determined uniquely by the conditions that it has the differential principal part $\mathfrak{p}$ and purely imaginary periods, so it is called the *Green's abelian differential of the second kind* with the differential principal part $\mathfrak{p}$. The disadvantage of this normalization is that the differential $\breve{\mu}_{\mathfrak{p}}$ is not a complex linear function of the differential principal part, since the associated abelian differential $\omega_{\mathfrak{p}}$ is a conjugate linear function of the differential principal part $\mathfrak{p}$; consequently it cannot be expected to depend analytically on other parameters such as the location of the poles. The advantage of this normalization on the other hand is that the period class of the differential $\breve{\mu}_{\mathfrak{p}}$ is purely imaginary, so the real part $g_{\mathfrak{p}}(z) = \Re\big(\breve{u}_{\mathfrak{p}}(z)\big)$ of the integral $\breve{u}_{\mathfrak{p}}(z)$ of the closed differential form $\breve{\mu}_{\mathfrak{p}}$ is a well defined harmonic function on the Riemann surface M with the singularities of the real part of the meromorphic function $\breve{u}_{\mathfrak{p}}(z)$. This function, called the *Green's function* of the differential principal part $\mathfrak{p}$, is determined uniquely up to a real additive constant by the specified principal part and determines the differential $\breve{\mu}_{\mathfrak{p}}$ since it is easy to see that $\breve{\mu}_{\mathfrak{p}} = 2\,\partial g_{\mathfrak{p}}$. Indeed if $\breve{u}_{\mathfrak{p}} = g_{\mathfrak{p}} + i\,h_{\mathfrak{p}}$ where $g_{\mathfrak{p}} = \Re(\breve{u}_{\mathfrak{p}})$ and $h_{\mathfrak{p}} = \Im(\breve{u}_{\mathfrak{p}})$ then by the Cauchy-Riemann equations $0 = \overline{\partial}\breve{u}_{\mathfrak{p}} = \overline{\partial}g_{\mathfrak{p}} + i\,\overline{\partial}h_{\mathfrak{p}}$ so $\overline{\partial}h_{\mathfrak{p}} = i\,\overline{\partial}g_{\mathfrak{p}}$ or equivalently $\partial h_{\mathfrak{p}} = -i\,\partial g_{\mathfrak{p}}$ and therefore $\breve{\mu}_{\mathfrak{p}} = d\breve{u}_{\mathfrak{p}} = \partial\breve{u}_{\mathfrak{p}} = \partial g_{\mathfrak{p}} + i\,\partial h_{\mathfrak{p}} = 2\partial g_{\mathfrak{p}}$.

4.5 Meromorphic Double Differentials of the Second Kind

The simplest abelian differentials of the second kind are those having a single double pole with residue zero; and when viewed also as functions of the pole they will be shown to be meromorphic double differentials. A *meromorphic double differential* $\mu(z,\zeta)$ on a compact Riemann surface M is an expression that is a well defined meromorphic differential form in each variable separately. More explicitly, if $\{U_\alpha, z_\alpha\}$ and $\{V_\beta, \zeta_\beta\}$ are two coordinate coverings of M then in each product $U_\alpha \times V_\beta$ a meromorphic double differential $\mu(z,\zeta)$ has the form

$$\mu(z,\zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha d\zeta_\beta \tag{4.25}$$

where $f_{\alpha\beta}(z_\alpha, \zeta_\beta)$ is a meromorphic function of the variables $(z_\alpha, \zeta_\beta) \in U_\alpha \times V_\beta$, and in an intersection $(U_\alpha \times V_\beta) \cap (U_\gamma \times V_\delta)$

$$f_{\alpha\beta}(z_\alpha, \zeta_\beta) = \kappa_{\alpha\gamma}(z)\kappa_{\beta\delta}(\zeta)f_{\gamma\delta}(z_\gamma, \zeta_\delta) \tag{4.26}$$

where $\kappa_{\alpha\gamma}(z) = (dz_\alpha/dz_\gamma)^{-1}$ and $\kappa_{\beta\delta}(\zeta) = (d\zeta_\beta/d\zeta_\delta)^{-1}$ are cocycles describing the canonical bundle κ over M in the two coordinate coverings; moreover it is required that (4.25) is a well defined meromorphic differential form in each variable separately, so that for each fixed point $z_\alpha \in U_\alpha$ the expression $f_{\alpha\beta}(z_\alpha, \zeta_\beta)d\zeta_\beta$ is a well defined meromorphic differential form in the variable $\zeta_\beta \in V_\beta$ and correspondingly when the two variables are reversed. A consequence is that *for a meromorphic double differential* (4.25) *the polar locus of the meromorphic function* $f_{\alpha\beta}(z_\alpha, \zeta_\beta)$ *of two complex variables does not contain a product* $a_\alpha \times V_\beta \subset U_\alpha \times V_\beta$ or $U_\alpha \times b_\beta \subset U_\alpha \times V_\beta$ *for any points* $a_\alpha \in U_\alpha,\ b_\beta \in V_\beta$;

this restriction on the polar loci of meromorphic double differentials is significant and should be kept in mind throughout the subsequent discussion. In particular if $\omega(z) = f_\alpha(z)dz_\alpha$ is a holomorphic abelian differential on M and $\mu(z) = g_\alpha(z)dz_\alpha$ is a meromorphic abelian differential on M with nontrivial singularities then the product $\omega(z)\mu(z) = f_\alpha(z)g_\alpha(\zeta)dz_\alpha d\zeta_\alpha$ is not a meromorphic double differential with the definition adopted here. A double differential can be viewed alternatively as a meromorphic differential form

$$\mu^*(z,\zeta) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)\ dz_\alpha \wedge d\zeta_\beta \tag{4.27}$$

on the complex manifold $M \times M$ expressed in terms of local product coordinate neighborhoods $U_\alpha \times U_\beta$, when $\{U_\alpha, z_\alpha\}$ and $\{V_\beta, \zeta_\beta\}$ are viewed as coordinate coverings of the two separate factors and formally $f_{\beta\alpha}(z_\alpha,\zeta_\beta) = -f_{\alpha\beta}(z_\alpha,\zeta_\beta)$ since (4.26) is just the formula for the way a differential form $\mu^*(z,\zeta)$ on $M \times M$ transforms under a change of coordinates $(z_\alpha,\zeta_\beta) = \big(f(z_\gamma), g(\zeta_\delta)\big)$ where f, g are holomorphic functions of a single complex variable; again though the singularities of the coordinates are restricted by the condition that the polar locus of the meromorphic function $f_{\alpha\beta}(z_\alpha,\zeta_\beta)$ does not contain a product $a_\alpha \times V_\beta \subset U_\alpha \times V_\beta$ or $U_\alpha \times b_\beta \subset U_\alpha \times V_\beta$ for any points $a_\alpha \in U_\alpha,\ b_\beta \in V_\beta$. However the real interest here lies in the double differential $\mu(z,\zeta)$ as an entity defined on the Riemann surface M itself, so this alternative viewpoint generally will not be used. A meromorphic double differential $\mu(z,\zeta)$ is *symmetric* if $\mu(z,\zeta) = \mu(\zeta,z)$, so that if the point $z \in U_\alpha$ has the local coordinate z_α and the point $\zeta \in V_\beta$ has the local coordinate ζ_β then $f_{\alpha\beta}(z_\alpha,\zeta_\beta) = f_{\beta\alpha}(\zeta_\beta,z_\alpha)$ when $\mu(z,\zeta)$ is written explicitly as in (4.25). In particular if points $z', z'' \in U_\alpha$ have local coordinates z'_α, z''_α then $f_{\alpha\alpha}(z'_\alpha,z''_\alpha) = f_{\alpha\alpha}(z''_\alpha,z'_\alpha)$. Similarly the double differential $\mu(z,\zeta)$ is *skew symmetric* if $\mu(z,\zeta) = -\mu(\zeta,z)$, with the corresponding interpretation. Of course any meromorphic double differential $\mu(z,\zeta)$ can be written uniquely as the sum $\mu(z,\zeta) = \frac{1}{2}\left(\mu(z,\zeta)+\mu(\zeta,z)\right) + \frac{1}{2}\left(\mu(z,\zeta)-\mu(\zeta,z)\right)$ of a symmetric and a skew-symmetric double differential.

The singularities of a meromorphic double differential can be described formally in the same way as the singularities of a meromorphic abelian differential. The sheaf of germs of holomorphic double differentials on $M \times M$ is a subsheaf of the sheaf of germs of meromorphic double differentials, and the quotient sheaf is the sheaf of double differential principal parts on $M \times M$; the image of a meromorphic double differential in the quotient sheaf at a point of $M \times M$ is its principal part at that point. However the quotient sheaf cannot be described as simply as in the case of ordinary meromorphic abelian differentials, since the singularities of meromorphic functions of several variables lie on holomorphic subvarieties of codimension one rather than just on isolated points. There is at least one important case in which a very simple intrinsic description of the singularities is possible, though, that in which the singular locus of the meromorphic double differential $\mu(z,\zeta)$ is the diagonal subvariety $D = \{\ (z,z) \mid z \in M\ \} \subset M \times M$ and the differential has a double pole with zero residue along the diagonal. More explicitly, for a coordinate neighborhood $U_\alpha \subset M$ in which the local coordinate is denoted by either z_α or ζ_α, the restriction of $\mu(z,\zeta)$ to the product neighborhood $U_\alpha \times U_\alpha \subset M \times M$ has the form

$$\mu(z,\zeta) = \left(\frac{1}{(z_\alpha - \zeta_\alpha)^2} + f_\alpha(z_\alpha,\zeta_\alpha)\right) dz_\alpha d\zeta_\alpha \tag{4.28}$$

where $f_\alpha(z_\alpha,\zeta_\alpha)$ is holomorphic in $U_\alpha \times U_\alpha$. Under a change of the local coordinate $z_\alpha = h_\alpha(t_\alpha),\ \zeta_\alpha = h_\alpha(\tau_\alpha)$ in the coordinate neighborhood U_α

$$\begin{aligned}\mu(t,\tau) &= \left(\frac{1}{\big(h_\alpha(t_\alpha) - h_\alpha(\tau_\alpha)\big)^2} + f_\alpha\big(h_\alpha(t_\alpha),h_\alpha(\tau_\alpha)\big)\right) h'_\alpha(t_\alpha)h'_\alpha(\tau_\alpha)dt_\alpha d\tau_\alpha \\ &= \left(\frac{f_{1\alpha}(t_\alpha,\tau_\alpha)}{\big(t_\alpha-\tau_\alpha\big)^2} + f_{2\alpha}(t_\alpha,\tau_\alpha)\right) dt_\alpha d\tau_\alpha\end{aligned}$$

where

$$\begin{aligned}f_{1\alpha}(t_\alpha,\tau_\alpha) &= \left(\frac{t_\alpha-\tau_\alpha}{h_\alpha(t_\alpha)-h_\alpha(\tau_\alpha)}\right)^2 h'_\alpha(t_\alpha)h'_\alpha(\tau_\alpha) \quad \text{and} \\ f_{2\alpha}(t_\alpha,\tau_\alpha) &= f_\alpha\big(h_\alpha(t_\alpha),h_\alpha(\tau_\alpha)\big)\, h'_\alpha(t_\alpha)h'_\alpha(\tau_\alpha)\end{aligned}$$

are holomorphic functions in $U_\alpha \times U_\alpha$. Since $\lim_{\tau_\alpha\to t_\alpha} f_{1\alpha}(t_\alpha,\tau_\alpha) = 1$ it follows that $f_{1\alpha}(t_\alpha,\tau_\alpha) = 1 + (t_\alpha - \tau_\alpha)f_{3\alpha}(t_\alpha,\tau_\alpha)$ for some holomorphic function $f_{3\alpha}(t_\alpha,\tau_\alpha)$ in $U_\alpha\times U_\alpha$. From its definition it is clear that the function $f_{1\alpha}(t_\alpha,\tau_\alpha)$ is symmetric in the variables t_α,τ_α; the function $f_{3\alpha}(t_\alpha,\tau_\alpha)$ hence must be skew-symmetric in these two variables, and consequently $f_{3\alpha}(t_\alpha,\tau_\alpha) = (t_\alpha-\tau_\alpha)f_{4\alpha}(t_\alpha,\tau_\alpha)$ for some holomorphic function $f_{4\alpha}(t_\alpha,\tau_\alpha)$ in $U_\alpha\times U_\alpha$. Thus in terms of the new local coordinates

$$\mu(t,\tau) = \left(\frac{1}{(t_\alpha-\tau_\alpha)^2} + g_\alpha(t_\alpha,\tau_\alpha)\right) dt_\alpha d\tau_\alpha \tag{4.29}$$

for the holomorphic function $g_\alpha(t_\alpha,\tau_\alpha) = f_{4\alpha}(t_\alpha,\tau_\alpha) + f_{2\alpha}(t_\alpha,\tau_\alpha)$. A comparison of (4.28) and (4.29) shows that the principal part of this meromorphic double differential has the same form for any local coordinate system on M, so its singularities can be specified completely merely by saying that it has the differential principal part $(z_\alpha-\zeta_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal of the manifold $M\times M$.

A *meromorphic double differential of the second kind* is a meromorphic double differential that is a differential of the second kind in each variable separately. For example, if there is a meromorphic double differential with principal part $(z_\alpha-\zeta_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal of the manifold $M\times M$ and no other singularities then it is a meromorphic double differential of the second kind. When a meromorphic double differential of the second kind is written explicitly as in (4.25), the expression $f_{\alpha\beta}(z_\alpha,\zeta^\circ_\beta)dz_\alpha$ is a well defined meromorphic abelian differential of the second kind in the variable z_α for each fixed point $\zeta^\circ_\beta \in V_\beta$, and it has well defined periods $g'_\beta(\tau;\zeta^\circ_\beta) = \int_\tau f_{\alpha\beta}(z_\alpha,\zeta^\circ_\beta)dz_\alpha$ on any homology class $\tau \in H_1(M)$. If τ is represented by a closed path that avoids the poles of the differential $f_{\alpha\beta}(z_\alpha, \zeta_\beta^\circ)dz_\alpha$ then that path also avoids the poles of the differential $f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ for all points ζ_β sufficiently near ζ_β°; hence $f_{\alpha\beta}(z_\alpha, \zeta_\beta)$ is continuous in the variable $z_\alpha \in \tau$ and holomorphic in the variable ζ_β for all points in an open neighborhood of the point ζ_β°, so the period $g'_\beta(\tau; \zeta_\beta) = \int_\tau f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ is a holomorphic function of the variable ζ_β. This is one point at which the restriction on the polar loci of meromorphic double differentials is crucial. Since $f_{\alpha\beta}(z_\alpha, \zeta_\beta) = \kappa_{\beta\gamma}(\zeta)f_{\alpha\gamma}(z_\alpha, \zeta_\gamma)$ for $\zeta \in V_\beta \cap V_\gamma$ it follows that $g'_\beta(\tau; \zeta_\beta) = \kappa_{\beta\gamma}(\zeta)g'_\gamma(\tau; \zeta_\gamma)$ for $\zeta \in V_\beta \cap V_\gamma$ and consequently that

$$\mu'(\tau;\zeta) = g'_\beta(\tau;\zeta_\beta)d\zeta_\beta = \Big(\int_{z_\alpha\in\tau} f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha\Big)d\zeta_\beta = \int_{z\in\tau}\mu(z,\zeta) \tag{4.30}$$

is a holomorphic abelian differential on M; it is called the *first period class* of the double differential $\mu(z, \zeta)$. The *second period class* $\mu''(\tau; z)$ is defined correspondingly by reversing the roles of the two variables, so

$$\mu''(\tau;z) = \int_{\zeta\in\tau}\mu(z,\zeta). \tag{4.31}$$

If the double differential $\mu(z, \zeta)$ is symmetric then clearly $\mu'(\tau; z) = \mu''(\tau; z)$; this common value is denoted by $\mu(\tau; z)$ and is called merely the *period class* of the symmetric double differential $\mu(z, \zeta)$. The first period class is a linear function of the homology class τ since clearly

$$\mu'(n'\tau' + n''\tau''; z) = n'\mu'(\tau'; z) + n''\mu'(\tau''; z) \tag{4.32}$$

for any integers $n', n'' \in \mathbb{Z}$ and any homology classes $\tau', \tau'' \in H_1(M)$; hence the first period class can be viewed as a homomorphism

$$\mu' \in \operatorname{Hom}\big(H_1(M), \Gamma(M, \mathcal{O}^{(1,0)})\big) = \operatorname{Hom}\big(\Gamma, \Gamma(M, \mathcal{O}^{(1,0)})\big). \tag{4.33}$$

The corresponding result of course holds for the second period class. Each of these period classes can be integrated again to yield the *double period classes* that associate to any homology classes $\tau', \tau'' \in H_1(M)$ the values

$$\begin{cases} \mu'(\tau',\tau'') = \int_{\zeta\in\tau''}\mu'(\tau';\zeta) = \int_{\zeta\in\tau''}\int_{z\in\tau'}\mu(z,\zeta), \\ \mu''(\tau',\tau'') = \int_{z\in\tau''}\mu''(\tau';z) = \int_{z\in\tau''}\int_{\zeta\in\tau'}\mu(z,\zeta). \end{cases} \tag{4.34}$$

If the double differential $\mu(z, \zeta)$ is symmetric the two double period classes coincide; the common period class is denoted by $\mu(\tau', \tau'')$ and is called the *double period class* of the symmetric double differential. Since the double differential is meromorphic the order of the iterated integrals is significant; the difference

$$\int_{\zeta\in\tau''}\int_{z\in\tau'}\mu(z,\zeta) - \int_{z\in\tau'}\int_{\zeta\in\tau''}\mu(z,\zeta) = \mu'(\tau',\tau'') - \mu''(\tau'',\tau') \tag{4.35}$$

does not necessarily vanish, even for a symmetric double differential $\mu(z,\zeta)$.

In terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M and $\tau_j \in H_1(M)$ for the homology of M, the first period class can be written

$$\mu'(\tau_j;\zeta) = \sum_{k=1}^{g} \lambda'_{kj}\omega_k(\zeta) \quad \text{for} \quad 1 \le j \le 2g \tag{4.36}$$

for some complex constants λ'_{kj}; the coefficients in this expansion form a $g \times 2g$ matrix $\Lambda' = \{\lambda'_{kj}\}$ called the *first period matrix* of the double differential $\mu(z,\zeta)$ in terms of these bases. The indexing convention is chosen so that this is a $g \times 2g$ matrix, hence has the same shape as the other matrices that have been called period matrices. The second period class can be written correspondingly

$$\mu''(\tau_j;z) = \sum_{k=1}^{g} \lambda''_{kj}\omega_k(z) \quad \text{for} \quad 1 \le j \le 2g, \tag{4.37}$$

where the coefficients form the *second period matrix* $\Lambda'' = \{\lambda''_{kj}\}$ of the double differential $\mu(z,\zeta)$ in terms of these bases. The values of the double period classes on the basis τ_i can be viewed as entries in the *double period matrices*, the $2g \times 2g$ matrices

$$N' = \big\{\mu'(\tau_i,\tau_j)\big\} \quad \text{and} \quad N'' = \big\{\mu''(\tau_i,\tau_j)\big\}; \tag{4.38}$$

and since

$$\begin{cases} \mu'(\tau_i,\tau_j) &= \int_{\zeta\in\tau_j} \sum_{k=1}^{g} \lambda'_{ki}\omega_k(\zeta) &= \sum_{k=1}^{g} \lambda'_{ki}\omega_{kj} \\[2ex] \mu''(\tau_i,\tau_j) &= \int_{z\in\tau_j} \sum_{k=1}^{g} \lambda''_{ki}\omega_k(z) &= \sum_{k=1}^{g} \lambda''_{ki}\omega_{kj} \end{cases} \tag{4.39}$$

the double period matrices can be expressed in terms of the first and second period period matrices as

$$N' = {}^t\Lambda'\,\Omega, \quad N'' = {}^t\Lambda''\,\Omega. \tag{4.40}$$

If the double differential is symmetric $\Lambda' = \Lambda''$, which common value is denoted by Λ and is called simply the *period matrix* of the double differential, and $N' = N''$, which common value is denoted by N and is called the *double period matrix* of the symmetric double differential; thus for symmetric double differentials the double period matrix is

$$N = {}^t\Lambda\Omega. \tag{4.41}$$

In terms of other bases $\tilde{\omega}_i = \sum_{k=1}^{g} a_{ik}\omega_k$ and $\tilde{\tau}_j = \sum_{l=1}^{2g} \tau_l q_{lj}$, where $A = \{a_{ij}\} \in \mathrm{Gl}(g,\mathbb{C})$ and $Q = \{q_{lj}\} \in \mathrm{Gl}(2g,\mathbb{Z})$ are arbitrary invertible matrices, the first period class is described by a matrix $\tilde{\Lambda}' = \{\tilde{\lambda}'_{ij}\}$; and by the linearity property (4.32)

$$\begin{aligned}\mu'(\tilde{\tau}_j;\zeta) &= \sum_{i=1}^{g} \tilde{\lambda}'_{ij}\tilde{\omega}_i(\zeta) = \sum_{i,k=1}^{g} \tilde{\lambda}'_{ij}a_{ik}\omega_k(\zeta) \\ &= \mu'\Big(\sum_{l=1}^{2g}\tau_l q_{lj};\zeta\Big) = \sum_{l=1}^{2g} q_{lj}\mu'(\tau_l;\zeta) = \sum_{k=1}^{g}\sum_{l=1}^{2g} q_{lj}\lambda'_{kl}\omega_k(\zeta)\end{aligned}$$

so that ${}^t\!A\tilde{\Lambda}' = \Lambda' Q$ and hence

$$\tilde{\Lambda}' = {}^t\!A^{-1}\Lambda' Q \tag{4.42}$$

Thus *the first period matrices of the double differential $\mu(z,\zeta)$ in terms of any bases for the space of holomorphic abelian differentials and for the homology of the surface are equivalent*[2] *period matrices.* The second period matrix is described in the corresponding way, so the analogous formula holds for a change of bases for the second period matrix and consequently the second period matrices for any bases also are equivalent period matrices. Since $\tilde{\Omega} = A\Omega Q$ it follows from (4.40) that the associated double period matrices are related by

$$\tilde{N}' = {}^t\tilde{\Lambda}'\tilde{\Omega} = {}^tQN'Q \tag{4.43}$$

and similarly for $\tilde{N}''$. Of course for a symmetric double differential correspondingly

$$\tilde{\Lambda} = {}^t\!A^{-1}\Lambda\, Q \quad \text{and} \quad \tilde{N} = {}^tQN\,Q \tag{4.44}$$

under a change of basis for the holomorphic abelian differentials on M described by the matrix A and a change of basis for the homology of M described by the matrix Q.

4.6 The Intrinsic Double Differential of the Second Kind

With these general observations about meromorphic double differentials out of the way, the discussion can turn to the special meromorphic double differentials arising as the simplest intrinsic meromorphic differentials of the second kind on a compact Riemann surface.

Theorem 4.17 (i) *On a compact Riemann surface M of genus $g > 0$ there is a unique symmetric meromorphic double differential of the second kind $\mu_M(z,\zeta)$ with the differential principal part $(z_\alpha - \zeta_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal of the product manifold $M \times M$, such that $\mu_M(z,\zeta)$ is the intrinsic abelian differential of the second kind with that principal part in each variable separately.*

[2] The equivalence of period matrices is defined in equation (F.1) in Appendix F.1.

(ii) *If Ω is the period matrix of the Riemann surface M and P is the intersection matrix of that surface in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ then the period matrix Λ of the symmetric double differential $\mu_M(z,\zeta)$ in terms of these bases is*

$$\Lambda = -2\pi G\overline{\Omega} \tag{4.45}$$

where $G = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\,{}^t\overline{\Omega}$; thus the period matrix Λ is equivalent to the complex conjugate $\overline{\Omega}$ of the period matrix Ω of the Riemann surface.
(iii) *The double period matrix of the symmetric double differential $\mu_M(z,\zeta)$ in terms of these bases is*

$$N = -2\pi\,{}^t\overline{\Omega}\,{}^tG\,\Omega, \tag{4.46}$$

so that

$$N - {}^tN = 2\pi i P^{-1}. \tag{4.47}$$

Proof: (i) Select a coordinate covering of the surface M by simply connected coordinate neighborhoods $U_\gamma \subset M$ with local coordinates ζ_γ; these neighborhoods and their coordinates will be held fixed throughout the proof. Select a point in one of these coordinate neighborhoods U_α at which the local coordinate ζ_α takes the particular value z_α and consider the intrinsic abelian differential of the second kind $\mu_\mathfrak{p}$ with the differential principal part

$$\mathfrak{p} = \frac{d\zeta_\alpha}{(\zeta_\alpha - z_\alpha)^2}$$

at that point and with no other singularities on M. This differential principal part is described fully by specifying the coordinate neighborhood U_α and the value z_α of the local coordinate ζ_α at the singularity; hence the differential $\mu_\mathfrak{p}$ can be denoted unambiguously by μ_{α,z_α}. In any coordinate neighborhood U_γ of the covering this differential can be written $\mu_{\alpha,z_\alpha} = f_{\alpha\gamma}(z_\alpha,\zeta_\gamma)d\zeta_\gamma$, where $f_{\alpha\gamma}(z_\alpha,\zeta_\gamma)$ is a meromorphic function of the variable $\zeta_\gamma \in U_\gamma$ with a pole at the point z_α if that point also lies in the coordinate neighborhood U_γ but with no other poles. In particular in the coordinate neighborhood U_α itself the function $f_{\alpha\alpha}(z_\alpha,\zeta_\alpha)$ has a Laurent expansion at the point z_α beginning

$$f_{\alpha\alpha}(z_\alpha,\zeta_\alpha) = \frac{1}{(\zeta_\alpha - z_\alpha)^2} + \cdots .$$

The associated integral $u_{\alpha,z_\alpha}(\zeta) = \int_{\zeta_0}^{\zeta} \mu_{\alpha,z_\alpha}$ is a meromorphic function of the variable $\zeta \in \widetilde{M}$ with simple poles at the points of $\widetilde{M}$ having image z_α under the covering projection $\pi : \widetilde{M} \longrightarrow M$; indeed in any connected component of the inverse image $\pi^{-1}(U_\alpha) \subset \widetilde{M}$ and in terms of the local coordinate induced by ζ_α under the covering projection π, the integral $u_{\alpha,z_\alpha}(\zeta_\alpha)$ has a Laurent expansion at the point z_α in terms of the local coordinate ζ_α beginning

$$u_{\alpha,z_\alpha}(\zeta_\alpha) = -\frac{1}{\zeta_\alpha - z_\alpha} + \cdots .$$

There is yet another intrinsic abelian differential of the second kind $\mu_{\beta,z_\beta} = = f_{\beta\gamma}(z_\beta,\zeta_\gamma)d\zeta_\gamma$ with a principal part of the corresponding form at a point of the coordinate neighborhood U_β at which the local coordinate ζ_β takes the value z_β. It then it follows from equation (4.13) in Theorem 4.8 (iii) that

$$\begin{aligned}
0 &= \sum_{a\in M} \mathrm{res}_a\big(u_{\alpha,z_\alpha}\mu_{\beta,z_\beta}\big)\\
&= \mathrm{res}_{z_\alpha}\Big(-\frac{1}{\zeta_\alpha - z_\alpha} f_{\beta\alpha}(z_\beta,\zeta_\alpha)d\zeta_\alpha\Big) + \mathrm{res}_{z_\beta}\Big(u_{\alpha,z_\alpha}(\zeta_\beta)\frac{d\zeta_\beta}{(\zeta_\beta - z_\beta)^2}\Big)\\
&= -f_{\beta\alpha}(z_\beta,z_\alpha) + \frac{d}{d\zeta_\beta}u_{\alpha,z_\alpha}(\zeta_\beta)\Big|_{\zeta_\beta=z_\beta} = -f_{\beta\alpha}(z_\beta,z_\alpha) + f_{\alpha\beta}(z_\alpha,z_\beta).
\end{aligned}$$

As a consequence of this symmetry the functions $f_{\alpha\beta}(z_\alpha,\zeta_\beta)$ also are meromorphic functions of the variable $z_\alpha \in U_\alpha$, so by Rothstein's Theorem[3] these functions are meromorphic functions in the product coordinate neighborhoods $U_\alpha \times V_\beta \subset M \times M$; and consequently $\mu(z,\zeta) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)\, dz_\alpha d\zeta_\beta$ is a symmetric meromorphic double differential of the second kind with the principal part $(\zeta_\alpha - z_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal and no other singularities. Furthermore by construction $f_{\alpha\beta}(z_\alpha,\zeta_\beta)d\zeta_\beta$ is the intrinsic abelian differential of the second kind in the variable ζ with this principal part at the point $\zeta_\alpha = z_\alpha$; and by symmetry the same is true in the other variable as well. These properties determine this double differential uniquely.

(ii) For a fixed point $z_\alpha \in U_\alpha$ it follows from (i) that the differential form $\mu_M(z_\alpha,\zeta)$ in the variable ζ is the intrinsic abelian differential of the second kind with the differential principal part $(\zeta_\alpha - z_\alpha)^{-2}d\zeta_\alpha$; therefore by (4.16) in Corollary 4.9 its period on the homology class τ_j is

$$\begin{aligned}
\mu_M(\tau_j;z_\alpha) &= -2\pi \sum_{m,n=1}^{g} g_{mn}\overline{\omega}_{nj}\mathrm{res}_{\zeta_\alpha=z_\alpha}\left(w_m(\zeta_\alpha)\frac{d\zeta_\alpha}{(\zeta_\alpha - z_\alpha)^2}\right)dz_\alpha\\
&= -2\pi \sum_{m,n=1}^{g} g_{mn}\overline{\omega}_{nj}w'_m(z_\alpha)dz_\alpha = -2\pi \sum_{m,n=1}^{g} g_{mn}\overline{\omega}_{nj}\omega_m(z_\alpha);
\end{aligned}$$

and since $\mu_M(\tau_j,z_\alpha) = \sum_{m=1}^{g}\lambda_{mj}\omega_m(z_\alpha)$ by (4.36), for the special case of a symmetric double differential for which $\Lambda' = \Lambda$, it follows that

$$\lambda_{mj} = -2\pi\sum_{n=1}^{g} g_{mn}\overline{\omega}_{nj},$$

which in matrix terms is (4.45).

(iii) The double period matrix is expressed in terms of the period matrix Λ as in (4.41), so from (4.45) it follows that

$$N = {}^t\Lambda\Omega = -2\pi\,{}^t\overline{\Omega}\,{}^tG\,\Omega$$

[3]Rothstein's Theorem is that a function of n complex variables that is meromorphic in each variable separately is a meromorphic function of all n variables; the theorem is an extension of Hartogs's Theorem from holomorphic function to meromorphic functions, and is discussed on page 409 in Appendix A.1.

and consequently

$$N - {}^t\!N = -2\pi\Big({}^t\overline{\Omega}\,{}^tG\,\Omega - {}^t\Omega\,G\,\overline{\Omega}\Big) = -2\pi\Big({}^t\overline{\Omega}\,H^{-1}\,\Omega - {}^t\Omega\,{}^tH^{-1}\overline{\Omega}\Big) = 2\pi i\,P^{-1}$$

by the inverse of the Riemann equality in the form of equation (F.39) in Appendix F.4. That concludes the proof.

The double differential $\mu_M(z,\zeta)$ of the preceding theorem is the *intrinsic double differential of the second kind* on the Riemann surface M; it is the analogue for Riemann surfaces of genus $g > 0$ of the familiar meromorphic double differential $(z-\zeta)^{-2}dz\,d\zeta$ on the Riemann sphere $\mathbb{P}^1$. In view of (4.47) the double period matrix of the second kind is not symmetric, so the difference in (4.35) is nonzero.

Corollary 4.18 *If $\tau_j \in H_1(M)$ is a basis for the homology of a compact Riemann surface M of genus $g > 0$ for $1 \le j \le 2g$ then the $2g$ periods $\mu(\tau_j; z) = \int_{\zeta\in\tau_j} \mu_M(z,\zeta)$ of the intrinsic double differential of the second kind $\mu_M(z,\zeta)$ span the g-dimensional space of holomorphic abelian differentials on M.*

Proof: For bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, in terms of which the period matrix of the surface is the matrix Ω, the period matrix of the symmetric double differential $\mu_M(z,\zeta)$ is $\Lambda = -2\pi G\overline{\Omega}$ by (4.45), where G is nonsingular and $\operatorname{rank}\overline{\Omega} = g$; hence $\operatorname{rank}\Lambda = g$. The periods of the double differential are the differential forms $\mu(\tau_j; z) = \sum_{k=1}^{g} \lambda_{kj}\omega_k(z)$ as in (4.37) for the case of a symmetric double differential; and since $\operatorname{rank}\Lambda = g$ then among these $2g$ periods there are g linearly independent holomorphic abelian differentials. That suffices for the proof.

If $\mu_M(z,\zeta) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha d\zeta_\beta$ is the intrinsic double differential of the second kind on a compact Riemann surface M of genus $g > 0$ then in terms of the local coordinate ζ_β in a coordinate neighborhood $U_\beta \subset M$ set

$$\mu_M^{(\nu)}(z;\,\zeta_\beta) = \left(\frac{\partial^\nu f_{\alpha\beta}(z_\alpha,\zeta_\beta)}{\partial^\nu \zeta_\beta}\right) dz_\alpha \tag{4.48}$$

for any integer $\nu \ge 0$. It is evident from (4.26) that this is a well defined meromorphic differential form in the variable z for any fixed value of the local coordinate ζ_β; in particular for $\nu = 0$ it is just the abelian differential of the second kind on M that arises as the restriction of the double differential $\mu(z,\zeta)$. For $\nu > 1$ though this differential form depends not just on the particular point represented by the coordinate ζ_β but also on the choice of the local coordinate system.

Corollary 4.19 *The differential form $\mu^{(\nu)}(z;\zeta_\beta)$ in the variable z on a compact Riemann surface M of genus $g > 0$ is the intrinsic abelian differential of the second kind with the differential principal part*

$$\frac{(\nu+1)!}{(z_\beta-\zeta_\beta)^{\nu+2}}\;dz_\beta. \tag{4.49}$$

If Ω is the period matrix and P is the intersection matrix of M, in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, the periods of the differential form $\mu^{(\nu)}(z;\zeta_\beta)$ are

$$\mu_M^{(\nu)}(T;\zeta_\beta) = -2\pi \sum_{m,n=1}^{g} g_{mn} w_m^{(\nu+1)}(\zeta_\beta)\overline{\omega_n(T)} \tag{4.50}$$

for any covering translation $T \in \Gamma$, where $G = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\, {}^t\overline{\Omega}$.

Proof: By Theorem 4.17 (i) the differential form $\mu_M^{(0)}(z;\zeta_\beta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta)dz_\alpha$ in the variable z for any fixed value $\zeta_\beta \in U_\beta$ is the intrinsic abelian differential of the second kind with the differential principal part $\mathfrak{p}_{\zeta_\beta} = (z_\beta - \zeta_\beta)^{-2}dz_\beta$, and by Corollary 4.9 its periods are

$$\begin{aligned} \mu_M(T;\zeta_\beta) &= -2\pi \sum_{m,n=1}^{g} g_{mn}\mathrm{res}_{\zeta_\beta}(w_m\mathfrak{p}_{\zeta_\beta})\,\overline{\omega_n(T)} \\ &= -2\pi \sum_{m,n=1}^{g} g_{mn} w_m'(\zeta_\beta)\,\overline{\omega_n(T)} \end{aligned} \tag{4.51}$$

for any covering translation $T \in \Gamma$, where $w'(\zeta_\beta) = dw(\zeta_\beta)/d\zeta_\beta$; thus $\mu_M^{(0)}(z;\zeta_\beta)$ has the desired properties for the case $\nu = 0$. The derivative $\mu_M^{(\nu)}(z;\zeta_\beta)$ then has the differential principal part

$$\frac{\partial^\nu}{\partial\zeta_\beta^\nu}\frac{1}{(z_\beta - \zeta_\beta)^2} = \frac{(\nu+1)!}{(z_\beta - \zeta_\beta)^{\nu+2}};$$

and since the periods for a covering translation $T \in \Gamma$ are just the integrals from a point $a \in \widetilde{M}$ to the image Ta along any path in $\widetilde{M}$ that avoids the singularities of the differential form it follows that

$$\begin{aligned} \mu_M^{(\nu)}(T;\zeta_\beta) &= \int_a^{Ta} \frac{\partial^\nu f_{\alpha\beta}(z_\alpha, \zeta_\beta)}{\partial\zeta_\beta^\nu dz_\alpha} = \frac{\partial^\nu}{\partial\zeta_\beta^\nu}\int_a^{Ta} f_{\alpha,\beta}(z_\alpha\zeta_\beta)dz_\alpha \\ &= -2\pi\frac{\partial^\nu}{\partial\zeta_\beta^\nu}\sum_{m,n=1}^{g} g_{mn}w_m'(\zeta)\overline{\omega_n(T)} \\ &= -2\pi \sum_{m,n=1}^{g} g_{mn}w_m^{(\nu+1)}(\zeta_\beta)\overline{\omega_n(T)}. \end{aligned}$$

These periods are the complex conjugates of the periods of the holomorphic abelian differential $-2\pi\sum_{m,n=1}^{g}\overline{g_{mn}w_m^{\nu+1}(\zeta_\beta)}\omega_n(z)$, so by Theorem 4.8 (i) the differential form $\mu^{(n)}(z;\zeta_\beta)$ is an intrinsic abelian differential of the second kind. That suffices to conclude the proof.

4.7 Double Differentials of the Second kind as Integral Kernels

The intrinsic double differential of the second kind $\mu_M(z,\zeta)$ can be be viewed as an integral kernel on the Riemann surface, somewhat like the Cauchy integral but for evaluating the derivative of functions rather than the functions themselves.

Theorem 4.20 *If $f(z)$ is a holomorphic function in an open neighborhod of a closed contractible subset $\overline{U} \subset M$ of a compact Riemann surface M and if the boundary $\gamma = \partial\overline{U}$ is a simple closed path then*

$$df(z) = \frac{1}{2\pi i}\int_\gamma \mu_M(z,\zeta)f(\zeta) \quad \textit{for all } z \in U. \tag{4.52}$$

Proof: If $z \in U$ and $\overline{\Delta} \subset U$ is a closed disc in a local coordinate ζ centered at the point z and contained in U then in Δ the intrinsic double differential $\mu_M(z,\zeta)$ can be written as the sum of its differential principal part $\mu_z(\zeta) = (\zeta - z)^{-2}d\zeta dz$ and a holomorphic double differential $\phi(z,\zeta)$, as in (4.28); consequently by the Cauchy integral formula

$$\begin{aligned}\frac{1}{2\pi i}\int_{\zeta\in\gamma}\mu_M(z,\zeta)f(\zeta) &= \frac{1}{2\pi i}\int_{\zeta\in\gamma}\frac{f(\zeta)}{(\zeta - z)^2}d\zeta dz + \frac{1}{2\pi i}\int_{\zeta\in\gamma}\phi(z,\zeta)f(\zeta)\\ &= f'(z)dz,\end{aligned} \tag{4.53}$$

since $\int_{\zeta\in\gamma}\phi(z,\zeta)f(\zeta) = 0$ as the integral around the boundary $\gamma = \partial\Delta$ of a holomorphic differential in Δ. By assumption the path $\partial\overline{U} - \partial\overline{\Delta}$ is the boundary of a region in U in which the differential $\mu_M(z,\zeta)f(\zeta)$ in the variable ζ is holomorphic hence closed; so by Stokes's Theorem $\int_{\zeta\in\partial\overline{U}}\mu_M(z,\zeta)f(\zeta) = \int_{\zeta\in\partial\overline{\Delta}}\mu_M(z,\zeta)f(\zeta)$ and the asserted result follows from (4.53) which suffices for the proof.

For a meromorphic function $f(z)$ rather than a holomorphic function an obvious modification of the preceding theorem holds, where the integral (4.52) is equal to the sum of the differential $df(z)$ and of the residues of the meromorphic differential $\mu_M(z,\zeta)f(\zeta)$ at the singularities of the function $f(\zeta)$. A more interesting variant though is the following.

Theorem 4.21 *If $f(z)$ is a meromorphic function on the compact Riemann surface M with the principal part $p(f) = \sum_{\nu=1}^n \mathfrak{p}_{a_\nu}$ for some distinct points $a_\nu \in M$ then for any point $z \in \Delta$ other than the points a_ν*

$$df(z) = -\frac{1}{2\pi i}\sum_{\nu=1}^n \operatorname{res}_{\zeta=a_\nu}\mu_M(z,\zeta)\mathfrak{p}_{a_\nu}(\zeta) \tag{4.54}$$

where $\mathfrak{p}_{a_\nu}(\zeta)$ indicates that the principal part $\mathfrak{p}_{a_\nu}$ is viewed as a principal part in the coordinate ζ.

Proof: Choose a fundamental domain $\Delta \subset \widetilde{M}$ for the action of the covering translation group Γ of the universal covering space $\widetilde{M}$ when the surface M is represented as the quotient $M = \widetilde{M}/\Gamma$, as in Figure D.2 of Appendix D.1, such that none of the points $\widetilde{a}_\nu \in \widetilde{M}$ representing the points $a_\nu \in M$ lie on the boundary $\partial\Delta$. Then by the Cauchy integral formua for any point $z \in \Delta$ other than the points a_ν

(4.55)

$$\begin{aligned}\frac{1}{2\pi i}\int_{\zeta\in\partial\Delta} \mu_M(z,\zeta)f(\zeta) &= \frac{1}{2\pi i}\mathrm{res}_{\zeta=z}\mu_M(z,\zeta)f(\zeta) + \sum_\nu \frac{1}{2\pi i}\mathrm{res}_{\zeta=\widetilde{a}_\nu}\mu_M(z,\zeta)f(\zeta)\\ &= f'(z)dz + \frac{1}{2\pi i}\sum_\nu \mathrm{res}_{\zeta=\widetilde{a}_\nu}\mu_M(z,\zeta)\mathfrak{p}_{a_\nu}(\zeta)\end{aligned}$$

$\blacksquare$

where $f'(z)$ arises just as in the proof of the preceding Theorem 4.20. For any fixed point $z \in \widetilde{M}$ the product $\mu_M(z,\zeta)f(\zeta)$ is a Γ-invariant meromorphic abelian differential on $\widetilde{M}$ in the variable ζ, so since the boundary integral in (4.55) involves integration over pairs of paths in $\widetilde{M}$, where each pair represents a closed path in M and the same path with a reversed orientation, it follows that the boundary integral vanishes. The asserted result follows from this observation and (4.55), which suffices for the proof.

What is interesting in this last result is that the differential $df(z)$ is determined quite explicitly in terms just of the principal part $p(f) = \sum_\nu \mathfrak{p}_{a_\nu}$ and the intrinsic meromorphic double differential $\mu_M(z,\zeta)$. It is a straightforward calculation to show that (4.54) can be rewritten alternatively as a formula involving the partial derivative of the double differential of the second kind rather than a residue calculation; the result is a rather simpler and more useful representation of $df(z)$ in terms of its principal part, and amounts to a special case of the representation through the differential forms $\mu_M^{(\nu)}(z;\zeta)$ defined in (4.48) in the preceding Section 4.6.

4.8 Basic and Canonical Double Abelian Differentials of the Second Kind

For any choice of local coordinates ζ_β at its poles, any differential principal part of the second kind can be written as a unique linear combination of the differential principal parts (4.49); and since an intrinsic differential form of the second kind is determined uniquely by its principal part it follows from the preceding corollary that any intrinsic abelian differential of the second kind can be written uniquely as a linear combination of the intrinsic abelian differentials $\mu^{(n)}(z;\zeta_\beta)$. Thus the intrinsic abelian differentials of the second kind on a compact Riemann surface of genus $g > 0$ are holomorphic functions of their differential principal parts, in the sense that they are holomorphic functions of the local coordinates ζ_β describing the locations of the poles and of the coefficients in the Laurent expansion of the differential principal parts at these poles. Double differentials with the same principal part as the intrinsic double differential $\mu_M(z,\zeta)$ differ from $\mu_M(z,\zeta)$ by a double differential that is everywhere holomorphic. If $\mu(z,\zeta) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha\, d\zeta_\beta$ is a holomorphic double differential and $\omega_i \in \Gamma(M,\mathcal{O}^{(1,0)})$ is a basis for the space of holomorphic abelian differentials on M then since $f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha$ is a holomorphic abelian differential in the variable z for any fixed point ζ_β it can be written as the sum $f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha = \sum_{i=1}^{g} e_{i\beta}(\zeta_\beta)\omega_i(z)$ for some coefficients $e_{i\beta}(\zeta_\beta)$ depending on the point ζ_β. If $\tau_j \in H_1(M)$ is a basis for the homology of the surface M and $\omega_{ij} = \int_{\tau_j}\omega_i$ is the period matrix of the surface in terms of these bases then

$$\int_{\tau_j}\mu(z,\zeta) = \Big(\int_{\tau_j} f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha\Big)d\zeta_\beta = \sum_{j=1}^{2g}\omega_{ij}e_{i\beta}(\zeta_\beta)d\zeta_\beta;$$

the periods of the double differential $\mu(z,\zeta)$ are holomorphic abelian differentials, and since the period matrix $\Omega = \{\omega_{ij}\}$ has rank g it follows that the expressions $e_{i\beta}(\zeta_\beta)d\zeta_\beta$ are holomorphic abelian differentials hence can be written $e_{i\beta}(\zeta_\beta)d\zeta_\beta = \sum_{i,j=1}^{g} e_{ij}\omega_i(z)\omega_j(\zeta)$ for some complex constants e_{ij}, and consequently a holomorphic double differential must be of the form

$$f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha\, d\zeta_\beta = \sum_{i,j=1}^{g} e_{ij}\omega_i(z)\omega_j(\zeta)$$

for some complex constants e_{ij}. Thus any meromorphic double differential on M with the same differential principal part as $\mu_M(z,\zeta)$ must be of the form

$$\mu_{M,E}(z,\zeta) = \mu_M(z,\zeta) + \sum_{k,l=1}^{g} e_{kl}\omega_k(z)\omega_l(\zeta) \tag{4.56}$$

for $g \times g$ complex matrix $E = \{e_{ij}\}$. Such a double differential is called the *basic double differential of the second kind* on the compact Riemann surface M described by the matrix E; the symmetric basic double differentials are those described by symmetric matrices E.

Corollary 4.22 *If $\Omega = \{\omega_{ij}\}$ is the period matrix of a compact Riemann surface M of genus $g > 0$ in terms of bases ω_i for the holomorphic abelian differentials on M and τ_j for the first homology of M, the period matrices of the basic double differential $\mu_{M,E}(z,\zeta)$ are*

$$\Lambda'_E = \Lambda + {}^t E\Omega, \quad \Lambda''_E = \Lambda + E\Omega \tag{4.57}$$

where Λ is the period matrix of the intrinsic double differential of the second kind $\mu_M(z,\zeta)$; correspondingly the double period matrices of the basic double differential $\mu_{M,E}(z,\zeta)$ are

$$N'_E = N + {}^t\Omega E\Omega, \quad N''_E = N + {}^t\Omega\, {}^t E\Omega \tag{4.58}$$

where N is the double period matrix of the intrinsic double differential of the second kind $\mu_M(z,\zeta)$, so

$$N'_E - {}^tN''_E = N - {}^tN = 2\pi i P^{-1}, \tag{4.59}$$

which is independent of the matrix E.

Proof: From (4.56) it follows that the first period class of the basic double differential $\mu_E(z,\zeta)$ is

$$\begin{aligned}
\mu'_{M,E}(\tau_j;\zeta) &= \int_{z\in\tau_j} \mu_{M,E}(z,\zeta) \\
&= \mu'_M(\tau_j;\zeta) + \int_{z\in\tau_j} \sum_{k,l=1}^{g} e_{kl}\omega_k(z)\omega_l(\zeta) \\
&= \sum_{l=1}^{g} \lambda'_{lj}\omega_l(\zeta) + \sum_{k,l=1}^{g} e_{kl}\omega_{kj}\omega_l(\zeta),
\end{aligned}$$

so its first period matrix is $\Lambda'_E = \Lambda' + {}^tE\,\Omega$; and the second period class correspondingly is

$$\mu''_{M,E}(\tau_j;z) = \sum_{k=1}^{g} \lambda''_{kj}\omega_k(z) + \sum_{k,l=1}^{g} e_{kl}\omega_k(z)\omega_{lj}$$

so its second period matrix is $\Lambda''_E = \Lambda'' + E\Omega$. Since the double differential $\mu_M(z,\zeta)$ is symmetric $\Lambda' = \Lambda'' = \Lambda$. Then by (4.40) the double period matrices are $N'_E = {}^t\Lambda'_E\Omega = {}^t(\Lambda + {}^tE\Omega)\Omega = N + {}^t\Omega E\Omega$ and $N''_E = {}^t\Lambda''_E\Omega = {}^t(\Lambda + E\Omega)\Omega = N + {}^t\Omega\,{}^tE\Omega$; and (4.59) then follows from (4.47), which suffices to conclude the proof.

That $N'_E - {}^tN''_E = N - {}^tN = 2\pi i P^{-1}$ reflects the fact that this difference is really determined by the topology of the singular locus of the double differential, rather than by the particular choice of the matrix E, although that point will not be pursued in the discussion here. On a marked Riemann surface one of the basic double differentials represents the canonical abelian differentials of the second type with a single double pole.

Theorem 4.23 (i) *On a marked Riemann surface M of genus $g > 0$, with the marking described by generators $A_j, B_j \in \Gamma$ of the covering translation group of M, there is a unique symmetric meromorphic double differential of the second kind $\hat{\mu}_M(z,\zeta)$ that has the principal part $(z_\alpha - \zeta_\alpha)^{-2}dz_\alpha d\zeta_\alpha$ along the diagonal of the product manifold $M \times M$ and that is the canonical abelian differential of the second kind with that principal part in each variable separately.*
(ii) *The double differential $\hat{\mu}_M(z,\zeta)$ is the basic symmetric double differential of the second kind $\hat{\mu}_M(z,\zeta) = \mu_{M,E}(z,\zeta)$ described by the matrix $E = \pi Y^{-1}$, where the period matrix of M in terms of the generators $A_j, B_j \in \Gamma$ describing*

the marking and of the associated canonical holomorphic abelian differentials ω_i *is* $\Omega = (\mathrm{I} \quad Z)$ *for a matrix* $Z = X + iY \in \mathfrak{H}_g$.
(iii) *The period matrix of the double differential* $\hat{\mu}_M(z,\zeta)$ *is* $\hat{\Lambda} = (0 \quad 2\pi i\,\mathrm{I})$ *and its period class is determined by*

$$\hat{\mu}_M(A_j;z) = 0, \quad \hat{\mu}_M(B_j;z) = 2\pi i\,\omega_j(z), \tag{4.60}$$

where $\omega_j \in \Gamma(M, \mathcal{O}^{(1,0)})$ *are the canonical holomorphic abelian differentials on the marked Riemann surface* M.
(iv) *The double period matrix of the double differential* $\hat{\mu}_M(z,\zeta)$ *is*

$$\hat{N} = 2\pi i \begin{pmatrix} 0 & 0 \\ \mathrm{I} & Z \end{pmatrix}, \tag{4.61}$$

so that $\hat{N} - {}^t\hat{N} = -2\pi i \mathrm{J}$ *where* J *is the basic skew-symmetric matrix.*

Proof: The period matrix of the marked Riemann surface M in terms of the generators $A_j, B_j \in \Gamma$ describing the marking and of the associated canonical holomorphic abelian differentials ω_i has the form $\Omega = (\,\mathrm{I} \quad Z)$ as in Theorem 3.22, where $Z = X + iY \in \mathfrak{H}_g$, the Siegel upper half-space of rank g; and the intersection matrix is the basic skew-symmetric matrix $\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}$. Consequently the matrices G and H in Theorem 4.17 (ii) are $H = i\,\Omega \mathrm{J}\,{}^t\overline{\Omega} = i(\overline{Z} - Z) = 2Y$ and $G = {}^tH^{-1} = \frac{1}{2}Y^{-1}$, so the period matrix Λ of the intrinsic double differential of the second kind $\mu_M(z,\zeta)$ is $\Lambda = -2\pi G\overline{\Omega} = -\pi Y^{-1}\overline{\Omega}$. By (4.57) the symmetric basic double differential $\mu_{M,E}(z,\zeta)$ described by a symmetric matrix E has the period matrix

$$\Lambda_E = \Lambda + E\Omega = -\pi Y^{-1}\overline{\Omega} + E\Omega = (\Lambda'_E \quad \Lambda''_E)$$

for the $g \times g$ matrix blocks

$$\Lambda'_E = E - \pi Y^{-1} \quad \text{and} \quad \Lambda''_E = (E - \pi Y^{-1})X + i(E + \pi Y^{-1})Y).$$

There is a unique symmetric matrix E for which Λ'_E vanishes, the matrix $E = \pi Y^{-1}$; and for this choice of the matrix E clearly $\Lambda''_E = 2\pi i\,\mathrm{I}$, so the period matrix is $\Lambda_E = (0 \quad 2\pi i\,\mathrm{I})$ for the $g \times g$ identity matrix I and by (4.36) the periods of the double differential $\mu_{M,E}(z,\zeta)$ are as in (4.60). It follows that $\mu_{M,E}(z,\zeta)$ is the canonical abelian differential of the second kind with the given principal part in each variable separately. Finally by (4.41) the double period matrix of this double differential is $\hat{N} = {}^t\hat{\Lambda}\Omega = {}^t(0 \quad 2\pi i\mathrm{I})(\mathrm{I} \quad Z)$, which is (4.61). That suffices to conclude the proof.

The double differential $\hat{\mu}_M(z,\zeta)$ is called the *canonical double differential* of the second kind on the marked Riemann surface M. The canonical double differentials of the second kind for all the markings of a Riemann surface thus are just basic double differentials for various matrices E so are expressible directly in terms of the intrinsic double differential and the holomorphic abelian differentials as in (4.56). The basic double differentials in general can be described somewhat more intrinsically as follows.

Theorem 4.24 *If M is a compact Riemann surface of genus $g > 1$ then the only meromorphic double differentials of the second kind with singularities along the diagonal subvariety of $M \times M$ and nowhere else are constant multiples of the basic double differentials of M.*

Proof: If $\mu(z,\zeta)$ is a meromorphic double differential of the second kind with singularities only along the diagonal subvariety of $M \times M$ then it can be written $\mu(z,\zeta) = f_{\alpha\beta}(z_\alpha,\zeta_\beta)dz_\alpha d\zeta_\beta$ in terms of a covering of the surface M by coordinate neighborhoods U_α, in which the local coordinate is denoted by either z_α or ζ_α; the coefficients $f_{\alpha\beta}(z_\alpha,\zeta_\beta)$ are meromorphic functions in the coordinate neighborhood $U_\alpha \times U_\beta$ with singularities only along the diagonal, and in a product neighborhood $U_\alpha \times U_\alpha$ containing the diagonal these coefficients have the form

$$f_{\alpha\alpha}(z_\alpha,\zeta_\alpha) = \frac{h_\alpha(z_\alpha,\zeta_\alpha)}{(z_\alpha - \zeta_\alpha)^n} dz_\alpha d\zeta_\alpha$$

for some integer $n \geq 2$ where $h_\alpha(z_\alpha,\zeta_\alpha)$ is holomorphic in $U_\alpha \times U_\alpha$ and $h_\alpha(z_\alpha,z_\alpha)$ is not identically zero. In an intersection $(U_\alpha \times U_\alpha) \cap (U_\beta \times U_\beta)$

$$h_\alpha(z_\alpha,\zeta_\alpha) = \left(\frac{z_\alpha - \zeta_\alpha}{z_\beta - \zeta_\beta}\right)^n \left(\frac{dz_\alpha}{dz_\beta}\right)^{-1} \left(\frac{d\zeta_\alpha}{d\zeta_\beta}\right)^{-1} h_\beta(z_\beta,\zeta_\beta);$$

and upon taking the limit in this identity as ζ_β approaches z_β and hence ζ_α approaches z_α it follows that

$$h_\alpha(z_\alpha,z_\alpha) = \left(\frac{dz_\alpha}{dz_\beta}\right)^{n-2} h_\beta(z_\beta,z_\beta).$$

Thus the functions $h_\alpha(z_\alpha,z_\alpha)$ describe a nontrivial holomorphic cross-section of the line bundle κ^{2-n}, where κ is the canonical bundle of M, and the characteristic class of this bundle is $c(\kappa^{2-n}) = (2-n)c(\kappa) = -(n-2)(2g-2)$. If $n > 2$ then $c(\kappa^{2-n}) < 0$ since $g > 1$ and consequently $h_\alpha(z_\alpha,z_\alpha) = 0$, which implies that the meromorphic functions $f_{\alpha\beta}(z_\alpha,\zeta_\beta)$ have poles of order strictly less than n along the diagonal, in contradiction to the assumption that the singularities are of order n. If $n = 2$ then $\kappa^{2-n} = 1$ is the identity line bundle so the functions $h_\alpha(z_\alpha,z_\alpha)$ must be constants, and consequently the differential is a multiple of a basic double differential on M. That suffices to conclude the proof.

There are meromorphic double differentials of the second kind on Riemann surfaces of genus $g > 1$ other than the basic double differentials of the second kind; but their singularities lie along holomorphic subvarieties $V \subset M \times M$ other than the diagonal subvariety, and the discussion of such double differentials will not be pursued here.

Chapter 5

Meromorphic Differentials of the Third Kind

5.1 Meromorphic Abelian Integrals of the Third Kind

Any differential principal part $\mathfrak{p}$ on a compact Riemann surface M can be written as the sum $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$ of a differential principal part $\mathfrak{p}_1$ consisting of simple poles at a finite number of points of M and a differential principal part $\mathfrak{p}_2$ of the second kind on M. By Corollary 4.4 (ii) the differential principal part $\mathfrak{p}$ is the principal part of a meromorphic differential form on M if and only if the sum of the residues at the poles of $\mathfrak{p}_1$ is zero; and in that case $\mathfrak{p}_1$ can be written as a sum of differential principal parts, each of which consists of simple poles at two distinct points of M with residues that are negatives of one another. That suggests that the examination of abelian differentials of the third kind on a compact Riemann surface M can begin by examining a differential principal part $\mathfrak{p}(a_+, a_-)$ consisting of a simple pole with residue $+1$ at the point a_+, a simple pole with residue -1 at the point a_-, and no other singularities on M.

It follows from Corollary 4.4 (ii) that there is a meromorphic abelian differential ν on M with the differential principal part $\mathfrak{p}(a_+, a_-)$; and of course the other meromorphic abelian differentials on M with this differential principal part differ from ν by holomorphic abelian differentials. To define the period class of the differential ν, choose an oriented simple path $\delta \subset M$ from the point a_- to the point a_+ and a point $z_- \in \widetilde{M}$ such that $\pi(z_-) = a_-$, where $\pi : \widetilde{M} \longrightarrow M$ is the covering projection from the universal covering space $\widetilde{M}$ to M with covering translation group Γ. There is a unique path $\tilde{\delta} \in \widetilde{M}$ beginning at the point z_- such that $\pi(\tilde{\delta}) = \delta$, and this path ends at a point $z_+ \in \widetilde{M}$ for which $\pi(z_+) = a_+$. The inverse image of the path δ under the covering projection π is the collection of paths $\pi^{-1}(\delta) = \Gamma\tilde{\delta} = \bigcup_{T\in\Gamma} T\tilde{\delta}$, where the paths $T\tilde{\delta} \subset \widetilde{M}$ for distinct covering translations $T \in \Gamma$ are disjoint. The complement $M \sim \delta \subset M$ is a connected set, since the path δ does not intersect itself; and $\widetilde{M} \sim \Gamma\tilde{\delta} \subset \widetilde{M}$ is a connected covering space over $M \sim \delta$. When ν is viewed as a Γ-invariant differential form on $\widetilde{M}$ its integral around any closed path $\gamma \subset \widetilde{M} \sim \Gamma\tilde{\delta}$ is zero. Indeed since $\widetilde{M}$ is simply connected γ is the boundary $\gamma = \partial\Delta$ of a domain $\Delta \subset \widetilde{M}$, and for any component $T\tilde{\delta} \subset \Delta$ it is possible to choose a closed path $\gamma_T \subset \Delta$ that encircles $T\tilde{\delta}$ once in such a manner that the paths γ_T are disjoint and have disjoint interiors; since ν is a closed differential form in the complement of the paths $T\tilde{\delta}$ it follows from Stokes's Theorem that $\int_\gamma \nu = \sum_T \int_{\gamma_T} \nu = 0$, for the integral over γ_T is the sum of the residues of the differential form ν at the two poles Tz_+ and Tz_- enclosed by that path hence is zero. Therefore the integral

$$v^\delta(z,a) = \int_a^z \nu \tag{5.1}$$

along any path that avoids the sets $T\tilde{\delta}$ is a well defined holomorphic function of the variables $z, a \in \widetilde{M} \sim \Gamma\tilde{\delta}$ independent of the path of integration; this function is called the *integral* of the abelian differential of the third kind ν with respect to the path δ, although of course it is really a holomorphic function on the open subset $\widetilde{M} \sim \Gamma\tilde{\delta} \subset \widetilde{M}$ in both variables. This integral clearly satisfies the symmetry condition $v^\delta(z,a) = -v^\delta(a,z)$, and $v^\delta(z,z) = 0$. It is more convenient in many circumstances to view this integral as a function of the first variable only, and to allow it to be modified by an arbitrary additive constant; in that case the simpler notation $v^\delta(z)$ will be used, with the same cautions as in the cases of abelian integrals of the first and second kinds. For any covering translation $T \in \Gamma$ the difference

$$v^\delta(Tz) - v^\delta(z) = \nu^\delta(T) \tag{5.2}$$

is a constant since $dv^\delta(z) = \nu(z)$ is invariant under T and $\widetilde{M} \sim \Gamma\tilde{\delta}$ is connected; the mapping $\nu^\delta : T \longrightarrow \nu^\delta(T)$ is a group homomorphism $\nu^\delta \in \mathrm{Hom}(\Gamma, \mathbb{C}) = \mathrm{Hom}(H_1(M), \mathbb{C}) = H^1(\Gamma, \mathbb{C})$ called the *period class* of the abelian differential ν with respect to the path δ. The period class clearly is unchanged by adding an arbitrary constant to the function $v^\delta(z)$, so it depends only on the abelian differential ν.

Theorem 5.1 *On a compact Riemann surface M of genus $g > 0$ let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{jk}\}$ be the intersection matrix of the surface in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$.*
(i) *If $\delta \subset M$ is a simple path from a_- to a_+ the periods $\nu^\delta(\tau_j)$ with respect to the path δ of an abelian differential of the third kind ν with the differential principal part $\mathfrak{p}(a_+, a_-)$ satisfy*

$$\sum_{j,k=1}^{2g} \omega_{ij}\, p_{jk}\, \nu^\delta(\tau_k) = 2\pi i \int_\delta \omega_i \tag{5.3}$$

for $1 \le i \le g$.
(ii) *If $\delta', \delta'' \subset M$ are two disjoint paths from a_- to a_+ the periods $\nu'^{\delta'}(\tau_j)$*

with respect to the path δ' of an abelian differential of the third kind ν' with the differential principal part $\mathfrak{p}(a'_+, a'_-)$ and the periods $\nu''^{\delta''}(\tau_k)$ with respect to the path δ'' of an abelian differential of the third kind ν'' with the differential principal part $\mathfrak{p}(a''_+, a''_-)$ satisfy

$$\sum_{j,k=1}^{2g} \nu'^{\delta'}(\tau_j)\, p_{jk}\, \nu''^{\delta''}(\tau_k) = 2\pi i\Big(\int_{\delta''}\nu' - \int_{\delta'}\nu''\Big). \tag{5.4}$$

(iii) *If $\delta \subset M$ is a simple path from a_- to a_+ the periods $\nu^\delta(\tau_k)$ with respect to the path δ of an abelian differential ν of the third kind with the differential principal part $\mathfrak{p}(a_+, a_-)$ and the periods $\mu(\tau_j)$ of a meromorphic abelian differential μ of the second kind with poles at points $a_i \notin \delta$ satisfy*

$$\sum_{j,k=1}^{2g} \mu(\tau_j) p_{jk} \nu^\delta(\tau_k) = 2\pi i \int_\delta \mu - 2\pi i \sum_{a_i} \mathrm{res}_{a_i}(v^\delta \mu) \tag{5.5}$$

where $v^\delta(z) = \int_a^z \nu$ is an integral of the differential form ν on $\widetilde{M} \sim \Gamma\tilde{\delta}$.

Proof: (i) Let $v^\delta(z) = \int_a^z \nu$ be the integral of the abelian differential of the third kind ν on $\widetilde{M} \sim \Gamma\tilde{\delta}$; the periods of ν thus are given by $\nu^\delta(T) = v^\delta(Tz) - v^\delta(z)$. Choose a contractible open neighborhood $\Delta \subset M$ of the path $\delta \subset M$ and let $\tilde{\Delta}$ be that component of the inverse image $\pi^{-1}(\Delta) \in \widetilde{M}$ for which $\tilde{\delta} \subset \tilde{\Delta}$, where $\pi : \widetilde{M} \longrightarrow M$ is the covering projection. The complete inverse image $\pi^{-1}(\Delta) \subset \widetilde{M}$ consists of disjoint open sets $T\tilde{\Delta}$ for all $T \in \Gamma$, and the set $T\tilde{\Delta}$ contains the component $T\tilde{\delta}$ of the path $\pi^{-1}(\tilde{\delta})$. Choose a $\mathcal{C}^\infty$ real-valued function r on M that is identically one on an open neighborhood of $M \sim \Delta$ and is identically zero in an open neighborhood of the path δ in Δ; this function also will be viewed as a Γ-invariant function on $\widetilde{M}$. In terms of this auxiliary function introduce the smoothed integral

$$\tilde{v}^\delta(z) = \begin{cases} v^\delta(z) & \text{for} \quad z \in \widetilde{M} \sim \Gamma\tilde{\Delta} \\ r(z)v^\delta(z) & \text{for} \quad z \in \tilde{\Delta} \\ \tilde{v}^\delta(T^{-1}z) + \nu^\delta(T) & \text{for} \quad z \in T\tilde{\Delta}, \quad T \neq I. \end{cases}$$

Thus $\tilde{v}^\delta(z)$ is a $\mathcal{C}^\infty$ function on $\widetilde{M}$, $\tilde{v}^\delta(Tz) = \tilde{v}^\delta(z) + \tilde{v}^\delta(T)$ for any covering translation $T \in \Gamma$, and $\tilde{v}^\delta(z) = v^\delta(z)$ whenever $z \notin \Gamma\tilde{\Delta}$. The differential form $\tilde{\nu}^\delta = d\,\tilde{v}^\delta$ then is a $\mathcal{C}^\infty$ closed Γ-invariant differential form on $\widetilde{M}$, or equivalently is a $\mathcal{C}^\infty$ closed differential form on M, that is holomorphic outside the set Δ and that has the periods $\tilde{\nu}^\delta(T) = \nu^\delta(T)$ for all covering transformations $T \in \Gamma$. If ϕ_k is a basis for the first deRham group of M that is dual to the chosen basis for the homology of M then $\tilde{\nu}^\delta \sim \sum_{k=1}^{2g} \nu^\delta(\tau_k)\phi_k(z)$, where $\sim$ denotes cohomologous differential forms; and the abelian differentials of the first kind can be written correspondingly as $\omega_i \sim \sum_{j=1}^{2g} \omega_{ij}\phi_j$. Then

$$\int_M \omega_i \wedge \tilde{\nu}^\delta = \sum_{j,k=1}^{2g} \int_M \omega_{ij}\phi_j \wedge \nu^\delta(\tau_k)\phi_k = \sum_{j,k=1}^{2g} \omega_{ij} p_{jk} \nu^\delta(\tau_k)$$

where $p_{jk} = \int_M \phi_j \wedge \phi_k$ are the entries of the intersection matrix P of the surface M in terms of these bases. On the other hand the differential form $\tilde{\nu}^\delta(z)$ is holomorphic outside Δ, so that $\omega_i \wedge \tilde{\nu}^\delta = 0$ there, and consequently by Stokes's Theorem and the residue theorem

$$\begin{aligned}
\int_M \omega_i \wedge \tilde{\nu}^\delta &= \int_\Delta \omega_i \wedge \tilde{\nu}^\delta = \int_{\tilde{\Delta}} d(w_i \tilde{\nu}^\delta) \\
&= \int_{\partial\tilde{\Delta}} w_i \tilde{\nu}^\delta = \int_{\partial\tilde{\Delta}} w_i \nu^\delta \\
&= 2\pi i \Big(w_i(a_+) - w_i(a_-) \Big) = 2\pi i \int_\delta \omega_i
\end{aligned}$$

since $\nu^\delta = \tilde{\nu}^\delta$ on the boundary of the disc Δ. Combining these two equations yields (5.3).

(ii) For meromorphic abelian differentials of the third kind ν' with the differential principal part $\mathfrak{p}(a'_+, a'_-)$ and ν'' with the differential principal part $\mathfrak{p}(a''_+, a''_-)$, and for disjoint paths δ' from a'_- to a'_+ and δ'' from a''_- to a''_+, introduce the smoothed functions $\tilde{v}'^{\delta'}$ and $\tilde{v}''^{\delta''}$ in disjoint open neighborhoods Δ' and Δ'' of the paths δ' and δ'' as in the proof of part (i). Then for the $\mathcal{C}^\infty$ differential forms $\tilde{\nu}'^{\delta'} = d\,\tilde{v}'^{\delta'}$ and $\tilde{\nu}''^{\delta''} = d\,\tilde{v}''^{\delta''}$ on $\widetilde{M}$ it follows that

$$\int_M \tilde{\nu}'^{\delta'} \wedge \tilde{\nu}''^{\delta''} = \sum_{i,j=1}^{2g} \int_M \nu'^{\delta'}(\tau_i)\phi_i \wedge \nu''^{\delta''}(\tau_j)\phi_j = \sum_{i,j=1}^{2g} \nu'^{\delta'}(\tau_i) p_{ij} \nu''^{\delta''}(\tau_j).$$

Again since both $\tilde{\nu}'^{\delta'}$ and $\tilde{\nu}''^{\delta''}$ are holomorphic outside $\Delta' \cup \Delta''$ it follows that $\tilde{\nu}'^{\delta'} \wedge \tilde{\nu}''^{\delta''} = 0$ outside $\Delta' \cup \Delta''$, and consequently from Stokes's Theorem and the residue theorem it follows that

$$\begin{aligned}
\int_M \tilde{\nu}'^{\delta'} \wedge \tilde{\nu}''^{\delta''} &= \int_{\Delta' \cup \Delta''} \tilde{\nu}'^{\delta'} \wedge \tilde{\nu}''^{\delta''} \\
&= \int_{\tilde{\Delta}''} d\big(\tilde{v}'^{\delta'} \tilde{\nu}''^{\delta''}\big) - \int_{\tilde{\Delta}'} d\big(\tilde{v}''^{\delta''} \tilde{\nu}'^{\delta'}\big) \\
&= \int_{\partial\tilde{\Delta}''} v'^{\delta'} \nu'' - \int_{\partial\tilde{\Delta}'} v''^{\delta''} \nu' \\
&= 2\pi i \big(v'^{\delta'}(a''_+) - v'^{\delta'}(a''_-) \big) - 2\pi i \big(v''^{\delta''}(a'_+) - v''^{\delta''}(a'_-) \big) \\
&= 2\pi i \Big(\int_{\delta''} \nu' - \int_{\delta'} \nu'' \Big).
\end{aligned}$$

Combining these two equations yields (5.4).

(iii) Finally for an abelian differential ν of the third kind with the differential principal part $\mathfrak{p}(a_+, a_-)$, for any simple path δ from a_- to a_+, and for an abelian differential μ of the second kind with poles at points $a_i \notin \delta$, let Δ_i be open neighborhoods of the poles a_i and Δ be an open neighborhood of the path δ such that all of these open sets have disjoint closures. Introduce the smoothed integrals $\tilde{v}_\delta(z)$ as in the preceding part of the proof of the present theorem and $\tilde{u}(z)$ as in the proof of Theorem 4.7, where $\tilde{u}(z)$ is modified within the open sets $T\tilde{\Delta}_i$ covering Δ_i and $\tilde{v}_\delta$ is modified within the open sets $T\tilde{\Delta}$ covering Δ. Since $\tilde{\mu} \sim \sum_{j=1}^{2g} \mu(\tau_j)\phi_j$ and $\tilde{\nu}_\delta \sim \sum_{k=1}^{2g} \nu^\delta(\tau_k)\phi_k$ it follows that

$$\int_M \tilde{\mu} \wedge \tilde{\nu}_\delta = \sum_{j,k=1}^{2g} \int_M \mu(\tau_j)\phi_j \wedge \nu^\delta(\tau_k)\phi_k = \sum_{i,j=1}^{2g} \mu(\tau_j) p_{jk} \nu^\delta(\tau_k).$$

Since $\tilde{\mu} \wedge \tilde{\nu}^\delta = 0$ outside the set $(\cup_i \Delta_i) \cup \Delta$ it further follows from Stokes's Theorem and the residue theorem that

$$\begin{aligned}
\int_M \tilde{\mu} \wedge \tilde{\nu}^\delta &= \int_{\Delta \cup (\cup_i \Delta_i)} \tilde{\mu} \wedge \tilde{\nu}^\delta \\
&= \int_{\tilde{\Delta}} d\big(\tilde{u}\tilde{\nu}^\delta\big) - \sum_i \int_{\tilde{\Delta}_i} d\big(\tilde{\mu}\tilde{v}^\delta\big) \\
&= \int_{\partial\tilde{\Delta}} u\nu - \sum_i \int_{\partial\tilde{\Delta}_i} \mu v^\delta \\
&= 2\pi i \int_\delta \mu - 2\pi i \sum_i \operatorname{res}_{a_i}\big(v^\delta \mu\big).
\end{aligned}$$

Combining these two equations yields (5.5) and thereby concludes the proof.

Although the integral and the periods of an abelian differential of the third kind ν with the differential principal part $\mathfrak{p}(a_+, a_-)$ depend on the choice of a simple path δ from the point a_- to the point a_+, this dependence is rather limited.

Lemma 5.2 *If ν is a meromorphic abelian differential of the third kind on a compact Riemann surface M of genus $g > 0$ and has the differential principal part $\mathfrak{p}(a_+, a_-)$, and if δ' and δ'' are any two simple paths on M from the point a_- to the point a_+, then for any points $z, a \in \widetilde{M} \sim \pi^{-1}(\delta' \cup \delta'')$ the integrals $v^{\delta'}(z,a)$ and $v^{\delta''}(z,a)$ of ν with respect to these two paths satisfy*

(5.6) $$v^{\delta'}(z,a) - v^{\delta''}(z,a) = 2\pi i\, n_{\delta',\delta''}(z,a) \quad \textit{where} \quad n_{\delta',\delta''}(z,a) \in \mathbb{Z};$$

and the period classes of the differential ν for these two paths satisfy

(5.7) $$\nu^{\delta'}(T) - \nu^{\delta''}(T) = 2\pi i\, n_{\delta',\delta''}(T) \quad \textit{where} \quad n_{\delta',\delta''}(T) \in \mathbb{Z}.$$

Proof: For any points $z, a \in \widetilde{M} \sim \pi^{-1}(\delta' \cup \delta'') \subset \widetilde{M}$ the complex number $v^{\delta'}(z,a) - v^{\delta''}(z,a)$ is the difference between the integrals of the meromorphic abelian differential ν along two paths from the point a to the point z in the complement $\widetilde{M} \sim \pi^{-1}(a_+ \cup a_-)$; thus it is the integral of ν along a closed path in $\widetilde{M} \sim \pi^{-1}(a_+ \cup a_-)$ and consequently it is equal to the sum of the residues of ν at the poles enclosed by that path. Since the residue of ν at each pole is ± 1 the integral is $2\pi i$ times an integer. Furthermore for any covering translation $T \in \Gamma$

$$\begin{aligned}\nu^{\delta'}(T) - \nu^{\delta''}(T) &= \big(v^{\delta'}(Tz,a) - v^{\delta'}(z,a)\big) - \big(v^{\delta''}(Tz,a) - v^{\delta''}(z,a)\big) \\ &= 2\pi i\big(n_{\delta',\delta''}(Tz,a) - n_{\delta',\delta''}(z,a)\big)\end{aligned}$$

so this too is an integer, and that suffices for the proof.

Theorem 5.3 *Let ν be a meromorphic abelian differential of the third kind on a compact Riemann surface M of genus $g > 0$, with the differential principal part $\mathfrak{p}(a_+, a_-)$.*
(i) *For any choice of a simple path δ on M from a_- to a_+ the holomorphic function*

$$q_\nu(z,a) = \exp v^\delta(z,a) \tag{5.8}$$

in the variables $z, a \in \widetilde{M} \sim \tilde{\delta}$ extends to a meromorphic function $q_\nu(z,a)$ of the variables $z, a \in \widetilde{M}$ that is independent of the choice of the path δ. The extended function is multiplicatively skew-symmetric, in the sense that $q_\nu(z,a) = q_\nu(a,z)^{-1}$; and as a function of the variable $z \in \widetilde{M}$ for a fixed point $a \in \widetilde{M}$ it has simple zeros at the points $\pi^{-1}(a_+)$, simple poles at the points $\pi^{-1}(a_-)$ and no other zeros or poles on $\widetilde{M}$.
(ii) *For any choice of a simple path δ on M from a_- to a_+ and for any covering translation $T \in \Gamma$ the exponential*

$$e_\nu(T) = \exp \nu^\delta(T) \tag{5.9}$$

is independent of the choice of the path δ; and the mapping $T \longrightarrow e_\nu(T)$ is a group homomorphism $e_\nu \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$.
(iii) *The function $q_\nu(z,a)$ as a function of the variable $z \in \widetilde{M}$ for a fixed point $a \in \widetilde{M}$ is a meromorphic relatively automorphic function for the flat factor of automorphy defined by the homomorphism e_ν.*

Proof: (i) It is evident from the definition (5.1) that the function $q_\nu(z,a) = \exp v^\delta(z,a)$ can be extended holomorphically across the interior points of the paths $\Gamma\tilde{\delta}$ in both variables; and as a consequence of the preceding lemma the extension is a single valued nowhere vanishing holomorphic function of the variables $z, a \in \widetilde{M}$ in the complement of the points of $\widetilde{M}$ covering a_+ and a_-. It follows immediately from (5.1) that $\nu^\delta(z,a) = -\nu^\delta(a,z)$ and hence that $q_\nu(z,a)$ is multiplicatively skew-symmetric. Since the differential ν has a simple pole at a_+ with residue $+1$ its integral $v^\delta(z,a)$ as a function of the variable z has a logarithmic singularity at any point $z_+ \in \widetilde{M}$ for which $\pi(z_+) = a_+$; in an open neighborhood of such a point z_+ the integral can be written

$$v^\delta(z,a) = \log(z - z_+) + h(z)$$

for a holomorphic function $h(z)$, and consequently $q_\nu(z,a) = (z - z_+)\exp h(z)$ so this function has a simple zero at z_+ but is holomorphic and nonvanishing in this neighborhood otherwise. Correspondingly the differential ν has a simple pole at a_- with residue -1, so its integral $v^\delta(z,a)$ has a logarithmic singularity at any point $z_- \in \widetilde{M}$ for which $\pi(z_-) = a_-$; in an open neighborhood of such a point z_- the integral an be written

$$v^\delta(z,a) = -\log(z - z_-) + h(z)$$

for a holomorphic function $h(z)$, and consequently $q_\nu(z,a) = (z-z_-)^{-1}\exp h(z)$ so this function has a simple pole at z_-. Altogether then $q_\nu(z,a)$ is a well defined meromorphic function of the variable $z \in \widetilde{M}$; and from the multiplicative skew-symmetry property already demonstrated it follows that $q_\nu(z,a)$ also is meromorphic in the variable $a \in \widetilde{M}$, so by Rothstein's Theorem[1] it is a meromorphic function of the two variables $(z,a) \in \widetilde{M} \times \widetilde{M}$.

(ii) The exponential $e_\nu(T) = \exp \nu^\delta(T)$ of the additive group homomorphism $\nu^\delta \in \operatorname{Hom}(\Gamma, \mathbb{C})$ is a multiplicative group homomorphism $e_\nu \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$; and it follows from the preceding lemma that this homomorphism is independent of the choice of the path δ from a_- to a_+.

(iii) Finally $q_\nu(Tz,a)/q_\nu(z,a) = \exp\big(v^\delta(Tz,a) - v^\delta(z,a)\big) = \exp \nu_\delta(T) = e_\nu(T)$ for any covering translation $T \in \Gamma$; that is just the condition that $q_\nu(z,a)$ as a function of the variable $z \in \widetilde{M}$ is a relatively automorphic function for the factor of automorphy $e_\nu \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$, and that suffices to conclude the proof.

5.2 Intrinsic Abelian Differentials of the Third Kind

An abelian differential of the third kind with the differential principal part $\mathfrak{p}(a_+, a_-)$ is determined only up to the addition of arbitrary holomorphic abelian differentials. It is possible to normalize the abelian differentials of the third kind in terms of their period classes so that there is a unique abelian differential with that differential principal part; but the normalization depends on the choice of the path δ from a_- to a_+ since it involves the periods $\nu^\delta(T)$ and not just their exponentials $e_\nu(T) = \exp \nu^\delta(T)$.

Theorem 5.4 (i) *For any simple path δ from a point a_- to a point a_+ on a compact Riemann surface M of genus $g > 0$ there are a unique meromorphic abelian differential ν_δ and a unique holomorphic abelian differential ω_δ such*

[1]For Rothstein's Theorem see Appendix A.1.

that ν_δ has the differential principal part $\mathfrak{p}(a_+, a_-)$ and that its period class ν_δ^δ with respect to the path δ is equal to the period class of the complex conjugate differential $\overline{\omega_\delta}$.
(ii) *The holomorphic abelian differential ω_δ is characterized by the condition that*

$$\int_M \omega \wedge \overline{\omega_\delta} = 2\pi i \int_\delta \omega \tag{5.10}$$

for all holomorphic abelian differentials ω.
(iii) *If δ' is a simple path from a point a'_- to a point a'_+ on M and δ'' is a simple path from a point a''_- to a point a''_+ on M, where the paths δ' and δ'' are disjoint, then the meromorphic abelian differentials $\nu_{\delta'}$ and $\nu_{\delta''}$ satisfy*

$$\int_{\delta''} \nu_{\delta'} = \int_{\delta'} \nu_{\delta''}. \tag{5.11}$$

(iv) *The abelian differential ν_δ and the intrinsic abelian differential of the second kind $\mu_\mathfrak{p}$ with poles at points $a_i \notin \delta$ satisfy*

$$\int_\delta \mu_\mathfrak{p} = \sum_{a_i} \mathrm{res}_{a_i}(v_\delta^\delta \mathfrak{p}), \tag{5.12}$$

where $v_\delta^\delta(z) = \int_a^z \nu_\delta$ is the integral of the meromorphic abelian differential ν_δ on $\widetilde{M} \sim \Gamma\tilde{\delta}$.

Proof: (i) Let $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ be a basis for the space of holomorphic abelian differentials on the surface M and $\tau_j \in H_1(M)$ be a basis for the homology of the surface M, and in terms of these bases let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{ij}\}$ be the intersection matrix of M. As in (F.9) in Appendix F.1 there is the direct sum decomposition $\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g$, in which the subspace ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ consists of the period vectors $\{\omega(\tau_j)\}$ of the holomorphic abelian differentials ω on the basis τ_j and the subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$ consists of the period vectors $\{\overline{\omega(\tau_j)}\}$ of the complex conjugates $\overline{\omega}$ of the holomorphic abelian differentials on the basis τ_j. If $\{\nu^\delta(\tau_j)\} \in \mathbb{C}^{2g}$ is the period vector of an abelian differential ν with the differential principal part $\mathfrak{p}(a_+, a_-)$ there is a unique holomorphic abelian differential ω such that the period vector of the sum $\nu_\delta = \nu + \omega$ is contained in the linear subspace $\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$, hence such that period class $\nu_\delta^\delta \in \mathrm{Hom}(\Gamma, \mathbb{C})$ of the differential ν_δ is the same as the period class of the complex conjugate of some holomorphic abelian differential ω_δ.

(ii) If ϕ_j are closed real differential forms of a basis for the first deRham group of M dual to the basis τ_j, then from the homologies $\omega_i \sim \sum_{j=1}^g \omega_{ij}\phi_j$ and $\omega_\delta \sim \sum_{j=1}^g \omega_\delta(\tau_k)\phi_k$ it follows that

$$\int_M \omega_i \wedge \overline{\omega_\delta} = \int_M \sum_{j,k=1}^g \omega_{ij}\phi_j \wedge \overline{\omega_\delta(\tau_k)}\phi_k = \sum_{j,k=1}^g \omega_{ij} p_{jk} \overline{\omega_\delta(\tau_k)};$$

and since $\overline{\omega(\tau_k)} = \nu_\delta^\delta(\tau_k)$ as in (i) it follows from (5.3) in Theorem 5.1 that

$$\sum_{j,k=1}^{g} \omega_{ij} p_{jk} \overline{\omega_\delta(\tau_k)} = \sum_{j,k=1}^{g} \omega_{ij} p_{jk} \nu_\delta^\delta(\tau_k) = 2\pi\, i \int_\delta \omega_i. \tag{5.13}$$

Combining the two preceding equations shows that (5.10) holds for the basis ω_i, and consequently it holds for all holomorphic abelian differentials ω.

(iii) If $\nu_{\delta'}$ is the meromorphic abelian differential of the third kind with the differential principal part $\mathfrak{p}(a'_+, a'_-)$ and $\nu_{\delta''}$ is the meromorphic abelian differential of the third kind with the differential principal part $\mathfrak{p}(a'_+, a'_-)$ as in (i), then for disjoint paths δ' from a'_- to a'_+ and δ'' from a''_- to a''_+, the associated holomorphic differentials $\omega_{\delta'}$ and $\omega_{\delta''}$ satisfy $\omega_{\delta'} \wedge \omega_{\delta''} = 0$, since the product is a differential form of type $(2,0)$ on the Riemann surface M. From the homologies $\omega_{\delta'} \sim \sum_{j=1}^{g} \omega_{\delta'}(\tau_j)\phi_j$ and $\omega_{\delta''} \sim \sum_{k=1}^{g} \omega_{\delta''}(\tau_k)\phi_k$ it follows that

$$0 = \int_M \overline{\omega_{\delta'}} \wedge \overline{\omega_{\delta''}} = \int_M \sum_{j,k=1}^{g} \overline{\omega_{\delta'}(\tau_j)}\phi_j \wedge \overline{\omega_{\delta''}(\tau_k)}\phi_k = \sum_{j,k=1}^{g} \overline{\omega_{\delta'}(\tau_j)} p_{jk} \overline{\omega_{\delta''}(\tau_k)}.$$

Since $\overline{\omega_{\delta_i}(\tau_j)} = \nu_\delta^\delta(\tau_j)$ by (i) it follows from (5.4) in Theorem 5.1 that

$$\sum_{j,k=1}^{g} \overline{\omega_{\delta'}(\tau_j)} p_{jk} \overline{\omega_{\delta''}(\tau_k)} = \sum_{j,k=1}^{g} \nu_{\delta'}^{\delta'}(\tau_j) p_{jk} \nu_{\delta''}^{\delta''}(\tau_k) = 2\pi\, i\Big(\int_{\delta''} \nu_{\delta'} - \int_{\delta'} \nu_{\delta''} \Big).$$

Combining these two equations shows that (5.11) holds.

(iv) The holomorphic abelian differentials $\omega_\mathfrak{p}$ and ω_δ with period classes conjugate to the period classes of the meromorphic abelian differentials $\mu_\mathfrak{p}$ and ν_δ satisfy $\omega_\mathfrak{p} \wedge \omega_\delta = 0$, since the product is a differential form of type $(2,0)$ on the Riemann surface M; so from the homologies $\overline{\omega_\mathfrak{p}} \sim \sum_{j=1}^{g} \mu_\mathfrak{p}(\tau_j)\phi_j$ and $\overline{\omega_\delta} \sim \sum_{k=1}^{g} \nu_\delta^\delta(\tau_k)\phi_k$ it follows that

$$0 = \int_M \overline{\omega_\delta} \wedge \overline{\omega_\mathfrak{p}} = \int_M \sum_{j,k=1}^{g} \mu_\mathfrak{p}(\tau_j)\phi_j \wedge \nu_\delta^\delta(\tau_k)\phi_k = \sum_{j,k=1}^{g} \mu_\mathfrak{p}(\tau_j) p_{jk} \nu_\delta^\delta(\tau_k).$$

Since $\overline{\omega_\mathfrak{p}(\tau_j)} = \nu_\mathfrak{p}(\tau_j)$ and $\overline{\omega_\delta(\tau_k)} = \nu_\delta^\delta(\tau_k)$ it follows from (5.5) in Theorem 5.1 that

$$\sum_{j,k=1}^{g} \mu_\mathfrak{p}(\tau_j) p_{jk} \nu_\delta^\delta(\tau_k) = 2\pi i \int_\delta \mu - 2\pi i \sum_{a_i} \operatorname{res}_{a_i}(v^\delta \mu).$$

Combining these two equations yields (5.12), and that suffices to conclude the proof.

The meromorphic abelian differential ν_δ of part (i) of the preceding theorem with the differential principal part $\mathfrak{p}(a_+, a_-)$ is called the *intrinsic abelian differential of the third kind* with respect to the path δ, and the holomorphic abelian differential ω_δ is called the *associated holomorphic abelian differential*; both are determined uniquely by the path δ from the pole a_- to the pole a_+, and their period classes can be written explicitly in terms of that path.

Corollary 5.5 *On a compact Riemann surface M of genus $g > 0$ let $\Omega = \{\omega_{ij}\}$ be the period matrix and $P = \{p_{ij}\}$ be the intersection matrix in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$. For a differential principal part $\mathfrak{p}(a_+, a_-)$ and any simple path δ from a_- to a_+ the periods of the intrinsic abelian differential of the third kind ν_δ with respect to the path δ are*

$$\nu_\delta^\delta(T) = -2\pi \sum_{m,n=1}^{g} g_{mn}\overline{\omega_n(T)}\left(\int_\delta \omega_m\right) \tag{5.14}$$

for any covering translation $T \in \Gamma$, where $G = \{g_{ij}\} = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega\, P\, {}^t\overline{\Omega}$.

Proof: The associated holomorphic abelian differential ω_δ can be written as the sum $\omega_\delta = \sum_{l=1}^{g} c_l\omega_l$ for some complex constants c_l, so its periods are $\omega_\delta(\tau_k) = \sum_{l=1}^{g} c_l\omega_l(\tau_k) = \sum_{l=1}^{g} c_l\omega_{lk}$. Substituting this into (5.13) yields the identity

$$2\pi\, i \int_\delta \omega_m = \sum_{j,k=1}^{g} \omega_{mj}p_{jk}\overline{\omega_\delta(\tau_k)} = \sum_{j,k,l=1}^{g} \omega_{mj}p_{jk}\overline{\omega}_{lk}\overline{c}_l = -i\sum_{l=1}^{g} h_{ml}\overline{c}_l$$

where h_{ml} are the entries in the $g \times g$ matrix $H = i\Omega\, P\, {}^t\overline{\Omega}$. The matrix H is positive definite Hermitian by Riemann's inequality, Theorem 3.20 (ii), so $G = {}^tH^{-1}$ exists; and if $G = \{g_{mn}\}$ then upon multiplying the preceding equation by g_{mn} and summing over m it follows that

$$\overline{c_n} = -2\pi \sum_{m=1}^{g} g_{mn} \int_\delta \omega_m$$

hence that

$$\nu_\delta^\delta(\tau_j) = \overline{\omega_\delta(\tau_j)} = \sum_{n=1}^{g} \overline{c_n\omega_{nj}} = -2\pi \sum_{m,n=1}^{g} g_{mn}\overline{\omega_{nj}} \int_\delta \omega_m. \tag{5.15}$$

If the covering translation $T \in \Gamma$ corresponds to a homology class $\tau \in H_1(M)$ and $\tau \sim \sum_{j=1}^{g} n_j\tau_j$ for some integers n_j then $\nu_\delta^\delta(T) = \nu_\delta^\delta(\tau) = \sum_{j=1}^{g} n_j\nu_\delta^\delta(\tau_j)$ and $\overline{\omega_n(T)} = \overline{\omega_n(\tau)} = \sum_{j=1}^{g} n_j\overline{\omega_n(\tau_j)} = \sum_{j=1}^{g} n_j\overline{\omega_{nj}}$; multiplying both sides of (5.15) by n_j and summing over $j = 1, \ldots, g$ yields (5.14) and thereby concludes the proof.

Although the explicit formula (5.14) depends on the choice of bases ω_i for the holomorphic abelian differentials on M and τ_j for the homology of M, it is clear that the value $\nu_\delta^\delta(T)$ is independent of these choices. It may be comforting, just as in the case of the corresponding result for meromorphic abelian differentials of the second kind discussed on page 107, to prove that directly; and for that purpose it is convenient to rewrite (5.14). Choose a point $z_- \in \widetilde{M}$ such that $\pi(z_-) = a_-$, where $\pi : \widetilde{M} \longrightarrow M$ is the covering projection. A path δ from a_- to a_+ in M has a unique lifting to a path in $\tilde{\delta} \subset \widetilde{M}$ beginning at the point

z_-, and the lifting is a simple path that ends at a point $z_+ \in \widetilde{M}$ for which $\pi(z_+) = a_+$. If $w_m(z,a) = \int_a^z \omega_m$ are the integrals of the differential forms ω_m then

$$\int_\delta \omega_m = \int_{\tilde{\delta}} d\, w_m = w_m(z_+, z_-);$$

hence (5.14) can be written

$$\nu_\delta^\delta(T) = -2\pi \sum_{m,n=1}^{g} w_m(z_+, z_-)\, g_{mn}\, \overline{\omega_n(T)} \tag{5.16}$$

for any covering translation $T \in \Gamma$. To rewrite this formula in matrix terms, introduce the column vector $\tilde{w}(z,a) = \{w_j(z,a)\}$ consisting of the integrals $w_j(z,a) = \int_a^z \omega_j$ of the holomorphic abelian differentials ω_j and the homomorphism $\omega \in \operatorname{Hom}(\Gamma, \mathbb{C}^g)$ where $\omega(T) = \{\omega_j(T)\} \in \mathbb{C}^g$ is the column vector of period classes of the holomorphic abelian differentials. In these terms (5.16) takes the form

$$\nu_\delta^\delta(T) = -2\pi\, {}^t\tilde{w}(z_+, z_-)\, G\, \overline{\omega(T)} \tag{5.17}$$

for all $T \in \Gamma$. A change of basis for the holomorphic abelian differentials on M has the effect of replacing the vector $\tilde{w}(z,a)$ by $A\tilde{w}(z,a)$, the vector $\overline{\omega(T)}$ by $\overline{A\omega(T)}$, and the form matrix G by ${}^tA^{-1}G\overline{A^{-1}}$ as in equation (F.41) in Appendix F.4; this change clearly leaves (5.17) unchanged.

5.3 The Intrinsic Cross-Ratio Function

Since the holomorphic abelian differential ω_δ associated to the intrinsic abelian integral of the third kind ν_δ has periods that are the complex conjugate of the periods $\nu_\delta^\delta(T)$ of the intrinsic abelian differential of the third kind it follows from (5.16) that

$$\omega_\delta(z) = -2\pi \sum_{m,n=1}^{g} \overline{w_m(z_+, z_-)\, g_{mn}}\ \omega_n(z); \tag{5.18}$$

thus ω_δ is determined just by the points z_+ and z_-. Equivalently the differential ω_δ depends only on the homotopy type of the path δ; for any two homotopic paths in M from a_- to a_+ when lifted to paths in $\widetilde{M}$ beginning at the point z_- have the same end point, and since $\widetilde{M}$ is simply connected conversely any two paths in $\widetilde{M}$ from z_- to z_+ are homotopic so their projections under the covering projection $\pi : \widetilde{M} \longrightarrow M$ are homotopic paths from a_- to a_+ in M. To make the dependence on the points z_+ and z_- quite explicit, set

$$\omega_\delta(z) = \omega_{z_+,z_-}(z). \tag{5.19}$$

As for the dependence on the choice of the initial point z_- for which $\pi(z_-) = a_-$, it is evident from (5.18) that

$$\omega_{Tz_+,Tz_-}(z) = \omega_{z_+,z_-}(z) \quad \text{for any } T \in \Gamma. \tag{5.20}$$

Since the intrinsic abelian differential of the third kind $\nu_\delta(z)$ is determined uniquely by its principal part $\mathfrak{p}(a_+, a_-)$ and its associated holomorphic abelian differential $\omega_\delta(z)$, by Theorem 5.4 (i), it too is determined uniquely by the points z_+ and z_- and consequently can be denoted unambiguously by

$$\nu_\delta(z) = \nu_{z_+,z_-}(z); \tag{5.21}$$

and of course (5.20) holds for this abelian differential as well. The integral

$$v^\delta_{z_+,z_-}(z, a) = \int_a^z \nu_{z_+,z_-} \tag{5.22}$$

is defined as a holomorphic function on the complement $\widetilde{M} \sim \Gamma\tilde{\delta}$. so in that sense still depends on the choice of the path δ; but by Theorem 5.3 its exponential is a meromorphic function

$$q(z, a; z_+, z_-) = q_{z_+,z_-}(z, a) = \exp v^\delta_{z_+,z_-}(z, a) = \exp \int_a^z \nu_{z_+,z_-} \tag{5.23}$$

of the variables z, $a \in \widetilde{M}$, which is called the *cross-ratio function* of the Riemann surface M or sometimes the *intrinsic cross-ratio function* of the Riemann surface M to be more specific. The group homomorphism or flat factor of automorphy associated to the abelian differential $\nu_\delta(z) = \nu_{z_+,z_-}(z)$ as in Theorem 5.3 (ii) is denoted correspondingly by $e_{z_+,z_-}(T)$. If Ω is the period matrix and P is the intersection matrix of M in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ then by (5.16)

$$\begin{aligned} e_{z_+,z_-}(T) &= \exp -2\pi \sum_{m,n=1}^{g} w_m(z_+, z_-)\, g_{mn}\, \overline{\omega_n(T)} \\ &= \exp -2\pi\, {}^t\tilde{w}(z_+, z_-)\, G\, \overline{\omega(T)} \end{aligned} \tag{5.24}$$

for any covering translation $T \in \Gamma$. This flat factor of automorphy can be described alternatively as

$$e_{z_+,z_-}(T) = \rho_{t(z_+,z_-)}(T) \tag{5.25}$$

in terms of the canonical parametrization of flat factors of automorphy (3.27) associated to the basis $\tau_j \in H_1(M)$, where $t(z_+, z_-) = \{t_j(z_+, z_-)\} \in \mathbb{C}^{2g}$ is the complex vector for which $e_{z_+,z_-}(\tau_j) = \exp 2\pi i t_j(z_+, z_-)$; in view of (5.24) this is is the vector with components

$$t_j(z_+, z_-) = i \sum_{m,n=1}^{g} w_m(z_+, z_-) g_{mn} \overline{\omega_{nj}} \tag{5.26}$$

for $1 \le j \le 2g$, or in matrix terms, when all vectors are viewed as column vectors, is the vector

$$t(z_+, z_-) = i\, {}^t\overline{\Omega}\, {}^tG\tilde{w}(z_+, z_-). \tag{5.27}$$

Again although the preceding explicit formulas involve the choice of bases for the holomorphic abelian differentials on M and the homology of M, the factor of automorphy $e_{z_+,z_-}(T) = \rho_{t(z_+,z_-)}(T)$ is independent of these choices. The basic properties of the cross-ratio function can be summarized as follows.

Theorem 5.6 *If M is a compact Riemann surface M of genus $g > 0$ with the universal covering space $\widetilde{M}$ and the covering translation group Γ, the cross-ratio function of M is a meromorphic function $q(z_1, z_2; z_3, z_4)$ on the complex manifold $\widetilde{M} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M}$ that is characterized uniquely by the following properties:*
(i) *The function $q(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$, simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, but is otherwise holomorphic and nowhere vanishing; and $q(z_1, z_2; z_3, z_4) = 1$ if $z_1 = z_2$ or $z_3 = z_4$.*
(ii) *The function $q(z_1, z_2; z_3, z_4)$ has the symmetries*

$$\begin{aligned}(5.28)\qquad q(z_1, z_2; z_3, z_4) &= q(z_3, z_4; z_1, z_2) = q(z_2, z_1; z_4, z_3) = \\ &= q(z_2, z_1; z_3, z_4)^{-1} = q(z_1, z_2; z_4, z_3)^{-1}.\end{aligned}$$

(iii) *The function $q(z_1, z_2; z_3, z_4)$ as a function of the variable $z_1 \in \widetilde{M}$ for any fixed points $z_2, z_3, z_4 \in \widetilde{M}$ is a meromorphic relatively automorphic function for the canonically parametrized factor of automorphy $\rho_{t(z_3,z_4)}(T)$ described by the vector $t(z_3, z_4) \in \mathbb{C}^{2g}$ that for any bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ has the explicit form*

$$(5.29)\qquad t(z_3, z_4) = i\ {}^t\overline{\Omega}\,{}^tG\tilde{w}(z_3, z_4)$$

in which Ω is the period matrix, P is the intersection matrix, and $G = {}^tH^{-1}$ for the positive definite symmetric matrix $H = i\,\Omega P\,{}^t\overline{\Omega}$ in terms of these bases.

Proof: Whenever z_3, z_4 are points of $\widetilde{M}$ that are not equivalent under Γ it follows from Theorem 5.3 that the cross-ratio function $q(z_1, z_2; z_3, z_4) = q_{z_3,z_4}(z_1, z_2)$ is a well defined meromorphic function of the variable $z_1 \in \widetilde{M}$ that is relatively automorphic for the factor of automorphy $e_{z_3,z_4} = \rho_{t(z_3,z_4)}$ and that has simple zeros at the points Γz_3, simple poles at the points Γz_4, and no other zeros or poles on $\widetilde{M}$. It is clear from the definition (5.23) that $q(z_1, z_2; z_3, z_4) = q(z_2, z_1; z_3, z_4)^{-1}$, from which the corresponding analyticity properties as a function of the variable $z_2 \in \widetilde{M}$ follow immediately; and it is also clear from the definition that $q(z_1, z_1; z_3, z_4) = 1$. For two intrinsic abelian differentials ν_{z_1,z_2} and ν_{z_3,z_4} of the third kind, and for disjoint paths δ_1 from $\pi(z_2)$ to $\pi(z_1)$ and δ_3 from $\pi(z_4)$ to $\pi(z_3)$, it follows from Theorem 5.4 (iii) that $\int_{\delta_1} \nu_{z_3,z_4} = \int_{\delta_3} \nu_{z_1,z_2}$; consequently

$$\begin{aligned} q(z_1, z_2; z_3, z_4) &= \exp \int_{z_2}^{z_1} \nu_{z_3,z_4} = \exp \int_{\delta_1} \nu_{z_3,z_4} \\ &= \exp \int_{\delta_3} \nu_{z_1,z_2} = \exp \int_{z_4}^{z_3} \nu_{z_1,z_2} = q(z_3, z_4; z_1, z_2). \end{aligned}$$

This symmetry, together with that already demonstrated, implies all the symmetries of (ii). From these symmetries and the already established analyticity properties of $q(z_1, z_2; z_3, z_4)$ as a function of the variables z_1 and z_2 it follows that $q(z_1, z_2; z_3, z_4)$ also is a meromorphic function of the variable z_3 and the variable z_4, hence by Rothstein's Theorem[2] it is a meromorphic function on $\widetilde{M} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M}$; and altogether it has the zeros and poles as in (i). Finally the quotient of any two functions satisfying the conditions of the theorem is necessarily a holomorphic and nowhere vanishing function on $\widetilde{M} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M}$ that is invariant under the group of covering translations in each factor, hence is a constant; and from the normalization condition of (i) it is evident that this constant is 1. That suffices to conclude the proof.

The preceding theorem also holds for surfaces of genus $g = 0$ to the extent possible. On the Riemann sphere $\mathbb{P}^1$ the classical cross-ratio function is

$$q(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}. \tag{5.30}$$

This clearly is a meromorphic function on $\mathbb{P}^1$ in all four variables, with the singularities as in Theorem 5.6 (i); and equally clearly it satisfies the symmetry conditions of Theorem 5.6 (ii). Theorem 5.6 (iii) is not applicable, since the covering translation group is trivial in this case; but the remaining properties obviously characterize the cross-ratio function uniquely. The terminology of course is suggested by this special case; but the role that the cross-ratio function plays for general Riemann surfaces differs in many ways from its role in classical projective geometry. However some of the simple relations among the classical cross-ratio functions hold as well for the cross-ratio functions for surfaces of genus $g > 0$.

Corollary 5.7 *The cross-ratio for any compact Riemann surface satisfies the product formula*

$$q(z_1, z_2; z_3, z_4)\; q(z_1, z_2; z_4, z_5) = q(z_1, z_2; z_3, z_5) \tag{5.31}$$

Proof: Introduce the meromorphic function

$$f(z_1, z_2, z_3, z_4, z_5) = q(z_1, z_2; z_3, z_4)\; q(z_1, z_2; z_4, z_5)\; q(z_1, z_2; z_5, z_3) \tag{5.32}$$

of the variables $(z_1, z_2, z_3, z_4, z_5) \in \widetilde{M}^5$. This is a relatively automorphic function in each variable for the appropriate factor of automorphy, since by part (iii) of the preceding theorem $q(Tz_1, z_2; z_3, z_4) = \rho_{t(z_3,z_4)}(T) q(z_1, z_2; z_3, z_4)$ for all $T \in \Gamma$. In particular then

(5.33)
$$\begin{aligned} f(Tz_1, z_2, z_3, z_4, z_5) &= \rho_{t(z_3,z_4)}(T)\rho_{t(z_4,z_5)}(T)\rho_{t(z_5,z_4)}(T) f(z_1, z_2, z_3, z_4, z_5) \\ &= f(z_1, z_2, z_3, z_4, z_5) \end{aligned}$$

[2]For Rothstein's Theorem see Appendix A.1.

for all $T \in \Gamma$ since it is evident from (5.29) that $t(z_3, z_4)+t(z_4, z_5)+t(z_5, z_30 = 0$, and similarly for the other variables inview of the symmetries (5.28); thus the function $f(z_1, z_2, z_3, z_4, z_5)$ is invariant under the action of the group Γ in each variable. On the other hand by (i) of the preceding theorem the divisor of $f(z_1, z_2, z_3, z_4, z_5)$ as a Γ-invariant function just of the variable $z_1 \in \widetilde{M}$, hence as a meromorphic function of the variable $z_1 \in M$ on the compact Riemann surface M, is

$$\mathfrak{d}f(z_1) = (z_3 - z_4) + (z_4 - z_5) + (z_5 - z_3) = 0. \tag{5.34}$$

Therefore $f(z_1, z_2, z_3, z_4, z_5)$ as a function of the variable $z_1 \in M$ is holomorphic and nowhere vannishing on the compact Riemann surface M so is constant; and that means that the function $f(z_1, z_2, z_3, z_4, z_5)$ is actuallly independent of the variable z_1. By the symmetry (5.28) in the variables z_1 and z_2 the function $f(z_1, z_2, z_3, z_4, z_5)$ also is independent of the variable z_2. Similarly as a function of the variable z_3

$$\mathfrak{d}f(z_3) = (z_1 - z_2) + (z_2 - z_1) = 0; \tag{5.35}$$

so $f(z_1, z_2, z_3, z_4, z_5)$ is independent of the variable z_3, and from the symmetries (5.28) it is independent of the variables z_4 amd z_5, hence $f(z_1, z_2, z_3, z_4, z_5)$ is a constant. From the normalization (i) it follows that $f(z_1, z_1, z_3, z_4, z_5) = 1$ so actually $f(z_1, z_2, z_3, z_4, z_5) = 1$ for all points $z_j \in M$, and that suffices for the proof.

Further relations similar to that of the preceding corollary will be discussed in connections with generalizations of the cross-ratio function in Chapter 13. In another direction, all meromorphic abelian differentials can be expressed in terms of the cross-ratio function and its derivatives. It follows immediately from the definition (5.23) of the cross-ratio function that the intrinsic abelian differential of the third kind with the principal part $\mathfrak{p}(z_+, z_-)$ can be written explicitly in terms of the cross-ratio function as

$$\nu_{z_+,z_-} = d\,\log q(z, a; z_+, z_-) \tag{5.36}$$

for any point $a \in \widetilde{M}$; and consequently its integral can be written

$$v^{\delta}_{z_+,z_-}(z, a) = \int_a^z \nu_{z_+,z_-} = \log q(z, a; z_+, z_-) \tag{5.37}$$

for that branch of the logarithm for which $v^{\delta}_{z_+,z_-}(a, a) = \log q(a, a; z_+, z_-) = \log 1 = 0$. The intrinsic double differential of the second kind on a compact Riemann surface of genus $g > 0$ also can be written explicitly in terms of the cross-ratio function of the surface as follows.

Corollary 5.8 *The intrinsic double differential of the second kind $\mu_M(z, \zeta)$ on compact Riemann surface M of genus $g > 0$ can be expressed in terms of the* *cross-ratio function* $q(z_1, z_2; z_3, z_4)$ *of that surface as*

$$\mu_M(z,\zeta) = \frac{\partial^2}{\partial z_\alpha \partial \zeta_\beta} \log q(z_\alpha, a; \zeta_\beta, b)\ dz_\alpha d\zeta_\beta \tag{5.38}$$
$$= d_z d_\zeta \log q(z, a; \zeta, b)$$

for any points $a, b \in \widetilde{M}$.

Proof: When the intrinsic double differential of the second kind is written $\mu_M(z,\zeta) = f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha d\zeta_\beta$ in terms of a covering of the surface M by coordinate neighborhoods U_α, in which the local coordinates are denoted by either z_α or ζ_α, then $\mu_{\mathfrak{p}}(z) = f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha$ is the intrinsic meromorphic abelian differential of the second kind with the principal part $\mathfrak{p} = (z_\beta - \zeta_\beta)^{-2} dz_\beta$ at the fixed point $\zeta_\alpha \in U_\beta$. From Theorem 5.4 (iv) it then follows that for any simple path δ from z_- to z_+ on $\widetilde{M}$ and for the integral (5.37)

$$\begin{aligned}
\int_{z_-}^{z_+} f_{\alpha\beta}(z_\alpha, \zeta_\beta) dz_\alpha = \int_\delta \mu_{\mathfrak{p}} \quad &= \quad \operatorname{res}_{\zeta_\beta}\left(v^\delta_{z_+,a}\mathfrak{p}\right) \\
&= \operatorname{res}_{z_\beta=\zeta_\beta}\left(\log q(z_\beta, a; z_+, z_-) \frac{dz_\beta}{(z_\beta - \zeta_\beta)^2}\right) \\
&= \frac{\partial}{\partial \zeta_\beta} \log q(\zeta_\beta, a; z_+, z_-).
\end{aligned}$$

Differentiating the preceding equation with respect to the variable z_+ at the point $z_+ = z_\alpha \in U_\alpha$ shows that

$$f_{\alpha\beta}(z_\alpha, \zeta_\beta) = \frac{\partial^2}{\partial z_\alpha \partial \zeta_\beta} \log q(\zeta_\beta, a; z_\alpha, z_-);$$

this is equivalent to the desired result and that suffices to conclude the proof.

5.4 Abel's Theorem

The cross-ratio function encodes the basic relation between flat line bundles and holomorphic line bundles, or equivalently, the divisors describing holomorphic line bundles.

Theorem 5.9 *If M is a compact Riemann surface of genus $g > 0$ with the universal covering projection $\pi : \widetilde{M} \longrightarrow M$, if Ω is the period matrix and P is the intersection matrix of M in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, and if $G = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\, {}^t\overline{\Omega}$, then for any points $z_+, z_- \in \widetilde{M}$ the canonically parametrized flat line bundle $\rho_{t(z_+,z_-)}$ described by the vector $t(z_+, z_-) = i\, {}^t\overline{\Omega}\, {}^tG\tilde{w}(z_+, z_-)$, where $\tilde{w}(z, a) \in \mathbb{C}^g$ is the vector with entries $w_i(z, a) = \int_a^z \omega_i$, is holomorphically equivalent to the holomorphic line bundle $\zeta_{\pi(z_+)}\zeta^{-1}_{\pi(z_-)}$ over M.*

Proof: For any fixed points $a, z_+, z_- \in \widetilde{M}$ it follows from Theorem 5.6 that the cross-ratio function $q(z, a; z_+, z_-)$ as a function of the variable $z \in \widetilde{M}$ is a meromorphic relatively automorphic function for the canonically parametrized flat factor of automorphy $\rho_{t(z_+,z_-)}$ and has the divisor $\mathfrak{d} = 1 \cdot a_+ - 1 \cdot a_-$ on M where $a_+ = \pi(z_+),\ a_- = \pi(z_-)$; consequesntly the flat line bundle $\rho_{t(z_+,z_-)}$ represents the holomorphic line bundle $\zeta_{a_+}\zeta_{a_-}^{-1}$, and that suffices for the proof.

This observation can be extended quite naturally in terms of the Abel-Jacobi mapping (3.4), the holomorphic mapping $w_a : M \longrightarrow J(M)$ from the Riemann surface M to its Jacobi variety $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ induced by the holomorphic mapping $\tilde{w}_a : \widetilde{M} \longrightarrow \mathbb{C}^g$ that associates to a point z in the universal covering surface $\widetilde{M}$ of M the point $\tilde{w}_a(z) = \{w_i(z, a)\} \in \mathbb{C}^g$, where $w_i(z, a) = \int_a^z \omega_i$ for the base point $a \in \tilde{M}$. The Jacobi variety can be viewed not just as a complex manifold but also as a complex Lie group, with the identity element $0 \in J(M)$ represented by the origin $0 \in \mathbb{C}^g$ when the Jacobi variety is viewed as the quotient $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$. With this structure as a complex Lie group the Jacobi variety $J(M)$ will be called the *Jacobi group.* The Abel-Jacobi mapping thus takes the base point $a \in M$ to the identity $0 \in J(M)$ in the Jacobi group. The Abel-Jacobi mapping then extends naturally to the *Abel-Jacobi homomorphism*, the group homomorphism

$$w_a : \Gamma(M, \mathcal{D}) \longrightarrow J(M) \tag{5.39}$$

from the additive group $\Gamma(M, \mathcal{D})$ of divisors on M to the Jacobi group $J(M)$ that associates to any divisor $\mathfrak{d} = \sum_{j=1}^{r} \nu_j \cdot a_j \in \Gamma(M, \mathcal{D})$ the point

$$w_a(\mathfrak{d}) = \sum_{j=1}^{r} \nu_j w_a(a_j) \in J(M); \tag{5.40}$$

thus the image $w_a(\mathfrak{d})$ is the point in the quotient space $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ represented by the vector $\sum_{j=1}^{r} \nu_j \tilde{w}(z_j, a) \in \mathbb{C}^g$ for any points $z_j \in \tilde{M}$ such that $\pi(z_j) = a_j$ under the universal covering projection $\pi : \tilde{M} \longrightarrow M$. In these terms the result of the preceding Theorem5.9 can be restated as follows.

Theorem 5.10 (Abel's Theorem) *Two divisors $\mathfrak{d}'$ and $\mathfrak{d}''$ of the same degree on a compact Riemann surface M of genus $g > 0$ are linearly equivalent if and only if they have the same image $w_a(\mathfrak{d}') = w_a(\mathfrak{d}'') \in J(M)$ under the Abel-Jacobi homomorphism $w_a : \Gamma(M, \mathcal{D}) \longrightarrow J(M)$.*

Proof: If Ω is the period matrix and P is the intersection matrix of M in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, and if $G = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\,{}^t\overline{\Omega}$, then by Theorem 5.9 for any points $z_+, z_- \in \widetilde{M}$ the canonically parametrized flat line bundle $\rho_{t(z_+,z_-)}$ described by the vector $t(z_+, z_-) = i\,{}^t\overline{\Omega}\,{}^tG\tilde{w}(z_+, z_-)$, where $\tilde{w}(z, a) \in \mathbb{C}^g$ is the vector with entries $w_i(z, a) = \int_a^z \omega_i$, is holomorphically equivalent to the holomorphic line bundle $\zeta_{\pi(z_+)}\zeta^{-1}_{\pi(z_-)}$ over M, where $\pi : \widetilde{M} \longrightarrow M$ is the coveering projection mapping. Therefore if $\mathfrak{d}' = \sum_{j=1}^{r} a'_j$ and $\mathfrak{d}'' = \sum_{j=1}^{r} a''_j$ the canonically parametrized flat line bundle $\rho_{t(\mathfrak{d}',\mathfrak{d}'')}$ described by the vector

$$t(\mathfrak{d}',\mathfrak{d}'') = \sum_{j=1}^{n} t(a'_j, a''_j) = i\,{}^t\overline{\Omega}\,{}^tG\widetilde{w}(\mathfrak{d}',\mathfrak{d}''), \tag{5.41}$$

where $\widetilde{w}(\mathfrak{d}',\mathfrak{d}\text{‘}'') = \sum_{j=1}^{r}(\widetilde{w}(a'_j, a''_j)$, is holomorphically equivalent to the line bundle $\zeta_{\mathfrak{d}'}\zeta^{-1}_{\mathfrak{d}''}$. The point in the Picard variety represented by the flat line bundle $\rho_{t(\mathfrak{d}',\mathfrak{d}'')}$ is described by the image of the vector $t(\mathfrak{d}',\mathfrak{d}'')$ in the quotient $P(M) = \mathbb{C}^{2g}/(\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g)$. The isomorphism between the Picard and Jacobi varieties of Theorem 3.23 takes the element of the Picard variety represented by the vector $t(\mathfrak{d}',\mathfrak{d}'')$ to the element of the Jacobi variety $J(M) = \mathbb{C}^{2g}/\Omega\mathbb{Z}^{2g}$ represented by the point $\Omega P t(\mathfrak{d}',\mathfrak{d}'')$. Since $i\,\Omega P\,{}^t\overline{\Omega}\,{}^tG = H\,{}^tG = \mathrm{I}$ it follows that $\Omega P t(\mathfrak{d}',\mathfrak{d}'') = \tilde{w}(\mathfrak{d}',\mathfrak{d}'')$; therefore the divisors $\mathfrak{d}'$ and $\mathfrak{d}''$ are linearly equivalent if and only if $w_{z_0}(\mathfrak{d}' - \mathfrak{d}\text{“}) = 0$ which suffices for the proof.

It is quite common to rephrase Abel's Theorem to avoid the necessity of choosing a base point in the complex torus $J(M)$ by using the alternative mapping $\widetilde{w} : \widetilde{M} \times \widetilde{M} \longrightarrow \mathbb{C}^g$ from the universal covering space $\widetilde{M}$ of a Riemann surface M that associates to any points $z', z'' \in \widetilde{M}$ the point $\widetilde{w}(z', z'') = \{w_i(z', z'')\} \in \mathbb{C}^g$ for the integrals $w_i(z', z'') = \int_{z''}^{z'} \omega_i$ of a basis for the holomorphic abelian differentials ω_i on M, and viewing the Jacobi variety as the quotient $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$ for the lattice subgroup $\mathcal{L}(\Omega)$ defined by the period matrix Ω of the abelian differentials ω_i.

Corollary 5.11 *Two divisors $\mathfrak{d}'$ and $\mathfrak{d}''$ of degree r on a compact Riemann surface M, represented by divisors $\widetilde{\mathfrak{d}}' = \sum_{j=1}^{r} z'_i$ and $\widetilde{\mathfrak{d}}'' = \sum_{j=1}^{r} z''_i$ for points $z'_i, z''_i \in \widetilde{M}$, are linearly equivalent if and only if*

$$\widetilde{w}(z'_J, z''_j) \in \mathcal{L}(\Omega). \tag{5.42}$$

Proof: This is merely a rephrasing of the results of the preceding Theorem 5.10, so no further proof is necessary.

One important application of Abel's Theorem is to provide an imbedding of a Riemann surface of genus $g > 0$ as a nonsingular holomorphic subvariety of its Jacobi variety; that complements the imbeddings of a Riemann surface in complex projective spaces given by Theorem 2.17.

Theorem 5.12 *The Abel-Jacobi mapping of a compact Riemann surface M of genus $g > 0$ is a nonsingular holomorphic mapping $w_a : M \longrightarrow J(M)$ that defines a biholomorphic mapping $w_a : M \longrightarrow w_a(M)$ from the Riemann surface M to a connected one-dimensional holomorphic submanifold $w_a(M) \subset J(M)$.*

Proof: It follows from Abel's Theorem, Theorem 5.10, that $w_a(p) = w_a(q)$ for two points $p, q \in M$ if and only if the divisors $1\cdot p$ and $1\cdot q$ are linearly equivalent, or what is the same thing, if and only if $\zeta_p = \zeta_q$; but by Corollary 2.5 for a Riemann surface of genus $g > 0$ that is the case if and only if $p = q$. That shows that the Abel-Jacobi mapping $w_a : M \longrightarrow J(M$ is an injective holomorphic mapping. If $\omega_i = f_{\alpha,i}(z_\alpha)dz_\alpha$ is a basis for the holomorphic abelian differentials in terms of a local coordinate z_α at a point $p \in M$ then the derivative of the Abel-Jacobi mapping at the point p is the matrix $\{f'_{\alpha,i}(p)\}$. This is a nonsingular matrix; for if all the g holomorphic abelian differentials vanish at the point p then $\gamma(\kappa\zeta_p^{-1}) = g$, but by the Riemann-Roch Theorem $\gamma(\kappa\zeta_p^{-1}) = \gamma(\zeta_p)+2g-3+1-g = g-1$ since $\gamma(\zeta_p) = 1$ for any point p of a Riemann surface of genus $g > 0$. Therefore the Abel-Jacobi mapping is a nonsingular injective holomorphic mapping $w_a : M \longrightarrow J(M)$; and since the image is a holomorphic subvariety of $J(M)$ by Remmert's Proper Mapping Theorem[3], that suffices for the proof.

Corollary 5.13 *If M is a compact Riemann surface of genus $g = 1$ then the Abel-Jacobi mapping is a biholomorphic mapping $w_a : M \longrightarrow J(M)$.*

Proof: Since $\dim J(M) = 1$ for a Riemann surface of genus $g = 1$ this is an immediate consequence of the preceding theorem, so no further proof is required.

5.5 Basic and Canonical Cross-Ratio Functions

In addition to the intrinsic cross-ratio function it is convenient to introduce, in parallel to the discussion of double differentials in the preceding chapter, the *basic cross-ratio function* associated to an arbitrary symmetric matrix $E = \{e_{kl}\}$ defined in terms of a basis ω_i for the holomorphic abelian differentials on M by

$$q_E(z_1, z_2; z_3, z_4) = q(z_1, z_2; z_3, z_4) \exp \sum_{k,l=1}^{g} e_{kl} w_k(z_1, z_2) w_l(z_3, z_4) \tag{5.43}$$

where $w_i(z_1, z_2) = \int_{z_2}^{z_1} \omega_i$ are the integrals of the holomorphic abelian differentials ω_i. These functions can be characterized as follows.

Theorem 5.14 *If M is a compact Riemann surface M of genus $g > 0$ with the universal covering space $\widetilde{M}$ and the covering translation group Γ, the basic cross-ratio functions of M are meromorphic functions $\tilde{q}(z_1, z_2; z_3, z_4)$ on the complex manifold $\widetilde{M} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M}$ that are characterized uniquely by the following properties:*
(i) *The function $\tilde{q}(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$ and simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, but is otherwise holomorphic and nowhere vanishing on $\widetilde{M}^4$; and $\tilde{q}(z_1, z_2; z_3, z_4) = 1$ if $z_1 = z_2$ or $z_3 = z_4$.*

[3]Remmert's Proper Mapping Theorem is discussed on page 423 of Appendix A.3

(ii) *The function* $\tilde q(z_1, z_2; z_3, z_4)$ *has the symmetries*

$$
\begin{aligned}
&\tilde q(z_1, z_2; z_3, z_4) = \tilde q(z_3, z_4; z_1, z_2) = \tilde q(z_2, z_1; z_4, z_3) = \\
&= \tilde q(z_2, z_1; z_3, z_4)^{-1} = \tilde q(z_1, z_2; z_4, z_3)^{-1}.
\end{aligned}
\tag{5.44}
$$

(iii) *The function* $\tilde q(z_1, z_2; z_3, z_4)$ *as a function of the variable* $z_1 \in \widetilde M$ *for any fixed points* $z_2, z_3, z_4 \in \widetilde M$ *is a relatively automorphic function for a factor of automorphy described under the canonical parametrization of flat factors of automorphy in terms of any basis* $\tau_j \in H_1(M)$ *by* $\rho_{s(z_3,z_4)}$ *for some vector* $s(z_3, z_4)$ *depending only on the parameters* z_3, z_4.

Proof: It is clear from the definition (5.43) of the basic cross-ratio function $q_E(z_1, z_2; z_3, z_4)$, in which E is required to be a symmetric matrix, and from the characterization of the intrinsic cross-ratio function in Theorem 5.6, that a basic cross-ratio function satisfies (i) and (ii), and that for any covering translation $T \in \Gamma$

$$
\begin{aligned}
q_E(Tz_1, z_2; z_3, z_4) &= q(Tz_1, z_2; z_3, z_4) \cdot \exp \sum_{k,l=1}^{g} e_{kl} w_k(Tz_1, z_2) w_l(z_3, z_4) \\
&= \rho_{t(z_3,z_4)}(T)\; q(z_1, z_2; z_3, z_4) \cdot \\
&\qquad \cdot \exp \sum_{k,l=1}^{g} e_{kl} \big(w_k(z_1, z_2) + \omega_k(T)\big) w_l(z_3, z_4) \\
&= \rho_{t(z_3,z_4)}(T)\; \Big(\exp \sum_{k,l=1}^{g} e_{kl} \omega_k(T) w_l(z_3, z_4)\Big) q_E(z_1, z_2; z_3, z_4);
\end{aligned}
$$

in particular for $T = T_j$

$$
\begin{aligned}
q_E(T_j z_1, z_2; z_3, z_4) &= \exp \Big(2\pi i t_j(z_3, z_4) + \sum_{k,l=1}^{g} e_{kl} \omega_{kj} w_l(z_3, z_4)\Big) q_E(z_1, z_2; z_3, z_4) \\
&= \rho_{s(z_3,z_4)}(T_j) q_E(z_1, z_2; z_3, z_4)
\end{aligned}
$$

where

$$
s_j(z_3, z_4) = t_j(z_3, z_4) + \frac{1}{2\pi i} \sum_{k,l=1}^{g} e_{kl} \omega_{kj} w_l(z_3, z_4),
\tag{5.45}
$$

so a basic cross-ratio function also satisfies (iii).

Conversely suppose that $\tilde q(z_1, z_2; z_3, z_4)$ is an arbitrary meromorphic function on $\widetilde M \times \widetilde M \times \widetilde M \times \widetilde M$ that satisfies (i), (ii), and (iii). It follows from (i) and Theorem 5.6 (i) that

$$
\tilde q(z_1, z_2; z_3, z_4) = q(z_1, z_2; z_3, z_4) \exp h(z_1, z_2; z_3, z_4)
$$

for a holomorphic function $h(z_1, z_2; z_3, z_4)$ on the simply connected complex manifold $\widetilde{M} \times \widetilde{M} \times \widetilde{M} \times \widetilde{M}$, since the two cross-ratio functions have the same zeros and poles. In addition it follows that $\exp h(z_1, z_1; z_3, z_4) = 1$, and consequently $h(z_1, z_1; z_3, z_4) = 2\pi i n$ for some integer n; so after replacing the function $h(z_1, z_2; z_3, z_4)$ by $h(z_1, z_2; z_3, z_4) - 2\pi i n$ it can be assumed that $h(z_1, z_1; z_3, z_4) = 0$. The functions $\tilde{q}(z_1, z_2; z_3, z_4)$ and $q(z_1, z_2; z_3, z_4)$ both satisfy the symmetries of part (ii), so $\exp h(z_1, z_2; z_3, z_4)$ does as well. Therefore $h(z_1, z_2; z_3, z_4) - h(z_3, z_4; z_1, z_2) = 2\pi i n_1$ for some integer n_1; and setting $z_1 = z_2$ and $z_3 = z_4$ shows that actually $n_1 = 0$. Furthermore $h(z_1, z_2; z_3, z_4) + h(z_2, z_1; z_3, z_4) = 2\pi i n_2$ for another integer n_2; and setting $z_1 = z_2$ shows that $n_2 = 0$ as well. It follows from these observations that the function $h(z_1, z_1; z_3, z_4)$ satisfies

$$\begin{aligned} h(z_1, z_2; z_3, z_4) &= h(z_3, z_4; z_1, z_2) = h(z_2, z_1; z_4, z_3) \\ &= -h(z_2, z_1; z_3, z_4) = -h(z_1, z_2; z_4, z_3). \end{aligned}$$

For any covering translation $T \in \Gamma$ it follows from (iii) that

$$h(Tz_1, z_2; z_3, z_4) = h(z_1, z_2; z_3, z_4) + g(T; z_3, z_4)$$

for some holomorphic function $g(T; z_3, z_4)$ of the variables $z_3, z_4 \in \widetilde{M}$; thus $h(z_1, z_2; z_3, z_4)$ as a function of the variable z_1 alone is a holomorphic abelian integral, and since it vanishes when $z_1 = z_2$ it can be written in terms of a basis $w_k(z, z_2)$ for the holomorphic abelian integrals as

$$h(z_1, z_2; z_3, z_4) = \sum_{k=1}^{g} e_k(z_2, z_3, z_4) w_k(z_1, z_2)$$

for some uniquely determined functions $e_k(z_2, z_3, z_4)$, which consequently must be holomorphic functions of the variables z_2, z_3, z_4. In that case

$$\begin{aligned} g(T; z_3, z_4) &= h(Tz_1, z_2; z_3, z_4) - h(z_1, z_2; z_3, z_4) \\ &= \sum_{k=1}^{g} e_k(z_2, z_3, z_4)\omega_k(T), \end{aligned}$$

and since this is the case for all $T \in \Gamma$ it follows that the coefficients $e_k(z_2, z_3, z_4)$ must be independent of the variable z_2. Upon interchanging the two pairs of variables it follows from the symmetry of the function $h(z_1, z_2; z_3, z_4)$ that

$$\sum_{k=1}^{g} e_k(z_3, z_4) w_k(z_1, z_2) = \sum_{k=1}^{g} e_k(z_1, z_2) w_k(z_3, z_4)$$

and consequently that $e_k(z_3, z_4)$ is a holomorphic abelian integral in each variable as well; and since this integral vanishes when $z_3 = z_4$ it follows that $e_k(z_3, z_4) = \sum_{l=1}^{g} e_{kl} w_l(z_3, z_4)$, which shows that the function $\tilde{q}(z_1, z_2; z_4, z_3)$ is a basic cross-ratio function as defined in (5.43) and thereby concludes the proof.

Corollary 5.15 *Let Ω be the period matrix and P be the intersection matrix of a compact Riemann surface M of genus $g > 0$ in terms of any arbitrary bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$; and let $\tilde{w}(z, a) = \{w_i(z, a)\}$ be the column vector with entries the integrals $w_i(z, a) = \int_a^z \omega_i$. The basic cross-ratio function $q_E(z_1, z_2; z_3, z_4)$ for a symmetric matrix E is a relatively automorphic function for the flat factor of automorphy described under the canonical parametrization of flat factors of automorphy in terms of these bases by $\rho_{s(z_3,z_4)}$ for the vector*

$$s(z_3, z_4) = i\,{}^t\Big({}^t\overline{\Omega}\,{}^tG - \frac{1}{2\pi}\,{}^t\Omega\,{}^tE\Big)\tilde{w}(z_3, z_4) \tag{5.46}$$

where $G = {}^tH^{-1}$ for the positive definite symmetric matrix $H = i\Omega P\,{}^t\overline{\Omega}$.

Proof: The intrinsic cross-ratio function is a relatively automorphic function for the flat factor of automorphy $\rho_t(z_3, z_4)$ for the vector $t(z_3, z_4)$ given by (5.27), and the basic cross-ratio function associated to a symmetric matrix $E = {}^tE$ is a relatively automorphic function for the flat factor of automorphy $\rho_s(z_3, z_4)$ for the vector $s(z_3, z_4)$ given by (5.45); so

$$\begin{aligned} s(z_3, z_4) &= t(z_3, z_4) + \frac{1}{2\pi i}\,{}^t\Omega E\tilde{w}(z_3, z_4) \\ &= i\,{}^t\overline{\Omega}\,{}^tG\tilde{w}(z_3, z_4) + \frac{1}{2\pi i}\,{}^t\Omega E\tilde{w}(z_3, z_4), \end{aligned}$$

which reduces to (5.46), and that suffices to conclude the proof.

The characterization of basic cross-ratio functions in Theorem 5.14 is more appealing than the characterization of intrinsic cross-ratio functions in Theorem 5.6 in that the factor of automorphy is stated in a rather more general form; the explicit description of the factor of automorphy in Corollary 5.15 shows that the intrinsic cross-ratio function is characterized among the basic cross-ratio functions by having a factor of automorphy described by the complex conjugate period matrix alone. The basic double differential of the second kind on the Riemann surface M described by a symmetric matrix E can be expressed in terms of the basic cross-ratio function associated to that matrix, in an extension of Theorem 5.8; indeed it is evident from the definitions (4.56) and (5.43) that

$$\begin{aligned} \frac{\partial^2}{\partial z \partial \zeta} \log q_E(z, a; \zeta, b) dz d\zeta &= \frac{\partial^2}{\partial z \partial \zeta} \log\Big(q(z, a; \zeta, b) \cdot \exp \sum_{k,l=1}^{g} e_{kl} w_k(z, a) w_l(\zeta, b)\Big) \\ &= \mu_M(z, \zeta) + \sum_{k,l=1}^{g} e_{kl}\omega_k(z)\omega_l(\zeta) \qquad (5.47) \\ &= \mu_{M,E}(z, \zeta) \end{aligned}$$

for any symmetric matrix E. In particular the canonical double differential of the second kind $\hat{\mu}(z, \zeta)$ on a marked Riemann surface of genus $g > 0$, the basic double differential with the properties given in Theorem 4.23, can be expressed in terms of the corresponding basic cross-ratio function, called the *canonical cross-ratio function* on the marked Riemann surface; this function is denoted by $\hat{q}(z_1, z_2; z_3, z_4)$, so

$$\frac{\partial^2}{\partial z \partial \zeta} \log \hat{q}(z, a; \zeta, b) dz d\zeta = \hat{\mu}_M(z, \zeta). \tag{5.48}$$

The canonical cross-ratio function can be characterized as follows.

Theorem 5.16 *On a marked compact Riemann surface M of genus $g > 0$, with the marking described by generators $A_j, B_j \in \Gamma$, the canonical cross-ratio function $\hat{q}(z_1, z_2; z_3, z_4)$ is characterized by the following properties.*
(i) *The function $\hat{q}(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$ and simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, but is otherwise holomorphic and nowhere vanishing on $\widetilde{M}^4$; and $\hat{q}(z_1, z_2; z_3, z_4) = 1$ if $z_1 = z_2$ or $z_3 = z_4$.*
(ii) *The function $\hat{q}(z_1, z_2; z_3, z_4)$ has the symmetries*

$$\begin{aligned} &\hat{q}(z_1, z_2; z_3, z_4) = \hat{q}(z_3, z_4; z_1, z_2) = \hat{q}(z_2, z_1; z_4, z_3) = \\ &\qquad = \hat{q}(z_2, z_1; z_3, z_4)^{-1} = \hat{q}(z_1, z_2; z_4, z_3)^{-1}. \end{aligned} \tag{5.49}$$

(iii) *For any fixed points z_2, z_3, z_4*

$$\begin{aligned} \hat{q}(A_j z_1, z_2; z_3, z_4) &= \hat{q}(z_1, z_2; z_3, z_4), \\ \hat{q}(B_j z_1, z_2; z_3, z_4) &= \hat{q}(z_1, z_2; z_3, z_4) \exp 2\pi i\, w_j(z_3, z_4) \end{aligned} \tag{5.50}$$

where $w_j(z_3, z_4) = \int_{z_4}^{z_3} \omega_j$ are the integrals of the canonical abelian differentials ω_j on the marked surface.

Proof: The intersection matrix of the surface M in terms of the basis for $H_1(M)$ described by the generators $A_j, B_j \in \Gamma$ of the marking of M is the basic skew-symmetric matrix J, and by Theorem 4.23 the period matrix Ω of M in terms of this basis and of the canonical holomorphic abelian differentials ω_i on the marked surface M is the matrix $\Omega = (\,\mathrm{I} \quad Z)$ where $Z = X + iY \in \mathfrak{H}_g$, the Siegel upper half-space of rank g. By Theorem 4.23 the canonical double differential of the second kind on M is the basic double differential described by the matrix $E = \pi Y^{-1}$. The basic cross-ratio function associated to this matrix E is a relatively automorphic function for the canonically parametrized factor of automorphy described by the vector $s(z_3, z_4)$ given explicitly by (5.46). Since $G = \frac{1}{2} Y^{-1} = \frac{1}{2\pi} E$, as in the proof of Theorem 4.23, it follows that

$$\begin{aligned} i\Big({}^t\overline{\Omega}\, {}^tG - \frac{1}{2\pi}\, {}^t\Omega E\Big) &= \frac{i}{2\pi} \begin{pmatrix} \mathrm{I} \\ X - i\,Y \end{pmatrix} E - \frac{i}{2\pi} \begin{pmatrix} \mathrm{I} \\ X + i\,Y \end{pmatrix} E \\ &= \begin{pmatrix} 0 \\ \mathrm{I} \end{pmatrix}; \end{aligned}$$

consequently $s(z_3, z_4) = {}^t\begin{pmatrix} 0 & \mathrm{I}\end{pmatrix}\tilde{w}(z_3, z_4)$ so that

$$\rho_{s(z_3,z_4)}(A_j) = 1, \qquad \rho_{s(z_3,z_4)}(B_j) = \exp 2\pi i w_j(z_3, z_4)$$

and therefore the basic cross-ratio function associated to this matrix E satisfies (iii). Since this cross-ratio function is characterized uniquely by (i), (ii) and (iii), that suffices to conclude the proof.

The *Green's* cross-ratio function is quite useful for some purposes, even though it is not a holomorphic function in all variables; it is defined by

$$\check{q}(z_1, z_2; a, b) = q(z_1, z_2; a, b) \exp 2\pi \sum_{k,l=1}^{g} g_{kl} w_k(z_1, z_2)\overline{w_l(a, b)} \tag{5.51}$$

in terms of bases $\omega_i(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials and $\tau_j \in H_1(M)$ for the homology of a compact Riemann surface M of genus $g > 0$, where Ω is the period matrix and P is the intersection matrix of M, and $G = {}^tH^{-1}$ for the positive definite symmetric matrix $H = i\,\overline{\Omega} P\, {}^t\Omega$. Although the definition (5.51) is expressed in terms of these bases it follows just as in the demonstration of the invariance of (5.17) that the function $\check{q}(z_1, z_2; a, b)$ itself is intrinsically defined on M, independent of the choice of bases.

Theorem 5.17 *On a compact Riemann surface M of genus $g > 0$ the Green's cross-ratio function $\check{q}(z_1, z_2; a, b)$ is a meromorphic function of the variables $(z_1, z_2) \in \widetilde{M} \times \widetilde{M}$ and a conjugate meromorphic function of the variables $(a, b) \in \widetilde{M} \times \widetilde{M}$ with the following properties:*
(i) *The function $\check{q}(z_1, z_2; a, b)$ has simple zeros along the subvarieties $z_1 = Ta$ and $z_2 = Tb$ and simple poles along the subvarieties $z_1 = Tb$ and $z_2 = Ta$ for all $T \in \Gamma$, but no other zeros or poles in the variables (z_1, z_2), for any fixed point (a, b); and it takes the value 1 if $z_1 = z_2$ or $a = b$.*
(ii) *The function $\check{q}(z_1, z_2; a, b)$ has the symmetries*

$$\check{q}(z_1, z_2; a, b) = \check{q}(z_2, z_1; a, b)^{-1} = \check{q}(z_1, z_2; b, a)^{-1}.$$

(iii) *The function $\check{q}(z_1, z_2; a, b)$ as a function of the variable $z_1 \in \widetilde{M}$ for any fixed points $z_2, a, b \in \widetilde{M}$ is a meromorphic relatively automorphic function for the canonically parametrized factor of automorphy $\rho_{r(a,b)}(T)$ described by the real vector $r(a, b) \in \mathbb{R}^{2g}$ that for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ is given by*

$$r(a, b) = 2\Im\Big({}^t\Omega\, G\, \overline{\tilde{w}(a, b)}\Big) = \frac{1}{2\,i}\Big({}^t\Omega\, G\, \overline{\tilde{w}(a, b)} - {}^t\overline{\Omega}\, \overline{G}\, \tilde{w}(a, b)\Big) \tag{5.52}$$

in which Ω is the period matrix, P is the intersection matrix, $G = {}^tH^{-1}$ for the positive definite symmetric matrix $H = i\,\Omega P\, {}^t\overline{\Omega}$ in terms of these bases, and $\Im(z) = y$ is the imaginary part of the complex number $z = x + i\,y$; hence

$$|\rho_{r(a,b)}(T)| = 1 \quad \textit{for all} \quad T \in \Gamma. \tag{5.53}$$

Proof: Properties (i) and (ii) are immediate consequences of the corresponding properties of the intrinsic cross-ratio function in Theorem 5.6 and of the definition (5.51) of the Green's cross-ratio function, since the exponential is a nowhere vanishing function that is holomorphic in the variables z_1, $z_2 \in \widetilde{M}$ and conjugate meromorphic in the variables a, $b \in \widetilde{M}$, and $w_k(z_1, z_2) = -w_k(z_2, z_1)$. For the generator $T_j \in \Gamma$ corresponding to the basis element $\tau_j \in H_1(M)$ it follows from the definition (5.51) and Theorem 5.6 (iii) that

$$\begin{aligned}
\check{q}(T_j z_1, z_2; a, b) &= q(T_j z_1, z_2; a, b) \cdot \exp 2\pi \sum_{k,l=1}^{g} g_{kl} w_k(T_j z_1, z_2) \overline{w_l(a, b)} \\
&= \rho_{t(a,b)}(T_j) q(z_1, z_2; a, b) \cdot \\
&\quad \cdot \exp 2\pi \sum_{k,l=1}^{g} g_{kl} \big(w_k(z_1, z_2) + \omega_{kj} \big) \overline{w_l(a, b)} \\
&= \check{q}(z_1, z_2; a, b) \cdot \exp 2\pi \sum_{k,l=1}^{g} \Big(- g_{lk} \overline{\omega_{kj}} w_l(a, b) + g_{kl} \omega_{kj} \overline{\omega_l(a, b} \Big) \\
&= \check{q}(z_1, z_2; a, b) \exp 2\pi \sum_{k,l=1}^{g} \Big(- \overline{g_{kl} \omega_{kj}} w_l(a, b) + g_{kl} \omega_{kj} \overline{w_l(a, b)} \Big) \\
&= \check{q}(z_1, z_2; a, b) \exp 2\pi i \; r_j(a, b) = \rho_{r(a,b)}(T_j) \check{q}(z_1, z_2; a, b),
\end{aligned}$$

and therefore $q(T z_1, z_2; a, b) = \rho_{r(a,b)}(T) \check{q}(z_1, z_2; a, b)$ for all $T \in \Gamma$. Since $r(a, b)$ is a real vector $|\rho_{r(a,b)}(T_j)| = 1$ for each generator $T_j \in \Gamma$ and therefore $|\rho_{r(a,b)}(T)| = 1$ for all $T \in \Gamma$; and that suffices for the proof.

Chapter 6

Abelian Factors of Automorphy

6.1 Definitions

A holomorphic factor of automorphy for the action of the covering translation group Γ on the universal covering space $\widetilde{M}$ of a compact Riemann surface M of genus $g > 0$ represents a holomorphic line bundle over the surface M, as in Theorem 3.11; and topologically trivial holomorphic line bundles can be represented by flat factors of automorphy, as in Theorem 3.13. It will be demonstrated in this chapter that all holomorphic line bundles can be represented by intrinsically and explicitly defined factors of automorphy. These factors of automorphy are modeled on those that appear in the classical theory of elliptic functions[1]. The sigma function of Weierstrass for a one-dimensional complex torus $M = \mathbb{C}/\mathcal{L}$ is a holomorphic function $\sigma(z)$ on the complex plane $\mathbb{C}$ that transforms under translations $T \in \mathcal{L}$ by functional equations of the form $\sigma(Tz) = \sigma(z) \cdot \exp \sigma_T(z)$ for some linear functions $\sigma_T(z)$; hence $\sigma(z)$ is a holomorphic relatively automorphic function for the factor of automorphy $\exp \sigma_T(z)$. The Jacobian theta function $\vartheta(z)$ for the complex torus M is another example of a holomorphic relatively automorphic function for a factor of automorphy given by exponentials of linear functions. Linear functions on a complex torus M are holomorphic abelian integrals on M; so the analogues of these classical factors of automorphy for a general compact Riemann surface M of genus $g > 1$ are factors of automorphy of the form

$$\zeta(T, z) = \exp 2\pi i\, \sigma(T, z), \tag{6.1}$$

where for each covering translation $T \in \Gamma$ the function $\sigma(T, z)$ is a holomorphic abelian integral on the universal covering space $\widetilde{M}$ of the surface M. This

[1] For the properties of the classical elliptic functions see for instance Whittaker and Watson, *Modern Analysis*, (Cambridge, 1902).

integral can be written

$$\sigma(T,z) = \sigma_0(T) + \sum_{i=1}^{g} \sigma_i(T) w_i(z,a) \tag{6.2}$$

for some mappings $\sigma_i : \Gamma \longrightarrow \mathbb{C}$ for $0 \le i \le g$, where $w_i(z,a) = \int_a^z \omega_i$ in terms of a basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M and of a base point $a \in \widetilde{M}$.

Theorem 6.1 *The functions $\zeta(T,z)$ form a factor of automorphy for the action of the covering translation group Γ on the universal covering space $\widetilde{M}$ of a compact Riemann surface M of genus $g > 0$ if and only if*
(i) *the mappings σ_i are group homomorphisms for $1 \le i \le g$ and*
(ii) *the mapping σ_0 satisfies the condition that*

$$\sigma_0(S) + \sigma_0(T) - \sigma_0(ST) + \sum_{i=1}^{g} \sigma_i(S)\omega_i(T) \in \mathbb{Z} \tag{6.3}$$

for all $S, T \in \Gamma$, where $\omega_i \in \mathrm{Hom}(\Gamma, \mathbb{C})$ are the period classes of the holomorphic abelian differentials $\omega_i(z) = dw_i(z,a)$.

Proof: The functions $\zeta(T,z)$ form a factor of automorphy if and only if

$$\zeta(S,Tz)\zeta(T,z)\zeta(ST,z)^{-1} = 1$$

for all $S, T \in \Gamma$; and when $\zeta(T,z)$ has the form (6.1) that is equivalent to the condition that

$$\sigma(S,Tz) + \sigma(T,z) - \sigma(ST,z) \in \mathbb{Z} \tag{6.4}$$

for all $S, T \in \Gamma$. Hence to prove the theorem it suffices to show that condition (6.4) is equivalent to conditions (i) and (ii) of the theorem. For this purpose note that in terms of the more explicit form (6.2) for the functions $\sigma(T,z)$

$$\begin{aligned} \sigma(S,Tz) &+ \sigma(T,z) - \sigma(ST,z) \\ &= \sigma_0(S) + \sigma_0(T) - \sigma_0(ST) \\ &\quad + \sum_{i=1}^{g} \Big(\sigma_i(S) w_i(Tz,a) + \sigma_i(T) w_i(z,a) - \sigma_i(ST) w_i(z,a)\Big) \\ &= \sigma_0(S) + \sigma_0(T) - \sigma_0(ST) + \sum_{i=1}^{g} \sigma_i(S)\omega_i(T) \\ &\quad + \sum_{i=1}^{g} \Big(\sigma_i(S) + \sigma_i(T) - \sigma_i(ST)\Big) w_i(z,a). \end{aligned} \tag{6.5}$$

If (6.4) holds then (6.5) is an integer for all $S, T \in \Gamma$; and since the functions $w_i(z, a)$ are linearly independent for all $S, T \in \mathbb{Z}$ it must be the case that

$$\sum_{i=1}^{g} \Big(\sigma_i(S) + \sigma_i(T) - \sigma_i(ST) \Big) = 0 \quad \text{for all} \quad S, T \in \mathbb{Z}, \tag{6.6}$$

so the mappings σ_i for $1 \le i \le g$ are homomorphisms as in condition (i). Substituting (6.6) into (6.5) reduces it to (6.3) which is thus an integer yielding condition (ii) of the theorem. Conversely if (i) and (ii) are satisfied the mappings σ_i are group homomorphisms for $1 \le i \le g$ so (6.6) holds and hence (6.5) reduces to (6.4). That suffices for the proof.

The factors of automorphy $\zeta(T, z)$ of the preceding theorem are called *abelian factors of automorphy*. It is evident that the product of any two abelian factors of automorphy again has the form of an abelian factor of automorphy, as does the inverse $\zeta(T, z)^{-1}$ of an abelian factor of automorphy; the collection of abelian factors of automorphy thus form a multiplicative group, which is called the *group of abelian factors of automorphy* on the Riemann surface M and is denoted by $\mathcal{A}(M)$. The product of an abelian factor of automorphy $\zeta(T, z)$ and a representation $\rho \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ of the fundamental group Γ of the Riemann surface M clearly also is an abelian factor of automorphy; consequently, in view of Corollary 3.9, if an abelian factor of automorphy $\zeta(T, z)$ describes a holomorphic line bundle ζ over the Riemann surface M then all holomorphic line bundles over M of characteristic class $c(\zeta)$ can be described by abelian factors of automorphy of the form $\rho(T)\zeta(T, z)$ for suitable representations $\rho(T)$ of the group Γ.

Condition (6.3) really has the form of a condition in the cohomology group[2] $H^2(\Gamma, \mathbb{C})$ of the covering translation group Γ with coefficients in the field $\mathbb{C}$ on which Γ is viewed as acting trivially. The function $f(S, T) = \sum_{i=1}^{g} \sigma_i(S)\omega_i(T)$ of pairs (S, T) of elements of the group Γ actually is a two-cocycle in $Z^2(\Gamma, \mathbb{C})$ and the function $\delta\lambda_0(S, T) = \lambda_0(S) + \lambda_0(T) - \lambda_0(ST)$ is a two-cocycle that is the coboundary of the one-cochain $\lambda_0(T) \in C^1(\Gamma, \mathbb{C})$. In these terms (6.3) is just the condition that the cocycle $f(S, T)$ is cohomologous to an integral cocycle, a cocycle in the subgroup $Z^2(\Gamma, \mathbb{Z})$. That is one approach to the more detailed study of abelian factors of automorphy. However the approach that will be followed here is rather more concrete and explicit, and does not require any use of the cohomological machinery.

The homomorphisms $\sigma_i \in \operatorname{Hom}(\Gamma, \mathbb{C})$ can be represented as the period mappings of the holomorphic abelian differentials $\omega_j(z)$ on M and their complex conjugates, as discussed in Section 3.1; consequently it can be supposed that

$$\sigma_i(T) = \sum_{j=1}^{g} \Big(u_{ij}\overline{\omega_j(T)} + v_{ij}\omega_j(T) \Big) \quad \text{for} \quad 1 \le i \le g \tag{6.7}$$

[2] Some general properties of the cohomology of groups, and some specific properties for the special case of groups that are the fundamental groups of orientable surfaces, can be found in Appendix E.

for some $g \times g$ complex matrices $U = \{u_{ij}\}$ and $V = \{v_{ij}\}$ and all $T \in \Gamma$, where $\omega_j \in \mathrm{Hom}(\Gamma, \mathbb{C})$ are the period classes of the holomorphic abelian differentials $\omega_j(z)$. Then (6.2) takes the form

$$\sigma(T,z) = \sigma_0(T) + \sum_{i,j=1}^{g} \Big(u_{ij}\overline{\omega_j(T)} + v_{ij}\omega_j(T)\Big) w_i(z,a), \tag{6.8}$$

or in matrix notation

$$\sigma(T,z) = \sigma_0(T) + {}^t\tilde{w}(z,a)\Big(U\overline{\omega(T)} + V\omega(T)\Big) \tag{6.9}$$

where $\tilde{w}(z,a) = \{w_i(z,a)\}$ is the vector of abelian integrals and $\omega(T) = \{\omega_j(T)\}$ is the vector of the periods of these integrals, both viewed as column vectors in $\mathbb{C}^g$ as on page 66. In these terms the abelian factor of automorphy (6.1) has the form

$$\zeta(T,z) = \exp 2\pi i\Big(\sigma_0(T) + {}^t\tilde{w}(z,a)\big(U\overline{\omega(T)} + V\omega(T)\big)\Big), \tag{6.10}$$

which can be viewed as the product

$$\zeta(T,z) = \lambda_0(T)\xi_{U,V,a}(T,z) \tag{6.11}$$

of an *auxiliary mapping* for that factor of automorphy, the mapping

$$\lambda_0 : \Gamma \longrightarrow \mathbb{C}^* \quad \text{given by} \quad \lambda_0(T) = \exp 2\pi i\sigma_0(T), \tag{6.12}$$

and a *root factor* for that factor of automorphy, given by

$$\xi_{U,V,a}(T,z) = \exp 2\pi i\, {}^t\tilde{w}(z,a)\Big(U\overline{\omega(T)} + V\omega(T)\Big). \tag{6.13}$$

The condition in Theorem 6.1 that (6.10) is an abelian factor of automorphy then can be expressed alternatively as follows.

Corollary 6.2 *There is an abelian factor of automorphy with the root factor $\xi_{U,V,a}(T,z)$ if and only if there is a mapping $\lambda_0 : \Gamma \longrightarrow \mathbb{C}^*$ such that*

$$\lambda_0(ST) = \lambda_0(S)\lambda_0(T)\phi_{U,V}(S,T) \tag{6.14}$$

for all $S, T \in \Gamma$, where $\phi_{U,V} : \Gamma \times \Gamma \longrightarrow \mathbb{C}^*$ is the mapping defined by

$$\phi_{U,V}(S,T) = \exp 2\pi i\, {}^t\omega(T)\Big(U\overline{\omega(S)} + V\omega(S)\Big); \tag{6.15}$$

the mapping λ_0 then is an auxiliary mapping for this abelian factor of automorphy.

Proof: Condition (i) of Theorem 6.1 is automatically satisfied by the explicit form of the root factor. Condition (ii) can be rewritten in terms of the mapping $\lambda_0(T) = \exp 2\pi i\sigma_0(T)$ as

$$\lambda_0(S)\lambda_0(T)\lambda_0(ST)^{-1} \exp 2\pi i \sum_{i=1}^{g} \sigma_i(S)\omega_i(T) = 1, \tag{6.16}$$

which is just (6.14) since $\sigma_i(S)$ has the form (6.7); and that suffices for the proof.

The auxiliary mapping (6.15) plays a significant role in the subsequent discussion. It is useful to note here that

$$\begin{aligned}
\phi_{U,V}(S_1S_2,T) &= \phi_{U,V}(S_1,T)\ \phi_{U,V}(S_2,T),\\
\phi_{U,V}(S,T_1T_2) &= \phi_{U,V}(S,T_1)\ \phi_{U,V}(S,T_2), \quad \text{and}\\
\phi_{U,V}(S^{-1},T) &= \phi_{U,V}(S,T^{-1}) = \phi_{U,V}(S,T)^{-1}
\end{aligned} \tag{6.17}$$

for all $S,T \in \Gamma$, since $\omega(ST) = \omega(S) + \omega(T)$ and $\omega(T^{-1}) = -\omega(T)$.

6.2 Root Factors

If T_i are generators of the covering translation group Γ of a Riemann surface M of genus $g > 0$ then any mapping $\lambda_0 : \Gamma \longrightarrow \mathbb{C}^*$ that satisfies (6.14) is determined fully by the values $\lambda_0(T_i)$ and $\lambda_0(T_i^{-1})$, since any element in Γ is a product of the generators T_i and their inverses and the value of the mapping λ_0 on a product of two elements of Γ is determined by its values on those two elements through the product formula (6.14). Actually since $\phi_{U,V}(I,I) = 1$ for the identity $I \in \Gamma$ it follows from the product formula (6.14) that $\lambda_0(I)\lambda_0(I) = \lambda_0(I)$ hence $\lambda_0(I) = 1$; and then by the product formula again $1 = \lambda_0(TT^{-1}) = \lambda_0(T)\lambda_0(T^{-1})\phi_{U,V}(T,T^{-1})$ for any $T \in \Gamma$ while $\phi_{U,V}(T,T^{-1}) = \phi_{U,V}(T,T)^{-1}$ by (6.17) so that

$$\lambda_0(T^{-1}) = \lambda_0(T)^{-1}\phi_{U,V}(T,T) \quad \text{for any } T \in \Gamma. \tag{6.18}$$

Thus the values $\lambda_0(T_i)$ already fully determine the mapping λ_0.

Now consider the free semigroup $\mathcal{F}^*$ generated by the symbols T_i and T_i^{-1}, the set of all words $T^* = T_{i_1}^{\pm1}T_{i_2}^{\pm1}\cdots T_{i_n}^{\pm1}$ where the product of two words T_1^* and T_2^* is the word $T_1^*T_2^*$. The natural mapping that associates to any word $T^* = T_{i_1}^{\pm1}T_{i_2}^{\pm1}\cdots T_{i_n}^{\pm1} \in \mathcal{F}^*$ the element $T = T_{i_1}^{\pm1}T_{i_2}^{\pm1}\cdots T_{i_n}^{\pm1} \in \Gamma$ is a surjective semigroup homomorphism $\pi : \mathcal{F}^* \longrightarrow \Gamma$. The period classes ω of the holomorphic abelian integrals on M are group homomorphisms $\omega : \Gamma \longrightarrow \mathbb{C}$, which can be lifted to semigroup homomorphisms $\widetilde{\omega}^* : \mathcal{F}^* \longrightarrow \mathbb{C}$ by setting $\widetilde{\omega}^*(T^*) = \omega(\pi T^*)$; the mappings $\phi_{U,V}(S,T)$ defined by (6.15) can be lifted correspondingly to mappings $\widetilde{\phi}^*_{U,V} : \mathcal{F}^* \times \mathcal{F}^* \longrightarrow \mathbb{C}$ defined as in (6.15) but in terms of the mappings $\widetilde{\omega}^*$, and the liftings satisfy the analogues of (6.17). Since $\mathcal{F}^*$ is the free semigroup generated by T_i and T_i^{-1} there is a mapping $\widetilde{\lambda}_0^* : \mathcal{F}^* \longrightarrow \mathbb{C}^*$ satisfying the analogue of the product formula (6.14) beginning with any chosen values $\widetilde{\lambda}_0^*(T_i) \in \mathbb{C}^*$ and $\widetilde{\lambda}_0^*(T_i^{-1})$ and extending this to any word in $\mathcal{F}^*$ by applying successively the product formula (6.14) in the semigroup $\mathcal{F}^*$;

for instance if $T = T_{i_1}T_{i_2}^{-1}T_{i_3}T_{i_4}\cdots$ then

$$
\begin{aligned}
\widetilde{\lambda}_0^*(T) &= \widetilde{\lambda}_0^*(T_{i_1})\widetilde{\lambda}_0^*(T_{i_2}^{-1}T_{i_3}T_{i_4}\cdots)\cdot\widetilde{\phi}_{U,V}^*(T_{i_1},T_{i_2}^{-1}T_{i_3}T_{i_4}\cdots) \\
&= \widetilde{\lambda}_0^*(T_{i_1})\widetilde{\lambda}_0^*(T_{i_2}^{-1})\widetilde{\lambda}_0^*(T_{i_3}T_{i_4}\cdots) \\
&\quad\cdot\widetilde{\phi}_{U,V}^*(T_{i_1},T_{i_2}^{-1}T_{i_3}T_{i_4}\cdots)\widetilde{\phi}_{U,V}^*(T_{i_2}^{-1},T_{i_3}T_{i_4}\cdots) \\
&= \cdots .
\end{aligned}
\tag{6.19}
$$

This is one way of applying the product formula (6.14) to calculate the value of $\widetilde{\lambda}_0^*(T)$; of course there are many different ways in which the terms of a product can be grouped to apply the product formula, but the result of the calculation actually is independent of the particular grouping chosen, as a consequence of the following.

Lemma 6.3 *If a mapping $\widetilde{\lambda}_0^* : \mathcal{F}^* \longrightarrow \mathbb{C}^*$ satisfies the product formula* (6.14) *for the products R^*S^* and S^*T^* of the words $R^*, S^*, T^* \in \mathcal{F}^*$ then $\widetilde{\lambda}_0^*(R^*S^*T^*)$ can be defined equivalently by*

$$
\widetilde{\lambda}_0^*\big(R^*S^*T^*\big) = \widetilde{\lambda}_0^*\big(R^*\cdot(S^*T^*)\big) = \widetilde{\lambda}_0^*\big((R^*S^*)\cdot T^*\big).
\tag{6.20}
$$

Proof: Applying the product formula (6.14) directly would yield the values

$$
\begin{aligned}
\widetilde{\lambda}_0^*\big(R^*\cdot(S^*T^*)\big) &= \widetilde{\lambda}_0^*(R^*)\widetilde{\lambda}_0^*(S^*T^*)\widetilde{\phi}_{U,V}^*(R^*,S^*T^*) \\
&= \widetilde{\lambda}_0^*(R^*)\cdot\widetilde{\lambda}_0^*(S^*)\widetilde{\lambda}_0^*(T^*)\widetilde{\phi}_{U,V}^*(S^*,T^*)\cdot\widetilde{\phi}_{U,V}^*(R^*,S^*T^*)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\lambda}_0^*\big((R^*S^*)\cdot T^*\big) &= \widetilde{\lambda}_0^*(R^*S^*)\widetilde{\lambda}_0^*(T^*)\widetilde{\phi}_{U,V}^*(R^*S^*,T^*) \\
&= \widetilde{\lambda}_0^*(R^*)\widetilde{\lambda}_0^*(S^*)\widetilde{\phi}_{U,V}^*(R^*,S^*)\cdot\widetilde{\lambda}_0^*(T^*)\widetilde{\phi}_{U,V}^*(R^*S^*,T^*);
\end{aligned}
$$

so (6.20) is equivalent to

$$
\widetilde{\phi}_{U,V}^*(S^*,T^*)\widetilde{\phi}_{U,V}^*(R^*,S^*T^*) = \widetilde{\phi}_{U,V}^*(R^*,S^*)\widetilde{\phi}_{U,V}^*(R^*S^*,T^*),
\tag{6.21}
$$

which follows immediately from (6.17), and that suffices for the proof.

What is particularly relevant is the following consequence of the preceding Lemma 6.3.

Lemma 6.4 *The mapping $\widetilde{\lambda}_0^* : \mathcal{F}^* \longrightarrow \mathbb{C}^*$ satisfies the product formula* (6.14).

Proof: If $T^* = T_1^*T_2^* \in \mathcal{F}^*$ it follows from the preceding lemma that the value $\widetilde{\lambda}_0^*(T^*)$ can be calculated by grouping the successive application of the product formula (6.14) in any way, the analogue of the corresponding argument for the consequence of the associative law for multiplicatioin. In particular the values of $\widetilde{\lambda}_0^*(T_1^*)$ and $\widetilde{\lambda}_0^*(T_2^*)$ can be calculated separately and then the value $\widetilde{\lambda}_0^*(T^*)$ can be calculated by applying the product formula to the product $T_1^*T_2^*$, yielding $\widetilde{\lambda}_0^*(T_1^*T_2^*) = \widetilde{\lambda}_0^*(T_1^*) \cdot \widetilde{\lambda}_0^*(T_2^*) \cdot \widetilde{\phi}_{U,V}(T_1^*, T_2^*)$. That suffices for the proof.

An alternative proof of the preceding Lemma 6.4 follows by direct calculation from the obvious general form of the product formula (6.19) when the mappings $\widetilde{\phi}_{U,V}^*(S^*, T^*)$ are decomposed by an application of (6.17) into products of terms $\widetilde{\phi}_{U,V}(T_i^*, T_j^*)$ in which T_i^*, T_j^* are generators of the free semigroup $\mathcal{F}^*$.

The free group $\mathcal{F}$ on the symbols T_i is the quotient of the free semigroup $\mathcal{F}^*$ upon identifying the words $T_i^*(T_i^*)^{-1}$ and $(T_i^*)^{-1}T_i^*$ with the identity; the natural mapping $\mathcal{F}^* \longrightarrow \widetilde{\mathcal{F}}$ is a semigroup homomorphism which associates to each word $T^* \in \mathcal{F}^*$ a word $\widetilde{T} \in \mathcal{F}$. The period homomorphisms $\widetilde{\omega}^*$ and the mappings $\widetilde{\phi}_{U,V}^*$ respect this identification so induce group homomorphisms $\widetilde{\omega} : \mathcal{F} \longrightarrow \mathbb{C}$ and mappings $\widetilde{\phi}_{U,V} : \mathcal{F} \longrightarrow \mathbb{C}$ which satisfy the product formula (6.14) and the formula (6.17) for mappings from $\mathcal{F}$. When the values $\widetilde{\lambda}_0^*(T_i)$ and $\widetilde{\lambda}_0^*(T_i^{-1})$ in terms of which the mapping $\widetilde{\lambda}_0^*$ is defined are chosen so that they satisfy (6.18) then the product formula (6.14) shows that

$$\begin{aligned}
\widetilde{\lambda}_0^*(\widetilde{T} \cdot \widetilde{T}^{-1}) &= \widetilde{\lambda}_0^*(\widetilde{T})\widetilde{\lambda}_0^*(\widetilde{T}^{-1})\widetilde{\phi}_{U,V}^*(\widetilde{T}, \widetilde{T}^{-1}) \\
&= \widetilde{\lambda}_0^*(\widetilde{T}) \cdot \widetilde{\lambda}_0^*(\widetilde{T})^{-1}\widetilde{\phi}_{U,V}^*(\widetilde{T}, \widetilde{T}) \cdot \widetilde{\phi}_{U,V}^*(\widetilde{T}, \widetilde{T}^{-1}) \\
&= 1
\end{aligned}$$

since $\widetilde{\phi}_{U,V}^*(\widetilde{T}, \widetilde{T}^{-1}) = \widetilde{\phi}_{U,V}^*(\widetilde{T}, \widetilde{T})^{-1}$ by (6.17), and similarly for the product $\widetilde{\lambda}_0^*(\widetilde{T}^{-1} \cdot \widetilde{T})$. Thus with these choices for the values $\widetilde{\lambda}_0^*(T_i^{-1})$ the mapping $\widetilde{\lambda}_0^*$ also respects the identification leading to the free group and consequently induces a mapping $\widetilde{\lambda}_0 : \mathcal{F} \longrightarrow \mathbb{C}^*$ which satisfies the product formula (6.14) in terms of the mapping $\widetilde{\phi}_{U,V} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{C}^*$.

On a marked[3] Riemann surface M of genus $g > 0$ the group Γ is generated by the $2g$ covering translations A_j, B_j associated to the marking; the group Γ is the quotient $\Gamma = \mathcal{F}/\mathcal{K}$ of the free group $\mathcal{F}$ generated by $\widetilde{A}_j$ and $\widetilde{B}_j$ modulo the subgroup $\mathcal{K} \subset \mathcal{F}$ generated by the single element

$$\widetilde{C} = \widetilde{C}_1\widetilde{C}_2 \cdots \widetilde{C}_g \in \mathcal{F} \quad \text{where} \quad \widetilde{C}_j = [\widetilde{A}_j, \widetilde{B}_j] = \widetilde{A}_j\widetilde{B}_j\widetilde{A}_j^{-1}\widetilde{B}_j^{-1}. \tag{6.22}$$

In that case Lemma 6.2 can be applied quite easily to yield the following result.

Lemma 6.5 *On a marked Riemann surface M of genus $g > 0$ there is an abelian factor of automorphy with the root factor $\xi_{U,V,a}(T, z)$ and the auxiliary mapping with specified values $\lambda_0(A_j)$ and $\lambda_0(B_j)$ if and only if $\widetilde{\lambda}_0(\widetilde{C}) = 1$.*

Proof: The mappings $\widetilde{\omega} : \mathcal{F} \longrightarrow \mathbb{C}$, $\widetilde{\phi}_{U,V} : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{C}^*$ and $\widetilde{\lambda}_0 : \mathcal{F} \longrightarrow \mathbb{C}^*$ are all well defined as in the preceding discussion, and $\widetilde{\lambda}_0$ satisfies the product formula (6.14). The mappings $\widetilde{\omega}$ and $\widetilde{\phi}_{U,V}$ are invariant under $\widetilde{C}$ and induce

[3]The definition and properties of markings of a surface are discussed in Appendix D.1.

the corresponding mappings ω and $\phi_{U,V}$ of the group Γ. If $\widetilde{\lambda}_0(\widetilde{C}) = 1$ then by the product formula (6.14)

$$\widetilde{\lambda}_0(\widetilde{T}\widetilde{C}) = \widetilde{\lambda}_0(\widetilde{T})\widetilde{\lambda}_0(\widetilde{C})\widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{C}) = \widetilde{\lambda}_0(\widetilde{T})\widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{C})$$

and correspondingly for the product $\widetilde{C}\widetilde{T}$; and $\widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{C}) = 1$ since $\widetilde{C} \in [\Gamma,\Gamma]$ so actually $\widetilde{\lambda}_0(\widetilde{C}\widetilde{T}) = \widetilde{\lambda}_0(\widetilde{T}\widetilde{C}) = \widetilde{\lambda}_0(\widetilde{T})$. Thus the mapping $\widetilde{\lambda}_0$ is invariant under $\widetilde{C}$ and consequently induces a mapping $\lambda_0 : \Gamma \longrightarrow \mathbb{C}^*$ that satisfies the product formula (6.14). It then follows from Corollary 6.2 that there is an abelian factor of automorphy with the root factor $\xi_{U,V,a}(T,z)$ and the auxiliary mapping λ_0. Conversely if there is such an abelian factor of automorphy the mapping $\lambda_0 : \Gamma \longrightarrow \mathbb{C}^*$ induces a mapping $\widetilde{\lambda}_0 : \mathcal{F} \longrightarrow \mathbb{C}^*$ for which $\widetilde{\lambda}_0(\widetilde{C}) = 1$. That suffices for the proof,

Perhaps surprisingly the value $\widetilde{\lambda}_0(\widetilde{C})$ is independent of the choice of the values $\widetilde{\lambda}_0(A_j)$ and $\widetilde{\lambda}_0(B_j)$ that define the mapping $\widetilde{\lambda}_0$.

Lemma 6.6 *If a mapping* $\widetilde{\lambda}_0 : \mathcal{F} \longrightarrow \mathbb{C}^*$ *satisfies* (6.14) *then the restriction of* $\widetilde{\lambda}_0$ *to the commutator subgroup* $[\mathcal{F},\mathcal{F}] \subset \mathcal{F}$ *is the group homomorphism* $\widetilde{\lambda}_0\Big|[\mathcal{F},\mathcal{F}] \in \mathrm{Hom}([\mathcal{F},\mathcal{F}],\mathbb{C}^*)$ *given explicitly by*

$$\widetilde{\lambda}_0([\widetilde{S},\widetilde{T}]) = \frac{\widetilde{\phi}_{U,V}(\widetilde{S},\widetilde{T})}{\widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{S})} \tag{6.23}$$

for any $\widetilde{S},\widetilde{T} \in \mathcal{F}$.

Proof: It is clear from the definition (6.15) of the auxiliary mapping $\widetilde{\phi}_{U,V}(\widetilde{S},\widetilde{T})$ that

$$\widetilde{\phi}_{U,V}(\widetilde{C},\widetilde{T}) = \widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{C}) = 1 \text{ for all } \widetilde{T} \in \mathcal{F},\ \widetilde{C} \in [\mathcal{F},\mathcal{F}], \tag{6.24}$$

since $\omega(\widetilde{C}) = 0$ for any commutator $\widetilde{C} \in [\mathcal{F},\mathcal{F}]$; and it then follows from the product formula (6.14) that $\widetilde{\lambda}_0(\widetilde{S}\widetilde{T}) = \widetilde{\lambda}_0(\widetilde{S})\widetilde{\lambda}_0(\widetilde{T})$ whenever either $\widetilde{S} \in [\mathcal{F},\mathcal{F}]$ or $\widetilde{T} \in [\mathcal{F},\mathcal{F}]$, so in particular the restriction of the mapping $\widetilde{\lambda}_0$ to the commutator subgroup $[\mathcal{F},\mathcal{F}]$ is a group homomorphism. Then by (6.14) and (6.18)

$$\begin{aligned} \widetilde{\lambda}_0([\widetilde{S},\widetilde{T}]) &= \widetilde{\lambda}_0\big(\widetilde{S}\widetilde{T}\,(\widetilde{T}\widetilde{S})^{-1}\big) \\ &= \widetilde{\lambda}_0(\widetilde{S}\widetilde{T})\widetilde{\lambda}_0\big((\widetilde{T}\widetilde{S})^{-1}\big)\widetilde{\phi}_{U,V}(\widetilde{S}\widetilde{T},(\widetilde{T}\widetilde{S})^{-1}) \\ &= \widetilde{\lambda}_0(\widetilde{S}\widetilde{T})\cdot\widetilde{\lambda}_0(\widetilde{T}\widetilde{S})^{-1}\widetilde{\phi}_{U,V}(\widetilde{T}\widetilde{S},\widetilde{T}\widetilde{S})\cdot\widetilde{\phi}_{U,V}(\widetilde{S}T,TS)^{-1}. \end{aligned} \tag{6.25}$$

Note that by (6.14)

$$\widetilde{\lambda}_0(\widetilde{S}\widetilde{T})\widetilde{\lambda}_0(\widetilde{T}\widetilde{S})^{-1} = \frac{\widetilde{\lambda}_0(\widetilde{S})\widetilde{\lambda}_0(\widetilde{T})\widetilde{\phi}_{U,V}(\widetilde{S},\widetilde{T})}{\widetilde{\lambda}_0(\widetilde{T})\widetilde{\lambda}_0(\widetilde{S})\widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{S})} = \frac{\widetilde{\phi}_{U,V}(\widetilde{S},\widetilde{T})}{\widetilde{\phi}_{U,V}(\widetilde{T},\widetilde{S})}$$

while it is clear from (6.18) that $\widetilde{\phi}_{U,V}(\widetilde{S}\widetilde{T},\widetilde{T}\widetilde{S}) = \widetilde{\phi}_{U,V}(\widetilde{T}\widetilde{S},\widetilde{T}\widetilde{S})$ hence that

$$\widetilde{\phi}_{U,V}(\widetilde{T}\widetilde{S},\widetilde{T}\widetilde{S})\cdot\widetilde{\phi}_{U,V}(\widetilde{S}\widetilde{T},\widetilde{T}\widetilde{S})^{-1} = 1;$$

and substituting the preceding two observations into (6.25) yields (6.23), thereby concluding the proof.

The preceding Lemma 6.6 shows that the condition $\widetilde{\lambda}_0(\widetilde{C})$ in Lemma6.5 depends only on the matrices U, V and not on the particular choices of the values $\widetilde{\lambda}_0(\widetilde{A}_j)$ and $\widetilde{\lambda}_0(\widetilde{B}_j)$ defining the mapping $\widetilde{\lambda}_O$; so if this condition is satisfied then $\xi_{U,V,a}(T,z)$ is the root factor of an abelian factor of automorphy and the associated auxiliary mappings can be specified to take any chosen values on the generators A_j, B_j of the group Γ.

To investigate this condition further it is convenient to decompose the period matrix of a marked Riemann surface M into blocks

$$\Omega = \begin{pmatrix}\Omega_A & \Omega_B\end{pmatrix} \quad \text{where} \quad (\Omega_A)_{ij} = \{\omega_i(A_j)\},\ (\Omega_B)_{ij} = \{\omega_i(B_j)\} \tag{6.26}$$

associated to the choice of generators A_j, B_j of the group Γ, where the period vectors of a basis $\omega_i(z)$ of the abelian differentials are the column vectors

$$\omega(A_j) = \{\omega_i(A_j)\} \quad \text{and} \quad \omega(B_j) = \{\omega_i(B_j)\}. \tag{6.27}$$

In these terms the condition of Lemma 6.5 can be rephrased as follows.

Theorem 6.7 *On a marked Riemann surface $M = \widetilde{M}/\Gamma$ of genus g there is an abelian factor of automorphy $\zeta(T,z)$ with the root factor $\xi_{U,V,a}(T,z)$ if and only if the matrices U, V satisfy*

$$\operatorname{tr}\Big({}^t\Omega_B(V - {}^tV)\Omega_A + {}^t\Omega_B U\overline{\Omega_A} - {}^t\overline{\Omega_B}\,{}^tU\Omega_A\Big) \in \mathbb{Z} \tag{6.28}$$

where $\operatorname{tr}(X)$ denotes the trace of the matrix X. The auxiliary mappings λ_0 for this abelian factor of automorphy can be required to have any desired initial values $\lambda_0(A_j)$ and $\lambda_0(B_j)$.

Proof: From Lemma 6.6 and the definition (6.15) of the mapping $\widetilde{\phi}_{U,V}(\widetilde{S},\widetilde{T})$ it follows that

$$\widetilde{\lambda}_0(\widetilde{C}) = \widetilde{\lambda}(\widetilde{C}_1\widetilde{C}_2\cdots\widetilde{C}_g) = \prod_{j=1}^{g}\widetilde{\lambda}_0(\widetilde{C}_j) = \prod_{j=1}^{g}\widetilde{\lambda}_0([\widetilde{A}_j,\widetilde{B}_j]) = \prod_{j=1}^{g}\frac{\widetilde{\phi}_{U,V}(\widetilde{A}_j,\widetilde{B}_j)}{\widetilde{\phi}_{U,V}(\widetilde{B}_j,\widetilde{A}_j)}$$

$$= \exp 2\pi i\sum_{j=1}^{g}\Big({}^t\omega(B_j)\big(U\overline{\omega(A_j)} + V\omega(A_j)\big) - {}^t\omega(A_j)\big(U\overline{\omega(B_j)} + V\omega(B_j)\big)\Big);$$

therefore $\widetilde{\lambda}_0(\widetilde{C}) = 1$ if and only if

$$\sum_{j=1}^{g}\Big({}^t\omega(B_j)\big(U\overline{\omega(A_j)} + V\omega(A_j)\big) - {}^t\omega(A_j)\big(U\overline{\omega(B_j)} + V\omega(B_j)\big)\Big) \in \mathbb{Z}. \tag{6.29}$$

Since $\sum_{j=1}^{g} {}^t\omega(B_j)U\overline{\omega(A_j)} = \sum_{jkl=1}^{g}(\Omega_B)_{kj}U_{kl}(\overline{\Omega_A})_{lj} = \mathrm{tr}({}^t\Omega_B U\overline{\Omega_A})$ and similarly for the other terms, it is easy to see that equation (6.29) can be rewritten as

$$\mathrm{tr}\Big({}^t\Omega_B U\overline{\Omega_A} + {}^t\Omega_B V\Omega_A - {}^t\Omega_A U\overline{\Omega_B} - {}^t\Omega_A V\Omega_B\Big) \in \mathbb{Z} \tag{6.30}$$

or equivalently in the form given in equation (6.28). That suffices for the proof.

The characteristic class of the holomorphic line bundle described by an abelian factor of automnorphy $\lambda_0(T)\xi_{U,V,a}(T,z)$ also is determined fully just by the matrices U and V, as follows.

Theorem 6.8 *The characteristic class of the holomorphic line bundle ζ over a marked Riemann surface M of genus $g > 0$ described by the abelian factor of automorphy $\zeta(T,z) = \lambda_0(T)\xi_{U,V,a}(T,z)$ is*

$$c(\zeta) = \mathrm{tr}\Big({}^t\Omega_B(V - {}^tV)\Omega_A + {}^t\Omega_B U\overline{\Omega_A} - \overline{{}^t\Omega_B}\,{}^tU\Omega_A\Big) \tag{6.31}$$

in terms of the decomposition 6.26 *of the period matrix Ω of the surface M.*

Proof: By the basic Existence Theorem, Theorem 2.19, the holomorphic line bundle described by an abelian factor of automorphy $\zeta(T,z) = \lambda_0\xi_{U,V,a}(T,z)$ has a nontrivial meromorphic cross-section, hence this abelian factor of automorphy has a nontrivial meromorphic relatively automorphic function, a meromorphic function $f(z)$ on $\widetilde{M}$ such that $f(Tz) = \zeta(T,z)f(z)$ for all $T \in \Gamma$. For this function

$$\begin{aligned} d\log f(Tz) &= d\log f(z) + d\log\zeta(T,z) = d\log f(z) + d\log\xi_{U,V,a}(T,z) \\ &= d\log f(z) + 2\pi i\,{}^t\tilde{\omega}(z)\Big(U\overline{\omega(T)} + V\omega(T)\Big) \end{aligned} \tag{6.32}$$

for all $T \in \Gamma$, since the auxiliary mapping $\lambda_0(T)$ is a constant. When the Riemann surface $M = \widetilde{M}/\Gamma$ is identified as the quotient of the fundamental domain $\Delta \subset \widetilde{M}$ when the edges of Δ are identified as in figure (D.2) in Appendix D, the characteristic class of the line bundle ζ is the degree of the divisor of the meromorphic function $f(z)$ in Δ as in (1.14) so

$$\begin{aligned}
c(\zeta) &= \frac{1}{2\pi i}\int_{\partial\Delta} d\log f(z) \\
&= \frac{1}{2\pi i}\sum_{j=1}^{g}\int_{C_1\cdots C_{j-1}\tilde{\alpha}_j - C_1\cdots C_j B_j\tilde{\alpha}_j + C_1\cdots C_{j-1}A_j\tilde{\beta}_j - C_1\cdots C_j\tilde{\beta}_j} d\log f(z) \\
&= \frac{1}{2\pi i}\sum_{j=1}^{g}\int_{\tilde{\alpha}_j} d\log f(C_1\cdots C_{j-1}z) - d\log f(C_1\cdots C_j B_j z) \\
&\quad + \frac{1}{2\pi i}\sum_{g}\int_{\beta_j} d\log f(C_1\cdots C_{j-1}A_j z) - d\log f(C_1\cdots C_j z) \\
&= -\sum_{j=1}^{g}\int_{\tilde{\alpha}_j} {}^t\tilde{\omega}(z)\Big(U\overline{\omega(B_j)} + V\omega(B_j)\Big) + \sum_{j=1}^{g}\int_{\tilde{\beta}_j} {}^t\tilde{\omega}(z)\Big(U\overline{\omega(A_j)} + V\omega(A_j)\Big) \\
&= -\sum_{j=1}^{g} {}^t\omega(A_j)\Big(U\overline{\omega(B_j)} + V\omega(B_j)\Big) + \sum_{j=1}^{g} {}^t\omega(B_j)\Big(U\overline{\omega(A_j)} + V\omega(A_j)\Big) \\
&= -\operatorname{tr} {}^t\Omega_A\Big(U\overline{\Omega_B} + V\Omega_B\Big) + \operatorname{tr} {}^t\Omega_B\Big(U\overline{\Omega_A} + V\Omega_A\Big) \qquad \blacksquare
\end{aligned}$$

which can be rewritten as (6.31), thereby concluding the proof.

On a marked Riemann surface there is a canonical basis for the holomorphic abelian differentials for which the component Ω_A of the period matrix Ω is the identity matrix, as in Theorem 3.22; and the Riemann matrix conditions in the form in Theorem F.22 in Appendix F.3 assert that the component Ω_B of the period matrix is a matrix $\Omega_B = Z$ in the Siegel upper half-space $\mathfrak{H}_g$, so that Z is a $2g \times 2g$ symmetric matrix $Z = X + iY$ where the imaginary part Y is a positive definite matrix; roughly speaking, the symmetry of Z expresses Riemann's equality while the positive definiteness of Y expresses Riemann's inequality. In these terms the preceding two theorems can be rephrased as follows.

Corollary 6.9 *In terms of the canonical basis for the holomorphic abelian differentials on a marked Riemann surface M of genus $g > 0$, for which the period matrix has the form $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ where $Z = X + iY \in \mathfrak{H}_g$, there is an abelian factor of automorphy with the root factor $\xi_{U,V,a}(T, z)$ if and only if*

$$\operatorname{tr}\Big(iY(U + {}^tU)\Big) = n \in \mathbb{Z}; \tag{6.33}$$

and n is the characteristic class of the holomorphic line bundle described by this abelian factor of automorphy.

Proof: When $\Omega_A = I$ and $\Omega_B = Z$ for a symmetric $g \times g$ complex matrix

$Z = X + iY$, condition (6.28) reduces to the condition that $n \in \mathbb{Z}$ where

$$\begin{aligned} n &= \operatorname{tr}\Big(Z(V - {}^tV) + ZU - \overline{Z}\,{}^tU\Big) \\ &= \operatorname{tr}\Big(Z(V - {}^tV) + X(U - {}^tU) + iY(U + {}^tU)\Big). \end{aligned} \tag{6.34}$$

Since Z and X are symmetric matrices

$$\operatorname{tr}\, Z(V - {}^tV) = \operatorname{tr}\, X(U - {}^tU) = 0,$$

so (6.34) reduces to (6.33). By Theorem 6.8 the characteristic class of the line bundle describe by this abelian factor of automorphy is n, and that suffices for the proof.

Since the matrix Y is positive definite it is invertible, and its inverse Y^{-1} is a symmetic real matrix; the matrix $U = {}^tU = \frac{1}{2gi}Y^{-1}$ then is a symmetric matrix for which $\operatorname{tr}\Big(iY(U + {}^tU)\Big) = \operatorname{tr}\left(iY \cdot \frac{1}{gi}Y^{-1}\right) = \operatorname{tr}\left(\frac{1}{g}I\right) = 1$. The factor of automorphy with the root factor $\xi_{U,V,a}(T, z)$ for this matrix U and the matrix $V = 0$ and for any auxiliary mapping λ then describes a holomorphic line bundle of characteristic class 1; so for a suitable choice λ_a of the auxiliary mapping the abelian factor of automorphy

$$\zeta_a(T, z) = \lambda_a(T)\xi_{\frac{1}{2gi}Y^{-1},0,a}(T, z) = \lambda_a(T) \exp \tfrac{\pi}{g}\, {}^t\tilde{w}(z, a)Y^{-1}\overline{\omega(T)} \tag{6.35}$$

describes the point bundle ζ_a, or equivalently admits a holomorphic relatively automorphic function $p_a(z)$ having a simple zero at the point $a \in \widetilde{M}$ and equivalent points $\Gamma a \subset \widetilde{M}$ but nowhere else on $\widetilde{M}$. The factor of automorphy (6.35) is called the *intrinsic abelian factor of automorphy* at the point $a \in \widetilde{M}$, since the root factor is determined explicitly by the period matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ of the marked Riemann surface in the normal form; it is determined uniquely up to a holomorphically trivial flat line bundle. The relatively automorphic function $p_a(z)$ is called the *local prime function* at the point $a \in \widetilde{M}$. Abelian factors of automorphy representing any divisor on M then can be written as products of powers of the intrinsic abelian factors of automorphy for the appropriate points $a \in \widetilde{M}$.

There are other abelian factors of automorphy on a marked Riemann surface that describe point bundles; perhaps the simplest is one that is expressed in terms of the choice of a holomorphic abelian differential $\omega_1(z)$ on M. Since the matrix Y is positive definite its leading term y_{11} is nonzero. If Δ^{11} is the matrix with entries $\left(\Delta^{11}\right)_{jk} = \delta^1_j \delta^1_k$ and if $U = \frac{1}{2iy_{11}}\Delta^{11}$ then

$$iY(U + {}^tU) = \frac{1}{y_{11}} Y \Delta^{11} = \frac{1}{y_{11}} \begin{pmatrix} y_{11} & 0 & 0 & \cdots \\ y_{21} & 0 & 0 & \cdots \\ \cdots & & & \cdots \\ y_{g1} & 0 & 0 & \cdots \end{pmatrix}$$

so $\operatorname{tr}\Big(iY(U+{}^tU)\Big)=1$ as in the preceding case. Thus for a suitable choice $\lambda_{a,\omega_1}(T)$ of the auxiliary mapping the factor of automorphy

$$\begin{aligned}\zeta_{a,\omega_1}(T,z) &= \lambda_{a,\omega_1}(T)\xi_{\frac{1}{2iy_{11}}\Delta^{11},0,a}(T,z)\\ &= \lambda_{a,\omega_1}(T)\exp\tfrac{\pi}{y_{11}}w_1(z,a)\,\overline{\omega_1(T)}\end{aligned}\tag{6.36}$$

also describes the point bundle ζ_a. This abelian factor of automorphy is called the *minimal abelian factor of automorphy* at the point $a\in\widetilde{M}$ for the abelian differential $\omega_1(z)$, and it is defined uniquely up to a holomorphically trivial flat line bundle.

Corollary 6.10 *In terms of the canonical basis for the holomorphic abelian differentials on a marked Riemann surface M of genus $g>0$, for which the period matrix has the form $\Omega=\begin{pmatrix}I & Z\end{pmatrix}$ where $Z=X+iY\in\mathfrak{H}_g$, if U_0,V_0 are any $g\times g$ complex matrices for which* $\operatorname{tr} Y(U_0+{}^tU_0)=0$ *then an arbitrary abelian factor of automorphy $\zeta(T,z)=\lambda(T)\xi_{U,V,a}(T,z)$ is holomorphically equivalent to the abelian factor of automorphy*

$$\zeta^*(T,z)=\lambda^*(T)\xi_{U+U_0,V+V_0,a}(T,z)\tag{6.37}$$

for a suitable auxiliary mapping $\lambda^*(T)$. In particular it can always be assumed that $V=0$ in any abelian factor of automorphy.

Proof: It follows from the preceding Corollary 6.9 that if U_0,V_0 are $g\times g$ complex matrices for which $\operatorname{tr} Y(U_0+{}^tU_0)=0$ there is an auxiliary mapping $\lambda_0(T)$ for which the abelian factor of automorphy $\zeta_0(T,z)=\lambda_0(T)\xi_{U_0,V_0,a}(T,z)$ describes a holomorphic line bundle of characteristic class 0; moreover after multiplying by a flat line bundle it can be assumed that the abelian factor of automorphy $\zeta_0(T,z)=\lambda_0(T)\xi_{U_0,V_0,a}(T,z)$ is holomorphically trivial so describes the identity line bundle. Consequently any abelian factor of automorphy $\zeta(T,z)=\lambda(T)\xi_{U,V,a}(T,z)$ is holomorphically equivalent to the abelian factor of automorphy $\zeta(T,z)\zeta_0(T,z)$, which has the form (6.37) where $\lambda^*(T)=\lambda_0(T)\lambda(T)$, and that suffices for the proof.

6.3 Hyperabelian Factors of Automorphy

A particularly interesting special class of abelian factors of automorphy for a compact Riemann surface M consists of those for which the auxiliary mappings are trivial on the commutator subgroup $[\Gamma,\Gamma]\subset\Gamma$; they are called the *hyperabelian factors of automorphy* for the Riemann surface M. It is clear that the hyperabelian factors of automorphy form a subgroup of the group of abelian factors of automorphy on M; that subgroup will be denoted by $\mathcal{A}_0(M)\subset\mathcal{A}(M)$. Since the restriction to the commutator subgroup of any auxiliary mapping for an abelian factor of automorphy is determined fully by the root factor it follows that the condition that an abelian factor of automorphy be hyperabelian depends solely on the root factor; the root factors of hyperabelian factors of automorphy hence are called *hyperabelian root factors*.

Lemma 6.11 *A function $\xi_{U,V,a}(T,z)$ is a hyperabelian root factor if and only if*

$$\phi_{U,V}(S,T) = \phi_{U,V}(T,S) \quad \textit{for alll } S,T \in \Gamma \tag{6.38}$$

where $\phi_{U,V}(S,T)$ is the function (6.15).

Proof: It follows from Theorem 6.6 that any tentative auxililary mapping for the root factor $\xi_{U,V,a}(T,z)$ vanishes on the commutator subgroup $[\Gamma,\Gamma]$ if and only if it satisfies (6.38); and since (**??**) is automatically satisfied

Any hyperabelian root factor automatically satisfies the condition (**??**) since $C \in [\Gamma,\Gamma]$; so any expression $\xi_{U,V,a}(T,z)$ that satisfies the condition that the associated mapping must vanish on the commutator subgroup automatically satisfies the condition that it be the root factor of an abelian factor of automorphy.

Theorem 6.12 *A root factor $\xi_{U,V,a}(T,z)$ is hyperabelian if and only if*

$$^{t}U = -\overline{U}, \quad ^{t}V = V, \quad 2\Re\Big(\,^{t}\Omega U\overline{\Omega}\Big) \textit{ is an integral matrix.} \tag{6.39}$$

Proof: If $T_k \in \Gamma$ for $1 \le k \le 2g$ are generators of the covering translation group that represent a basis for $H_1(M)$ then since $[\Gamma,\Gamma] \subset \Gamma$ is the normal subgroup of Γ generated by the commutators $[T_k,T_l]$ and the auxiliary mappings for any root factor restrict to homomorphisms on the commutator subgroup it follows from Theorem 6.6 that a root factor $\xi_{U,V,a}(T,z)$ is hyperabelian if and only if

$$\phi_{U,V}(T_k,T_l) = \phi_{U,V}(T_l,T_k) \text{ for } 1 \le k,l \le 2g \tag{6.40}$$

in terms of the mapping (6.15). If Ω is the period matrix of the Riemann surface in terms of the generators T_k and the abelian differentials $\omega_i(z)$ then $\omega(T_k) = \Omega\delta_k$ where δ_k are the column vectors of the identity matrix; hence

$$\begin{aligned}\phi_{U,V}(T_k,T_l) &= \exp 2\pi i\, ^{t}\omega(T_l)\Big(U\overline{\omega(T_k)} + V\omega(T_k)\Big)\\ &= \exp 2\pi i\, ^{t}\delta_l\, ^{t}\Omega\Big(U\overline{\Omega}\delta_k + V\Omega\delta_k\Big)\end{aligned}$$

so (6.40) is equivalent to the condition that the difference

$$n_{lk} = \,^{t}\delta_l\, ^{t}\Omega\Big(U\overline{\Omega}\delta_k + V\Omega\delta_k\Big) - \,^{t}\delta_k\, ^{t}\Omega\Big(U\overline{\Omega}\delta_l + V\Omega\delta_l\Big) \tag{6.41}$$

is an integer for any indices k,l, or in matrix terms that the $2g \times 2g$ matrix $N = \{n_{lk}\}$ defined by

$$\begin{aligned}N &= \,^{t}\Omega(U\overline{\Omega} + V\Omega) - \,^{t}\Big(\,^{t}\Omega(U\overline{\Omega} + V\Omega)\Big)\\ &= \,^{t}\Omega U\overline{\Omega} + \,^{t}\Omega V\Omega - \,^{t}\overline{\Omega}\,^{t}U\Omega - \,^{t}\Omega\,^{t}V\Omega\\ &= \begin{pmatrix} ^{t}\Omega & ^{t}\overline{\Omega}\end{pmatrix}\begin{pmatrix} V - \,^{t}V & U \\ -\,^{t}U & 0\end{pmatrix}\begin{pmatrix}\Omega \\ \overline{\Omega}\end{pmatrix}\end{aligned} \tag{6.42}$$

is an integral matrix. It is evident from its definition (6.41) that N is a skew symmetric matrix. Now if Π is the inverse period matrix to the period matrix Ω, the $g \times 2g$ matrix Π for which

$$\begin{pmatrix} \overline{\Pi} \\ \Pi \end{pmatrix} = \begin{pmatrix} {}^t\Omega & {}^t\overline{\Omega} \end{pmatrix}^{-1}$$

as in Theorem F.12 in Appendix F.1, then equation (6.42) can be rewritten

$$\begin{pmatrix} V - {}^tV & U \\ -{}^tU & 0 \end{pmatrix} = \begin{pmatrix} \overline{\Pi} \\ \Pi \end{pmatrix} N \begin{pmatrix} {}^t\overline{\Pi} & {}^t\Pi \end{pmatrix} = \begin{pmatrix} \overline{\Pi} N\, {}^t\overline{\Pi} & \overline{\Pi} N\, {}^t\Pi \\ \Pi N\, {}^t\overline{\Pi} & \Pi N\, {}^t\Pi \end{pmatrix} \tag{6.43}$$

or equivalently

$$V - {}^tV = \overline{\Pi} N\, {}^t\overline{\Pi}, \qquad U = \overline{\Pi} N\, {}^t\Pi. \qquad -{}^tU = \Pi N\, {}^t\overline{\Pi}, \qquad 0 = \Pi N\, {}^t\Pi. \tag{6.44}$$

Using these observations, if N is an integral matrix it follows from (6.44) that

$$0 = \overline{\Pi N\, {}^t\Pi} = \overline{\Pi} N\, {}^t\overline{\Pi} = V - {}^tV \quad \text{so} \quad {}^tV = V$$

and also that

$$\overline{U} = \overline{\overline{\Pi} N\, {}^t\Pi} = \Pi N\, {}^t\overline{\Pi} = -{}^tU \quad \text{so} \quad {}^tU = -\overline{U},$$

thus demonstrating the first two assertions in (6.39). Substituting these observations in the matrix (6.42) shows that

$$\begin{aligned} N &= \begin{pmatrix} {}^t\Omega & {}^t\overline{\Omega} \end{pmatrix} \begin{pmatrix} 0 & U \\ \overline{U} & 0 \end{pmatrix} \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \\ &= {}^t\Omega U \overline{\Omega} + {}^t\overline{\Omega}\overline{U}\Omega = 2\Re\Big({}^t\Omega U \overline{\Omega}\Big), \end{aligned} \tag{6.45}$$

which is the third assertion in (6.39). Conversely if the three conditions (6.39) are satisfied then substituting the first two conditions in the matrix N of (6.42) shows as in the preceding calculation (6.45) that N has the form (6.45), hence N is integral as a consequence of the third condition, to conclude the proof.

Corollary 6.13 *If M is a marked Riemann surface with the period matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ in terms of a canonical basis for the holomorphic abelian differentials, where $Z = X + iY \in \mathfrak{H}_g$, a root factor $\xi_{U,V,a}(T,z)$ is hyperabelian if and only if*

(6.46)

(i) ${}^tU = -\overline{U}$ *and* ${}^tV = V$,

(ii) $\Re(U) = N = -{}^tN \in \frac{1}{2}\mathbb{Z}^{g\times g}$,

(iii) $\Im(U) = (N_1 - NX)Y^{-1} = Y^{-1}({}^tN_1 + XN)$ *where* $N_1 \in \frac{1}{2}\mathbb{Z}^{g\times g}$,

(iv) $-XNX + YNY + XN_1 - {}^tN_1X \in \frac{1}{2}\mathbb{Z}^{g\times g}$.

Proof: The preceding theorem shows that the root factor $\xi_{U,V,a}(T,z)$ is hyperabelian if and only if ${}^t U = -\overline{U}$, ${}^t V = V$ and $2\Re({}^t\Omega U\overline{\Omega})$ is an integral matrix. The first two of these conditions are listed in (i); the third condition for the period matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$, where $Z \in \mathfrak{H}_g$ so that ${}^t Z = Z$, can be written

$$(6.47) \qquad 2\Re({}^t\Omega U\overline{\Omega}) = 2\Re\begin{pmatrix} I \\ Z \end{pmatrix} U \begin{pmatrix} I & \overline{Z} \end{pmatrix} = 2\Re\begin{pmatrix} U & U\overline{Z} \\ ZU & ZU\overline{Z} \end{pmatrix} \in \mathbb{Z}^{2g\times 2g}.$$

If $U = N + iT$ for some real $g\times g$ matrices N, T, where ${}^t N = -N$ and ${}^t T = T$ since ${}^t U = -U$, the preceding condition (6.47) can be written as the set of conditions

(a) $2\Re(U) = 2N \in \mathbb{Z}^{g\times g}$

(b) $2\Re(U\overline{Z}) = 2(NX + TY) \in \mathbb{Z}^{g\times g}$

(c) $2\Re(ZU) = 2(XN - YT) \in \mathbb{Z}^{g\times g}$

(d) $2\Re(ZU\overline{Z}) = 2(XNX + YNY + XTY - YTX) \in \mathbb{Z}^{g\times g}$

Condition (ii) in the corollary is condition (a), where ${}^t N = -N$ as already noted. Since ${}^t(NX + TY) = -XN + YT$ conditions (b) and (c) are equivalent; and (b) is really just the assertion that $NX + TY = N_1 \in \mathbb{Z}^{g\times g}$ so that $\Im(U) = T = (N_1 - NX)Y^{-1}$, which is the first part of (iii); since ${}^t T = T$ as already noted that yields the second part of (iii). Replacing T in (d) by the expressions for T in terms of the matrices N and N_1 easily leads to condition (iv), and that suffices for the proof.

The preceding corollary gives necessary and sufficient conditions for the existence of nontrivial hyperabelian factors of automorphy $\xi_{U,V,a}(T,z))$ on a marked Riemann surface in terms of the period matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ of that surface in the standard form. These conditions involve integral relations between the real and imaginary parts of the matrix Z; so they are either further restrictions on the period matrix of a surface or special sets of auxiliary integral matrices N_1 and N_2 that describe hyperabelian factors of automorphy for general Riemann surfaces. The simplest solutions of these equations for general Riemann surfaces are those for which the the matrix U is purely imaginary; in that case the conditions (6.46) reduce to

$$(6.48) \qquad \begin{aligned} &U = i\,N_1 Y^{-1} = i\,Y^{-1}\,{}^t N_1 \quad \text{where} \quad N_1 \in \tfrac{1}{2}\mathbb{Z}^{g\times g} \quad \text{and} \\ &XN_1 - {}^t N_1 X = N_2 \in \tfrac{1}{2}\mathbb{Z}^{g\times g}. \end{aligned}$$

The first set of equalities imply that $YN_1 = {}^t N_1 Y$. Generally the only half-integral matrices N_1 satisfying this condition are scalar matrices, so integral multiples of the matrix $N_1 = -\frac{1}{2}I$; and since this matrix commutes with any matrix the second condition is satisfied for $N_2 = 0$. Thus the matrices $U = iN_1Y^{-1} = \frac{1}{2i}Y^{-1}, V = 0$ satisfy the conditions of the preceding corollary so there is a hyperabelian root factor of the form

$$(6.49) \qquad \xi_{\frac{1}{2i}Y^{-1},0,a}(T,z) = \exp \pi\,{}^t\widetilde{w}(z,a)Y^{-1}\overline{\omega(T)}.$$

The associated factor of automorphy for a suitable auxiliaryi mapping is just the g-th power of the intrinsic abelian factor of automorphy $\zeta_a(T,z)$ of (6.35), the factor of automorphy describing the point bundle ζ_a.

When a compact Riemann surface M of genus $g > 0$ is represented as the quotient $M = \widetilde{M}/\Gamma$ of its universal covering space $\widetilde{M}$ by the covering translation group Γ then since the commutator subgroup $[\Gamma,\Gamma] \subset \Gamma$ is a normal subgroup the covering projection $\widetilde{\pi} : \widetilde{M} \longrightarrow M = \widetilde{M}/\Gamma$ can be decomposed as the composition $\widetilde{\pi} = \widehat{\pi} \circ \pi_a$ of two mappings in the chain of covering projections

$$\widetilde{M} \xrightarrow{\pi_a} \widehat{M} = \widetilde{M}/[\Gamma,\Gamma] \xrightarrow{\widehat{\pi}} M = \widehat{M}/\Gamma_a \tag{6.50}$$

where $\Gamma_a = \Gamma/[\Gamma,\Gamma]$ is the abelianization of the group Γ. The group Γ can be generated by $2g$ generators with the single relation (D.4) so its abelianization Γ_a is a free abelian group on $2g$ generators. The subgroup $[\Gamma,\Gamma] \subset \Gamma$ is not of finite index, since the quotient $\Gamma_a = \Gamma/[\Gamma,\Gamma]$ is an infinite group, so $\widehat{M}$ is not a compact Riemann surface. The surface $\widehat{M}$ is not simply connected; indeed its fundamental group is isomorphic to $[\Gamma,\Gamma]$. The fundamental group of any noncompact connected surface is a free group[4]; so the group $[\Gamma,\Gamma]$ actually is a free group, a result which though interesting will not be used here. The Riemann surface $\widetilde{M}$ can be identified with the unit disc, through the general uniformization theorem. The Riemann surface $\widehat{M}$ however appears to be an example of a non-continuable[5] Riemann surface, a noncompact Riemann surface that cannot be realized as a proper subset of another Riemann surface; but that topic will not be pursued further here.

The holomorphic abelian differentials on M are represented by Γ-invariant holomorphic differential 1-forms ω_i on $\widetilde{M}$, and their integrals $w_i(z,z_0) = \int_{z_0}^{z} \omega_i$ are holomorphic functions on $\widetilde{M}$ such that

$$w_i(Tz,z_0) = w_i(z,z_0) + \omega_i(T) \quad \text{for all} \quad T \in \Gamma. \tag{6.51}$$

The set of period vectors $\omega(T) = \{\omega_i(T)\} \in \mathbb{C}^g$ for all $T \in \Gamma$ form the lattice subgroup $\mathcal{L}(\Omega) \subset \mathbb{C}^g$; and the set of integrals $w_i(z,z_0)$ describe a holomorphic mapping

$$\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \mathbb{C}^g \quad \text{where} \quad \widetilde{w}_{z_0}(z) = \{w_i(z,z_0)\} \in \mathbb{C}^g. \tag{6.52}$$

It follows from (13.25) that the mapping (13.26) commutes with the covering projections $\widetilde{\pi} : \widetilde{M} \longrightarrow M$ and $\pi : \mathbb{C}^g \longrightarrow J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$, so it induces the Abel-Jacobi mapping $w_{z_0} : M \longrightarrow J(M)$ as in the commutative diagram (3.4). Recall from the earlier discussion that the Abel-Jacobi mapping is a nonsingular biholomorphic mapping from the Riemann surface M to its image $W_1 = w_{z_0}(M) \subset J(M)$, which is an irreducible holomorphic submanifold of the

---

[4]See the discussion the book *Riemann Surfaces* by Lars Ahlfors and Leo Sario, section 44.

[5]See the discussion in the paper by S. Bochner, "Fortsetzung Riemannscher Flächen" *Math. Annalen* 98(1928), pp. 406-421.

complex torus $J(M)$. The holomorphic mapping (13.26) and the Abel-Jacobi mapping have the same local expression; so if the image of the mapping (13.26) is denoted by

$$\widetilde{W}_1 = \widetilde{w}_{z_0}(\widetilde{M}) \subset \mathbb{C}^g \tag{6.53}$$

then the mapping (13.26) is a nonsingular holomorphic mapping, hence is a locally biholomorphic mapping

$$\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \widetilde{W}_1. \tag{6.54}$$

This situation can be summarized in the commutative diagram of holomorphic mappings

$$\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{w}_{z_0}"] \arrow[d, "\widetilde{\pi}"'] & \widetilde{W}_1 \arrow[r, "\iota", "\subset"'] \arrow[d, "\pi"'] & \mathbb{C}^g \arrow[d, "\pi"'] \\
M = \widetilde{M}/\Gamma \arrow[r, "w_{z_0}", "\cong"'] & W_1 = \widetilde{W}_1/\mathcal{L}(\Omega) \arrow[r, "\iota", "\subset"'] & J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)
\end{tikzcd} \tag{6.55}$$

in which ι is the natural inclusion mapping. Although the subset $\widetilde{W}_1$ is defined as the image (13.27) it also can be characterized by

$$\widetilde{W}_1 = \pi^{-1}(W_1) \quad \text{so} \quad \widetilde{W}_1 + \lambda = \widetilde{W}_1 \quad \text{for all } \lambda \in \mathcal{L}(\Omega). \tag{6.56}$$

Indeed if $t \in \widetilde{W}_1 \subset \mathbb{C}^g$ then by definition $t = \widetilde{w}_{z_0}(z)$ for some point $z \in \widetilde{M}$. and if $\lambda \in \mathcal{L}(\Omega)$ then $\lambda = \omega(T)$ for some $T \in \Gamma$; it then follows from (13.25) that $w_{z_0}(Tz) = w_{z_0}(z) + \lambda = t + \lambda$, so $t + \lambda \in \widetilde{W}_1$. This also shows that the mapping $\widetilde{w}_{z_0}$ is a covering projection. Since $\widetilde{W}_1$ is the inverse image of the holomorphic submanifold W_1 by the holomorphic mapping π it follows that $\widetilde{W}_1$ is a holomorphic submanifold of $\mathbb{C}^g$.

The holomorphic mapping (13.28) is locally biholomorphic but it is not globally biholomorphic. Indeed if $\widetilde{w}_{z_0}(z_1) = \widetilde{w}_{z_0}(z_2)$ for two points $z_1, z_2 \in \widetilde{M}$ then by the commutativity of the diagram (13.29) the images $a_1 = \widetilde{\pi}(z_1)$ and $a_2 = \widetilde{\pi}(z_2)$ in M have the same image under the Abel-Jacobi mapping w_{z_0}; and since the mapping w_{z_0} is injective it follows that $a_1 = a_2$. Consequently $z_1 = Tz_2$ for some $T \in \Gamma$; and then $w_{z_0}(z_1) = w_{z_0}(Tz_2) = w_{z_0}(z_1) + \omega(T)$ so that $\omega(T) = 0$, which by Corollary 3.6 is equivalent to the condition that $T \in [\Gamma, \Gamma]$. The converse clearly holds, so

$$\widetilde{w}_{z_0}(z_1) = \widetilde{w}_{z_0}(z_2) \quad \text{if and only if} \quad z_1 = Tz_2 \quad \text{where} \quad T \in [\Gamma, \Gamma]. \tag{6.57}$$

That means that the mapping $\widetilde{w}_{z_0}$ in the diagram (13.29) is a covering projection, with the covering translation group $[\Gamma, \Gamma]$, and that this mapping can be factored through the quotient surface $\widehat{M} = \widetilde{M}/[\Gamma, \Gamma]$ so the diagram (13.29) can be factored into the commutative diagram of holomorphic mappings

(6.58)
$$
\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\widetilde{w}_{z_0}} & \widetilde{w}_{z_0}(\widetilde{M}) = \widetilde{W}_1 & \xrightarrow[\subset]{\iota} & \mathbb{C}^g \\
\pi_a\big\downarrow & & \big\| & & \big\| \\
\widehat{M} = \widetilde{M}/[\Gamma,\Gamma] & \xrightarrow[\cong]{\widehat{w}_{z_0}} & \widehat{w}_{z_0}(\widehat{M}) = \widetilde{W}_1 & \xrightarrow[\subset]{\iota} & \mathbb{C}^g \\
\widehat{\pi}\big\downarrow & & \pi\big\downarrow & & \pi\big\downarrow \\
M = \widehat{M}/\Gamma_a & \xrightarrow[\cong]{w_{z_0}} & w_{z_0}(M) = W_1 = \widetilde{W}_1/\mathcal{L}(\Omega) & \xrightarrow[\subset]{\iota} & J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)
\end{array}
$$

where all the vertical arrows are covering projections, as also is the mapping $\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \widetilde{W}_1$. The holomorphic mapping $\widehat{w}_{z_0}$ clearly is surjective, it is injective as a consequence of (13.31), and it is locally biholomorphic since it has the same local expression as the Abel-Jacobi mapping w_{z_0}; hence it is a biholomorphic mapping, as indicated in the diagram. The image $\widehat{w}_{z_0}(\widehat{M}) = \widetilde{W}_1$ thus is an irreducible holomorphic submanifold of $\mathbb{C}^g$ that is biholomorphic to $\widetilde{M}$.

The holomorphic mapping $\widehat{w}_{z_0}$ is defined as the mapping induced by the mapping $\widetilde{w}_{z_0}$; but it also can be described somewhat independently. Indeed it follows from (13.25) that the holomorphic abelian integrals $w_i(z, z_0)$ are invariant under the covering translation group $[\Gamma, \Gamma]$ so they can be viewed as holomorphic functions $\widehat{w}_i(\widehat{z}, z_0)$ of points $\widehat{z}$ in the complex manifold $\widehat{M}$. Of course the holomorphic abelian differentials can be viewed as holomorphic differential forms on the Riemann surface $\widehat{M}$, which is not simply connected; but their integrals actually also are well defined global holomorphic functions $\widehat{w}_i(\widehat{z}, z_0)$ on the manifold $\widehat{M}$. In terms of these integrals the mapping $\widehat{w}_{z_0}$ can be viewed as the mapping defined by

$$\widehat{w}_{z_0}(\widehat{z}) = \{\widehat{w}_i(\widehat{z}, z_0)\} \in \mathbb{C}^g; \tag{6.59}$$

and

$$\widehat{w}_i(\widehat{T}\widehat{z}, z_0) = \widehat{w}_i(\widehat{z}, z_0) + \widehat{\omega}_i(\widehat{T}) \quad \text{for all} \quad \widehat{T} \in \Gamma_a \tag{6.60}$$

where $\widehat{\omega}_i(\widehat{T}) \in \mathbb{C}$ is the period $\omega_i(T)$ for any $T \in \Gamma$ representing $\widehat{T} \in \Gamma_a$. The set of period vectors $\widehat{\omega}(\widehat{T})$ for all $\widehat{T} \in \Gamma_a$ also form the lattice subgroup $\mathcal{L}(\Omega) \subset \mathbb{C}^g$.

6.4 The Prime Function

Since the associated mappings for any of these root factors are determined only up to the multiplication by an arbitrary flat factor of automorphy it is possible to choose an associated mapping $\lambda_a(T)$ for the intrinsic root factor $\xi_{1,a}(T, z)$ so that the factor of automorphy $\zeta_a(T, z) = \lambda_a(T)\xi_{1,a}(z)$ represents the point bundle ζ_a. With this choice there will exist a holomorphic relatively automorphic function $p_a(z)$ for the factor of automorphy $\zeta_a(T,z)$ such that $p_a(z)$ has a simple zero at the point $a \in \widetilde{M}$ and the equivalent points Γa but is nonzero otherwise; and the automorphy condition has the form

$$p_a(Tz) = p_a(z)\lambda_a(T)\exp\Big(\frac{\pi}{g}\,{}^t\tilde{w}(z,a)Y^{-1}\overline{\omega(T)}\Big) \text{ for all } T \in \Gamma. \tag{6.61}$$

If $q(z, z_0; w, a)$ is the intrinsic cross-ratio function as characterised in Theorem 5.6, a relatively automorphic function on $\widetilde{M}^4$ that as a function of $z \in \widetilde{M}$ has a simple zero at $z = w$ and a simple pole at $z = a$, then the product

$$p(z,w) = p_a(z)q(z, z_0; w, a) \tag{6.62}$$

is a holomorphuic function of the variables $z, w \in \widetilde{M}$ that has a simple zero on the subvariety $z = w$ and no other zeros. By Theorem 5.6 the intrinsic cross-ratio function on any Riemann surface is a relatively automorphic function of the variable z for the flat factor of automorphy

$$\zeta_p(T,z) = \exp -2\pi\,{}^t\tilde{w}(w,a)G\overline{\omega(T)}),$$

using the form given in (5.24). For a marked Riemann surface with the standard basis for the holomorphic abelian differentials the period matrix has the form $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ where $Z = X + iY \in \mathfrak{H}_g$ and $G = \frac{1}{2}Y^{-1}$, so the factor of automorphy actually has the form

$$\zeta_p(T,z) = \exp \pi\,{}^t\tilde{w}(a,w)Y^{-1}\bar{[}\Omega(T). \tag{6.63}$$

Chapter 7

Families of Holomorphic Line Bundles

PRELIMIINAARY VERSION

7.1 Local Bases for Bounded Spaces of Meromorphic Automorphic Functions

When considering the spaces of holomorphic cross-sections of families of holomorphic line bundles over a compact Riemann surface M from an analytic perspective, it is convenient and relevant to the analytic interpretation to represent line bundles by factors of automorphy for the action of the covering translation group Γ on the universal covering space $\tilde{M}$ of that Riemann surface and to describe the holomorphic cross-sections of these line bundles by holomorphic relatively automorphic functions for the factors of automorphy. Every holomorphic line bundle over M actually can be described by a simple explicit factor of automorphy, indeed by an abelian factor of automorphy. For the discussion in this chapter the specific form of the factor of automorphy generally is not relevant. What is very relevant though is that if $\eta(T, z)$ is a factor of automorphy describing a holomorphic line bundle η of characteristic class $c(\eta) = r$ over M, all holomorphic line bundles of characteristic class r can be represented by factors of automorphy of the form $\rho_t(T)\eta(T, z)$ for parameters $t \in \mathbb{C}^{2g}$, where ρ_t is the flat line bundle parametrized by the point $t \in \mathbb{C}^{2g}$ under the canonical parametrization (3.27) of flat line bundles associated to generators $T_1, \ldots, T_{2g} \in \Gamma$ representing a basis $\tau_j \in H_1(M)$; explicitly $\rho_t \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is the representation of the group Γ for which $\rho_t(T_j) = \exp 2\pi i t_j$ for $1 \le j \le 2g$. Families of holomorphic line bundles of characteristic class g then can be described by factors of automorphy $\rho_t(T)\eta(T, z)$ for parameter values $t \in V$ lying in subsets $V \subset \mathbb{C}^{2g}$; and holomorphic cross-sections of these families of line bundles can be described by relatively automorphic functions $f(z, t) \in \Gamma(M, \mathcal{O}(\rho_t\eta)$

of the variable $z \in \tilde{M}$ that also depend on the variable $t \in \mathbb{C}^{2g}$. When the subset $V \subset \mathbb{C}^{2g}$ is a holomorphic subvariety the factors of automorphy $\rho_t(T)\eta(T, z)$ are holomorphic functions of the parameter $t \in V$, and it is possible to consider relatively automorphic functions $f(z, t)$ for these factors of automorphy that are also holomorphic or meromorphic functions of the variables $(z, t) \in \tilde{M} \times V$.

Theorem 7.1 *Let η be a factor of automorphy describing a holomorphic line bundle of characteristic class $c(\eta) = r$ on a compact Riemann surface M of genus $g > 0$; let $a_1, \ldots, a_n$ be n points of the universal covering space $\tilde{M}$ of the surface M, not necessarily distinct; and let $t_0 \in \mathbb{C}^{2g}$ be a point in the parameter space for the canonical parametrization of flat line bundles over M associated to generators $T_1, \ldots, T_{2g} \in \Gamma$ of the covering translation group of M. If $r + n > 2g - 2$ there are open neighborhoods $\tilde{U}_j \subset \tilde{M}$ of the points a_j, an open neighborhood $\tilde{U}_0 \subset \mathbb{C}^{2g}$ of the point t_0, and $r + n + 1 - g$ meromorphic functions $f_i(z, u, t)$ of the variables $z \in \tilde{M}$, $u_j \in \tilde{U}_j$, $t \in \tilde{U}_0$ with singularities at most simple poles along the subvarieties $z = Tu_j$ for $T \in \Gamma$ and $1 \le j \le n$, such that for any fixed points $u_j \in \tilde{U}_j$, $t \in \tilde{U}_0$ these functions are a basis for the vector space*

$$\Lambda_\eta(u, t) = \Big\{ f \in \Gamma(M, \mathcal{M}(\rho_t\eta)) \,\Big|\, \mathfrak{d}(f) + u_1 + \cdots + u_n \ge 0 \Big\}. \tag{7.1}$$

Proof: To simplify the calculations slightly, choose a marking of the surface M, that is, a base point $a \in \tilde{M}$ and generators $A_j = T_j$, $B_j = T_{g+j} \in \Gamma$ for $1 \le j \le g$; and suppose that the canonical parametrization of flat line bundles is expressed in terms of these generators $T_j \in \Gamma$. Let $\omega_i(z)$ be the canonical basis for the holomorphic abelian differentials on the marked Riemann surface M and let $w_i(z, a)$ be the associated holomorphic abelian integrals at the base point $a \in \tilde{M}$. The $g \times 2g$ period matrix of the differentials $\omega_i(z)$ has the form $\Omega = (\mathrm{I} \quad Z)$ for a $g \times g$ symmetric matrix $Z = \{z_{ij}\}$ in the Siegel upper half-space $\mathfrak{H}_g$ of rank g, as in Theorem 3.22. Let $\tilde{U}_j \subset \tilde{M}$ be open coordinate neighborhoods of the points $a_j \in \tilde{M}$ for $1 \le j \le n$. Choose g auxiliary points $b_k \in \tilde{M}$ that represent distinct points of the surface M that are also distinct from any of the points of M represented by the points a_j and are such that the $g \times g$ matrix $W' = \{w_i'(b_k, a)\}$ is nonsingular, where $w_i'(b_k, a) = \partial w_i(z_k, a)/\partial z_k|_{z_k = b_k}$ in terms of a local coordinate z_k centered at the point b_k; and let $\tilde{V}_k \subset \tilde{M}$ be open coordinate neighborhoods of the points b_k. By shrinking the neighborhoods $\tilde{U}_j$ and $\tilde{V}_k$ if necessary it can be supposed that the open subsets $\tilde{V}_1, \ldots, \tilde{V}_g, \bigcup_{j=1}^n \tilde{U}_j$ of $\tilde{M}$ represent $g+1$ disjoint open subsets of the surface M. The holomorphic line bundles $\zeta_{\mathfrak{a}} = \zeta_{a_1 + \cdots + a_n}$ and $\zeta_{\mathfrak{b}} = \zeta_{b_1 + \cdots + b_g}$ have characteristic classes $c(\zeta_{\mathfrak{a}}) = n$ and $c(\zeta_{\mathfrak{b}}) = g$ respectively; and since $c(\eta\zeta_{\mathfrak{a}}) = r + n > 2g - 2$ by assumption it follows from the Riemann-Roch Theorem that $\gamma(\rho_{t_0}\eta\zeta_{\mathfrak{a}}) = r + n + 1 - g$ and $\gamma(\rho_{t_0}\eta\zeta_{\mathfrak{a}}\zeta_{\mathfrak{b}}) = r + n + 1$. Consider then the vector space

$$X = \Big\{ g \in \Gamma(M, \mathcal{M}(\rho_{t_0}\eta\zeta_{\mathfrak{a}})) \,\Big|\, \mathfrak{d}(g) + b_1 + \cdots + b_g \ge 0 \Big\}$$

consisting of meromorphic cross-sections of the line bundle $\rho_{t_0}\eta\zeta_{\mathfrak{a}}$ with singularities at most simple poles at the points $\Gamma b_k \in \tilde{M}$. Since X and $\Gamma(M, \mathcal{O}(\rho_{t_0}\eta\zeta_{\mathfrak{a}}\zeta_{\mathfrak{b}}))$

are isomorphic vector spaces under the isomorphism that arises by multiplying a cross-section $g \in X$ by a nontrivial cross-section $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{b}}))$ that vanishes at the points b_k, as in the argument on page 53, it follows that $\dim X = \gamma(\rho_{t_0}\eta\zeta_{\mathfrak{a}}\zeta_{\mathfrak{b}}) = r + n + 1$. Let $g_i(z)$ be a basis for the vector space X for $0 \le i \le r+n$, where the cross-sections $g_i(z)$ are meromorphic functions on $\tilde{M}$ that are relatively automorphic functions for the factor of automorphy $\rho_{t_0}\eta\zeta_{\mathfrak{a}}$; and if $\hat{q}(z_1, z_2; z_3, z_4)$ is the canonical cross-ratio function of the marked Riemann surface M and $h_{a_j}(z) \in \Gamma(M, \mathcal{O}(\zeta_{a_j}))$ is a nontrivial holomorphic cross-section vanishing at the points Γa_j, set

$$\begin{aligned} \hat{g}_i(z,u,x,s) &= g_i(z) \cdot \prod_{j=1}^{n} \Big(h_{a_j}(z)^{-1} \hat{q}(z,p;a_j,u_j) \Big) \cdot \prod_{k=1}^{g} \hat{q}(z,q;b_k,x_k) \cdot \\ &\qquad\qquad \cdot \exp 2\pi i \sum_{l=1}^{g} s_l w_l(z,b) \end{aligned}$$

for $0 \le i \le r+n$, where $z \in \tilde{M}$, $u_j \in \tilde{U}_j \subset \tilde{M}$, $x_k \in \tilde{V}_k \subset \tilde{M}$, $s_l \in \mathbb{C}$, and $p, q \in \tilde{M}$ are fixed points of $\tilde{M}$ such that $p, q \notin \big(\bigcup_{j=1}^{n} \Gamma\tilde{U}_j\big) \cup \big(\bigcup_{k=1}^{g} \Gamma\tilde{V}_k\big)$; thus $Tu_j \ne p$ and $Tx_k \ne q$ for any points $u_j \in \tilde{U}_j$, $x_k \in \tilde{V}_k$, and any $T \in \Gamma$, so in particular $Ta_j \ne p$ and $Tb_k \ne q$. Recall from Theorem 5.16 (i) that the cross-ratio function $\hat{q}(z_1, z_2; z_3, z_4)$ has simple zeros along the subvarieties $z_1 = Tz_3$ and $z_2 = Tz_4$ and simple poles along the subvarieties $z_1 = Tz_4$ and $z_2 = Tz_3$ for all $T \in \Gamma$, and no other zeros or poles on the surface $\tilde{M}$. It follows that the product $h_{a_j}(z)^{-1}\hat{q}(z,p;a_j,u_j)$ is a nontrivial meromorphic function of the variables $(z, u_j) \in \tilde{M} \times \tilde{U}_j$ with at most simple poles along the subvarieties $z = Tu_j$ for $1 \le j \le n$ and all $T \in \Gamma$ and no other singularities, since the poles of the function $h_{a_j}(z)^{-1}$ at $z = Ta_j$ are cancelled by the zeros of the cross-ratio function there; and the product $g_i(z) \prod_{k=1}^{g} \hat{q}(z,q;b_k,x_k)$ is a nontrivial meromorphic function of the variables $(z,x) \in \tilde{M} \times \prod_{k=1}^{g} \tilde{V}_k$ with at most simple poles along the subvarieties $z = Tx_k \in \tilde{M}$ for $1 \le k \le g$ and all $T \in \Gamma$ and no other singularities, since the poles of the function $g_i(z)$ at the points Tb_k are cancelled by the zeros of the cross-ratio functions at those points. Altogether the functions $\hat{g}_i(z,u,x,s)$ are meromorphic functions of the variables $(z,u,x,s) \in \tilde{M} \times \prod_{j=1}^{n} \tilde{U}_j \times \prod_{k=1}^{g} \tilde{V}_k \times \mathbb{C}^g$ with singularities at most simple poles along the subvarieties $z = Tu_j$ and $z = Tx_k$ and no other singularities; and for any fixed points (u,x,s) they are $r+n+1$ linearly independent meromorphic functions of the variable $z \in \tilde{M}$. Furthermore it follows from Theorem 5.16 (iii) that

$$\begin{aligned} \hat{g}_i(A_m z, u, x, s) &= \hat{g}_i(z,u,x,s) \rho_{t_0}(A_m) \eta(A_m, z) \exp 2\pi i s_m \\ \hat{g}_i(B_m z, u, x, s) &= \hat{g}_i(z,u,x,s) \rho_{t_0}(B_m) \eta(B_m, z) \cdot \\ &\qquad \cdot \exp 2\pi i \Big(-\sum_{j=1}^{n} w_m(u_j, a_j) - \sum_{k=1}^{g} w_m(x_k, b_k) + \sum_{l=1}^{g} s_l z_{lm} \Big) \end{aligned}$$

for $1 \le m \le g$; hence for each fixed point (u,x,s) these functions are $r+n+1$ linearly independent meromorphic relatively automorphic functions for the factor of automorphy $\rho_t\eta$ where

(7.2)

$$t_m = \begin{cases} t_{0m} + s_m \\ \qquad \text{for} \quad 1 \le m \le g, \\ \\ t_{0m} - \sum_{j=1}^{n} w_{m-g}(u_j, a_j) - \sum_{k=1}^{g} w_{m-g}(x_k, b_k) + \sum_{l=1}^{g} s_l z_{l\,m-g} \\ \qquad \text{for} \quad g+1 \le m \le 2g, \end{cases}$$

and t_{0m} are the coordinates of the initial point $t_0 \in \mathbb{C}^{2g}$. In particular since $\hat{q}(z,p;a_j,a_j) = \hat{q}(z,q;b_k,b_k) = 1$ by Theorem 5.16 (i) it follows that

$$\hat{g}_i(z,a,b,0) = g_i(z) \prod_{j=1}^{n} h_{a_j}(z)^{-1} \in \Gamma(M, \mathcal{M}(\rho_{t_0}\eta));$$

these are $r+n+1$ linearly independent meromorphic relatively automorphic functions for the factor of automorphy $\rho_{t_0}\eta$ with poles at most at the divisors $\mathfrak{a} = a_1 + \cdots + a_n$ and $\mathfrak{b} = b_1 + \cdots + b_g$ and no other singularities, so they are a basis for the vector space

$$Y = \left\{ g \in \Gamma(M, \mathcal{M}(\rho_{t_0}\eta)) \;\middle|\; \mathfrak{d}(g) + a_1 + \cdots + a_n + b_1 + \cdots + b_g \ge 0 \right\}$$

since as in the earlier argument $\dim Y = \gamma(\rho_{t_0}\eta\zeta_{\mathfrak{a}}\zeta_{\mathfrak{b}}) = r+n+1$.

The residue of the meromorphic function $\hat{g}_i(z,u,x,s)$ of the variable $z \in \tilde{M}$ at the simple pole $z = x_p$ is

$$r_{ip}(u,x,s) = \frac{1}{2\pi i} \int_{z \in \partial \tilde{V}_p} \hat{g}_i(z,u,x,s)dz,$$

which is a holomorphic function of the variables $u_j \in \Gamma\tilde{U}_j$, $x_k \in \Gamma\tilde{V}_k$, $s_l \in \mathbb{C}$. Let $R(u,x,s) = \{r_{ip}(u,x,s)\}$ be the $(r+n+1) \times g$ matrix composed of these functions. For any vector $c = (c_0, \dots, c_{r+n}) \in \mathbb{C}^{r+n+1}$ it is evident that $\sum_{l=0}^{r+n} c_l r_{lp}(a,b,0) = 0$ for $1 \le p \le g$ if and only if the linear combination $\hat{g}_c(z) = \sum_{l=0}^{r+n} c_l \hat{g}_l(z,a,b,0)$ is a meromorphic relatively automorphic function $\hat{g}_c \in \Gamma(M, \mathcal{M}(\rho_{t_0}\eta))$ for which $\mathfrak{d}(\hat{g}_c) + \mathfrak{a} \ge 0$. Since the functions $\hat{g}_i(z,a,b,0)$ are a basis for the vector space Y it follows that the functions $\hat{g}_c(z)$ for vectors c for which $\sum_{l=0}^{r+n} c_l r_{lp}(a,b,0) = 0$ for $1 \le p \le g$ span the $(r+n+1-g)$-dimensional space of relatively automorphic functions for the factor of automorphy $\rho_{t_0}\eta$ with singularities at most along the divisor $\mathfrak{a}$; consequently there are precisely $r+n+1-g$ linearly independent such vectors c, so the $(r+n+1) \times g$ matrix $R(u,x,s)$ must have rank g at the point $(u,x,s) = (a,b,0)$. This matrix then also has rank g at all nearby points, so it follows from familiar arguments that after shrinking the neighborhoods $\tilde{U}_j$, $\tilde{V}_k$ further if necessary and choosing a sufficiently small open neighborhood $\tilde{W}$ of the origin in $\mathbb{C}^g$ there will exist $r+n+1-g$ holomorphic mappings

$$c^i : \prod_{j=1}^{n} \tilde{U}_j \times \prod_{k=1}^{g} \tilde{V}_k \times \tilde{W} \longrightarrow \mathbb{C}^{r+n+1}$$

that are linearly independent vectors at each point (u,x,s) and are such that $\sum_{l=0}^{r+n} c^i_l(u,x,s) r_{lp}(u,x,s) = 0$ for $1 \le p \le g$, where $c^i(u,x,s) = \{c^i_l(u,x,s)\}$. In case the sort of arguments required for this conclusion are not altogether familiar, a detailed proof is included in Lemma 7.2 following the proof of the present theorem. The $r+n+1-g$ linear combinations

$$f_i(z,u,s,x) = \sum_{l=0}^{r+n} c^i_l(u,x,s)\hat{g}_l(z,u,x,s)$$

therefore are meromorphic functions of the variables $z \in \tilde{M}$, $u_j \in \tilde{U}_j$, $x_k \in \tilde{V}_k$, $s \in \tilde{W}$ with singularities at most simple poles along the subvarieties $z = Tu_j$, since the singularities along the subvarieties $z = Tx_k$ have been eliminated; and for any fixed point (u,x,s) they are linearly independent meromorphic relatively automorphic functions for the factor of automorphy $\rho_t\eta$, so are a basis for the vector space $\Lambda_\eta(u,t)$ where the parameter $t \in \mathbb{C}^{2g}$ is the function of the parameters (u,x,s) given explicitly by (7.2).

To examine t as a function of the variables (u,x,s) consider the holomorphic mapping

$$F : \prod_{j=1}^{n} \tilde{U}_j \times \prod_{k=1}^{g} \tilde{V}_k \times \tilde{W} \longrightarrow \prod_{j=1}^{n} \tilde{U}_j \times \mathbb{C}^{2g}$$

defined by $F(u,x,s) = \big(u, t(u,x,s)\big)$, and note that in particular $F(a,b,0) = (a,t_0)$. It is natural to write $t(u,x,s) = \big(t'(u,x,s),\, t''(u,x,s)\big) \in \mathbb{C}^g \times \mathbb{C}^g$, since $t(u,x,s)$ is defined separately in (7.2) for the first and second g components; it then follows from (7.2) that the Jacobian matrix J of this mapping at the point $(a,b,0)$ is

$$\left.\begin{pmatrix} \dfrac{\partial u}{\partial u} & \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial s} \\[2ex] \dfrac{\partial t'}{\partial u} & \dfrac{\partial t'}{\partial x} & \dfrac{\partial t'}{\partial s} \\[2ex] \dfrac{\partial t''}{\partial u} & \dfrac{\partial t''}{\partial x} & \dfrac{\partial t''}{\partial s} \end{pmatrix}\right|_{\substack{u=a\\ x=b\\ t=0}} = \begin{pmatrix} \mathrm{I} & 0 & 0 \\ * & 0 & \mathrm{I} \\ * & -W' & Z \end{pmatrix}.$$

The matrix W', consisting of the derivatives of the holomorphic abelian integrals at the points b_k, is nonsingular by the choice of the auxiliary points b_k; consequently the Jacobian matrix J also is nonsingular, so after shrinking the neighborhoods $\tilde{U}_j$, $\tilde{V}_k$ and $\tilde{W}$ further if necessary F will be a biholomorphic mapping onto an open neighborhood of the point $(a, t_0) \in \prod_{j=1}^{n} \tilde{U}_j \times \mathbb{C}^{2g}$. Through this mapping the cross-sections $f_i(z, u, x, s) \in \Gamma(M, \mathcal{O}(\rho_t\eta))$ can be viewed alternatively as cross-sections $f_i(z, u, t) \in \Gamma(M, \mathcal{O}(\rho_t\eta))$ that are meromorphic functions of the parameters (u, t) rather than of the parameters (u, x, s). Although it was assumed for convenience that the canonical parametrization of flat line bundles over M was taken with respect to generators for Γ arising from a marking of the surface, it is clear once the theorem has been proved for these generators that it is valid for the canonical parametrization of flat line bundles with respect to any generators $T_j \in \Gamma$ representing a basis for the homology $H_1(M)$. That suffices to conclude the proof of the theorem.

The auxiliary result used in the proof of the preceding theorem is demonstrated by the following slightly more general lemma.

Lemma 7.2 *If $F(z)$ is an $m \times n$ matrix of holomorphic functions in an open neighborhood U of a point z_0 in a holomorphic variety, and if $\operatorname{rank} F(z) = r$ at all points $z \in U$, then after shrinking the neighborhood U if necessary there are $n-r$ holomorphic mappings $g_i : U \longrightarrow \mathbb{C}^n$ that have linearly independent values at each point $z \in U$ and that satisfy $F(z)g_i(z) = 0$ at each point $z \in U$.*

Proof: After rearranging rows and columns as necessary it can be assumed that the leading $r \times r$ minor of the matrix $F(z_0)$ is nonsingular; and after shrinking the neighborhood U if necessary that also will be the case for the matrices $F(z)$ at all points $z \in U$. The matrix $F(z)$ thus can be decomposed into matrix blocks

$$F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}$$

where the $r \times r$ submatrix $F_{11}(z)$ is nonsingular for all points $z \in U$. Since $\operatorname{rank} F(z) = r$ the last $m - r$ rows of the matrix $F(z)$ are unique linear combinations of the first r rows, where the coefficients in these linear combinations must be holomorphic functions in U; thus

$$\begin{pmatrix} F_{21}(z) & F_{22}(z) \end{pmatrix} = A(z) \begin{pmatrix} F_{11}(z) & F_{12}(z) \end{pmatrix}$$

for some $(m-r) \times r$ matrix $A(z)$ of holomorphic functions of the variable $z \in U$. If $c_i \in \mathbb{C}^{n-r}$ for $1 \le i \le n - r$ are $n - r$ linearly independent constant vectors then the $n - r$ vectors

$$g_i(z) = \begin{pmatrix} -F_{11}(z)^{-1} F_{12}(z) c_i \\ c_i \end{pmatrix} \in \mathbb{C}^n$$

for $1 \le i \le n - r$ are linearly independent holomorphic functions of the variable $z \in U$ such that

$$\begin{pmatrix} F_{11}(z) & F_{12}(z) \end{pmatrix} g_i(z) = 0;$$

and in addition

$$\begin{pmatrix} F_{21}(z) & F_{22}(z) \end{pmatrix} g_i(z) = A(z) \begin{pmatrix} F_{11}(z) & F_{12}(z) \end{pmatrix} g_i(z) = 0$$

as well, so altogether $F(z)g_i(z) = 0$ for $1 \le i \le n-r$. That suffices to conclude the proof.

In the definition (7.1) of the vector space $\Lambda_\eta(u,t)$ the meromorphic cross-sections $f \in \Gamma(M, \mathcal{M}(\rho_t\eta))$ are required to satisfy the condition that

$$\mathfrak{d}(f) + u_1 + \cdots + u_n \ge 0$$

for points $u_i \in \tilde{M}$, a condition that arose naturally in the course of the proof through the use of cross-ratio functions. However the cross-sections f have the same singularities at all points of $\tilde{M}$ representing the same point of M, so it is evident that this condition actually can be viewed as a condition involving points $u_i \in M$ rather than points on the universal covering space; that is the most convenient interpretation for most applications of the theorem. The hypothesis in the preceding theorem that $c(\eta) + n > 2g - 2$ can be replaced by the hypothesis that the family of holomorphic line bundles involved and the singularities allowed for the cross-sections are restricted so that $\dim \Lambda_\eta(u,t)$ is independent of u and t. With these modifications the preceding theorem can be restated as follows.

Corollary 7.3 *Let η be a factor of automorphy describing a holomorphic line bundle on a compact Riemann surface M of genus $g > 0$; let W be a holomorphic subvariety of an open subset of the complex manifold M^n; and consider the family of flat line bundles ρ_t parametrized by a holomorphic subvariety $\tilde{V}$ of an open subset of the parameter space $\mathbb{C}^{2g}$ for the canonical parametrization of flat line bundles over M associated to generators $T_1, \ldots, T_{2g} \in \Gamma$ of the covering translation group Γ of M. If $\dim \Lambda_\eta(u,t) = \nu$ for all points $u = (u_1, \ldots, u_n) \in W$ and $t \in \tilde{V}$ then for any points $a \in W$ and $t_0 \in \tilde{V}$ there are open neighborhoods $U' \subset M^n$ of the point a and $\tilde{U}'' \subset \mathbb{C}^{2g}$ of the point t_0 and ν meromorphic functions $f_i(z,u,t)$ on the holomorphic variety $\tilde{M} \times (U' \cap W) \times (\tilde{U}'' \cap V)$ such that for any fixed points $u \in U' \cap W$ and $t \in \tilde{U}'' \cap V$ these functions are a basis for the vector space $\Lambda_\eta(u,t)$.*

Proof: If $c(\eta) + n > 2g - 2$ the desired result is just that of the preceding theorem, upon recalling that in the conclusions of that theorem it can be assumed that the parameters u_i lie in the Riemann surface M rather than in its universal covering space. If $c(\eta)+n \le 2g-2$ choose $m = 2g - c(\eta) - n - 1$ points $b_1, \ldots, b_m$ on M which represent distinct points of M which are also distinct from the points a_j; and let $h_\mathfrak{b} \in \Gamma(M, \mathcal{O}(\zeta_{b_1 + \cdots + b_m}))$ be a holomorphic cross-section such that $\mathfrak{d}(h_\mathfrak{b}) = b_1 + \cdots + b_m$. Since $c(\eta\zeta_\mathfrak{a}\zeta_\mathfrak{b}) = c(\eta) + n + m = 2g - 1$ it follows from the preceding theorem applied to the line bundle $\eta\zeta_\mathfrak{b}$ that there are open neighborhoods $U'_j \subset M$ of the points a_j, an open neighborhood $\tilde{U}'' \subset \mathbb{C}^{2g}$ of the point t_0, and g meromorphic functions $g_l(z,u,t)$ on the variety $\tilde{M} \times U'_1 \times \cdots \times U'_n \times \tilde{U}''$ with singularities at most simple poles along the subvarieties $z = Tu_j$ for $T \in \Gamma,\ 1 \le j \le n$, such that for any fixed points $u_j \in U'_j,\ t \in \tilde{U}''$ these functions are a basis for the g-dimensional vector space

$$X = \left\{ g \in \Gamma(M, \mathcal{M}(\rho_t\eta\zeta_\mathfrak{b})) \,\middle|\, \mathfrak{d}(g) + u_1 + \cdots + u_n \ge 0 \right\}.$$

Assume that the neighborhoods U'_j are sufficiently small that they do not contain any of the points b_k, and set $U' = U'_1 \times \cdots \times U'_n \subset M^n$. If $(u_1, \ldots, u_n) \in U' \cap W$ and $t \in \tilde{U}'' \cap V$ and if $c = (c_1, \ldots, c_g) \in \mathbb{C}^g$ is a vector such that $\sum_{l=1}^{g} c_l g_l(b_k, u, t) = 0$ for $1 \le k \le m$ then $g_c(z, u, t) = \sum_{l=1}^{g} h_{\mathfrak{b}}(z)^{-1} c_l g_l(z, u, t)$ is an element of the vector space $\Lambda_\eta(u, t)$. Conversely whenever $g(z)$ is an element of the vector space $\Lambda_\eta(u, t)$ where $(u_1, \ldots, u_n) \in U' \cap W$ and $t \in \tilde{U}'' \cap V$ then $h_{\mathfrak{b}}(z) g(z)$ is an element of the vector space X that vanishes at the distinct points b_k, hence it can be written $g(z) = \sum_{l=1}^{g} c_l g_l(b_k, u, t)$ for some vector $c = (c_1, \ldots, c_g) \in \mathbb{C}^g$ for which $\sum_{l=1}^{g} c_l g_l(b_k, u, t) = 0$ for $1 \le k \le m$. Since $\dim \Lambda_\eta(u, t) = \nu$ it follows that the set of vectors c such that $\sum_{l=1}^{g} c_l g_l(b_k, u, t) = 0$ for $1 \le k \le m$ is a vector space of dimension ν; consequently the $g \times m$ matrix $G(u, t) = \{g_l(b_k, u, t)\}$ has rank $g - \nu$ for any points $(u_1, \ldots, u_n) \in U' \cap W$ and $t \in \tilde{U}'' \cap V$. It then follows from Lemma 7.2 that, after shrinking the neighborhoods U'_j and $\tilde{U}''$ sufficiently, there are ν holomorphic mappings $c^i : (U' \cap W) \times (\tilde{U}'' \cap V) \longrightarrow \mathbb{C}^g$ that have linearly independent values at each point and satisfy $\sum_{l=1}^{g} c^i_l(u, t) g_l(b_k, u, t) = 0$ for $1 \le k \le m$ and for all points $u \in U' \cap W$ and $t \in \tilde{U}'' \cap V$, where $c^i(u, t) = \{c^i_l(u, t)\}$. The functions $f_i(z, u, t) = \sum_{l=1}^{g} c^i_l(u, t) h_{\mathfrak{b}}(z)^{-1} g_l(z, u, t)$ are meromorphic functions on the variety $\tilde{M} \times (U' \cap W) \times (\tilde{U}'' \cap V)$, and for any fixed point $(u, t) \in (U' \cap W) \times (\tilde{U}'' \cap V)$ they are a basis for the vector space $\Lambda_\eta(u, t)$. That suffices to conclude the proof of the corollary.

7.2 Bases for Automorphic Functions for Equivalent Factors of Automorphy

Two factors of automorphy $\rho_{t'}\eta$ and $\rho_{t''}\eta$ are holomorphically equivalent whenever $t' - t'' \in {}^t\Omega\mathbb{C}^g + \mathbb{Z}^{2g}$, by Corollary 3.14; and in that case the preceding results can be extended somewhat to show that the relatively automorphic functions for these two bundles are related through this holomorphic equivalence, when that equivalence is described fairly explicitly. For this purpose select generators $T_j \in \Gamma$ associated to a homology basis on the Riemann surface M and a basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ for the holomorphic abelian differentials on M; so the associated period matrix of the surface is the $g \times 2g$ complex matrix Ω with entries $\omega_{ij} = \omega_i(T_j)$. In these terms introduce the auxiliary function $\phi(z, s)$ of points $(z, s) \in \tilde{M} \times \mathbb{C}^g$ defined by

$$\phi(z, s) = \exp 2\pi i \sum_{k=1}^{g} s_k w_k(z, p) \tag{7.3}$$

where $w_k(z, p) = \int_p^z \omega_i$ are the integrals of the abelian differentials ω_i for a base point $p \in \tilde{M}$; as a function of the variable $z \in \tilde{M}$ this is a holomorphic and nowhere vanishing function on the universal covering surface $\tilde{M}$, and as a function of the variable $s = (s_1, \ldots, s_g) \in \mathbb{C}^g$ it is a homomorphism from the additive group $\mathbb{C}^g$ to the multiplicative group $\mathbb{C}^*$. Note that

$$\begin{aligned}\phi(T_j z, s) &= \exp 2\pi i \sum_{k=1}^{g} s_k w_k(T_j z, p) = \exp 2\pi i \sum_{k=1}^{g} s_k \big(w_k(z,p) + \omega_{kj}\big) \\ &= \rho_{{}^t\Omega s}(T_j)\phi(z,s)\end{aligned}$$

for the flat line bundle $\rho_{{}^t\Omega s} \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ described by vector ${}^t\Omega s \subset \mathbb{C}^{2g}$. Since $\rho_{{}^t\Omega s}$ is a group homomorphism iterating the preceding formula shows that

$$\begin{aligned}\phi(T_j T_k z, s) &= \rho_{{}^t\Omega s}(T_j)\phi(T_k z, s) = \rho_{{}^t\Omega s}(T_j)\rho_{{}^t\Omega s}(T_k)\phi(z,s) \\ &= \rho_{{}^t\Omega s}(T_j T_k)\phi(z,s);\end{aligned}$$

and since any element $T \in \Gamma$ can be written as a product of the elements T_j and their inverses it follows that

$$\phi(Tz, s) = \rho_{{}^t\Omega s}(T)\phi(z,s) \tag{7.4}$$

for all $T \in \Gamma$. This exhibits explicitly the holomorphic triviality of the flat line bundles parametrized by these points $t \in {}^t\Omega\mathbb{C}^g$, and hence the holomorphic equivalence of the flat line bundles parametrized by points $t', t'' \in \mathbb{C}^{2g}$ for which $t' - t'' \in {}^t\Omega\mathbb{C}^g$. Of course it is clear from the definition of the canonically parametrized flat line bundle ρ_t is the identity bundle whenever $t \in \mathbb{Z}^{2g}$.

There is a convenient interpretation of this holomorphic equivalence that is worth noting here. The holomorphic equivalence of flat line bundles of Corollary 3.14 was expressed by the exact sequence

$$0 \longrightarrow {}^t\Omega\mathbb{C}^g + \mathbb{Z}^{2g} \xrightarrow{\iota} {}^t\mathbb{C}^g \xrightarrow{p} P(M) \longrightarrow 0 \tag{7.5}$$

where ι is the inclusion mapping and p is the mapping that associates to a point $t \in \mathbb{C}^{2g}$ the holomorphic line bundle represented by the canonically parametrized flat line bundle $\rho_t \in P(M)$ for the Picard variety $P(M)$ of the Riemann surface M. In this exact sequence (7.5) each coset $t + {}^t\Omega\mathbb{C}^g + \mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ consists of all the points of $\mathbb{C}^{2g}$ parametrizing an holomorphic equivalence class of flat line bundles; but for present purposes it is convenient to describe these cosets slightly differently. The full period matrix $\binom{\Omega}{\overline{\Omega}}$ of the Riemann surface M is a nonsingular $2g \times 2g$ matrix, and the inverse of its transpose conjugate is a matrix of the form $\binom{\Pi}{\overline{\Pi}}$ where the $g \times 2g$ matrix Π is the *inverse period matrix* to Ω; thus Π is a $g \times 2g$ matrix such that

$$\Pi\,{}^t\Omega = 0, \quad \Pi\,{}^t\overline{\Omega} = \mathrm{I}_g, \quad {}^t\Omega\overline{\Pi} + {}^t\overline{\Omega}\Pi = \mathrm{I}_{2g}, \tag{7.6}$$

where I_r denotes the identity matrix of rank r. The last identity in (7.6) shows that any point $s \in \mathbb{C}^{2g}$ can be written $s = {}^t\Omega(\overline{\Pi}s) + {}^t\overline{\Omega}(\Pi s)$, which is an explicit formula for the direct sum decomposition

$$\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g \tag{7.7}$$

of the vector space $\mathbb{C}^{2g}$ into the two complementary linear subspaces ${}^t\Omega\mathbb{C}^g$ and ${}^t\overline{\Omega}\mathbb{C}^g$. Points $t \in {}^t\overline{\Omega}\mathbb{C}^g$ thus can be viewed as representing cosets $t + {}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$; and since parameter values in ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ describe holomorphically trivial flat line bundles it follows that any holomorphic equivalence class of flat line bundles can be parametrized by a point $t \in {}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$. Different points of ${}^t\overline{\Omega}\mathbb{C}^g$ still may parametrize holomorphically equivalent flat line bundles though. The first two identities in (7.6) show that the linear mapping ${}^t\overline{\Omega}\Pi : \mathbb{C}^{2g} \longrightarrow \mathbb{C}^{2g}$ defined by the $2g\times 2g$ matrix ${}^t\overline{\Omega}\Pi$ is the zero mapping on the linear subspace ${}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ and is the identity mapping on the linear subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$; thus it is the natural projection of the direct sum (7.7) to its second factor. Applying this projection to the exact sequence (7.5) hence yields the exact sequence

$$0 \longrightarrow {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \xrightarrow{\iota} {}^t\overline{\Omega}\mathbb{C}^g \xrightarrow{p_0} P(M) \longrightarrow 0 \tag{7.8}$$

in which ι again is the natural inclusion mapping and p_0 is just the restriction $p_0 = p\big|\,{}^t\overline{\Omega}\mathbb{C}^g$ of the mapping p in the sequence (7.5) to the subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$. The resulting description of the Picard variety $P(M)$ as the quotient $P(M) = {}^t\overline{\Omega}\mathbb{C}^g / {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$ clearly is equivalent to the customary description as the quotient $P(M) = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$; so the mapping p_0 in (7.8) is a covering projection, and two points of ${}^t\overline{\Omega}\mathbb{C}^g$ parametrize holomorphically equivalent flat line bundles precisely when they differ by a point in the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \subset {}^t\overline{\Omega}\mathbb{C}^g$. If $U \subset P(M)$ is a contractible open subset and $\hat{U} \subset {}^t\overline{\Omega}\mathbb{C}^g$ is a connected component of the inverse image $p_0^{-1}(U)$ under the covering projection p_0 of (7.8) the restriction of the mapping p_0 to the set $\hat{U}$ is a one-to-one mapping $p_0 : \hat{U} \longrightarrow U$; so parameters $t \in \hat{U}$ can be used as local coordinates in the subset $U \subset P(M)$. The complete inverse image of the subset U under the covering projection p_0 is the disjoint union

$$p_0^{-1}(U) = \bigcup_{\nu\in\mathbb{Z}^{2g}} \Big(\hat{U} + {}^t\overline{\Omega}\Pi\nu\Big) \subset {}^t\overline{\Omega}\mathbb{C}^g \tag{7.9}$$

of translates of the subset $\hat{U} \subset {}^t\overline{\Omega}\mathbb{C}^g$ by points in the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \subset {}^t\overline{\Omega}\mathbb{C}^g$. The complete inverse image of the subset U under the mapping p of the exact sequence (7.5), the set of all points of $\mathbb{C}^{2g}$ parametrizing holomorphic line bundles in the subset $U \subset P(M)$, is the disjoint union

$$p^{-1}(U) = \bigcup_{\nu\in\mathbb{Z}^{2g}} \Big(\hat{U} + {}^t\Omega\mathbb{C}^g + {}^t\overline{\Omega}\Pi\nu\Big) \subset \mathbb{C}^{2g} \tag{7.10}$$

of translates of the connected open subset $\hat{U} + {}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$ by points of the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \subset {}^t\overline{\Omega}\mathbb{C}^g$.

Corollary 7.4 *Let η be a factor of automorphy describing a holomorphic line bundle on a compact Riemann surface M of genus $g > 0$, let W be a holomorphic subvariety of an open subset of the complex manifold M^n, let V be a holomorphic subvariety of an open subset of the Picard variety $P(M)$ and set $\tilde{V} = p^{-1}(V) \subset \mathbb{C}^{2g}$; and suppose that* $\dim \Lambda_\eta(u,t) = \nu$ *for all points $u = (u_1, \dots, u_n) \in W$ and*

$t \in \tilde{V} \subset \mathbb{C}^{2g}$. For any points $u_0 \in W$ and $\xi_0 \in V$ there are open neighborhoods $U' \subset M^n$ of the point u_0 and $U'' \subset P(M)$ of the point ξ_0 and ν meromorphic functions $f_i(z,u,t)$ on the holomorphic variety $\tilde{M} \times (W \cap U') \times (\tilde{V} \cap \tilde{U}'')$ for $1 \le i \le \nu$, where $\tilde{U}'' = p^{-1}(U'') \subset \mathbb{C}^{2g}$, such that these functions are a basis for the vector space $\Lambda_\eta(u,t)$ at any point $(u,t) \in (W \cap U') \times (\tilde{V} \cap \tilde{U}'')$ and satisfy

$$f_i(z,u,t + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu) = \phi(z, s - \overline{\Pi}\nu) f_i(z,u,t) \tag{7.11}$$

for all points $z \in \tilde{M},\ u \in W \cap U',\ t \in \tilde{V} \cap \tilde{U}'',\ s \in \mathbb{C}^g,\ \nu \in \mathbb{Z}^{2g}$.

Proof: If $(u_0, \xi_0) \in W \times V$ and $t_0 \in p_0^{-1}(\xi_0) \subset {}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$ it follows from Corollary 7.3 that there are open neighborhoods $U' \subset M^n$ of u_0 and $\tilde{U} \subset \mathbb{C}^{2g}$ of t_0 and meromorphic functions $\tilde{f}_i(z,u,t)$ on the holomorphic variety $\tilde{M} \times (W \cap U') \times (\tilde{V} \cap \tilde{U})$ that are a basis for the vector space $\Lambda_\eta(u,t)$ at any point $(u,t) \in (W \cap U') \times (\tilde{V} \cap \tilde{U})$; these functions thus satisfy

$$\tilde{f}_i(Tz,u,t) = \rho_t(T)\eta(T,z)\tilde{f}_i(z,u,t) \tag{7.12}$$

for all $T \in \Gamma$ and all points $(z,u,t) \in \tilde{M} \times (W \cap U') \times (\tilde{V} \cap \tilde{U})$. By restricting the neighborhood $\tilde{U}$ suitably it can be assumed that the intersection $\hat{U} = \tilde{U} \cap {}^t\overline{\Omega}\mathbb{C}^g$ is a contractible open neighborhood of t_0 in the linear subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$, so is homeomorphic to an open neighborhood $U'' \subset P(M)$ of the point ξ_0 under the restriction $p_0 : \hat{U} \longrightarrow U''$ of the covering projection p_0. The restriction of the function $\tilde{f}_i(z,u,t)$ then is a meromorphic function $\hat{f}_i(z,u,t)$ on $\tilde{M} \times (W \cap U') \times (\tilde{V} \cap \hat{U})$. For any points $z \in \tilde{M},\ u \in W \cap U',\ \hat{t} \in \tilde{V} \cap \hat{U},\ s \in \mathbb{C}^g,\ \nu \in \mathbb{Z}^{2g}$ set

$$f_i(z,u,\hat{t} + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu) = \phi(z, s - \overline{\Pi}\nu)\hat{f}_i(z,u,\hat{t}).$$

If $\tilde{U}'' = p^{-1}(U'') \subset \mathbb{C}^{2g}$ then since any point $t \in \tilde{V} \cap \tilde{U}''$ can be written uniquely as the sum $t = \hat{t} + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu$ for points $\hat{t} \in \tilde{V} \cap \hat{U},\ s \in \mathbb{C}^g,\ \nu \in \mathbb{Z}^{2g}$ in view of the decomposition (7.10) it is evident that these functions are meromorphic functions $f_i(z,u,t)$ of the variables $(z,u,t) \in \tilde{M} \times (W \cap U') \times (\tilde{V} \cap \tilde{U}'')$. If $t = \hat{t} + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu$ and $T \in \Gamma$ it follows from (7.12) and (7.4) that

$$\begin{aligned} f_i(Tz,u,t) &= \phi(Tz, s - \overline{\Pi}\nu)\hat{f}_i(Tz,u,\hat{t}) \\ &= \rho_{{}^t\Omega(s-\overline{\Pi}\nu)}(T)\phi(z,s-\overline{\Pi}\nu) \cdot \rho_{\hat{t}}(T)\eta(T,z)\hat{f}_i(z,u,\hat{t}) \\ &= \rho_t(T)\eta(T,z)f_i(z,u,t), \end{aligned}$$

since $\hat{t} + {}^t\Omega s - {}^t\Omega\overline{\Pi}\nu = \hat{t} + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu - \nu = t - \nu$ by the second identity in (7.6) and $\rho_{t-\nu}(T) = \rho_t(T)$; thus $f_i(z,u,t) \in \Lambda_\eta(u,t)$ for all $u \in W \cap U'$ and $t \in \tilde{V} \cap \tilde{U}''$, and since these functions are linearly independent they are a basis for the vector space $\Lambda_\eta(u,t)$. Furthermore for any $s_0 \in \mathbb{C}^g$ and $\nu_0 \in \mathbb{Z}^{2g}$

$$\begin{aligned} f_i(z,u,t + {}^t\Omega s_0 + {}^t\overline{\Omega}\Pi\nu_0) &= f_i\big(z,u,\hat{t} + {}^t\Omega(s+s_0) + {}^t\overline{\Omega}\Pi(\nu+\nu_0)\big) \\ &= \phi\big(z, s + s_0 - \overline{\Pi}(\nu+\nu_0)\big)\hat{f}_i(z,u,\hat{t}) \\ &= \phi(z, s_0 - \overline{\Pi}\nu_0) \cdot \phi(z, s - \overline{\Pi}\nu)\hat{f}_i(z,u,\hat{t}) = \phi(z, s_0 - \overline{\Pi}\nu_0) f_i(z,u,t). \end{aligned}$$

That suffices to conclude the proof.

When the subvarieties $W \subset M^n$ and $V \subset P(M)$ are sufficiently regular the local result of the preceding corollary can be extended to the following global assertion.

Theorem 7.5 *Let η be a factor of automorphy describing a holomorphic line bundle on a compact Riemann surface M of genus $g > 0$, let W be a holomorphic submanifold of an open subset of the complex manifold M^n, let V be a holomorphic submanifold of an open subset of the Picard variety $P(M)$ and set $\tilde{V} = p^{-1}(V) \subset \mathbb{C}^{2g}$; and suppose that $\dim \Lambda_\eta(u,t) = \nu$ for all points $u = (u_1, \ldots, u_n) \in W$ and $t \in \tilde{V}$. Then the union*

$$\Lambda_\eta(W, \tilde{V}) = \bigcup_{u \in W,\ t \in \tilde{V}} \Lambda_\eta(u,t) \tag{7.13}$$

has a uniquely determined structure as a holomorphic vector bundle of rank ν over the complex submanifold $W \times \tilde{V}$ such that for any sufficiently fine coordinate coverings $\{U'_a\}$ of the complex manifold M^n and $\{U''_l\}$ of the complex manifold $P(M)$ there are meromorphic functions $f_{al,i}(z,u,t)$ on the holomorphic submanifolds $\tilde{M} \times (W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l)$ for $1 \le i \le \nu$, where $\tilde{U}''_l = p^{-1}(U''_l) \subset \mathbb{C}^{2g}$, that are a basis for the vector space $\Lambda_\eta(u,t)$ for any fixed point $(u,t) \in (W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l)$ and that for any fixed point $z \in \tilde{M}$ are a meromorphic cross-section of the vector bundle $\Lambda_\eta(W, \tilde{V})$. Under translation of the parameter $t \in \tilde{V}$ through vectors in ${}^t\Omega\mathbb{C}^g + {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$ the vector bundle $\Lambda_\eta(W, \tilde{V})$ is invariant while the functions $f_{al,i}(z,u,t)$ satisfy 7.11.

Proof: For any sufficiently fine open coverings $\{U'_a\}$ of the product M^n and $\{U''_l\}$ of the Picard variety $P(M)$ the conclusions of Corollary 7.4 hold for the products $U'_a \times U''_l$; thus there are ν meromorphic functions $f_{al,i}(z,u,t)$ on the holomorphic submanifolds $\tilde{M} \times (W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l)$ for $1 \le i \le \nu$, where $\tilde{U}''_l = p^{-1}(U''_l) \subset \mathbb{C}^{2g}$, that are a basis for the vector space $\Lambda_\eta(u,t)$ at any point $(u,t) \in (W \cap U'_a) \times (\tilde{V} \cap U''_l)$ and that satisfy (7.11). The union of the subspaces $\Lambda_\eta(u,t)$ for points $(u,t) \in (W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l)$ can be identified with the product $(W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l) \times \mathbb{C}^\nu$ by associating to any point $(u,t,x) \in (W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l) \times \mathbb{C}^\nu$ the element $\sum_{i=1}^{\nu} x_i f_{al,i}(z,u,t) \in \Lambda_\eta(u,t)$; the union (7.13) thus has a local product structure over each subset $(W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l) \subset W \times \tilde{V}$. To combine these local product structures into a vector bundle note that there are two sets of meromorphic functions $f_{al,i}(z,u,t)$ and $f_{bm,j}(z,u,t)$ of the variable $z \in \tilde{M}$ that are bases for the vector space $\Lambda_\eta(u,t)$ at any point (u,t) in an intersection $(W \cap U'_a \cap U'_b) \times (\tilde{V} \cap \tilde{U}''_l \cap \tilde{U}''_m)$; hence there are uniquely determined complex values $\lambda_{ab,lm;ij}(u,t)$ depending on the point (u,t) such that

$$f_{al,i}(z,u,t) = \sum_{j=1}^{\nu} \lambda_{ab,lm;ij}(u,t) f_{bm,j}(z,u,t). \tag{7.14}$$

It follows from Cramer's rule that these values $\lambda_{ab,lm;ij}(u,t)$ are determinants of matrices having as entries various of the functions $f_{al,i}(z,u,t)$ and $f_{bm,j}(z,u,t)$, so they are meromorphic functions of the variables (u,t) on the complex manifold $(W \cap U'_a \cap U'_b) \times (\tilde{V} \cap \tilde{U}''_l \cap \tilde{U}''_m)$; and since they take finite values on this complex manifold they are actually holomorphic functions. The nonsingular matrices $\Lambda_{ab,lm}(u,t) = \{\lambda_{ab,lm;ij}(u,t)\}$ are thus holomorphic matrix-valued functions on the complex manifold $(W \cap U'_a \cap U'_b) \times (\tilde{V} \cap \tilde{U}''_l \cap \tilde{U}''_m)$; and it follows immediately from their uniqueness that $\Lambda_{ab,lm}(u,t) \cdot \Lambda_{bc,mn}(u,t) = \Lambda_{ac,ln}(u,t)$ whenever $(u,t) \in (W \cap U'_a \cap U'_b \cap U'_c) \times (\tilde{V} \cap \tilde{U}''_l \cap \tilde{U}''_m \cap \tilde{U}''_n)$, so they are the coordinate transition functions describing a holomorphic vector bundle $\Lambda_\eta(W,\tilde{V})$ of rank ν over the complex manifold $W \times \tilde{V}$. Equation(7.14) is just the condition that for any fixed point $z \in \tilde{M}$ the functions $f_{al,i}(z,u,t)$ are a meromorphic cross-section of the vector bundle $\Lambda_\eta(W,\tilde{V})$. Any other local meromorphic functions $g_{al,i}(z,u,t)$ that are bases of the vector spaces $\Lambda_\eta(u,t)$ can be expressed in terms of the functions $f_{al,i}(z,u,t)$ as

$$g_{al,i}(z,u,t) = \sum_{j=1}^{\nu} c_{al;ij}(u,t) \cdot f_{al,j}(z,u,t)$$

for some nonsingular complex matrices $C_{al}(u,t) = \{c_{al;ij})u,t)\}$; again the entries in these matrices are bounded meromorphic functions and consequently holomorphic functions on the submanifolds $(W \cap U'_a) \times (\tilde{V} \cap \tilde{U}''_l)$. The coordinate transition functions for the vector bundle defined in terms of this alternative basis are $C_{al}(u,t)\ \Lambda_{ab,lm}(u,t)\ C_{al}(u,t)^{-1}$, so they describe the same holomorphic vector bundle $\Lambda_\eta(W,\tilde{V})$; and that demonstrates the uniqueness of this vector bundle. For any vector ${}^t\Omega s + {}^t\overline{\Omega}\Pi\nu \in {}^t\Omega\mathbb{C}^g + {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$ equation (7.14) takes the form

$$\begin{aligned} f_{al,i}(z,u,t + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu) \quad &= \quad \sum_{j=1}^{\nu} \lambda_{ab,lm;ij}(u,t + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu) \cdot \\ &\qquad \cdot f_{bm,j}(z,u,t + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu); \end{aligned}$$

and since the functions $f_{al,i}(z,u,t)$ and $f_{bm,j}(z,u,t)$ satisfy (7.11) with the same factor $\phi(z, s - \overline{\Pi}\nu)$ it follows from the preceding equation that

$$f_{al,i}(z,u,t) \quad = \quad \sum_{j=1}^{\nu} \lambda_{ab,lm;ij}(u,t + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu) f_{bm,j}(z,u,t).$$

Comparing this last equation with (7.14) and using the uniqueness of the coefficients $\lambda_{ab,lm;ij}(u,t)$ shows that

$$\lambda_{ab,lm;ij}(u,t + {}^t\Omega s + {}^t\overline{\Omega}\Pi\nu) = \lambda_{ab,lm;ij}(u,t), \tag{7.15}$$

so the vector bundle $\Lambda_\eta(W,\tilde{V})$ is invariant under translation of the variable $t \in \mathbb{C}^{2g}$ by vectors in ${}^t\Omega\mathbb{C}^g + {}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$ and that suffices to conclude the proof.

The proof of the preceding theorem required the subsets $W \subset M^n$ and $V \subset P(M)$ to be submanifolds rather than merely holomorphic subvarieties in order to show that the coordinate transition functions for the holomorphic vector bundle $\Lambda_\eta(W, \tilde{V})$ are holomorphic functions rather than merely bounded meromorphic functions (weakly holomorphic functions)[1]. For most applications this additional hypothesis is not a problem. Without this additional hypothesis the vector bundle $\Lambda_\eta(W, \tilde{V})$ at least is a weakly holomorphic vector bundle over the subvariety $W \times \tilde{V}$. The invariance of the vector bundle $\Lambda_\eta(W, \tilde{V})$ under translation of the parameter $t \in \tilde{V}$ through vectors in ${}^t\Omega\mathbb{C}^g + \mathbb{Z}^{2g}$, means that this vector bundle induces a holomorphic vector bundle $\Lambda_\eta(W, V)$ over the quotient submanifold $W \times V$ when the Picard variety $P(M)$ is described as in (7.5) as the quotient space of $\mathbb{C}^{2g}$ under the group of such translations. The meromorphic functions $f_{al,i}(z, u, t)$ are not invariant under such translations so they do not describe a meromorphic cross-section of the induced vector bundle $\Lambda_\eta(W, V)$ over $W \times V$; but they do satisfy the relative invariance condition (7.11), which has a related interpretation. To see this assume further that the open subsets $U_l'' \subset P(M)$ are contractible and that the intersections $U_l'' \cap U_m''$ are connected, and choose a connected component $\hat{U}_l \subset {}^t\overline{\Omega}\mathbb{C}^g$ of the inverse image $p_0^{-1}(U_l'')$ of each set U_l'' under the covering projection p_0 of (7.8). In view of the decomposition (7.7) the bundle $\Lambda_\eta(V, W)$ can be described equivalently as the quotient of the restriction $\lambda_\eta(V, \tilde{W})|(V \times \tilde{W} \cap {}^t\overline{\Omega}\Pi\mathbb{C}^g$ under the action of the covering translation group of the covering p_0 of (7.8), the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g} \subset {}^t\overline{\Omega}\mathbb{C}^g$. The restriction is defined by the coordinate transition functions $\Lambda_{ab,lm}(u, t)$ in the intersections

$$(7.16)\quad (W \cap U_a' \cap U_b') \times (\tilde{V} \cap U_l'' \cap U_m'' \cap {}^t\overline{\Omega}\mathbb{C}^g = (W \cap U_a' \cap U_b') \times (\tilde{V} \cap p_0^{-1}(U_l \cap U_m),$$

and these coordinate functions are invariant under translation through the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$. In view of (7.9) the intersection (7.16) is a union of disjoint components, translates of the intersection $(W \cap U_a' \cap U_b') \times (\tilde{V} \cap \hat{U}_l \cap \hat{U}_m)$ by vectors in the lattice subgroup ${}^t\overline{\Omega}\Pi\mathbb{Z}^{2g}$; consequently the bundle $\Lambda_\eta(V, W)$ can be described by the restrictions of the coordinate transition functions $\Lambda_{ab,lm}(u, t)$ to the intersections $(W \cap U_a' \cap U_b') \times (\tilde{V} \cap \hat{U}_l \cap \hat{U}_m)$ where the subsets $\hat{U}_l$ are viewed as a coordinate covering of the Picard variety $P(M)$. If $t_l \in \hat{U}_l$ and $t_m \in \hat{U}_m$ have the same image $p_0(t_l) = p_0(t_m) \in U_l'' \cap U_m''$ then it follows from the exact sequence (7.8) that

$$(7.17)\qquad t_l - t_m = {}^t\overline{\Omega}\Pi\nu_{lm} \quad \text{where} \quad \nu_{lm} \in \mathbb{Z}^{2g},$$

and the integer ν_{lm} is independent of the point $p_0(t_l) = p_0(t_m) \in U_l'' \cap U_m''$ since this intersection is assumed to be connected. If points $t_l \in \hat{U}_l,\ t_m \in \hat{U}_m,\ t_n \in \hat{U}_n$ have the same image $p_0(t_l) = p_0(t_m) = p_0(t_n) \in U_l'' \cap U_m'' \cap U_n''$ then the analogue of (7.17) holds for any pair of these points, so

$$0 = (t_l - t_m) + (t_m - t_n) + (t_n - t_l) = {}^t\overline{\Omega}(\Pi\nu_{lm} + \Pi\nu_{mn} + \Pi\nu_{nl});$$

[1]Weakly holomorphic functions are discussed on page 417 in Appendix A.

and since the linear mapping ${}^t\overline{\Omega} : \mathbb{C}^g \longrightarrow \mathbb{C}^{2g}$ is injective it follows that $\Pi\nu_{lm} + \Pi\nu_{mn} + \Pi\nu_{nl} = 0$, so taking complex conjugates $\overline{\Pi}\nu_{lm} + \overline{\Pi}\nu_{mn} + \overline{\Pi}\nu_{nl} = 0$ since $\overline{\nu}_{lm} = \nu_{lm}$. Therefore for any fixed point $z \in \tilde{M}$

$$
\begin{aligned}
&\phi(z,\ -\overline{\Pi}\nu_{lm})\,\phi(z,\ -\overline{\Pi}\nu_{mn})\,\phi(z,\ -\overline{\Pi}\nu_{nl}) \\
&= \quad \phi(z,\ -\overline{\Pi}\nu_{lm} - \overline{\Pi}\nu_{mn} - \overline{\Pi}\nu_{nl}) = \phi(z,0) = 1,
\end{aligned}
\tag{7.18}
$$

and the complex constants $\phi(z,\ -\overline{\Pi}\nu_{lm})$ for that fixed point $z \in \tilde{M}$ consequently can be viewed as the coordinate transition functions describing a flat line bundle Φ_z over the Picard variety $P(M)$ in terms of the coordinate covering $\{\hat{U}_l\}$.

Corollary 7.6 *The functions $f_{al,i}(z,u,t)$ of the preceding theorem when restricted to the subspace ${}^t\overline{\Omega}\mathbb{C}^g \subset \mathbb{C}^{2g}$ describe a meromorphic cross-section of the vector bundle $\Phi_z \otimes \Lambda_\eta(W,V)$ over $W \times V$.*

Proof: If $u \in U'_a \cap U'_b$ and the points $t_l \in \hat{U}_l$ and $t_m \in \hat{U}_m$ have the same image $p_\eta(t_l) = p_\eta(t_m) \in U''_l \cap U''_m$ then from (7.14), (7.17) and the invariance conditions (7.15) and (7.11) it follows that

$$
\begin{aligned}
f_{al,i}(z,u,t_l) &= \sum_{j=1}^{\nu} \lambda_{ab,lm;ij}(u,t_l)\ f_{bm,j}(z,u,t_l) \\
&= \quad \sum_{j=1}^{\nu} \lambda_{ab,lm;ij}(u, t_m + {}^t\overline{\Omega}\Pi\nu_{lm})\ f_{bm,j}(z,u,t_m + {}^t\overline{\Omega}\Pi\nu_{lm}) \\
&= \quad \sum_{j=1}^{\nu} \lambda_{ab,lm;ij}(u,t_m)\cdot\phi(z,-\overline{\Pi}\nu_{lm})\ f_{bm,j}(z,u,t_m);
\end{aligned}
$$

that is just the condition that for a fixed point $z \in \tilde{M}$ the functions $f_{al,i}(z,u,t_l)$ of the variables $(u,t_l) \in (U'_a \times \hat{U}_l) \subset (M^n \times {}^t\overline{\Omega}\mathbb{C}^g)$ are a meromorphic cross-section of the product bundle $\Phi_z \otimes \Lambda_\eta(W,V)$ over $W \times V$, which concludes the proof.

For many applications of the results in this section the primary interest is in cross-sections $f \in \Lambda_\eta(u,t)$ for a compact Riemann surface M of genus $g > 0$ viewed as functions of the parameters $u \in M^n$ and $\xi = \rho_t\eta \in P_r(M)$, where $P_r(M)$ is the connected component of the extended Picard variety consisting of holomorphic line bundles ξ of characteristic class $c(\xi) = r$; and there is somewhat less interest in the explicit description of the line bundles ξ as being represented by canonically parametrized flat line bundles. Thus in these circumstances it is more convenient to describe the vector space $\Lambda_\eta(u,t)$ in the equivalent form

$$
\Lambda(u,\xi) = \Big\{\ f \in \Gamma(M,\mathcal{M}(\xi))\ \Big|\ \mathfrak{d}(f) + u_1 + \cdots + u_n \geq 0\ \Big\}
\tag{7.19}
$$

where $u = (u_1,\ldots,u_n) \in M^n$ and $\xi \in P_r(M)$; these two forms are clearly equivalent under the identification $\xi = \rho_t\eta$. Coordinate coverings $\hat{U}_l$ of the universal covering space ${}^t\overline{\Omega}\mathbb{C}^g$ of the Picard variety $P(M)$, viewed as coordinate coverings of $P(M)$ itself, can be viewed as coordinate coverings of the component $P_r(M)$ of the extended Picard variety through the same identification $\xi = \rho_t \eta \in P_r(M)$; and unless it is necessary to be more explicit this identification can be ignored. The flat line bundle Φ_z over the Picard variety $P(M)$ can be identified correspondingly with a flat line bundle over $P_r(M)$. Since all flat line bundles over M can be described by local coordinate bundles in a sufficiently fine coordinate covering of the surface M the line bundles ξ in any family of line bundles can be represented by coordinate bundles $\xi_{\alpha\beta}$ in a fixed suitably fine coordinate covering $\{U_\alpha\}$ of M. Then for coordinate coverings U'_a of M^n and $\hat{U}_l$ of $P_r(M)$ the meromorphic cross-sections of Theorem 7.5 can be viewed as local meromorphic functions $f_{\alpha a l}(z_\alpha, u_a, \xi_l)$ of the variables $z \in U_\alpha \subset M$, $u \in U'_a \subset M^n$ and $\xi \in \hat{U}_l \subset P_r(M)$ such that

$$f_{\alpha a l}(z, u, \xi) = \xi_{\alpha\beta}(z) f_{\beta a l}(z, u, \xi) \tag{7.20}$$

for $z \in U_\alpha \cap U_\beta$. In these terms Theorem 7.5 and its corollary can be restated as follows.

Corollary 7.7 *If M is a compact Riemann surface of genus $g > 0$, W is a holomorphic submanifold of an open subset of the compact manifold M^n, and V is a holomorphic submanifold of an open subset of the component $P_r(M)$ of the extended Picard variety of M, and if* $\dim \Lambda(u, \xi) = \nu$ *for all points $(u, \xi) \in W \times V$, then the union*

$$\Lambda(W, V) = \bigcup_{u \in W\ \xi \in V} \Lambda(u, \xi)$$

has a uniquely determined structure as a holomorphic vector bundle of rank ν over the manifold $W \times V$ such that for any sufficiently fine open coordinate coverings $\{U_\alpha\}$ of the surface M, $\{U'_a\}$ of the product M^n, and $\{U''_l\}$ of the complex manifold $P_r(M)$ there are ν meromorphic functions $f_{\alpha a l, i}(z_\alpha, u_a, \xi_l)$ on the subsets $U_\alpha \times (W \cap U'_a) \times (V \cap U''_l)$ that are a basis for the vector space $\Lambda(u, \xi)$ at any point $(u, \xi) \in (W \cap U'_a) \times (V \cap U''_l)$ and that for any fixed point $z_\alpha \in U_\alpha$ are a meromorphic cross-section of the product bundle $\Phi_{z_\alpha} \otimes \Lambda(W, V)$ over $W \times V$.

Proof: This is merely a restatement of the result of the preceding theorem and its corollary, so no further proof is required.

In the special case that $n + r > 2g - 2$ it is possible to take $W = M^n$ and $V = P_r(M)$ in Theorem 7.5 and its corollaries, and the conclusions then take a slightly simpler form.

Chapter 8

Prym Differentials and Prym Cohomology

A *holomorphic Prym differential* for a flat line bundle ρ over a Riemann surface M is a holomorphic cross-section $\sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho))$. Holomorphic differential forms that are cross-sections of an arbitrary holomorphic line bundle were introduced on page 16; holomorphic Prym differentials are just the special case in which the line bundle is a flat line bundle. A holomorphic Prym differential can be viewed as a collection of holomorphic differential forms σ_α in the open neighborhoods of a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the surface M such that $\sigma_\alpha = \rho_{\alpha\beta}\sigma_\beta$ in any nonempty intersection $U_\alpha \cap U_\beta$, when the bundle ρ is described by a flat cocycle $\rho_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathbb{C}^*)$. When these local differential forms are written $\sigma_\alpha = f_\alpha(z_\alpha)dz_\alpha$ in terms of local coordinates z_α in the coordinate neighborhoods U_α, the coefficients $f_\alpha(z_\alpha)$ describe a holomorphic cross-section $f \in \Gamma(M, \mathcal{O}(\rho\kappa))$ where κ is the canonical bundle of the surface M; thus holomorphic Prym differentials can be viewed equivalently as holomorphic cross-sections of the holomorphic line bundle $\rho\kappa$ of characteristic class $c(\rho\kappa) = 2g-2$, and the sheaf $\mathcal{O}^{(1,0)}(\rho)$ of germs of holomorphic Prym differentials for the flat line bundle ρ can be identified in this way with the sheaf $\mathcal{O}(\rho\kappa)$. Correspondingly a *meromorphic Prym differential* is a cross-section $\sigma \in \Gamma(M, \mathcal{M}^{(1,0)}(\rho))$ and can be viewed as a meromorphic cross-section of the holomorphic line bundle $\rho\kappa$. On the other hand a $\mathcal{C}^\infty$ *Prym differential* is defined to be a cross-section $\sigma \in \Gamma(M, \mathcal{E}^1_c(\rho))$, where $\mathcal{E}^1_c(\rho)$ is the sheaf of germs of closed complex-valued $\mathcal{C}^\infty$ differential forms of total degree 1 that are cross-sections of ρ. Holomorphic and meromorphic Prym differentials are automatically closed differential forms on M; $\mathcal{C}^\infty$ Prym differentials are closed by definition, and do not correspond simply to $\mathcal{C}^\infty$ cross-sections of the line bundle $\rho\kappa$. There are no nontrivial flat line bundles over the Riemann sphere $\mathbb{P}^1$ since it is simply connected. There are nontrivial flat line bundles over a compact Riemann surface M of genus $g = 1$; but since the canonical bundle of M is trivial by Corollary **??**, Prym differentials are just cross-sections of a flat line bundle over M. For these reasons the discussion of Prym differentials generally is limited to Riemann surfaces of genus $g > 1$.

Holomorphic Prym differentials have well defined period classes, analogous to the period classes of holomorphic abelian differentials. There are two equivalent definitions of these period classes, a sheaf-theoretic definition which will be discussed first and a group-theoretic definition which in some ways is more convenient. Since the coordinate transition functions $\rho_{\alpha\beta}$ of a flat line bundle ρ over a Riemann surface M are constant, exterior differentiation leads to the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C}(\rho) \longrightarrow \mathcal{O}(\rho) \xrightarrow{d} \mathcal{O}^{(1,0)}(\rho) \longrightarrow 0$$

over M, where $\mathbb{C}(\rho) \subset \mathcal{O}(\rho)$ is the subsheaf of locally constant cross-sections of the flat line bundle ρ. The associated exact cohomology sequence includes the segment

$$\Gamma(M, \mathcal{O}(\rho)) \xrightarrow{d} \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)),$$

which can be rewritten equivalently

$$0 \longrightarrow \frac{\Gamma(M, \mathcal{O}^{(1,0)}(\rho))}{d\,\Gamma(M, \mathcal{O}(\rho))} \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)). \tag{8.1}$$

The image $\delta_s\sigma \in H^1(M, \mathbb{C}(\rho))$ of a Prym differential $\sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho))$ is the *sheaf-theoretic period class* of that differential. If the bundle ρ is not analytically trivial, indicated by writing $\rho \nsim 1$, then $\Gamma(M, \mathcal{O}(\rho)) = 0$ by Corollary 1.4 so the exact sequence (8.1) reduces to the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)) \quad \text{if } \rho \nsim 1. \tag{8.2}$$

Correspondingly for $\mathcal{C}^\infty$ Prym differentials exterior differentiation leads to the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C}(\rho) \longrightarrow \mathcal{E}(\rho) \xrightarrow{d} \mathcal{E}^1_c(\rho) \longrightarrow 0$$

over M, where $\mathcal{E}(\rho)$ is the sheaf of germs of complex-valued $\mathcal{C}^\infty$ cross-sections of the flat line bundle ρ; it is to obtain this exact sequence, paralleling the corresponding sequence for holomorphic Prym differentials, that $\mathcal{C}^\infty$ Prym differentials are defined to be closed differential forms. The exact cohomology sequence associated to this exact sequence of sheaves contains the segment

$$\Gamma(M, \mathcal{E}(\rho)) \xrightarrow{d} \Gamma(M, \mathcal{E}^1_c(\rho)) \xrightarrow{\delta_s} H^1(M, \mathbb{C}(\rho)) \longrightarrow H^1(M, \mathcal{E}(\rho));$$

but $H^1(M, \mathcal{E}(\rho)) = 0$ since $\mathcal{E}(\rho)$ is a fine sheaf[1], so this exact sequence can be written as the isomorphism

$$\delta_s : \frac{\Gamma(M, \mathcal{E}^1_c(\rho))}{d\Gamma(M, \mathcal{E}(\rho))} \xrightarrow{\cong} H^1(M, \mathbb{C}(\rho)). \tag{8.3}$$

[1]Fine sheaves and their cohomological properties are discussed in Appendix C.2.

The image $\delta_s\sigma \in H^1(M, \mathbb{C}(\rho))$ of a $\mathcal{C}^\infty$ Prym differential $\sigma \in \Gamma(M, \mathcal{E}^1_c(\rho))$ is the sheaf-theoretic period class of that differential. It follows from the isomorphism (8.3) that every cohomology class in $H^1(M, \mathbb{C}(\rho))$ is the period class of a $\mathcal{C}^\infty$ Prym differential; that is not the case for holomorphic Prym differentials though, as will be seen as the discussion continues.

For a more explicit description of the exact cohomology sequence leading to the sheaf-theoretic period class, a $\mathcal{C}^\infty$ Prym differential σ for a flat line bundle ρ is represented by closed differential forms $\sigma_\alpha(z)$ in the open subsets of a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of M; and $\sigma_\alpha(z) = \rho_{\alpha\beta}\sigma_\beta(z)$ in any intersection $U_\alpha \cap U_\beta$, where the cocycle $\rho_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathbb{C}^*)$ is a coordinate bundle describing the flat line bundle ρ. After a refinement of the coordinate covering if necessary, the local differentials $\sigma_\alpha(z)$ can be written as the exterior derivatives $\sigma_\alpha(z) = df_\alpha(z)$ of $\mathcal{C}^\infty$ functions $f_\alpha(z)$ in the coordinate neighborhoods U_α; and these functions form a zero-cochain $f \in C^0(\mathfrak{U}, \mathcal{E}(\rho))$. Since $d(f_\beta(z) - \rho_{\beta\alpha}f_\alpha(z)) = \sigma_\beta(z) - \rho_{\beta\alpha}\sigma_\alpha(z) = 0$ in any intersection $U_\alpha \cap U_\beta$ it follows that

$$\sigma_{\alpha\beta} = f_\beta(z) - \rho_{\beta\alpha}f_\alpha(z) \tag{8.4}$$

is a complex constant; these constants form the one-cochain $\sigma \in C^1(\mathfrak{U}, \mathbb{C}(\rho))$ for which $\sigma = \delta f$, as in (1.42), so this one-cochain is a one-cocycle $\sigma \in Z^1(\mathfrak{U}, \mathbb{C}(\rho))$. The cohomology class of this cocycle is independent of the choice of the local integrals $f_\alpha(z)$, since adding constants c_α to the functions $f_\alpha(z)$ adds the coboundary $c_\beta - \rho_{\beta\alpha}c_\alpha$ to the cocycle (8.4). This cohomology class is the period class $\delta_s\sigma$ of the Prym differential σ. For a holomorphic Prym differential $\sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \subset \Gamma(M, \mathcal{E}^1_c(\rho))$ the same construction with holomorphic functions f_α yields the corresponding period class.

The period classes of Prym differentials can be interpreted alternatively and more conveniently in terms of the cohomology[2] of the covering translation group Γ of the Riemann surface M. Since the universal covering space $\tilde{M}$ of the surface M is simply connected, any holomorphic differential form on $\tilde{M}$ can be written as the exterior derivative of a holomorphic function defined on all of $\tilde{M}$; thus there is the exact sequence of complex vector spaces

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(\tilde{M}, \mathcal{O}) \xrightarrow{d} \Gamma(\tilde{M}, \mathcal{O}^{(1,0)}) \longrightarrow 0. \tag{8.5}$$

A flat line bundle ρ over the Riemann surface M can be represented by a flat factor of automorphy $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$, that is, by a one-dimensional representation of the covering translation group Γ, as in Theorem 3.11. The group Γ acts as a group of operators on the right on the vector space $\Gamma(\tilde{M}, \mathcal{O}^{(1,0)})$ by associating to an element $T \in \Gamma$ and a holomorphic differential form $\sigma \in \Gamma(\tilde{M}, \mathcal{O}^{(1,0)})$ the holomorphic differential form $\sigma|_\rho T \in \Gamma(\tilde{M}, \mathcal{O}^{(1,0)})$ defined by

$$(\sigma|_\rho T)(z) = \rho(T)^{-1}\sigma(Tz); \tag{8.6}$$

for it is readily verified that $\sigma|_\rho(ST) = (\sigma|_\rho S)|_\rho T$ for any $S, T \in \Gamma$. The group Γ acts as a group of operators on the right on the vector space $\Gamma(\tilde{M}, \mathcal{O})$ in

[2] The machinery of the cohomology of groups used here is discussed in Appendix E.

the same way, with the group operation (8.6) on holomorphic functions rather than on holomorphic differential forms. The subspace $\mathbb{C} \subset \Gamma(\tilde{M}, \mathcal{O})$ of constant functions under this group operation will be denoted by $\mathbb{C}_\rho$ for clarity in the subsequent discussion; this is just the complex numbers under the group operation that associates to an element $T \in \Gamma$ and a complex number $c \in \mathbb{C}$ the complex number

$$c|_\rho T = \rho(T)^{-1}\, c. \tag{8.7}$$

It is evident that these group actions commute with the mappings in the exact sequence (8.5); so, as in the discussion on page 503 in Appendix E.1, associated to this exact sequence is the exact sequence of cohomology groups of the group Γ, which contains the segment

$$H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O})) \xrightarrow{d} H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O}^{(1,0)})) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho).$$

By (E.15) the cohomology group $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O}))$ is the subspace of $\Gamma(\tilde{M}, \mathcal{O})$ consisting of elements $f \in \Gamma(\tilde{M}, \mathcal{O})$ that are invariant under the action of the group Γ, hence that satisfy $f(z) = (f|_\rho T)(z) = \rho(T)^{-1} f(Tz)$ for each $T \in \Gamma$; consequently there is the natural identification $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O})) \cong \Gamma(M, \mathcal{O}(\rho))$, and correspondingly $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{O}^{(1,0)})) \cong \Gamma(M, \mathcal{O}^{(1,0)}(\rho))$. Thus the preceding segment of the exact cohomology sequence can be rewritten as the exact sequence

$$\Gamma(M, \mathcal{O}(\rho)) \xrightarrow{d} \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho),$$

or equivalently as the exact sequence

$$0 \longrightarrow \frac{\Gamma(M, \mathcal{O}^{(1,0)}(\rho))}{d\,\Gamma(M, \mathcal{O}(\rho))} \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho). \tag{8.8}$$

The image $\delta\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ of a Prym differential $\sigma \in \Gamma(M, \mathcal{O}^{(1,0)}(\rho))$ under the coboundary mapping δ is the *group-theoretic period class* of that differential. Again if the bundle ρ is not analytically trivial, that is if $\rho \nsim 1$, then $\Gamma(M, \mathcal{O}(\rho)) = 0$ by Corollary 1.4 so the preceding exact sequence takes the form

$$0 \longrightarrow \Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho) \quad \text{if } \rho \nsim 1. \tag{8.9}$$

For $\mathcal{C}^\infty$ Prym differentials paralleling (8.5) is the exact sequence of complex vector spaces

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(\tilde{M}, \mathcal{E}) \xrightarrow{d} \Gamma(\tilde{M}, \mathcal{E}^1_c) \longrightarrow 0 \tag{8.10}$$

on which the group Γ acts in the same way, and hence there is the associated exact cohomology sequence containing the segment

$$H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E})) \xrightarrow{d} H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^1_c)) \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho).$$

It follows from (E.15) again that there are the isomorphisms $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E})) \cong \Gamma(M, \mathcal{E}(\rho))$ and $H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^1_c)) \cong \Gamma(M, \mathcal{E}^1_c(\rho))$, so the preceding exact sequence can be rewritten as the exact sequence

$$(8.11) \qquad 0 \longrightarrow \frac{\Gamma(M, \mathcal{E}^1_c(\rho))}{d\Gamma(M, \mathcal{E}(\rho))} \xrightarrow{\delta} H^1(\Gamma, \mathbb{C}_\rho).$$

The image $\delta\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ of a $\mathcal{C}^\infty$ Prym differential $\sigma \in \Gamma(M, \mathcal{E}^1_c(\rho))$ under the coboundary mapping δ is its group-theoretic period class, and the exact sequence (8.8) is just the restriction of the exact sequence (8.11) to holomorphic Prym differentials.

The cohomology group $H^1(\Gamma, \mathbb{C}_\rho)$, called the *Prym cohomology group* of the Riemann surface M for the flat line bundle ρ, is the quotient

$$(8.12) \qquad H^1(\Gamma, \mathbb{C}_\rho) = \frac{Z^1(\Gamma, \mathbb{C}_\rho)}{B^1(\Gamma, \mathbb{C}_\rho)},$$

where as in (E.17) the group $Z^1(\Gamma, \mathbb{C}_\rho)$ of cocycles consists of those mappings $\sigma : \Gamma \longrightarrow \mathbb{C}$ for which

$$(8.13) \qquad \sigma(ST) = \sigma(S)|_\rho T + \sigma(T) = \rho(T)^{-1}\sigma(S) + \sigma(T)$$

for all $S, T \in \Gamma$ and as in (E.18) the subgroup $B^1(\Gamma, \mathbb{C}_\rho) \subset Z^1(\Gamma, \mathbb{C}_\rho)$ of coboundaries consists of cocycles of the form

$$(8.14) \qquad (\delta c)(T) = c|_\rho T - c = c\,\big(\rho(T)^{-1} - 1\big)$$

for all $T \in \Gamma$ and a complex constant c. For a more explicit description of the exact cohomology sequence leading to the group-theoretic period class, a $\mathcal{C}^\infty$ Prym differential σ for the flat line bundle ρ is represented by a closed differential form $\sigma(z)$ on the universal covering space $\tilde{M}$ such that $(\sigma|_\rho T)(z) = \sigma(z)$, or equivalently such that $\sigma(Tz) = \rho(T)\sigma(z)$, for all $T \in \Gamma$. Since $\tilde{M}$ is simply connected $\sigma(z)$ is the exterior derivative of a $\mathcal{C}^\infty$ function $f(z)$ on $\tilde{M}$; the function $f(z)$ is called a *Prym integral*, and is determined uniquely up to an arbitrary additive constant. If $f(z)$ is any choice of a Prym integral then $d\big((f|_\rho T)(z) - f(z)\big) = (\sigma|_\rho T)(z) - \sigma(z) = 0$ for any $T \in \Gamma$, so

$$(8.15) \qquad \sigma(T) = (f|_\rho T)(z) - f(z) = \rho(T)^{-1} f(Tz) - f(z)$$

is a complex constant, and clearly $\sigma(I) = 0$ for the identity $I \in \Gamma$; equivalently

$$(8.16) \qquad f(Tz) = \rho(T)\big(f(z) + \sigma(T)\big)$$

for any $T \in \Gamma$ for a complex constant $\sigma(T)$, and $\sigma(I) = 0$ for the identity $I \in \Gamma$. The mapping $\sigma : \Gamma \longrightarrow \mathbb{C}$ that associates to any $T \in \Gamma$ the value $\sigma(T)$ is a one-cochain $\sigma \in C^1(\Gamma, \mathbb{C}_\rho)$; and since (8.15) amounts to the condition that $\sigma = \delta f$, when the function $f(z)$ is viewed as a zero-cochain $f \in C^0(\Gamma, \Gamma(M, \mathcal{O}(\rho))$, it follows that σ actually is a one-cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$. When the Prym integral

$f(z)$ is replaced by $f(z) + c$ for a complex constant c the cocycle σ is modified by adding to it the coboundary $c\,\big(\rho(T)^{-1} - 1\big)$; thus the cohomology class of the cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ is independent of the particular choice of the Prym integral $f(z)$. This cohomology class is the period class $\delta\sigma$ of the Prym differential $\sigma(z)$. For a holomorphic Prym differential the same construction yields its period class, and the Prym integral is a holomorphic function on $\tilde{M}$.

Lemma 8.1 *Every cohomology class in $H^1(\Gamma, \mathbb{C}_\rho)$ is the period class of a $\mathcal{C}^\infty$ Prym differential, so the period mapping*

$$\delta : \frac{\Gamma(M, \mathcal{E}^1_c(\rho))}{d\Gamma(M, \mathcal{E}(\rho))} \xrightarrow{\cong} H^1(\Gamma, \mathbb{C}_\rho) \tag{8.17}$$

is an isomorphism of complex vector spaces.

Proof: Choose a $\mathcal{C}^\infty$ partition of unity $\{r_\alpha(z)\}$ on the Riemann surface M subordinate to a finite covering $\{U_\alpha\}$ of M by contractible open subsets, and view the functions $r_\alpha(z)$ as Γ-invariant functions on the universal covering surface $\tilde{M}$. For each set U_α choose a connected component $\tilde{U}_\alpha \subset \tilde{M}$ of the inverse image $\pi^{-1}(U_\alpha)$ under the universal covering $\pi : \tilde{M} \longrightarrow M$; the restrictions $\pi : \tilde{U}_\alpha \longrightarrow U_\alpha$ then are biholomorphic mappings, the sets $S\tilde{U}_\alpha$ for $S \in \Gamma$ are pairwise disjoint, and the full inverse image of the set U_α is the union $\pi^{-1}(U_\alpha) = \bigcup_{S\in\Gamma} S\tilde{U}_\alpha \subset \tilde{M}$. For any cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ and for each index α introduce the $\mathcal{C}^\infty$ function $f_\alpha(z)$ on $\tilde{M}$ defined by

$$f_\alpha(z) = \begin{cases} \rho(S)\sigma(S)r_\alpha(z) & \text{if} \quad z \in S\tilde{U}_\alpha, \\[2ex] 0 & \text{if} \quad z \notin \bigcup_{S\in\Gamma} S\tilde{U}_\alpha; \end{cases}$$

it is evident that this is a well defined $\mathcal{C}^\infty$ function on $\tilde{M}$ that vanishes outside the sets $S\tilde{U}_\alpha$. If $z \in S\tilde{U}_\alpha$ and $T \in \Gamma$ then $Tz \in TS\tilde{U}_\alpha$ so it follows from the definition of the functions $f_\alpha(z)$ and(8.13) that

$$\begin{aligned} f_\alpha(Tz) &= \rho(TS)\sigma(TS)r_\alpha(Tz) \\ &= \rho(TS)\Big(\sigma(S) + \rho(S)^{-1}\sigma(T)\Big)r_\alpha(z) \\ &= \rho(T)f_\alpha(z) + \rho(T)\sigma(T)r_\alpha(z); \end{aligned}$$

and of course this holds trivially if $z \notin \bigcup_{S\in\Gamma} S\tilde{U}_\alpha$ since all the terms vanish. The sum $f(z) = \sum_\alpha f_\alpha(z)$ is a $\mathcal{C}^\infty$ function on $\tilde{M}$; and since $1 = \sum_\alpha r_\alpha(z)$, summing the preceding identity shows that $f(Tz) = \rho(T)\big(f(z) + \sigma(T)\big)$. Then $\sigma(z) = d\,f(z)$ is a $\mathcal{C}^\infty$ Prym differential on M, and it is evident from (8.16) that the period class of this Prym differential is represented by the cocycle σ; that suffices to conclude the proof.

The sheaf-theoretic and group-theoretic period classes of Prym differentials can be identified through the isomorphisms δ_s of (8.3) and δ of (8.17), which when combined provide the commutative diagram

(8.18)
$$\begin{tikzcd} H^1(\Gamma, \mathbb{C}_\rho) \arrow[rr, "\phi"] & & H^1(M, \mathbb{C}(\rho)) \\ & \dfrac{\Gamma(M, \mathcal{E}^1_c(\rho))}{d\Gamma(M, \mathcal{E}(\rho))} \arrow[ul, "\delta"] \arrow[ur, "\delta_s"'] & \end{tikzcd}$$

in which δ and δ_s are isomorphisms and $\phi = \delta_s \cdot \delta^{-1}$ consequently also is an isomorphism. When the period mappings δ and δ_s are restricted to the subspace $\Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \subset \Gamma(M, \mathcal{E}^1_c(\rho))$ it follows that the restriction of the isomorphism ϕ is an isomorphism

$$\phi : \delta\Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \xrightarrow{\cong} \delta_s\Gamma(M, \mathcal{O}^{(1,0)}(\rho)) \tag{8.19}$$

between the subspaces of the cohomology groups $H^1(\Gamma, \mathbb{C}_\rho)$ and $H^1(M, \mathbb{C}(\rho))$ consisting of the group-theoretic and sheaf-theoretic period classes of holomorphic Prym differentials. The isomorphism ϕ of (8.18) can be described alternatively and rather more explicitly. Choose a coordinate covering of the Riemann surface M by finitely many contractible coordinate neighborhoods U_α such that the intersections $U_\alpha \cap U_\beta$ are connected; and for each subset U_α choose a connected component $\tilde{U}_\alpha \subset \tilde{M}$ of the inverse image $\pi^{-1}(U_\alpha) \subset \tilde{M}$ under the covering projection $\pi : \tilde{M} \longrightarrow M$. The restricted covering projection $\pi_\alpha : \tilde{U}_\alpha \longrightarrow U_\alpha$ then is a biholomorphic mapping between these two coordinate neighborhoods. For any point $z \in U_\alpha \cap U_\beta$ the two points $\pi_\alpha^{-1}(z) \in \tilde{U}_\alpha$ and $\pi_\beta^{-1}(z) \in \tilde{U}_\beta$ are related by $\pi_\alpha^{-1}(z) = T_{\alpha\beta} \cdot \pi_\beta^{-1}(z)$ for a uniquely determined covering translation $T_{\alpha\beta} \in \Gamma$ that is independent of the choice of the point $z \in U_\alpha \cap U_\beta$. If $\sigma(z)$ is a closed $\mathcal{C}^\infty$ differential form on the universal covering surface $\tilde{M}$ representing a Prym differential $\sigma \in \Gamma(M, \mathcal{E}^1_c(\rho))$ then $\sigma(Tz) = \rho(T)\sigma(z)$ for each $T \in \Gamma$. Introduce the associated differential forms σ_α in the coordinate neighborhoods U_α defined by $\sigma_\alpha(z) = \sigma(\pi_\alpha^{-1}(z))$ for $z \in U_\alpha$. If $z \in U_\alpha \cap U_\beta$ then $\sigma_\alpha(z) = \sigma(\pi_\alpha^{-1}(z)) = \sigma(T_{\alpha\beta} \cdot \pi_\beta^{-1}(z)) = \rho(T_{\alpha\beta})\sigma(\pi_\beta^{-1}(z)) = \rho(T_{\alpha\beta})\sigma_\beta(z)$; thus the local differentials $\sigma_\alpha(z)$ also represent the Prym differential σ when the line bundle ρ is represented by the cocycle $\rho_{\alpha\beta} = \rho(T_{\alpha\beta})$ for the coordinate covering $\mathfrak{U} = \{U_\alpha\}$. If $f(z)$ is a Prym integral of the Prym differential $\sigma(z)$, so that $df(z) = \sigma(z)$, then the group-theoretic period class $\delta\sigma$ of the Prym differential is represented by the cocycle $\sigma(T) = \rho(T)^{-1}f(Tz) - f(z) \in Z^1(\Gamma, \mathbb{C}_\rho))$ as in (8.15); and if $f_\alpha(z) = f(\pi_\alpha^{-1}(z))$ then these local functions satisfy $df_\alpha(z) = \sigma_\alpha(z)$ so the sheaf-theoretic period class $\delta_s\sigma$ of the Prym differential σ is represented by the cocycle $\sigma_{\alpha\beta} = \rho_{\beta\alpha}f_\alpha(z) - f_\beta(z) \in Z^1(\mathfrak{U}, \mathbb{C}(\rho))$ as in (8.4). Thus

$$\begin{aligned} \sigma_{\alpha\beta} &= \rho_{\beta\alpha} f(\pi_\alpha^{-1}(z)) - f(\pi_\beta^{-1}(z)) = \rho(T_{\beta\alpha}) f(T_{\alpha\beta} \cdot \pi_\beta^{-1}(z)) - f(\pi_\beta^{-1}(z)) \\ &= \rho(T_{\beta\alpha})\rho(T_{\alpha\beta})\Big(f(\pi_\beta^{-1}(z)) + \sigma(T_{\alpha\beta})\Big) - f(\pi_\beta^{-1}(z)) = \sigma(T_{\alpha\beta}); \end{aligned}$$

consequently the image $\phi(\sigma) \in H^1(\Gamma, \mathbb{C}(\rho))$ under the isomorphism (8.18) of the cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ represented by a cocycle $\sigma(T) \in Z^1(\Gamma, \mathbb{C}_\rho)$ is the cohomology class represented by the cocycle $\sigma_{\alpha\beta} \in Z^1(M, \mathbb{C}(\rho))$ for which

$$\sigma_{\alpha\beta} = \sigma(T_{\alpha\beta}). \tag{8.20}$$

Before continuing with a more detailed discussion of the Prym cohomology group it is convenient first to list some general properties of Prym cocycles, properties that hold for an arbitrary group Γ.

Lemma 8.2 *If $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is a representation of a group Γ then for any cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ and any elements $S, T \in \Gamma$*
(i) $\sigma(I) = 0$ *for the identity $I \in \Gamma$,*
(ii) $\sigma(T^{-1}) = -\rho(T)\sigma(T)$,
(iii) $\sigma(STS^{-1}) = \rho(S)\sigma(T) + \big(\rho(T)^{-1} - 1\big)\rho(S)\sigma(S)$,
(iv) $\sigma([S,T]) = \big(1 - \rho(T)\big)\rho(S)\sigma(S) - \big(1 - \rho(S)\big)\rho(T)\sigma(T)$
for the commutator $[S,T] = STS^{-1}T^{-1}$; and
(v) *if $R \in \Gamma$ is an element for which $\rho(R) \neq 1$ the cocycle σ is cohomologous to a unique cocycle σ_R for which $\sigma_R(R) = 0$.*

Proof: The first four results follow from the defining equation (8.13) by straightforward calculations, so no details need be given here. As for (v), the cocycle σ is cohomologous to the cocycle σ_R defined by

$$\sigma_R(T) = \sigma(T) - \frac{\sigma(R)}{\rho(R)^{-1} - 1}\ \big(\rho(T)^{-1} - 1\big),$$

as is evident from (8.14), and $\sigma_R(R) = 0$. A cocycle $\tau \in Z^1(\Gamma, \mathbb{C}_\rho)$ cohomologous to σ must be of the form $\tau(T) = \sigma_R(T) + c\,\big(\rho(T)^{-1} - 1\big)$; and if $\tau(R) = c\,\big(\rho(R)^{-1} - 1\big) = 0$ then $c = 0$ so $\tau = \sigma_R$, which suffices for the proof.

It follows from (v) of the preceding lemma that if $\rho(R) \neq 1$ for an element $R \in \Gamma$ then any cohomology class in $H^1(\Gamma, \mathbb{C}_\rho)$ can be represented by a unique cocycle σ_R for which $\sigma_R(R) = 0$, a cocycle called a *normalized cocycle* with respect to the element $R \in \Gamma$; thus if $Z^1_R(\Gamma, \mathbb{C}_\rho) \subset Z^1(\Gamma, \mathbb{C}_\rho)$ is the subgroup of normalized cocycles with respect to R then

$$H^1(\Gamma, \mathbb{C}_\rho) \cong Z^1_R(\Gamma, \mathbb{C}_\rho). \tag{8.21}$$

This provides an explicit description of the Prym cohomology group $H^1(\Gamma, \mathbb{C}_\rho)$ for a nontrivial representation ρ, and it is useful in various circumstances; but this description involves the choice of a particular element $R \in \Gamma$ for which $\rho(R) \neq 1$, so to that extent it is not intrinsic. However there is a more intrinsic description of the Prym cohomology group $H^1(\Gamma, \mathbb{C}_\rho)$ for a nontrivial representation ρ of an arbitrary group Γ. Since the representation ρ is trivial on the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ it is clear from the cocycle condition (8.13) that the restriction of a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ to the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ is a group homomorphism $\sigma\big|[\Gamma, \Gamma] \in \mathrm{Hom}([\Gamma, \Gamma], \mathbb{C})$; and it follows from Lemma 8.2 (iii) that this homomorphism satisfies $\sigma(TCT^{-1}) = \rho(T)\sigma(C)$ for any elements $T \in \Gamma,\ C \in [\Gamma,\Gamma]$ since $\rho(C) = 1$. That suggests introducing for an arbitrary homomorphism $\rho \in \operatorname{Hom}(\Gamma,\mathbb{C}^*)$ of a group Γ the set of homomorphisms

(8.22)
$$\operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C}) = \left\{ \sigma \in \operatorname{Hom}([\Gamma,\Gamma],\mathbb{C}) \;\middle|\; \begin{array}{c} \sigma(TCT^{-1}) = \rho(T)\sigma(C) \\ \\ \text{for all } \ T \in \Gamma, \ \ C \in [\Gamma,\Gamma] \end{array} \right\}.$$

It is clear that the set $\operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ has the natural structure of a complex vector space, since if $\sigma_1,\sigma_2 \in \operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ and $c_1,c_2 \in \mathbb{C}$ then $c_1\sigma_1 + c_2\sigma_2 \in \operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ as well. The individual homomorphisms in $\operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ have the following properties.

Lemma 8.3 *If $\sigma \in \operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ then for any $R,S,T \in \Gamma$*
(i) $\sigma([ST,R]) = \sigma([S,R]) + \rho(S)\sigma([T,R])$,
(ii) $\sigma([T^{-1},R]) = -\rho(T)^{-1}\sigma([T,R])$,
(iii) $\big(1-\rho(R)\big)\sigma([S,T]) + \big(1-\rho(T)\big)\sigma([R,S]) + \big(1-\rho(S)\big)\sigma([T,R]) = 0$;
(iv) *if $\rho(R) \neq 1$ then σ is the restriction $\sigma = \sigma_R|[\Gamma,\Gamma]$ to the commutator subgroup of a unique normalized cocycle $\sigma_R \in Z^1_R(\Gamma,\mathbb{C}_\rho)$, that given by*

$$\sigma_R(T) = \frac{\sigma([T,R])}{\rho(T)\big(1-\rho(R)\big)}. \tag{8.23}$$

Proof: (i) From the standard commutator identity

$$\begin{aligned} [ST,R] &= STRT^{-1}S^{-1}R^{-1} \\ &= S\cdot TRT^{-1}R^{-1}\cdot S^{-1}\cdot SRS^{-1}R^{-1} \\ &= S[T,R]S^{-1}\cdot[S,R] \end{aligned} \tag{8.24}$$

and the assumption that $\sigma \in \operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ it follows that

$$\begin{aligned} \sigma([ST,R]) &= \sigma(S[T,R]S^{-1}) + \sigma([S,R]) \\ &= \rho(S)\sigma([T,R]) + \sigma([S,R]). \end{aligned}$$

(ii) This follows immediately from (i) upon setting $S = T^{-1}$, since $\sigma([I,R]) = \sigma(I) = 0$.
(iii) From the commutator identity

$$\begin{aligned} R[S,T]R^{-1} &= RSTS^{-1}T^{-1}R^{-1} \\ &= [R,S]SR\cdot TS^{-1}\cdot R^{-1}T^{-1}[T,R] \\ &= [R,S]S[R,T]TR\cdot S^{-1}R^{-1}T^{-1}[T,R] \\ &= [R,S]\cdot S[R,T]S^{-1}\cdot[S,T]\cdot T[S,R]T^{-1}\cdot[T,R] \end{aligned}$$

and the assumption that $\sigma \in \operatorname{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ it follows that

$$\begin{aligned} \rho(R)\sigma([S,T]) = {} & \sigma([R,S]) + \rho(S)\sigma([R,T]) + \sigma([S,T]) \\ & + \rho(T)\sigma([S,R]) + \sigma([T,R]); \end{aligned}$$

and since $\sigma([S,R]) = -\sigma([R,S])$ and $\sigma([T,R]) = -\sigma([R,T])$ this is equivalent to (iii).
(iv) For the mapping $\sigma_R : \Gamma \longrightarrow \mathbb{C}$ defined by (8.23) it follows from (i) that

$$\begin{aligned}\sigma_R(ST) &= \frac{\sigma([ST,R])}{\rho(ST)\big(1-\rho(R)\big)} = \frac{\rho(S)\sigma([T,R]) + \sigma([S,R])}{\rho(ST)\big(1-\rho(R)\big)}\\ &= \sigma_R(T) + \rho(T)^{-1}\sigma_R(S);\end{aligned}$$

hence σ_R is a cocycle, and since it is clear from (8.23) that $\sigma_R(R) = 0$ it is even a normalized cocycle with respect to R. Next for any commutator $C \in [\Gamma,\Gamma]$

$$\begin{aligned}\sigma([C,R]) &= \sigma(C\cdot RC^{-1}R^{-1}) = \sigma(C) + \sigma(RC^{-1}R^{-1})\\ &= \big(1-\rho(R)\big)\sigma(C),\end{aligned}$$

or equivalently

$$\sigma(C) = \frac{\sigma([C,R])}{\rho(C)\big(1-\rho(R)\big)} = \sigma_R(C)$$

since $\rho(C) = 1$; thus the normalized cocycle σ_R restricts to the homomorphism σ on commutators. Finally if $\sigma'_R \in Z^1_R(\Gamma,\mathbb{C}_\rho)$ is a normalized cocycle that vanishes on commutators then $\sigma'(R) = \sigma'([R,T]) = 0$ for any $T \in \Gamma$ so by Lemma 8.2 (iv)

$$\begin{aligned}0 = \sigma'_R([R,T]) &= \big(1-\rho(T)\big)\rho(R)\sigma'_R(R) - \big(1-\rho(R)\big)\rho(T)\sigma'_R(T)\\ &= 0 - \big(1-\rho(R)\big)\rho(T)\sigma'_R(T);\end{aligned}$$

and since $\rho(R) \neq 1$ it follows that $\sigma'_R(T) = 0$. Thus there is a unique normalized cocycle that restricts to σ on the commutator subgroup $[\Gamma,\Gamma] \subset \Gamma$, and that suffices to conclude the proof.

The symmetry expressed in part (iii) of the preceding lemma is a form of Lie identity for cocycles in the group $Z^1(\Gamma,\mathbb{C}_\rho)$. The result in part (iv) leads to the following intrinsic description of the cohomology group $H^1(\Gamma,\mathbb{C}_\rho)$.

Theorem 8.4 *If $\rho \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ is a nontrivial representation of a group Γ, the mapping that associates to a cocycle $\sigma \in Z^1(\Gamma,\mathbb{C}_\rho)$ its restriction to the commutator subgroup $[\Gamma,\Gamma] \subset \Gamma$ induces an isomorphism*

$$H^1(\Gamma,\mathbb{C}_\rho) \cong \mathrm{Hom}_\rho([\Gamma,\Gamma],\mathbb{C}) \tag{8.25}$$

of complex vector spaces.

Proof: It was already observed that the restriction of a cocycle $\sigma \in Z^1(\Gamma,\mathbb{C}_\rho)$ to the commutator subgroup is a homomorphism $\sigma|[\Gamma,\Gamma] \in \mathrm{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$; so restricting normalized cocycles to the commutator subgroup is a homomorphism of complex vector spaces

$$Z^1_R(\Gamma,\mathbb{C}_\rho) \longrightarrow \mathrm{Hom}_\rho([\Gamma,\Gamma],\mathbb{C}).$$

By Lemma 8.3 (iv) this homomorphism actually is an isomorphism; and since $Z^1_R(\Gamma, \mathbb{C}_\rho) \cong H^1(\Gamma, \mathbb{C}_\rho)$ by (8.21) that suffices to conclude the proof of the theorem.

Let $\Gamma_\rho \subset \Gamma$ be the kernel of the representation $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$, so that

$$\Gamma_\rho = \left\{ T \in \Gamma \;\middle|\; \rho(T) = 1 \right\}. \tag{8.26}$$

Of course $[\Gamma, \Gamma] \subset \Gamma_\rho$ since any homomorphism to a commutative group vanishes on the commutator subgroup. For general representations $\Gamma_\rho = [\Gamma, \Gamma]$; but for special representations such as those for which the image $\rho(\Gamma)$ is a finite group the subgroup $\Gamma_\rho \subset \Gamma$ is of finite index. In analogy with (8.22) let

$$\mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C}) = \left\{ \sigma \in \mathrm{Hom}(\Gamma_\rho, \mathbb{C}) \;\middle|\; \begin{array}{l} \sigma(TST^{-1}) = \rho(T)\sigma(S) \\ \\ \text{for all } \; S \in \Gamma_\rho, \;\; T \in \Gamma \end{array} \right\}. \tag{8.27}$$

Clearly this too has the natural structure of a complex vector space. It follows from the cocycle condition (8.13) that the restriction of a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ to the subgroup Γ_ρ is a homomorphism $\sigma|\Gamma_\rho \in \mathrm{Hom}(\Gamma_\rho, \mathbb{C})$; and it follows from Lemma 8.2 (iii) that this homomorphism satisfies $\sigma(TST^{-1}) = \rho(T)\sigma(S)$ so it belongs to the subgroup $\mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C})$. In addition the restriction of a homomorphism $\sigma \in \mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C})$ to the subgroup $[\Gamma, \Gamma] \subset \Gamma_\rho$ is an element $\phi(\sigma) = \sigma|[\Gamma, \Gamma] \in \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$.

Theorem 8.5 *If* $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ *is a nontrivial representation of a group* Γ *the restriction mapping*

$$\phi : \mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C}) \longrightarrow \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}) \tag{8.28}$$

is an isomorphism of complex vector spaces, and consequently

$$\mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C}) \cong \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}) \cong H^1(\Gamma, \mathbb{C}_\rho). \tag{8.29}$$

Proof: If $\sigma \in \mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C})$ and $\sigma|[\Gamma, \Gamma] = 0$ then for any $T \in \Gamma_\rho$ and $R \notin \Gamma_\rho$

$$\begin{aligned} 0 = \sigma([T, R]) = \sigma(T \cdot RT^{-1}R^{-1}) \;\; &= \;\; \sigma(T) + \sigma(RT^{-1}R^{-1}) \\ &= \big(1 - \rho(R)\big)\sigma(T); \end{aligned}$$

and since $\rho(R) \neq 1$ it follows that $\sigma(T) = 0$, so the restriction mapping (8.28) is an injective linear mapping. On the other hand if $\sigma \in \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ then it follows from Lemma 8.3 (iv) that σ is the restriction $\sigma = \sigma_R|[\Gamma, \Gamma]$ of a normalized cocycle $\sigma_R \in Z^1_R(\Gamma, \mathbb{C}_\rho)$; and since $\sigma_R|\Gamma_\rho \in \mathrm{Hom}_\rho(\Gamma_\rho, \mathbb{C})$ that shows that the restriction mapping (8.28) also is surjective, and consequently is an isomorphism. In view of Theorem 8.4 that suffices to conclude the proof.

The further study of the Prym cohomology groups requires more use of the detailed structure of the covering translation group Γ of a compact Riemann surface M of genus $g > 0$; this structure can be described most conveniently in terms of a marking[3] of the surface. For the purposes of the present discussion a marking of a compact Riemann surface M of genus $g > 1$ is a set $\mathcal{T} = (T_1, T_2, \ldots, T_{2g})$ of generators of the group Γ, often labeled alternatively $A_i = T_i,\ B_i = T_{g+i}$ for $1 \le i \le g$, subject to the relation $C_1 \cdots C_g = I$ where $C_i = [A_i, B_i] = [T_i, T_{g+i}]$. A representation $\rho \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ can be described fully in terms of this marking by the parameters $\zeta_i = \rho(T_i)$ for $1 \le i \le 2g$, since any representation is determined uniquely by its values on the generators T_i; and all the relations among these generators are contained in the commutator subgroup so these values can be specified arbitrarily, leading to the identification

$$\operatorname{Hom}(\Gamma, \mathbb{C}^*) = (\mathbb{C}^*)^{2g}. \tag{8.30}$$

Theorem 8.6 *If M is a marked Riemann surface of genus $g > 1$ with the marking $\mathcal{T} = (T_1, \ldots, T_{2g})$, and if $\rho \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ is a representation described by parameters $\zeta_i = \rho(T_i)$, then for any cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ the values $z_i = \sigma(T_i)$ satisfy*

$$\sum_{i=1}^{g} \Big((1 - \zeta_{g+i})\zeta_i z_i - (1 - \zeta_i)\zeta_{g+i} z_{g+i} \Big) = 0. \tag{8.31}$$

Conversely if z_i are any $2g$ complex numbers satisfying this identity there is a uniquely determined cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ for which $z_i = \sigma(T_i)$.

Proof: If $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$ then since $\sigma|[\Gamma, \Gamma] \in \operatorname{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ and the generators T_i of the marking satisfy $\prod_{i=1}^{g} [T_i, T_{g+i}] = I$ it follows from Lemma 8.2 (iv) that

$$0 = \sigma\Big(\prod_{i=1}^{g} [T_i, T_{g+i}] \Big) = \sum_{i=1}^{g} \sigma\big([T_i, T_{g+i}]\big)$$

$$= \sum_{i=1}^{g} \Big((1 - \zeta_{g+i})\zeta_i \sigma(T_i) - (1 - \zeta_i)\zeta_{g+i} \sigma(T_{g+i}) \Big),$$

which is (8.31). For the converse assertion suppose that z_i are complex constants satisfying (8.31). The covering translation group Γ is the quotient of the free group F on $2g$ generators $\tilde{T}_1, \ldots, \tilde{T}_{2g}$ by the normal subgroup $K \subset F$ generated by the product $\tilde{C} = \tilde{C}_1 \cdots \tilde{C}_g$ where $\tilde{C}_i = [\tilde{T}_i, \tilde{T}_{g+i}]$. The composition of the representation ρ and the natural homomorphism $F \longrightarrow \Gamma$ is a representation of the free group F, which to simplify the notation also will be denoted by ρ. There is a cocycle $\tilde{\sigma} \in Z^1(F, \mathbb{C}_\rho)$ such that $\tilde{\sigma}(\tilde{T}_i) = z_i$; to avoid a digression in the proof here, this will be established in the following Lemma 8.7. Since the restriction of this cocycle is an element $\tilde{\sigma}|[F, F] \in \operatorname{Hom}_\rho([F, F], \mathbb{C})$ it also

[3] The definition and properties of markings of surfaces are discussed in Appendix D.1.

follows from Lemma 8.2 (iv) and (8.31) that

$$\tilde\sigma(\tilde C) = \tilde\sigma\Big(\prod_{i=1}^{g}[\tilde T_i, \tilde T_{g+i}]\Big) = \sum_{i=1}^{g}\tilde\sigma\big([\tilde T_i, \tilde T_{g+i}]\big)$$

$$= \sum_{i=1}^{g}\Big((1-\zeta_{g+i})\zeta_i\tilde\sigma(\tilde T_i) - (1-\zeta_i)\zeta_{g+i}\tilde\sigma(\tilde T_{g+i})\Big)$$

$$= \sum_{i=1}^{g}\Big((1-\zeta_{g+i})\zeta_i z_i - (1-\zeta_i)\zeta_{g+i}z_{g+i}\Big) = 0.$$

Moreover it follows from Lemma (8.2) (iii) that $\tilde\sigma(\tilde T\tilde C\tilde T^{-1}) = \tilde\rho(\tilde T)\tilde\sigma(\tilde C) = 0$ for all $\tilde T \in \Gamma$ as well, since $\tilde\rho(\tilde C) = 1$; so since $K \subset F$ is the normal subgroup generated by $\tilde C$ then $\tilde\sigma(\tilde S) = 0$ for all $\tilde S \in K$. From this and the cocycle condition (8.13) it follows that $\tilde\sigma(\tilde S\tilde T) = \tilde\rho(\tilde T)^{-1}\tilde\sigma(\tilde S) + \tilde\sigma(\tilde T) = \tilde\sigma(\tilde T)$ for all $\tilde S \in K$ and $\tilde T \in F$, which means that $\tilde\sigma(\tilde T_1) = \tilde\sigma(\tilde T_2)$ for any elements $\tilde T_1,\ \tilde T_2 \in F$ that represent the same element of Γ under the natural homomorphism $F \longrightarrow \Gamma$; it is thus possible to define a mapping $\sigma : \Gamma \longrightarrow \mathbb{C}$ by setting $\sigma(T) = \tilde\sigma(\tilde T)$ for any $\tilde T \in F$ representing $T \in \Gamma$. The result is a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$, since whenever $\tilde S,\ \tilde T \in F$ represent $S,\ T \in \Gamma$ then $\tilde S\tilde T \in F$ represents $ST \in \Gamma$ so from the cocycle condition (8.13) for the group F it follows that $\sigma(ST) = \tilde\sigma(\tilde S\tilde T) = \rho(\tilde T)^{-1}\tilde\sigma(\tilde S)+\tilde\sigma(\tilde T) = \rho(T)^{-1}\sigma(S)+\sigma(T)$. The cocycle thus constructed satisfies $\sigma(T_i) = z_i$, and is uniquely determined by this condition; and that suffices to conclude the proof.

Lemma 8.7 *If F is a free group generated by finitely many elements T_i, and if $\rho \in \mathrm{Hom}(F, \mathbb{C}^*)$ is a representation of that group, then for any complex constants z_i there is a unique cocycle $\sigma \in Z^1(F, \mathbb{C}_\rho)$ such that $\sigma(T_i) = z_i$.*

Proof: To any word T in the formal symbols T_i and T_i^{-1}, that is, to any finite sequence of these symbols with possible repetitions but no cancellation of terms, associate a value $\sigma(T)$ by setting $\sigma(T_i) = z_i$ and $\sigma(T_i^{-1}) = -\rho(T_i)z_i$ and then inductively setting $\sigma(T_iT) = \rho(T)^{-1}\sigma(T_i) + \sigma(T)$ and $\sigma(T_i^{-1}T) = \rho(T)^{-1}\sigma(T_i^{-1}) + \sigma(T)$ for any word T. It is easy to see by induction on the length of the word ST, the number of symbols in that word, that σ satisfies the cocycle condition $\sigma(ST) = \rho(T)^{-1}\sigma(S) + \sigma(T)$ for any words S and T. Indeed that follows immediately from the definition of the mapping σ if the word ST is of length 2; and if it true for the word ST then

$$\begin{aligned}\sigma(T_iS\cdot T) &= \rho(ST)^{-1}\sigma(T_i) + \sigma(ST)\\ &= \rho(ST)^{-1}\sigma(T_i) + \rho(T)^{-1}\sigma(S) + \sigma(T)\\ &= \rho(T)^{-1}\Big(\rho(S)^{-1}\sigma(T_i) + \sigma(S)\Big) + \sigma(T)\\ &= \rho(T)^{-1}\sigma(T_iS) + \sigma(T),\end{aligned}$$

and similarly for T_i^{-1} in place of T_i. It further follows that the value $\sigma(T)$ is unchanged when the pairs $T_iT_i^{-1}$ and $T_i^{-1}T_i$ are deleted from any word. Indeed $\sigma(T_iT_i^{-1}) = \rho(T_i)\sigma(T_i) + \sigma(T_i^{-1}) = \rho(T_i)z_i - \rho(T_i)z_i = 0$ and correspondingly for the other order $T_i^{-1}T_i$; then from the cocycle condition it follows that $\sigma(AT_iT_i^{-1}B) = \rho(T_iT_i^{-1}B)^{-1}\sigma(A) + \rho(B)^{-1}\sigma(T_iT_i^{-1}) + \sigma(B) = \rho(B)^{-1}\sigma(A) + \sigma(B) = \sigma(AB)$. Thus the mapping σ can be viewed as defined on the free group F, and is a cocycle $\sigma \in Z^1(F, \mathbb{C}_\rho)$; and that suffices for the proof.

For the trivial representation $\rho = 1$ condition (8.31) is vacuous and Theorem 8.6 reduces to the familiar assertion that for any complex numbers z_i there is a cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C})$ such that $\sigma(T_i) = z_i$.

Corollary 8.8 *Let M be a marked Riemann surface M of genus $g > 1$ with the marking $\mathcal{T} = (T_1, \ldots, T_{2g})$, let $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ be a representation described by parameters $\zeta_i = \rho(T_i)$, and assume that $\rho(T_l) = \zeta_l \neq 1$ for some index l. Then the linear mapping*

$$Z_l : \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C}) \longrightarrow \mathbb{C}^{2g} \tag{8.32}$$

that associates to a homomorphism $\sigma \in \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ the vector

$$Z_l(\sigma) = \Big\{ z_{i,l} = \sigma([T_i, T_l]) \Big| 1 \le i \le 2g \Big\} \in \mathbb{C}^{2g} \tag{8.33}$$

is an injective linear mapping; its image is the linear subspace $H_{\rho,l} \subset \mathbb{C}^{2g}$ consisting of vectors $\{z_{i,l}\} \in \mathbb{C}^{2g}$ such that
(i) $z_{l,l} = 0$
(ii) $\sum_{i=1}^{g} \Big((1 - \zeta_{g+i}) z_{i,l} - (1 - \zeta_i) z_{g+i,l} \Big) = 0,$
and

$$\dim H_{\rho,l} = 2g - 2. \tag{8.34}$$

Proof: A homomorphism $\sigma \in \mathrm{Hom}_\rho([\Gamma, \Gamma], \mathbb{C})$ is determined completely by the values $z_{i,j} = \sigma([T_i, T_j])$ for $1 \le i, j \le 2g$, since any elements $S, T \in \Gamma$ can be written as words in the generators T_i and their inverses and $\sigma([S, T])$ then can be expressed in terms of the values $z_{i,j} = \sigma([T_i, T_j])$ by repeated use of Lemma 8.3 (i) and (ii). By Lemma 8.3 (iii)

$$(1 - \zeta_i) z_{j,l} + (1 - \zeta_l) z_{i,j} + (1 - \zeta_j) z_{l,i} = 0, \tag{8.35}$$

and since $\zeta_l \neq 1$ by assumption the values $z_{i,j}$ for all i, j are determined completely by the values $z_{i,l}$ for $1 \le i \le 2g$. It follows from these two observations that the linear mapping Z_l is injective. Clearly $z_{l,l} = \sigma([T_l, T_l]) = \sigma(I) = 0$, so the vectors in the image of the linear mapping Z_l satisfy (i). Since $C_1 \cdots C_g = I$

then for any mapping $\sigma \in \mathrm{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$

$$\begin{aligned} 0 = \sigma(C_1 \cdots C_g) &= \sum_{i=1}^{g} \sigma(C_i) = \sum_{i=1}^{g} \sigma([T_i, T_{g+i}]) = \sum_{i=1}^{g} z_{i,g+i} \\ &= \frac{1}{1-\zeta_l} \sum_{i=1}^{g} \Big((1-\zeta_{g+i}) z_{i,l} - (1-\zeta_i) z_{g+i,l} \Big), \end{aligned}$$

by (8.35), so the vectors in the image of the linear mapping Z_l also satisfy (ii). On the other hand for any vector $\{z_{i,l}\} \in H_{\rho,l}$ it follows from (ii) that the values $z_i = z_{i,l}/\zeta_i(1-\zeta_l)$ satisfy (8.31), so by Theorem 8.6 there is a cocycle $\sigma \in Z^1(\Gamma,\mathbb{C}_\rho)$ such that

$$\sigma(T_i) = z_i = \frac{z_{i,l}}{\zeta_i(1-\zeta_l)};$$

it then follows from (i) and Lemma 8.2 (iv) that

$$\sigma([T_i,T_l]) = (1-\zeta_l)\zeta_i\sigma(T_i) - (1-\zeta_i)\zeta_l\sigma(T_l) = z_{i,l} - z_{l,l} = z_{i,l},$$

hence $\{z_{i,l}\}$ is in the image of the linear mapping Z_l so the image of Z_l is the full linear subspace $H_{\rho,l}$. Finally since

(8.36)
$$(1-\zeta_{g+i})z_{i,l}-(1-\zeta_i)z_{g+i,l} = \begin{cases} -(1-\zeta_l)z_{l+g,l} & \text{for } i = l \text{ if } 1 \le l \le g, \\ (1-\zeta_l)z_{l-g,l} & \text{for } i = l-g \text{ if } g+1 \le l \le 2g \end{cases}$$

and $\zeta_l \ne 1$, it is clear that the linear equations (i) and (ii) are linearly independent, and consequently that $\dim H_{\rho,l} = 2g-2$, which suffices to conclude the proof.

For any index l in the range $1 \le l \le 2g$, the unique index l' in that range such that $|l-l'| = g$ is called the *dual index* to l; thus if $1 \le l \le g$ then $l' = l+g$ while if $g+1 \le l \le 2g$ then $l' = l-g$. If $\{z_{i,l}\} \in H_{\rho,l} \subset \mathbb{C}^{2g}$ it is evident from (8.36) that equation (ii) of Corollary 8.8 can be used to express the entry $z_{l',l}$ as a linear function of the remaining entries of that vector, while $z_{l,l} = 0$ by (i); so since $\dim H_{\rho,l} = 2g-2$ a vector in $H_{\rho,l}$ is determined uniquely by the entries $z_{i,l}$ for all indices $i \ne l, l'$, and these values can be assigned arbitrarily. If ρ is a representation for which $\rho(T_l) \ne 1$, the composition of the isomorphism $H^1(\Gamma,\mathbb{C}_\rho) \xrightarrow{\cong} \mathrm{Hom}_\rho([\Gamma,\Gamma],\mathbb{C})$ of Theorem 8.4 and the isomorphism $Z_l : \mathrm{Hom}_\rho([\Gamma,\Gamma],\mathbb{C}) \xrightarrow{\cong} H_{\rho,l}$ of Corollary 8.8 is an isomorphism

$$\hat{Z}_l : H^1(\Gamma,\mathbb{C}_\rho) \xrightarrow{\cong} H_{\rho,l}, \tag{8.37}$$

so a cohomology class $\sigma \in H^1(\Gamma,\mathbb{C}_\rho)$ can be described uniquely by the values

$$\{z_{i,l}\} = \{\sigma([T_i,T_l])\} \in H_{\rho,l} \quad \text{for } 1 \le i \le 2g,\ i \ne l, l', \tag{8.38}$$

for the image $\hat{Z}_l(\sigma) \in H_{\rho,l}$; these values are called the *canonical coordinates* of the cohomology class σ with respect to the generator T_l in the marking $\mathcal{T}$, and any set of $2g-2$ complex numbers $z_{i,l}$ for $1 \le i \le 2g,\ i \ne l, l'$ are the canonical coordinates for some Prym cohomology class. This coordinatization of the Prym cohomology classes of course depends on the choice of the marking $\mathcal{T}$ and of a generator $T_l \in \mathcal{T}$ for which $\rho(T_l) \ne 1$; it is convenient just to say that the canonical coordinates depend on the *indexed marking* $\mathcal{T}(l)$ of the surface M, where an indexed marking is defined to be a marking $\mathcal{T} = (T_1, \ldots, T_{2g})$ together with the choice of a particular generator $T_l \in \mathcal{T}$. Of course only those indexed markings for which $\rho(T_l) \ne 1$ can yield a coordinatization of the Prym cohomology group for the representation ρ.

To describe the relations between different systems of canonical coordinates, it is useful to introduce some auxiliary algebraic observations. For any nontrivial representation $\rho \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ of the covering translation group Γ of the Riemann surface M let $\mathbb{Z}[\rho] \subset \mathbb{C}$ be the subring defined by

$$\mathbb{Z}[\rho] = \mathbb{Z}\left[\rho(T), \frac{1-\rho(T)}{1-\rho(S)}\right] \text{ for all } S, T \in \Gamma,\ \rho(S) \ne 1; \tag{8.39}$$

thus $\mathbb{Z}[\rho]$ is the ring generated by $\rho(T)$ and $\big(1-\rho(T)\big)/\big(1-\rho(S)\big)$ for all $S, T \in \Gamma$ for which $\rho(S) \ne 1$. It is an integral domain, as a subring of the field $\mathbb{C}$, and its field of quotients is denoted by $\mathbb{Q}(\rho)$. A representation $\rho \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ is said to be *finite of order* q if its image $\rho(\Gamma) \subset \mathbb{C}^*$ is a finite subgroup of order q in the multiplicative group $\mathbb{C}^*$; the image $\rho(\Gamma)$ then is the cyclic group of order q generated by a primitive q-th root of unity. For finite representations the ring $\mathbb{Z}(\rho)$ takes a particularly simple form.

Lemma 8.9 (i) *If ρ is a finite representation of order q of a group Γ and ϵ is a primitive q-th root of unity then the ring $\mathbb{Z}[\rho]$ is generated by ϵ and the quotients $(1-\epsilon)(1-\epsilon^d)^{-1}$ for all integers d such that $1 < d < q$ and $d|q$, and the field of quotients $\mathbb{Q}(\rho)$ of the ring $\mathbb{Z}[\rho]$ is the cyclotomic field $\mathbb{Q}(\epsilon)$ of q-th roots of unity;*
(ii) $\mathbb{Z}[\rho] = \mathbb{Z}[\epsilon]$ *if $q = p$ is prime; and*
(iii) $\mathbb{Z}[\rho] = \mathbb{Z}$ *if $q = 2$.*

Proof: If ρ is a finite representation of order q and ϵ is a primitive q-th root of unity then for any $T \in \Gamma$ it is the case that $\rho(T) = \epsilon^n$ for some integer n in the range $1 \le n \le q$; so it follows immediately from the definition (8.39) that the ring $\mathbb{Z}[\rho]$ is generated by ϵ and the quotients $(1-\epsilon^m)/(1-\epsilon^n)$ for integers m, n in the range $1 \le m, n < q$. If $m > 1$ then since

$$\frac{1-\epsilon^m}{1-\epsilon^n} = P(\epsilon) \cdot \frac{1-\epsilon}{1-\epsilon^n}$$

where $P(\epsilon) = 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{m-1} \in \mathbb{Z}(\epsilon)$ it follows that $(1-\epsilon^m)/(1-\epsilon^n)$ can be replaced by $(1-\epsilon)/(1-\epsilon^n)$ in the list of generators of $\mathbb{Z}[\epsilon]$. If $n = 1$ then

$(1-\epsilon)/(1-\epsilon^n) = 1$, so it can be assumed that $1 < n < q$. Then since $\epsilon^q = 1$

$$\begin{aligned}
0 &= 1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{q-1} \\
&= (1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}) + \epsilon^n(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}) + \cdots \\
&\qquad \cdots + \epsilon^{kn}(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{r-1}) \\
&= (1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1})(1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n}) \\
&\qquad + \epsilon^{kn}(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{r-1})
\end{aligned}$$

where $r \le n < q$ and $q = kn + r$, so that $\epsilon^{kn} = \epsilon^{-r}$; hence

$$\begin{aligned}
\frac{1-\epsilon}{1-\epsilon^n} &= \frac{1}{1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1}} \\
&= \frac{1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n}}{(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{n-1})(1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n})} \\
&= \frac{1 + \epsilon^n + \epsilon^{2n} + \cdots + \epsilon^{(k-1)n}}{-\epsilon^{kn}(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^{r-1})} = -\epsilon^r P(\epsilon)\frac{(1-\epsilon)}{(1-\epsilon^r)}
\end{aligned}$$

where $P(\epsilon) = 1 + \epsilon^n + \cdots + \epsilon^{(k-1)n} \in \mathbb{Z}[\epsilon]$. If $r = n$ then $q = kn + r = (k+1)n$ so that $n|q$; hence if n is not a divisor of q then $r < n$ and the quotient $(1-\epsilon)/(1-\epsilon^n)$ can be replaced in the list of generators of the ring $\mathbb{Z}[\rho]$ by the quotient $(1-\epsilon)/(1-\epsilon^r)$. The argument can be repeated, and eventually the only generators left of the form $(1-\epsilon)/(1-\epsilon^n)$ are those for which $1 < n < q$ and $n|q$, as desired. All of these generators are contained in the cyclotomic field $\mathbb{Q}(\epsilon)$, which therefore must be the field of quotients of the ring $\mathbb{Z}[\rho]$. If $q = p$ is a prime it follows immediately from (i) that the ring $\mathbb{Z}[\rho]$ is generated by ϵ alone, so that $\mathbb{Z}[\rho] = \mathbb{Z}[\epsilon]$; and if $p = 2$ then $\epsilon = -1$ and $\mathbb{Z}[\rho] = \mathbb{Z}$. That suffices to conclude the proof.

Part (iii) of the preceding lemma can be demonstrated more directly by noting that if ρ is a finite representation of order two then $\rho(T) = \pm 1 \in \mathbb{Z}$ for all $T \in \Gamma$; consequently $\big(1-\rho(T)\big)\big(1-\rho(S)\big)^{-1} \in \mathbb{Z}$ for all $S, T \in \Gamma$ for which $\rho(S) \ne 1$, since this quotient obviously is either 0 or 1. If q is not a prime the ring $\mathbb{Z}[\rho]$ can be properly larger than just the ring $\mathbb{Z}[\epsilon]$ generated by a primitive q-th root of unity. For example if ρ is a finite representation of order 4 and $\epsilon = i$ then

$$\mathbb{Z}[\rho] = \mathbb{Z}\left[i, \frac{1-i}{1-i^2}\right] = \mathbb{Z}\left[i, \frac{1-i}{2}\right] = \mathbb{Z}\left[\frac{i}{2}\right]$$

since $\left(\frac{1-i}{2}\right)^2 = -\frac{i}{2}$; and if ρ is a finite representation of order 6 and $\epsilon = \exp(2\pi i/6) = \frac{1}{2}(1 + i\sqrt{3})$ then

$$\mathbb{Z}[\rho] = \mathbb{Z}\left[\epsilon, \frac{1-\epsilon}{1-\epsilon^2}, \frac{1-\epsilon}{1-\epsilon^3}\right] = \mathbb{Z}\left[\epsilon, \frac{1+\bar\epsilon}{3}, \frac{1-\epsilon}{2}\right] = \mathbb{Z}\left[\frac{i}{2\sqrt{3}}\right] = \mathbb{Z}\left[\frac{\epsilon}{3}\right]$$

since $\left(\frac{1+\bar\epsilon}{3}\right)\left(\frac{1-\epsilon}{2}\right) = \frac{i}{2\sqrt{3}}$. In both of these cases, and in general, the ring $\mathbb{Z}[\rho]$ is not a finite Z-module; but an evident consequence of the preceding lemma is that ring $\mathbb{Z}[\rho]$ is a finite Z-module whenever ρ is finite of prime order, and that is one of the reasons that special case is so interesting.

For the application of these algebraic observations, consider first the various systems of indexed markings $\mathcal{T}(l)$ for a fixed marking $\mathcal{T}$ of a compact Riemann surface M of genus $g > 0$. If $\rho : \Gamma \longrightarrow \mathbb{C}^*$ is a homomorphism for which $\rho(T_l) = \zeta_l \neq 1$ and $\rho(T_k) = \zeta_k \neq 1$ equation (8.35) can be rewritten

$$z_{i,k} = \frac{1-\zeta_k}{1-\zeta_l} z_{i,l} - \frac{1-\zeta_i}{1-\zeta_l} z_{k,l}, \tag{8.40}$$

which expresses the canonical coordinates $z_{i,k}$ of Prym cohomology classes in terms of the indexed marking $\mathcal{T}(k)$ as linear functions of the canonical coordinates $z_{i,l}$ of Prym cohomology classes in terms of the indexed marking $\mathcal{T}(l)$; but this formula also involves the additional variable $z_{l',l}$ if either $i = l'$ or $k = l'$, and that variable is not one of the canonical coordinates with respect to the indexed marking $\mathcal{T}(l)$. A straightforward calculation shows that equation (ii) of Theorem 8.8 is equivalent to

$$z_{l',l} = \epsilon(l) \sum_{\substack{1 \le i \le g \\ i \ne l, l'}} \Big(\frac{1-\zeta_{i+g}}{1-\zeta_l}\ z_{i,l} - \frac{1-\zeta_i}{1-\zeta_l}\ z_{i+g,l} \Big) \tag{8.41}$$

where

$$\epsilon(l) = \begin{cases} +1 & \text{if} \quad l < l' \\ -1 & \text{if} \quad l > l', \end{cases} \tag{8.42}$$

and this expresses $z_{l',l}$ as a linear function of the canonical coordinates in terms of the indexed marking $\mathcal{T}(l)$. Equations (8.40) and (8.41) taken together therefore express the canonical coordinates $z_{i,k}$ of a Prym cohomology class in terms of the indexed marking $\mathcal{T}(k)$ as linear functions of the canonical coordinates $z_{i,l}$ of that Prym cohomology class in terms of the indexed marking $\mathcal{T}(l)$.

The coefficients of these linear equations are rational functions of the parameters $\zeta_i = \rho(T_i)$ describing the representation ρ, so they can be viewed as elements $R^{ij}_{kl}(\zeta) \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g})$; and at the same time it is also clear from the equations (8.40) and (8.41) that for any fixed value of the parameters ζ_i these coefficients are contained in the ring $\mathbb{Z}[\rho]$ for the representation ρ described by those values of the parameters ζ_i. Thus the systems of canonical coordinates for Prym cohomology classes in terms of the indexed markings $\mathcal{T}(k)$ and $\mathcal{T}(l)$ are related by linear equations of the form

$$z_{i,k} = \sum_{\substack{1 \le j \le g \\ j \ne l, l'}} R^{ij}_{kl}(\zeta) z_{j,l} \quad \text{for} \quad 1 \le i \le 2g,\ i \ne k, k' \tag{8.43}$$

where $R^{ij}_{kl}(\zeta) \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g}) \cap \mathbb{Z}[\rho]$. If the vectors $z_k = \{z_{i,k}\}$ and $z_l = \{z_{i,l}\}$ are viewed as column vectors of length $2g-2$ these linear equations can be written more succinctly in matrix form as $z_k = R_{kl}(\zeta)\, z_l$ where $R_{kl}(\zeta) = \{\, R^{ij}_{kl}(\zeta)\}$.

When these coefficients are viewed as rational functions in $Q(\zeta_1, \dots, \zeta_{2g})$ they are uniquely determined by these linear equations, since the canonical coordinates are independent variables; and from the uniqueness it follows that $R_{kl}(\zeta)R_{lk}(\zeta) = \mathrm{I}$, so that $R_{kl}(\zeta) \in \mathrm{Gl}\big(2g-2, \mathbb{Q}(\zeta_1, \dots, \zeta_{2g})\big)$, and further that $R_{kl}(\zeta)R_{lm}(\zeta) = R_{km}(\zeta)$.

The set of all values $\zeta_i \ne 0$ for $1 \le i \le 2g$ parametrize the full group $H^1(M, \mathbb{C}^*)$ of flat line bundles over the Riemann surface M, and the subset of values ζ_i not all of which are equal to 1 parametrize the subset $U \subset H^1(M, \mathbb{C}^*)$ of all nontrivial flat line bundles, those flat line bundles other than the identity flat line bundle. The set U can be written as the union

$$U = \bigcup_{l=1}^{2g} U_l \ \text{ where } \ U_l = \Big\{ (\zeta_1, \zeta_2, \dots, \zeta_{2g}) \in \mathbb{C}^*)^{2g} \ \Big| \ \zeta_l \ne 1 \Big\}. \tag{8.44}$$

The matrix $R_{kl}(\zeta)$ takes well defined numerical values on the intersection $U_k \cap U_l$ so is actually a holomorphic nonsingular matrix-valued function in that subset The condition that $R_{kl}(\zeta)R_{lm}(\zeta) = R_{km}(\zeta)$ means that these matrices describe a holomorphic vector bundle of rank $2g-2$ over the complex manifold U, a bundle called the *Prym cohomology bundle* of the marked Riemann surface M, an interesting if rather simple bundle that will not be examined further just here.

Theorem 8.10 *If M is a marked Riemann surface with the marking $\mathcal{T} = (T_1, \dots, T_{2g})$, and if $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is a representation for which $\rho(T_l) \ne 1$, then for any commutator $[R, S] \in [\Gamma, \Gamma]$ there are complex numbers $\epsilon_j(R, S) \in \mathbb{Z}[\rho]$ depending only on the elements $R, S \in \Gamma$ such that*

$$\sigma([R, S]) = \sum_{j=1}^{2g} \epsilon_j(R, S)\sigma([T_j, T_l]) \tag{8.45}$$

for every cocycle $\sigma \in Z^1(\Gamma, \mathbb{C}_\rho)$.

Proof: When a covering translation $S \in \Gamma$ is expressed in terms of the generators T_i as $S = T_{k_1}^{\nu_1} T_{k_2}^{\nu_2} T_{k_3}^{\nu_3} \cdots$ it follows from Lemma 8.3 (i) that

$$\begin{aligned}\sigma([S, T_l]) &= \sigma([T_{k_1}^{\nu_1}, T_l]) + \rho(T_{k_1}^{\nu_1})\sigma([T_{k_2}^{\nu_2}, T_l]) + \rho(T_{k_1}^{\nu_1}T_{k_2}^{\nu_2})\sigma([T_{k_3}^{\nu_3}, T_l]) + \cdots \\ &= \sum_i \epsilon'_i(S)\sigma([T_{k_i}^{\nu_i}, T_l])\end{aligned}$$

where $\epsilon'_i(S) \in \mathbb{Z}[\rho]$ depends only on the element $S \in \Gamma$. Next it follows inductively from Lemma 8.3 (i) that if $\nu > 0$ then

$$\sigma([T_k^\nu, T_l]) = \Big(1 + \rho(T_k) + \cdots + \rho(T_k^{\nu-1})\Big)\sigma([T_k, T_l]) = \epsilon''(T_k^\nu)\sigma([T_k, T_l])$$

where $\epsilon''(T_k^\nu) \in \mathbb{Z}[\rho]$ depends only on the element $T_k^\nu \in \Gamma$; and if $\nu < 0$ it follows from this and Lemma 8.3 (ii) that

$$\begin{aligned}\sigma([T_k^\nu, T_l]) &= -\rho(T_k^\nu)\sigma([T_k^{-\nu}, T_l]) = -\rho(T_k^\nu)\epsilon''(T_k^{-\nu})\sigma([T_k, T_l]) \\ &= \epsilon''(T_k^\nu)\sigma([T_k, T_l])\end{aligned}$$

where $\epsilon''(T_k^{\nu}) = -\rho(T_k^{\nu})\epsilon''(T_k^{-\nu}) \in \mathbb{Z}[\rho]$ depends only on the element T_k^{ν}. Combining these observations shows that

$$\begin{aligned}\sigma([S,T_l]) &= \sum_i \epsilon_i'(S)\epsilon''(T_{k_i}^{\nu_i})\sigma([T_{k_i},T_l]) \\ &= \sum_{j=1}^{2g} \epsilon_j(S)\sigma([T_j,T_l])\end{aligned}$$

where $\epsilon_j(S) \in \mathbb{Z}[\rho]$ depends only on the element $S \in \Gamma$. Finally for any two covering translations R, $S \in \Gamma$ it follows from Lemma 8.3 (iii) and the preceding observation that

$$\begin{aligned}\sigma([R,S]) &= \frac{1-\rho(S)}{1-\rho(T_l)}\ \sigma([R,T_l]) - \frac{1-\rho(R)}{1-\rho(T_l)}\ \sigma([S,T_l]) \\ &= \sum_{j=1}^{2g}\left(\frac{1-\rho(S)}{1-\rho(T_l)}\ \epsilon_j(R) - \frac{1-\rho(R)}{1-\rho(T_l)}\ \epsilon_j(S)\right)\sigma([T_j,T_l]) \\ &= \sum_{j=1}^{2g} \epsilon_j(R,S)\sigma([T_j,T_l])\end{aligned}$$

where $\epsilon_j(R,S) \in \mathbb{Z}[\rho]$ depends only on the elements $R, S \in \Gamma$, and that suffices to conclude the proof.

Corollary 8.11 *If $\mathcal{S} = (S_1,\ldots,s_{2g})$ and $\mathcal{T} = (T_1,\ldots,T_{2g})$ are two markings of a compact Riemann surface M of genus $g > 1$, and if $\rho \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ is a representation for which both $\rho(S_l) \neq 1$ and $\rho(T_m) \neq 1$, then there is a uniquely determined $(2g-2)\times(2g-2)$ matrix*

$$E(\rho) = \{\epsilon_{ij}\} \in \mathrm{Gl}(2g-2,\mathbb{Z}[\rho]) \tag{8.46}$$

with entries $\epsilon_{ij} \in \mathbb{Z}[\rho]$ such that for any Prym cohomology class $\sigma \in H^1(\Gamma,\mathbb{C}_\rho)$ the canonical coordinates $w_{i,l} = \sigma([S_i,S_l])$ and $z_{j,m} = \sigma([T_j,T_m])$ of this class in terms of the generators S_l and T_m in these two markings are related by

$$w_{i,l} \quad = \sum_{\substack{1\le j\le 2g \\ j\neq m,m'}} \epsilon_{ij}\ z_{j,m} \tag{8.47}$$

for all indices $1 \le i \le 2g, i \neq l, l'$.

Proof: By the preceding theorem there are values $\epsilon_{ij}^* \in \mathbb{Z}[\rho]$ for all indices $1 \le i,j \le 2g$ such that

$$\sigma([S_i,S_l]) = \sum_{1\le j\le 2g} \epsilon_{ij}^*\ \sigma([T_j,T_m])$$

for every cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$; thus in terms of the canonical coordinates $w_{i,l} = \sigma([S_i, S_l])$ and $z_{j,m} = \sigma([T_j, T_m])$ of that cohomology class

$$\begin{aligned} w_{i,l} &= \sum_{1 \le j \le 2g} \epsilon^*_{ij}\ \sigma([T_j, T_m]) \\ &= \sum_{\substack{1 \le j \le g \\ j \ne m, m'}} \epsilon^*_{ij}\ z_{j,m} + \epsilon^*_{im}\ \sigma([T_m, T_m]) + \epsilon^*_{im'}\ \sigma([T_{m'}, T_m]) \end{aligned}$$

for $1 \le i \le 2g,\ i \ne l, l'$. Of course $\sigma([T_m, T_m]) = 0$, while by (8.41)

$$\sigma([T_{m'}, T_m]) = \epsilon(m) \sum_{\substack{1 \le j \le g \\ j \ne m, m'}} \left(\frac{1 - \rho(T_{j+g})}{1 - \rho(T_m)}\ z_{j,m} - \frac{1 - \rho(T_j)}{1 - \rho(T_m)}\ z_{j+g,m} \right)$$

and the coefficients of the canonical coordinates $z_{i,m}$ in this equation also lie in the ring $\mathbb{Z}[\rho]$. These observations taken together yield (8.47). Since the canonical coordinates are linearly independent the complex matrix $E(\rho)$ is invertible. The same arguments hold when the roles of the two markings are interchanged, so the inverse matrix also must have coefficients in the ring $\mathbb{Z}[\rho]$ and hence $E(\rho) \in \mathrm{Gl}(2g-2, \mathbb{Z}[\rho])$. That suffices to conclude the proof.

The matrix $E(\rho)$ of the preceding corollary can be viewed as a function of the flat line bundle $\rho \in U$. To be explicit, choose a marking of the surface M by generators $R_i \in \Gamma$ and use the values $\zeta_i = \rho(R_i)$ to parametrize all nontrivial flat representations $\rho \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ by points in the complex manifold U defined as on page 209 in terms of this marking, so that the representations ρ can be viewed as functions $\rho(\zeta)$. Since any element $T \in \Gamma$ can be written in terms of the generators R_i as a product $T = \prod_i R_{k(i)}^{\nu_i}$ where $1 \le k(i) \le 2g$ and $\nu_i \in \mathbb{Z}$, it is evident that any element $\epsilon \in \mathbb{Z}[\rho]$ can be written as a quotient of polynomials in the variables ζ_i with integer coefficients, and consequently can be viewed as an element $\epsilon \in \mathbb{Q}(\zeta_1, \ldots, \zeta_{2g})$.

Corollary 8.12 *If $\{S_i\}$ and $\{T_j\}$ are two sets of generators of the covering translation group Γ of a compact Riemann surface M of genus $g > 1$, describing two markings of M, there is a uniquely determined $(2g-2) \times (2g-2)$ matrix*

$$E(\zeta) = \{e_{ij}(\zeta)\} \in \mathrm{Gl}(2g-2, \mathbb{Q}(\zeta)) \tag{8.48}$$

of rational functions $e_{ij}(\zeta)$ of the variables $\zeta = (\zeta_1, \ldots \zeta_{2g}) \in U$ parametrizing flat line bundles over M such that for any flat line bundle $\rho = \rho(\zeta) \in$ $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ for which $\rho(S_l) \ne 1$ and $\rho(T_m) \ne 1$ and for any Prym cohomology class $\sigma \in H^1(\Gamma, \mathbb{C}_\rho)$ the canonical coordinates $w_{i,l} = \sigma([S_i, S_l])$ and $z_{j,m} = \sigma([T_j, T_m])$ of σ in terms of the generators S_l and T_m in these two *markings are related by*

$$w_{i,l} \quad = \sum_{\substack{1\le j\le 2g \\ j\ne m,m'}} e_{ij}(\zeta)\; z_{j,m} \tag{8.49}$$

for $1 \le i \le 2g, i \ne l, l'$.

Proof: This is merely a restatement of the conclusion of the preceding corollary, since the coefficients ϵ_{ij} of that corollary can be written $\epsilon_{ij} = e_{ij}(\zeta)$ for rational functions $e_{ij}(\zeta) \in \mathbb{Q}(\zeta_1, \ldots \zeta_{2g})$, so no further proof is required.

Thus the matrix $E(\rho)$ of Corollary 8.11 when viewed as a function of the flat line bundle ρ described by the parameters $\zeta_i = \rho(R_i)$ is a rational function $E(\zeta)$ of the variables ζ_i; and the values taken by this matrix function for a fixed value of the parameters ζ_i lie in the ring $\mathbb{Z}[\rho]$ for the representation ρ described by the parameters ζ_i. It is evident from this that

$$E(\overline{\rho}) = \overline{E(\rho)} \tag{8.50}$$

since the rational functions $e_{ij}(\zeta)$ have rational integral coefficients.

Addendum: Higher Prym Cohomology Groups

The discussion in this chapter so far has involved only the first Prym cohomology groups. It is perhaps natural to ask about the higher-dimensional Prym cohomology groups, so a brief discussion of these groups is included here; but since they play only a very limited role in the study of Riemann surfaces the discussion is somewhat condensed and is relegated to this addendum, which can be ignored altogether if desired. If M is a compact Riemann surface of genus $g > 0$ with covering translation group Γ then by using a $\mathcal{C}^\infty$ partition of unity on M, as in the demonstration of (E.33) in Appendix E.2 or the natural extension of the proof of Lemma 8.1, it is easy to see that $H^q(\Gamma, \Gamma(\tilde{M}, \mathcal{E})) = 0$ whenever $q > 0$; consequently from the exact cohomology sequence associated to the exact sequence (8.10) of vector spaces on which the group Γ acts as in (8.6) it follows that

$$H^q(\Gamma, \mathbb{C}_\rho) \cong H^{q-1}(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^1_c)) \quad \text{for} \quad q \geq 2.$$

On the other hand there is also the exact sequence of vector spaces

$$0 \longrightarrow \Gamma(\tilde{M}, \mathcal{E}^1_c)) \longrightarrow \Gamma(\tilde{M}, \mathcal{E}^1)) \xrightarrow{d} \Gamma(\tilde{M}, \mathcal{E}^2)) \longrightarrow 0$$

on which the group Γ acts in the same way; and from the associated exact cohomology sequence it follows first that

$$H^1(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^1_c)) \cong \frac{H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^2))}{d\, H^0(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^1))} \cong \frac{\Gamma(M, \mathcal{E}^2(\rho))}{d\, \Gamma(M, \mathcal{E}^1(\rho))}$$

and second that

$$H^q(\Gamma, \Gamma(\tilde{M}, \mathcal{E}^1_c)) = 0 \quad \text{for} \quad q \geq 2.$$

Combining these observations shows that

$$(8.51) \qquad H^q(\Gamma, \mathbb{C}_\rho) \cong \begin{cases} \dfrac{\Gamma(M, \mathcal{E}^2(\rho))}{d\, \Gamma(M, \mathcal{E}^1(\rho))} & \text{for} \quad q = 2, \\[2ex] 0 & \text{for} \quad q \geq 3; \end{cases}$$

consequently for the study of Riemann surfaces only the second cohomology group $H^2(\Gamma, \mathbb{C}_\rho)$ really is of interest.

To trace through the isomorphism (8.51) for $q = 2$ explicitly, recall the description of the coboundary operators on inhomogeneous cochains in the cohomology of groups in (E.13) in Appendix E.1 and consider a differential form $\phi \in \Gamma(M, \mathcal{E}^2(\rho))$, viewed as a $\mathcal{C}^\infty$ differential form $\phi(z)$ on the universal covering surface $\tilde{M}$ such that $\phi(Tz) = \rho(T)\phi(z)$ for all $T \in \Gamma$. This differential form can be written as the exterior derivative $\phi(z) = d\psi(z)$ of a $\mathcal{C}^\infty$ one-form $\psi(z)$ on $\tilde{M}$ since $\tilde{M}$ is contractible. Then for any $T \in \Gamma$ it follows that $d\big(\rho(T)^{-1}\psi(Tz) - \psi(z)\big) = \rho(T)^{-1}\phi(Tz) - \phi(z) = 0$, so since $\tilde{M}$ is contractible there is a $\mathcal{C}^\infty$ function $f(T, z)$ on $\tilde{M}$, depending also on the element $T \in \Gamma$, such that $\rho(T)^{-1}\psi(Tz) - \psi(z) = df(T,z)$. In particular $df(I,z) = 0$ for the identity element $I \in \Gamma$, so it always can be assumed that $f(I,z) = 0$. For these functions and for any elements $T_1, T_2 \in \Gamma$

$$\begin{aligned} d\Big(\rho(T_2)^{-1} f(T_1, T_2 z) + f(T_2, z) - f(T_1T_2, z)\Big) \\ = \rho(T_2)^{-1}\Big(\rho(T_1)^{-1}\psi(T_1T_2z) - \psi(T_2z)\Big) \\ + \Big(\rho(T_2)^{-1}\psi(T_2z) - \psi(z)\Big) \\ - \Big(\rho(T_1T_2)^{-1}\psi(T_1T_2z) - \psi(z)\Big) \\ = 0; \end{aligned}$$

consequently the expression

$$v(T_1, T_2) = \rho(T_2)^{-1} f(T_1, T_2 z) + f(T_2, z) - f(T_1T_2, z) \tag{8.52}$$

is a constant in the variable $z \in \tilde{M}$. In particular since $f(I,z) = 0$ by assumption it follows that

$$\begin{aligned} v(I, T_2) &= \rho(T_2)^{-1} f(I, T_2 z) + f(T_2, z) - f(T_2, z) &= 0, \\ v(T_1, I) &= f(T_1, z) + f(I, z) - f(T_1, z) &= 0, \end{aligned}$$

so the mapping $v : \Gamma \times \Gamma \longrightarrow \mathbb{C}$ is an inhomogeneous two-cochain $v \in C^2(\Gamma, \mathbb{C}_\rho)$ as defined in equation (E.9) of Appendix E.2. Equation (8.52) exhibits the two-cochain v as the coboundary of the one-cohain $f(T,z) \in C^1(\Gamma, \Gamma(\tilde{M}, \mathcal{E}))$, so it must be a two-cocycle $v \in Z^2(\Gamma, \mathbb{C}_\rho)$. Actually it is a straightforward calculation to verify directly from (8.52) that the cochain $v(T_1, T_2)$ satisfies the cocycle condition

$$\rho(T_3)^{-1} v(T_1, T_2) - v(T_2, T_3) + v(T_1T_2, T_3) - v(T_1, T_2T_3) = 0, \tag{8.53}$$

keeping in mind that the expression (8.52) is independent of the variable $z \in \tilde{M}$. This cocycle of course depends on the choices of the differential form $\psi(z)$ and of the functions $f(T,z)$. If $d\psi_1(z) = \phi(z)$ and $df_1(T,z) = \rho(T)^{-1}\psi_1(Tz) - \psi_1(z)$ then since $d\big(\psi_1(z) - \psi(z)\big) = 0$ it must be the case that $\psi_1(z) - \psi(z) = dg(z)$ for some $\mathcal{C}^\infty$ function $g(z)$ on $\tilde{M}$, so $\rho(T)^{-1}\psi_1(Tz) - \psi_1(z) = \rho(T)^{-1}\psi(Tz) - \psi(z) + d\big(\rho(T)^{-1}g(Tz) - g(z)\big) = d\big(f(T,z) + \rho(T)^{-1}g(Tz) - g(z)\big)$ and consequently $f_1(T,z) = f(T,z) + \rho(T)^{-1}g(Tz) - g(z) + c(T)$ for some constants $c(T)$; in order that $f_1(I,z) = 0$ it is necessary that $c(I) = 0$, so the mapping $c : \Gamma \longrightarrow \mathbb{C}$ can be viewed as an inhomogeneous one-cochain $c \in C^0(\Gamma, \mathbb{C}_\rho)$. It is a straightforward calculation to verify that for these functions $f_1(T,z)$ the definition (8.52) yields the expression $v_1(T_1,T_2) = v(T_1,T_2) + \rho(T_2)^{-1}c(T_1) + c(T_2) - c(T_1T_2)$, which is a cocycle cohomologous to $v(T_1,T_2)$. Thus the cohomology class in $H^2(\Gamma, \mathbb{C}_\rho)$ represented by the cocycle $v(T_1,T_2)$ is independent of these choices so depends only on the initial differential form $\phi(z)$; this cohomology class is the *period class* $\delta\phi \in H^2(\Gamma, \mathbb{C}_\rho)$ of the differential form ϕ.

The second cohomology group $H^2(\Gamma,\mathbb{C}_\rho)$ has an explicit description given by the theorem of H. Hopf, which for the simpler situation in which the group Γ acts trivially is Theorem E.2 in Appendix E.2. This description does not involve any special properties of the group Γ as the covering translation group of a compact Riemann surface, so will be demonstrated in Theorem 8.14 for more general groups as well; its application to Riemann surfaces of course does depend on the special structure of the covering translation group of a surface, so will be discussed separately in the subsequent Corollary 8.15. Note that any representation $\rho \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ of any group Γ can be used to exhibit that group as a group of operators on the complex numbers as in (8.7), in terms of which the cohomology groups $H^p(\Gamma,\mathbb{C}_\rho)$ are well defined.

Lemma 8.13 *If F is a finitely generated free group and $\rho \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$ then $H^2(F,\mathbb{C}_\rho)=0$.*

Proof: The proof is a straightforward extension of the proof of Lemma E.25 in Appendix E.2. First setting $T_1=T_3=T$ and $T_2=T^{-1}$ in the cocycle condition (8.53) shows that $v(T^{-1},T)=\rho(T)^{-1}v(T,T^{-1})$ for any cocycle $v\in Z^2(F,\mathbb{C}_\rho)$, since $v(T_1,T_2)=0$ if $T_1=I$ or $T_2=I$. Then for any cocycle $v\in Z^2(F,\mathbb{C}_\rho)$ and any free generator T_i of the group F choose a value $\sigma(T_i)\in\mathbb{C}$, set $\sigma(T_i^{-1})=v(T_i,T_i^{-1})-\rho(T_i)\sigma(T_i)=\rho(T_i)v(T_i^{-1},T_i)-\rho(T_i)\sigma(T_i)$, and define $\sigma(T)$ for any formal product of the generators T_i and their inverses, without cancellation, by

$$\sigma(ST)=\rho(T)^{-1}\sigma(S)+\sigma(T)-v(S,T). \tag{8.54}$$

It follows readily from these definitions that $\sigma(T_iT_i^{-1})=\sigma(T_i^{-1}T_i)=0$, and it is a straightforward calculation to verify from the definition (8.54) and the cocycle condition (8.53) that $\sigma(R\cdot ST)=\sigma(RS\cdot T)$; consequently σ is a well defined mapping $\sigma: F\longrightarrow\mathbb{C}$, and since $\sigma(I)=0$ it actually is a one-cochain $\sigma\in C^0(F,\mathbb{C}_\rho)$. Equation (8.54) shows that the cocycle v is the coboundary of the cochain σ, and consequently that $H^1(F,\mathbb{C}_\rho)=0$. That suffices to conclude the proof.

Next supppose that Γ is any finitely generated group, so can be described by an exact sequence

$$0\longrightarrow K\xrightarrow{\iota}F\xrightarrow{p}\Gamma\longrightarrow 0 \tag{8.55}$$

in which F is a finitely generated free group, ι is an inclusion mapping and p is the projection to the quotient group. A representation $\rho\in\mathrm{Hom}(\Gamma,\mathbb{C}^*)$ lifts to the representation $\rho\circ p\in\mathrm{Hom}(F,\mathbb{C}^*)$, which to simplify the notation also will be denoted just by ρ; so if $\tilde{T}\in F$ then by definition $\rho(\tilde{T})=\rho(p(\tilde{T}))$. For any cocycle $v\in Z^2(\Gamma,\mathbb{C}_\rho)$ the composition $v\circ p=p^*(v)$ is a cocycle $p^*(v)\in Z^2(F,\mathbb{C}_\rho)$, so by the preceding lemma is the coboundary of a one-cochain $\sigma\in C^1(F,\mathbb{C}_\rho)$; thus σ is a mapping $\sigma: F\longrightarrow\mathbb{C}$ such that $\sigma(I)=0$ and

$$v(\tilde{T}_1,\tilde{T}_2)=\rho(\tilde{T}_2)^{-1}\sigma(\tilde{T}_1)+\sigma(\tilde{T}_2)-\sigma(\tilde{T}_1\tilde{T}_2) \tag{8.56}$$

for all $\tilde{T}_i \in F$. Since $v(T_1, T_2) = 0$ if either $T_1 = I$ or $T_2 = I$ then $p^*(v)(\tilde{T}_1, \tilde{T}_2) = 0$ if either $\tilde{T}_1 \in K$ or $\tilde{T}_2 \in K$. It follows from this and from the cocycle condition (8.56) that if $\tilde{S} \in K$ and $\tilde{T} \in F$ then

$$\begin{aligned}\sigma(\tilde{S}\tilde{T}) &= \rho(\tilde{T})^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T}),\\ \sigma(\tilde{T}\tilde{S}) &= \sigma(\tilde{T}) + \sigma(\tilde{S});\end{aligned}$$

consequently the restriction $\sigma|K$ is a homomorphism from the group K to the additive group $\mathbb{C}$ and

$$\begin{aligned}\sigma(\tilde{T}) + \sigma(\tilde{S}) = \sigma(\tilde{T}\tilde{S}) &= \sigma(\tilde{T}\tilde{S}\tilde{T}^{-1} \cdot \tilde{T})\\ &= \rho(\tilde{T})^{-1}\sigma(\tilde{T}\tilde{S}\tilde{T}^{-1}) + \sigma(\tilde{T})\end{aligned}$$

so that $\sigma(\tilde{T}\tilde{S}\tilde{T}^{-1}) = \rho(\tilde{T})\sigma(\tilde{S})$. Therefore the restriction $\sigma|K$ is an element of the group

$$\text{(8.57)} \qquad \operatorname{Hom}_\rho(K, \mathbb{C}) = \left\{ \sigma \in \operatorname{Hom}(K, \mathbb{C}) \;\middle|\; \begin{array}{c} \sigma(\tilde{T}\tilde{S}\tilde{T}^{-1}) = \rho(p(\tilde{T}))\sigma(\tilde{S}) \\ \\ \text{for all } \ \tilde{S} \in K, \ \tilde{T} \in F \end{array} \right\}$$

analogous to (8.22). On the other hand for any cocycle $\tau \in Z^1(F, \mathbb{C}_\rho)$ the cocycle condition shows that $\tau(\tilde{S}\tilde{T}) = \rho(\tilde{T})^{-1}\tau(\tilde{S}) + \tau(\tilde{T})$ for all $\tilde{S}, \tilde{T} \in F$, and consequently that $\tau|K \in \operatorname{Hom}(K, \mathbb{C})$; and it follows from Lemma 8.2 that $\tau(\tilde{T}\tilde{S}\tilde{T}^{-1}) = \rho(\tilde{T})\tau(\tilde{S}) + \big(\rho(\tilde{S})^{-1} - 1\big)\rho(\tilde{T})\tau(\tilde{T})$ for all $\tilde{S}, \tilde{T} \in F$, so if $\tilde{S} \in K$ then $\rho(\tilde{S}) = 1$ and $\tau(\tilde{T}\tilde{S}\tilde{T}^{-1}) = \rho(\tilde{T})\tau(\tilde{S})$. Thus

$$Z^1(F, \mathbb{C}_\rho)\Big|K \subset \operatorname{Hom}_\rho(K, \mathbb{C}),$$

so it is possible to introduce the quotient space

$$\text{(8.58)} \qquad \operatorname{Hom}_{\rho,Z}(K, \mathbb{C}) = \frac{\operatorname{Hom}_\rho(K, \mathbb{C})}{Z^1(F, \mathbb{C}_\rho)\Big|K}.$$

Theorem 8.14 *If Γ is a finitely generated group described by the exact sequence (8.55) for a finitely generated free group F, and if $\rho \in \operatorname{Hom}(\Gamma, \mathbb{C}^*)$ is a representation of the group Γ, then the mapping ϕ that associates to the cohomology class represented by a cocycle $v \in Z^2(\Gamma, \mathbb{C}_\rho)$ the restriction $\sigma|K$ of any cochain $\sigma \in C^1(F, \mathbb{C}_\rho)$ with the coboundary $\delta\sigma = p^*(v) \in Z^2(F, \mathbb{C}_\rho)$ determines an isomorphism*

$$H^2(\Gamma, \mathbb{C}_\rho) \cong \operatorname{Hom}_{\rho,Z}(K, \mathbb{C}).$$

Proof: If $v \in B^2(\Gamma, \mathbb{C}_\rho)$ then $v = \delta\sigma$ for a cochain $\sigma \in C^1(\Gamma, \mathbb{C}_\rho)$ and $p^*(v) = \delta p^*(\sigma)$ where $p^*(\sigma)|K = 0$. If $v \in Z^2(\Gamma, \mathbb{C}_\rho)$ is the coboundary of two cochains $\sigma_1, \ \sigma_2 \in C^1(F, \mathbb{C}_\rho)$ then $\sigma_1 - \sigma_2 \in Z^1(F, \mathbb{C}_\rho)$ so that $(\sigma_1 - \sigma_2)|K \in Z^1(F, \mathbb{C}_\rho)|K$ and consequently σ_1 and σ_2 determine the same element in the quotient $\mathrm{Hom}_{\rho,Z}(K,\mathbb{C})$. It follows from these observations that the mapping ϕ yields a well defined homomorphism

$$\phi^* : H^2(\Gamma,\mathbb{C}_\rho) \longrightarrow \mathrm{Hom}_{\rho,Z}(K,\mathbb{C}).$$

If a cocycle $v \in Z^2(\Gamma,\mathbb{C}_\rho)$ represents an element in the kernel of the homomorphism ϕ^* then $p^*(v) = \delta\sigma$ for a cochain $\sigma \in C^1(F,\mathbb{C}_\rho)$ for which $\sigma|K \in Z^1(F,\mathbb{C}_\rho)\Big|K$, so there is a cocycle $\tau \in Z^1(F,\mathbb{C}_\rho)$ such that $\sigma|K = \tau|K$; thus $\sigma - \tau \in C^1(\Gamma,\mathbb{C}_\rho)$, and since $p^*(v) = \delta(\sigma-\tau)$ it must be the case that $v \in B^2(\Gamma,\mathbb{C}_\rho)$ and therefore the mapping ϕ^* is injective. Finally consider any element $\sigma \in \mathrm{Hom}_\rho(K,\mathbb{C})$. Choose a coset decomposition $F = \bigcup_i K\tilde{L}_i$ for some elements $\tilde{L}_i \in F$ representing elements $L_i \in \Gamma$, select arbitrary values $\sigma(\tilde{L}_i) \in \mathbb{C}$, and in terms of these values define the mapping $\sigma : F \longrightarrow \mathbb{C}$ by setting $\sigma(\tilde{S}\tilde{L}_i) = \rho(\tilde{L}_i)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{L}_i)$ for all $S \in K$. For any elements $\tilde{R},\ \tilde{S} \in K$ and for $\tilde{T} = \tilde{S}\tilde{L}_i \in F$ it follows that

$$\begin{aligned}\sigma(\tilde{R}\tilde{T}) &= \sigma(\tilde{R}\tilde{S}\tilde{L}_i) = \rho(\tilde{L}_i)^{-1}\sigma(\tilde{R}\tilde{S}) + \sigma(\tilde{L}_i)\\ &= \rho(\tilde{L}_i)^{-1}\sigma(\tilde{R}) + \rho(\tilde{L}_i)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{L}_i)\\ &= \rho(\tilde{T})^{-1}\sigma(\tilde{R}) + \sigma(\tilde{T})\end{aligned}$$

and

$$\begin{aligned}\sigma(\tilde{T}\tilde{R}) &= \sigma(\tilde{T}\tilde{R}\tilde{T}^{-1}\cdot\tilde{T}) = \rho(\tilde{T})^{-1}\sigma(\tilde{T}\tilde{R}\tilde{T}^{-1}) + \sigma(\tilde{T})\\ &= \rho(\tilde{T})^{-1}\rho(\tilde{T})\sigma(\tilde{R}) + \sigma(\tilde{T})\\ &= \sigma(\tilde{R}) + \sigma(\tilde{T}).\end{aligned}$$

The mapping $v : F \times F \longrightarrow \mathbb{C}$ defined by

$$v(\tilde{T}_1,\,\tilde{T}_2) = \rho(\tilde{T}_2)^{-1}\sigma(\tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{T}_1\tilde{T}_2)$$

is a cocycle $v \in Z^2(F,\mathbb{C}_\rho)$ that is the coboundary of the cochain $\sigma \in C^1(F,\mathbb{C}_\rho)$. For any $\tilde{S} \in K$ it follows from the preceding observations that

$$\begin{aligned}v(\tilde{S}\tilde{T}_1,\,\tilde{T}_2) &= \rho(\tilde{T}_2)^{-1}\sigma(\tilde{S}\tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{S}\tilde{T}_1\tilde{T}_2)\\ &= \rho(\tilde{T}_2)^{-1}\big(\rho(\tilde{T}_1)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T}_1)\big) + \sigma(\tilde{T}_2)\\ &\qquad - \rho(\tilde{T}_1\tilde{T}_2)^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T}_1\tilde{T}_2)\\ &= \rho(\tilde{T}_2)^{-1}\sigma(\tilde{T}_1) + \sigma(\tilde{T}_2) - \sigma(\tilde{T}_1\tilde{T}_2) = v(\tilde{T}_1,\,\tilde{T}_2)\end{aligned}$$

and similarly $v(\tilde{T}_1, \tilde{S}\tilde{T}_2) = v(\tilde{T}_1,\,\tilde{T}_2)$; thus $v \in Z^2(\Gamma,\mathbb{C}_\rho)$, and since $\sigma = \phi^*(v)$ that shows that the mapping ϕ^* is surjective and thereby concludes the proof.

Corollary 8.15 *If Γ is the covering translation group of a compact Riemann surface M of genus $g > 1$ then $H^2(\Gamma,\mathbb{C}_\rho) = 0$ for any nontrivial representation $\rho \in \mathrm{Hom}(\Gamma,\mathbb{C}^*)$.*

Proof: Choose a marking of M described by $2g$ generators $T_i = A_i,\ T_{g+i} = B_i$ of the group Γ for $1 \le i \le g$. The group Γ then is described by an exact sequence (8.55) in which F is the free group generated by $2g$ symbols $\tilde{T}_i$ for which $p(\tilde{T}_I) = T_i$; and the kernel $K \subset F$ is the normal subgroup generated by the single element $C \in [F, F]$ where $C = C_1 \cdots C_g$ for $C_i = [A_, B_i]$. Thus an element $\sigma \in \mathrm{H}om_\rho(K, \mathbb{C})$ is determined fully by the value $\sigma(C)$ alone, and consequently $\dim \mathrm{Hom}_\rho(K, \mathbb{C}) \le 1$. On the other hand there is a cocycle $\sigma \in Z^1(F, \mathbb{C}_\rho)$ taking any specified values $\sigma(\tilde{T}_i)$ on the generators $\tilde{T}_i$; for given any values $\sigma(\tilde{T}_i)$ the cocyle condition $\sigma(\tilde{S}\tilde{T}) = \rho(\tilde{T})^{-1}\sigma(\tilde{S}) + \sigma(\tilde{T})$ can be used to define the value $\sigma(\tilde{T})$ for any word $\tilde{T} \in F$. Then as in Lemma8.2

$$\sigma([\tilde{S}, \tilde{T}]) = \big(1 - \rho(\tilde{T})\big)\rho(\tilde{S})\sigma(\tilde{S}) - \big(1 - \rho(\tilde{S})\big)\rho(\tilde{T})\sigma(\tilde{T}).$$

It is clear from this that if the representation ρ is nontrivial there is a cocycle σ such that $\sigma(C) \ne 0$, and consequently that $Z^1(\Gamma, \mathbb{C}_\rho)|K \ne 0$; and then $0 < \dim\ Z^1(\Gamma, \mathbb{C}_\rho)|K \le \dim\ \mathrm{Hom}_\rho(K, \mathbb{C}) \le 1$ and therefore $\mathrm{Hom}_{\rho,Z}(K, \mathbb{C}) = 0$. It then follows from the preceding theorem that $H^2(\Gamma, \mathbb{C}_\rho) = 0$, and that suffices for the proof.

For the trivial representation $\rho = 1$ of course $Z^1(F, \mathbb{C}_1) = \mathrm{Hom}(F, \mathbb{C})$; and since any homomorphism $\sigma \in \mathrm{Hom}(F, \mathbb{C})$ is trivial on the commutator subgroup $[F, F] \subset F$ necessarily $Z^1(F, \mathbb{C}_1)|K = 0$ so $\mathrm{Hom}_{1,Z}(K, \mathbb{C}) = \mathrm{Hom}(K, \mathbb{C}) \cong \mathbb{C}$. Thus in this special case Theorem 8.14 reduces to the isomorphism $H^2(\Gamma, \mathbb{C}) \cong \mathbb{C}$, as expected since $H^2(\Gamma, \mathbb{C}) = \mathbb{C}$.

Part II

Mappings Between Riemann Surfaces

Chapter 9

Mappings to the Riemann Sphere

PRELIMINARY VERSION

9.1 Topological Properties of Mappings Between Riemann Surfaces

A *holomorphic mapping* $\phi : M \longrightarrow N$ between Riemann surfaces M and N is a continuous mapping between these topological manifolds that is holomorphic when expressed in terms of the local coordinates on M and N. If $U_\alpha \subset M$ is a coordinate neighborhood of a point $a \in M$ with local coordinate z_α and if $V_\beta \subset N$ is a coordinate neighborhood of the image $b = \phi(a) \in N$ with local coordinate w_β, the image of a point $z_\alpha \in U_\alpha$ sufficiently near a under the mapping $\phi : M \longrightarrow N$ is a point $\phi(z_\alpha) = w_\beta \in V_\beta$; in this way the coordinate w_β is expressed as a complex-valued function $w_\beta(z_\alpha)$ of the complex variable z_α, and the mapping ϕ is holomorphic if all of these local representations are holomorphic. If the function $w_\beta(z_\alpha)$ is constant the entire neighborhood U_α is mapped to the single point $b \in N$, and it follows from the identity theorem for holomorphic functions that the entire surface M is mapped to the point b; such trivial mappings generally will be excluded from consideration. A nonconstant holomorphic function of a complex variable is an open mapping, so the image of a nontrivial holomorphic mapping $\phi : M \longrightarrow N$ between Riemann surfaces is an open subset of N; and if M is compact its image is a compact and hence closed subset of N, so since N is connected $\phi(M) = N$. Therefore a nontrivial holomorphic mapping between compact Riemann surfaces is always surjective. Of course that is not the case if M is a noncompact Riemann surface; for instance the inclusion mapping of an open subset $M \subset N$ is a holomorphic mapping $\phi : M \longrightarrow N$ between two Riemann surfaces. If the local coordinates z_α and w_β are chosen so that $a \in U_\alpha$ corresponds to the origin $z_\alpha = 0$ and $b = \phi(a) \in V_\beta$

corresponds to the origin $w_\beta = 0$, then the local holomorphic function $w_\beta(z_\alpha)$ vanishes at the origin $z_\alpha = 0$ and is determined uniquely up to nonsingular holomorphic changes of coordinates in its domain and range preserving the origin; since the non-negative integer $\mathrm{ord}_0\big(w_\beta(z_\alpha)\big) - 1$ is invariant under such changes of coordinates it is an intrinsic property of the mapping ϕ, called the *ramification order* of the mapping ϕ at the point $a \in M$ and denoted by $\mathfrak{r}_a(\phi)$. For a nontrivial holomorphic mapping ϕ the local coordinates z_α and w_β can be chosen so that

$$w_\beta(z_\alpha) = z_\alpha^{r+1} \quad \text{where} \quad r = \mathfrak{r}_a(\phi) \geq 0, \tag{9.1}$$

thus providing a local normal form of the mapping $\phi : M \longrightarrow N$. A point $a \in M$ at which $\mathfrak{r}_a(\phi) > 0$ is called a *ramification point* of the mapping ϕ, or equivalently the mapping ϕ is said to be *ramified* at the point a. The ramification points are a discrete set of points on M, since they are just the points at which the derivative of the mapping ϕ in terms of any local coordinate is zero; so since M is compact the set of ramification points is a finite subset of M, which is called the *ramification locus* of the mapping ϕ and is denoted by R or by $R(\phi)$ when it is useful to be specific. The divisor

$$\mathfrak{r}(\phi) = \sum_{a \in M} \mathfrak{r}_a(\phi) \cdot a \tag{9.2}$$

hence is a well defined positive divisor on M, called the *ramification divisor* of the holomorphic mapping ϕ. The mapping ϕ is said to be *simply ramified* at a point $a \in M$ if $\mathfrak{r}_a(\phi) = 1$, and such a ramification point is called a *simple ramification* point; the mapping ϕ itself is said to be *simply ramified* if all of its ramification points are simple ramification points. A point $a \in M$ that is not a ramification point, a point at which $\mathfrak{r}_a(\phi) = 0$, is called a *regular point* or an *unramified point* of the mapping ϕ; the regular points are those points of M at which the mapping ϕ is locally biholomorphic.

The images $\phi(a_i) \in N$ of the ramification points $a_i \in M$ of a nontrivial holomorphic mapping $\phi : M \longrightarrow N$ between compact Riemann surfaces are called the *branch points* of the mapping ϕ; the finite set of branch points is called the *branch locus* of the mapping ϕ and is denoted by B or by $B(\phi)$ when it is useful to be more specific. The inverse image $\phi^{-1}(b) \subset N$ of any point $b \in N$ consists of a finite number of points of M. If b is not a branch point none of these points are ramification points, so for a suitably small open neighborhood W of the point b in N the inverse image $\phi^{-1}(W)$ is a collection of δ disjoint open subsets of M, each of which is mapped biholomorphically to W under the mapping ϕ; thus the mapping ϕ exhibits the inverse image $\phi^{-1}(W)$ as a covering space of δ sheets over W. The number δ is independent of the point $b \notin B$, since it is a locally constant function on the connected manifold $N \sim B$; it is called the *degree* of the mapping ϕ, and it is denoted by $\deg \phi$. The restriction

$$\phi : \big(M \sim R(\phi)\big) \longrightarrow \big(N \sim B(\phi)\big) \tag{9.3}$$

of a holomorphic mapping $\phi : M \longrightarrow N$ of degree δ with branch locus $B \subset N$ thus is a δ-sheeted covering space in the usual sense, called the *regular part* of the holomorphic mapping ϕ. On the other hand if $b \in B$ then at least some of the points $a_i \in \phi^{-1}(b) \subset M$ are ramification points. However for each point $a_i \in \phi^{-1}(b)$ there are local coordinates z_i centered at a_i and w_i centered at b such that the mapping ϕ is described locally by the holomorphic function $w_i(z_i) = z_i^{r_i+1}$ where $r_i = r_{a_i}(\phi)$; so if U_i is a sufficiently small open neighborhood of a_i the restriction of the mapping ϕ to the complement $U_i \sim a_i$ is a covering mapping of $r_i + 1$ sheets and a_i is the only point in U_i that has the point b as its image under ϕ. It is evident from this that

$$\text{(9.4)} \qquad \deg \phi = \sum_{a \in \phi^{-1}(b)} \big(\mathfrak{r}_a(\phi) + 1\big) \quad \text{for any point } b \in N,$$

so the branch points of the mapping ϕ can be characterized as those points $b \in N$ at which $\phi^{-1}(b)$ consists of fewer than deg ϕ distinct points; the difference between deg ϕ and the number of distinct points in $\phi^{-1}(b)$ is called the *local branch order* of the mapping ϕ over the point b and is denoted by $\mathfrak{b}_b(\phi)$, so

$$\text{(9.5)} \qquad \mathfrak{b}_b(\phi) = \sum_{a \in \phi^{-1}(b)} \mathfrak{r}_a(\phi).$$

The branch points are precisely those points $b \in N$ at which $\mathfrak{b}_b(\phi) > 0$, and the *branch divisor* is defined to be the divisor

$$\text{(9.6)} \qquad \mathfrak{b}(\phi) = \sum_{b \in N} \mathfrak{b}_b(\phi) \cdot b$$

on the image surface N; thus the branch locus of the mapping ϕ is the support $B(\phi) = |\mathfrak{b}(\phi)|$ of the branch divisor of that mapping. The *branch order* of the mapping ϕ is the integer $\operatorname{br}(\phi)$ defined by

$$\text{(9.7)} \qquad \operatorname{br}(\phi) = \sum_{b \in N} \mathfrak{b}_b(\phi) = \sum_{a \in M} \mathfrak{r}_a(\phi),$$

and it is evident from (9.2) and (9.6) that

$$\text{(9.8)} \qquad \operatorname{br}(\phi) = \deg \mathfrak{r}(\phi) = \deg \mathfrak{b}(\phi).$$

The mapping ϕ is said to be *simply branched over a point* $b \in N$ if $\mathfrak{b}_b(\phi) = 1$, and is said to be *simply branched* if it is simply branched over each point of N; clearly ϕ is simply branched precisely when it has $\operatorname{br}\phi$ branch points altogether, and in that case there is precisely one simply ramified point of M over each of these branch points in N. The mapping ϕ is said to be *fully branched* over a point $b \in N$ if $\phi^{-1}(b) = a$ is a single point $a \in M$, and in that case it is also said to be *fully ramified* at the point $a \in M$; thus these are equivalent notions, and it is clear that the mapping ϕ is fully ramified at a point $a \in M$ whenever $\mathfrak{r}_a(\phi) = \deg \phi - 1$ and it is fully branched over a point $b \in N$ whenever $\mathfrak{b}_b(\phi) = \deg \phi - 1$. A mapping $\phi : M \longrightarrow N$ for which $\deg \phi = 2$ is both simply branched and fully branched over each point of N, while these notions are quite distinct if $\deg \phi \geq 3$.

Theorem 9.1 (Riemann-Hurwitz Formula) *If $\phi : M \longrightarrow N$ is a nontrivial holomorphic mapping from a compact Riemann surface M of genus g to a compact Riemann surface N of genus h then*

$$2g - 2 = (2h - 2)\deg\,\phi + \mathrm{br}\,(\phi). \tag{9.9}$$

Proof: It is always possible to triangulate[1] the manifold N in such a way that the branch points of the mapping ϕ are among the vertices of the triangulation. If there are n_i simplices of dimension i in the triangulation of N and m_i simplices of dimension i in the induced triangulation of M then

$$m_0 = \delta\, n_0 - \beta, \quad m_1 = \delta\, n_1, \quad m_2 = \delta\, n_2,$$

where $\delta = \deg\,\phi$ and $\beta = \mathrm{br}\,\phi$, since each simplex of N of dimension 1 or 2 is covered by δ simplices of M while each simplex of N of dimension 0 is covered by δ simplices of M except for the branch points, at which the number of simplices in M is reduced by the local branch order of the mapping ϕ at that point of N. Euler's formula (D.11) asserts that the Euler characteristic $\chi(M)$ of the surface M, the alternating sum of the ranks of the homology groups, is equal to the alternating sum of the numbers of simplices of each dimension in a triangulation of the surface; for the surface M with Betti numbers $b_i = \mathrm{rank}\, H_i(M)$ the Euler characteristic is $\chi(M) = b_0 - b_1 + b_2 = 2 - 2g$, so by Euler's formula $2 - 2g = m_0 - m_1 + m_2$, and correspondingly $2 - 2h = n_0 - n_1 + n_2$ for the surface N. Consequently

$$\begin{aligned} 2 - 2g &= m_0 - m_1 + m_2 \\ &= (\delta\, n_0 - \beta) - \delta\, n_1 + \delta\, n_2 = \delta(2 - 2h) - \beta, \end{aligned}$$

which yields the desired formula and thereby concludes the proof.

Corollary 9.2 (i) *If there is a nontrivial holomorphic mapping $\phi : M \longrightarrow N$ from a compact Riemann surface M of genus g to a compact Riemann surface N of genus h the branch order of ϕ is an even integer and $g \ge h$.*
(ii) *If $g = h > 1$ the mapping ϕ is a biholomorphic mapping between the two surfaces. If $g = h = 1$ the mapping ϕ is a holomorphic covering mapping in the usual sense; there are holomorphic covering mappings of δ sheets for any $\delta > 0$.*

Proof: It is clear from the Riemann-Hurwitz formula (9.9) that the branch order $\mathrm{br}\,(\phi)$ is an even integer. That formula can be rewritten

$$g - h = (h - 1)(\delta - 1) + \frac{\beta}{2} \tag{9.10}$$

where $\delta = \deg\,\phi$ and $\beta = \mathrm{br}\,(\phi)$. If $h = 0$ then of course $g \ge h$ for any $g \ge 0$. If $h > 0$ then both terms on the right-hand side are non-negative, hence $g - h \ge 0$;

[1]Some basic topological properties of surfaces are discussed in Appendix D. A triangulation of a surface is a decomposition of the surface into a collection of 2-dimensional simplices that intersect only in 1-dimensional boundary simplices.

and $g = h$ if and only if $(h-1)(\delta-1) = \beta = 0$, so if and only if ϕ is an unbranched covering and either $\delta = 1$ or $h = 1$. If $g = h > 1$ then $\delta = 1$ and the mapping ϕ is a biholomorphic mapping. If $g = h = 1$ the mapping ϕ is an unbranched covering mapping between two Riemann surfaces of genus 1. To see that there are unbranched coverings of δ sheets between two Riemann surfaces of genus 1 for any $\delta \geq 1$, by Corollary 12.9 (iii) a compact Riemann surface N of genus $g = 1$ can be identified with a complex torus $N = \mathbb{C}/\Omega\mathbb{Z}^2$ for a 1×2 period matrix $\Omega = (\omega_1 \ \ \omega_2)$, and for any integer $\delta \geq 1$ the period matrix $\Omega' = (\delta\omega_1 \ \ \omega_2)$ describes another complex torus $M = \mathbb{C}/\Omega'\mathbb{Z}$ and the identity mapping $z \longrightarrow z$ induces a holomorphic mapping $M \longrightarrow N$ that is an unbranched covering of degree δ. That suffices for the proof.

9.2 Mappings Defined by Meromorphic Functions

The Riemann sphere $\mathbb{P}^1$ can be described explicitly as the union $\mathbb{P}^1 = V_0 \cup V_1$ of two coordinate neighborhoods $V_0 = \{\, w_0 \in \mathbb{C} \,\}$ and $V_1 = \{\, w_1 \in \mathbb{C} \,\}$, where the local coordinates are related by

$$(9.11) \quad w_1 = 1/w_0 \quad \text{in} \quad V_0 \cap V_1 = \{\, w_0 \in \mathbb{C} \,\big|\, w_0 \neq 0 \,\} = \{\, w_1 \in \mathbb{C} \,\big|\, w_1 \neq 0 \,\}.$$

Except for the origins in these two coordinate neighborhoods there is a one-to-one correspondence between the points $w_0 \in V_0$ and $w_1 \in V_1$. As a point $w_1 \in V_1$ approaches the origin the corresponding point $w_0 \in V_0$ increases in modulus without limit; so $\mathbb{P}^1$ can be viewed as the completion of the coordinate neighborhood $V_0 = \mathbb{C}$ of the variable w_0 by adjoining the "point at infinity" of the plane of the variable w_0, represented by the origin in the coordinate neighborhood of the variable w_1. This interpretation of the Riemann sphere is indicated by writing $\mathbb{P}^1 = \mathbb{C} \cup \infty$. If f is a meromorphic function on a Riemann surface M and $U_0 \subset M$ is the set of points $z \in M$ at which $f(z)$ is holomorphic while $U_1 \subset M$ is the set of points $z \in M$ at which $f(z) \neq 0$ then the function f determines a holomorphic mapping $\phi_f : M \longrightarrow \mathbb{P}^1$ by associating to any point $z \in U_0$ the point $\phi_f(z) \in V_0$ with the coordinate $w_0(z) = f(z)$ and to any point $z \in U_1$ the point $\phi_f(z) \in V_1$ with the coordinate $w_1(z) = 1/f(z)$; clearly these definitions are compatible in the intersection $U_0 \cap U_1$ and define a holomorphic mapping of M to $\mathbb{P}^1$. Conversely if $\phi : M \longrightarrow \mathbb{P}^1$ is a holomorphic mapping and $U_0 = \phi^{-1}(V_0)$ while $U_1 = \phi^{-1}(V_1)$ then the function f on M defined by setting $f(z) = w_0(z)$ if $z \in U_0$ and $f(z) = 1/w_1(z)$ if $z \in U_1$ clearly is a well defined meromorphic function on M for which $\phi = \phi_f$. This establishes a formal one to-one correspondence between meromorphic functions f on M and holomorphic mappings $\phi_f : M \longrightarrow \mathbb{P}^1$, which will be used systematically in the subsequent discussion.

Since the mapping ϕ_f associated to a meromorphic function is described locally by either the function $f(z)$ or the function $(1/f)(z)$ it follows that the ramification divisor of the mapping ϕ_f can be described by

$$\mathfrak{r}_a(\phi_f) = \begin{cases} \deg_a f'(z) & \text{if} \quad a \in U_0, \\ \\ \deg_a (1/f)'(z) & \text{if} \quad a \in U_1, \end{cases} \tag{9.12}$$

where these two definitions are easily seen to be compatible for points in $U_0 \cap U_1$. When the divisor of a meromorphic function f is decomposed as the difference $\mathfrak{d}(f) = \mathfrak{d}_+(f) - \mathfrak{d}_-(f)$ of two disjoint positive divisors, the degree of the function f is the common value $\deg f = \deg \mathfrak{d}_+ f = \deg \mathfrak{d}_- f$ as defined on page 5. The function $f - c$ for any complex number c has the same degree as the function f, so its zero divisor $\mathfrak{d}_+(f - c)$ is a divisor of degree $\deg f$ and consists of those points at which the function f takes the value c, where these points are counted with the appropriate multiplicities. So long as c is not one of the finitely many values taken by the function f at points of the ramification divisor $\mathfrak{r}(\phi_f)$ the divisor $\mathfrak{d}_+(f - c)$ consists of $\deg f$ distinct points; hence the mapping ϕ_f is a mapping of degree $\deg f$, so that

$$\deg \phi_f = \deg f. \tag{9.13}$$

Consequently by the Rieman-Hurwitz Theorem the branch order of the mapping $\phi_f : M \longrightarrow \mathbb{P}^1$ from a compact Riemann surface M of genus g to the Riemann sphere described by a meromorphic function f on the surface M is

$$\operatorname{br}(\phi_f) = 2 \deg f + 2g - 2. \tag{9.14}$$

For example, there are rational functions of any degree $\delta > 0$ on $\mathbb{P}^1$ so it follows that there are nontrivial holomorphic mappings $\phi_f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of any positive degree δ, for which $\operatorname{br}(\phi_f) = 2(\delta - 1)$ by (9.14). A holomorphic mapping $\phi_f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ of degree 1 is a biholomorphic mapping, also called an *automorphism*, of the Riemann sphere; and the set of automorphisms clearly form a group under the composition of mappings. An automorphism ϕ_f is described by a meromorphic function f of degree 1, a function with a single simple pole and a single simple zero on $\mathbb{P}^1$, so a function of the form $f(z) = (az + b)/(cz + d)$ for an invertible matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ when the Riemann sphere is viewed as the union $\mathbb{P}^1 = \mathbb{C} \cup \infty$ with the coordinate z in $\mathbb{C}$; such functions traditionally are called *linear fractional mappings*. It is perhaps worth noting incidentally that a linear fractional mapping $f(z)$ actually is just the cross-ratio function $q(z, \alpha; -b/a - d/c)$ for the Riemann sphere, as characterized in Theorem 5.6; in this case it is the uniquely determined meromorphic function on $\mathbb{P}^1$, which is its own universal covering space, having a simple zero at the point $z = -b/a$ and a simple pole at the point $z = -d/c$ and taking the value 1 at the point $\alpha = (d - b)/(a - c)$. At any rate, it is evident from this that there is a uniquely determined automorphism of $\mathbb{P}^1$ that takes any three distinct points to the three points $1, 0, \infty$ on $\mathbb{P}^1$, or equivalently that takes any three distinct points to any other three distinct points of $\mathbb{P}^1$. Clearly two matrices A and B determine the same automorphism if and only if $A = \epsilon B$ for some nonzero complex constant

ϵ; so the group of automorphisms of the Riemann sphere $\mathbb{P}^1$ can be identified with the projective linear group $\mathrm{Pl}(2,\mathbb{C}) = \mathrm{Gl}(2,\mathbb{C})/\mathbb{C}^*$, the quotient of the general linear group by the subgroup of diagonal matrices. A rational function $f(z) = (az+b)/(cz+d)$ on $\mathbb{P}^1$ can be written as the quotient $f(z) = f_0(z)/f_1(z)$ of the two affine functions $f_0(z) = az + b$ and $f_1(z) = cz + d$ of the variable z, which are unique up to a common constant factor; but this decomposition is really valid just on the coordinate neighborhood of the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$ defined by the variable z. In terms of the variable $z_1 = 1/z$ in the other coordinate neighborhood the function $f(z)$ has the corresponding representation $f(z) = f(1/z_1) = (a + bz_1)/(c + dz_1)$ on $\mathbb{P}^1$, and it too can be written as the quotient $f(1/z_1) = g_0(z_1)/g_1(z_1)$ of two affine functions $g_0(z_1) = bz_1 + a$ and $g_1(z_1) = dz_1 + c$. For these functions $f_0(z) = az + b = z(bz_1 + a) = zg_0(z_1)$ and similarly $f_1(z) = zg_1(z_1)$, so the pairs of functions $\big(f_0(z), g_0(z_1)\big)$ and the pair of functions $\big(f_1(z), g_1(z_1)\big)$ really describe two holomorphic cross-sections of the holomorphic line bundle over $\mathbb{P}^1$ defined by the cocycle $\lambda_{01} = z$ in terms of the local coordinate z in V_0 and z_1 in V_1. That is also the case for mappings of the Riemann sphere $\mathbb{P}^1$ to itself of degree $\delta > 1$ as well as for mappings from compact Riemann surfaces of genus $g > 1$ to the Riemann sphere.

Theorem 9.3 *A meromorphic function f of degree $\delta > 0$ on a compact Riemann surface M can be written as the quotient $f = f_{\alpha 1}/f_{\alpha 0}$ of two holomorphic cross-sections with no common zeros for the holomorphic line bundle $\zeta_{\mathfrak{d}_-(f)}$ of the polar divisor of the function f, where $c(\zeta_{\mathfrak{d}_-(f)}) = \delta$; and the cross-sections $f_{\alpha 0}$ and $f_{\alpha 1}$ are determined uniquely up to a common constant factor. The line bundle $\zeta_{\mathfrak{d}_-(f)}$ is base-point-free, and any base-point-free holomorphic line bundle on M of characteristic class $\delta > 0$ arises in this way.*

Proof: If f is a meromorphic function of degree $\delta > 0$ on the surface M and f has the divisor $\mathfrak{d}(f) = \mathfrak{d}_+(f) - \mathfrak{d}_-(f)$, where $\mathfrak{d}_+(f)$ and $\mathfrak{d}_-(f)$ are disjoint positive divisors, these two divisors are linearly equivalent so $\zeta_{\mathfrak{d}_+(f)} = \zeta_{\mathfrak{d}_-(f)}$ as in (1.10). The holomorphic line bundle $\zeta_{\mathfrak{d}_-(f)}$ has characteristic class $c(\zeta_{\mathfrak{d}_-(f)}) = \deg \mathfrak{d}_-(f) = \delta$ as in (1.15); and it has a holomorphic cross-section $f_{\alpha-}$ with the divisor $\mathfrak{d}(f_{\alpha-}) = \mathfrak{d}_-(f)$, where that cross-section is determined uniquely up to a constant factor. The product $f\,f_{\alpha-} = f_{\alpha+}$ is another holomorphic cross-section of the line bundle $\zeta_{\mathfrak{d}_-(f)}$ and has the divisor $\mathfrak{d}(f_{\alpha+}) = \mathfrak{d}_+(f)$; and the function f is the quotient $f = f_+/f_-$. Since the divisors $\mathfrak{d}_+(f)$ and $\mathfrak{d}_-(f)$ are disjoint it follows from Lemma 2.9 that the line bundle $\zeta_{\mathfrak{d}_-(f)}$ is base-point-free. Conversely if λ is a base-point-free holomorphic line bundle over M with $c(\lambda) = \delta$ then λ has two holomorphic cross-sections $f_{\alpha 0}, f_{\alpha 1}$ with no common zeros by Lemma 2.9; the quotient $f = f_{\alpha 0}/f_{\alpha 1}$ is a meromorphic function with polar divisor $\mathfrak{d}_-(f) = \mathfrak{d}(f_{\alpha_1})$, where $\deg f = \deg \mathfrak{d}(f_{\alpha 1}) = c(\lambda) = \delta$, and $f_{\alpha 0}$ is a holomorphic cross-section of the line bundle $\zeta_{\mathfrak{d}_-(f)}$. That suffices for the proof.

An alternative description of the Riemann sphere $\mathbb{P}^1$ is as the quotient of the space $\mathbb{C}^2 \sim (0,0)$, the complement of the origin in $\mathbb{C}^2$, under the equivalence relation for which $(z_0, z_1) \asymp (t\,z_0, t\,z_1)$ for any nonzero complex number $t \in \mathbb{C}$. The equivalence class of the point $(z_0, z_1) \in \mathbb{C}^2$ is denoted by $[z_0, z_1] \in \mathbb{P}^1$. If

$z_0 \cdot z_1 \neq 0$ then $(z_0, z_1) \asymp \left(1, \frac{z_1}{z_0}\right) \asymp \left(\frac{z_0}{z_1}, 1\right)$ so $[z_0, z_1]$ can be described uniquely by either $w_0 = \frac{z_0}{z_1}$ or $w_1 = \frac{z_1}{z_0}$ where $w_0 w_1 = 1$; that essentially identifies this description of the space $\mathbb{P}^1$ with the earlier description in this section. When a meromorphic function f on a Riemann surface M is written as the quotient $f = f_{\alpha 0}/f_{\alpha 1}$ of two holomorphic cross-sections of a holomorphic line bundle λ and these cross-sections have no common zeros then $[f_{\alpha 0}(z), f_{\alpha 1}(z)]$ is a well-defined point in the projective space $\mathbb{P}^1$ for each point $z \in U_\alpha$ for the coordinate neighborhood U_α. In an intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods $[f_{\alpha 0}(z), f_{\alpha 1}(z)] = [\lambda_{\alpha\beta}(z) f_{\beta 0}(z), \lambda_{\alpha\beta}(z) f_{\beta 1}(z)] = [f_{\beta 0}(z), f_{\beta 1}(z)]$; therefore the mapping that associates to any point $z \in U_\alpha$ the point $F_\lambda(z) = [f_{\alpha 0}(z), f_{\alpha 1}(z)]$ is a well defined holomorphic mapping $F_\lambda : M \longrightarrow \mathbb{P}^1$. Of course since $[f_{\alpha 0}, f_{\alpha 1}] = [f_{\alpha 0}/f_{\alpha 1}, 1]$ the mapping F_λ is really the same as the mapping ϕ_f defined by the meromorphic function $f = f_{\alpha 0}/f_{\alpha 1}$. This alternative description of the mapping is sometimes quite convenient. Introduce the holomorphic functions

$$D_\alpha(z) = \det \begin{pmatrix} f_{\alpha 0}(z) & f'_{\alpha 0}(z) \\ f_{\alpha 1}(z) & f'_{\alpha 1}(z) \end{pmatrix} \tag{9.15}$$

in the coordinate neighborhoods $U_\alpha \subset M$. Since $f_{\alpha\, i} = \lambda_{\alpha,\beta} f_{\beta\, i}$ in an intersection $U_\alpha \cap U_\beta$ it follows that

$$f'_{\alpha\, i} = \kappa_{\alpha\beta} \lambda_{\alpha,\beta} f'_{\beta\, i} + \kappa_{\alpha\beta} \lambda'_{\alpha,\beta} f_{\beta\, i}$$

in an intersection $U_\alpha \cap U_\beta$, hence that

$$D_\alpha(z) = \lambda^2_{\alpha\beta}(z) \kappa_{\alpha\beta}(z) D_\beta(z) \tag{9.16}$$

in an intersection $U_\alpha \cap U_\beta$, where $\lambda_{\alpha\beta}(z)$ are the coordinate transition functions for the line bundle λ and $\kappa_{\alpha\beta}(z)$ are the coordinate transition functions for the canonical bundle κ. The functions $\{D_\alpha\}$ thus form a holomorphic cross-section of the line bundle $\kappa\lambda^2$.

Corollary 9.4 *On a compact Riemann surface M the ramification divisor of the holomorphic mapping $F_\lambda : M \longrightarrow \mathbb{P}^1$ described by two holomorphic cross-sections $f_{\alpha 0}, f_{\alpha 1} \in \Gamma\big(M, \mathcal{O}(\lambda)\big)$ with no common zeros is the divisor $\mathfrak{r}(F_\lambda) = \mathfrak{d}(D_\alpha)$ of the holomorphic cross-section $D_\alpha \in \Gamma\big(M, \mathcal{O}(\kappa\lambda^2)\big)$.*

Proof: The mapping F_λ is just the mapping ϕ_f described by the meromorphic function $f = f_{\alpha 0}/f_{\alpha 1}$, so for any point $a \in M$ the ramification order of the mapping ϕ_f is given by (9.12); hence $\mathfrak{r}_a = \operatorname{ord}_a f'(z)$ if $f(z)$ is holomorphic at the point $a \in M$, or equivalently if $f_{\alpha 1}(a) \neq 0$, and $\mathfrak{r}_a = \operatorname{ord}_a (1/f)'(z)$ if $(1/f)(z)$ is holomorphic at the point $a \in M$, or equivalently if $f_{\alpha 0}(a) \neq 0$. Therefore if $f_{\alpha 1}(a) \neq 0$ then since

$$f'(z) = \left(\frac{f_{\alpha 0}}{f_{\alpha 1}}\right)' = \frac{f_{\alpha 1} f'_{\alpha 0} - f_{\alpha 0} f'_{\alpha 1}}{f^2_{\alpha 1}} = -f^{-2}_{\alpha 1} D_\alpha(z)$$

it follows that $\mathfrak{r}_a = \mathrm{ord}_a f'(z) = \mathrm{ord}_a D_\alpha(z)$; on the other hand if $f_{\alpha 0}(a) \ne 0$ then since

$$\big(1/f(z)\big)' = \left(\frac{f_{\alpha 1}}{f_{\alpha 0}}\right)' = \frac{f_{\alpha 0}f'_{\alpha 1} - f_{\alpha 1}f'_{\alpha 0}}{f^2_{\alpha 0}} = f^{-2}_{\alpha 0}D_\alpha(z)$$

it follows that $\mathfrak{r}_a = \mathrm{ord}_a f'(z) = \mathrm{ord}_a D_\alpha(z)$, in that case as well, which suffices for the proof.

Since $D_\alpha \in \Gamma\big(M, \mathcal{O}(\kappa\lambda^2)\big)$ it follows that

$$\deg \mathfrak{d}(D_\alpha) = c(\kappa\lambda^2) = 2c(\lambda) + 2g - 2, \tag{9.17}$$

and since $f_{\alpha 1} \in \Gamma\big(M, \mathcal{O}(\lambda)\big)$ it follows that

$$c(\lambda) = \deg \mathfrak{d}(f_{\alpha 1}) = \deg F_\lambda \tag{9.18}$$

where again F_λ is just the mapping ϕ_f described by the meromorphic function $f = f_{\alpha 0}/f_{\alpha 1}$; so altogether $\deg \mathfrak{d}(D_\alpha) = 2 \deg F_\lambda + 2g - 2$. It then follows from the preceding theorem that

$$\deg \mathfrak{r}(F_\lambda) = \deg \mathfrak{d}(D_\alpha) = 2 \deg F_\lambda + 2g - 2, \tag{9.19}$$

which provides an alternative proof of (9.14) in view of (9.8).

The degrees of holomorphic mappings $\phi : M \longrightarrow \mathbb{P}^1$ actually are integers that appear in the Lüroth semigroup $\mathcal{L}(M)$ of the compact Riemann surface M; the Lüroth semigroup was defined on page 39 as the additive semigroup of nonnegative integers consisting of the characteristic classes of base-point-free holomorphic line bundles over M. The identity bundle is a base-point free holomorphic line bundle of characteristic class 0, so the Lüroth semigroup begins with the integer 0; and the integers in the Lüroth semigroup can be labeled $0 = \delta_0(M) < \delta_1(M) < \delta_2(M) < \cdots$.

Theorem 9.5 *There exists a meromorphic function of degree $\delta > 0$ on a compact Riemann surface M, or equivalently there exists a nontrivial holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree $\delta > 0$, if and only if δ is an integer in the Lüroth semigroup $\mathcal{L}(M)$ of the surface M.*

Proof: Theorem 9.3 shows that the base-point-free holomorphic line bundles λ on M are precisely the line bundles of the polar divisors of meromorphic functions on M, where the degree δ of the meromorphic function is the characteristic class $\delta = c(\lambda)$ of the line bundle; and these are precisely the meromorphic functions describing holomorphic mappings $\phi_f : M \longrightarrow \mathbb{P}^1$ of degree δ, which suffices for the proof.

The least integer $\delta > 0$ such that there is a meromorphic function of degree δ on the compact Riemann surface M, or equivalently such that there is a nontrivial holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree δ, is known rather uneuphoniously as the *gonality* of the Riemann surface M; so by Corollary 9.5

the gonality of the surface M is the integer $\delta = \delta_1(M)$ in the Lüroth semigroup of M. On an elliptic curve, a compact Riemann surface M of genus $g = 1$, it follows from the Riemann-Roch Theorem that $\gamma(\zeta_a) = 1$ and $\gamma(\zeta_a^2) = 2$ for any point $a \in M$; so by Lemma 2.10 (i) the line bundle ζ_a^2 is base-point-free; and since no nontrivial line bundle λ with $c(\lambda) = 1$ is base-point-free it follows that $\delta_1(M) = 2$. In general a compact Riemann surface M of genus $g > 1$ is said to be *hyperelliptic* if $\delta_1(M) = 2$, so it it admits a nontrivial holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree 2. Hyperelliptic Riemann surfaces have a number of quite special properties, and were the first general class of surfaces beyond elliptic surfaces to be examined in considerable detail. A compact Riemann surface M of genus $g > 1$ is said to be *trigonal* if it admits a nontrivial holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree 3 but is not hyperelliptic, so if $\delta_1(M) = 3$; and M is said to be *quadrigonal* if it admits a nontrivial holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree 4 but is not trigonal or hyperelliptic, so if $\delta_1(M) = 4$. The term gonality is thus a rather natural extension of this classical terminology.Lüroth semigroup An alternative characterization is occasionally useful.

Theorem 9.6 *Let M be a compact Riemann surface of genus $g > 1$.*
(i) *The gonality $\delta = \delta_1(M)$ of M is the least integer δ such that there is a holomorphic line bundle λ over M with $c(\lambda) = \delta$ and $\gamma(\lambda) = 2$.*
(ii) *If λ is a holomorphic line bundle over M with $c(\lambda) = \delta_1(M)$ then $\gamma(\lambda) \le 2$, and $\gamma(\lambda) = 2$ if and only if λ is base-point-free.*

Proof: (i) Let λ be a holomorphic line bundle over M such that $\gamma(\lambda) = 2$ and $\gamma(\sigma) \le 1$ whenever $c(\sigma) < c(\lambda)$; then $\gamma(\lambda\zeta_a^{-1}) \le 1$ for any point $a \in M$, so it follows from Lemma 2.10 that λ is base-point-free, hence $c(\lambda) \ge \delta_1(M)$. On the other hand no line bundle σ over M for which $0 < c(\sigma) < c(\lambda)$ is base-point-free, since no holomorphic line bundle σ with $c(\sigma) > 0$ and $\gamma(\sigma) \le 1$ can be base-point-free on a compact Riemann surface of genus $g > 1$; hence $\delta_1(M) \ge c(\lambda)$ as well, so actually $\delta_1(M) = c(\lambda)$.
(ii) It follows from (i) that $\gamma(\sigma) \le 1$ whenever $c(\sigma) < \delta_1(M)$. Therefore if λ is a line bundle for which $c(\lambda) = \delta_1(M)$ then for any point $a \in M$ it follows that $c(\lambda\zeta_a^{-1}) < c(\lambda) = \delta_1(M)$ so $\gamma(\lambda\zeta_a^{-1}) \le 1$; and it then follows from Lemma 2.6 that $\gamma(\lambda) \le \gamma(\lambda\zeta_a^{-1}) + 1 = 2$. If $\gamma(\lambda) = 2$ then $\gamma(\lambda\zeta_a^{-1}) = 1 = \gamma(\lambda) - 1$ for any point $a \in M$ and it follows from Lemma 2.10 that λ is base-point-free; and of course if $\gamma(\lambda) = 1$ the bundle λ is not base-point-free. That suffices for the proof.

9.3 The Local Maximal Function

The degrees of special classes of holomorphic mappings $f : M \longrightarrow \mathbb{P}^1$ from a compact Riemann surface M to the Riemann sphere form various special subsets of the Lüroth semigroup of M. The simplest mappings are probably those defined by a meromorphic functions f on M having a single pole; such a mapping is fully ramified at the pole of f and fully branched over the point ∞.

A convenient tool in examining such mappings is the *local maximal function* of the surface M at the point $a \in M$, the mapping $\mu_a : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by

$$\mu_a(r) = \gamma(\zeta_a^r) - 1 \quad \text{for any} \quad r \in \mathbb{Z} \tag{9.20}$$

for the point bundle ζ_a.

Theorem 9.7 *The local maximal function $\mu_a(r)$ of a compact Riemann surface M of genus $g > 1$ at a point $a \in M$ satisfies the following conditions:*

$$\mu_a(r) \le \mu_a(r+1) \le \mu_a(r) + 1 \quad \textit{for any } r \in \mathbb{Z}. \tag{9.21}$$

$$\mu_a(r) = \begin{cases} -1 & \textit{for} \quad r \le -1, \\ 0 & \textit{for} \quad r = 0, 1 \\ r - g & \textit{for} \quad r \ge 2g - 1. \end{cases} \tag{9.22}$$

Proof: Lemma 2.6 asserts that $\gamma(\lambda) \le \gamma(\lambda\zeta_a) \le \gamma(\lambda) + 1$ for any holomorphic line bundle λ and point bundle ζ_a on M; in particular for $\lambda = \zeta_a^r$ that yields (9.21). Since $\gamma(\zeta_a^r) = 0$ for $r < 0$ it follows that $\mu_a(r) = -1$ for $r < 0$; and since $\gamma(\zeta_a^0) = \gamma(1) = 1$ for any compact Riemann surface and $\gamma(\zeta_a^1) = 1$ for any compact Riemann surface of genus $g > 1$ by Theorem 2.7 it follows that $\mu_a(0) = \mu_a(1) = 0$. By the Riemann-Roch Theorem $\gamma(\zeta_a^r) = \gamma(\kappa\zeta_a^{-r}) + c(\zeta_a^r) + 1 - g$; here $c(\zeta_a^r) = r$ hence $c(\kappa\zeta_a^{-r}) = 2g-2-r < 0$ for $r > 2g-2$ so $\gamma(\zeta_a^r) = r+1-g$ for $r > 2g-2$ and $\mu_a(r) = r - g$ for $r > 2g-2$, which suffices for the proof.

For a surface of genus $g = 1$ the local maximal function is fully determined by the preceding theorem and is actually independent of the choice of the point $a \in M$, since Theorem 9.7 shows that

$$\text{if } g = 1 \text{ then } \mu_a(r) = \begin{cases} -1 & \text{if} \quad r \le -1, \\ 0 & \text{if} \quad r = 0, \\ r - 1 & \text{if} \quad r \ge 1; \end{cases} \tag{9.23}$$

thus in this case $\mu_a(r)$ is actually even independent of the particular Riemann surface of genus $g = 1$ being considered. For $g = 2$ the preceding Theorem 9.7 only shows that

$$\text{if } g = 2 \text{ then } \mu_a(r) = \begin{cases} -1 & \text{if} \quad r \le -1, \\ 0 & \text{if} \quad r = 0, 1, \\ 0 \text{ or } 1 & \text{if} \quad r = 2, \\ r - 2 & \text{if} \quad r \ge 3. \end{cases} \tag{9.24}$$

By the Riemann-Roch Theorem if $g = 2$ then $\gamma(\zeta_a^2) = \gamma(\kappa\zeta_a^{-2}) + 1$, and since $c(\kappa\zeta_a^{-2}) = 0$ then $\gamma(\kappa\zeta_a^{-2}) = 1$ if $\kappa\zeta_a^{-2}$ is the trivial line bundle while otherwise $\gamma(\kappa\zeta_a^{-2}) = 0$, so

$$\text{if } g = 2 \text{ then } \mu_a(2) = \begin{cases} 1 & \text{if} \quad \kappa = \zeta_a^2, \\ 0 & \text{if} \quad \kappa \ne \zeta_a^2; \end{cases} \tag{9.25}$$

thus generally $\mu_a(2) = 0$, but if M is a Riemann surface for which the canonical bundle has the form $\kappa = \zeta_a^2$ for some point $a \in M$ then $\mu_a(2) = 1$ for that particular point a. This is a model for the values of the local maximal function $\mu_a(r)$ for Riemann surfaces of genus $g > 2$ in the range $2 \le r \le 2g-2$ in which the local critical values are not fully determined by Theorem 9.7. A discussion of the essential properties of mappings satisfying the basic property (9.21) of the local lmaximal sequience will be taken up in Chapter 11 in connection with the Brill-Noether diagram. For the simple case of the local maximal function the essential results can be derived directly quite easily though, as will be done here. The basic tool in the more explicit description of the local maximal function is the collection of indices

$$r_i(a) = \inf \Big\{ r \in \mathbb{Z} \Big| \mu_a(r) \ge i \Big\}, \tag{9.26}$$

called the *local critical values* of the Riemann surface M at the point $a \in M$.

Theorem 9.8 *The local critical values at a point a of a compact Riemann surface M of genus $g > 0$ take the following values:*

$$r_i(a) = \begin{cases} -\infty & \textit{for} \quad i \le -1, \\ 0 & \textit{for} \quad i = 0, \\ g+i & \textit{for} \quad i \ge g, \end{cases} \tag{9.27}$$

so all indices $r \ge 2g$ are local critical values; and

$$r_1(a) \ge 2. \tag{9.28}$$

For any index $i \ge 0$

$$r_i(a) < r_{i+1}(a) \quad \textit{and} \quad \mu_a(r) = i \quad \textit{for} \quad r_i(a) \le r < r_{i+1}(a). \tag{9.29}$$

The local critical values at a point $a \in M$ are characterized by the condition that

$$\mu_a(r) - \mu_a(r-1) = \begin{cases} 1 & \textit{if } r \textit{ is a local critical value at } a, \\ \\ 0 & \textit{otherwise.} \end{cases} \tag{9.30}$$

Proof: Since $\mu_a(r) = -1$ for $r < 0$ by (9.22) it follows that $r_i(a) = -\infty$ whenever $i < 0$; and since $\mu_a(-1) = -1$ and $\mu_a(0) = 0$ by (9.22) it further follows that $r_0(a) = 0$. The equation $\mu_a(r) = r - g$ for all $r \ge 2g-1$ in (9.22) can be rewritten by setting $i = r - g$ as the equation $\mu_a(g+i) = i$ for all $i \ge g-1$; hence $\mu_a(2g-1) = g-1$, $\mu_a(2g) = g$, $\mu_a(2g+1) = g+1, \cdots$, which shows that $r_g(a) = 2g$ and more generally that $r_i(a) = g+i$ for all $i \ge g$, thus demonstrating (9.27). Since $r_0(a) = 0$ then $r_1(a) \ge 1$; but if $r_1(a) = 1$ then $\gamma(\zeta_a^1) = \mu_a(1) + 1 = 2$ and by Theorem 2.4 the Riemann surface M would be the Riemann sphere $\mathbb{P}^1$ of genus $g = 0$, which is excluded. From the definition of the local critical values it is evident that $\mu_a\big(r_i(a)\big) \ge i$ and $\mu_a\big(r_i(a) - 1\big) < i$

for any finite values of $r_i(a)$, so for $i \ge 0$; and from (9.24) it is evident that $\mu_a\big(r_i(a)\big) \le \mu_a\big(r_i(a)-1\big)+1$, hence

$$i \le \mu_a\big(r_i(a)\big) \le \mu_a\big(r_i(a)-1\big)+1 < i+1 \le \mu_a\big(r_{i+1}(a)\big),$$

which shows that $\mu_a\big(r_i(a)\big) = i$ and $\mu_a\big(r_i(a)\big) < \mu_a\big(r_{i+1}(a)\big)$ hence that $r_i(a) < r_{i+1}(a)$. In view of this it is apparent that $\mu_a(r) = i$ for $r_i(a) \le r < r_{i+1}(a)$, thus demonstrating (9.29). Finally if $\mu_a(r) = \mu_a(r-1)$ it is clear from the definition of the local critical values that r cannot be a local critical value; on the other hand it follows from (9.29) that $\mu_a\big(r_i(a)\big) - \mu_a\big(r_i(a)-1\big) = 1$ for any local critical value $r_i(a)$, and that suffices for the proof.

The preceding Theorem 9.8 shows that the finite local critical values at a point $a \in M$ are nonnegative integers, beginning with

$$0 = r_0(a) < 1 < r_1(a) < \cdots < r_{g-1}(a) < r_g(a) = 2g \tag{9.31}$$

and continuing with all integers $r > 2g$; but the theorem does not determine the $g-1$ local critical values $r_1(a), \ldots, r_{g-1}(a)$ in the range $[2, 2g-1]$ explicitly. The integers that are not local critical values at the point $a \in M$ are called the *local gap values* at the point $a \in M$. The local gap values thus are all integers $\nu < 0$ together with the g integers ν in the range $[1, 2g-1]$ that are not local critical values; the positive local gap values also are called the *Weierstrass gaps* at the point a of the Riemann surface M. Since $\mu_a(r) = 0$ for $r = 0$ and $r = 1$ it follows that $\nu_1(a) = 1$ is a gap value; thus the Weierstrass gaps are g integers in the range

$$1 = \nu_1(a) < \nu_2(a) < \ldots < \nu_g(a) \le 2g-1, \tag{9.32}$$

and $g-1$ of these are not specified explicitly by the preceding theorem. It follows immediately from these observations that the characterization (9.30) of the local critical values can be extended to assert that

$$\mu_a(r) - \mu_a(r-1) = \begin{cases} 1 & \text{if } r \text{ is a local critical value at } a \\ \\ 0 & \text{if } r \text{ is a local gap value at } a. \end{cases} \tag{9.33}$$

The local maximal function at the point $a \in M$ thus can be described either in terms of the local gap values or in terms of the local critical values at the point $a \in M$.

Theorem 9.9 *The holomorphic line bundle ζ_a^r on a compact Riemann surface M is base-point-free if and only if r is a local critical value at the point a; therefore the line bundle ζ_a^r is not base-point-free if and only if r is a local gap value at the point a.*

Proof: The line bundle ζ_a^r is base-point-free if and only if $\gamma(\zeta_a^r\zeta_x^{-1}) = \gamma(\zeta_a^r) - 1$ for all $x \in M$, by Lemma 2.10; hence ζ_a^r fails to be base-point-free if and only if

$\gamma(\zeta_a^r\zeta_x^{-1}) = \gamma(\zeta_a^r)$ for some point $x \in M$, and by Lemma 2.6 that is equivalent to the condition that all holomorphic cross-sections of the bundle ζ_a^r vanish at the point x. There is always at least one holomorphic cross-section of ζ_z^r that vanishes only at the point a, the r-th power of a holomorphic cross-section of the bundle ζ_a; hence the only point x at which all the holomorphic cross-sections of the bundle ζ_a^r can possibly vanish is the point $x = a$, and the condition that all the holomorphic cross-sections of the line bundle ζ_a^r vanish at the point a is that $\gamma(\zeta_a^r) = \gamma(\zeta_a^{r-1})$, or equivalently that $\mu_a(r) = \mu_a(r-1)$. Therefore the line bundle ζ_a^r is not base-point-free if and only if $\mu_a(r) = \mu_a(r-1)$, which by (9.33) is just the condition that r is a local gap value at the point a. That suffices for the proof.

Corollary 9.10 *The local critical values of a compact Riemann surface M form an additive semigroup of the integers $\mathbb{Z}$ that is contained in the Lüroth semigroup $\mathcal{L}(M) \subset \mathbb{Z}$ of the surface M.*

Proof: This follows immediately from the preceding theorem and the definition of the Lüroth semigroup, so no further proof is required.

The semigroup of local critical values at a point a of a compact Riemann surface M is called the *Weierstrass semigroup*[2] at the point a and is denoted by $\mathcal{W}_a(M)$. The Weierstrass semigroups at the points $a \in M$ thus are subsemigroups $\mathcal{W}_a(M) \subset \mathcal{L}(M)$ of the Lüroth semigroup of the surface M associated to all the points $a \in M$; and the integers $\nu \geq 1$ that are not contained in the semigroup $\mathcal{W}_a(M)$ are precisely the g Weierstrass gap values $\nu_i(a)$ at the point a for $1 \leq i \leq g$, integers in the range $1 \leq \nu_i(a) \leq 2g-1$. Altogether this is a fascinating structure associated to the Riemann surface M, an intricate refinement of the Lüroth semigroup of the surface.

9.4 Weierstrass Points

Another approach to the local critical values at points of a Riemann surface M is through the holomorphic abelian differentials and their integrals on M. For each point $a \in M$ of a compact Riemann surface M of genus $g > 1$ it is possible to choose a basis w_i for the holomorphic abelian integrals so that

$$\text{(9.34)} \qquad \mathrm{ord}_a w_{i,a}(z) = \rho_i(a) \quad \text{where} \quad 1 = \rho_1(a) < \rho_2(a) < \cdots < \rho_g(a).$$

Indeed since not all the holomorphic abelian differentials vanish at any point there is an abelian integral $w_{1,a}(z)$ of order 1 at the point a. After subtracting suitable multiples of $w_{1,a}(z)$ from the remaining integrals it can be assumed that

[2] Hurwitz pointed out the problem of describing precisely which semigroups of nonnegative integers can be such a semigroup $\mathcal{W}_a(M)$ for some point a on some compact Riemann surface M. That not all semigroups of nonnegative integers can be such semigroups was first established by R.-O. Buchweitz in his Hanover PhD thesis, 1976; the problem has been studied extensively but is still not fully solved.

they all vanish to at least order 2 at a; so let $w_{2,a}(z)$ be one of the integrals of the least order $\rho_2(a)$ among them. After subtracting suitable multiples of $w_{2,a}(z)$ from the remaining integrals it can be assumed that they all vanish to at least order $\rho_2(a)+1$; so let $w_{3,a}(z)$ can one of the integrals of least order $\rho_3(a)$ among them, and so on. The integers $\rho_i(a)$ thus are well defined analytic invariants intrinsically attached to the point $a \in M$; they are called the *local orders* of the holomorphic abelian integrals at the point $a \in M$. It follows immediately from (9.34) that for the holomorphic abelian differentials $\omega_{i,a}(z) = dw_{i,a}(z)$ correspondingly

$$\text{ord}_a \omega_{i,a}(z) = \rho_i(a) - 1 \quad \text{where} \quad 1 = \rho_1(a) < \rho_2(a) < \cdots < \rho_g(a). \tag{9.35}$$

To analyze this further it is convenient to set

$$N_a(r) = \text{the number of integers } \rho_i(a) \text{ such that } \rho_i(a) \le r. \tag{9.36}$$

It is evident from this definition that

$$N_a(r) - N_a(r-1) = \begin{cases} 1 & \text{if } \ r = \rho_i(a) \ \text{ for some } i \\ \\ 0 & \text{otherwise.} \end{cases} \tag{9.37}$$

It is also the case that

$$\mu_a(r) = r - N_a(r). \tag{9.38}$$

Indeed to verify (9.38) note that $N_a(r)$ can be described equivalently as the number of the integers $\rho_i(a)$ in the preceding list such that $\rho_i(a)-1 < r$; so since there are g of the integers $\rho_i(a)$ altogether it follows that $g - N_a(r)$ is number of the integers $\rho_i(a)$ for which $\rho_i(a) - 1 \ge r$, or equivalently is the number of the holomorphic abelian differentials $\omega_{i,a}(z)$ for which $\text{ord}_a\omega_{i,a}(z) \ge r$. The differentials $\omega_{i,a}$ are a basis for the vector space $B = \{b_0, b_1, \ldots, b_n\} \subset \mathbb{P}^1$of holomorphic abelian differentials on M so $g - N_r(a)$ actually is the dimension of the space of those holomorphic abelian differentials $\omega(z)$ on M for which $\text{ord}_a\omega_{i,a}(z) \ge r$; and since that dimension is $\gamma(\kappa\zeta_a^{-r})$ it follows that $g - N_a(r) = \gamma(\kappa\zeta^{-r})$. By the Riemann-Roch Theorem $\mu_a(r) = \gamma(\zeta_a^r) - 1 = \gamma(\kappa\zeta_a^r) + r - g = r - N_a(r)$, and that suffices for the proof.

Theorem 9.11 *The Weierstrass gaps $\nu_i(a)$ are precisely the local orders $\rho_i(a)$ of the holomorphic abelian differentials at a point a of a compact Riemann surface M of genus $g > 1$, that is,*

$$\rho_i(a) = \nu_i(a) \quad \textit{for} \quad 1 \le i \le g. \tag{9.39}$$

Proof: From (9.38) and and (9.37) it follows that

$$\begin{aligned} \mu_a(r) - \mu_a(r-1) &= \big(r - N_a(r)\big) - \big(r - 1 - N_a(r-1)\big) \\ &= 1 - \Big(N_a(r) - N_a(r-1)\Big) \\ &= \begin{cases} 0 & \text{if} \quad r = \rho_i(a) \quad \text{for some } i \\ \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand from (9.30) it follows that r is a local gap value $r = \nu_j(a)$ if and only if $\mu_a(r) - \mu_a(r-1) = 0$, hence the preceding equation shows that $\rho_i(a)$ are precisely the finite local gap values at the point a, the Weierstrass gaps, and that suffices for the proof.

If the abelian differentials $\omega_{i,a}(z)$ are written in terms of local coordinates z_α in a coordinate covering $\{U_\alpha\}$ of M as $\omega_{i,a}(z) = f_{i,a}(z_\alpha)dz_\alpha$ for some functions $f_{i,a}(z_\alpha)$ the $g \times g$ matrix

$$\Omega_a(z_\alpha) = \begin{pmatrix} f_{1,a}(z_\alpha) & f'_{1,a}(z_\alpha) & \cdots & f^{(g-1)}_{i,a}(z_\alpha) \\ f_{2,a}(z_\alpha) & f'_{2,a}(z_\alpha) & \cdots & f^{(g-1)}_{2,a})(z_\alpha) \\ \cdots & \cdots & \cdots & \cdots \\ f_{g,a}(z_\alpha) & f'_{g,a}(z_\alpha) & \cdots & f^{(g-1)}_{g,a}(z_\alpha) \end{pmatrix}, \tag{9.40}$$

where $f^{(n)}_{i,a}(z_\alpha)$ denotes the n-th derivative with respect to the variable z_α, is called the **local Brill-Noether matrix** of the surface M. Of course it can be defined in terms of any basis for the space of holomorphic abelian differentials on M; but it is particularly interesting for the basis (9.34), for which the Taylor expansions in a local coordinate z_α centered at the point a begin $f_{i,\alpha}(z) = z_\alpha^{\rho_i(a)-1} + \cdots$. The Taylor expansion of the function $\det \Omega_a(z_\alpha)$ at the point $a \in M$ can be calculated by replacing the functions $f_{i,\alpha}(z)$ by their Taylor expansions; so the initial term of the Taylor expansion of the function $\det \Omega_a(z_\alpha)$ is given aside from a nonzero constant factor by

$$\det \begin{pmatrix} z_\alpha^{\rho_1(a)-1} & z_\alpha^{\rho_1(a)-2} & z_\alpha^{\rho_1(a)-3} & \cdots & z_\alpha^{\rho_1(a)-g} \\ z_\alpha^{\rho_2(a)-1} & z_\alpha^{\rho_2(a)-2} & z_\alpha^{\rho_2(a)-3} & \cdots & z^{\rho_2(a)-g} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_\alpha^{\rho_g(a)-1} & z_\alpha^{\rho_g(a)-2} & z_\alpha^{\rho_g(a)-3} & \cdots & z_\alpha^{\rho_g(a)-g} \end{pmatrix}, \tag{9.41}$$

where any terms involving a negative power of the variable z_α should be replaced by 0. The matrix in (9.41) is essentially the Wronskian of the polynomials $z_\alpha^{\rho_i(a)-1}$; and since these polynomials are linearly independent the determinant (9.41) does not vanish identically[3]. The determinant (9.41) explicitly is the sum of products of one element from each separate row and column; so since the entry in row i and column j is $z_\alpha^{\rho_i(a)-j}$ it follows that the determinant is a nonzero constant multiple of the variable z_α to the power $\sum_{i=1}^g \rho_i(a) - \sum_{j=1}^g j = \sum_{i=1}^g (\rho_i(a) - i)$. Therefore if

$$\omega(a) = \operatorname{ord}_a \det \Omega_\alpha(z_\alpha) \tag{9.42}$$

then

$$\omega(a) = \sum_{i=1}^g (\rho_i(a) - i) = \sum_{i=1}^g (\nu_i(a) - i), \tag{9.43}$$

[3]For the relevant property of the Wronskian matrix see for instance the paper by Alin Bostan and Philippe Dumas "Wronsksians and Linear Independence " in *The American Mathematical Monthly*, vol. 117 (2010), pp. 722-272.

since $\nu_i(a) = \rho_i(a)$ as already observed. The integer $\omega(a) \geq 0$ is called the *Weierstrass weight* of the point $a \in M$, and a point $a \in M$ is called a *Weierstrass point* of M if $\omega(a) > 0$. The Weierstrass points form a particularly interesting intrinsically defined set of points on any compact Riemann surface. It is convenient to note explicitly the characterizations of Weierstrass points in terms of the indices $r_i(a), \nu_i(a), \rho_i(a)$.

Theorem 9.12 (i) *A point $a \in M$ on a compact Riemann surface M of genus $g > 1$ is not a Weierstrass point if and only if $\nu_i(a) = \rho_i(a) = i$ for $1 \leq i \leq g$, hence is a Weierstrass point if and only if $\nu_i(a) = \rho_i(a) > i$ for some index $1 \leq i \leq g$,*
(ii) *A point $a \in M$ on a compact Riemann surface M of genus $g > 1$ is not a Weierstrass point if and only if $r_1(a) = g + 1$, hence is a Weierstrass point if and only if $r_1(a) \leq g$.*

Proof: (i) Since the Weierstrass gaps are in the range (9.32) it follows that $\nu_i(a) \geq i$ for $1 \leq i \leq g$; and since the Weierstrass weight is defined by (9.43) clearly $\omega(a) = 0$ if and only if $\nu_i(a) = i$ for $1 \leq i \leq g$, which is therefore the condition that a not be a Weierstrass point. The point a then is a Weierstrass point precisely when it is not the case that $\nu_i(a) = i$ for all indices i, hence if and only if $\nu_i(a) > i$ for at least one of the indices i. The same conditions hold for the indices $\rho_i(a)$ since $\rho_i(a) = \nu_i(a)$ by (9.39).
(ii) Since the local critical values are the complement of the local gap values it follows directly from (i) that a is a Weierstrass point if and only if $r_1(a)$ occurs somewhere among the indices $1, 2, \ldots, g$, hence if and only if $r_1(a) \leq g$; and since the local critical values are in the range (9.31) it follows that a is not a Weierstrass point if and only if $r_1(a) = g + 1$, and that suffices for the proof.

In the intersection $U_\alpha \cap U_\beta$ of two coordinate neighborhoods on M, with local coordinates z_{alpha} and z_{beta}, the coefficients $f_{i,a}(z_\alpha)$ and $f_{i,a}(z_\beta)$ of the holomorphic abelian differentials satisfy $f_{i,a}(z_\alpha) = \kappa_{\alpha\beta} f_{i,a}(z_\beta)$ where $\{\kappa_{\alpha\beta}\}$ is the holomorphic coordinate bundle describing the canonical bundle κ in terms of these coordinates; and since $d/dz_\alpha = \kappa_{\alpha\beta}\, d/dz_\beta$ it follows from the chain rule for differentiation that the coefficient functions $f_{i,\alpha}(p)$ and $f_{i,\beta}(p)$ and their derivatives at the point p with respect to the local coordinates z_α and z_β respectively are related by

$$
\begin{aligned}
f_{i,a}(z_\alpha) &= \kappa_{\alpha\beta} f_{i,a}(z_\beta) \\
f'_{i,a}(z_\alpha) &= \kappa^2_{\alpha\beta} f'_{i,a}(z_\beta) + \kappa_{\alpha\beta}\kappa'_{\alpha\beta} f_{i,a}(z_\beta) \\
f''_{i,a}(z_\alpha) &= \kappa^3_{\alpha\beta} f''_{i,a}(z_\beta) + 3\kappa^2_{\alpha\beta}\kappa'_{\alpha\beta} f'_{i,a}(z_\beta) \\
&\quad + \left(\kappa_{\alpha\beta}{\kappa'_{\alpha\beta}}^2 + \kappa^2_{\alpha\beta}\kappa''_{\alpha\beta}\right) f_{i,a}(z_\beta) \\
&\quad \text{and so on,}
\end{aligned} \tag{9.44}
$$

where $\kappa^{(n)}_{\alpha\beta}$ denotes the n-th derivative of the function $\kappa_{\alpha\beta}$ with respect to the variable z_β and correspondingly for the higher derivatives. It is a straightforward matter to verify that

$$\Omega_a(z_\alpha) = \Omega_a(z_\beta)\cdot K_{\alpha\beta} \tag{9.45}$$

for the nonsingular $g\times g$ matrix

$$K_{\alpha\beta} = \begin{pmatrix} \kappa_{\alpha\beta} & \kappa_{\alpha\beta}\kappa'_{\alpha\beta} & \kappa_{\alpha\beta}(\kappa'_{\alpha\beta})^2 + \kappa^2_{\alpha\beta}\kappa''_{\alpha\beta} & \cdots \\ 0 & \kappa^2_{\alpha\beta} & 3\kappa^2_{\alpha\beta}\kappa'_{\alpha\beta} & \cdots \\ 0 & 0 & \kappa^3_{\alpha\beta} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

for which

$$\det K_{\alpha\beta} = \kappa_{\alpha\beta}^{\frac{1}{2}g(g+1)}. \tag{9.46}$$

Consequently

$$\det\Omega_a(\alpha) = \kappa_{\alpha\beta}^{g(g+1)/2}(z)\det\Omega_a(z_\beta) \quad \text{in } U_\alpha\cap U_\beta, \tag{9.47}$$

so the determinants of the local Brill-Noether matrices describe a holomorphic cross-section

$$\det\Omega_a(z) \in \Gamma\Big(M, \mathcal{O}(\kappa^{g(g+1)/2})\Big). \tag{9.48}$$

As noted before the local Brill-Nother matrix is essentially the Wronskian matrix of the local functions $f_{i,a}(z_\alpha)$, so since these functions are linearly independent it is a nontrivial holomorphic cross-section. That is the key to a basic property of the Weierstrass points of M.

Theorem 9.13 *There are only finitely many Weierstrass points on a compact Riemann surface M of genus $g>1$, and the Weierstrss weights at these points satisfy*

$$\sum_{a\in M}\omega(a) = (g-1)g(g+1). \tag{9.49}$$

Proof: The functions $\det\Omega_a(z_\alpha)$ describe a holomorphic cross-section (9.48) of the line bundle $\kappa^{g(g+1)/2}$ so the degree of the divisor of this cross-section is equal to the characteristic class $c(\kappa^{g(g+1)/2}) = (2g-2)\frac{g(g+1)}{2} = (g-1)g(g+1)$ of the line bundle $\kappa^{g(g+1)/2}$. By (9.42) the degree of the divisor of the function $\det\Omega_a(z_\alpha)$ at a point $a\in M$ is just the Weierstrass weight $\omega(a)$ of the point a, so the sum of the weights $\omega(a)$ at all points $a\in M$ is equal to $c(\kappa^{g(g+1)/2})$, and that yields equation (9.49). It follows from this equation that there are only finitely many points $a\in M$ at which $\omega(a)>0$, that is, there are only finitely many Weierstrass points, and that suffices for the proof.

Theorem 9.14 *(*(i) *The Weierstrass weight $\omega(a)$ of a Weierstrass point $a \in M$ on a compact Riemann surface M of genus $g > 1$ satisfies the inequalities*

$$1 \le \omega(a) \le \frac{1}{2}g(g-1). \tag{9.50}$$

(ii) *The minimal value $\omega(a) = 1$ is attained when the Weierstrass gap sequence at the point a is*

$$(1,\ 2,\ \dots,\ g-1,\ g+1). \tag{9.51}$$

(iii) *The maximal value $\omega(a) = \frac{1}{2}g(g-1)$ is attained when the Weierstrass gap sequence at the point a is*

$$(1,\ 3,\ 5,\ \dots,\ 2g-3,\ 2g-1). \tag{9.52}$$

(iv) *The total number N of Weierstrass points on M satisfies the inequalities*

$$2(g+1) \le N \le (g-1)g(g+1). \tag{9.53}$$

The minimum value of N is taken on a Riemann surface for which all the Weierstrass points have the maximum Weierstrass weight while the maximum value of N is taken on a Riemann surface for which all the Weierstrass points have the minimum Weierstrass weight.

Proof: (i) and (ii) By (9.43) the Weierstrass weight at a point $a \in M$ is given by $\omega(a) = \sum_{i=1}^{g}(\nu_i(a) - i)$. If $\nu_j(a) \ge j+1$ for some index $1 \le j \le g$ then $\nu_i(a) \ge i+1$ for all $i \ge j$ so that $\omega(a) \ge g - j$; the least Weierstrass weight at a Weierstrass point thus is $\omega(a) = 1$, which is the first inequality in (9.50, and that occurs when $\nu_j = j$ for $1 \le j \le g$ and $\nu_g = g+1$, which shows that the associated Weierstrass gap sequence is (9.51). (1) and (iii) If $r = r_1(a) \ge 2$ is the least local critical value at a Weierstrass point $a \in M$ then $2r, 3r, \dots$ are also local critical values at $a \in M$, since the local critical values form a semigroup in $\mathbb{Z}$ by Corollary 9.10. Each integer i in the range $1 \le i \le r-1$ then is a Weierstrass gap at the point $a \in M$; and for each of these integers there is a further integer $\lambda_i \ge 0$ such that

$$i,\ (i+r),\ \ (i+2r),\ \ \dots,\ \ (i+\lambda_i r) \quad \text{are Weierstrass gaps at } a \tag{9.54}$$

while

$$i + (\lambda_i + 1)r,\ \ i + (\lambda_i + 2)r,\ \ \dots \quad \text{are local critical values at } a. \tag{9.55}$$

Any integer is congruent to one of the integers $1 \le i \le r-1$ modulo r so all the Weierstrass gaps are included in the lists (9.54); hence the total number of Weierstrass gaps at a is

$$g = \sum_{i=1}^{r-1}(\lambda_i + 1) = r - 1 + \sum_{i=1}^{r-1}\lambda_i. \tag{9.56}$$

By (9.43) the Weierstrass weight of the point a is

$$
\begin{aligned}
\omega(a) &= \sum_{i=1}^{g}(\nu_i(a) - i) = \sum_{i=1}^{g}\nu_i(a) - \frac{1}{2}g(g+1) \\
&= \sum_{i=1}^{r-1}\sum_{j=0}^{\lambda_i}(i + jr) - \frac{1}{2}g(g+1) \\
&= \sum_{i=1}^{r-1}\Big(i(\lambda_i+1) + \frac{1}{2}r\lambda_i(\lambda_i+1)\Big) - \frac{1}{2}g(g+1) \\
&= \frac{1}{2}r(r-1) + \sum_{i=1}^{r-1}\frac{1}{2}\lambda_i(2i + r\lambda_i + r) - \frac{1}{2}g(g+1).
\end{aligned} \tag{9.57}
$$

Here $i \le r-1$, and since the largest Weierstrass gap is $\nu_g(a) \le 2g-1$ by (9.32) then $i + \lambda_i r \le 2g-1$; consequently

$$
\begin{aligned}
2i + r\lambda_i + r &= i + (i + r\lambda_i) + r \\
&\le (r-1) + (2g-1) + r = 2(g+r-1)
\end{aligned}
$$

so recalling (9.56)

$$
\sum_{i=1}^{r-1}\frac{1}{2}\lambda_i(2i+r\lambda_i+r) \le (g+r-1)\sum_{i=1}^{r-1}\lambda_i = (g+r-1)(g-r+1) = g^2-(r-1)^2.
$$

Substituting this inequality in (9.57) shows that

$$
\begin{aligned}
\omega(a) &\le \frac{1}{2}r(r-1) + g^2 - (r-1)^2 - \frac{1}{2}g(g+1) \\
&\le \frac{1}{2}g(g-1) - \frac{1}{2}(r-1)(r-2).
\end{aligned} \tag{9.58}
$$

It is evident from this inequality that the largest Weierstrass weight at a Weierstrass point is $\frac{1}{2}g(g-1)$, which is the second inequality in (9.50); and since $r \ge 2$ that occurs when $r = 2$, so that all even numbers are local critical values at the point a and hence the Weierstrass gap sequence consists just of the odd integers (9.53). (iv) If all of the Weierstrass points on M have the minimal Weierstrass weight $\omega(a) = 1$ it is evident from (9.49) in Theorem 9.13 that the number of Weierstrass points on M is $(g-1)g(g+1)$, and that of course is the largest possible number of Weierstrass points on M. On the other hand if all the Weierstrass points on M have the maximal Weierstrass weight $\frac{1}{2}g(g-1)$ it is evident from (9.49) that the number of Weierstrass points on M is $2g+2$, and that is the minimal number of Weierstrass points on M. That demonstrates (9.53) and concludes the proof.

A slight refinement of the preceding theorem is occasionally useful.

Corollary 9.15 *On a compact Riemann surface M of genus $g > 1$ the Weierstrass weight at a point $a \in M$ at which the first local critical value is $r_1(a)$ satisfies*

$$\omega(a) \le \frac{1}{2}g(g-1) - \frac{1}{2}\big(r_1(a)-1\big)\big(r_1(a)-2\big). \tag{9.59}$$

Proof: The inequality (9.59) is just the inequality (9.58) in the proof of the preceding theorem, so no further proof is required here.

The collection of Weierstrass points on a compact Riemann surface M of genus $g > 1$ is an intrinsically defined finite set of exceptional points of the Riemann surface M. Actually the Weierstrass points can be distinguished by their weights, or rather more precisely by their Weierstrass gap sequences; and each of these separate subsets of Weierstrass points is an intrinsically defined set of exceptional points on the Riemann surface M.

9.5 Hyperelliptic Riemann Surfaces

The role of Weierstrass points can be illustrated by examining one of the classical special classes of Riemann surfaces, the hyperelliptic Riemann surfaces defined on page 229.

Theorem 9.16 *Hyperelliptic Riemann surfaces of genus $g > 1$ are characterized by any of the following equivalent conditions:*
(i) *There is a point a at which the first critical value is $r_1(a) = 2$.*
(ii) *The surface has a point at which the Weierstrass gap sequence is* (9.52).
(iii) *The surface has a Weierstrass point of the maximal possible Weierstrass weight $\omega(a) = \frac{1}{2}g(g-1)$.*
(iv) *The surface has the minimal possible number $2(g+1)$ of Weierstrass points.*

Proof: (i) If M is a hyperelliptic then by definition there is a holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree 2, which as on page 225 is the mapping ϕ_f described by a meromorphic function f of degree 2. The mapping ϕ_f is not a homeomorphim since $g > 1$ so it must have at least one ramification point $a \in M$, which must be of ramification order 1; and after replacing the function f by $1/\big(f - f(a)\big)$ if necessary it can be assumed that the function f has a pole at the ramification point a, which must be a double pole. Then by Theorem 9.3 the function f can be written as the quotient of two holomorphic cross-sections of the holomorphic line bundle ζ_a^2 of the polar divisor of f; so $\gamma(\zeta_a^2) \ge 2$ hence $r_1(a) = 2$. Conversely if $r_1(a) = 2$ then 2 is in the Lüroth semigroup by Corollary 9.10 hence M is hyperelliptic.
(ii) If the Weierstrass gap sequence at a point a is (9.52) then the first local critical value at the point a is $r_1(a) = 2$, so the surface is hyperelliptic by (i). Conversely if the surface is hyperelliptic then $r_1(a) = 2$ for some point $a \in M$ by (ii); and since the local critical values form a semigroup under addition, all even values are local critical values hence the Weierstrass gap sequence must be

(9.52).
(iii) If the surface has a Weierstrass point of the maximal possible Weierstrass weight then by Theorem 9.14 the Weierstrass gap sequence at that point is (9.52), and conversely; and by (ii) that is equivalent to the surface being hyperelliptic.
(iv) If the surface has the minimal possible number of Weierstrass points then each Weierstrass point has the maximal Weierstrass weight, and conversely; so by (iii) that is equivalent to the surface being hyperelliptic. That suffices for the proof.

Theorem 9.17 *A hyperelliptic Riemann surface M of genus $g > 1$ has a unique representation as a two sheeted branched covering of $\mathbb{P}^1$, up to biholomorphic mappings of $\mathbb{P}^1$ to itself; the $2g+2$ ramification points are precisely the Weierstrass points of M, each of which has the Weierstrass gap sequence* (9.52)*.*

Proof: By definition a hyperelliptic Riemann surface M has a holomorphic mapping $\phi : M \longrightarrow \mathbb{P}^1$ of degree 2, which can be taken as the mapping $\phi = \phi_f$ defined by a meromorphic function f of degree 2 on M. For such a mapping all ramification points $a \in M$ clearly have the ramification order $\mathfrak{r}_a(\phi_f) = 1$; and since the Riemann-Hurwitz formula takes the form (9.14), when deg $\phi = 2$ the total branch order is $\text{br}(\phi_f) = 2g+2$ so there are altogether $2g+2$ ramification points of the mapping. At any ramification point $a \in M$ the function f defining the mapping $\phi = \phi_f$ can be assumed to have a double pole, by replacing f by $1/\big(f - f(a)\big)$ if necessary; that amounts to composing the mapping ϕ_f with the biholomorphic mapping of $\mathbb{P}^1$ to itself defined by $t \longrightarrow 1/\big(t - f(a)\big)$. As in Theorem 9.3 the function f then can be written as a quotient $f = f_{\alpha 0}/f_{\alpha 1}$ of two holomorphic cross-sections of the holomophic line bundle of the polar divisor of the function f, the line bundle ζ_a^2, which is base-point free and for which $\gamma(\zeta_a^2) = 2$ since the bundle has the two holomorphic cross-sections $f_{\alpha 0},\ f_{\alpha 1}$ and $\gamma(\zeta_a^2) \le 2$ by Theorem 2.7. On the one hand that means that the first local critical value at the point a is $r_1(a) = 2$, so a is a Weierstrass point with the Weierstrass gap sequence (9.52). That is the case for any of the $2g+2$ ramification points of the mapping f, so all the ramification points are Weierstrass points; and since by (9.49) there are also altogether $2g+2$ Weierstrass points, all of which must be ramification points. On the other hand the cross-sections $f_{\alpha 0}, f_{\alpha 1}$ are a basis for the holomorphic cross-sections of the line bundle ζ_a^2, and $f_{\alpha 1}$ has a double zero at the point $a \in M$. Any other meromorphic function g on the surface M having a double pole at the point $a \in M$ and defining a mapping $\phi_g : M \longrightarrow \mathbb{P}^1$ of degree 2 can also be expressed in the same way as a quotient $g = g_{\alpha 0}/g_{\alpha 1}$ of two holomorphic cross-sections of the line bundle ζ_a^2, where $g_{\alpha_1} = c f_{\alpha_1}$ for some constant $c \ne 0$ since g_{α_1} also has a double zero at the point $a \in M$ and $g_{\alpha_0} = c_0 f_{\alpha_0} + c_1 f_{\alpha_1}$ for some further constants c_0, c_1; thus $g = (c_0/c)f + (c_1/c)$, so the mapping ϕ_g is the composition of the mapping ϕ_f and the biholomorphic mapping $t \longrightarrow (c_0/c)t + (c_1/c)$ of $\mathbb{P}^1$ to itself. That suffices for the proof.

If M is a hyperelliptic Riemann surface and $\phi : M \longrightarrow \mathbb{P}^1$ is a holomorphic mapping exhibiting M as a two sheeted branched covering of $\mathbb{P}^1$ then for each point $z \in M$ that is not a ramification point of the mapping ϕ there is a unique point $Tz \in M$ other than z for which $\phi(Tz) = \phi(z)$; the mapping $T : z \longrightarrow Tz$, extended by setting $Tz = z$ for each ramification point $z \in M$, is called the *hyperelliptic involution* on M. Since the mapping ϕ is uniquely defined up to a biholomorphic mapping of $\mathbb{P}^1$ to itself it is clear that the hyperelliptic involution is uniquely determined. Aside from the ramification points of the mapping ϕ it is clear that the hyperelliptic involution is a one-to-one holomorphic mapping of the surface M to itself, that its square is the identity mapping $T^2 = I$ on M, and that it is locally biholomorphic; but it follows from the Riemann removable singularities theorem that the mapping is also holomorphic at the ramification points, since it is continuous at these points and holomorphic except at these points. Thus the mapping $T : M \longrightarrow M$ is an *automorphism* of of the Riemann surface M, a biholomorphic mapping of the surface M to itself. The quotient of the Riemann surface M under the group of transformations $\{T, I\}$ is just the Riemann sphere; and conversely any compact Riemann surface M with an automorphism T for which $T^2 = I$ and $M/\{T, I\} = \mathbb{P}^1$ obvioușly is a hyperelliptic Riemann surface. The description of other possible automorphisms of a hyperelliptic Riemann surface rests on their action on the Weierstrass points of the surface.

Corollary 9.18 *An automorphism of a hyperelliptic Riemann surface M of genus $g > 1$ permutes the Weierstrass points of M; and an automorphism fixes all the Weierstrass points if and only if the automorphism is the hyperelliptic involution or the identity mapping.*

Proof: Since the Weierstrass points on M are intrinsically determined they are clearly mapped to themselves by any automorphism of the surface. If $f : M \longrightarrow \mathbb{P}^1$ is a mapping exhibiting the Riemann surface M as a two-sheeted cover of the Riemann sphere and if $\phi : M \longrightarrow M$ is an automorphism of the Riemann surface M then the composite mapping $f \circ \phi^{-1} : M \longrightarrow \mathbb{P}^1$ also exhibits the surface M as a two-sheeted cover of the Riemann sphere, so by the preceding theorem $f \circ \phi^{-1} = \theta \circ f$ for an automorphism $\theta : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$, a linear fractional transformation of the projective space $\mathbb{P}^1$. If the automorphism ϕ preserves the Weierstrass points, the ramification points of the mapping f, then the branch points of f must be preserved by the mapping $\theta : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$; but there are $2g + 2 > 3$ of these points, so the mapping θ is linear fractional transformation leaving at least 3 points fixed hence θ is the identity transformation. Consequently the automorphism ϕ at most interchanges the two points $f^{-1}(a)$ over each point $a \in \mathbb{P}^1$ so ϕ is either the identity mapping or the hyperelliptic involution, and that suffices for the proof.

Corollary 9.19 *If M is a hyperelliptic Riemann surface of genus g with the hyperelliptic involution $T : M \longrightarrow M$ then $\omega(Tz) = -\omega(z)$ for any holomorphic abelian differential on M; hence the canonical divisors on M are precisely the divisors of the form $\mathfrak{k} = \sum_{j=1}^{g-1}(1 \cdot a_j + 1 \cdot Ta_j)$ for any points $a_1, \ldots, a_{g-1} \in M$.*

Proof: If ω is a holomorphic abelian differential on M then $\omega(Tz)$ also is a holomorphic abelian differential on M, since the hyperelliptic involution $T : M \longrightarrow M$ is a biholomorphic mapping. The sum $\omega(z) + \omega(Tz)$ then is a holomorphic abelian differential on M that is invariant under the hyperelliptic involution T, so determines a holomorphic abelian differential on the quotient space $M/\{I, T\} = \mathbb{P}^1$; but there are no nontrivial holomorphic abelian differentials on $\mathbb{P}^1$ and consequently $\omega(z) + \omega(Tz) = 0$. It follows from this that if $\omega(a) = 0$ for a point $a \in M$ then $\omega(Ta) = 0$ as well, so the divisor of the holomorphic abelian differential $\omega(z)$ must be of the form $\mathfrak{d}(\omega) = \sum_{j=1}^{g-1}(1 \cdot a_j + 1 \cdot Ta_j)$ for some points $a_1, \ldots, a_{g-1}$ on M. Conversely for any divisor of the form $\mathfrak{d} = \sum_{j=1}^{g-1}(1 \cdot a_j + 1 \cdot Ta_j)$ there is a nontrivial holomorphic abelian differential $\omega(z)$ such that $\omega(a_j) = 0$ for $1 \le j \le g-1$, since there are g linearly independent holomorphic abelian differentials on M; and since $\omega(Tz) = \omega(z)$ the divisor of this differential is $\mathfrak{d}(\omega) = \sum_{j=1}^{g-1}(1 \cdot a_j + 1 \cdot Ta_j)$, and that suffices for the proof.

A hyperellipic Riemann surface M of genus $g > 1$ is a two-sheeted branched covering of the Riemann sphere $\mathbb{P}^1$ with $2g + 2$ branch points $a_i \in \mathbb{P}^1$; so M is the Riemann surface represented by the holomorphic subvariety

$$V = \left\{ (w, z) \in \mathbb{P}^1 \times \mathbb{P}^1 \;\middle|\; w^2 = \prod_{j=1}^{2g+2} (z - a_i) \right\}, \subset \mathbb{P}^1 \times \mathbb{P}^1, \tag{9.60}$$

since V has the natural structure of a compact Riemann surface for which the natural projection $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ defined by $\pi(w, z) = z$ exhibits V as a branched covering of $\mathbb{P}^1$ branched over the $2g + 2$ pairwise distinct points $a_i \in \mathbb{P}^1$. The $2g + 2$ points $(w_i, z_i) \in V$ are the ramification points of the mapping $\pi : V \longrightarrow \mathbb{P}^1$, and are precisely the Weierstrass points on $V = M$, each of which has the Weierstrass gap sequence (9.52) as in Theorem 9.16.

9.6 Automorphisms of Riemann Surfaces

For Riemann surfaces other than hyperelliptic surfaces the Weierstrass points also are useful in the examination of automorphisms of the surface.

Theorem 9.20 *An automorphism of a non hyperelliptic compact Riemann surface of genus $g > 1$ permutes the Weierstrass points of the surface and fixes all the Weierstrass points if and only if it is the identity mapping.*

Proof: The Weierstrass points on any compact Riemann surface M are intrinsically defined so they are mapped to themselves by any automorphism $T : M \longrightarrow M$ of the surface. Suppose T is a nontrivial automorphism of the Riemann surface M that fixes the Weierstrass points of M. There must be a point $a \in M$ that is not a Weierstrass point and that is not preserved by the mapping T, since there are only finitely many Weierstrass points so the complement of the set of Weierstrass points is a dense subset of M and any automorphism that is the identity on a dense subset is the identity automorphiosm. Since a is not a Weierstrass point $r_1(a) = g+1$ by Lemma 9.12, so $\gamma(\zeta_a^{g+1}) = 2$. One cross-section f_0 of ζ_a^{g+1} can be taken to be a power of a nontrivial cross-section of ζ_a, so it has a zero of order $g+1$ at the point a, while the other cross-section f_1 is necessarily nonzero at the point a, since ζ_a^{g+1} is base-point-free; the quotient $f = f_1/f_0$ is a nontrivial meromorphic function on M with a pole of order $g+1$ at the point a and no other singularities. Since $Ta \ne a$ the function $g = f - f \circ T$ is a nontrivial meromorphic function on M with poles of order $g+1$ at the points a and Ta but no other singularities on M, so it is a meromorphic function of degree $2g+2$; its zero divisor then is a positive divisor of degree $2g+2$, so consists of at most $2g+2$ points of M. Howwever $Tz = z$ for each Weierstrass point $z \in M$ so the the function $g = f - f \circ T$ must vanish at each Weierstrass point of M; and if M is not hyperelliptic then $g > 1$ and by Theorem 9.16 (iii) there are $N > 2g+2$ Weierstrass points on M, so the function g has at least $N > 2g+2$ zeros. That is a contradiction, so any automorphism θ of M that preserves the Weierstrass points must be the identity mapping, and that concludes the proof

Corollary 9.21 *The group of automorphisms of a compact Riemann surface of genus $g > 1$ is finite.*

Proof: For a hyperelliptic Riemann surface of genus $g > 1$ this follows from Corollary 9.18, since every automorphism preserves the finite set of Weierstrass points and the only automorphisms that fix the Weierstrass points are the identity mapping and the hyperelliptic involution. For a non-hyperelliptic Riemann surface of genus $g > 1$ this follows from Theorem 9.20, since every automorphism preserves the finite set of Weierstrass points and the only automorphism that fixes the finitely many Weierstrass points is the identity mapping. That suffices for the proof.

The quotient space M/G of a Riemann surface M under any group G of automorphisms of M is another Riemann surface that is closely related to the surface M, as follows.

Theorem 9.22 (i) *If G is a group of automorphisms of a compact Riemann surface M of genus $g > 1$ the quotient space $N = M/G$ has the natural structure of a compact Riemann surface for which the quotient mapping $\pi : M \longrightarrow N$ is a holomorphic mapping of degree $\nu = |G|$, the order of the group G.*
(ii) *For any point $p \in M$ the ramification order of the mapping π at the point p is $\mathfrak{r}_p(\pi) = \nu_p - 1$ where $\nu_p = |G_p|$ is the order of the subgroup $G_p \subset G$ consisting of those automorphisms that fix the point p.*
(iii) *For any point $q \in N$ the subgroups G_p for all the points $p \in \pi^{-1}(q)$ are conjugate subgroups of G so they all have the same order, which will be denoted by ν_q^*; and there are ν/ν_q^* points in $\pi^{-1}(q)$.*
(iv) *The branch divisor of the mapping π is*

$$\mathfrak{b}(\pi) = \sum_{q \in N} \nu \left(1 - \frac{1}{\nu_q^*}\right) \cdot q, \tag{9.61}$$

and the branch order of the mapping π is

$$\mathrm{br}(\pi) = \sum_{q \in N} \nu \left(1 - \frac{1}{\nu_q^*}\right). \tag{9.62}$$

(v) *The genus h of the quotient surface N is determined by the equation*

$$2g - 2 = (2h-2)\nu + \nu \sum_{q \in N} \left(1 - \frac{1}{\nu_q^*}\right). \tag{9.63}$$

Proof: (i) The group G of automorphisms of a compact Riemann surface M of genus $g > 1$ is a finite group by Corollary 9.21. Let $\pi : M \longrightarrow N$ be the natural mapping from M to the quotient space $N = M/G$ and let $X \subset M$ be the set of those points of M that are fixed under an automorphism in G other than the identity. Each automorphism is a holomorphic mapping so its fixed points are a discrete subset of the compact manifold M, hence a finite set of points of M; and since the group G is finite the set X also is finite. For a sufficiently small open neighborhood U_p of a point $p \in (M \sim X)$ each of the finitely many automorphisms $T \in G$ determines a biholomorphic mapping $T : U_p \longrightarrow TU_p$ and the images TU_p for distinct automorphisms $T \in G$ are disjoint subsets of M; therefore the neighborhood U_p can be identified with an open neighborhood V_p in the quotient space N, thus providing the structure of a Riemann surface on the quotient $(M \sim X)/G$. The quotient mapping π then is a holomorphic mapping exhibiting $M \sim X$ as a covering map of $\nu = |G|$ sheets over the quotient manifold $(M \sim X)/G$. On the other hand if $p \in X$ the set of automorphisms $T \in G$ such that $Tp = p$ form a finite subgroup $G_p \subset G$. If U_p is an open neighborhood of p then so is the finite intersection $U'_p = \bigcap_{T \in G_p} TU_p$, and $TU'_p = U'_p$ for all $T \in G_p$. If U_p is sufficiently small the images TU'_p coincide with U'_p when $T \in G_P$ and are disjoint from U'_p otherwise; so the quotient U'_p/G_p can be identified with a subset of the quotient space $N = M/G$. If z is a local coordinate in M centered at the point p then $g(z) = \prod_{T \in G_p} Tz$ is a holomorphic function in U'_p that is invariant under the action of the subgroup G_p; it describes a holomorphic mapping $g : U'_p \longrightarrow V_p$ to an open subset $V_p \in \mathbb{C}$ that induces a bijective mapping $g : U'_p/G_p \longrightarrow V_p$, so V_p can be identified with a local coordinate neighborhood on the quotient space N. The same construction at all points of X extends the Riemann surface $(M \sim X)/G$ to a Riemann surface structure on the quotient space $N = M/G$ for which the natural mapping $\pi : M \longrightarrow N$ is a holomorphic mapping of degree $\nu = |G|$.
(ii) It clear from the construction in (i) that the mapping π is ramified just at the points $p \in X$, and that the ramification order at any point $p \in M$ is $\nu_p - 1$ where $\nu_p = |G_p|$ is the order of the subgroup G_p.
(iii) It is evident from the definition of the subgroups $G_p \subset G$ that $TG_pT^{-1} = G_{Tp}$ for any $T \in G$; thus the subgroups G_{Tp} for all $T \in G$ are conjugate subgroups of G so all of them have the same order $\nu_{Tp} = \nu_q^*$. The number of distinct points in $\pi^{-1}(q)$ for any point $q \in N$ is the number of cosets of the subgroup G_p so is the index $\nu/\nu_p = \nu/\nu_q^*$ of the subgroup G_p.

(iv) A point $q \in N$ is a branch point of the mapping π precisely when $q = \pi(p)$ for a ramification point $p \in M$; and since by (ii) the ramification order of a point $p \in M$ is $\nu_p - 1$ while by (iii) there are ν/ν_q^* points $p \in \pi^{-1}(q) \subset M$ and $\nu_p = \nu_q^*$ for all of them it follows from (9.5) that the local branch order at the point q is

$$\mathfrak{b}_q(\pi) = \frac{\nu}{\nu_q^*}(\nu_q^* - 1) = \nu\left(1 - \frac{1}{\nu_q^*}\right)$$

so the branch divisor is

$$\mathfrak{b}(\pi) = \sum_{q\in N} \mathfrak{b}_q(\pi)\cdot q = \sum_{q\in N} \nu\left(1 - \frac{1}{\nu_q^*}\right)\cdot q,$$

which is (9.62). By (9.7) the total branch order is

$$\mathrm{br}(\pi) = \sum_{q\in N} \mathfrak{b}_q(\pi) = \sum_{q\in N} \nu\left(1 - \frac{1}{\nu_q^*}\right),$$

which is (9.62).
(v) Substituting the results in (iv) into the Riemann-Hurwitz equation (9.9) then yields (9.63), and that suffices for the proof.

The preceding result can be applied to yield bounds on the sizes of the groups of automorphisms of compact Riemann surfaces, following Hurwitz[4].

Theorem 9.23 (Hurwitz's Theorem) *The order of the group of all automorphisms of a compact Riemann surface of genus $g > 1$ is bounded above by $84(g-1)$.*

Proof: It follows from the preceding Theorem 9.22 that if G is the group of automorphisms of a compact Riemann surface M of genus $g > 1$ and if the quotient space $N = M/G$ is a Riemann surface of genus h then the order $\nu = |G|$ of the group satisfies

$$2g - 2 = \nu\Big(2h - 2 + H\Big) \tag{9.64}$$

where $H = \sum_{i=1}^{r}\left(1 - \frac{1}{\nu_i}\right)$ for some $r \ge 1$ and $\nu_i \ge 2$. Here $2g-2 > 0$ since $g > 1$ so it follows from (9.64) that $2h - 2 + H > 0$; a positive lower bound to $2h - 2 + H$ in (9.64) provides an upper bound for the order ν of the group G depending on the explicit values of h and H.
(i) First if $H = 0$ then since $2h-2+H > 0$ necessarily $h \ge 2$ hence $2h-2+H \ge 2$ and it follows from (9.64) that $\nu \le g - 1$.
(ii) Next if $H > 0$ since $r \ge 1$ and $\nu_i \ge 2$ so $1 - \frac{1}{\nu_i} \ge \frac{1}{2}$ then $H \ge \frac{1}{2}$; and if it also the case that $h \ge 1$ then $2h - 2 + H \ge \frac{1}{2}$ and it follows from (9.64) that

[4] This result can be found in the paper by Adolph Hurwitz, "Ueber algebraische Gebilde mit Eindeutigen Transformationen in sich", *Mathematische Annalen* vol. 41, 1893, pp. 403 - 442.

$\nu \le 4(g-1)$.
(iii) That leaves just the case that $H > 0$ and $h = 0$, for which (9.64) takes the form

$$2g - 2 = (H-2)\nu. \tag{9.65}$$

Here $H > 2$ since $g > 1$; so to finish the proof it is sufficient to find the least value of H subject to the restriction that $H > 2$. Since $\left(1 - \frac{1}{\nu_i}\right) < 1$ then $H = \sum_{i=1}^{r}(1 - \frac{1}{\nu_i}) < r$, so since $H > 2$ then necessarily $r \ge 3$. There are various lower bounds for H depending on the value of r.
(iii,a) If $r \ge 5$ then since $\nu_1 \ge 2$ and therefore $\left(1 - \frac{1}{\nu_i}\right) \ge \frac{1}{2}$ it follows that $H \ge \frac{5}{2}$ so $H - 2 = \frac{1}{2}$ and hence $\nu \le 4(g-1)$ again.
(iii,b) If $r = 4$ it cannot be the case that $\nu_i = 2$ for all i since then $\left(1 - \frac{1}{\nu_i}\right) = \frac{1}{2}$ so $H = 4\frac{1}{2} = 2$; therefore it must be the case that $\nu_i \ge 3$ for at least one index i so $\left(1 - \frac{1}{\nu_i}\right) \ge \frac{2}{3}$ for at least one index i therefore $H \ge 3 \cdot \frac{1}{2} + 1 \cdot \frac{2}{3} = \frac{13}{6}$ so $H - 2 \ge \frac{1}{6}$ hence $\nu \le 12(g-1)$.
(iii,c) If $r = 3$ there are a few cases to consider but after a final analysis that will be left to the reader the result is that in this case the lower bound is attained for $\nu_1 = 2, \nu_2 = 3, \nu_3 = 7$ and is $H = \frac{85}{42}$ so $H - 2 = \frac{1}{42}$ hence $\nu \le 84(g-1)$. This is the largest of the upper bounds for ν in all the cases and it provides the Hurwitz bounds and therefore concludes the proof.

The Riemann surfaces M of genus $g > 1$ for which the group G of automorphisms has the maximum order $84(g-1)$ are called *Hurwitz surfaces*, and their groups of automorphisms are called *Hurwitz groups*. Hurwitz surfaces are characterized by the values of the invariants listed in part (iii,c) of the proof of the preceding theorem; with the notation as in that proof, Hurwitz surfaces are characterized by the conditions that $h = 0$, $H > 2$, and $\nu_1 = 2, \nu_2 = 3, \nu_3 = 7$. Thus the quotient surface M/G is the Riemann sphere; and the branch order of the mapping $\pi : M \longrightarrow M/G = \mathbb{P}^1$ is $\nu = 84$ while by (9.61) where ν_q^* are the values ν_i the branch divisor of the mapping π is

$$\mathfrak{b}(\pi) = 2 \cdot z_1 + 3 \cdot z_2 + 7 \cdot z_3 \tag{9.66}$$

for three points $z_1, z_2, z_3 \in \mathbb{P}^1$. The automorphism group G of order $84(g-1)$ has three conjugacy classes of subgroups of G, where the i-th conjugacy class is formed by the ν/ν_i subgroups of order ν_i consisting of the transformations that leave one of the ν/ν_i points of $\pi^{-1}(z_i) \subset M$ fixed.

The proof of the Hurwitz upper bound for the order of the group of automorphisms of a Riemann surface of genus $g > 1$ was derived from general topological properties that any group of automorphisms of a Riemann surface of genus g necessarily has; but the proof does not show that there actually are Riemann surfaces for which the group of automorphisms has order $84(g-1)$. Fortunately about the same time that Hurwitz was deriving these results Felix

Klein[5] was investigating a quartic curve with remarkable symmetries, a curve of genus $g = 3$ with an automorphism group of order $168 = 84 \cdot 2$, so a Hurwitz surface of genus $g = 2$ showing that there do exist some Hurwitz surfaces. The next example of a Hurwitz surface was discovered by Robert Fricke [6], a surface of genus 7 with an automorphism group of order 504; and until 1960 those were the only known examples of Hurwitz surfaces. However in 1961 Alexander Murray Macbeath[7] rediscovered the Fricke surface independently (so the surface is generally called the Fricke-Macbeath surface) and demonstrated that there actually are infinitely many Hurwitz surfaces. On the other hand there are values of the genus g for which there are no Riemann surfaces with an automorphism group of order $84(g-1)$. For instance it has been shown that the maximal order of the automorphism group of a Riemann surface of genus $g = 2$ is 48 rather than 84; the unique example of such a surface is the Bolza curve[8]. Hurwitz surfaces and Hurwitz groups have been extensively investigated but still remain somewhat mysterious[9]

[5]See the paper by F. Klein, "Ueber die Transformation siebenter Ordnung der elliptischen Functionen" in *Mathematische Annalen* vol. 14, 1878, pp. 428 - 471.

[6]See the paper by R. Fricke, R. (1899), "Ueber eine einfache Gruppe von 504 Operationen", *Mathematische Annalen*, vol.52, 1899, pp.321-329.

[7]See the paper by A. Macbeath, "On a Theorem of Hurwitz", in *Proc. Glasgow Math. Assoc.*, vol. 5 (1961), pp 90-96.

[8]See the paper by Oskar Bolza, "On Binary Sextics with Linear Transformations into Themselves", in *American Journal of Mathematics* vol. 10, 1887, pp. 47-70.

[9]For a general discussion of Hurwitz surfaces and groups see the book *The Eightfold Way: The Beauty of Klein's Quartic Curve*, MSRI Publications, Vol. 35, edited by Silvio Levy.

Chapter 10

The Hurwitz Parameters

10.1 The Varieties of Positive Divisors

The branch divisor of a holomorphic mapping between two Riemann surfaces is an example of a positive divisor on a Riemann surface, a special class of divisors that play a significant role in the study of Riemann surfaces. It is convenient to digress briefly here to examine in more detail the sets of positive divisors on Riemann surfaces. A positive divisor of degree r on a Riemann surface M can be viewed as an unordered set of r not necessarily distinct points of M; so the set of all positive divisors of degree r can be identified with the quotient $M^{(r)} = M^r/\mathfrak{S}_r$ of the r-dimensional complex manifold M^r by the symmetric group $\mathfrak{S}_r$ acting as the group of permutations of the factors, a set called the *r-th symmetric product* of the surface with itself. The mapping that associates to an ordered set of r points the corresponding unordered set of those points is the natural quotient mapping

$$\pi_r : M^r \longrightarrow M^{(r)} = M^r/\mathfrak{S}_r. \tag{10.1}$$

The quotient space $M^{(r)}$ has the natural quotient topology, in which a subset $U \subset M^{(r)}$ is open precisely when the inverse image $\pi_r^{-1}(U) \subset M^r$ is open. The restriction of the quotient mapping π_r to the subset

$$M^{r*} = \left\{ (z_1, \ldots, z_r) \in M^r \;\middle|\; z_i \neq z_j \quad \text{for} \quad i \neq j \right\} \subset M^r \tag{10.2}$$

clearly is a locally homeomorphic mapping.

Lemma 10.1 *The symmetric product $\mathbb{C}^{(r)}$ has the structure of a complex manifold of dimension r such that the natural quotient mapping $\pi_r : \mathbb{C}^r \longrightarrow \mathbb{C}^{(r)}$ is a holomorphic mapping and its restriction to the subset $\mathbb{C}^{r*}$ is locally biholomorphic.*

Proof: Consider the mapping $\tau_r : \mathbb{C}^r \longrightarrow \mathbb{C}^r$ defined by

$$\tau_r(z_1, z_2, \ldots, z_r) = \big(e_1(z_1, \ldots, z_r), e_2(z_1, \ldots, z_r), \ldots, e_r(z_1, \ldots, z_r)\big),$$

where $e_i(z_1, \ldots, z_r)$ are the elementary symmetric functions in r variables. It will be shown first that there is a one-to-one mapping $\sigma_r : \mathbb{C}^{(r)} \longrightarrow \mathbb{C}^r$ such that the following diagram

(10.3)
$$\begin{tikzcd} \mathbb{C}^r \arrow[rr, "\tau_r"] \arrow[dr, "\pi_r"'] & & \mathbb{C}^r \\ & \mathbb{C}^{(r)} \arrow[ur, "\sigma_r"'] & \end{tikzcd}$$

is a commutative diagram of mappings. The elementary symmetric functions $e_i = e_i(z_1, \ldots, x_r)$ are the polynomials in the variables $z_1, \ldots, z_r$ defined as the coefficients of the polynomial

$$\prod_{i=1}^{r}(X - z_i) = X^r - e_1 X^{r-1} + e_2 X^{r-2} - \cdots + (-1)^r e_r. \tag{10.4}$$

These coefficients clearly are invariant under permutations of the variables $z_1, \ldots, z_r$, so the mapping τ_r factors through the quotient mapping π_r as the composition $\tau_r = \sigma_r \circ \pi_r$ for some mapping σ_r for which the resulting diagram (12.20) is a commutative diagram of mappings. The mapping τ_r is surjective since any monic polynomial of degree r can be written as the product $\prod_{i=1}^{r}(X - z_i)$ where z_i are its roots; consequently the mapping σ_r also is surjective. Since a monic polynomial is determined uniquely by its roots and conversely determines the roots uniquely up to order, two points of M^r have the same image under τ_r if and only if they have the same image under π_r, so the mapping σ_r is injective. Altogether then the mapping σ_r is one-to-one, so can be used to identify the symmetric product $\mathbb{C}^{(r)}$ with the image $\mathbb{C}^r$, and thereby to give the symmetric product the structure of a complex manifold. The mapping τ_r clearly is holomorphic, and its restriction to $\mathbb{C}^{r*}$ is locally biholomorphic; so from the commutativity of the diagram (12.20) the same is true for the mapping σ_r, and that suffices for the proof.

For some purposes it is more convenient to use the power sums

$$s_i(z_1, \ldots, z_r) = z_1^i + \cdots + z_r^i \tag{10.5}$$

for $1 \le i \le r$ in place of the elementary symmetric functions in the definition of the mapping τ_r in the preceding lemma; Newton's formulas expressing the elementary symmetric functions in terms of the power sums and conversely show that the two choices lead to equivalent results.

Theorem 10.2 *The symmetric product $M^{(r)}$ of a compact Riemann surface M has the natural structure of a compact complex manifold of dimension r. The quotient mapping $\pi_r : M^r \longrightarrow M^{(r)}$ is a holomorphic mapping that is a locally biholomorphic mapping from the dense open subset $M^{r*} \subset M^r$ consisting of sets of r distinct points of M to its image $M^{(r)*} = \pi_r(M^{r*})$.*

Proof: For any divisor $\mathfrak{d} = \nu_1 \cdot a_1 + \cdots + \nu_s \cdot a_s \in M^{(r)}$ for which $a_1, \ldots, a_s$ are distinct points of M choose disjoint open neighborhoods $U_i \subset M$ of the points a_i. The quotient spaces $U_i^{(\nu_i)}$ have the natural structures of complex manifolds by the preceding lemma; and the product $U_{\mathfrak{d}} = U_1^{(\nu_1)} \times \cdots \times U_s^{(\nu_s)}$ then provides the structure of a complex manifold on an open neighborhood of the divisor $\mathfrak{d} \in M^{(r)}$. If the divisor $\mathfrak{d}$ consists of distinct points, so that $\nu_i = 1$ for all indices i, the quotient spaces $U_i^{(\nu_i)} = U_i^{(1)}$ are just the neighborhoods U_i themselves, so the quotient mapping $\pi_r : M^r \longrightarrow M^{(r)}$ then is a locally biholomorphic mapping in an open neighborhood of the divisor $\mathfrak{d}$. That suffices to conclude the proof.

The set $M^{(r)}$ often is called the *variety of positive divisors of degree* r on the Riemann surface M since it has the natural structure of a holomorphic variety, actually of a compact complex manifold. The quotient mapping (12.18) clearly is a finite, proper, surjective holomorphic mapping from M^r onto $M^{(r)}$. The complement $V^r = (M^r \sim M^{r*})$ is the union $V^r = \bigcup_{i \neq j} V^{i,j}$ of the proper holomorphic subvarieties $V^{i,j} = \{\, (z_1, \ldots, z_r) \in M^r \mid z_i = z_j \,\}$, so is a proper holomorphic subvariety of M^r; and by Remmert's Proper Mapping Theorem[1]its image $\pi_r(V^r) = V^{(r)} \subset M^{(r)}$ is a proper holomorphic subvariety of the complex manifold $M^{(r)}$. The inverse image $\pi_r^{-1}(V^{(r)}) \subset M^r$ then is a proper holomorphic subvariety of M^r and the restriction

$$\pi_r : \left(M^r \sim \pi_r^{-1}(V^{(r)})\right) \longrightarrow \left(M^{(r)} \sim V^{(r)}\right)$$

clearly is a locally biholomorphic covering mapping of r sheets; but the inverse image under π_r of any point in $V^{(r)}$ consists of strictly fewer than r points of M^r, so the mapping π_r is a branched holomorphic covering[2]of r sheets over the complex manifold $M^{(r)}$, branched over the subvariety $V^{(r)} \subset M^{(r)}$.

10.2 Special Branch Divisors

The branch divisors of holomorphic mappings $\phi : M \longrightarrow \mathbb{P}^1$ from compact Riemann surfaces to the Riemann sphere are examples of positive divisors in $\mathbb{P}^1$. A particular class of these divisors play a significant role in the study of holomorphic mappings to the Riemann sphere, the divisors of mappings that are fully branched over a point of $\mathbb{P}^1$, usually the point ∞ when the Riemann sphere is viewed as the union $\mathbb{P}^1 = \mathbb{C} \cup \infty$.

---

[1]Remmert's Proper Mapping Theorem asserts that if $f : V \longrightarrow W$ is a proper holomorphic mapping from a holomorphic variety V to a holomorphic variety W then the image $f(V)$ is a holomorphic subvariety of W with $\dim f(V) \le \dim V$; in particular if f is a holomorphic mapping from a compact complex manifold V to a complex manifold W then the image $f(V)$ is a holomorphic subvariety of W with $\dim f(V) \le \dim V$. This is discussed in more detail in Appendix A.3.

[2]The definitions and general properties of branched holomorphic coverings are given on page 423 in Appendix A.3.

Theorem 10.3 *If M is a compact Riemann surface of genus $g > 1$ then for any point $a \in M$ there is a holomorphic mapping $\phi_a : M \longrightarrow \mathbb{P}^1$ of degree $r > 1$ that is fully ramified at the point a if and only if r is a local critical value of the Riemann surface M at the point a.*

Proof: If $\phi : M \longrightarrow \mathbb{P}^1$ is a holomorphic mapping of degree r that is fully ramified at the point a it can be assumed by choosing suitable coordinates on $\mathbb{P}^1$ that $\phi(a) = \infty$ where $\mathbb{P}^1 = \mathbb{C} \cup \infty$; so $\phi = \phi_f$ actually is the holomorphic mapping described by a meromorphic function f on M that is of degree r and that has a pole of order r at the point a as its sole singularity. Conversely any meromorphic function f on M of degree r that has a single pole at the point a describes a holomophic mapping $\phi_f : M \longrightarrow \mathbb{P}^1$ that is fully ramified at the point a. By Theorem 9.3 a meromorphic function f on M of degree r with a pole just at the point $a \in M$ can be written as the quotient $f = f_{\alpha 1}/f_{\alpha 0}$ of two holomorphic cross-sections $f_{\alpha 0}, f_{\alpha 1} \in \Gamma\big(M, \mathcal{O}(\zeta_a^r)\big)$ with no common zeros for the holomorphic line bundle ζ_a^r of the polar divisor $r \cdot a$ of the function f. The line bundle ζ_a^r therefore is base-point-free, and it then follows from Theorem 9.9 that r is a local critical value of the Riemann surface M at the point a. That suffices for the proof.

To study these mappings in more detail it is necessary to examine the local maximal function a bit more closely.

Lemma 10.4 *If M is a compact Riemann surface of genus $g > 1$ then for any point $a \in M$ there are holomorphic cross-sections $f_{i,\alpha} \in \Gamma\big(M, \mathcal{O}(\zeta_a^{r_i(a)})\big)$ for $i \ge 0$ such that $f_{0,\alpha}(a)$ has a simple zero at a while $f_{i,\alpha}(a) \ne 0$ if $i > 0$. For any integer r in the interval $r_i(a) \le r < r_{i+1}(a)$ for $i > 0$ the vector space $\Gamma\big(M, \mathcal{O}(\zeta_a^r)\big)$ has dimension $i+1$ and has a basis consisting of the holomorphic cross-sections $f_{0,a}^r$ and $f_{j,a} f_{0,a}^{r - r_j(a)}$ for $1 \le j \le i$.*

Proof: The vector space $\Gamma\big(M, \mathcal{O}(\zeta_a)\big)$ of holomorphic cross-sections of the line bundle ζ_a is one-dimensional and is spanned by a holomorphic cross-section $f_{0,a} \in \Gamma\big(M, \mathcal{O}(\zeta_a)\big)$ that has a simple zero at the point $a \in M$. On the other hand for any local critical value $r_i(a)$ for $i > 0$ the line bundle $\zeta_a^{r_i(a)}$ is base-point-free, by Theorem 9.9, so there is a holomorphic cross-section $f_{i,a} \in \Gamma\big(M, \mathcal{O}(\zeta_a^{r_i(a)})\big)$ for which $f_{i,a}(a) \ne 0$. It follows from Theorem 9.8 that $\dim \Gamma\big(M, \mathcal{O}(\zeta_a^r)\big) = i + 1$ if $r_i(a) \le r < r_{i+1}(a)$. The $i+1$ cross-sections $f_{0,a}^r$ and $f_{j,a} f_{0,a}^{r - r_j(a)}$ in $\Gamma\big(M, \mathcal{O}(\zeta_a)\big)$ for $1 \le j \le i$ (equivalently for $r_j(a) \le r$) are linearly independent, since their orders at the point a are distinct, so they form a basis, and that suffices for the proof.

By the preceding Lemma the holomorphic cross-sections of the line bundle $\zeta_a^{r_i(a)}$ for $i > 0$ form the complex vector space of dimension $i+1$ consisting of the cross-sections

$$f_{t,\alpha} = t_0 f_{0,\alpha}^{r_i(a)} + \sum_{j=1}^{i} t_j f_{j,\alpha} f_{0,\alpha}^{r_i(a) - r_j(a)} \in \Gamma\big(M, \mathcal{O}(\zeta_a^{r_i(a)})\big) \tag{10.6}$$

for vectors $t = (t_0, t_1, \ldots, t_i) \in \mathbb{C}^{i+1}$; and $f_{i,\alpha}(a) \neq 0$ for $i > 0$ while $f_{0,\alpha}(a) = 0$ and $f_{0,\alpha}(z) \neq 0$ at any point $z \neq a$. The quotients $f_t = f_{t,\alpha}/f_{0,\alpha}^{r_i(a)}$ thus are meromorphic functions on M of degree at most $r_i(a)$ and with a pole only at the point $a \in M$; and by Theorem 9.3 all such meromorphic functions f on M are of this form so all of them can be written

$$f_t = t_0 + t_1 f_{1,\alpha} f_{0,\alpha}^{-r_1(a)} + \cdots + t_{i-1} f_{i-1,\alpha} f_{0,\alpha}^{-r_{i-1}(a)} + t_i f_{i,\alpha} f_{0,\alpha}^{-r_i(a)} \tag{10.7}$$

for vectors $t = (t_0, t_1, \ldots, t_i) \in \mathbb{C}^{i+1}$. These are the meromorphic functions that describe holomorphic mappings $\phi_{f_t} : M \longrightarrow \mathbb{P}^1$ of degree at most $r_i(a)$ that are fully ramified at the point $a \in M$ and for which $\phi_{f_t}(a) = \infty \in \mathbb{P}^1 = \mathbb{C} \cup \infty$. If $t_i \neq 0$ the mapping ϕ is of degree precisely r_i; but if $t_i = 0$ the meromorphic function $f_{t,a}$ has a pole of degree strictly less than $r_i(a)$ at the point a, indeed of degree $r_j(a)$ if $t_j \neq 0$ but $t_{j+1} = t_{j+2} = \cdots = t_i = 0$. Thus the set of those holomorphic mappings $\phi : M \longrightarrow \mathbb{P}^1$ of degree $r_i(a)$ that are fully ramified at the point $a \in M$ and for which $\phi_{f_t}(a) = \infty \in \mathbb{P}^1 = \mathbb{C} \cup \infty$ is parametrized by the set of vectors $t = (t_0, t_1, \ldots, t_i) \in \mathbb{C}^{i+1}$ for which $t_i \neq 0$. In particular there are meromorphic functions f_t of degree $r_1(a)$ that have a pole only at the point $a \in M$ and all of them can be written

$$f_t = t_0 + t_1 f_{1,a}/f_{0,a}^{r_1(a)} \text{ for constants } t_0, t_1 \in \mathbb{C},\ t_1 \neq 0. \tag{10.8}$$

These functions describe holomorphic mappings $\phi_{f_t} : M \longrightarrow \mathbb{P}^1$ of degree $r_1(a)$ that are fully ramified at the point $a \in M$ and for which $\phi_{f_t}(a) = \infty \in \mathbb{P}^1$.

Theorem 10.5 *If M is a compact Riemann surface of genus $g > 1$ then for any point $a \in M$ there are meromorphic functions f_a on M that have a pole of order $r_1(a)$ at the point $a \in M$ as their only singularity; and the holomorphic mappings $\phi_a = \phi_{f_a} : M \longrightarrow \mathbb{P}^1$ described by such functions are of degree $r_1(a)$, are fully ramified at the point $a \in M$, map that point to the point $\phi_a(a) = \infty \in \mathbb{P}^1 = \mathbb{C} \cup \infty$ and are determined uniquely up to arbitrary holomorphic automorphisms $T : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ that preserve the point $\infty \in \mathbb{P}^1 = \mathbb{C} \cup \infty$.*

Proof: The meromorphic functions f_t on M that have a pole of order $r_1(a)$ just at the point $a \in M$ all have the form (10.8) for constants $t_0, t_1 \in \mathbb{C}$ where $t_1 \neq 0$, as just noted; and the mappings $\phi_t = \phi_{f_t}$ described by these functions are precisely the holomorphic mappings of M to the Riemann sphere of degree $r_1(a)$ that are fully branched at the point $a \in M$. If f_0 is the particular function for the parameters $t_0 = 0, t_1 = 1$ then all the other functions of the form (10.8) have the form $f_{(t_0,t_1)} = t_1 f_a + t_0$ for some values $t_0, t_1 \in \mathbb{C}$ for which $t_1 \neq 0$. The mapping $\phi_{(t_0,t_1)} : M \longrightarrow \mathbb{P}^1$ defined by the function $f_{(t_0,t_1)}$ then is the composition $\phi_{(t_0,t_1)} = T \circ \phi_0$ where ϕ_0 is the mapping described by the function f_0 and $T : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the automorphism of the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \infty$ given by $Tz = t_1 z + t_0$; these are precisely the automorphisms of $\mathbb{P}^1 = \mathbb{C} \cup \infty$ that leave the point ∞ fixed, as discussed on page 226, and that suffices for the proof.

Corollary 10.6 *If M is a compact Riemann surface of genus $g > 1$ and if $a \in M$ is not a Weierstrass point of M then $r_1(a) = g+1$ and there are holomorphic mappings*

$$\phi_a : M \longrightarrow \mathbb{P}^1 \tag{10.9}$$

of degree $g+1$ that are fully ramified at the point $a \in M$ and for which $\phi_a(a) = \infty \in \mathbb{P}^1 = \mathbb{C} \cup \infty$; and these mappings are determined uniquely up to arbitrary holomorphic automorphisms $T : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ that preserve the point $\infty \in \mathbb{P}^1 = (\mathbb{C} \cup \infty)$.

Proof: If M is a compact Riemann surface of genus $g > 1$ the Riemann-Roch Theorem shows that $\gamma(\zeta_a^{g+1}) = \gamma(\kappa\zeta_a^{-g-1}) + 2 \geq 2$, so that $r_1(a) \leq g+1$. On the other hand Theorem 9.12 shows that $a \in M$ is a Weierstrass point of M if and only if $r_1(a) \leq g$. Therefore for all points $a \in M$ except for the finitely many Weierstrass points it must be the case that $r_1(a) = g+1$. Then by the preceding Theorem 10.5 there are holomorphic mappings $\phi_a : M \longrightarrow \mathbb{P}^1$ of degree $g+1$ that are fully ramified at the point a, that satisfy $\phi_a(a) = \infty$, and that are determined uniquely up to holomorphic automorphisms of $\mathbb{P}^1$ that preserve the point ∞, which suffices for the proof.

A *triply generally pointed* Riemann surface $M_{a;b,c}$ of genus $g > 1$ is defined to be a compact Riemann surface M of genus $g > 1$ together with the choice of a base point $a \in M$ that is not a Weierstrass point of M and of two other points $b, c \in M$ such that $\phi_a(a), \phi_a(b), \phi_a(c)$ are distinct points of $\mathbb{P}^1$ for some choice of the mapping ϕ_a of Corollary 10.6. Since the mappings ϕ_a of Corollary 10.6 differ by holomorphic automorphisms of $\mathbb{P}^1$ that preserve the point ∞ it then follows of course that $\phi_a(a), \phi_a(b), \phi_a(c)$ are distinct points of $\mathbb{P}^1$ for any choice of the mapping ϕ_a. The collection of all triply generally pointed compact Riemann surfaces of genus g will be denoted by $\mathfrak{M}_{g,3}$. That these surfaces are triply pointed just indicates that they are surfaces together with a choice of three distinct points on the surface; that these surfaces are generally pointed indicates that the point a is in a natural sense a general point of the surface, a point other than a Weierstrass point, and that the three points have distinct images on $\mathbb{P}^1$.

Corollary 10.7 *If $M_{a;b,c} \in \mathfrak{M}_{g,3}$ is a triply generally pointed Riemann surface there is a uniquely determined holomorphic mapping*

$$\phi_{a;b,c} : M_{a;b,c} \longrightarrow \mathbb{P}^1 \tag{10.10}$$

of degree $g+1$ that is fully ramified at the point $a \in M_{a;b,c}$ and for which $\phi_{a;b,c}(a) = \infty$, $\phi_{a;b,c}(b) = 0$ and $\phi_{a;b,c}(c) = 1$.

Proof If $M_{a;b,c} \in \mathfrak{M}_{g,3}$ then a is not a Weierstrass point of M so by the preceding Corollary 10.6 there is a holomorphic mapping $\phi_a : M_{a;b,c} \longrightarrow \mathbb{P}^1$ of degree $g+1$ that is fully ramified at the point a and for which $\phi(a) = \infty \in \mathbb{P}^1$; and that mapping is determined uniquely up to holomorphic automorphisms $T : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ for which $T(\infty) = \infty$. Since by the definition of a triply generally pointed Riemann surface the images $\phi_a(a), \phi_a(b), \phi_a(c)$ are three distinct points of $\mathbb{P}^1$, and since as discussed on page 226 there is a uniquely determined holomorphic automorphism $T : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ for which $T\phi_a(a) = \infty$, $T\phi_a(b) = 0$ and $T\phi_a(c) = 1$, it follows that the composition $T \cdot \phi_a : M \longrightarrow \mathbb{P}^1$ is the unique holomorphic mapping as asserted in the statement of the corollary, which suffices for the proof.

The branch divisor of the mapping $\phi_{a;b,c}$ of the preceding Corollary 10.7 has the special form

$$\mathfrak{b}(\phi_{a;b,c}) = g \cdot \infty + \mathfrak{b}(a, b, c) \tag{10.11}$$

where the residual part $\mathfrak{b}(a, b, c)$ of that divisor actually is a divisor of degree $3g$ in the finite part $\mathbb{C} \subset \mathbb{P}^1$ of the projective plane so it can be viewed as a point $\mathfrak{b}(a, b, c) \in \mathbb{C}^{(3g)}$; this point is called the *Hurwitz parameter* of the Riemann surface $M_{a;b,c}$. The divisor $\mathfrak{b}(a, b, c) = \sum_{i=1}^{n} \nu_i \cdot z_i$ automatically satisfies the additional condition that $\nu_i \le g$ for each index i, so the Hurwitz parameters really lie in the proper subset

$$\mathcal{P}_{3g} = \Big\{ \mathfrak{d} = \sum_i \nu_i \cdot z_i \in \mathbb{C}^{(3g)} \Bigm| \nu_i \le g \Big\} \subset \mathbb{C}^{(3g)}. \tag{10.12}$$

It is easy to see that the complement $\mathbb{C}^{(3g)} \sim \mathcal{P}_{3g}$ is a proper holomophic subvariety of the complex manifold $\mathbb{C}^{(3g)}$, so that $\mathcal{P}_{3g}$ is a dense open subset of the complex manifold $\mathbb{C}^{(3g)}$.

All Riemann surfaces of genus $g > 1$ appear among the triply generally pointed Riemann surfaces $\mathfrak{M}_{g,3}$, so all surfaces of genus $g > 1$ can be represented as branched coverings of degree $g + 1$ over the Riemann sphere that are fully ramified at any chosen point $a \in M$ that is not a Weierstrass point; and the branched covering can be specifed uniquely by the choice of two other points $b, c \in M$. At least for some Weierstrass points $a \in M$ the surface can be represented as a branched covering of degree $g + 1$ over the Riemann sphere that is fully ramified at the point $a \in M$, but with varying additional conditions to ensure the uniqueness of the representation. A *specially pointed* Riemann surface of genus $g > 1$ of *type* i is a Riemann surface of genus $g > 1$ together with the choice of a Weierstrass point $a \in M$ for which $r_i(a) = g + 1$; since $r_1(a) < g + 1$ for any Weierstrass point $a \in M$ by Lemma 9.12 it follows that the the type must be an integer $i > 1$. Not all Riemann surface of genus $g > 1$ have Weierstrass points of this special type; that is a topic that will be discussed later in connection with more general holomorphic mappings between Riemann surfaces. However if M is a specially pointed surface and if $r_i(a) = g + 1$ for a Weierstrass point $a \in M$ then by Lemma 10.4 as in equation (10.7) the space of meromorphic functions on M with a pole of order $g + 1$ at the point a and no other singularities has dimension $i + 1$; these functions describe representations of M as a branched covering of degree $g + 1$ over $\mathbb{P}^1$ fully ramified at the point a with image $\infty \in \mathbb{P}^1$. However to specify a unique such representation, as in Corollary10.6, it is necessary to specify $i + 1$ points of M and their images on $\mathbb{P}^1$. This is not a topic that merits much further discussion just here though.

10.3 Monodromy

If a compact Riemann surface M is represented as a holomorphic branched covering $\pi : M \longrightarrow \mathbb{P}^1$ of degree d over the Riemann sphere $\mathbb{P}^1$ it is possible to describe that surface in a convenient and useful way in terms of its branch divisor $\mathfrak{d}$. The branch locus $B = |\mathfrak{d}| \subset \mathbb{P}^1$ is a set of $n+1$ distinct points

$$B = \{b_0, b_1, \ldots, b_n\} \subset \mathbb{P}^1. \tag{10.13}$$

The branched covering π restricts to an unbranched covering $\pi_0 : M_0 \longrightarrow \mathbb{P}^1_0$ of degree d over the complememt $\mathbb{P}^1_0 = \mathbb{P}^1 \sim B$ of the branch locus, where $M_0 = M \sim \pi^{-1}(B)$. For any base point $p \in \mathbb{P}^1_0$ choose $n+1$ circles γ_i in $\mathbb{P}^1_0$ that bound disjoint discs centered at the points b_i and are oriented in the clockwise direction; and choose $n+1$ paths λ_i from the base point $p \in \mathbb{P}^1_0$ to the circle γ_i, then around the circle γ_i, and finally back to p along the original segment of that path, as sketched in the accompanying Figure 10.1. The paths λ_i represent elements of the fundamental group $\pi_1(\mathbb{P}^1_0, p)$ of the punctured Riemann sphere $\mathbb{P}^1_0$ at the base point p; these elements of the fundamental group also will be denoted by λ_i. If $\lambda_i \cdot \lambda_j$ denotes the path obtained by traversing first λ_i then λ_j it is clear that the path $\lambda_0 \cdot \lambda_1 \cdots \lambda_{n-1} \cdot \lambda_n$ is contractible in $\mathbb{P}^1_0$, since it can be deformed to a simple closed curve which contains the points b_i in its interior hence it can be shrunk to a point in its exterior; consequently

$$\lambda_0 \cdot \lambda_1 \cdot \ \cdots \ \cdot \lambda_{n-1} \cdot \lambda_n = 1 \quad \text{in the group} \quad \pi_1(\mathbb{P}^1_0, p). \tag{10.14}$$

If X is the union of the paths λ_i for $1 \le i \le n$, excluding the path λ_0, the complement of X can be deformed continuously to X by expanding the holes $b_1, \ldots, b_n$ to the full interiors of the circles $\gamma_1, \ldots, \gamma_n$ and by expanding the hole b_0 to the full exterior of X. It follows that the fundamental group $\pi_1(\mathbb{P}^1_0, p)$ is isomorphic to the fundamental group of the subset X, which is easily seen to be a free group on the homotopy classes $\lambda_1, \ldots, \lambda_n$. Consequently there are no further relations among the homotopy classes $\lambda_0, \lambda_1, \ldots, \lambda_n$ other than (10.14), which represents the homotopy class λ_0 in terms of the free generators $\lambda_1, \ldots, \lambda_n$ of the fundamental group $\pi_1(\mathbb{P}^1_0, p)$.

The inverse image $\pi^{-1}(p) \subset M_0$ consists of d points in M_0, where d is the degree of the mapping π. A path $\lambda \subset \mathbb{P}^1_0$ from the base point p and back to p lifts to a unique path $\tilde{\lambda} \subset M_0$ beginning at any chosen point $\tilde{p} \in \pi^{-1}(p)$ and returning to another point in $\pi^{-1}(p)$, which will be denoted by $\tilde{\sigma}_\lambda(\tilde{p})$ and may or may not be the original point $\tilde{p}$. The mapping that associates to each point $\tilde{p} \in \pi^{-1}(p)$ the point $\tilde{\sigma}_\lambda(\tilde{p}) \in \pi^{-1}(p)$ thus is a permutation $\tilde{\sigma}_\lambda$ of the set of d points $\pi^{-1}(p)$, an element of the group $\mathfrak{S}_{\pi^{-1}(p)}$ of permuations of the set $\pi^{-1}(p)$. It is clear that this permutation depends only on the homotopy class of the path λ; so the mapping that associates to any homotopy class $\lambda \in \pi_1(\mathbb{P}^1_0, p)$ the permutation $\tilde{\sigma}_\lambda$ can be viewed as a mapping

$$\tilde{\sigma} : \pi_1(\mathbb{P}^1_0, p) \longrightarrow \mathfrak{S}_{\pi^{-1}(p)}. \tag{10.15}$$

a

Figure 10.1: Branch point diagram for the covering mapping $\pi : M \longrightarrow \mathbb{P}^1$

The permutation associated to the path $\lambda' \cdot \lambda''$, the path defined by proceeding first along λ' and then along λ'', is the permutation $\tilde{\sigma}_{\lambda'}$ followed by the permutation $\tilde{\sigma}_{\lambda''}$, the product permutation $\tilde{\sigma}_{\lambda''} \cdot \tilde{\sigma}_{\lambda'}$ in the group $\mathfrak{S}_{\pi^{-1}(p)}$; hence the mapping (10.15) is a group homomorphism. If the points of $\pi^{-1}(p)$ are numbered $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_d$ a permutation $\tilde{\sigma}_\lambda$ in the group $\mathfrak{S}_{\pi^{-1}(p)}$ can be identified with the permutation σ_λ in the symmetric group $\mathfrak{S}_d$ of permutations of the set of integers $\{1, 2, \ldots, d\}$ for which $\tilde{\sigma}_\lambda(\tilde{p}_i) = \tilde{p}_{\sigma_\lambda(i)}$. The composition of the homomorphism (10.15) with this identification can be used to interpret the homomorphism $\tilde{\sigma}$ of (10.15) as a homomorphism

$$\sigma : \pi_1(\mathbb{P}^1_0, p) \longrightarrow \mathfrak{S}_d; \tag{10.16}$$

this homomorphism is called the *monodromy homomorphism* or just the *monodromy* of the covering space[3]. The numbering of the points in $\psi^{-1}(p)$ is possible in $d!$ different ways, and this must be kept in mind in counting the numbers of covering spaces. Since λ_i for $1 \le i \le n$ are free generators of the fundamental group a homomorphism (10.16) is determined fully by the permutations $\sigma(\lambda_i)$ for $1 \le i \le n$; and any choice of permutations $\sigma(\lambda_i)$ for $1 \le i \le n$ determines a homomorphism (10.16). For some purposes it is more natural to describe monodromies by specifying the permutations $\sigma(\lambda_i) \in \mathfrak{S}_d$ for all indices $0 \le i \le d$; and then a set of permuations $\sigma(\lambda_i)$ describes a group homomorphism if and only if a

$$\sigma(\lambda_0) \cdot \sigma(\lambda_1) \cdot \cdots \cdot \sigma(\lambda_{n-1}) \cdot \sigma(\lambda_n) = 1 \text{ in } \mathfrak{S}_d, \tag{10.17}$$

corresponding to the relation (10.14) expressing λ_0 in terms of the free generators $\lambda_1, \ldots, \lambda_n$ of the fundamental group.

[3] The classical discussion of branched coverings of two-dimensional manifolds through monodromy can be found in the book by H. Seifert and W. Threlfall, *Lehrbüch der Topologie*; some other books that cover such topics are those by W. S. Massey, *Algebraic Topology: An Introduction*, and S. S. Cairns, *Introductory Topology*, for instance.

Theorem 10.8 *Any homomorphism $\sigma : \pi_1(\mathbb{P}^1_0, p) \longrightarrow \mathfrak{S}_d$ for a base point p in the complement $\mathbb{P}^1_0 = \mathbb{P}^1 \sim B$ of a finite subset $B \subset \mathbb{P}^1$ is the monodromy of a branched covering $\pi : M_0 \longrightarrow \mathbb{P}^1_0$ of degree d with the branch locus B.*

Proof: Let $\phi : \widehat{\mathbb{P}^1_0} \longrightarrow \mathbb{P}^1_0$ be the universal covering space of $\mathbb{P}^1_0$ and G be the associated covering translation group. For any point $\widehat{p} \in \phi^{-1}(p)$ there is the natural isomorphism $\theta_{\widehat{p}} : G \longrightarrow \pi_1(\mathbb{P}^1_0, p)$ that associates to any covering translation $T \in G$ the homotopy class $\lambda_T \in \pi_1(\mathbb{P}^1_0, p)$ of the image $\phi(\widehat{\lambda_T})$ under the covering projection ϕ of any path $\widehat{\lambda_T} \subset \widehat{\mathbb{P}^1_0}$ from $\widehat{p}$ to $T\widehat{p}$. If $\widehat{\lambda_S}$ is a path from $\widehat{p}$ to $S\widehat{p}$ in $\widehat{\mathbb{P}^1_0}$ and $\widehat{\lambda_T}$ is a path from $\widehat{p}$ to $T\widehat{p}$ in $\widehat{\mathbb{P}^1_0}$ the path $\widehat{\lambda_S} \cdot S\widehat{\lambda_T}$, the path $\widehat{\lambda_S}$ followed by the path $S\widehat{\lambda_T}$, is a path from $\widehat{P}$ to $ST\widehat{p}$; consequently $\theta_{\widehat{p}}(ST) = \phi(\widehat{\lambda_S} \cdot S\widehat{\lambda_T}) = \theta_{\widehat{p}}(S) \cdot \theta_{\widehat{p}}(T)$ so $\theta_{\widehat{p}}$ is a group homomorphism. It is a standard result in topology that the homomorphism $\theta_{\hat{p}}$ thus defined actually is an isomorphism. For any homomorphism $\sigma : \pi_1(\mathbb{P}^1_0, p) \longrightarrow \mathfrak{S}_d$ the composite homomorphism $\sigma \circ \theta_{\widehat{p}} : G \longrightarrow \mathfrak{S}_d$ also can be viewed as the monodromy homomorphism, since $(\sigma \circ \theta_{\widehat{p}})(\widehat{\lambda_T})$ is the permutation of $\pi^{-1}(p)$ obtained by lifting the path $\phi(\widehat{\lambda_T})$ to the universal covering space $\widehat{\mathbb{P}^1_0}$ as the path $\widehat{\lambda_T}$. Next let $\widehat{M_i}$ for $1 \le i \le d$ be d separate copies of the Riemann surface $\widehat{\mathbb{P}^1_0}$ and let $\widehat{M} = \bigcup_{i=1}^d \widehat{M_i}$ be their disjoint union, which is thus a noncompact Riemann surface. The identity mapping $\widehat{M_i} \longrightarrow \widehat{\mathbb{P}^1_0}$ for each copy $\widehat{M_i}$ defines a mapping $\rho : \widehat{M} \longrightarrow \widehat{\mathbb{P}^1_0}$ that is a d-sheeted unbranched covering. Any permutation $\sigma \in \mathfrak{S}_d$ acts as a permutation of the sheets of this covering by the natural mapping from the manifold $\widehat{M_i}$ to the manifold $\widehat{M}_{\sigma(i)}$; and clearly $\rho \circ \sigma = \rho$ for this action. The action of a covering translation $T \in G$ on $\widehat{\mathbb{P}^1_0}$ lifts to an action on each of the copies $\widehat{M_i}$, hence to an action on $\widehat{M}$ that commutes with the mapping $\rho : \widehat{M} \longrightarrow \widehat{\mathbb{P}^1_0}$; the composition $T_\sigma = (\sigma \circ \theta_{\widehat{p}}) \circ T$, the action of T on $\widehat{M}$ followed by the permutation $\sigma \circ \theta_{\widehat{p}}$ of the sheets of $\widehat{M}$, then is a holomorphic mapping $T_\sigma : \widehat{M} \longrightarrow \widehat{M}$ that commutes with the action of G on $\widehat{\mathbb{P}^1_0}$ as in the commuative diagram

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{T_\sigma} & \widehat{M} \\ \rho \big\downarrow & & \rho \big\downarrow \\ \widehat{\mathbb{P}^1_0} & \xrightarrow{T} & \widehat{\mathbb{P}^1_0}. \end{array} \tag{10.18}$$

The holomorphic mapping ρ then clearly induces an unbranched covering mapping $\rho : M \longrightarrow \mathbb{P}^1_0$ between the quotient Riemann surfaces $M = \widehat{M}/G$ and $\mathbb{P}^1_0 = \widetilde{\mathbb{P}^1_0}/G$, and it is clear from the construction that σ is the monodromy of this covering. That suffices for the proof.

The construction in the preceding Theorem 10.8 identified the symmetric group $\mathfrak{S}_d$ as a group of permutations of the sheets in $\widehat{M}$ by chosing a numbering of the sheets $\widehat{M_i}$ and setting $\sigma(\widehat{M_i}) = \widehat{M}_{\sigma(i)}$. A renumbering of the sheets changes this identification by an inner automorphism of the symmetric group $\mathfrak{S}_d$ but yields the same covering mapping; thus any two homomorphisms (10.15) that differ by an inner automorphism of the symmetric group $\mathfrak{S}_d$ yield the same branched covering. The relation between unbranched coverings of $\mathbb{P}^1_0 = (\mathbb{P}^1 \sim B)$ and homomorphisms $\sigma : \pi_1(\mathbb{P}^1_0, p) \longrightarrow \mathfrak{S}_d$ generated by the monodromy construction consequently is a bijective corresondence between unbranched coverings and equivalence classes of homomorphisms σ, where two homomorphisms are equivalent if and only if when they differ by an inner automorphism of the symmetric group $\mathfrak{S}_d$.

The unbranched covering $\pi_0 : M_0 \longrightarrow \mathbb{P}^1_0$ is derived from the branched covering $\pi : M \longrightarrow \mathbb{P}^1$, and the monodromy of π_0 explicitly describes the branching of the mapping π. Indeed for any branch point $b_i \in B$ the restriction of the covering mapping π to the inverse image $\pi^{-1}(\gamma_i)$ of the circle $\gamma_i \subset \mathbb{P}^1_0$ is an unbranched covering of degree d of γ_i, the monodromy of which is the cyclic group generated by the permutation $\sigma(\lambda_i)$. The connected components of this unbranched covering of γ_i clearly are the orbits of the action of the permutation $\sigma(\lambda_i)$. If the number of elements in the j-th orbit of the permutation $\sigma(\lambda_i)$ is denoted by d^i_j then d^i_j is the degree of the restriction of the mapping π to the associated component of $\pi^{-1}(\gamma_i)$ so $\sum_j d^i_j = d$ for each index i. A connected component of d^i_j sheets of this covering of γ_i can be completed to a branched covering of the interior of the circle with a single ramification point of order d^i_j over b_i; this branched covering is homeomorphic to the branched acovering described by the holomorphic function $z^{d^i_j}$ at the origin. When this completion is applied to all components of the coverings of all the circles γ_i the result is an extension of the unbranched covering M_0 of $\mathbb{P}^1_0$ to the branched covering M of the Riemann sphere $\mathbb{P}^1$ branched at the points $B \subset \mathbb{P}^1$; and the divisor of this branched covering thus is $\mathfrak{d} = \sum_{i=1}^n \sum_j d^i_j \cdot b_i$.

10.4 The Hurwitz Moduli Space

INCOMPLETE VERSION – TO BE REVISED

For any triply generally pointed Riemann surface $M_{a;b,c}$ of genus $g > 1$ there is the branched covering mapping $\phi_{a;b,c} : M_{a;b,c} \longrightarrow \mathbb{P}^1$ of degree $g+1$ used in Corollary 10.7 to define the Hurwitz parameter $\mathfrak{b}(a,b,c) \in \mathbb{C}^{(3g)}$ of that surface; this mapping exhibits the surface as a branched covering of $\mathbb{P}^1$ of degree $g+1$ with the branch divisor

$$\mathfrak{d} = g \cdot \infty + \mathfrak{b}(a,b,c). \tag{10.19}$$

The surface $M_{a;b,c}$ can be reconstructed from the monodromy of the branched covering (10.19). To examine branched coverings of this form but for general divisors $\mathfrak{b} \in \mathbb{C}^{(3g)}$ it is convenient to begin with the special case in which $\mathfrak{b}$ is a divisor in the set of divisors

$$\mathcal{S}_g = \left\{ b_1 + b_2 + \ldots + b_{3g-1} + b_{3g} \in \mathbb{C}^{(3g)} \;\middle|\; b_i \neq b_j \ \text{ for } \ i \neq j \right\} \tag{10.20}$$

consisting of positive divisors formed by $3g$ distinct points in $\mathbb{C} \subset \mathbb{P}^1$. Since any branched covering with a branch divisor of this form is fully branched over ∞ the covering space must be a connected surface; and since the genus of that covering space is determined by the degree and branch order of the branched covering through the Riemann-Hurwitz Formula, Theorem 9.1, the covering space is a Riemann surface of genus g. The possible monodromies for such branched coverings, if there are any, can be described as in (10.14) by collections $\boldsymbol{\sigma} \in \mathfrak{S}_{g+1}^{3g+1}$ of $3g+1$ permutations $\sigma_i \in \mathfrak{S}_{g+1}$ belonging to the set

$$\widehat{\mathcal{T}}_g = \left\{ \boldsymbol{\sigma} = \{\sigma_0, \sigma_1, \ldots, \sigma_{3g}\} \;\middle|\; \sigma_i \in \mathfrak{S}_{g+1} \right\} \subset \mathfrak{S}_{g+1}^{3g+1} \quad \text{where}$$
$$\sigma_{3g} \cdot \sigma_{3g-1} \cdot \quad \cdots \quad \cdot \sigma_1 \cdot \sigma_0 = 1$$
$$\sigma_0 = (1, 2, \ldots, g+1),$$
$$\sigma_i \;\text{ are transpositions for }\; 1 \le i \le n; \tag{10.21}$$

here σ_0 is the permutation associated to the point $b_0 = \infty$ while σ_i is the permutation associated to the point b_i for $1 \le i \le 3g$. As noted in the earlier discussion, sets of permutations that are equivalent under the equivalence relation

$$\boldsymbol{\sigma}_1 \sim \boldsymbol{\sigma}_2 \;\text{ if }\; \boldsymbol{\sigma}_1 = \tau \boldsymbol{\sigma}_2 \tau^{-1} \;\text{ for a permutation }\; \tau \in \mathfrak{S}_{g+1} \tag{10.22}$$

determine the same branched coverings, but coverings described by different numberings of the sheets in the covering space; so the monodromy really is described by elements in the quotient space $\mathcal{T}_g = \widehat{\mathcal{T}}_g$ under this equivalence relation. The set $\widehat{\mathcal{T}}_g$ is actually a nonempty set, which can be seen by exhibiting an element in that set. Indeed by direct calculation

$$\pi_g = (g+1, g)(g, g-1) \cdots (4,3)(3,2)(2,1)(1,2,3,\cdots,g,g+1)$$

is the identity permutation, where the product $\sigma_2 \cdot \sigma_1$ of two permuations is the permutation σ_1 followed by the permutation σ_2. Since $\tau_j^2 = 1$ for any transpositions τ_j the product $\left(\prod_{j=1}^{g}(\tau_j \cdot \tau_j)\right) \cdot \pi_g$ then is also the identity permutation; it is a representation of the identity as the product of the cyclic permutation $\sigma_0 = (1,2,3,\cdots,g,g+1)$ and $3g$ transpositions, as in (10.21), so the permutations in this product represent a nontrivial element of the set $\widehat{\mathcal{T}}_g$. The number of elements in $\widehat{\mathcal{T}}_g$ clearly is the number of ways in which the cyclic permutation $\sigma_0 \in \mathfrak{S}_{g+1}$ can be expressed as a product of $3g$ transpositions; that number divided by $(g+1)!$, which is the number of elements in the set $\mathcal{T}_g$, will be denoted by $\mathfrak{h}_g$ and will be called the *Hurwitz number*[4]. The explicit value of $\mathfrak{h}_g$ will not be needed in the discussion here.

[4]The number $\mathfrak{h}_g$ is actually a special case of the general Hurwitz numbers, which count the number of ways in which permutations can be factored in specific manners. See the papers by A. Hurwitz "Ueber Riemann'sche Flächen mit gegebenen Verzweigungspunkten", in Math. Ann. 39 (1891), pp. 1-60, and "Ueber die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten", in Math. Ann. 55 (1901), pp. 53 - 66. There has been a good deal of work on this problem since Hurwitz's papers. See for instance the paper by A. Okounkov and R. Pandharipande "Gromov-Witten theory, Hurwitz theory, and completed cycles', in Annals of Math. 163, (1996), pp.517-606, and the paper by T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein "Hurwitz numbers and intersections on moduli spaces of curves", in Invent. Math. 146, (2001), pp.297-327. These results provide formulas for $\mathfrak{h}_g$ but not simple expressions.

The set $\mathcal{S}_g$ has the natural structure of a complex manifold of dimension $3g$, as an open subset of the connected complex manifold $\mathcal{C}^{(3g)}$; and since the complement $\mathbb{C}^{(3g)} \sim \mathcal{S}_g$ is a holomorphic subvariety of $\mathbb{C}^{(3g)}$ it follows that the complex manifold $\mathcal{S}_g$ is connected. The universal covering space $\widetilde{\mathcal{S}}_g$ of the manifold $\mathcal{S}_g$ inherits a complex structure from that of $\mathcal{S}_g$ so it too is a connected complex manifold; as is customary it can be identified with the space of homotopy classes of paths in $\mathcal{S}_g$ beginning at a base point $\widetilde{\mathfrak{b}}_0 \in \widetilde{\mathcal{S}}_g$. The product space

$$\widetilde{\mathfrak{H}}_{g,0} = \left\{ (\widetilde{\mathfrak{b}},\ \boldsymbol{\sigma}) \ \middle|\ \widetilde{\mathfrak{b}} \in \widetilde{\mathcal{S}}_g,\ \boldsymbol{\sigma} \in \mathcal{T}_g \right\} \tag{10.23}$$

where $\mathcal{T}_g$ has the discrete topology is another complex manifold of dimension $3g$, really a disjoint union of $\mathfrak{h}_g(g+1)!$ copies of the manifold $\widetilde{\mathcal{S}}g$. A monodromy $\boldsymbol{\sigma}_0$ at the base point $\widetilde{\mathfrak{b}}_0 = (b_{1,0}, b_{2,0}, \ldots, b_{3g,0})$ is described by $3g+1$ permutations $\sigma_{i,0} \in \mathfrak{S}_{g+1}$ associated to the points $b_{i,0}$. If $\widetilde{\mathfrak{b}}_t$ is a path in $\widetilde{\mathcal{S}}_g$ beginning at the base point $\mathfrak{b}_0$ when $t = 0$ then there is a natural continuation $\boldsymbol{\sigma}_t$ of the monodromy $\boldsymbol{\sigma}_0$ along that path determined by associating to the points $b_{i,t}$ the permutation $\sigma_{i,t} = \sigma_{i,0}$. If the path $\widetilde{\mathfrak{b}}_t$ returns to the base point $\widetilde{\mathfrak{b}}_0$ when $t = t_1$ then since the entire path can be deformed to the constant path in the simply connected manifold $\widetilde{\mathcal{S}}_g$ it follows that the permutations $\sigma_{i,t}$ return to their initial values $\sigma_{i,0}$. However if $\widetilde{\mathfrak{b}}_{t_1}$ represents the same point of $\mathcal{S}_g$ as $\widetilde{\mathfrak{b}}_0$ it is not necessarily the case that $\sigma_{i,t_1} = \sigma_{t,0}$, but only that $\sigma_{i,t_1} \sim \sigma_{t,0}$ since although the covering space is the same the numbering of the points of the covering space may change. For example if the point $b_{1,t}$ moves once around the point $b_{2,0}$ as t ranges from t_0 to t_1 but all the other points remain fixed then the effects of the permutation $\boldsymbol{\sigma}_{t_1}$ are those of the permutation $\tau_1\boldsymbol{\sigma}_0\tau_1^{-1}$.

, which is called the *Hurwitz special moduli space* of Riemann surfaces of genus g, special since the branch divisors are of the form (10.19) for the special case in which $\mathfrak{b} \in \mathcal{S}_g$. To each point $(\mathfrak{b},\ \boldsymbol{\sigma}) \in \mathfrak{H}_{g,0}$ there can be associated the Riemann surface $M_{\mathfrak{b},\ \boldsymbol{\sigma}}$ that is a branched covering

$$\phi_{\mathfrak{b},\ \boldsymbol{\sigma}} : M_{\mathfrak{b},\boldsymbol{\sigma}} \longrightarrow \mathbb{P}^1 \tag{10.24}$$

of degree $g+1$ over the Riemann sphere having the branch divisor $g \cdot \infty + \mathfrak{b}$ and the monodromy $\boldsymbol{\sigma}$. If $a = \phi^{-1}_{\mathfrak{b},\boldsymbol{\sigma}}(\infty)$ is not a Weierstrass point of $M_{\mathfrak{b},\boldsymbol{\sigma}}$ while $b = \phi^{-1}_{\mathfrak{b},\boldsymbol{\sigma}}(0)$ and $c = \phi^{-1}_{\mathfrak{b},\boldsymbol{\sigma}}(1)$ then $M_{\mathfrak{b},\boldsymbol{\sigma}} = M_{a;b,c} \in \mathfrak{M}_{g,3}$ and $\mathfrak{b}$ is its Hurwitz parameter. On the other hand if $a = \phi^{-1}_{\mathfrak{b},\boldsymbol{\sigma}}(\infty)$ is a Weierstrass point of the Riemann surface $M_{\mathfrak{b},\boldsymbol{\sigma}}$ then $g+1 > r_1(a)$ by Lemma 9.12, and since the mapping $\phi_{\boldsymbol{\sigma},\mathfrak{b}}$ is of degree $g+1$ and is fully ramified at the point a it is also the case that $g+1 = r_i(a)$ for some $i > 1$ by Theorem 10.3; thus the point a is a somewhat special Weierstrass point of the Riemann surface $M_{\mathfrak{b},\boldsymbol{\sigma}}$. Altogether the Riemann surfaces parametrized by points in $\mathfrak{H}_{g,0}$ are all the triply generally pointed Riemann surfaces $M \in \mathfrak{M}_{g,3}$ with Hurwitz parameters in $\mathcal{S}_g$, and in addition those Riemann surfaces of genus g with Weierstrass points a for which

$g+1 = r_i(a)$ for some $i > 1$, called the *exceptional* Riemann surfaces of genus g. In the first case some of the Riemann surfaces $M_{\mathfrak{b},\boldsymbol{\sigma}}$ are holomorphically equivalent but are distinguished by the points a, b, c of the marking; in the second case some of the surfaces are holomorphically equivalent but can be distinguished by a choice of $i+1$ rather than just of 3 points for a marking, in view of (10.7). The exceptional Riemann surfaces will be examined later in the discussion of more general mappings between Riemann surfaces.

This representation of Riemann surfaces can be extended to more general cases of the Hurwitz parameters through continuity. A Hurwitz parameter $\mathfrak{b} \in \mathcal{S}_g$ is a divisor $\mathfrak{b} = \sum_{i=1}^{3g}$ formed by $3g$ distinct points $b_i \in \mathbb{C}$, and it can be modified continuously by m0oving the points b_i in $\mathbb{C}$ so long as they remain distinct. It is also possible to consider the limits as some of the points are merged. For example suppose that two points b_j and b_{j+1} in Figure**??**The number of elements in the nonempty set $\mathcal{T}_g$ of course is just the number of ways in which the cyclic permutation $\sigma_0 \in \mathfrak{S}_{g+1}$ can be expressed as a product of $3g$ transpositions, a number that will be denoted by $\mathfrak{h}_g$. The explicit determination of this number is a rather difficult matter, which was initially explored by Hurwitz so the number $\mathfrak{h}_g$ is called

are near and are to be merged. Choose circles around these two points and touching at a single point q, as sketched in Figure 10.4. The path λ_j can be taken to proceed from the point p to the point q, then along the circles around b_j back to the point q and then back to the point p, and corresondingly for the path λ_k. In the product path $\lambda_j\lambda_{j+1}$ the segment of the path λ_j proceeding from q to p cancels the segment of the path λ_{j+1} from p to q, so the resulting path $\lambda_{j,j+1}$ proceeds from p to q, then follows the circle around b_j then the circle around b_{j+1} then back to p; so that path really amounts to a path $\lambda_{j,j+1}$ from p encircling both b_j and b_{j+1} before proceeding back to p. The permutation $\sigma_{j,j+1}$ associated to the path $\lambda_{j,j+1}$ is just the product $\sigma_{j,j+1} = \sigma_j\sigma_{j+1}$, so the result is to merge the branch points b_j and b_{j+1} to a single point with the associated permutation σ_{jk}. When the product $\sigma_j\sigma_{j+1}$ is replaced by the single permutation $\sigma_{j,j+1}$ the permutations still satisfy (**??**) so the result is still a branched covering. It may be the case that $\sigma_j\sigma_{j+1} = 1$, and the mapping is locally unbranched in the limit; or it may be the case that there are two simply ramified points over the limit point in $\mathbb{P}^1$. Ietone fewer branch points. The process can be reversed, replacing a single branch point b_{jk} with an associated permutation σ_{jk} to a pair of branch points b_j and b_k with associated permutations σ_j and σ_k so long as $\sigma_{jk} = \sigma_j\sigma_k$ in $\mathfrak{S}_d$.

Figure 10.2: Merger of branch points for a branched covering

The number of elements in the nonempty set $\mathcal{T}_g$ of course is just the number of ways in which the cyclic permutation $\sigma_0 \in \mathfrak{S}_{g+1}$ can be expressed as a product of $3g$ transpositions, a number that will be denoted by $\mathfrak{h}_g$. The explicit determination of this number is a rather difficult matter, which was initially explored by Hurwitz so the number $\mathfrak{h}_g$ is called

Chapter 11

The Brill-Noether Diagram

PRELIMINARY VERSION

11.1 Some Examples

The Riemann-Roch Theorem in the form of Theorem‘2.23 asserts that

$$\gamma(\lambda) = \gamma(\kappa\lambda^{-1}) + c(\lambda) + 1 - g \tag{11.1}$$

for any holomorphic line bundle λ over a compact Riemann surface of genus g. If $c(\lambda) < 0$ of course $\gamma(\lambda) = 0$ by Corollary 1.3. On the other hand if $c(\lambda) > 2g - 2 = c(\kappa)$ then $c(\kappa\lambda^{-1}) < 0$ so $\gamma(\kappa\lambda^{-1}) = 0$ and by (11.1)

$$\gamma(\lambda) = c(\lambda) + 1 - g \quad \text{if} \quad c(\lambda) > 2g - 2. \tag{11.2}$$

Thus $\gamma(\lambda)$ is fully determined as a function of $c(\lambda)$ whenever $c(\lambda) < 0$ or $c(\lambda) > 2g - 2$. In the intermediate or interesting range $0 \le c(\lambda) \le 2g - 2$ the dimension $\gamma(\lambda)$ generally is not uniquely determined by the characteristic class $c(\lambda)$, although a good deal can be said about the function $\gamma(\lambda)$ nonetheless, as will be illustrated as the discussion proceeds. Riemann surfaces of small genus are simple examples, since the interesting range is quite limited and something can be said easily about line bundles in that range.

Theorem 11.1 (i) *If M is a is a unique compact Riemann surface of genus $g = 0$, the Riemann sphere $\mathbb{P}^1$, for which $c(\kappa) = -2$..*
(ii) *For any integer n there is a unique holomorphic line bundle λ on $\mathbb{P}^1$ for which $c(\lambda) = n$, and*

$$\gamma(\lambda) = \begin{cases} c(\lambda) + 1 & \textit{if} \quad c(\lambda) \ge 0, \\ 0 & \textit{if} \quad c(\lambda) < 0. \end{cases} \tag{11.3}$$

(ii) *All holomorphic line bundles λ on $\mathbb{P}^1$ for which $c(\lambda) \ge 0$ are base-point-free, so the Lüroth semigroup of $\mathbb{P}^1$ is*

$$\mathcal{L}(\mathbb{P}^1) = \{\, n \in \mathbb{Z} \,|\, n \ge 0 \,\}. \tag{11.4}$$

(iii) *For any $d > 0$ a basis for the space of polynomials of degree d are the coefficients of a nonsingular biholomorphic mapping F from $\mathbb{P}^1$ to a connected one-dimensional complex submanifold $F(\mathbb{P}^1) \subset \mathbb{P}^d$.*

Proof: (i) If M is a compact Riemann surface of genus $g = 0$ then $c(\kappa) = -2$ by the Canonical Bundle Theorem, Theorem 2.24. Of course $\gamma(\lambda) = 0$ if $c(\lambda) < 0$; and if $c(\lambda) \ge 0$ then $c(\kappa\lambda^{-1}) = -2 - c(\lambda) < 0$ so $\gamma(\kappa\lambda^{-1}) = 0$ and substituting this into (11.1) yields the remainder of (11.3). In particular then (11.3) shows that $\gamma(\zeta_a) = 2$, so by Theorem 2.4 the Riemann surface is the Riemann sphere $M = \mathbb{P}^1$.
(ii) Since $2g = 0$ it follows from Theorem 2.28 (i) that all holomorphic line bundles λ on $\mathbb{P}^1$ for which $c(\lambda) \ge 0$ are base-point-free, while those bundles λ for which $\gamma(\lambda) < 0$ of course are not; hence the Lüroth semigroup of $\mathbb{P}^1$ has the form (11.4).
(iii) If λ is a holomorphic line bundle on $\mathbb{P}^1$ with $c(\lambda) = d > 0$ then $\lambda = \zeta_a^d$ for a point $a \in \mathbb{P}^1$ and the cross-sections $\Gamma\big(\mathbb{P}^1, \mathcal{O}(\zeta_a^d)\big)$ can be identified with polynomials of degree d in a coordinate system on $\mathbb{P}^1$ by Theorem 2.1 (iii). The holomorphic mapping described by basis for $\Gamma\big(\mathbb{P}^1, \mathcal{O}(\zeta_a^d)\big)$ then is a biholomorphic mapping F from $\mathbb{P}^1$ to its image $F(\mathbb{P}^1) \subset \mathbb{P}^d$ by Theorem 2.17 (iv), and that suffices for the proof.

Theorem 11.2 (i) *If M is a compact Riemann surface of genus $g = 1$ then $c(\kappa) = 0$ and*

$$\gamma(\lambda) = \begin{cases} c(\lambda) & \text{if } \ c(\lambda) > 0, \\ 1 & \text{if } \ c(\lambda) = 0 \ \text{ and } \ \lambda = 1 \\ 0 & \text{if } \ c(\lambda) = 0 \ \text{ and } \ \lambda =\neq 1 \\ 0 & \text{if } \ c(\lambda) < 0, \end{cases} \tag{11.5}$$

where $\lambda = 1$ indicates that λ is the identity bundle. (ii *Only the identity bundle and holomorphic line bundles λ for which $c(\lambda) \ge 2$ are base-point-free, hence the Lüroth semigroup for this surface is*

$$\mathcal{L}(M) = \Big\{ n \in \mathbb{Z} \,\Big|\, n = 0 \ \text{ or } \ n \ge 2 \Big\}. \tag{11.6}$$

(iii) *The set of holomorphic line bundles λ for which $c(\lambda) = n$ can be identified with the points of M.*

Proof: (i) If M is a compact Riemann surface of genus $g = 1$ then $c(\kappa) = 0$ by the Canonical Bundle Theorem, Theorem 2.24. Again if $c(\lambda) < 0$ then $\gamma(\lambda) = 0$. If $c(\lambda) = 0$ and $\gamma(\lambda > 0$ a holomorphic crosss-section of λ can have no zeros so must exhibit the reduction of λ to the identity bundle, for which $\lambda = 1$. If $c(\lambda) > 0$ then $c(\kappa\lambda^{-1}) = -c(\lambda) < 0$ so $\gamma(\kappa\lambda^{-1}) = 0$ and iti follows from (11.1) that $\gamma(\lambda) = c(\lambda)$.
(ii) Bundles λ for which $c(\lambda) < 0$ of course are not base-point-free. If λ is a line bundle for which $c(\lambda) = 0$ then either $\gamma(\lambda) = 0$ and λ is not base-point-free or

$\gamma(\lambda) \neq 0$ and λ is the identity bundle, which is base-point-free. If $c(\lambda) = 1$ then $\gamma(\lambda) = 1$ as in (i) and λ cannot be base-point-free. If $c(\lambda) \geq 2 = 2g$ then λ is base-point-free by Theorem 2.28 (i). Consequently the Lüroth semigroup has the form asserted.
(iii) On a compact Riemann surface of genus $g = 1$ the base-point-free holomorphic line bundles are the identity bundle and all bundles λ for which $c(\lambda) \geq 2$; consequently the Lüroth semigroup of such a surface is

$$\mathcal{L}(M) = \left\{\, n \in \mathbb{Z} \,\middle|\, n = 0 \;\text{ or }\; n \geq 2 \,\right\}. \tag{11.7}$$

Proof: (i) For a compact Riemann surface of genus $g = 0$ it follows from Theorem 2.28 (i) that all holomorphic line bundles λ for which $c(\lambda) \geq 0$ are base-point-free.
(ii) For a compact Riemann surface of genus $g = 0$ it follows from Theorem 2.28 (ii) that all holomorphic line bundles λ for which $c(\lambda) \geq 2$ are base-point-free. If $c(\lambda) = 1$ then it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = 1$; hence λ is a point bundle $\lambda = \zeta_a$ by Theorem 2.4, and therefore λ is not base-point-free. If $c(\lambda) = 0$ then either λ is the identity bundle, which is base-point-free, or $\gamma(\lambda) = 0$ and λ is not base-point-free. That suffices for the proof. (ii) On a compact Riemann surface of genus $g = 1$ the base-point-free holomorphic line bundles are the identity bundle and all bundles λ for which $c(\lambda) \geq 2$; consequently the Lüroth semigroup of such a surface is

$$\mathcal{L}(M) = \left\{\, n \in \mathbb{Z} \,\middle|\, n = 0 \;\text{ or }\; n \geq 2 \,\right\}. \tag{11.8}$$

Proof: (i) For a compact Riemann surface of genus $g = 0$ it follows from Theorem 2.28 (i) that all holomorphic line bundles λ for which $c(\lambda) \geq 0$ are base-point-free.
(ii) For a compact Riemann surface of genus $g = 0$ it follows from Theorem 2.28 (ii) that all holomorphic line bundles λ for which $c(\lambda) \geq 2$ are base-point-free. If $c(\lambda) = 1$ then it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = 1$; hence λ is a point bundle $\lambda = \zeta_a$ by Theorem 2.4, and therefore λ is not base-point-free. If $c(\lambda) = 0$ then either λ is the identity bundle, which is base-point-free, or $\gamma(\lambda) = 0$ and λ is not base-point-free. That suffices for the proof.

11.2 The Brill-Noether Matrix

The Riemann-Roch Theorem in the form given in Corollary 2.26 can be rewritten for the special case of nontrivial positive divisors in terms of a useful auxiliary expression involving holomorphic differential forms. On a compact Riemann surface M of genus $g > 0$ choose a coordinate covering $\{U_\alpha\}$ with local coordinates $\{z_\alpha\}$ and a basis $\omega_i = f_{i\alpha} dz_\alpha$ for the space of holomorphic differential forms for $1 \leq i \leq g$; that there are g differential forms in a basis follows from (2.28). If $\mathfrak{d} = \sum_{j=1}^{n} \nu_j \cdot p_j$ is a positive divisor of degree $r =$

$\sum_{j=1}^{n}\nu_j > 0$ in which p_j are distinct points and if $p_j \in U_{\alpha_j}$ the *Brill-Noether matrix* $\Omega_{\alpha_1\ldots\alpha_n}(\mathfrak{d})$ of this divisor in terms of the local coordinates z_{α_j} in U_{α_j} is the $g \times r$ matrix in which row i for $1 \le i \le g$ is

$$\begin{aligned}
&f_{i\alpha_1}(p_1), f'_{i\alpha_1}(p_1), \tfrac{1}{2}f''_{i\alpha_1}(p_1), \ldots, \tfrac{1}{(\nu_1-1)!}f^{(\nu_1-1)}_{i\alpha_1}(p_1),\\
&f_{i\alpha_2}(p_2), f'_{i\alpha_2}(p_2), \tfrac{1}{2}f''_{i\alpha_2}(p_2), \ldots, \tfrac{1}{(\nu_2-1)!}f^{(\nu_2-1)}_{i\alpha_2}(p_2),\\
&\quad\cdots \qquad \cdots \qquad \cdots \qquad\qquad \cdots\\
&f_{i\alpha_n}(p_n), f'_{i\alpha_n}(p_n), \tfrac{1}{2}f''_{i\alpha_n}(p_n), \ldots, \tfrac{1}{(\nu_n-1)!}f^{(\nu_n-1)}_{i\alpha_n}(p_n),
\end{aligned} \tag{11.9}$$

where the derivatives of the function $f_{i\alpha_j}$ at the point $p_j \in U_{\alpha_j}$ are with respect to the local coordinate z_{α_j}. One extreme case is that in which $\mathfrak{d}$ is a positive divisor consisting of r distinct points p_j, in which case

$$\text{row i of } \Omega_{\alpha_1\cdots\alpha_r}(p_1 + \cdots + p_r) = \Big\{f_{i\alpha_1}(p_1), \ldots, f_{i\alpha_r}(p_r)\Big\}. \tag{11.10}$$

Another extreme case is that in which $\mathfrak{d}$ is a multiple of a single point $p \in U_\alpha$, in which case

$$\text{row i of } \Omega_\alpha(r \cdot p) = \left\{f_{i\alpha}(p), f'_{i\alpha}(p), \ldots, \frac{1}{(r-1)!}f^{(r-1)}_{i\alpha}(p)\right\}. \tag{11.11}$$

If $\{U_\alpha, \kappa_{\alpha\beta}\}$ is the holomorphic coordinate bundle describing the canonical bundle κ in terms of the chosen coordinates and $p \in U_\alpha \cap U_\beta$ then upon differentiating (2.22) and noting that $d/dz_\alpha = \kappa_{\alpha\beta}\ d/dz_\beta$ by the chain rule for differentiation it follows that the coefficient functions $f_{i\alpha}(p)$ and $f_{i\beta}(p)$ and their derivatives at the point p with respect to the local coordinates z_α and z_β respectively are related by

$$\begin{aligned}
f_{i\alpha}(p) &= \kappa_{\alpha\beta}(p) f_{i\beta}(p)\\
f'_{i\alpha}(p) &= \kappa_{\alpha\beta}(p)^2 f'_{i\beta}(p) + \kappa_{\alpha\beta}(p)\kappa'_{\alpha\beta}(p) f_{i\beta}(p)\\
f''_{i\alpha}(p) &= \kappa_{\alpha\beta}(p)^3 f''_{i\beta}(p) + 3\kappa_{\alpha\beta}(p)^2\kappa'_{\alpha\beta}(p) f'_{i\beta}(p)\\
&\quad + \big(\kappa_{\alpha\beta}(p)\kappa'_{\alpha\beta}(p)^2 + \kappa_{\alpha\beta}(p)^2\kappa''_{\alpha\beta}(p)\big) f_{i\beta}(p)\\
&\quad \text{and so on,}
\end{aligned} \tag{11.12}$$

where $\kappa'_{\alpha\beta}$ denotes the derivative of the function $\kappa_{\alpha\beta}$ with respect to the variable z_β and correspondingly for the higher derivatives. It is a straightforward matter to verify using (11.9) and (11.12) that

$$\Omega_{\alpha_1\ldots\alpha_n}(\mathfrak{d}) = \Omega_{\beta_1\ldots\beta_n}(\mathfrak{d}) \cdot K_{\alpha\beta} \tag{11.13}$$

for the nonsingular $r \times r$ matrix

$$K_{\alpha\beta} = \begin{pmatrix}
\kappa_{\alpha_1\beta_1} & \kappa_{\alpha_1\beta_1}\kappa'_{\alpha_1\beta_1} & \kappa_{\alpha_1\beta_1}(\kappa'_{\alpha_1\beta_1})^2 + \kappa^2_{\alpha_1\beta_1}\kappa''_{\alpha_1\beta_1} & \cdots\\
0 & \kappa^2_{\alpha_1\beta_1} & 3\kappa^2_{\alpha_1\beta_1}\kappa'_{\alpha_1\beta_1} & \cdots\\
0 & 0 & \kappa^3_{\alpha_1\beta_1} & \cdots\\
\cdots & \cdots & \cdots & \cdots
\end{pmatrix}$$

for which

$$\det K_{\alpha\beta} = \kappa_{\alpha_1\beta_1}^{\frac{1}{2}\nu_1(\nu_1+1)}\kappa_{\alpha_2\beta_2}^{\frac{1}{2}\nu_2(\nu_2+1)}\cdots\kappa_{\alpha_n\beta_n}^{\frac{1}{2}\nu_n(\nu_n+1)}. \tag{11.14}$$

Since the matrix $K_{\alpha\beta}$ is nonsingular the rank of the matrix $\Omega_{\alpha_1\cdots\alpha_n}(\mathfrak{d})$ is independent of the choice of local coordinates; so when considering merely the rank of the Brill-Noether matrix the notation can be simplified by dropping the subscripts indicating the choice of local coordinates at the points of the divisor. The rank of the matrix $\Omega(\mathfrak{d})$ also clearly is independent of the choice of a basis for the space of holomorphic differential forms on M. The Riemann-Roch Theorem then takes the following form in terms of the Brill-Noether matrix $\Omega(\mathfrak{d})$.

Theorem 11.3 (Riemann-Roch Theorem) *If $\mathfrak{d}$ is a nontrivial positive divisor on a compact Riemann surface M of genus $g > 0$*

$$\gamma(\zeta_{\mathfrak{d}}) = \dim L(\mathfrak{d}) = \deg \mathfrak{d} - \operatorname{rank}\, \Omega(\mathfrak{d}) + 1, \tag{11.15}$$

where $\Omega(\mathfrak{d})$ is the Brill-Noether matrix of the divisor $\mathfrak{d}$.

Proof: For a positive divisor $\mathfrak{d}$ the vector space $L^{(1,0)}(-\mathfrak{d})$ of meromorphic differential forms $\omega = f_\alpha\, dz_\alpha$ such that $\mathfrak{d}(f_\alpha) - \mathfrak{d} \ge 0$ consists of the holomorphic differential forms that vanish on the divisor $\mathfrak{d}$. A holomorphic differential form $\omega = f_\alpha\, dz_\alpha$ can be written in terms of the basis $\omega_i = f_{i\alpha}\, dz_\alpha$ as the sum $\omega = \sum_i c_i\, \omega_i$ for some complex constants c_i, so $f_\alpha = \sum_{i=1}^g c_i\, f_{i\alpha}$. If $c = (c_1, \ldots, c_g)$ is the row vector formed from these constants and the divisor $\mathfrak{d}$ is nontrivial and has the Brill-Noether matrix $\Omega(\mathfrak{d})$ then the entries in the row vector $c \cdot \Omega(\mathfrak{d})$ are just the values of the function $f_\alpha(z_\alpha)$ and of its derivatives at the points of the divisor $\mathfrak{d}$, paralleling the entries in row i of the matrix $\Omega(\mathfrak{d})$. Consequently the holomorphic differential form $\omega = f\, dz$ vanishes on the divisor $\mathfrak{d}$ precisely when $c \cdot \Omega(\mathfrak{d}) = 0$, so

$$\dim L^{(1,0)}(-\mathfrak{d}) = \dim \left\{ c \in \mathbb{C}^g \;\middle|\; c \cdot \Omega(\mathfrak{d}) = 0 \right\} = g - \operatorname{rank}\, \Omega(\mathfrak{d}). \tag{11.16}$$

Substituting this into the Riemann-Roch formula of Corollary 2.26 yields the desired result and thereby concludes the proof.

11.3 Special Positive Divisors

The Riemann-Roch Theorem in the form of Theorem 11.3 yields an effective lower bound for the dimension $\gamma(\zeta_{\mathfrak{d}})$ of the space of holomorphic cross-sections of the line bundle $\zeta_{\mathfrak{d}}$ of a positive divisor $\mathfrak{d} \ge 0$. The following auxiliary lemma is useful for this purpose.

Lemma 11.4 *If $f_i(z)$ are g linearly independent holomorphic functions in a connected open subset $U \subset \mathbb{C}$ then rank $\{f_i(z_j)\} = \min(r, g)$ for all points $z = (z_1, z_2, \ldots, z_r) \in U^r$ outside a proper holomorphic subvariety of U^r.*

Proof: For convenience of notation let $f(z) = \{f_i(z)\}$ be the vector valued function formed from the functions $f_i(z)$. First it will be demonstrated by induction on r that if $f_i(z)$ are g linearly independent holomphic functions in U then

$$\text{rank}\begin{pmatrix} f(z_1) & f(z_2) & \cdots & f(z_r)\end{pmatrix} = \min(r, g) \tag{11.17}$$

for at least one point $(z_1, \ldots, z_r) \in U^r$. It is clearly enough just to show that for $r \le g$. The result is trivially true for $r = 1$, so assume it holds for $r - 1$ and consider the matrix in (11.17) where $r \le g$. By the inductive hypothesis there will be some points $z_1, \ldots, z_{r-1} \in U$ so that the vectors $f(z_1), \ldots, f(z_{r-1})$ are linearly independent. If the result for the case r were not true then for any point $z \in U$ the vector $f(z)$ would be a linear combination $f(z) = \sum_{i=1}^{r-1} a_i(z) f(z_i)$ for some $a_i(z)$ depending of course on the point z; but since $r - 1 < g$ there are constants $b_j \in \mathbb{C}$, not all zero, such that $\sum_{j=1}^{g} b_j f_j(z_i) = 0$ for $1 \le i \le r - 1$, and then $\sum_{j=1}^{g} b_j f_j(z) = 0$ for all z, a contradiction since the functions $f_i(z)$ are assumed to be linearly independent. That shows that the set of points $(z_1, \ldots, z_r) \in U^r$ at which rank $\{f_i(z_j)\} < r$ for any $r \le g$ is a proper subset of U^r. Since this set clearly is a holomorphic subvariety of U^r it must be a proper holomorphic subvariety, which suffices for the proof.

The converse of the preceding lemma is rather more difficult to show, indeed is not true just for $\mathcal{C}^\infty$ functions, as demonstrated by Böcher[1].

Theorem 11.5 *If $\mathfrak{d} \ge 0$ is a positive divisor on a compact Riemann surface M of genus $g > 0$ then*

$$\gamma(\zeta_{\mathfrak{d}}) - 1 = \dim L(\mathfrak{d}) - 1 \ge \max(0, \deg \mathfrak{d} - g), \tag{11.18}$$

and this lower bound is attained for some positive divisors $\mathfrak{d}$ of any degree.

Proof: For the trivial divisor $\mathfrak{d} = 0$ of course $\deg \mathfrak{d} = 0$ and $\gamma(\zeta_{\mathfrak{d}}) = \gamma(1) = 1$, so the asserted inequality holds trivially as an equality. The Brill-Noether matrix $\Omega_{\mathfrak{d}}$ for a positive divisor $\mathfrak{d}$ of degree $r > 0$ on a compact Riemann surface M of genus $g > 0$ is a $g \times r$ matrix, and consequently rank $\Omega_{\mathfrak{d}} \le \min(r, g)$; hence from the Riemann-Roch Theorem in the form of equation (11.15) it follows that $\gamma(\zeta_{\mathfrak{d}}) = \deg \mathfrak{d} - \text{rank } \Omega(\mathfrak{d}) + 1 \ge r - \min(r, g) + 1 = r + \max(-r, -g) + 1 = \max(0, r-g)+1$, which is (11.18). If $U \subset M$ is a coordinate neighborhood with a local coordinate z then r distinct points in U can be described by their r distinct coordinate values z_j; so if $f_i(z)dz$ is a basis for the holomorphic differentials on M for $1 \le i \le g$ then as in (11.10) the Brill-Noether matrix for the divisor $\mathfrak{d} = z_1 + \cdots + z_r$ is the $g \times r$ matrix $\{f_i(z_j)\}$. Since the holomorphic functions $f_i(z)$ are linearly independent, rank $\{f_i(z_j)\} = \min(r, g)$ for general sets of r distinct points of this coordinate neighborhood, by the preceding Lemma11.4. It follows

[1] See Böcher, The theory of linear dependence, *Annals of Math* vol 2 (1901), pages 81 - 96, and a simpler proof by Bostan and Dumas, Wronskians and Linear Independence, *American Mathematical Monthly*, vol 117 (2010), pages 722-727.

that in general $\gamma(\zeta_{\mathfrak{d}}) - 1 = r - \text{rank }\{f_i(z_j)\} = r - \min(r,g) = \max(0, r-g)$, which suffices to conclude the proof.

A positive divisor $\mathfrak{d} \ge 0$ for which the difference $\gamma(\zeta_{\mathfrak{d}}) - 1$ exceeds the lower bound of the preceding theorem is called a *special positive divisor*[2] while a divisor for which the difference $\gamma(\zeta_{\mathfrak{d}}) - 1$ attains that lower bound is called a *general positive divisor*; thus

$$(11.19)\quad \begin{aligned} &\mathfrak{d} \ge 0 \text{ is a special positive divisor if } \gamma(\zeta_{\mathfrak{d}}) - 1 > \max(0, \deg \mathfrak{d} - g),\\ &\mathfrak{d} \ge 0 \text{ is a general positive divisor if } \gamma(\zeta_{\mathfrak{d}}) - 1 = \max(0, \deg \mathfrak{d} - g),\end{aligned}$$

and any positive divisor $\mathfrak{d} \ge 0$ is either special or general. By the preceding theorem there are general positive divisors of any degree $r \ge 0$ on a compact Riemann surface of genus $g > 0$; indeed in the proof of that theorem it was demonstrated that general positive divisors actually are general in a fairly natural sense, which will be made more precise in the discussion of subvarieties of special positive divisors on page 319. It is worth noting explicitly here some common special and general positive divisors.

Corollary 11.6 *On a compact Riemann surface M of genus $g > 0$ the divisor $1 \cdot p$ for a point $p \in M$ is a general positive divisor. Equivalently not all holomorphic differential forms on M vanish at any point $p \in M$.*

Proof: For any point p of a compact Riemann surface M of genus $g > 0$ it follows from Theorem 2.4 that $\gamma(\zeta_p) - 1 = 0 = \max(0, 1-g)$ so the divisor $1 \cdot p$ is a general positive divisor. It follows from Theorem 11.3 that rank $\Omega(p) = 2 - \gamma(\zeta_p) = 1$ for the Brill-Noether matrix $\Omega(1 \cdot p)$ of this divisor; and since as in (11.11) the Brill-Noether matrix for this divisor is the $g \times 1$ matrix

$$\Omega(1 \cdot p) = \begin{pmatrix} f_{1\alpha}(p) \\ \cdots \\ f_{g\alpha}(p) \end{pmatrix}$$

where $f_{i\alpha} dz_\alpha$ are the holomorphic differential forms on M it follows that not all of these differential forms vanish at the point p, which suffices for the proof.

Corollary 11.7 *On a compact Riemann surface M of genus $g > 0$ a positive divisor $\mathfrak{d}$ with $\deg \mathfrak{d} > 2g - 2$ is a general positive divisor. Equivalently* rank $\Omega(\mathfrak{d}) = g$ *for the $g \times \deg \mathfrak{d}$ Brill-Noether matrix $\Omega(\mathfrak{d})$ of any positive divisor $\mathfrak{d}$ with $\deg \mathfrak{d} > 2g - 2$.*

[2] There is some variety in the literature in what is meant by the term "special positive divisor". Traditionally the *index of speciality* of a positive divisor $\mathfrak{d}$ on a compact Riemann surface of genus g is defined to be the difference $g - \text{rank }\Omega(\mathfrak{d}) = \dim L^{(1,0)}(-\mathfrak{d})$, and special positive divisors are defined to be those positive divisors for which this index is positive, hence those positive divisors $\mathfrak{d}$ for which rank $\Omega(\mathfrak{d}) < g$. On the other hand the definition adopted here seems quite commonly used in informal discussions of properties of positive divisors, and reflects more closely the most interesting aspect of the discussion of these divisors. The two notions obviously agree for divisors of degree at least g.

Proof: If $\mathfrak{d}$ is a positive divisor and $\deg \mathfrak{d} > 2g-2$ then $c(\kappa\zeta_{\mathfrak{d}}^{-1}) < 0$ so $\gamma(\kappa\zeta_{\mathfrak{d}}^{-1}) = 0$ by Corollary 1.3, and it then follows from the Riemann-Roch Theorem (2.29) that $\gamma(\zeta_{\mathfrak{d}}) - 1 = \deg \mathfrak{d} - g = \max(0, \deg \mathfrak{d} - g)$ so $\mathfrak{d}$ is a general positive divisor. From Theorem 11.3 it then follows that $\operatorname{rank} \Omega(\mathfrak{d}) = \deg \mathfrak{d} + 1 - \gamma(\zeta_{\mathfrak{d}}) = g$, and that suffices for the proof.

Corollary 11.8 *On a compact Riemann surface M of genus $g > 1$ a positive divisor of degree $2g-2$ is a special positive divisor if and only if it is a canonical divisor; all positive divisors of degree $2g-2$ other than canonical divisors are general positive divisors.*

Proof: A positive divisor $\mathfrak{d}$ of degree $2g-2$ is a special positive divisor if and only if $\gamma(\zeta_{\mathfrak{d}}) - 1 > \max(0, g-2)$, hence if and only if $\gamma(\zeta_{\mathfrak{d}}) \ge \max(2, g)$; thus $\gamma(\zeta_{\mathfrak{d}}) \ge g$ when $g > 1$, hence $\zeta_{\mathfrak{d}}$ is the canonical bundle by the Canonical Bundle Theorem, Theorem 2.24, so $\mathfrak{d}$ is a positive canonical divisor and that suffices for the proof.

This last corollary is a special case of the more general observation that the special positive divisors on compact Riemann surfaces of genus $g > 1$ arise from positive canonical divisors. To make this more precise, a nontrivial positive divisor $\mathfrak{d}$ is said to be *part of a positive canonical divisor* if its residual divisor $\mathfrak{d}'$ is also a positive divisor, that is, if there is a positive divisor $\mathfrak{d}'$ such that $\mathfrak{d} + \mathfrak{d}' = \mathfrak{k}$; in particular a positive canonical divisor itself is part of a positive canonical divisor.

Corollary 11.9 *A nontrivial special positive divisor on a compact Riemann surface of genus $g > 0$ is part of a positive canonical divisor; and conversely any positive divisor $\mathfrak{d}$ of $\deg \mathfrak{d} \ge g$ that is part of a positive canonical divisor is a special positive divisor.*

Proof: Combining the Riemann-Roch Theorem in the form of equation (11.15) with the definition (11.19) shows that a positive divisor $\mathfrak{d}$ of degree $r > 0$ with the Brill-Noether matrix $\Omega(\mathfrak{d})$ is a special positive divisor if and only if $r - \operatorname{rank} \Omega(\mathfrak{d}) = \gamma(\zeta_{\mathfrak{d}}) - 1 > \max(0, r-g)$, hence if and only if $\operatorname{rank} \Omega(\mathfrak{d}) < \min(r, g)$. Thus if $\mathfrak{d}$ is a special positive divisor then $\operatorname{rank} \Omega(\mathfrak{d}) < g$, so if the Brill-Noether matrix is defined in terms of a basis $f_i(z)dz$ for the holomorphic differential forms on M there is a nontrivial row vector $c \in \mathbb{C}^g$ such that $c \cdot \Omega(\mathfrak{d}) = 0$; then $\sum_i c_i f_i(z)dz$ is a nontrivial holomorphic differential form that vanishes at the divisor $\mathfrak{d}$, hence $\mathfrak{d}$ is part of the positive canonical divisor that is the divisor of this holomorphic differential form. Conversely if $\mathfrak{d}$ is part of a positive canonical divisor then there is a nontrivial holomorphic differential form that vanishes on $\mathfrak{d}$, so that $\operatorname{rank} \Omega(\mathfrak{d}) < g$; and if $r \ge g$ that is just the condition that $\mathfrak{d}$ is a special divisor. That suffices for the proof.

On a compact Riemann surface of genus $g > 0$ the line bundle of the trivial divisor $\mathfrak{d} = 0$ is the identity bundle $\zeta_0 = 1$, and since $\gamma(\zeta_0) - 1 = 0 = \max(0, -g)$ it follows that the trivial divisor is a general positive divisor. A divisor $\mathfrak{d}$ for which $\deg \mathfrak{d} = 1$ is a point bundle, so also is a general positive divisor by

Corollary 11.6. On the other hand any positive divisor $\mathfrak{d}$ for which $\deg \mathfrak{d} > 2g-2$ is a general divisor by Corollary 11.7. Consequently on a compact Riemann surface of genus $g > 0$

$$2 \le \deg \mathfrak{d} \le 2g-2 \quad \text{for any special positive divisor } \mathfrak{d} \ge 0. \tag{11.20}$$

The upper bound $2g-2$ for the degrees of the special positive divisors is effective for compact Riemann surfaces of genus $g > 1$ by Corollary 11.8. Thus the investigation of special positive divisors can be limited to an examination of special positive divisors with degrees limited to the values (11.20); this will be taken up again in the discussion of maximal sequences in Chapter‘11.

The Riemann-Roch Theorem is the basic result about the dimensions $\gamma(\lambda) = \dim \Gamma(M, \mathcal{O}(\lambda))$ of the spaces of holomorphic cross-sections of holomorphic line bundles λ over a compact Riemann surface M. For many purposes it is more convenient to focus on the difference $\gamma(\lambda) - 1$, the dimension of the complex projective space $\mathbb{P}\Gamma(M, \mathcal{O}(\lambda))$ associated to the vector space $\Gamma(M, \mathcal{O}(\lambda))$. Of course $\gamma(\lambda) \ge 0$ while from the Riemann-Roch Theorem it follows that $\gamma(\lambda) = \gamma(\kappa\lambda^{-1}) + c(\lambda) + 1 - g \ge c(\lambda) + 1 - g$, and consequently

$$\gamma(\lambda) - 1 \ge \max(-1, c(\lambda) - g). \tag{11.21}$$

On the other hand $\gamma(\lambda) = 0$ if $c(\lambda) < 0$ by Corollary 1.3 and $\gamma(\lambda) = 0$ or 1 if $c(\lambda) = 0$ by Corollary 1.4, while $\gamma(\lambda) \le c(\lambda) + 1$ if $c(\lambda) > 0$ by Theorem 2.7, so

$$\gamma(\lambda) - 1 \le \max\big(-1, c(\lambda)\big). \tag{11.22}$$

More precise upper bounds can be described in terms of the *maximal function* of the compact Riemann surface M, the function $\mu(r)$ of integers $r \in \mathbb{Z}$ defined by

$$\mu(r) = \sup\Big\{\gamma(\lambda) - 1 \,\Big|\, \lambda \in P_r(M)\Big\} \tag{11.23}$$

where $P_r(M)$ is the set of holomorphic line bundles over M of characteristic class r. As will become clear in the later discussion, the maximal function shares some of the basic properties of the local maximal function $\mu_a(r)$ defined in (9.20) in the preceding chapter, and is a somewhat related invariant. Some general properties of the maximal function, special cases of which were demonstrated for the local maximal function in Theorem 9.7, can be established quite easily.

Theorem 11.10 *The maximal function of a compact Riemann surface satisfies*

$$\mu(r) \le \mu(r+1) \le \mu(r) + 1 \tag{11.24}$$

and

$$\mu(2g-2-r) = \mu(r) + g - 1 - r \tag{11.25}$$

for all $r \in \mathbb{Z}$, while

$$\mu(r) = -1 \quad \textit{for} \quad r < 0 \quad \textit{and} \quad \mu(r) = r - g \quad \textit{for} \quad r > 2g-2. \tag{11.26}$$

In particular

$$\mu(0) = 0 \quad \textit{and} \quad \mu(2g-2) = g-1 \tag{11.27}$$

and if $g > 0$

$$\mu(1) = 0 \quad \textit{and} \quad \mu(2g-3) = g-2. \tag{11.28}$$

Proof: First let λ_r be a holomorphic line bundle for which $c(\lambda_r) = r$ and $\gamma(\lambda_r) - 1 = \mu(r)$. For any point bundle ζ_p clearly $c(\lambda_r\zeta_p) = r+1$ while $\gamma(\lambda_r\zeta_p) \ge \gamma(\lambda_r)$ by Lemma 2.6, so $\mu(r+1) \ge \gamma(\lambda_r\zeta_p) - 1 \ge \gamma(\lambda_r) - 1 = \mu(r)$, which is the first inequality in (11.24). On the other hand $c(\lambda_r\zeta_p^{-1}) = r-1$ while $\gamma(\lambda_r\zeta_p^{-1}) \ge \gamma(\lambda_r) - 1$ by Lemma 2.6 again, so $\mu(r-1) \ge \gamma(\lambda_r\zeta_p^{-1}) - 1 \ge \gamma(\lambda_r) - 2 = \mu(r) - 1$, which is equivalent to the second inequality in (11.24). By the Riemann-Roch Theorem $\gamma(\lambda) = \gamma(\kappa\lambda^{-1}) + r + 1 - g$ so

$$\begin{aligned}
\mu(r) &= \sup\left\{\, \gamma(\lambda) - 1 \,\middle|\, c(\lambda) = r \,\right\} \\
&= \sup\left\{\, \gamma(\kappa\lambda^{-1}) + (r+1-g) - 1 \,\middle|\, c(\lambda) = r \,\right\} \\
&= (r+1-g) + \sup\left\{\, \gamma(\lambda') - 1 \,\middle|\, c(\lambda') = 2g-2-r \,\right\} \\
&= (r+1-g) + \mu(2g-2-r)
\end{aligned}$$

where $\lambda' = \kappa\lambda^{-1}$, thus yielding (11.25). The first part of (11.26) follows immediately from (11.22) while from the Riemann-Roch Theorem again $\gamma(\lambda) = \gamma(\kappa\lambda^{-1}) + c(\lambda) + 1 - g = c(\lambda) + 1 - g$ if $c(\lambda) > c(\kappa) = 2g-2$, which yields the second part of (11.26). If $c(\lambda) = 0$ then $\gamma(\lambda) \le 1$ by (11.22) and $\gamma(\lambda) = 1$ when λ is the trivial bundle, so $\mu(0) = 0$; and it then follows from (11.25) that $\mu(2g-2) = \mu(0) + g - 1 = g - 1$, which yields (11.27). Finally if $g > 0$ and $c(\lambda) = 1$ then $\gamma(\lambda) \le 1$ by Theorem 2.7 while $\gamma(\lambda) = 1$ if λ is a point bundle, so $\mu(1) = 0$; and it then follows from (11.25) that $\mu(2g-3) = \mu(1) + g - 2 = g - 2$, which suffices to conclude the proof.

For Riemann surfaces of small genus the maximal function is fully determined by the preceding theorem. Indeed if $g = 0$ it follows immediately from (11.26) that

$$\mu(r) = \begin{cases} -1 & \text{for} \quad r < 0 \\ r & \text{for} \quad r \ge 0 \end{cases} \qquad \text{if } g = 0; \tag{11.29}$$

if $g = 1$ it follows immediately from (11.26) and (11.27) that

$$\mu(r) = \begin{cases} -1 & \text{for} \quad r < 0 \\ 0 & \text{for} \quad r = 0 \\ r-1 & \text{for} \quad r \ge 1 \end{cases} \qquad \text{if } g = 1; \tag{11.30}$$

and if $g = 2$ it follows immediately from (11.26), (11.27) and (11.28) that

$$\mu(r) = \begin{cases} -1 & \text{for} \quad r < 0 \\ 0 & \text{for} \quad r = 0, 1 \\ 1 & \text{for} \quad r = 2 \\ r-2 & \text{for} \quad r \ge 3 \end{cases} \qquad \text{if } g = 2. \tag{11.31}$$

For surfaces of genus $g \ge 3$ the value of the maximal function in the interval $2 \le g \le 2g-2$ depends on the particular Riemann surface while outside that range it follows from (11.26), (11.27) and (11.28) that

$$\mu(r) = \begin{cases} -1 & \text{for} \quad r < 0 \\ 0 & \text{for} \quad r = 0,1 \\ r-g & \text{for} \quad r > 2g-2 \end{cases} \qquad \text{if } g \ge 3. \tag{11.32}$$

Any integral-valued function $\mu(r)$ of the integers that satisfies (11.24), that is, that satisfies

$$\mu(r) \le \mu(r+1) \le \mu(r)+1 \quad \text{for all} \quad r \in \mathbb{Z}, \tag{11.33}$$

can be described fully by the parameters

$$\begin{aligned} n_+ &= \sup\Big\{ \mu(r) \Bigm| r \in \mathbb{Z} \Big\}, \\ n_- &= \inf\Big\{ \mu(r) \Bigm| r \in \mathbb{Z} \Big\}, \\ r_i &= \inf\Big\{ r \in \mathbb{Z} \Bigm| \mu(r) \ge i \Big\} \quad \text{for } i \le n_+ \ ; \end{aligned} \tag{11.34}$$

for it is clear from the preceding equation that

$$\mu(r) = i \ \text{ for } r_i \le r < r_{i+1} \tag{11.35}$$

and that

$$\mu(r) - \mu(r-1) = \begin{cases} 1 & \text{if } r = r_i \text{ for some } i \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{11.36}$$

It is convenient to set $r_i = +\infty$ for $i > n_+$, while it follows from the preceding definitions that $r_i = -\infty$ for $i \le n_-$. It is also clear that $r_i < r_{i+1}$ for $n_- \le i < n_+$ since $\mu(r_{i+1}) = i+1$ while $\mu(r_{i+1}) - 1 = i$ so $r_i \le r_{i+1} - 1$.

For some purposes it is also useful to consider the dual function

$$\mu^*(s) = s - \mu(s), \tag{11.37}$$

for which

$$\mu^*(s+1) - \mu^*(s) = 1 - \big(\mu(s+1) - \mu(s)\big) \tag{11.38}$$

and consequently

$$\mu^*(s) \le \mu^*(s+1) \le \mu^*(s)+1; \tag{11.39}$$

thus the dual function $\mu^*(s)$ satisfies the same basic equation as does the function $\mu(r)$, so in parallel with the preceding discussion introduce the corresponding basic parameters

$$\begin{aligned} n_+^* &= \sup\Big\{ \mu^*(s) \Bigm| s \in \mathbb{Z} \Big\}, \\ n_-^* &= \inf\Big\{ \mu^*(s) \Bigm| s \in \mathbb{Z} \Big\}, \\ s_j &= \inf\Big\{ s \in \mathbb{Z} \Bigm| \mu^*(s) \ge j \ \text{ for } j \le n_+^* \Big\}; \end{aligned} \tag{11.40}$$

it is clear from the preceding equation that

$$\mu^*(s) = j \ \text{ for } s_j \le s < s_{j+1} \tag{11.41}$$

and that

$$\mu^*(s) - \mu^*(s-1) = \begin{cases} 1 & \text{if } s = s_j \text{ for some } j \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{11.42}$$

It is convenient to set $s_j = +\infty$ for $j > n_+^*$, while it follows from the preceding equation that $s_j = -\infty$ for $j \le n_-^*$. It is clear that $s_j < s_{j+1}$ for $n_-^* \le j < n_+^*$ as before. Furthermore for any integer $n \in \mathbb{Z}$ it follows immediately from (11.38) that

$$\text{either } n = r_i \text{ for some } i \text{ or } n = s_j \text{ for some } j \text{ but not both,} \tag{11.43}$$

so the sets $\{r_i\}$ and $\{s_j\}$ are disjoint and cover $\mathbb{Z}$.

The maximal function of a compact Riemann surface satisfies (11.33) so it can be described fully by the paramters (11.34). The Riemann sphere is a somewhat anomalous Riemann surface in many ways, and its maximal function is fully determined by (11.29); so to avoid considering too many special cases the subsequent discussion in this chapter generally will be limited to compact Riemann surfaces of genus $g > 0$. It is evident from Theorem 11.10 that

$$n_+ = +\infty \quad \text{and} \quad n_- = -1; \tag{11.44}$$

the parameters r_i are called the *critical values* of the Riemann surface M. The dual function $\mu^*(r) = r - \mu(r)$ is called the *dual maximal function* of the Riemann surface M, and its invariants s_j are called the *gap values* of the Riemann surface M. From (11.43) it follows that the critical values and gap values are disjoint, and any integer is either a critical value or a gap value. When it is necessary or convenient to specify the Riemann surface M explicity the maximal function and dual maximal function will be denoted by $\mu_M(r)$ and $\mu_M^*(r)$, and the critical values and gap values will be denoted correspondingly by $r_i(M)$ and $s_j(M)$. The maximal function of the surface M is determined fully by either the critical values r_i or the gap values s_j of that surface. Since $\mu(r) = -1$ for $r < 0$ it follows from the definition of the critical values that

$$r_i = -\infty \quad \text{for} \quad i < 0; \tag{11.45}$$

and since $\mu(r) = r - g$ for $r > 2g - 2$ it further follows from the definition of the critical values that

$$r_i = g + i \quad \text{for} \quad i \ge g. \tag{11.46}$$

It was already observed in the preceding general discussion of consequences of the basic inequality (11.24) that $r_i < r_{i+1}$ for $n_- \le i < n_+$, so for the critical values of the Riemann surface

$$r_i < r_{i+1} \quad \text{for} \quad 0 \le i < +\infty, \tag{11.47}$$

while of course $r_i = r_{i+1} = -\infty$ for $i < -1$ as in (11.45). Since $\mu(0) = \mu(1) = 0$ by (11.27) and (11.28) it follows that

$$r_0 = 0, \quad \text{and} \quad r_1 > 1; \tag{11.48}$$

while since $\mu(2g-3) = g-2$ and $\mu(2g-2) = g-1$ by the same equations it also follows that

$$r_{g-1} = 2g-2. \tag{11.49}$$

The gap values of a compact Riemann surface of genus $g > 0$ are the complement of the critical values; so since all integers $r \geq 2g$ are critical values while no integers $r < 0$ are critical values it follows that all integers $s < 0$ are gap values and the remaining gap values are just those integers in the interval $(0, 2g-1)$ that are not critical values. In more detail, since $\mu(r) = -1$ for $r < 0$ it follows that $\mu^*(r) = r - \mu(r) = r + 1$ for $r < 0$ hence by the definition of the gap values

$$s_j = j - 1 \quad \text{for} \quad j \leq 0. \tag{11.50}$$

and since all integers $r \geq 2g$ are critical values by (11.46) none of these integers are gap values so

$$s_j = +\infty \quad \text{for} \quad j > g. \tag{11.51}$$

Further since $\mu(r_i) = i$ and $\mu(r_i - 1) = i - 1$ for $i \geq 0$ substituting these values into (11.25) shows that

$$\mu(2g-2-r_i) = i + g - 1 - r_i \quad \text{and} \quad \mu(2g-1-r_i) = i + g - 1 - r_i,$$

or in terms of the dual maximal function

$$\mu^*(2g-2-r_i) = g - i - 1 \quad \text{and} \quad \mu^*(2g-1-r_i) = g - i$$

for $i \geq 0$, and consequently $s_{g-i} = 2g - 1 - r_i$ for $i \geq 0$ or equivalently

$$s_j = 2g - 1 - r_{g-j} \quad \text{for} \quad j \leq g. \tag{11.52}$$

In particular since $r_0 = 0$ and $r_{g-1} = 2g - 2$

$$s_1 = 1 \quad \text{and} \quad s_g = 2g - 1. \tag{11.53}$$

For convenience the preceding results about the ranges of the critical and gap values are summarized as follows.

Theorem 11.11 *For a compact Riemann surface M of genus $g > 0$ the critical values satisfy*

$$0 = r_0 \leq 1 < r_1 < r_2 < \cdots < r_{g-1} = 2g - 2 \tag{11.54}$$

while

$$r_i = -\infty \ \textit{for} \ i < 0 \ \textit{and} \quad r_i = g + i \ \textit{for} \ i \geq g. \tag{11.55}$$

The complementary gap values s_j satisfy

$$1 = s_1 < s_2 < \cdots < s_g = 2g - 1 \tag{11.56}$$

while

$$s_j = +\infty \ \textit{for} \ j > g \ \textit{and} \ s_j = j - 1 \ \textit{for} \ j \leq 0 \tag{11.57}$$

Proof: Since this is just a summary of the preceding discussion no further proof is required.

It is traditional to call the g positive gap values (11.56) the *Weierstrass gap values* of the Riemann surface M. The maximal function of a compact Riemann surface M can be represented conveniently and usefully by the *Brill-Noether diagram*, as in the example in Figure 11.1. In this figure the characteristic classes $r = c(\lambda)$ of holomorphic line bundles over M extend along the horizontal axis while the projective dimensions $\nu = \gamma(\lambda) - 1$ of the spaces of holomorphic cross-sections of these line bundles extend along the vertical axis. The upper heavy broken line is the graph of the maximal function itself, consisting of the line segments connecting points $\big(r, \mu(r)\big)$; for short it is called the *maximal curve* of the Brill-Noether diagram. The lower heavy broken line is the graph of the lower bound $\mu_-(r) = \max(-1, r - g)$ for the values $\gamma(\lambda) - 1$ for holomorphic line bundles λ with $c(\lambda) = r$, as given in (11.21); for short this broken line is called the *minimal curve* of the Brill-Noether diagram, and the function $\mu_-(r)$ is called the *minimal function* of the Riemann surface M. The shaded region in the diagram, lying between the maximal and minimal curves, thus consists of those points (r, ν) for which there may be line bundles λ for which $r = c(\lambda)$ and $\nu = \gamma(\lambda) - 1$. The maximal and minimal curves coincide with the horizontal straight line of height -1 for $r < 0$, and coincide with the straight line of slope 1 if $r > 2g - 2$; those are the ranges of values the characteristic classes $c(\lambda)$ of holomorphic line bundles λ for which the dimension $\gamma(\lambda)$ is determined completely by the value $c(\lambda)$ through the Riemann-Roch Theorem, as in Theorem 11.11. The critical value r_i of M is that point on the horizontal axis at which the maximal curve first takes the value i; the critical values are indicated explicitly on the diagram in Figure 11.1, while the points on the horizontal axis that are not the critical values are the gap values of M. The Riemann-Roch Theorem takes the form of a symmetry of the Brill-Noether diagram about the axis $r = g - 1$, as is evident upon examining Figure 11.1 more closely. Indeed formula (11.25), which is a direct consequence of the Riemann-Roch Theorem, can be rewritten

$$\mu(2g - 2 - r) - \frac{2g - 2 - r}{2} = \mu(r) - \frac{r}{2}, \tag{11.58}$$

so it asserts that the difference $\mu(r) - \frac{r}{2}$ is symmetric about the axis $r = g - 1$. Similarly the minimal function $\mu_-(r) = \max(-1, r - g)$ for $0 \leq r \leq g - 1$ satisfies

$$\mu_-(2g - 2 - r) - \frac{2g - 2 - r}{2} = g - 2 - r - \frac{2g - 2 - r}{2} = -1 - \frac{r}{2} = \mu_-(r) - \frac{r}{2}$$

Figure 11.1: Example of a Brill-Noether diagram for a compact Riemann surface of genus $g = 16$.

so it too is symmetric about the axis $r = g - 1$. Consequently the difference $\mu(r) - \mu_{-}(r)$, the height of the maximal curve above the minimal curve, also is symmetric about the axis $r = g - 1$. The pattern of increases in the height of the maximal curve above the mininal curve to the left of this axis as r increases is reflected in a corresponding pattern of increases in the height of the maximal curve above the minimal curve to the right of that axis as r decreases. Horizontal line segments of the maximal curve to the left of the axis are reflected in line segments parallel to the minimal curve to the right of the axis. In terms of the critical values this symmetry takes the form

(11.59) $\quad r \geq 0$ is a critical value if and only if $2g - 1 - r$ is a gap value.

These symmetries hold in general cases as well as in the special case considered in Figure 11.1.

To each point (r, ν) in the Brill-Noether diagram of a compact Riemann surface M of genus $g > 0$ there can be associated the set of holomorphic line bundles λ for which $c(\lambda) = r$ and $\gamma(\lambda) - 1 = \nu$, the set

$$\hat{X}_r^{\nu} = \left\{ \lambda \in P_r(M) \,\middle|\, \gamma(\lambda) - 1 = \nu \right\} \subset P_r(M) \tag{11.60}$$

where $P_r(M)$ is the complex torus consisting of those holomorphic line bundles of characteristic class r. It is evident that

$$\hat{X}_r^{\nu} = \hat{W}_r^{\nu} \sim \hat{W}_r^{\nu+1} \tag{11.61}$$

where $\hat{W}_r^\nu = \Big\{ \lambda \in P_r(M) \Big| \gamma(\lambda) - 1 \ge \nu \Big\}$ are the holomorphic subvarieties of $P_r(M)$ defined in (12.42). The subset $\hat{X}_r^\nu \subset P_r(M)$ thus is not necessarily a holomorphic subvariety of the complex torus $P_r(M)$; but as the subset described by (11.61) in terms of the holomorphic subvarieties $\hat{W}_r^{\nu+1} \subset \hat{W}_r^\nu$ the set $\hat{X}_r^\nu$ at least has the structure of a holomorphic variety, since it is a holomorphic subvariety of the complex torus $P_r(M)$ in an open neighborhood of each of its points. The sets $\hat{X}_r^\nu$ thus are examples of sets that have natural complex analytic structures but do not have such natural structures as algebraic varieties. For convenience the sets of line bundles associated to points on the maximal curve are called the *maximal line bundles* and are denoted by $\hat{X}_r^{MAX}$, while the sets of line bundles associated to points on the minimal curve are called the *minimal line bundles* and are denoted by $\hat{X}_r^{MIN}$; thus

$$\hat{X}_r^{MAX} = \hat{X}_r^{\mu(r)} \quad \text{and} \quad \hat{X}_r^{MIN} = \hat{X}_r^{\mu_-(r)} = \hat{X}_r^{\max(-1,r-g)} \tag{11.62}$$

while

$$P_r(M) = \bigcup_{\nu=\mu_-(r)}^{\mu(r)} \hat{X}_r^\nu \quad = \quad \hat{X}_r^{MIN} \cup \cdots \cup \hat{X}_r^{MAX} \tag{11.63}$$

and the holomorphic subvariety $\hat{W}_r^\nu \subset P_r(M)$ is the union

$$\hat{W}_r^\nu = \bigcup_{\sigma=\nu}^{\mu(r)} \hat{X}_r^\sigma = \hat{X}_r^\nu \cup \cdots \cup \hat{X}_r^{MAX}. \tag{11.64}$$

The set $\hat{X}_r^{MAX} = \hat{W}_r^{\mu(r)}$ hence actually is a holomorphic subvariety of the complex torus $P_r(M)$ for any index r. Of course

$$\hat{X}_r^{MAX} = \hat{X}_r^{MIN} = P_r(M) \quad \text{if } r < 0 \text{ or } r > 2g-2. \tag{11.65}$$

It follows from Corollary 1.4 that $\hat{X}_0^{MAX} = \hat{X}_0^0$ consists of the identity bundle alone; and it follows from the Canonical Bundle Theorem, Theorem 2.24, that $\hat{X}_{2g-2}^{MAX} = \hat{X}_{2g-2}^{g-1}$ consists of the canonical bundle κ alone.

The symmetry about the axis $r = g-1$ of the maximal and minimal curves in the Brill-Noether diagram can be extended to a corresponding symmetry of the varieties $\hat{X}_r^\nu$. Indeed from the symmetric form of the Riemann-Roch Theorem, the Brill-Noether formula of Corollary 2.25 stating that $C(\lambda) = C(\kappa\lambda^{-1})$ where $C(\lambda) = c(\lambda) - 2\big(\gamma(\lambda) - 1\big)$ is the Clifford Index of a holomorphic line bundle λ, it follows that

$$\gamma(\lambda) - 1 - \frac{1}{2}c(\lambda) = -\frac{1}{2}C(\lambda) = -\frac{1}{2}C(\kappa\lambda^{-1}) = \gamma(\kappa\lambda^{-1}) - 1 - \frac{1}{2}c(\kappa\lambda^{-1}).$$

Hence whenever $\lambda \in \hat{X}_r^\nu$, so that $c(\lambda) = r$ and $\gamma(\lambda) - 1 = \nu$, then $c(\kappa\lambda^{-1}) = 2g-2-r$ and $\gamma(\kappa\lambda^{-1}) = g-1-(r-\nu)$ or equivalently $\kappa\lambda^{-1} \in \hat{X}_{2g-2-r}^{g-1-(r-\nu)}$. This observation can be expressed conveniently as the symmetry

$$\kappa \cdot \{\hat{X}_r^\nu\}^{-1} = \hat{X}_{2g-2-r}^{g-1-(r-\nu)} \tag{11.66}$$

where $\kappa\{\hat{X}_r^\nu\}^{-1}$ denotes the set of line bundles $\kappa\lambda^{-1}$ for all $\lambda \in \hat{X}_r^\nu$; the symmetry (11.66) actually is equivalent to the Riemann-Roch Theorem.

The symmetry relation (11.66) among holomorphic varieties in the complex manifold $P_r(M)$ appears more natural when these varieties are viewed as subsets of the Jacobi variety $J(M)$. It is possibly worth digressing here to discuss equivalent formulations of the Brill-Noether diagram and the varieties $\hat{X}_r^\nu$ in terms of the Jacobi manifold $J(M)$ or the the manifold $M^{(r)}$ of positive divisors. The mappings in the Abel-Jacobi diagram (12.41) of Theorem 12.21 associate to the varieties $\hat{X}_r^\nu$ contained in the complex manifold $P_r(M)$ corresponding varieties contained in the complex manifolds $J(M)$ and $M^{(r)}$; and these varieties can be grouped in analogues of the Brill-Noether diagram. Thus set

$$X_r^\nu = W_r^\nu \sim W_r^{\nu+1} \subset J(M) \quad \text{for} \quad r, \nu \in \mathbb{Z} \tag{11.67}$$

and correspondingly set

$$Y_r^\nu = G_r^\nu \sim G_r^{\nu+1} \subset M^{(r)} \quad \text{for} \quad r, \nu \in \mathbb{Z},\ r > 0, \tag{11.68}$$

noting that the sets Y_r^ν are defined only for intergers $r > 0$. As the complements of holomorphic subvarieties of holomorphic varieties all of these sets at least have the structures of holomorphic varieties. In analogy to (11.62) introduce the holomorphic varieties $X_r^{MAX} = X_r^{\mu(r)} \subset J(M)$ and $X_r^{MIN} = X_r^{\mu_-(r)} \subset J(M)$, and note that as in (11.63) and (11.64)

$$J(M) = \bigcup_{\nu=\mu_-(r)}^{\mu(r)} X_r^\nu \quad = \quad X_r^{MIN} \cup \cdots \cup X_r^{MAX} \tag{11.69}$$

and

$$W_r^\nu = \bigcup_{\sigma=\nu}^{\mu(r)} X_r^\sigma = X_r^\nu \cup \cdots \cup X_r^{MAX} \subset J(M). \tag{11.70}$$

Similarly for positive divisors let $Y_r^{MAX} = Y^{\mu(r)} \subset M^{(r)}$. However the varieties Y_r^ν are empty for points on the minimal curve for which $\nu = -1$, so when considering the varieties $Y_r^\nu \subset M^{(r)}$ it is more natural to consider in place of the minimal curve the *general curve* consisting of points in the Brill-Noether diagram associated to the sets of *general positive divisors* on the surface M as defined in (11.19); thus set

$$Y_r^{GEN} = Y_r^{\max(0,r-g)} \subset M^{(r)} \quad \text{for } r > 0. \tag{11.71}$$

The general curve differs from the minimal curve in that it is the horizontal line of height 0 for $r \le g-1$ rather than the horizontal line of height -1; but it coincides with the mininal curve for $r \ge g$. All points above the general curve in the Brill-Noether diagrams for the manifold $M^{(r)}$ then are *special positive divisors*, as defined in (11.19). It follows that

$$M^{(r)} = \bigcup_{\nu=\max(0,r-g)}^{\mu(r)} Y_r^\nu = Y_r^{GEN} \cup \cdots \cup Y_r^{MAX} \quad \text{for } r > 0 \tag{11.72}$$

and

$$G_r^\nu = \bigcup_{\sigma=\nu}^{\mu(r)} Y_r^\sigma = Y_r^\nu \cup \cdots \cup X_r^{MAX} \quad \text{for } r > 0 \text{ and } \nu \geq 0. \tag{11.73}$$

The relations between these three families of sets follow from the Abel-Jacobi diagrams (12.16) or (12.41), which are expressed in terms of the holomorphic mappings

$$\zeta = \hat{\zeta} \circ \psi : M^{(r)} \longrightarrow P_r(M) \quad \text{and} \quad w_{z_0} = \hat{w}_{z_0} \circ \psi : M^{(r)} \longrightarrow J(M), \tag{11.74}$$

where $\zeta : M^{(r)} \longrightarrow P_r(M)$ takes the holomorphic variety $Y_r^\nu \subset M^{(r)}$ to the holomorphic variety $\hat{X}_r^\nu \subset P_r(M)$ and $w_{z_0} : M^{(r)} \longrightarrow J(M)$ takes the subvariety $Y_r^\nu \subset M^{(r)}$ to the holomorphic variety $X_r^\nu \subset J(M)$. This is summarized in the following commutative diagram.

(11.75)

$$\begin{tikzcd} Y_r^\nu \subset M^{(r)} \arrow[rr, "\zeta"] \arrow[dr, "w_{z_0}"'] & & \hat{X}_r^\nu \subset P_r(M) \arrow[dl, "\phi_{a_0}"] \\ & X_r^\nu \subset J(M) & \end{tikzcd} \qquad \text{for } r > 0.$$

Theorem 11.12 *The diagram* (11.75) *is a commutative diagram of surjective holomorphic mappings between holomorphic varieties. The mapping ϕ_{a_0} is biholomorphic, but the inverse image of a point under either of the holomorphic mappings ζ or w_{z_0} is a complex submanifold of Y_r^ν that is biholomorphic to the complex projective space $\mathbb{P}^\nu$ and consequently*

$$\dim X_r^\nu = \dim \hat{X}_r^\nu = \dim Y_r^\nu - \nu \quad \textit{for } r > 0 \textit{ and } \nu \geq 0. \tag{11.76}$$

Proof: That (11.75) is a commutative diagram of holomorphic mappings in which ϕ_{a_0} is a biholomorphic mapping is clear from the diagrams (12.16) or (12.41), in which ϕ_{a_0} is a biholomorphic mapping. By Theorem 12.21 the inverse image $w_{z_0}^{-1}(t)$ of a point $t \in X_r^\nu$ is a complex submanifold of $M^{(r)}$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$; and the commutativity of the diagram (11.75) together with the fact that ϕ_{a_0} is a biholomorphic mapping show that the the inverse image $\zeta^{-1}(\lambda)$ of a point $\lambda \in \hat{X}_r^\nu$ also is a complex submanifold of $M^{(r)}$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$. If $V_r^\nu \subset Y_r^\nu$ is an irreducible component of the holomorphic variety Y_r^ν its image $w_{z_0}(V_r^\nu)$ is an irreducible component of the holomorphic variety X_r^ν, and since the fibres of this mapping have dimension ν it follows from Remmert's Proper Mapping Theorem that

$$\dim w_{z_0}(V_r^\nu) = \dim V_r^\nu - \nu \quad \text{for } r > 0 \text{ and } \nu \geq 0.$$

Since $\dim Y_r^\nu$ is the largest of the dimensions of its irreducible components, and correspondingly for $\dim X_r^\nu$, it follows that

$$\dim X_r^\nu = \dim Y_r^\nu - \nu \quad \text{for } r > 0 \text{ and } \nu \geq 0.$$

Of course $\dim \hat{X}_r^\nu = \dim X_r^\nu$ since the varieties $\hat{X}_r^\nu$ and X_r^ν are biholomorphic, and that suffices for the proof.

In view of the isomorphisms of the preceding theorem, it is generally sufficient to state and prove results about the varieties $\hat{X}_r^\nu$, X_r^ν and Y_r^ν just in terms of one of the three sets of varieties; hence much of the subsequent discussion will continue to be phrased in terms of the sets $\hat{X}_r^\nu \subset P_r(M)$ of holomorphic line bundles, although when another interpretation is more useful or more convenient it will be used. It is always possible to translate the results back and forth among these sets of varieties through Theorem 11.12.

Actually the symmetry relation (11.66) is one example in which an alternative description is more natural; for in terms of points of the Jacobi variety $J(M)$ the symmetry relation takes the form

$$k - X_r^\nu = X_{2g-2-r}^{g-1-(r-\nu)} \quad \text{for all} \quad \nu, r \tag{11.77}$$

where $k = \phi_{a_0}(\kappa) \in J(M)$ is the image of the canonical line bundle κ. This is a simple relation between two subsets of the same complex torus $J(M)$. Indeed the mapping of the Jacobi variety to itself that sends a point $t \in J(M)$ to the point $k - t \in J(M)$ is a biholomorphic mapping of the complex torus $J(M)$ to itself which takes the holomorphic variety X_r^ν to the holomorphic variety $X_{2g-2-r}^{g-1-(r-\nu)}$; so these two holomorphic varieties are biholomorphic, and consequently

$$\dim X_{2g-2-r}^{g-1-(r-\nu)} = \dim X_r^\nu \quad \text{for all} \quad \nu, r. \tag{11.78}$$

The sets X_r^ν and $X_{2g-2-r}^{g-1-(r-\nu)}$ are associated to symmetric points in the diagram of the maximal function, in the sense that these points have coordinates r that are symmetric with respect to the axis $r = g-1$ and have the same height above the minimal curve, or equivalently below the maximal curve. In particular

$$k - X_r^{MAX} = X_{2g-2-r}^{MAX} \quad \text{and} \quad k - X_r^{MIN} = X_{2g-2-r}^{MIN}, \tag{11.79}$$

showing that $\dim X_r^{MAX} = \dim X_{2g-2-r}^{MAX}$ and $\dim X_r^{MIN} = \dim X_{2g-2-r}^{MIN}$.

It is a useful preliminary to the further discussion to note that the varieties $X_r^{MIN} \subset J(M)$ on the minimal curve of the Brill-Noether diagram can be described quite explicitly for all Riemann surfaces. First

$$X_r^{MIN} = J(M) \sim W_r \quad \text{for} \quad 0 \le r \le g-1 \tag{11.80}$$

since if $0 \le r \le g-1$ then $X_r^{MIN} = X_r^{-1} = W_r^{-1} \sim W_r^0 = J(M) \sim W_r$; and by the symmetry relation (11.79) the preceding equation implies that

$$X_r^{MIN} = J(M) \sim (k - W_{2g-2-r}) \quad \text{for} \quad g-1 \le r \le 2g-2. \tag{11.81}$$

Since $X_r^{MIN} = J(M)$ for $r > 2g-2$ as in (11.65) it follows from this and the two preceding two equations that

$$\dim \hat{X}_r^{MIN} = g \quad \text{for all } g \ge 0. \tag{11.82}$$

This observation shows incidentally that the lower bound (11.21) actually is an effective lower bound; of course the upper bound (11.23) is effective by definition. More precisely the observations (11.80) and (11.81 show that the lower bound (11.21) is attained by most holomorphic line bundles, indeed by all holomorphic line bundles except those in a proper holomorphic subvariety of the complex torus $P_r(M)$. Since the varieties $X_r^\nu \subset J(M)$ for a fixed value of r and values of ν in the range $\mu_-(r) \le \nu \le \mu(r)$ are disjoint it follows from (11.69), (11.80) and (11.81) that

$$X_r^{\mu_-(r)+1} \cup \cdots \cup X^{\mu(r)} \subset \begin{cases} W_r \subset J(M) & \text{if} \quad 0 \le r \le g-1, \\ \\ k - (W_{2g-2-r}) & \text{if} \quad g-1 \le r \le 2g-2, \end{cases} \tag{11.83}$$

which bounds the varieties X_r^ν. Of course the corresponding observation holds in the other versions of the Brill-Noether diagram.

The first critical value r_1 of a compact Riemann surface of genus $g > 0$ is in many ways the most significant of the critical values of the surface M; and bounds on its possible values are of considerable interest. The basic result about these boundes is the following theorem, which has an extensive history[3] and for which there are a variety of proofs.

Theorem 11.13 *The first critical value r_1 of a compact Riemann surface M of genus $g > 0$ satisfies*

$$2 \le r_1 \le \left[\frac{g}{2}\right] + 1 \tag{11.84}$$

where as usual $\left[\frac{g}{2}\right]$ is the integer part of $\frac{g}{2}$.

Proof: The lower bound is just that of Theorem 11.11. For the upper bound, suppose to the contrary that $r_1 > \left[\frac{g}{2}\right] + 1$. If $g = 2h$ let $r = h+1$, $s = h$, and if $g = 2h+1$ let $r = h+1$, $s = h+1$, so in either case $r = \left[\frac{g}{2}\right]+1$ and consequently $r < r_1$; therefore $\gamma(\lambda) < 2$ for any holomorphic line bundle λ for which $c(\lambda) = r$, or equivalently $\hat{W}_r^1 = \emptyset$. It is more convenient for the rest of the argument to work with the Jacobi variety $J(M)$; thus $W_r^1 = \emptyset$, and then of course $W_s^1 = \emptyset$ also since $s \le r$. Choose a base point $z_0 \in \tilde{M}$ for the Abel-Jacobi mappping $w_{z_0} : M \longrightarrow J(M)$ such that the point $a \in M$ it represents is not a Weierstrass point of M. It then follows from Lemma 9.12 that $r_1(a) = g+1$ hence that ζ_a^{g+1} is base-point-free and $\gamma(\zeta_a^{g+1}) = 2$. Since $0 \in W_r \cap (-W_s)$ and $\dim W_r + \dim(-W_s) - g = r + s - g = 1$ the intersection $(W_r \cap (-W_s)$ is a holomorphic subvariety of the Jacobi variety of dimension at least 1 containing the origin 0; that is a general property of the intersection of holomorphic subvarieties of a complex manifold as discussed on page 418 in Appendix A.3. Consequently there is an irreducible one-dimensional holomorphic subvariety $W \subset J(M)$ for which $0 \in W \subset W_r \cap (-W_s)$. Since $W_s^1 = W_r^1 = \emptyset$, and hence $G_r^1 = G_s^1 = \emptyset$

[3]See for example the discussion in the book *Geometry of Algebraic Curves, I* by E. Arbarello, M. Cornalba, P. Griffiths and J. Harris.

as well, the surface M has no special positive divisors of degrees r or s, as in (12.40); consequently by Corollary 12.9 (i) the Abel-Jacobi mappings

$$w_{z_0} : M^{(r)} \longrightarrow W_r \quad \text{and} \quad w_{z_0} : M^{(s)} \longrightarrow W_s \tag{11.85}$$

are biholomorphic mappings. Any point $t \in W \subset W^{(r)} \cap W^{(s)}$ consequently can be written uniquely in the form

$$t = w_{z_0}(a_1 + \cdots + a_r) = -w_{z_0}(b_1 + \cdots + b_s) \tag{11.86}$$

for some divisors $a_1 + \cdots + a_r \in M^{(r)}$ and $b_1 + \cdots + b_s \in M^{(s)}$. Then

$$w_{z_0}(a_1 + \cdots + a_r + b_1 + \cdots + b_s) = 0, \tag{11.87}$$

so since $r + s = g + 1$ and it is also the case that $w_{z_0}\big((g+1)\cdot a_0\big) = 0$ it follows from Abel's Theorem, Corollary 5.10, that

$$a_1 + \cdots + a_r + b_1 + \cdots + b_s \sim (g+1)\cdot a_0; \tag{11.88}$$

the divisors $\mathfrak{d}_t = a_1 + \cdots + a_r + b_1 + \cdots + b_s$ thus are the divisors of holomorphic cross-sections of the holomorphic line bundle ζ_a^{g+1} for any point $t \in W$. If $X \subset M^{(r+s)}$ is the set of divisors of holomorphic cross-sections of the line bundle ζ_a^{g+1} then since $\gamma(\zeta_a^{g+1}) = 2$ and the line bundle ζ_a^{g+1} is base-point-free it follows that any divisor $\mathfrak{d} \in X$ is uniquely determined by specifying any of its points, and any point of M is in the divisor of some holomorphic cross-section of ζ_a^{g+1}; for if $f_0(z), f_1(z) \in \gamma\big(M, \mathcal{O}(\zeta_a^{g+1})\big)$ is a basis for this space of holomorphic cross-sections then $f_0(z)$ and $f_1(z)$ have no common zeros and $f_p(z) = f_1(p)f_0(z) - f_0(p)f_1(z)$ is the unique cross-section that vanishes at a point $p \in M$. The mapping that associates to a nontrivial cross-section $f_{x_0,x_1} = x_0 f_0(z) + x_1 f_1(z) \in \gamma\big(M, \mathcal{O}(\zeta_1^{g+1})\big)$ its divisor $\mathfrak{d}(f_{x_0,x_1}) \in M^{(r+s)}$ is a holomorphic mapping from the nonzero points of $\mathbb{C}^2$ to $M^{(r+s)}$; and since this mapping is the same for any pairs x_0, x_1 that represent the same point in $\mathbb{P}^1$ it induces a holomorphic mapping from $\mathbb{P}^1$ to $M^{(r+s)}$, a proper mapping since $\mathbb{P}^1$ is a compact manifold. It then follows from Remmert's Proper Mapping Theorem[4] that the image of this mapping, the set of divisors of holomorphic cross-sections of ζ_a^{g+1}, is an irreducible holomorphic subvariety $X \subset M^{(r+s)}$ and $\dim X = 1$. The mapping that associates to any point $t \in W$ the divisor $\mathfrak{d}_t = a_1 + \cdots + a_r + b_1 + \cdots + b_s \in X$ is then a well defined proper holomorphic mapping $\mathfrak{d} : W \longrightarrow X$; and since its image contains more than a single point the image must be a holomorphic subvariety of dimension 1 in X, so actually $\phi(W) = X$. Consequently any divisor $\mathfrak{d}_t$ also is uniquely determined by specifying any one of its points, and that can be an arbitrary point of M; hence the decomposition of the divisor $\mathfrak{d}_t$ as the sum of the two divisors $a_1 + \cdots + a_r$ and $b_1 + \cdots + b_s$ also is unique. On the other hand the divisors $\mathfrak{d}_t$ can be deformed continuously by moving the point a_1 along any path in M, and the decomposition of the divisors $\mathfrak{d}_t$ is preserved in this motion; but moving the point a_1 along a continuous path

[4]Remmert's Proper Mapping Theorem is discussed on page 423 of Appendix A.3.

to the point b_1 cannot preserve the decomposition of the divisors $\mathfrak{d}_t$, and that contradiction serves to conclude the proof.

Part of the significance of the first critical value on a compact Riemann surface M follows from the fact that it is also the smallest positive integer in the Lüroth semigroup of M; more generally though there is the following simple observation.

Theorem 11.14 *On a compact Riemann surface M of genus $g > 0$ all the holomorphic line bundles in $\hat{X}^{MAX}_{r_i}$ for any critical value r_i are base-point free. Consequently the critical values r_i of M belong to the Lüroth semigroup $\mathcal{L}(M)$ of the Riemann surface M, and in particular the critical value r_1 is the smallest positive integer in the Lüroth semigroup.*

Proof: If $\lambda \in X^{MAX}_{r_i}$ for $i \geq 0$ then $c(\lambda) = r_i$ and $\gamma(\lambda) - 1 = \mu(r_i) = i$. Since $c(\lambda\zeta_a^{-1}) = r_i - 1$ for any point $a \in M$ it follows that $\gamma(\lambda\zeta_a^{-1}) - 1 \leq \mu(r_i - 1) = \mu(r_i) - 1 = i - 1 < \gamma(\lambda) - 1$, hence λ is base-point-free by Lemma 2.10. By definition then its characteristic class $c(\lambda) = r_i$ belongs to the Lüroth semigroup. If $0 < r < r_1$ and λ is a holomorphic line bundle with $c(\lambda_) = r$ then $\gamma(\lambda) = 1$ so the bundle λ cannot be base-point-free; therefore there are no base-point-free line bundles λ such that $0 < c(\lambda) < r_1$, and that suffices for the proof.

A basic property of the Brill-Noether diagram is a convexity determined by the base-point-free holomorphic line bundles on M, a consequence of the following simple observation.

Theorem 11.15 *If τ is a base-point-free holomorphic line bundle on a compact Riemann surface M and if $h_0, h_1 \in \Gamma(M, \mathcal{O}(\tau))$ are two holomorphic cross-sections of τ with no common zeros then for any holomorphic line bundle λ on M for which $\gamma(\lambda) \neq 0$ there is the exact function of sheaves*

$$0 \longrightarrow \mathcal{O}(\lambda\tau^{-1}) \xrightarrow{p_1} \mathcal{O}(\lambda)^2 \xrightarrow{p_2} \mathcal{O}(\lambda\tau) \longrightarrow 0 \tag{11.89}$$

in which the sheaf homomorphisms p_1, p_2 are defined by

$$\begin{aligned} p_1(g) &= (h_0 g,\ h_1 g) \in \mathcal{O}_p(\lambda)^2 \quad \textit{for all} \quad g \in \mathcal{O}_p(\lambda\tau^{-1}), \\ p_2(g_0,\ g_1) &= h_1 g_0 - h_0 g_1 \in \mathcal{O}_p(\lambda\tau) \quad \textit{for all} \quad g_0, g_1 \in \mathcal{O}_p(\lambda) \end{aligned}$$

for any point $p \in M$.

Proof: It is evident that the sheaf homomorphisms p_1 and p_2 are well defined, that p_1 is injective, and that $p_2 p_1 = 0$. If $(g_0,\ g_1) \in \mathcal{O}_p(\lambda)^2$ and if $0 = p_2(g_0,\ g_1) = h_1 g_0 - h_0 g_1$ then $g_0/h_0 = g_1/h_1 = g \in \mathcal{M}_p(\lambda\tau^{-1})$; however either $h_0(p) \neq 0$ or $h_1(p) \neq 0$ so g actually is a holomorphic germ $g \in \mathcal{O}_p(\lambda\tau^{-1})$, and $(g_0, g_1) = (h_0 g, h_1 g) = p_1(g)$. Thus the kernel of p_2 is contained in the image of p_1, so the sheaf sequence is exact at the sheaf $\mathcal{O}(\lambda)^2$. If $f \in \mathcal{O}_p(\lambda\tau)$ and if for instance $h_0(p) \neq 0$ then $f = h_0 \cdot (f/h_0) = p_2(f/h_0,\ 0)$ so f is in the image of the homomorphism p_2. The corresponding argument holds if $h_1(p) \neq 0$, so the sheaf homomorphism p_2 is surjective, and that concludes the proof of the theorem.

The exact cohomology sequence arising from the exact sequence of sheaves (11.90) of the preceding theorem begins

$$0 \longrightarrow \Gamma(M, \mathcal{O}(\lambda\tau^{-1})) \xrightarrow{p_1} \Gamma(M, \mathcal{O}(\lambda))^2 \xrightarrow{p_2} \Gamma(M, \mathcal{O}(\lambda\tau)). \tag{11.90}$$

An immediate consequence of this exact sequence is the following corollary.

Corollary 11.16 *If τ is a base-point-free holomorphic line bundle on a compact Riemann surface M then for any holomorphic line bundle λ on M*

$$\gamma(\lambda) - \gamma(\lambda\tau^{-1}) \leq \gamma(\lambda\tau) - \gamma(\lambda). \tag{11.91}$$

and this is an equality if and only if the homomorphism p_2 in the exact sequence (11.90) *is surjective.*

Proof: If $\gamma(\lambda) = 0$ the corollary holds trivially. If $\gamma(\lambda) \neq 0$ then since $\gamma(\lambda\tau) = \dim \Gamma(M, \mathcal{O}(\lambda\tau))$ is at least equal to the dimension of the image of the homomorphism p_2 in (11.90) it follows from the exactness of the cohomology sequence (11.90) that

$$\gamma(\lambda\tau) \geq \dim \Gamma(M, \mathcal{O}(\lambda))^2 - \dim \Gamma(M, \mathcal{O}(\lambda\tau^{-1})) = 2\gamma(\lambda) - \gamma(\lambda\tau^{-1});$$

and this is an equality if and only if the sheaf homomorphism p_2 in the exact sequence (11.90) is surjective. That suffices for the proof.

Corollary 11.17 (Convexity Theorem for the Brill-Noether Diagram) ∎
For any integer $t \in \mathcal{L}(M)$ in the Lüroth semigroup of a compact Riemann surface M of genus $g > 0$ the maximal function of M satisfies the convexity condition

$$\mu(r) - \mu(r - t) \leq \mu(r + t) - \mu(r) \tag{11.92}$$

for all r.

Proof: If $t = 0$ the inequality is trivial. If $t \in \mathcal{L}(M)$ and $t \neq 0$ then by definition of the Lüroth semigroup $t = c(\tau)$ for a base-point-free holomorphic line bundle τ. For any integer r choose a maximal bundle $\lambda \in X_r^{\max}$, so that $c(\lambda) = r$ and $\mu(r) = \gamma(\lambda) - 1$. The inequality of the preceding Corollary can be rewritten $\gamma(\lambda\tau) + \gamma(\lambda\tau^{-1}) \geq 2\gamma(\lambda)$; and since by the definition of the maximal function $\mu(r + t) \geq \gamma(\lambda\tau) - 1$ and $\mu(r - t) \geq \gamma(\lambda\tau^1) - 1$ it follows that

$$\mu(r + t) + \mu(r - t) \geq \big(\gamma(\lambda\tau) - 1\big) + \big(\gamma(\lambda\tau^{-1}) - 1\big) \geq 2\gamma(\lambda) - 2 = 2\mu(r),$$

which suffices to prove the corollary.

To examine some of the conseqences of the preceding convexity theorem introduce the successive differences

$$\delta_i = r_i - r_{i-1} > 0 \quad \text{for} \quad i \ge 1 \tag{11.93}$$

of the critical values r_i of a compact Riemann surface M. It is clear that diffpart1

$$r_i = \delta_i + \delta_{i-1} + \cdots + \delta_1 \quad \text{for} \quad i \ge 1 \tag{11.94}$$

mudiffpart since $r_0 = 0$, so in particular $\delta_1 = r_1$; and it follows from Corollary **??** that

$$\delta_g = 2, \quad \text{and} \quad \delta_i = 1 \quad \text{for} \quad i > g. \tag{11.95}$$

The convexity condition of the preceding corollary then can be rephrased as follows. diffcor1

Corollary 11.18 *The successive differences of the critical values of a compact Riemann surface of genus $g > 0$ satisfy*

$$\delta_i \le r_1 \quad \textit{for all} \quad i \ge 1;$$

moreover if

$$\underbrace{\delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1}}_{\nu \textit{ terms}} < r_1$$

for some indices $i \ge \nu \ge 1$ then

$$\underbrace{\delta_{i+1} + \delta_{i+2} + \cdots + \delta_{i+\nu+1}}_{\nu+1 \textit{ terms}} \le r_1.$$

As in Corollary **??** the difference $\mu(r) - \mu(s)$ for any integers $r > s$ is equal to the number of critical values r_i in the half-open half-closed interval $(s, r]$, that is to say, such that $s < r_i \le r$. If $r_{i-\nu} > r_i - r_1$ where $i \ge \nu \ge 0$ then there are at least the $\nu + 1$ critical values $r_{i-\nu}, r_{i-\nu+1}, \ldots, r_i$ in the interval $(r_i - r_1, r_i]$ so that $\mu(r_i) - \mu(r_i - r_1) \ge \nu + 1$. It follows from the preceding theorem that $\mu(r_i + r_1) - \mu(r_i) \ge \nu + 1$, and consequently that there are at least $\nu + 1$ critical values in the interval $(r_i, r + r_1]$; these of course must be the critical values $r_{i+1}, r_{i+2}, \ldots, r_{i+\nu+1}$, so that $r_{i+\nu+1} \le r_i + r_1$. The first conclusion of the corollary is the result just ddemonstrated for the case $\nu = 0$, since $r_i - r_i < r_1$ for all indices $i \ge 1$, while the second conclusion is that for the case $\nu \ge 1$, and that suffices to conclude the proof of the corollary.

The property of the successive differences of the critical values described in the preceding lemma can be handled conveniently by introducing the *characteristics* of the maximal function of of the Riemann surface M, the integers k_ν defined by

char

(11.96)

$$k_\nu = \min\Big\{ i \ \Big| \ i \ge \nu, \ \underbrace{\delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1}}_{\nu \text{ terms}} < r_1 \Big\} \ \text{ for } 1 \le \nu < r_1.$$

Since $\delta_i = 1$ for all sufficiently large indices i it is clear that these characteristics are well defined. For some purposes it is convenient to extend this definition and to set $k_0 = 0$; although this extension is somewhat anomalous, it does reflect what is the most significant property of the characteristics, that they describe the intervals in the parameter i of the critical values r_i in which the maximal function $\mu(r)$ increases at different rates. This will become more apparent during the subsequent discussion. Initially though this will be taken as a reason for the terminology, since with this interpretation the characteristics are analogous to the relative characteristics $k_\nu(\lambda;\tau)$. However the behavior of the maximal function $\mu(r)$ is rather more complicated than that of the simple dimensions $\gamma(\lambda\tau^i)$, so that there are some further subtleties involved. All of this probably can be clarified best through a discussion of some illustrative examples. First, though, it is useful to note the following.

Lemma 11.19 *The characteristics of the maximal function of a compact Riemann surface of genus $g > 0$ satisfy*

$$1 < k_1 < k_2 < \cdots < k_{r_1 - 1} = r_1 + g - 1.$$

Proof: By definition k_1 is the least integer $i \ge 1$ such that $\delta_i < r_1$; so since $\delta_1 = r_1 \ge 2$ it follows that $k_1 > 1$. If $i < k_\nu$ for some integer $\nu < 1$ then

$$\underbrace{\delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1}}_{\nu \text{ terms}} \ge r_1;$$

and since $\delta_{i+1} \ge 1$ it is also the case that

$$\underbrace{\delta_{i+1} + \delta_i + \delta_{i-1} + \cdots + \delta_{i-\nu+1}}_{\nu+1 \text{ terms}} \ge r_1$$

hence that $i + 1 < k_{\nu+1}$. In particular for $i = k_\nu - 1$ it follows that $k_\nu < k_{\nu+1}$. Finally since $\delta_i \ge 1$ for all indices i

$$\begin{aligned} k_{r_1-1} &= \min\Big\{ i \ \Big| \ i \ge r_1 - 1 \text{ and } \underbrace{\delta_i + \delta_{i-1} + \cdots + \delta_{i-r_1+2}}_{r_1-1 \text{ terms}} < r_1 \Big\} \\ &= \min\Big\{ i \ \Big| \ i \ge r_1 - 1 \text{ and } \underbrace{\delta_i = \delta_{i-1} = \cdots = \delta_{i-r_1+2}}_{r_1-1 \text{ terms}} = 1 \Big\}. \end{aligned}$$

Now $\delta_g = 2$ and $\delta_i = 1$ for $i \ge g + 1$ by equation (11.94), so it is clear that $k_{r_1-1} = r_1 + g - 1$ and that suffices to conclude the proof of the lemma.

Possibly the clearest illustration of the significance of the characteristics is provided by examining the first few characteristics more closely. Since $\delta_1 = r_1$ and k_1 is the first index i for which $\delta_i < r_1$ then clearly diffseq1

$$\delta_1 = \delta_2 = \cdots = \delta_{k_1-1} = r_1, \quad \delta_{k_1} < r_1; \tag{11.97}$$

correspondingly the sequence of critical values begins diffseq2

$$r_2 = 2r_1, \quad r_3 = 3r_1, \quad \cdots, \quad r_{k_1-1} = (k_1-1)r_1, \quad r_{k_1} < k_1 r_1. \tag{11.98}$$

Now since $\delta_{k_1} < r_1$ then from Corollary 11.18 for the case $\nu = 1$ it follows that $\delta_{k_1+1} + \delta_{k_1+2} \le r_1$; and since $\delta_i \ge 1$ necessarily $\delta_{k_1+1} < r_1$ and $\delta_{k_1+2} < r_1$ as well. The argument can be repeated for k_1+1 and k_1+2 in place of k_1, so that $\delta_{k_1+3} < r_1$, $\delta_{k_1+4} < r_1$, and $\delta_{k_1+5} < r_1$, and so on. Altogether, since k_2 is the
diffseq3 least index i such that $\delta_i + \delta_{i+1} < r_1$, it follows that

$$\delta_i < r_1 \text{ for } i \ge k_1 \quad \text{and} \quad \delta_i + \delta_{i-1} = r_1 \text{ for } k_1 + 2 \le i < k_2. \tag{11.99}$$

Of course it may be the case that $k_2 \le k_1 + 2$, and then the last half of the preceding equation is vacuous; otherwise the successive differences δ_i in the range $k_1 + 2 \le i < k_2$ are determined uniquely by the value δ_{k_1+1} and the recursion relation $\delta_i = r_1 - \delta_{i-1}$. Thus the value of δ_{k_1+1} is a parameter that describes all the other successive differences up to δ_{k_2-1}, while the value of δ_{k_1} is a transitional term between the two ranges in which the successive differences behave quite regularly. The sequence of critical values r_i up to r_{k_2-1} is also determined by these parameters, although the explicit form is slightly more complicated so it is really more convenient just to consider the successive differences. It may be worth noting in passing that $\delta_i + \delta_{i-1} = 2r_1$ for $1 \le i < k_1$ while $\delta_{k_1} + \delta_{k_1-1} < 2r_1$ and $\delta_{k_1+1} + \delta_{k_1} < 2r_1 - 1$.

The argument can be repeated but with increasing complication; it may be sufficient here just to describe the next stage in detail, to indicate the general pattern. Since $\delta_{k_2} + \delta_{k_2-1} < r_1$ then from Corollary 11.18 for the case $\nu = 2$ it follows that $\delta_{k_2+1} + \delta_{k_2+2} + \delta_{k_2+3} \le r_1$; and since $\delta_i \ge 1$ necessarily $\delta_{k_2+2} + \delta_{k_2+1} < r_1$ and $\delta_{k_2+3} + \delta_{k_2+2} < r_1$. Consequently the argument can be repeated with k_2+2 and k_2+3 in place of k_2, although not with k_2+1 in place of k_2; more gaps of this sort arise as the process continues, explaining part of the increase in complication. However it is at least the case here that $\delta_{k_2+4} + \delta_{k_2+3} < r_1$, $\delta_{k_2+5} + \delta_{k_2+4} < r_1$ and $\delta_{k_2+6} + \delta_{k_2+5} \le r_1$, and that is enough to continue the argument to all the rest of the successive differences. Altogether, since k_3 is the least index i such that $\delta_i + \delta_{i-1} + \delta_{i-2} < r_1$, it follows
diffseq4 that

$$\delta_i + \delta_{i-1} < r_1 \quad \text{for} \quad i \ge k_2, \;\; i \ne k_2 + 1, \quad \text{and} \tag{11.100}$$

$$\delta_i + \delta_{i-1} + \delta_{i-2} = r_1 \quad \text{for} \quad k_2 + 3 \le i < k_3, \;\; i \ne k_2 + 4.$$

Again the last half of the preceding equation is vacuous if $k_3 \le k_2 + 3$; otherwise the successive differences δ_i in the range $k_2 + 5 \le i < k_3$ are determined uniquely by the values δ_{k_2+4} and δ_{k_2+3} and the recursion relation $\delta_i = r_1 - \delta_{i-1} - \delta_{i-2}$. Thus the values of δ_{k_2+3} and δ_{k_2+4} are parameters that describe all the remaining successive differences in this range, while the values δ_{k_2}, δ_{k_2+1}, and δ_{k_2+2} describe the transitional range, subject to the relation $\delta_{k_2+3} + \delta_{k_2+2} +$

$\delta_{k_2+1} = 0$. The process can be continued to describe the remaining successive differences, with corresponding patterns in subsequent cases.

To see the significance of these observations for the behavior of the maximal function itself, note as a consequence of (11.97) that each interval $(r, r + r_1]$ of length r_1 for $0 \le r < r_{k_1} - r_1$ includes precisely one critical value of the maximal function, so that rate1

$$\mu(r + r_1) - \mu(r) = 1 \quad \text{for} \quad 0 \le r < r_{k_1} - r_1. \tag{11.101}$$

Similarly as a consequence of (11.99) each interval $(r, r + r_1]$ of length r_1 for $r_{k_1+2} \le r < r_{k_2} - r_1$ includes precisely two critical values, so that rate2

$$\mu(r + r_1) - \mu(r) = 2 \quad \text{for} \quad r_{k_1+2} \le r < r_{k_2} - r_1. \tag{11.102}$$

There is a transitional region between these two intervals, in which the increase of the maximal function modulates between 1 and 2. Of course $\mu(r+r_1) - \mu(r) = 0$ whenever $r < r_{k_0} - r_1 = -r_1$, so that the separately defined characteristic $k_0 = 0$ also describes a point at which the rate of increase of the maximal function changes. There are corresponding results for higher characteristics, until finally $\mu(r + r_1) - \mu(r) = r_1$ whenever $r > 2g - 2$.

Rather than continuing this process, though, it may be better just to turn to the end of the sequence of successive differences of the critical values. Of course $\delta_i = 1$ whenever $i > g$ as in (11.94). The differences δ_i for $i \le g$ but i near g can be determined readily from the duality between the critical values and the gap values of the maximal function. By equation (11.97) it is evident that there is a string of $r_1 - 1$ consecutive gap values beginning with $s_1 = 1$ and continuing through s_{r_1-1}, followed by the critical value r_1 for a gap of 2 between the gap values s_{r_1-1} and s_{r_1}, followed by another string of $r_1 - 1$ consecutive gap values from $r_1 + 1$ to $r_2 - 1 = 2r_1 - 1 - 1$, and so on; consequently from the duality between the gap values and the critical values, as expressed in Corollary **??**, it follows that the string of successive differences of the critical values ending with $\delta_g = 2$ has the form diffseq6

$$\underbrace{1, 1, \ldots, 1, 2}_{r_1-1 \text{ terms}}, \underbrace{1, 1, \ldots, 1, 2}_{r_1-1 \text{ terms}}, \ldots, \underbrace{1, 1, \ldots, 1, 2}_{r_1-1 \text{ terms}} \tag{11.103}$$

where there are $k_1 - 1$ such blocks. This duality of course can be continued to the next string (11.99) of successive differences, and so on, although again with more complication than justifies any detailed treatment here. It may be more useful at this stage to examine some particular cases that arise naturally and in which the results obtained so far can be applied usefully.

Base-point free holomorphic line bundles play a significant role in the Brill-Noether diagram, beyond their appearance in the preceding theorem. The base divisior of a holomorphic line bundle λ on M was defined in (2.9), and by Theorem 2.11 any holomorphic line bundle can be written uniquely as the product

$\lambda = \lambda_0\zeta_{\mathfrak{b}(\lambda)}$ of a base-point-free holomorphic line bundle λ_0 and the line bundle $\zeta_{\mathfrak{b}(\lambda)}$ associated to the base divisor $\mathfrak{b}(\lambda)$ of λ, where $\gamma(\lambda_0) = \gamma(\lambda)$ and $\gamma(\zeta_{\mathfrak{b}(\lambda)}) = 1$; this is the base decomposition of the holomorphic line bundle λ. In this context the *base degree* of a holomorphic line bundle λ is defined by

$$\text{bdeg}(\lambda) = \deg \mathfrak{b}(\lambda); \tag{11.104}$$

so in the base decomposition $\lambda = \lambda_0\zeta_{\mathfrak{b}(\lambda)}$ of a line bundle, $\text{bdeg}(\lambda) = c(\zeta_{\mathfrak{b}(\lambda)}) = c(\lambda) - c(\zeta_0)$, hence $\text{bdeg}(\lambda) = 0$ if and only if λ is base-point-free. Let $\hat{X}_r^{\nu,o}$ denote the set of base-point-free line bundles in $\hat{X}_r^\nu$ and let $\hat{X}_r^{\nu,\sharp}$ denote the complementary set of line bundles in $\hat{X}_r^\nu$ with a nontrivial base divisor, so that

$$\hat{X}_r^{\nu,o} = \left\{ \lambda \in \hat{X}_r^\nu \;\middle|\; \mathfrak{b}(\lambda) = 0 \right\} \tag{11.105}$$

and

$$\hat{X}_r^{\nu,\sharp} = \left\{ \lambda \in \hat{X}_r^\nu \;\middle|\; \mathfrak{b}(\lambda) > 0 \right\}. \tag{11.106}$$

By definition then there is the decomposition

$$\hat{X}_r^\nu = \hat{X}_r^{\nu,o} \cup \hat{X}_r^{\nu,\sharp} \;\text{ where }\; \hat{X}_r^{\nu,o} \cap \hat{X}_r^{\nu,\sharp} = \emptyset. \tag{11.107}$$

The basic result about this decomposition is the following.

Theorem 11.20 *For any compact Riemann surface M of genus $g > 0$ the subset*

$$B_r^\nu = \left\{ (z,\lambda) \in M \times \hat{X}_r^\nu \;\middle|\; z \in \mathfrak{b}(\lambda) \;\right\} \tag{11.108}$$

is a holomorphic subvariety of the holomorphic variety $M \times \hat{X}_r^\nu$. The natural projection $\pi : M \times \hat{X}_r^\nu \longrightarrow \hat{X}_r^\nu$ induces a finite proper surjective holomorphic mapping

$$\pi : B_r^\nu \longrightarrow \hat{X}_r^{\nu,\sharp}, \tag{11.109}$$

so $\hat{X}_r^{\nu,\sharp}$ is a holomorphic subvariety of $\hat{X}_r^\nu$ and $\dim \hat{X}_r^{\nu,\sharp} = \dim B_r^\nu$.

Proof: As in the discussion in Chapter 7, holomorphic line bundles $\lambda \in \hat{X}_r^\nu$ can be described by factors of automorphy of the form $\rho_t\eta$, where η is a fixed factor of automorphy describing a fixed holomorphic line bundle of characteristic class r and ρ_t are canonically parametrized flat factors of automorphy for parameter values $t \in V$ for a suitable holomorphic subvariety $V \subset \mathbb{C}^{2g}$ of an open subset of the parameter space for flat line bundles; the subvariety V in this way parametrizes the variety $\hat{X}_r^\nu$ of holomorphic line bundles. Holomorphic cross-sections of a line bundle $\lambda \in \hat{X}_r^\nu$ correspond to holomorphic relatively automorphic functions for the factor of automorphy $\rho_t\eta$ for the parameter value $t \in V$ parametrizing the line bundle λ. The condition that $\lambda \in \hat{X}_r^\nu$ means that the dimension of the space of relatively automorphic functions is $\nu + 1$ for all

$t \in V$. It then follows from Corollary 7.3, for the special case that the relatively automorphic functions are holomorphic so the auxiliary parameter variety W is empty, that for any bundle λ_0 described by a parameter $t_0 \in V$ there is an open neighborhood $U \subset V$ of the point t_0 and there are $\nu + 1$ holomorphic relatively automorphc functions $f_{i,t}$ for the factor of automorphy $\rho_t\eta$ that are holomorphic functions of the paramter $t \in U$ and are a basis for space of relatively automorphic functions for for the factor of automorphy $\rho_t\eta$ for all $t \in U$. The subset

$$Y = \left\{ (z,t) \in M \times U \;\middle|\; f_{i,t}(z) = 0 \quad \text{for all } 1 \le i \le \nu+1 \right\} \tag{11.110}$$

consequently is a holomorphic subvariety of $M \times U$. If $(z,t) \in Y$ then all the relatively automorphic functions $f_{i,t}(z)$ vanish at the point $z \in M$; that means that all the holomorphic cross-sections of the line bundles λ parametrized by values $t \in U$ vanish at the point z, so by definition z is a point in the divisor $\mathfrak{b}(\lambda)$. Conversely if z is a point in the divisor $\mathfrak{b}(\lambda)$ then by definition all the holomorphic cross-sections of the line bundle λ vanish at the point $z \in M$ so all the relatively automorphic functions $f_{i,t}(z)$ vanish at the point z and consequently $(z,t) \in Y$. Thus locally the set B^ν_r is just the set Y; and since Y is a holomorphic variety that shows thats the set B^ν_r is a holomorphic subvariety in an open neighborhood of each of its points, so B^ν_r itself is a holomorphic variety. The natural projection mapping $\pi : M \times \hat{X}^\nu_r \longrightarrow \hat{X}^\nu_r$ is a proper holomorphic mapping, since M is compact; so there is the commutative diagram of holomorphic mappings

$$\begin{tikzcd} B^\nu_r \arrow[r, "\iota"] \arrow[d, "\pi"'] & M \times \hat{X}^\nu_r \arrow[d, "\pi"'] \\ \pi(B^\nu_r) \arrow[r, "\iota"] & \hat{X}^\nu_r \end{tikzcd} \tag{11.111}$$

where in both cases π is a proper holomorphic mapping and ι is the natural inclusion mapping. By Remmert's Proper Mapping Theorem, as discussed on page 423 in Appendix A.3, the image $\pi(B^\nu_r)$ is a holomorphic subvariety of $\hat{X}^\nu_r$, so it is a holomorphic variety. If $\lambda \in \pi(B^\nu_r)$ there is at least one point $z \in M$ for which $(z,\lambda) \in B^\nu_r$ hence for which z is a point of the divisor $\mathfrak{b}(\lambda)$, so the bundle λ is not base-point-free, while on the other hand if $\lambda \notin \pi(B^\nu_r)$ then there is no point $z \in M$ that is a base point for λ, so λ is base-point-free; therefore $\pi(B^\nu_r) = \hat{X}^{\nu,\sharp}_r$, the subset of non-base-point-free holomorphic line bundles in $\hat{X}^\nu_r$. The set of points $(z,\lambda) \in \hat{X}^\nu_r$ that have the same image $\pi(z,\lambda) = \lambda$ consists of those points $z \in M$ that are points in the base divisor of the line bndle λ so is a finite set of points; the mapping (11.109) thus also is a finite mapping. It then follows from the more detailed version (A.20) of Remmert's Proper Mapping Theorem that $\dim \hat{X}^{\nu,\sharp}_r = \dim B^\nu_r$, and that suffices for the proof.

For an alternative to the decomposition (11.107) introduce the set $\hat{X}^{\nu,*0}_r$ of those holomorphic line bundles in X^ν_r such that the line bundle $\kappa\lambda^{-1} \in$

$\hat{X}^{g-(r-\nu-1)}_{2g-2-r}$ is base-point-free and the set $\hat{X}^{\nu,*\sharp}_r$ of those holomorphic line bundles in X^ν_r such that the line bundle $\kappa\lambda^{-1} \in \hat{X}^{g-(r-\nu-1)}_{2g-2-r}$ has a nontrivial base divisor; of course these are defined only if $\hat{X}^{g-(r-\nu-1)}_{2g-2-r} \neq 0$. There is then the decomposition

---

Some basic relations between the various varieties $\hat{X}^\nu_r$ rest on the decomposition (11.107). Recall from the discussion in Chapter 2 that for any holomorphic line bundle λ with base divisor $\mathfrak{b}(\lambda)$ and for any point $a \in M$

$$\gamma(\lambda\zeta_a^{-1}) = \begin{cases} \gamma(\lambda) & \text{if} \quad a \in \mathfrak{b}(\lambda), \\ \gamma(\lambda) - 1 & \text{if} \quad a \notin \mathfrak{b}(\lambda); \end{cases} \tag{11.112}$$

Thus for any line bundle $\lambda \in \hat{X}^\nu_r$ and for any point $a \in \lambda$ it follows that $\gamma(\lambda) - 1 \leq \gamma(\lambda\zeta_a^{-1}) \leq \gamma(\lambda)$, so the mapping that associates to any pair $(a,\lambda) \in M \times \hat{X}^\nu_r$ the line bundle $\psi(a,\lambda) = \lambda\zeta_a^{-1}$ is a well defined holomorphic mapping

$$\psi : M \times \hat{X}^\nu_r \longrightarrow \hat{X}^{\nu-1}_{r-1} \cup \hat{X}^\nu_{r-1}. \tag{11.113}$$

If $(a,\lambda) \in B^\nu_r$ then $a \in \mathfrak{b}(\lambda)$ so $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda)$, while if $(a,\lambda) \notin B^\nu_r$ then $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) - 1$; thus

$$\psi\left(B^\nu_r\right) \subset \hat{X}^\nu_{r-1} \quad \text{while} \quad \psi\left((M \times \hat{X}^\nu_r) \sim B^\nu_r\right) \subset \hat{X}^{\nu-1}_{r-1}. \tag{11.114}$$

Note that from the definitions it follows that

$$B^\nu_r \subset M \times \hat{X}^{\nu,\sharp}_r \quad \text{while} \quad M \times X^{\nu,o}_r \subset (M \times \hat{X}^\nu_r) \sim B^\nu_r. \tag{11.115}$$

On the other hand if $\gamma(\kappa\lambda^{-1}) \neq 0$ and $\mathfrak{b}(\kappa\lambda^{-1})$ is the base divisor of the line bundle $\kappa\lambda^{-1}$ then by Corollary 2.30

$$\gamma(\lambda\zeta_a) = \begin{cases} \gamma(\lambda) + 1 & \text{if} \quad a \in \mathfrak{b}(\kappa\lambda^{-1}), \\ \gamma(\lambda) & \text{if} \quad a \notin \mathfrak{b}(\kappa\lambda^{-1}), \end{cases} \tag{11.116}$$

hence for any line bundle $\lambda \in \hat{X}^\nu_r$ and for any point $a \in \lambda$ it follows that $\gamma(\lambda) \leq \gamma(\lambda\zeta_a) \leq \gamma(\lambda) + 1$. Therefore the mapping that associates to any pair $(a,\lambda) \in M \times \hat{X}^\nu_r$ the line bundle $\phi(a,\lambda) = \lambda\zeta_a$ is a well defined holomorphic mapping

$$\phi : M \times \hat{X}^\nu_r \longrightarrow \hat{X}^\nu_{r+1} \cup \hat{X}^{\nu+1}_{r+1}. \tag{11.117}$$

Through the isomorphism (11.66) the decomposition (11.107) for a nonempty variety $\hat{X}^{\nu+g-1-r}_{2g-2-r}$ can be carried over to a different decomposition of the variety $\hat{X}^\nu_r$ by setting

$$\hat{X}^{\nu,*o}_r = \left\{ \lambda \in \hat{X}^\nu_r \;\middle|\; \mathfrak{b}(\kappa\lambda^{-1}) = 0 \right\} \tag{11.118}$$

and

$$\hat{X}^{\nu,*\sharp}_r = \left\{ \lambda \in \hat{X}^\nu_r \;\middle|\; \mathfrak{b}(\kappa\lambda^{-1}) > 0 \right\}; \tag{11.119}$$

this provides an alternative disjoint union decomposition

$$\hat{X}_r^\nu = \hat{X}_r^{\nu,*o} \cup \hat{X}_r^{\nu,*\sharp} \quad \text{where} \quad \hat{X}_r^{\nu,*o} \cap \hat{X}_r^{\nu,*\sharp} = \emptyset \tag{11.120}$$

and the alternative subset

$$B_r^{*\nu} = \Big\{ (z,\lambda) \in M \times \hat{X}_r^\nu \ \Big|\ z \in \mathfrak{b}(\kappa\lambda^{-1}) \ \Big\}\tag{11.121}$$

Corollary 11.21 *For any compact Riemann surface M of genus $g > 0$ the subset $B_r^{*\nu}$ is a holomorphic subvariety of the holomorphic variety $M \times \hat{X}_r^\nu$. The natural projection $\pi : M \times \hat{X}_r^\nu \longrightarrow \hat{X}_r^\nu$ induces a finite proper surjective holomorphic mapping*

$$\pi : B_r^\nu \longrightarrow \hat{X}_r^{\nu,*\sharp},\tag{11.122}$$

so $\hat{X}_r^{\nu,*\sharp}$ is a holomorphic subvariety of $\hat{X}_r^\nu$ and $\dim \hat{X}_r^{\nu,*\sharp} = \dim B_r^{*\nu}$.

This follows immediately from Theorem 11.20 applied to the variety $\hat{X}_{2g-2-r}^{\nu+g-1-r}$ if it is nonempty, so no further proof is required.

It follows from the preceding observations that in terms of the subvariety (11.108)

$$\phi\left(B_r^\nu\right) \subset \hat{X}_{r+1}^\nu \quad \text{while} \quad \phi\Big((M \times \hat{X}_r^\nu) \sim B_r^\nu\Big) \subset \hat{X}_{r-1}^\nu.\tag{11.123}$$

For line bundles on the maximal and minimal curves of the Brill-Noether diagram this decomposition of the holomorphic varieties $\hat{X}_r^\nu$ can be determined readily.

Theorem 11.22 *Let M be a compact Riemann surface of genus $g > 0$.*
(i) *For $g < r \le 2g-1$*

$$\hat{X}_r^{MIN,\flat} = \Big\{ \kappa\zeta_a\zeta_{\mathfrak{b}}^{-1} \ \Big|\ a \in M,\ \ \zeta_{\mathfrak{b}} \in \hat{X}_{2g-1-r}^0 \ \text{ and } \ a \notin \mathfrak{b} \ \Big\}.\tag{11.124}$$

(ii) *For $r > 2g-1$*

$$\hat{X}_r^{MIN,\flat} = \emptyset\tag{11.125}$$

so all line bundles in $\hat{X}_r^\nu$ for $r > 2g-1$ are base-point-free.

Proof: (i) If $g < r < 2g-1$ and $\lambda \in \hat{X}_r^{MIN} = \hat{X}_r^{r-g}$ is not base-point-free then there is a point $a \in M$ such that $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda) = r-g+1$; and by the Riemann-Roch Theorem that is equivalent to $\gamma(\kappa\lambda^{-1}\zeta_a) = 1$. Since $c(\kappa\lambda^{-1}\zeta_a) > 0$ then $\kappa\lambda^{-1}\zeta_a = \zeta_{\mathfrak{b}} \in \hat{X}_{2g-r-1}^0$ for a uniquely determined positive divisor $\mathfrak{b} \in M^{(2g-1-r)}$; thus $\lambda = \kappa\zeta_a\zeta_{\mathfrak{b}}^{-1}$ where $\zeta_{\mathfrak{b}} \in \hat{X}_{2g-r-1}^0$. There is a unique holomorphic cross-section of the line bundle $\zeta_{\mathfrak{b}}$, up to a constant factor, and if $a \in \mathfrak{b}$ that cross-section vanishes at a hence $1 = \gamma(\zeta_{\mathfrak{b}}\zeta_a^{-1}) = \gamma(\kappa\lambda^{-1})$; but then by the Rieman-Roch theorem $\gamma(\lambda) = r-g+2$, a contradiction. Consequently

$a \notin \mathfrak{b}$, which shows that the holomorphic line bundles $\lambda \in \hat{X}_r^{MIN}$ that are not base-point-free satisfy (11.124) in this case. Actually if $r = 2g-1$ it follows from Theorem 2.28 that the holomorphic line bundles $\lambda \in \hat{X}_{2g-1}^{g-1}$ that are not base-point-free are precisely those of the form $\lambda = \kappa\zeta_a$, which is the special case of (11.124) for which the divisor $\mathfrak{b}$ is the empty set.

Conversely to show that for $g < r \le 2g-1$ all the bundles of the form (11.124) actually are not base-poiont-free, suppose that λ is a holomorphic line bundle such that $\lambda = \kappa\zeta_a\zeta_{\mathfrak{b}}^{-1}$ where $\zeta_{\mathfrak{b}} \in \hat{X}_{2g-1-r}^0$ and $a \notin \mathfrak{b}$. Then $c(\lambda) = r$ and by the Riemann-Roch Theorem $\gamma(\lambda) = \gamma(\zeta_{\mathfrak{b}}\zeta_a^{-1}) + r + 1 - g$ and $\gamma(\lambda\zeta_a^{-1}) = \gamma(\zeta_{\mathfrak{b}}) + r - g = r + 1 - g$. Since $a \notin \mathfrak{b}$ and $\gamma(\zeta_{\mathfrak{b}}) = 1$ it must be the case that $\gamma(\zeta_{\mathfrak{b}}\zeta_a^{-1}) = 0$; the preceding equations then show $\gamma(\lambda) = r + 1 - g$ so $\lambda \in \hat{X}_r^{r-g} = \hat{X}_r^{MIN}$ and $\gamma(\lambda\zeta_a^{-1}) = \gamma(\lambda)$ hence λ is not base-point-free.

(ii) Finally if $r > 2g-1$ then $c(\kappa\lambda^{-1}\zeta_a) < 0$ so $\gamma(\kappa\lambda^{-1}\zeta_a) = 0$, which contradicts the condition that $\gamma(\kappa\lambda^{-1}\zeta_a) = 1$; thus there can be no holomorphic line bundles in $\hat{X}_r^{MIN}$ that are not base-point free if $r > 2g-1$. That suffices to conclude the proof.

Corollary 11.23 *For a compact Riemann surface M of genus $g > 0$ all integers $r > g$ belong to the Lüroth semigroup $\mathcal{L}(M)$ of M.*

Proof: Since the Lüroth semigroup $\mathcal{L}(M)$ by definition is the set of integers r such that there is a base-point-free holomorphic line bundle λ with $c(\lambda) = r$, the corollary will be proved by showing that the holomorphic varieties $\hat{X}_r^{MIN}$ for $r > g$ contain base-point-free holomorphic line bundles. That is of course the case for $r > 2g-1$ by part (ii) of the preceding theorem, so it suffices to consider the varieties $\hat{X}_r^{MIN}$ for $g < r \le 2g-1$. The subset $V \subset \hat{X}_r^{MIN} \subset P_r(M)$ consisting of holomorphic line bundles that are not base-point-free is a holomorphic subvariety by Theorem **??**; and if $g < r \le 2g-1$ the subvariety V consists of those holomorphic line bundles of the form $\lambda = \kappa\zeta_a\zeta_{\mathfrak{b}}^{-1}$ where $\zeta_{\mathfrak{b}} \in \hat{X}_{2g-r-1}^0$ and $a \notin \mathfrak{b}$ by (i) of the preceding theorem, so V is contained in the image of the holomorphic mapping $\phi : M \times \hat{X}_{2g-r-1} \longrightarrow P^r(M)$ that associates to any point $a \in M$ any line bundle $\zeta_{\mathfrak{b}} \in \hat{X}_{2g-r-1}^0$ the line bundle $\kappa\zeta_a\zeta_{\mathfrak{b}}^{-1} \in P_r(M)$. The product $M \times \hat{X}_{2g-r-1}$ is a compact complex manifold of dimension $2g-r$, since $\dim M = 1$ and $\dim \hat{X}_{2g-r-1} = 2g-r-1$; and by the detailed form (A.20) of Remmert's Proper Mapping Theorem it follows that $\dim \phi(M \times X_{2g-r-1}) \le 2g-r$ hence $\dim V \le 2g-r$. The holomorphic variety $\hat{X}_r^{MIN}$ has dimension g by (11.82), and since $2g-r < g$ for $r > g$ it follows that $\dim V < \dim \hat{X}_r^{MIN}$ so the complement $\hat{X}_r^{MIN} \sim V$, the set of base-point-free holomorphic line bundles in $\hat{X}_r^{MIN}$, is nonempty. That suffices for the proof.

For $g < r < 2g-1$ the proof of the preceding corollary really amounted to showing that both the subvariety $\hat{X}_r^{MIN,\flat}$ and its complement $\hat{X}_r^{MIN,o}$ are nonempty, so in that range the variety $\hat{X}_r^{\nu}$ contains both base-point-free line bundles and line bundles with nontrivial base divisors.

Theorem 11.24 *If M is a compact Riemann surface of genus $g > 0$ with the critical values r_i then for any integer r in the range $r_i < r < r_{i+1}$ for $i \ge 0$ a holomorphic line bundle $\lambda = \lambda_i \zeta_{\mathfrak{d}}$ where $\lambda_i \in X_{r_i}^{MAX}$ and $\mathfrak{d} \in M_r^{(r-r_i)}$ is a maximal bundle $\lambda \in \hat{X}_r^{MAX}$ and $\lambda = \lambda_i \zeta_{\mathfrak{d}}$ is its base decomposition; so* $\operatorname{bdeg}(\lambda) = \deg \mathfrak{d}$ *and λ is not base-point-free.*

Proof: If $0 \le r_i < r < r_{i+1}$ and $\lambda_i \in \hat{X}_{r_i}^{MAX}$ then λ_i is base-point-free by the preceding theorem. If $\lambda = \lambda_i \zeta_{\mathfrak{d}}$ where $\mathfrak{d}$ is a positive divisor with $\deg \mathfrak{d} = r - r_i$ then $\gamma(\lambda) \ge \lambda_i$ by Lemma 2.6 and $c(\lambda) = c(\lambda_i) + \deg \mathfrak{d} < r_{i+1}$ so $\gamma(\lambda) < \gamma(\lambda_i)$ as well and consequently $\gamma(\lambda) = \gamma(\lambda_i)$ so $\lambda \in \hat{X}_r^{MAX}$; and by Theorem 2.12 (i) in addition $\lambda = \lambda_i \zeta_{\mathfrak{d}}$ is the base decomposition of the line bundle λ. That suffices for the proof.

Theorem **??** shows that the varieties $\hat{X}_{r_i}^{MAX}$ for the critical values r_i contain only base-point-free holomorphic line bundles, while Theorem 11.24 shows that the varieties $\hat{X}_r^{MAX}$ contain line bundles that are not base-point-free if r is not a critical value; thus the critical values of a compact Riemann surface M of genus $g > 0$ can be characterized as those integers $r \ge 0$ such that $\hat{X}_r^{MAX}$ consists entirely of base-point-free holomorphic line bundles. For the general varieties X_r^{ν}, recall from the discussion in Chapter 2 that for any holomorphic line bundle λ with base divisor $\mathfrak{b}(\lambda)$ on a compact Riemann surface M of genus $g > 0$ and any point $a \in M$

$$\gamma(\lambda \zeta_a^{-1}) = \begin{cases} \gamma(\lambda) & \text{if} \quad a \in \mathfrak{b}(\lambda), \\ \gamma(\lambda) - 1 & \text{if} \quad a \notin \mathfrak{b}(\lambda); \end{cases} \tag{11.126}$$

and if $\gamma(\kappa\lambda^{-1}) \ne 0$ and $\mathfrak{b}(\kappa\lambda^{-1})$ is the base divisor of the line bundle $\kappa\lambda^{-1}$ then by Corollary 2.30

$$\gamma(\lambda \zeta_a) = \begin{cases} \gamma(\lambda) + 1 & \text{if} \quad a \in \mathfrak{b}(\kappa\lambda^{-1}), \\ \gamma(\lambda) & \text{if} \quad a \notin \mathfrak{b}(\kappa\lambda^{-1}). \end{cases} \tag{11.127}$$

For a special case of the preceding discussion, if $\lambda \in \hat{X}_r^1$ is base-point-free then for any point $z \in M$ clearly $\lambda \zeta_z^{-1} \in \hat{X}_{r-1}^0$ so

$$\lambda \zeta_z^{-1} = \zeta_{\mathfrak{d}}(z) \ \text{ for a unique divisor } \mathfrak{d}(z) \in M^{r-1} \tag{11.128}$$

thus describing a holomorphic mapping $\phi : M \longrightarrow M^{r-1}$. Assume first that the divisor $\mathfrak{d}(z)$ consists of $r-1$ distinct poins z_i for all points $z \in M \sim E$ for a finite subset $E \subset M$. Then in an open neighborhood U of any point $z_0 \in M \sim E$ there are well defined holomorphic mappings $\phi_i : U \longrightarrow M$ such that $\mathfrak{d}(z) = \sum_{i=1}^{r-1} 1 \cdot \phi_i(z)$. In any simply connected intersection of such neighborhoods U the local holomorphiuc mappings match so continue analytically to holomorphic mappings in the union of the neighborhoods. For any paths in the universal covering space $\widetilde{M \sim E}$ the local mappings continue analytically to single valued holomorphic mappings $\widetilde{\phi}_i : M \sim E \longrightarrow M$; but for any covering translation

$T \in \Gamma_E$ of the universal covering space over $M \sim E$ the local mappings satisfy $\phi_i(Tz) = \sum_{j=1}^{r-1} a_{ij}(T)\phi_j(z)$ for some complex constants $a_{ij}(T)$; describing a permutation matrix A_T; thus if $\phi = \{\widetilde{\phi}_i\}$ is the mapping $\widetilde{\phi} : \widetilde{M \sim E} \longrightarrow M^{r-1}$ then $\widetilde{\phi}(z) = A_{\widetilde{\phi}(z)}$ for some nonsingular matrices A_T describing a permutation representation $A \in \mathrm{Hom}(\Gamma_E, \mathrm{Gl}(r-1, \mathbb{C})$. This describes a branched covering space $\widehat{M}$ of degree $r-1$ over M, branched at the points of E. (The abelian differentials on M lift fo this covering in two ways; more Riemann relations?).

If M is a compact Riemann surface of genus $g > 0$ then for any point $a \in M$ there is the local maximal function $\mu_a(r)$ defined by $\mu_a(r) = \gamma(\zeta_a^r) - 1$, as discussed in Chapter 9. It is evident from the definitions that

$$\mu(r) \ge \mu_a(r) \quad \text{for all } a \in M. \tag{11.129}$$

The local critical values $r_i(a)$ are defined in terms of the local maximal function $\mu_a(r)$ just as the critical values r_i are defined in terms of the maximal function $\mu(r)$, both as special cases of the general discussion of functions similar to the maximal functions in (11.34); thus

$$r_i(a) = \inf \left\{ r \in \mathbb{Z} \;\middle|\; \mu_a(r) \ge i \right\}. \tag{11.130}$$

Since $\mu\big(r_i(a)\big) \ge \mu_a\big(r_i(a)\big) = i$ it follows from the definition of the critical value r_i that

$$r_i \le r_i(a) \quad \text{for all } a \in M. \tag{11.131}$$

The sequence $r_i(a)$ for any point $a \in M$ is also an additive semigroup in $\mathbb{Z}$ and a subsemigroup of of the Lüroth semigroup $\mathcal{L}(M)$ of the Riemann surface, by Corollary 9.10, properties which are shared with the sequence r_i. A somewhat different local version of the maximal function is also of interest.

The *semilocal maximal function* of M for a positive divisor $\mathfrak{a} \in M^{(n)}$ of degree n is defined as the function of integers

$$\mu_{\mathfrak{a}}(r) = \max \left\{ \gamma(\zeta_{\mathfrak{a}'}) - 1 \;\middle|\; \begin{array}{l} \mathfrak{a} = \mathfrak{a}' + \mathfrak{a}'', \\ \mathfrak{a}' \in M^{(r)},\ \mathfrak{a}'' \in M^{(n-r)} \end{array} \right\} \tag{11.132}$$

for integers $1 \le r \le n-1$, extended to be a function of all integers $r \in \mathbb{Z}$ by setting

$$\mu_{\mathfrak{a}}(r) = \begin{cases} -1 & \text{for } r < 0, \\ 0 & \text{for } r = 0, \\ \gamma(\zeta_{\mathfrak{a}}) - 1 & \text{for } r \ge n. \end{cases} \tag{11.133}$$

As a word of caution, the semilocal maximal function $\mu_{\mathfrak{a}}(r)$ differs significantly from both the maximal function $\mu(r)$ and the local maximal function $\mu_a(r)$ since unlike the latter two the function $\mu_{\mathfrak{a}}(r)$ involves the dimensions of the spaces of holomorphic cross-sections of only finitely many holomorphic line bundles so it is bounded above. Thus although the local maximal function for a point $a \in M$ and the semilocal maximal function for the divisor $n \cdot a$ for $n > 0$ coincide initially, since as is evident from their definitions $\mu_a(r) = \mu_{n\cdot a}(r)$ for $0 \le r \le n$, nonetheless $\mu_a(r) > \mu_{n\cdot a}(r) = \gamma(\zeta_a^n) - 1$ for $r > n$.

Theorem 11.25 *The semilocal maximal function of a compact Riemann surface M of genus $g > 0$ for a positive divisor $\mathfrak{a}$ satisfies*

$$\mu_{\mathfrak{a}}(r) \le \mu_{\mathfrak{a}}(r+1) \le \mu_{\mathfrak{a}}(r) + 1, \tag{11.134}$$

and in particular

$$\mu_{\mathfrak{a}}(1) = 0. \tag{11.135}$$

For any two positive divisors $\mathfrak{a}_1$ and $\mathfrak{a}_2$

$$\text{if } \ \mathfrak{a}_1 \ge \mathfrak{a}_2 \ \textit{ then } \ \mu_{\mathfrak{a}_1}(r) \ge \mu_{\mathfrak{a}_2}(r). \tag{11.136}$$

The semilocal maximal function and the maximal function are related by

$$\mu_{\mathfrak{a}}(r) \le \ \ \mu(r) \ \textit{ for all } \ r \tag{11.137}$$

and

$$\mu(r) = \sup\left\{ \mu_{\mathfrak{a}}(r) \,\middle|\, \deg \mathfrak{a} \ge r \right\} \ \textit{ for all } \ r \tag{11.138}$$

Proof: First since $\gamma(\zeta_p) = 1$ for any point $p \in M$ it is evident that $\mu_{\mathfrak{a}}(1) = 0$, which is (11.135).

If $1 \le r \le n-1$ and $\mathfrak{a} = \mathfrak{a}' + \mathfrak{a}''$ where $\mathfrak{a} \in M^{(n)}$ and $\mathfrak{a}' \in M^{(r)}$ then for any point $p \in \mathfrak{a}''$ it follows from Lemma 2.6 that $\gamma(\zeta_{\mathfrak{a}'}) \le \gamma(\zeta_{\mathfrak{a}})$ and

$$\gamma(\zeta_{\mathfrak{a}'}) - 1 \le \gamma(\zeta_{\mathfrak{a}'+p}) - 1 \le \gamma(\zeta_{\mathfrak{a}'}) \le \gamma(\zeta_{\mathfrak{a}}). \tag{11.139}$$

If $r \le n-2$ then $\gamma(\zeta_{\mathfrak{a}'+p}) - 1 \le \mu_{\mathfrak{a}}(r+1)$ by Definition (11.132), while if $r = n-1$ then $\mathfrak{a}' + p = \mathfrak{a}$ so $\gamma(\zeta_{\mathfrak{a}'+a}) - 1 = \gamma(\zeta_{\mathfrak{a}}) - 1 = \mu_{\mathfrak{a}}(r+1)$ by (11.133); thus in either case $\gamma(\zeta_{\mathfrak{a}'}) - 1 \le \mu_{\mathfrak{a}}(r+1)$, and since that is the case for all divisors $\mathfrak{a}' \in M^{(r)}$ it follows from (11.139) that $\mu_{\mathfrak{a}}(r) \le \mu_{\mathfrak{a}}(r+1) \le \mu_{\mathfrak{a}}(r) + 1$, which is (11.134) for $1 \le r \le n-1$. If $r = 0$ then since $\mu_{\mathfrak{a}}(0) = \mu_{\mathfrak{a}}(1) = 0$ by (11.133) and (11.135) that is enough to demonstrate (11.134) for $r = 0$; if $r = -1$ since also $\mu_{\mathfrak{a}}(-1) = -1$ by (11.133) that is enough to demonstrate (11.134) for $r = -1$; and finally since also $\mu_{\mathfrak{a}}(r) = -1$ for $r < -1$ by (11.133) that is enough to demonstrate (11.134) for $r < -1$ Since $\mu_{\mathfrak{a}}(r) = \gamma(\zeta_{\mathfrak{a}}) - 1$ for $r \ge n$ by (11.133) that is enough to demonstrate (11.134) for $r \ge n$, which establishes (11.134) for all r.

If $\mathfrak{a}_1 \ge \mathfrak{a}_2$ then $\mathfrak{a}_1 = \mathfrak{a}_2 + \mathfrak{a}_3$ for another positive divisor $\mathfrak{a}_3$, so if $n_i = \deg \mathfrak{a}_i$ then $n_1 = n_2 + n_3$. If $1 \le r \le n_2 - 1$ and if $\mathfrak{a}_2 = \mathfrak{a}_2' + \mathfrak{a}_2''$ for positive divisors $\mathfrak{a}_2'$ and $\mathfrak{a}_2''$ for which $\deg \mathfrak{a}_2' = r$ then $\mathfrak{a}_1 = \mathfrak{a}_2' + (\mathfrak{a}_2'' + \mathfrak{a}_3)$ and consequently it follows from the definition (11.132) for the divisor $\mathfrak{a}_1$ that $\gamma(\mathfrak{a}_2')-1\le\mu_r(\mathfrak{a}_1)$; and since that is the case for any choice of the divisor $\mathfrak{a}_2'$ it follows from the definition (11.132) for the divisor $\mathfrak{a}_2$ now that $\mu_{\mathfrak{a}_2}(r)\le\mu_{\mathfrak{a}_1}(r)$. On the other hand if $n_2\le r\le n_1-1$ and if $\mathfrak{a}_3=\mathfrak{a}_3'+\mathfrak{a}_3''$ for positive divisors $\mathfrak{a}_3'$ and $\mathfrak{a}_3''$ for which $\deg\mathfrak{a}_3'=r-n_2$ then $\mathfrak{a}_1=(\mathfrak{a}_2+\mathfrak{a}_3')+\mathfrak{a}_3''$ where $\deg(\mathfrak{a}_2+\mathfrak{a}_3')=r$ so from the definition (11.132) it follows that $\gamma_{\mathfrak{a}_2+\mathfrak{a}_3'}\le\mu_{\mathfrak{a}_1}(r)$; but from (11.133) and Lemma 2.6 it further follows that $\mu_{\mathfrak{a}_2}(r)=\gamma(\zeta_{\mathfrak{a}_2})-1\le\gamma(\zeta_{\mathfrak{a}_2+\mathfrak{a}_3'})$, and these two inequalities show that $\mu_{\mathfrak{a}_2}(r)\le\mu_{\mathfrak{a}_1}(r)$ also for $n_2\le r\le n_1-1$. Finally if $r\le 0$ then $\mu_{\mathfrak{a}_1}(r)=\mu_{\mathfrak{a}_2}(r)=0$ by (11.133) while if $r\ge n_1$ then from (11.132) and Lemma 2.6 it follows that $\mu_{\mathfrak{a}_2}(r)=\gamma(\zeta_{\mathfrak{a}_2}-1)\le\gamma(\zeta_{\mathfrak{a}_1}-1)=\mu_{\mathfrak{a}_1}(r)$, and that suffices to demonstrate (11.136).

Finally (11.137) is quite obvious from the definitions of the two maximal functions; and if $\mu(r)=\gamma(\zeta_{\mathfrak{a}})-1$ for some divisor $\mathfrak{a}\in M^{(r)}$ then it follows from definition (11.133) that $\mu_{\mathfrak{a}}(r)=\gamma(\zeta_{\mathfrak{a}})-1=\mu_{\mathfrak{a}}(r)$, and that suffices to establish (11.138) and thereby to conclude the proof.

Since the semilocal maximal function satisfies (11.33) it has all the properties discussed on page 277 and the following pages. For a positive divisor $\mathfrak{a}\in M^{(n)}$ of degree n the maximum and minimum values of the semilocal maximal function are $n_+(\mathfrak{a})=\gamma(\zeta_{\mathfrak{a}})-1$ and $n_-(\mathfrak{a})=-1$. The invariants r_i for the semilocal maximal function are defined by

$$r_i(\mathfrak{a})=\inf\Big\{\,r\in\mathbb{Z}\;\Big|\;\mu_{\mathfrak{a}}(r)\ge i\,\Big\},\tag{11.140}$$

following the general definition in (11.34), and are called the the *semilocal critical values* for the divisor $\mathfrak{a}$; there are altogether just the $n_+(\mathfrak{a})+1=\gamma(\zeta_{\mathfrak{a}})$ finite semilocal critical values

$$0=r_0(\mathfrak{a})<r_1(\mathfrak{a})<\cdots<r_{\gamma(\zeta_{\mathfrak{a}})-1}(\mathfrak{a})\tag{11.141}$$

extended for convenience in use in subsequent formulas by setting $r_{-1}(\mathfrak{a})=-\infty$ and $r_{\gamma(\zeta_{\mathfrak{a}})}(\mathfrak{a})=+\infty$. The complement of the set of semilocal critical values for a divisor $\mathfrak{a}$ is the set of *semilocal gap values* $s_j(\mathfrak{a})$ for the divisor $\mathfrak{a}$. An example of the graph of a semilocal maximal function, indicating the semilocal critical values, is sketched in Figure 11.2.

The basic properties of the semilocal critical values follow the pattern of the basic properties of the critrical values and local critical values. It follows from (11.35) that

$$\mu_{\mathfrak{a}}(r)=i\ \text{ for }\ r_i(\mathfrak{a})\le r<r_{i+1}(\mathfrak{a}),\tag{11.142}$$

and it follows from (11.36) that

$$\mu_{\mathfrak{a}}(r)-\mu_{\mathfrak{a}}(r-1)=\begin{cases}1 & \text{if } r=r_i(\mathfrak{a}) \text{ for some } i \text{ and}\\ 0 & \text{otherwise.}\end{cases}\tag{11.143}$$

The local maximal functions for different divisors are related as in (11.136), so if $\mathfrak{a}_1\ge\mathfrak{a}_2$ then $\mu_{\mathfrak{a}_1}\big((r_i(\mathfrak{a}_2)\big)\ge\mu_{\mathfrak{a}_2}\big((r_i(\mathfrak{a}_2)\big)=i$ and it follows from the definition

Figure 11.2: Example of a semilocal maximal function for a positive divisor $\mathfrak{a}$ of degree 13.

(11.140) of the semilocal critical value $r_i(\mathfrak{a}_1)$ that

$$\text{if } \ \mathfrak{a}_1 \geq \mathfrak{a}_2 \ \text{ then } \ r_i(\mathfrak{a}_1) \leq r_i(\mathfrak{a}_2) \ \text{ for any index } i. \tag{11.144}$$

in particular $r_i(\mathfrak{a}) \leq r_i(a)$ for any point a in the divisor $\mathfrak{a}$ since then $1 \cdot a \leq \mathfrak{a}$. In view of (**??**) the corresponding argument shows that $r_i \leq r_i(\mathfrak{a})$ for any positive divisor $\mathfrak{a}$. In summary, the semilocal critical values $r_i(\mathfrak{a})$, the local critical values $r_i(a)$ and the critical values r_i of a Riemann surface M are related by

$$r_i \leq r_i(\mathfrak{a}) \leq r_i(a) \quad \text{for any point } a \text{ in a positive divisor } \mathfrak{a}. \tag{11.145}$$

The interest of these critical values lies in part in the following observation.

Theorem 11.26 *The semilocal critical values at a divisor $\mathfrak{a}$ on a compact Riemann surface M of genus $g > 0$ belong to the Lüroth semigroup of M.*

Proof: Since $r_0(\mathfrak{a}) = 0$ does belong to the Lüroth semigroup of M it is enough just to demonstrate the theorem for strictly positive semilocal critical values. If $r = r_i(\mathfrak{a}) > 0$ is a semilocal critical value for the divisor $\mathfrak{a}$ then $\mu_{\mathfrak{a}}(r) = i$ while $\mu_{\mathfrak{a}}(s) < i$ if $s < r$. By the definition (11.132) of the semilocal maximal function there are positive divisors $\mathfrak{a}'$ and $\mathfrak{a}''$ of degrees $\deg \mathfrak{a}' = r$ and $\deg \mathfrak{a}'' = n - r$ such that $\mathfrak{a} = \mathfrak{a}' + \mathfrak{a}''$ and $\gamma(\zeta_{\mathfrak{a}'}) - 1 = i$. The theorem will be proved by showing that the line bundle $\zeta_{\mathfrak{a}'}$ of characteristic class $c(\zeta_{\mathfrak{a}'}) = r$ is base-point-free. If to the contrary $\zeta_{\mathfrak{a}'}$ is not base-point-free then there is some point $x \in M$ for which $\gamma(\zeta_{\mathfrak{a}'}\zeta_x^{-1}) = \gamma(\zeta_{\mathfrak{a}'}) = i$; and in that case it follows from Lemma 2.6 that all the holomorphic cross-sections of the line bundle $\zeta_{\mathfrak{a}'}$ vanish at the point x. If $\mathfrak{a}' = \sum_k \mu_k \cdot a_k$ and h_k is a nontrivial holomorphic cross-section of the bndle ζ_{a_k} then h_k has a simple zero at the point a_k as its sole zero; the product $h = \prod_k h_k^{\mu_k}$ then is a holomorphic cross-section of the bundle $\zeta_{\mathfrak{a}'}$ that vanishes only at points of $\mathfrak{a}'$, and since $h(x) = 0$ it follows that x is a point of the divisor $\mathfrak{a}'$. If $\mathfrak{a}' = \mathfrak{a}''' + 1 \cdot x$ for another positive divisor $\mathfrak{a}'''$ then $\mathfrak{a} = \mathfrak{a}' + \mathfrak{a}'' = \mathfrak{a}''' + (\mathfrak{a}'' + 1 \cdot x)$ where $\deg \mathfrak{a}''' = r - 1$, so from the definition (11.132) of the semilocal maximal function again it follows that $\mu_{\mathfrak{a}}(r-1) \geq \gamma(\zeta_{\mathfrak{a}'''}) = i$, which contradicts the assumption that r is the local critical value $r = r_i(\mathfrak{a})$. That contrdiction suffices to conclude the proof.

The semilocal maximal function for a divisor $\mathfrak{a}$ can be read directly from the Brill-Noether matrix $\Omega(\mathfrak{a})$ of $\mathfrak{a}$, extending to the semilocal maximal function the treatment of the local maximal function discussed on page **??** and the following pages. If $\mathfrak{a} = \nu_1 \cdot p_1 + \cdots + \nu_m \cdot p_m$ is a divisor of degree $n = \nu_1 + \cdots + \nu_m$ for distinct points $p_1, \ldots, p_m$ the Brill-Noether matrix is the $g \times n$ complex matrix with the rows as in (11.9). Explicitly if $\omega_i = f_{i\alpha_j}(z_{\alpha_j})dz_{\alpha_J}$ for $1 \le i \le g$ is a basis for the holomorphic abelian differentials, expressed in terms of local coordinates z_{α_j} at the points p_j, the entries in row i of the matrix $\Omega_{\alpha_1,\ldots,\alpha_m}(\mathfrak{a})$ are the functions $f^{(k_j)}_{i,\alpha_j}(p_j)/k_j!$ for $1 \le j \le m$ and $0 \le k_j \le \nu_j - 1$; the columns are indexed by j and k_j. The Riemann-Roch Theorem in terms of the Brill-Noether has the form (11.3) so

$$\gamma(\zeta_{\mathfrak{a}}) - 1 = n - \text{rank } \Omega_{\alpha_1,\ldots,\alpha_m}(\mathfrak{a}). \tag{11.146}$$

The divisors $\mathfrak{a}'$ for which there is a decomposition $\mathfrak{a} = \mathfrak{a}' + \mathfrak{a}''$ into a sum of positive divisors where $\deg \mathfrak{a}' = r$ are just the divisors $\mathfrak{a}' = \mu_1 \cdot p_1 + \cdots + \mu_m \cdot p_m$ for which $0 \le \mu_i \le \nu_i$ and $\mu_1 + \cdots + \mu_m = r$; and the Brill-Noether matrix for the divisor $\mathfrak{a}'$ is just the matrix $\Omega^{\mu_1,\ldots,\mu_m}_{\alpha_1,\ldots,\alpha_m}(\mathfrak{a})$ formed by the $r = \mu_1 + \cdots + \mu_m$ columns of the matrix $\Omega_{\alpha_1,\ldots,\alpha_m}(\mathfrak{a})$ for the column parameters restricted to the values $1 \le j \le m, \ \ 0 \le k_j \le \mu_j - 1$. Consequently from definition (11.132) it follows that

$$\mu_{\mathfrak{a}}(r) = \begin{cases} \max\left\{ r - \text{rank } \Omega^{\mu_1,\cdots\mu_m}_{\alpha_1,\ldots,\alpha_m}(\mathfrak{a}) \,\middle|\, 0 \le \mu_j \le \nu_j \right\} \\ r - \min \text{rank} \left\{ \Omega^{\mu_1,\cdots\mu_m}_{\alpha_1,\ldots,\alpha_m}(\mathfrak{a}) \,\middle|\, 0 \le \mu_j \le \nu_j \right\} \end{cases} \tag{11.147}$$

for integers $1 \le r \le n - 1$.

[refer to chap max2]

Theorem 11.27 *????? If M is a compact Riemann surface with the first critical value r_1 then*

$$\text{rank}\{f_i(z_j)\} \ge r_1 - 1 \tag{11.148}$$

for any divisor $\mathfrak{d} = \sum_{j=1}^{r_1-1} z_j \in M^{(r_1-1)}$, where the holomorphic abelian differentials on M are written $\omega_i(z) = f_i(z)dz$.

Proof: Since $\gamma(\zeta_{\mathfrak{d}}) < 1$ for any divisor $\mathfrak{d}$ of degree $\deg \mathfrak{d} < r_1$ it follows from (**??**) in the proof of the preceding theorem that rank $\{f_i(z_j)\} \ge r_1 - 1$, which suffices for the proof.

When there are at least 2 linearly independent holomorphic cross-sections of a holomorphic line bundle λ it is possible to use these cross-sections to obtain some further information about the spaces of holomorphic cross-sections of the line bndles λ^n for any $n > 0$. Since the first critical value r_1 is the least integer for which there are line bundles λ of characteristic class $c(\lambda) = r_1$ such that $\gamma(\lambda) \geq 2$, the value r_1 plays a particularly significant role in the study of the maximal function for Riemann surfaces.

Theorem 11.28 *If $f_0, f_1 \in \Gamma(M, \mathcal{O}(\lambda))$ are linearly independent holomorphic cross-sections of a holomorphic line bundle λ over a compact Riemann surface M then for any $n > 0$ the $n+1$ products $f_1^i f_2^{n-i}$ for $0 \leq i \leq n$ are linearly independent holomorphic crosss-sections of the line bundle λ^n.*

Proof: If there is a nontrivial linear relation $\sum_{i=0}^n c_i f_0^i f_1^{i-n} = 0$ and if $g = f_0/f_1$ then $\sum_{i=0}^n c_i g^i = 0$, so g is a constant, contradicting the assumption that the cross-sections f_0, f_1 are linearly independent; and that suffices for the proof.

Corollary 11.29 *The critical values r_i of a compact Riemann surface M of genus $g > 0$ satisfy $r_n \leq n r_1 \quad$ for all $\quad n > 0$.*

Proof: There is a holomorphic line bundle λ for which $c(\lambda) = r_1$ and $\gamma(\lambda) = 2$, and that line bundle has two linearly independent holomorphic cross-sections f_0, f_1. The preceding theorem shows that the $n+1$ cross-sections $f_1^i f_2^{n-i} \in \Gamma(M, \mathcal{O}(\lambda^n))$ for $0 \leq i \leq n$ are linearly independent, hence $\gamma(\lambda^n) - 1 \geq n$; and consequently $n\, r_1 = c(\lambda^n) \geq r_n$, which suffices for the proof.

Theorem 11.30 *If $f_0, f_1 \in \Gamma(M, \mathcal{O}(\lambda))$ and $g_0, g_1, \ldots, g_n \in \Gamma(M, \mathcal{O}(\sigma))$ are linearly independent holomorphic cross-sections of the holomorphic line bundles λ, σ over a compact Riemann surface M, where f_0, f_1 have no common zeros, then either the $2n+2$ holomorphic cross-sections $f_i g_j \in \Gamma(M, \mathcal{O}(\lambda\sigma))$ for the indices $0 \leq i \leq 1$ and $0 \leq j \leq n$ are linearly independent or $\sigma = \lambda \zeta_{\mathfrak{d}}$ for the line bundle $\zeta_{\mathfrak{d}}$ of a positive divisor $\mathfrak{d}$ on M.*

Proof: Any nontrivial linear relation $\sum_{i=0}^1 \sum_{j=0}^n c_{i,j} f_i g_j = 0$ among the cross-sections $f_i g_j$ can be rewritten as the identity $f_0 g_0' = f_1 g_1'$ for the nontrivial holomorphic cross-sections $g_0' = \sum_{j=0}^n c_{0,j} g_j$ and $g_1' = -\sum_{j=0}^n c_{1,j} g_j$ of the line bundle σ, or equivalently as the equality $f_0/f_1 = g_1'/g_0'$ of two meromorphic functions on M. If $\mathfrak{d} = \mathfrak{d}(g_0', g_1')$ is the divisor of common zeros of the cross-sections $g_0',\ g_1'$ and $h \in \Gamma(M, \mathcal{O}(\zeta_{\mathfrak{d}}))$ is a cross section for which $\mathfrak{d}(h) = \mathfrak{d}$ then $g_0' = h g_0''$ and $g_1' = h g_1''$ for holomorphic cross-sections $g_0'', g_1'' \in \Gamma(M, \mathcal{O}(\sigma\zeta_{\mathfrak{d}}^{-1})$ which have no common zeros; and $f_0/f_1 = g_1''/g_0''$. Since neither the cross-sections $g_0'',\ g_1''$ nor the cross-sections $f_0,\ f_1$ have any common zeros, the polar divisor of the meromorphic function $f_0/f_1 = g_1''/g_0''$ is $\mathfrak{d}(f_1) = \mathfrak{d}(g_0'')$; and since $\zeta_{\mathfrak{d}(f_1)} = \lambda$ and $\zeta_{\mathfrak{d}(g_0'')} = \sigma\zeta_{\mathfrak{d}}^{-1}$ it follows that $\lambda = \sigma\zeta_{\mathfrak{d}}^{-1}$, which suffices for the proof.

Corollary 11.31 *If* $\lambda_1 \in \hat{X}^{MAX}_{r_1} = \hat{X}^1_{r_1}$ *and* $\lambda_i \in \hat{X}^{MAX}_{r_i} = \hat{X}^i_{r_i}$ *for any* $i > 0$ *then either* (i) $r_{2i+1} - r_i \le r_1$ *or* (ii) $\lambda_i = \lambda_1\zeta_{\mathfrak{d}}$ *for a positive divisor* $\mathfrak{d}$*.*

Proof: If $\lambda_1 \in \hat{X}^{MAX}_{r_1}$ then λ_1 has two linearly independent holomorphic cross-sections, which have no common zeros since λ_1 is base-point-free by Theorem **??**; and if $\lambda_i \in \hat{X}^{MAX}_{r_i}$ then λ_i has $i+1$ linearly independent holomorphic cross-sections. It follows from the preceding theorem that either $\gamma(\lambda_1\lambda_i) \ge 2i+2$ or $\lambda_i = \lambda_1\zeta_{\mathfrak{d}}$ for some positive divisor $\mathfrak{d}$. If $\gamma(\lambda_1\lambda_i) - 1 \ge 2i+1$ then by the definition of the critical values $r_1 + r_i = c(\lambda_1\lambda_i) \ge r_{2i+1}$. That suffices for the proof.

If λ and σ are inequivalent base-point-free holomorphic line bundles of characteristic classes $c(\lambda) = r$ and $c(\sigma) = s$ then $r_3 \le r + s$.????????

(ii) If λ, σ are base-point-free holomorphic line bundles then by Lemma 2.9 there are linearly independent holomorphic cross-sections $f_1, f_2 \in \Gamma(M, \mathcal{O}(\lambda))$ with no common zeros and $g_1, g_2 \in \Gamma(M, \mathcal{O}(\sigma))$ also with no common zeros; the products of these cross-sections are the four holomorphic cross-sections $f_1g_1, f_1g_2, f_2g_1, f_2g_2 \in \Gamma(M, \mathcal{O}(\lambda\sigma))$. If there were a nontrivial linear relation between these four cross-sections it could be written as an identity of the form $f_1g_1' + f_2g_2' = 0$ where $g_1' = a_1g_1 + a_2g_2$ and $g_2' = b_1g_1 + b_2g_1$ for some constants a_i, b_i, not all of which are zero; and the cross-sections g_1' and g_2' would have to be linearly independent, since the cross-sections f_1 and f_2 are, so the cross-sections g_1' and g_2' also have no common zeros. But then $f_2/f_1 = -g_1'/g_2'$ is a meromorphic function with the polar divisor $\mathfrak{d}(f_1) = \mathfrak{d}(g_2')$ and consequently $\lambda = \zeta_{\mathfrak{d}_1} = \zeta_{\mathfrak{d}_2} = \sigma$, contradicting the assumption that the line bundles λ and σ are distinct. Therefore the four cross-sections $f_1g_1, f_1g_2, f_2g_1, f_2g_2 \in \Gamma(M, \mathcal{O}(\lambda\sigma))$ are linearly independent, so $\gamma(\lambda\sigma) - 1 \ge 3$ and threrefore $r + s = \gamma(\lambda\sigma) \ge r_3$, which suffices for the proof.

The preceding results can be used to describe fully the first part of the interesting region of the Brill-Noether diagram, at least for some cases.

Theorem 11.32 *If* M *is a compact Riemann surface of genus* $g > 0$ *for which* $X^1_{r_1}$ *is a finite set then whenever* $r_1 < r < r_2$ *the holomorphic variety* X^1_r *is a finite union of holomorphic varieties of dimension* $r - r_1$ *and contains no base-point-free holomorphic line bundles.*

Proof: If $r_1 < r < r_2$ then since $r_2 < 2r_1$ by Lemma **??** it follows that $0 < r - r_1 < r_2 - r_1 < r_1$ so $\gamma(\zeta_{\mathfrak{d}}) = 1$ for any divisor $\mathfrak{d}$ of degree $\deg \mathfrak{d} = r - r_1$. Any holomorphic line bundle $\lambda_0 \in X^1_{r_1}$ is base-point-free by Theorem **??**, and if $\deg \mathfrak{d} = r - r_1$ the product $\lambda = \lambda_0\zeta_{\mathfrak{d}}$ for a divisor $\mathfrak{d}$ of degree $\deg \mathfrak{d} = r - r_1$ is a line bundle of characteristic class $c(\lambda) = r$; and $\gamma(\lambda) \ge \gamma(\lambda_0 = 2$ while $\gamma(\lambda) \le 2$ since $c(\lambda) < r_2$, so actually $\gamma(\lambda) = \gamma(\lambda_0) = 2$. Therefore the decomposition $\lambda = \lambda_0\zeta_{\mathfrak{d}}$ is the base decomposition of that line bundle, so λ is not base-point-free. Conversely if $\lambda \in X^1_r$ is not base-point-free it must be a product $\lambda = \lambda_0\zeta_{\mathfrak{d}}$

for some base-point-free line bundle λ_0 with $c(\lambda_0) < r$, and the only possibility is $c(\lambda_0) = r_1$. For a fixed such line bundle λ_0 the set of line bundles $\lambda\zeta_{\mathfrak{d}}$ for divisors $\mathfrak{d}$ with $\deg \mathfrak{d} = r - r_1$ is the image of the compact complex manifold $M^{(r-r_1)}$ under the

11 ——————————————————-

——————————————————

Of interest in connection with the base-decomposition of holomorphic line bundles are relations between the subvarieties r, induced by the mappings

$$\pi_r : M \times M^{(r-1)} \longrightarrow M^{(r)} \text{ for which } \pi_r(a, \mathfrak{d}) = a + \mathfrak{d}. \tag{11.149}$$

For any divisor $\mathfrak{d}$ in the open subset $M^{(r)*} \subset M^{(r)}$ consisting of divisors of r distinct points of M it is evident that $\pi_r^{-1}(\mathfrak{d})$ consists of r distinct points of $M \times M^{(r-1)}$ and that the restriction of the mapping π_r is a covering projection of r sheets over $M^{(r)*}$. On the other hand for any divisor $\mathfrak{d}$ in the complementary holomorphic subvariety $(M^{(r)} \sim M^{(r)*}) \subset M^{(r)}$ the inverse image $\pi_r^{-1}(\mathfrak{d})$ consists of strictly fewer than r points. The mapping π_r thus is a finite branched holomorphic covering of r sheets over $M^{(r)}$, branched over the subvariety $M^{(r)} \sim M^{(r)*}$. In particular the mapping π_r is a finite proper surjective holomorphic mapping; so by Remmert's Proper Mapping Theorem the image under this mapping of the subvariety $M \times G^{\nu}_{r-1} \subset M \times M^{(r-1)}$ is a well defined holomorphic subvariety $\pi_r(M \times G^{\nu}_{r-1}) \subset M^{(r)}$.

Theorem 11.33 *If M is a compact Riemann surface of genus $g > 0$*

$$G^{\nu+1}_r \subset \pi_r(M \times G^{\nu}_{r-1}) \subset G^{\nu}_r \quad \textit{for } r \geq 2 \textit{ and all } \nu; \tag{11.150}$$

and if $\nu > 0$ and $G^{\nu+1}_r \neq G^{\nu}_r$ then $\pi_r(M \times G^{\nu}_{r-1}) = G^{\nu}_r$ if and only if none of the line bundles $\zeta_{\mathfrak{d}} \in P_r(M)$ is base-point-free for any divisor $\mathfrak{d} \in (G^{\nu}_r \sim G^{\nu+1}_r)$.

Proof: If $\nu \leq 0$ the inclusion (11.150) reduces to $G^{\nu+1}_r \subset \pi_r(M \times M^{(r-1)}) \subset M^{(r)}$ in view of (**??**), and that holds quite trivially; so it can be assumed for the remainder of the proof that $\nu > 0$. If $\mathfrak{d} \in G^{\nu+1}_r$ and if $a \in M$ is a point in the divisor $\mathfrak{d}$, so that $\mathfrak{d} = a + \mathfrak{d}' = \pi_r(a, \mathfrak{d}')$ for a divisor $\mathfrak{d}' \in M^{(r-1)}$, then $\gamma(\zeta_{\mathfrak{d}}) - 1 \geq \nu + 1$ and it follows from Lemma 2.6 that $\gamma(\zeta_{\mathfrak{d}'}) - 1 = \gamma(\zeta_a^{-1}\zeta_{\mathfrak{d}}) - 1 \geq \gamma(\zeta_{\mathfrak{d}}) - 2 \geq \nu$ and hence that $\mathfrak{d}' \in G^{\nu}_{r-1}$; thus $G^{\nu+1}_r \subset \pi_r(M \times G^{\nu}_{r-1})$. If $a \in M, \mathfrak{d}' \in G^{\nu}_{r-1}$ and $\mathfrak{d} = a + \mathfrak{d}' = \pi_r(a, \mathfrak{d}')$ then $\gamma(\zeta_{\mathfrak{d}'}) - 1 \geq \nu$ and it follows from Lemma 2.6 again that $\gamma(\zeta_{\mathfrak{d}}) - 1 = \gamma(\zeta_a\zeta_{\mathfrak{d}'}) - 1 \geq \gamma(\zeta_{\mathfrak{d}'}) - 1 \geq \nu$ so that $\mathfrak{d} \in G^{\nu}_r$; thus $\pi_r(M \times G^{\nu}_{r-1}) \subset G^{\nu}_r$. That demonstrates both inclulsions in (11.150). Next if $\mathfrak{d} \in (G^{\nu}_r \sim G^{\nu+1}_r)$ for $\nu > 0$ then $\gamma(\zeta_{\mathfrak{d}}) = \nu + 1 > 1$. If the line bundle $\zeta_{\mathfrak{d}}$ is not base-point-free then all the holomorphic cross-sections of the line bundle $\zeta_{\mathfrak{d}}$ vanish at some point a, which must be a point of the divisor $\mathfrak{d}$, so $\mathfrak{d} = a + \mathfrak{d}' = \pi_r(a, \mathfrak{d}')$ for some divisor $\mathfrak{d}' \in M^{(r-1)}$; and since all the holomorphic cross-sections of the bundle $\zeta_{\mathfrak{d}}$ vanish at the point a it follows from Lemma 2.6 that $\gamma(\zeta_{\mathfrak{d}'}) - 1 = \gamma(\zeta_a^{-1}\zeta_{\mathfrak{d}}) - 1 = \gamma(\zeta_{\mathfrak{d}}) - 1 \geq \nu$, hence that $\mathfrak{d}' \in G^{\nu}_{r-1}$ so $\mathfrak{d} \in \pi_r(M \times G^{\nu}_{r-1})$. Thus if none of the line bundles $\zeta_{\mathfrak{d}}$ is base-point-free for any divisor $\mathfrak{d} \in (G^{\nu}_r \sim G^{\nu+1}_r)$ then $(G^{\nu}_r \sim G^{\nu+1}_r) \subset \pi_r(M \times G^{\nu}_{r-1})$.

Since $G_r^{\nu+1} \subset \pi_r(M \times G_{r-1}^{\nu})$ by the first part of the proof then altogether $G_r^{\nu} \subset \pi_r(M \times G_{r-1}^{\nu})$, and this inclusion must be an equality since the reversed inclusion was demonstrated in the first part of the proof. Conversely if $G_r^{\nu} = \pi_r(M \times G_{r-1}^{\nu})$ and if $\mathfrak{d} \in G_r^{\nu} \sim G_r^{\nu+1}$ then $\gamma(\zeta_{\mathfrak{d}}) - 1 = \nu$ and $\mathfrak{d} = a + \mathfrak{d}'$ for some divisor $\mathfrak{d}' \in G_{r-1}^{\nu}$. Thus $\nu \le \gamma(\zeta_{\mathfrak{d}'}) - 1 = \gamma(\zeta_a^{-1}\zeta_{\mathfrak{d}}) - 1 \le \gamma(\zeta_{\mathfrak{d}}) - 1 = \nu$ by Lemma 2.6, and consequently $\gamma(\zeta_a^{-1}\zeta_{\mathfrak{d}}) = \gamma(\zeta_{\mathfrak{d}})$; so by Lemma 2.6 yet again all the holomorphic cross-sections of the bundle $\zeta_{\mathfrak{d}}$ must vanish at the point a, so the bundle $\zeta_{\mathfrak{d}}$ is not base-point-free. That suffices to conclude the proof of the theorem.

Corollary 11.34 *If M is a compact Riemann surface of genus $g > 0$*

$$\dim G_r^{\nu+1} \le 1 + \dim G_{r-1}^{\nu} \le \dim G_r^{\nu} \quad \textit{for } r \ge 2 \textit{ and all } \nu; \tag{11.151}$$

and if $\nu > 0$ and $G_r^{\nu+1} \neq G_r^{\nu}$ and none of the line bundles $\zeta_{\mathfrak{d}} \in P_r(M)$ is base-point-free for any divisor $\mathfrak{d} \in G_r^{\nu} \sim G_r^{\nu+1}$ then

$$1 + \dim G_{r-1}^{\nu} = \dim G_r^{\nu}. \tag{11.152}$$

Proof: Since the mapping π_r in (11.149) is finite and proper, Remmert's Proper Mapping Theorem implies not only that the image $\pi_r(M \times G_{r-1}^{\nu})$ is a holomorphic subvariety of $M^{(r)}$ but also that $\dim \pi_r(M \times G_{r-1}^{\nu}) = \dim(M \times G_{r-1}^{\nu})$; and of course $\dim(M \times G_{r-1}^{\nu}) = 1 + \dim G_{r-1}^{\nu}$ whenever G_{r-1}^{ν} is nonempty. The corollary follows immediately from these observations and the inclusion relations of the preceding theorem, and that suffices for the proof.

Corollary 11.35 *If M is a compact Riemann surface of genus $g < 0$ with the maximal function r_i and if $r_i \le r < r_{i+1}$ and there are no base-point-free holomorphic line bundles in $\dim X_r^{MAX}$ then $\dim X_r^{MAX} = \dim X_{r-1}^{MAX} + 1$* (ii) *If $r_i \le r < r_{i+1}$ then $\dim X_r^{MAX} > \dim X_{r-1}^{MAX} + 1$;*

Proof: If $r_i \le r \le r_{i+1}$ and there are no base-point-free holomorphic line bundles in $\dim X_r^{MAX}$ it follows from Corollary 11.34 that $\dim G_r^{MAX} = \dim G_{r-1}^{MAX} + 1$ and from this in view of Corollary **??** it follows that $\dim X_r^{MAX} = \dim X_{r-1}^{MAX} + 1$;
and that suffices for the proof.

[REMARK:] Deduce the consequences for the maximal function from this.

————————————-

NEXT MOVED TO CHAPTER 9

OOOOOOOOOOOOOOOOOOOOOOOOOO

Of course the equality of the dimensions is not of interest here; what is of interest though is that the holomorphic variety X_{g-1}^{ν} is mapped to itself by the automorphism

$$\epsilon_k : J(M) \longrightarrow J(M) \quad \text{defined by } \epsilon_k(t) = k - t, \tag{11.153}$$

a biholomorphic mapping of the Jacobi variety $J(M)$ to itself of period 2, and consequently of course that $\epsilon_k(W_{g-1}^{\nu}) = W_{g-1}^{\nu}$ as well. The automorphism ϵ_k

also has individual fixed points, which are just those points $t \in J(M)$ such that $t = k - t$ so which are the 2^{2g} points of the quotient torus $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ represented by the half-periods $\frac{1}{2}\Omega\mathbb{Z}^{2g}$ modulo the periods $\Omega\mathbb{Z}^{2g}$; they are called either the *points of order 2* or the *half-periods* of the torus $J(M)$. These fixed points are distributed among the disjoint fixed varieties $X^\nu_{g-1} \subset J(M)$; so if there are υ_ν fixed points in X^ν_{g-1} for $-1 \le \nu \le \mu(g-1)$ then

$$\sum_{\nu=-1}^{\mu(g-1)} \upsilon_\nu = 2^{2g}. \tag{11.154}$$

Under the biholomorphic mapping $\phi_{a_0} : P_{g-1}(M) \longrightarrow J(M)$ the fixed points correspond to holomorphic line bundles $\lambda \in P_S g - 1(M)$ for which $\lambda^2 = \kappa$; they are called the *semicanonical bundles* of the Riemann surface M, or alternatively the *theta characteristics* of M in view of their natural appearance in another form in the study of theta functions on Riemann surfaces. Perhaps the most interesting of the semicanonical line bundles are those that admit holomorphic cross-sections, so those that are contained in the varieties $\hat{X}^\nu_{g-1}$ for indices $\nu \ge 0$.

Theorem 11.36 *The semicanonical bundles $\lambda \in \hat{X}^\nu_{g-1}$ for $\nu \ge 0$ are the line bundles $\lambda = \zeta_{\mathfrak{d}}$ of positive divisors $\mathfrak{d} \in M^{(g-1)}$ of degree $g-1$ such that $2\,\mathfrak{d} = \mathfrak{k}$, the canonical divisor on M.*

Proof: A semicanonical line bundle λ such that $\gamma(\lambda) > 0$ has a nontrivial holomorphic cross-section $f \in \Gamma\big(M, \mathcal{O}(\lambda)\big)$ with a divisor $\mathfrak{d}(f)$; and since $f^2 \in \Gamma\big(M, \mathcal{O}(\lambda^2)\big) = \Gamma\big(M, \mathcal{O}(\kappa)\big)$ it follows that $2\,\mathfrak{d}(f) = \mathfrak{d}(f^2)$ is a canonical divisor on the Riemanan surface M. Conversely if there is a canonical divisor on M of the form $\mathfrak{k} = 2\mathfrak{d}$ for a positive divisor $\mathfrak{d} \in M^{(g-1)}$ then there is a holomorphic cross-section $h \in \Gamma\big(M, \mathcal{O}(\kappa)\big)$ of the canonical bundle κ of the Riemann surface M with the divisor $\mathfrak{d}(h) = 2\mathfrak{d}$. When the canonical line bundle κ on M is represented by a holomorphic factor of automorphy $\kappa(T, z)$ for the action of the covering translation group of M on the universal covering space $\tilde{M}$ of M, the cross-section h corresponds to a holomorphic function $h(z)$ on $\tilde{M}$ that is a relatively automorphic function for this factor of automorphy. Since the function $h(z)$ has a divisor of even order it has a well defined square root in an open neighborhood of each point of $\tilde{M}$; and since $\tilde{M}$ is simply connected any choice of a local square root at one point can be continued to the entire Riemann surface $\tilde{M}$ as a well defined holomorpic function $f(z) = \sqrt{h(z)}$ on $\tilde{M}$. The divisor $\mathfrak{d}\big(f(z)\big)$ of this function is invariant under the action of the covering translation group Γ, so the quotients $\lambda(T, z) = f(Tz)/f(z)$ are well defined holomorphic and nowhere vanishing functions on $\tilde{M}$; and it follows from their definition that they satisfy $\lambda(ST, z) = \lambda(S, Tz)\lambda(T, z)$ for any two covering translations $T \in \Gamma$, so they form a factor of automorphy for the action of the group Γ on $\tilde{M}$. This factor of automorphy describes a holomorphic line bundle λ over M, and $f(z)$ represents a holomorphic cross-section $f \in \Gamma\big(M, \mathcal{O}(\lambda)\big)$ of this bundle. Since $f^2 = h \in \Gamma(M, \mathcal{O}(\kappa))$ it follows that $\lambda^2 = \kappa$ and consequently that λ is a semicanonical bundle over M, which has the nontrivial holomorphic cross-section f. That suffices for the proof.

———————————————————— [Alternative lemma for proof of Theorem 11.13.]

Lemma 11.37 *On a compact Riemann surface M of genus $g > 0$ the line bundle ζ_a^{g+1} is base-point-free and $\gamma(\zeta_a^{g+1}) = 2$ for all but at most finitely many points $a \in M$.*

Proof: For any point $a \in M$ it follows from the Riemann-Roch Theorem in the form of Theorem 2.24 that $\gamma(\zeta_a^g) = \gamma(\kappa\zeta_a^{-g}) + 1 \geq 1$ and $\gamma(\zeta_a^{g+1}) = \gamma(\kappa\zeta_a^{-g-1}) + 2 \geq 2$. On the other hand it follows from the Riemann-Roch Theorem in the form of Theorem 11.3 that $\gamma(\zeta_a^g) = g + 1 - \operatorname{rank}\Omega(g \cdot a)$ where $\Omega(g \cdot a)$ is the Brill-Noether matrix of the divisor $g \cdot a$. When the holomorphic abelian differentials on M are written in terms of local coordinates z_α as $\omega_i = f_{i\alpha}(z_\alpha)$, the determinant of the Brill-Noether matrix $\Omega(g \cdot a)$ is just the Wronskian of the functions $f_{i\alpha}(z_\alpha)$, as in (11.11). Since the abelian differentials are linearly independent holomorphic functions their Wronskian does not vanish identically[5]; hence $\operatorname{rank}\Omega(g \cdot a) = g$ and $\gamma(\zeta_a^g) = 1$ at all but the finitely many points $a \in M$ at which $\det\Omega(g \cdot a) = 0$. If $\gamma(\zeta_a^g) = 1$ then $\gamma(\zeta_a^{g+1}) \leq 2$ by Lemma 2.6, and since it was already noted that $\gamma(\zeta_a^{g+1}) \geq 2$ it follows that $\gamma(\zeta_a^{g+1}) = 2$. Furthermore if $\gamma(\zeta_a^g) = 1$ the line bundle ζ_a^g is not base-point-free, indeed it has the base decomposition $\zeta_a^g = 1 \cdot \zeta_a^g$ for the identity bundle 1; and since $\gamma(\zeta_a^g\zeta_a) = 2$ it follows from Theorem 2.12 (iii) that ζ_a^r is base-point-free for some r in the range $1 \leq r \leq g + 1$, which can only be the case for $r = g + 1$. That suffices for the proof.

[5]It is obvious that if a finite number of functions are linearly dependent their Wronskian determinant is zero. The converse was long known to be false for $\mathcal{C}^\infty$ functions but it is true for holomorphic functions; see for instance the paper by M. Bôcher, The theory of linear dependence, *Annals of Math.* vol 2 (1900), pages 81-96; or more recently the paper by Alin Bostan and Philippe Dumas, Wronskians and Linear Independence, *Amer. Math. Monthly*, vol.117, (2010), pp. 722-727.

Chapter 12

The Abel-Jacobi Mapping

PRELIMINARY FORM

12.1 The Abel-Jacobi Diagram

If Ω is the period matrix of a compact Riemann surface M of genus $g > 0$, in terms of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$, the Abel-Jacobi mapping (3.4) is the holomorphic mapping $w_{z_0} : M \longrightarrow J(M)$ from the Riemann surface M to its Jacobi variety $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ induced by the holomorphic mapping $\tilde{w}_{z_0} : \tilde{M} \longrightarrow \mathbb{C}^g$ that associates to a point z in the universal covering surface $\tilde{M}$ of M the point $\tilde{w}_{z_0}(z) = \{w_i(z, z_0)\} \in \mathbb{C}^g$, where $w_i(z, z_0) = \int_{z_0}^{z} \omega_i$ for the base point $z_0 \in \tilde{M}$. For some purposes, and as in the earlier discussion of the Abel-Jacobi mapping in Section 3.1, it is convenient not to specify the base point $z_0 \in \tilde{M}$ but to allow the mapping $\tilde{w}_{z_0}$ to be modified by an arbitrary additive constant and hence to allow the mapping w_{z_0} to be modified by an arbitrary translation in the complex torus $J(M)$; in that case the mappings will be denoted just by $\tilde{w}$ and w. However for much of the discussion in the present chapter it will be assumed that there is a specified base point $z_0 \in \tilde{M}$ in terms of which the Abel-Jacobi mapping w_{z_0} is defined. Furthermore the Jacobi variety will be viewed not just as a complex manifold but also as a complex Lie group, so that it has a specified identity element $0 \in J(M)$ for the additive structure of the complex torus $J(M)$; the identity will be taken to be the point in the quotient space $\mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ represented by the origin $0 \in \mathbb{C}^g$, which is independent of the choice of bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$. With this structure as a complex Lie group the Jacobi variety $J(M)$ will be called the *Jacobi group*. The Abel-Jacobi mapping then extends naturally to the *Abel-Jacobi homomorphism*, the group homomorphism

$$w_{z_0} : \Gamma(M, \mathcal{D}) \longrightarrow J(M) \tag{12.1}$$

from the additive group $\Gamma(M,\mathcal{D})$ of divisors on M to the additive Jacobi group $J(M)$ that associates to any divisor $\mathfrak{d} = \sum_{j=1}^{r} \nu_j \cdot a_j \in \Gamma(M,\mathcal{D})$ the point

$$w_{z_0}(\mathfrak{d}) = \sum_{j=1}^{r} \nu_j w_{z_0}(a_j) \in J(M); \tag{12.2}$$

thus the image $w_{z_0}(\mathfrak{d})$ is the point in the quotient space $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ represented by the vector $\sum_{j=1}^{r} \nu_j \tilde{w}(z_j, z_0) \in \mathbb{C}^g$ for any points $z_j \in \tilde{M}$ such that $\pi(z_j) = a_j$ under the universal covering projection $\pi : \tilde{M} \longrightarrow M$.

Theorem 12.1 *If M is a compact Riemann surface of genus $g > 0$ with a base point $z_0 \in \tilde{M}$ and $a_0 = \pi(z_0) \in M$ is the image of that base point under the universal covering projection $\pi : \tilde{M} \longrightarrow M$, then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ there is a uniquely determined group isomorphism $\phi_{a_0} : P(M) \longrightarrow J(M)$ such that*

(12.3)

$$\begin{tikzcd} \Gamma_0(M,\mathcal{D}) \arrow[rr, "\zeta"] \arrow[dr, "w_{z_0}"'] & & P(M) \arrow[dl, "\phi_{a_0}"] \\ & J(M) & \end{tikzcd}$$

is a commutative diagram of surjective group homomorphisms.

Proof: By Theorem 3.14 any holomorphic line bundle $\lambda \in P(M)$ can be represented by the flat line bundle described through the canonical parametrization of flat line bundles (3.27) by a representation ρ_t for a vector $t \in \mathbb{C}^{2g}$; all vectors $t \in \mathbb{C}^{2g}$ describe line bundles in $P(M)$ in this way, and two vectors describe the same line bundle if and only if they differ by a vector in the linear subspace $\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g \subset \mathbb{C}^{2g}$. The mapping that associates to a line bundle $\lambda \in P(M)$ the point in the quotient space $\mathbb{C}^{2g}/(\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g)$ represented by any vector $t \in \mathbb{C}^g$ such that $\rho_t = \lambda$ thus is a group isomorphism

$$\phi^*_{a_0} : P(M) \longrightarrow \mathbb{C}^{2g}/(\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g)$$

from the Picard group to additive group of a complex torus. If P is the intersection matrix of the surface M in terms of the basis $\tau_j \in H_1(M)$ then by Theorem 3.23 the linear mapping $\Omega P : \mathbb{C}^{2g} \longrightarrow \mathbb{C}^g$ induces a biholomorphic mapping

$$(\Omega P)^* : \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g} \longrightarrow J(M)$$

between these two tori; and since it is induced by a linear mapping between their universal covering spaces it is also a group isomorphism. The composition

$$\phi_{a_0} = (\Omega P)^* \circ \phi^*_{a_0} : P(M) \longrightarrow J(M)$$

then is a group isomorphism from the Picard group to the Jacobi group $J(M)$ of M. Since all the mappings in the diagram (12.3) are group homomorphisms and the group of divisors of degree 0 is generated by divisors of the form $a'-a''$ for points $a', a'' \in M$, to show the commutativity of the diagram it is sufficient just to show that $\phi_{a_0}(\zeta_{a'-a''}) = w_{z_0}(a'-a'')$ for any two points $a', a'' \in M$. By Corollary 5.9 the line bundle $\zeta_{a'-a''} \in P(M)$ can be represented by the flat line bundle described through the canonical parametrization of flat line bundles by the representation $\rho_{t(z',z'')}$ for the vector $t(z',z'') = i\,{}^t\overline{\Omega}\,{}^tG\tilde{w}(z',z'') \in \mathbb{C}^{2g}$, where $G = {}^tH^{-1}$ for the positive definite Hermitian matrix $H = i\,\Omega P\,{}^t\overline{\Omega}$ and $z', z'' \in \tilde{M}$ are any points in the universal covering space of M that cover the points a', a'' respectively. Since $i\,\Omega P\,{}^t\overline{\Omega}\,{}^tG = H\,{}^tG = \mathrm{I}$ it follows that

$$
\begin{aligned}
\Omega P t(z',z'') &= i\,\Omega P\,{}^t\overline{\Omega}\,{}^tG\tilde{w}(z',z'') = \tilde{w}(z',z'') \\
&= \tilde{w}(z',z_0) - \tilde{w}(z',z_{z_0}) \in \mathbb{C}^g,
\end{aligned}
\tag{12.4}
$$

which represents the point $w_{z_0}(a'-a'')$ in the quotient torus $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$, thus establishing the commutativity of the diagram (12.3). The homomorphism ζ is surjective since any holomorphic line bundle is the line bundle of some divisor by Corollary 2.20; and since ϕ_{a_0} is an isomorphism and the diagram (12.3) is commutative it follows immediately that the homomorphism w_{z_0} is also surjective. From this and the commutativity of the diagram (12.3) it further follows that the isomorphism ϕ_{a_0} is uniquely determined, and that suffices to conclude the proof.

The commutative diagram (12.3) can be extended to a homomorphism from the full group $\Gamma(M,\mathcal{D})$ of divisors on M. First the homomorphism ϕ_{a_0} can be extended to a homomorphism

$$
\hat{\phi}_{a_0} : H^1(M,\mathcal{O}^*) \longrightarrow J(M)
\tag{12.5}
$$

defined on arbitrary line bundles by setting

$$
\hat{\phi}_{a_0}(\lambda) = \phi_{a_0}(\zeta_{a_0}^{-c(\lambda)}\,\lambda)
\tag{12.6}
$$

for any holomorphic line bundle λ. The restriction of this mapping to the subgroup $P(M) \subset H^1(M,\mathcal{O}^*)$ is just the homomorphism ϕ_{a_0}, and the extended mapping is also a homomorphism since if λ_1, λ_2 are line bundles with $c(\lambda_1) = r_1$, $c(\lambda_2) = r_2$ then by definition $\hat{\phi}_{a_0}(\lambda_1\lambda_2) = \phi_{a_0}(\zeta_{a_0}^{-r_1-r_2}\lambda_1\lambda_2) = \phi_{a_0}(\zeta_{a_0}^{-r_1}\lambda_1 \cdot \zeta_{a_0}^{-r_2}\lambda_2) = \phi_{a_0}(\zeta_{a_0}^{-r_1}\lambda_1)\cdot\phi_{a_0}(\zeta_{a_0}^{-r_2}\lambda_2) = \hat{\phi}_{a_0}(\lambda)\cdot\hat{\phi}_{a_0}(\lambda_2)$. It is clear that any holomorphic line bundle λ can be written uniquely as the product

$$
\lambda = \zeta_{a_0}^r\lambda_0 \quad \text{where } r = c(\lambda) \text{ and } \lambda_0 \in P(M);
\tag{12.7}
$$

and since $\hat{\phi}_{a_0}(\zeta_{a_0}^r\lambda_0) = \phi_{a_0}(\lambda_0)$ where ϕ_{a_0} is an isomorphism by Theorem 12.1 it is apparent that the kernel of the homomorphism $\hat{\phi}_{a_0}$ is the cylic subgroup

$$
\Gamma_{a_0} = \left\{ \zeta_{a_0}^r \,\middle|\, r \in \mathbb{Z} \right\} \subset H^1(M,\mathcal{O}^*).
\tag{12.8}
$$

The homomorphism $\hat{\phi}_{a_0}$ is surjective, since its restriction ϕ_{a_0} to the subgroup of holomorphic line bundles of characteristic class 0 is already surjective by Theorem 12.1, so there is the exact sequence of groups

$$0 \longrightarrow \Gamma_{a_0} \stackrel{\iota}{\longrightarrow} H^1(M, \mathcal{O}^*) \stackrel{\hat{\phi}_{a_0}}{\longrightarrow} P(M) \longrightarrow 0 \tag{12.9}$$

in which $\iota : \Gamma_{a_0} \longrightarrow H^1(M, \mathcal{O}^*)$ is the natural inclusion homomorphism. Actually in view of (12.7) this exact sequence splits, so the group of all holomorphic line bundles is just the product

$$H^1(M, \mathcal{O}^*) = \Gamma_{a_0} \cdot P(M). \tag{12.10}$$

It follows that the cosets of the subgroup $P(M) \subset H^1(M, \mathcal{O}^*)$ are the sets

$$P_r(M) = \zeta_{a_0}^r \cdot P(M), \tag{12.11}$$

and through this identification with the Picard group $P(M)$ the sets $P_r(M)$ have natural structures as complex tori biholomorphic to $P(M)$; they can be described alternatively as

$$P_r(M) = \left\{ \lambda \in H^1(M, \mathcal{O}^*) \,\middle|\, c(\lambda) = r \right\} \subset H^1(M, \mathcal{O}^*). \tag{12.12}$$

The Abel-Jacobi homomorphism (12.1) takes linearly equivalent divisors to the same point in the Jacobi variety by Abel's Theorem, Corollary 5.10, while the homomorphism (**??**) takes linearly linearly equivalent divisors to the same holomorphic line bundle by Theorem **??**; consequently both of these homomorphisms can be factored through the natural homomorphism

$$\psi : \Gamma(M, \mathcal{D}) \longrightarrow \Big(\Gamma(M, \mathcal{D}) / \sim \Big) \tag{12.13}$$

that associates to any divisor its linear equivalence class, and written as the compositions

$$\zeta = \hat{\zeta} \circ \psi \quad w_{z_0} = \hat{\zeta} \circ \hat{w}_{z_0} \tag{12.14}$$

for homomorphisms

$$\begin{aligned} &\hat{\zeta} : \Big(\Gamma(M, \mathcal{D}) / \sim \Big) \longrightarrow H^1(M, \mathcal{O}^*) \quad \text{and} \\ &\hat{w}_{z_0} : \Big(\Gamma(M, \mathcal{D}) / \sim \Big) \longrightarrow J(M). \end{aligned} \tag{12.15}$$

In these terms there is the following extension of Theorem 12.1.

Theorem 12.2 *If M is a compact Riemann surface of genus $g > 0$ with a base point $z_0 \in \tilde{M}$ and $a_0 = \pi(z_0) \in M$ is the image of that base point under the* *universal covering projection $\pi : \tilde{M} \longrightarrow M$, then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ there is the commutative diagram of group homomorphisms*

(12.16)

$$\begin{tikzcd}
\Gamma(M,\mathcal{D}) \arrow[r, "\psi"] & \big(\Gamma(M,\mathcal{D})/\sim\big) \arrow[rr, "\hat{\zeta}"] \arrow[dr, "\hat{w}_{z_0}"'] & & H^1(M,\mathcal{O}^*) \arrow[dl, "\hat{\phi}_{a_0}"] \\
 & & J(M) &
\end{tikzcd}$$

where $\hat{\zeta}$ is an isomorphism, $\hat{w}_{z_0}$ is a surjective group homomorphism the kernel of which is the image under the homomorphism ψ of the subgroup

$$\mathcal{D}_a = \Big\{\, n \cdot a_0 \;\Big|\; n \in \mathbb{Z} \,\Big\} \subset \Gamma(M, \mathcal{D}), \tag{12.17}$$

$\hat{\phi}_{a_0}$ is a surjective group homomorphism the kernel of which is the cyclic subgroup $\Gamma_{a_0} \subset H^1(M, \mathcal{O}^*)$, and for any $r \in \mathbb{Z}$ the restriction of $\hat{\phi}_{a_0}$ to the subset $P_r(M) \subset H^1(M, \mathcal{O}^*)$ is a biholomorphic mapping between complex tori.

Proof: If $\mathfrak{d} \in \Gamma(M, \mathcal{D})$ is a divisor of degree $r = \deg \mathfrak{d}$ then by definition $\hat{\phi}_{a_0}\big(\zeta(\mathfrak{d})\big) = \hat{\phi}_{a_0}(\zeta_{\mathfrak{d}}) = \phi_{a_0}(\zeta_{a_0}^{-r}\zeta_{\mathfrak{d}})$, and it follows from the commmutativity of the diagram (12.3) that $\phi_{a_0}(\zeta_{a_0}^{-r}\zeta_{\mathfrak{d}}) = w_{z_0}(\zeta_{a_0}^{-r}\zeta_{\mathfrak{d}}) = w_{z_0}(\zeta_{\mathfrak{d}})$ since $w_{z_0}(\zeta_{a_0}^{-r}) = 0$; that demonstrates the commutativity of the diagram (12.16), since both mappings ζ and w_{z_0} factor through the mapping (12.13). The homomorphism $\hat{\zeta}$ is an isomorphism by Theorem **??** since the line bundle of a divisor is the trivial line bundle precisely when the divisor is linearly equivalent to zero and every line bundle is the line bundle of some divisor. The homomorphism $\hat{\phi}_{a_0}$ is surjective since its restriction ϕ_{a_0} to the subgroup of line bundles of characteristic class zero is surjective by Theorem 12.1, so from the commutativity of (12.16) it follows that the homomorphism $\hat{w}_{z_0}$ also is surjective. The kernel of the homomorphism $\hat{\phi}_{a_0}$ is the subgroup $\Gamma_{a_0} \subset H^1(M, \mathcal{O}^*)$ by (12.9); and since Γ_a is the image under the isomorphism $\hat{\zeta}$ of the group of linear equivalence classes of divisors in $\mathcal{D}_a$ it follows from the commutativity of (12.16) that the kernel of the group homomorphism w_{z_0} is the subgroup $\psi(\mathcal{D}_a) \subset \Big(\Gamma(M,\mathcal{D})/\sim\Big)$. The complex structure on $P_r(M)$ is that as the coset $P_r(M) = \zeta_{a_0}^r P(M)$ in view of (12.6), so the mapping $\hat{\phi}_{a_0} : P_r(M) \longrightarrow J(M)$ is a biholomorphic mapping, and that suffices for the proof.

The diagram (12.16) is called the *Abel-Jacobi diagram* for the Riemann surface M. The homomorphisms ψ and $\hat{\zeta}$ along the top line are intrinsically defined mappings, while the mappings $\hat{w}_{z_0}$ and $\hat{\phi}_{a_0}$ involve the choice of bases for the holomorphic abelian differentials and for the homology of M and of a base point on $\tilde{M}$.

12.2 The Variety of Positive Divisors

The restriction of this diagram to the set of positive divisors of a fixed degree is particularly interesting and useful; but the examination of that restriction requires as a preliminary a somewhat more detailed discussion of the sets of positive divisors on a Riemann surface M. A positive divisor of degree r on M can be viewed as an unordered set of r points of M, not necessarily distinct points; so the set of all positive divisors of degree r can be identified with the quotient $M^{(r)} = M^r/\mathfrak{S}_r$ of the r-dimensional complex manifold M^r by the symmetric group $\mathfrak{S}_r$ acting as the group of permutations of the factors, a set called the *r-th symmetric product* of the surface with itself. The mapping that associates to an ordered set of r points the corresponding unordered set of those points is the natural quotient mapping

$$\pi_r : M^r \longrightarrow M^{(r)} = M^r/\mathfrak{S}_r. \tag{12.18}$$

The quotient space $M^{(r)}$ has the natural quotient topology, in which a subset $U \subset M^{(r)}$ is open precisely when the inverse image $\pi_r^{-1}(U) \subset M^r$ is open. The restriction of the quotient mapping π_r to the subset

$$M^{r*} = \Big\{ (z_1, \ldots, z_r) \in M^r \;\Big|\; z_i \ne z_j \quad \text{for} \quad i \ne j \Big\} \subset M^r \tag{12.19}$$

clearly is a locally homeomorphic mapping.

Lemma 12.3 *The symmetric product $\mathbb{C}^{(r)}$ has the structure of a complex manifold of dimension r such that the natural quotient mapping $\pi_r : \mathbb{C}^r \longrightarrow \mathbb{C}^{(r)}$ is a holomorphic mapping and its restriction to the subset $\mathbb{C}^{r*}$ is locally biholomorphic.*

Proof: Consider the mapping $\tau_r : \mathbb{C}^r \longrightarrow \mathbb{C}^r$ defined by

$$\tau_r(z_1, z_2, \ldots, z_r) = \big(e_1(z_1, \ldots, z_r), e_2(z_1, \ldots, z_r), \ldots, e_r(z_1, \ldots, z_r)\big),$$

where $e_i(z_1, \ldots, z_r)$ are the elementary symmetric functions in r variables. It will be shown first that there is a one-to-one mapping $\sigma_r : \mathbb{C}^{(r)} \longrightarrow \mathbb{C}^r$ such that

$$\begin{tikzcd} \mathbb{C}^r \arrow[rr, "\tau_r"] \arrow[dr, "\pi_r"'] & & \mathbb{C}^r \\ & \mathbb{C}^{(r)} \arrow[ur, "\sigma_r"'] & \end{tikzcd} \tag{12.20}$$

is a commutative diagram of mappings. The elementary symmetric functions $e_i = e_i(z_1, \ldots, x_r)$ are the polynomials in the variables $z_1, \ldots, z_r$ defined as the coefficients of the polynomial

$$\prod_{i=1}^{r}(X - z_i) = X^r - e_1X^{r-1} + e_2X^{r-2} - \cdots + (-1)^r e_r. \tag{12.21}$$

These coefficients clearly are invariant under permutations of the variables $z_1, \ldots, z_r$, so the mapping τ_r factors through the quotient mapping π_r as the composition $\tau_r = \sigma_r \circ \pi_r$ for some mapping σ_r for which the resulting diagram (12.20) is a commutative diagram of mappings. The mapping τ_r is surjective since any monic polynomial of degree r can be written as the product $\prod_{i=1}^{r}(X - z_i)$ where z_i are its roots; consequently the mapping σ_r also is surjective. Since a monic polynomial is determined uniquely by its roots and conversely determines the roots uniquely up to order, two points of M^r have the same image under τ_r if and only if they have the same image under π_r, so the mapping σ_r is injective. Altogether then the mapping σ_r is one-to-one, so can be used to identify the symmetric product $\mathbb{C}^{(r)}$ with the image $\mathbb{C}^r$, and thereby to give the symmetric product the structure of a complex manifold. The mapping τ_r clearly is holomorphic, and its restriction to $\mathbb{C}^{r*}$ is locally biholomorphic; so from the commutativity of the diagram (12.20) the same is true for the mapping σ_r, and that suffices for the proof.

For some purposes it is more convenient to use the power sums

$$s_i(z_1, \ldots, z_r) = z_1^i + \cdots + z_r^i \tag{12.22}$$

for $1 \le i \le r$ in place of the elementary symmetric functions in the definition of the mapping τ_r in the preceding lemma; Newton's formulas expressing the elementary symmetric functions in terms of the power sums and conversely show that the two choices lead to equivalent results.

Theorem 12.4 *The symmetric product $M^{(r)}$ of a compact Riemann surface M has the natural structure of a compact complex manifold of dimension r. The quotient mapping $\pi_r : M^r \longrightarrow M^{(r)}$ is a holomorphic mapping that is a locally biholomorphic mapping from the dense open subset $M^{r*} \subset M^r$ consisting of sets of r distinct points of M to its image $M^{(r)*} = \pi_r(M^{r*})$.*

Proof: For any divisor $\mathfrak{d} = \nu_1 \cdot a_1 + \cdots + \nu_s \cdot a_s \in M^{(r)}$ for which $a_1, \ldots, a_s$ are distinct points of M choose disjoint open neighborhoods $U_i \subset M$ of the points a_i. The quotient spaces $U_i^{(\nu_i)}$ have the natural structures of complex manifolds by the preceding lemma; and the product $U_{\mathfrak{d}} = U_1^{(\nu_1)} \times \cdots \times U_s^{(\nu_s)}$ then provides the structure of a complex manifold on an open neighborhood of the divisor $\mathfrak{d} \in M^{(r)}$. If the divisor $\mathfrak{d}$ consists of distinct points, so that $\nu_i = 1$ for all indices i, the quotient spaces $U_i^{(\nu_i)} = U_i^{(1)}$ are just the neighborhoods U_i themselves, so the quotient mapping $\pi_r : M^r \longrightarrow M^{(r)}$ then is a locally biholomorphic mapping in an open neighborhood of the divisor $\mathfrak{d}$. That suffices to conclude the proof.

12.3 The General Abel-Jacobi Mapping and the Brill-Noether Matrix

The restriction of the Abel-Jacobi homomorphism (12.1) to the subset $M^{(r)} \subset \Gamma_r(M, \mathcal{D})$ consisting of positive divisors of degree $r \geq 1$ is a mapping

$$w_{z_0} : M^{(r)} \longrightarrow J(M) \tag{12.23}$$

called the *general Abel-Jacobi mapping*, although sometimes for simplicity it is called just the Abel-Jacobi mapping again. It is a holomorphic mapping between these two complex manifolds, since it is induced by a holomorphic mapping $w_{z_0} : M^r \longrightarrow J(M)$ that commutes with permutations of the factors of the product M^r. As in the case of the Abel-Jacobi mapping itself, for some purposes it is convenient not to specify the base point $z_0 \in \tilde{M}$ but to allow the mapping to be modified by an arbitrary translation in the complex torus $J(M)$, and in that case the mapping is denoted just by w.

Theorem 12.5 *For a compact Riemann surface M of genus $g > 0$ the Brill-Noether matrix $\Omega(\mathfrak{d})$ at a divisor $\mathfrak{d} \in M^{(r)}$ can be identified with the Jacobian of the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ at the point $\mathfrak{d}$.*

Proof: If $\mathfrak{d}_a \in M^{(r)}$ is a divisor of the form $\mathfrak{d} = \nu_1 \cdot a_1 + \cdots + \nu_s \cdot a_s$, where $a_1, \ldots, a_s$ are s distinct points of the surface M, choose disjoint coordinate coordinate neighborhoods U_i of the points a_i with local coordinates z_i centered at the points a_i. A coordinate neighborhood of the divisor $\mathfrak{d}_a$ in the complex manifold $M^{(r)}$ is of the form $U = U_1^{(\nu_1)} \times \cdots \times U_s^{(\nu_s)}$ as in the preceding theorem. Explicitly the divisors in $M^{(r)}$ near $\mathfrak{d}$ are of the form $\mathfrak{d}_z = \sum_{i=1}^{s} \sum_{j=1}^{\nu_i} 1 \cdot z_{ij}$ where $z_{ij} \in U_i$ for $1 \leq j \leq \nu_i$ are the coordinate values of ν_i points of that neighborhood; and by using power sums in place of the elementary symmetric functions the local coordinates of such a divisor $\mathfrak{d}_z$ in the quotient space $U_i^{\nu_i}$ can be taken to be of the form

$$t_{im}(z_{i1}, \ldots, z_{i\nu_i}) = z_{i1}^m + \cdots + z_{i\nu_i}^m$$

for $1 \leq i \leq s$, $1 \leq m \leq \nu_i$. For points z_{ij} sufficiently near a_i in the coordinate neighborhoods U_i an abelian integral $w_k(z_{ij})$ has the Taylor expansion

$$w_k(z_{ij}) = w_k(a_i) + w_k'(a_i) z_{ij} + \frac{1}{2} w_k''(a_i) z_{ij}^2 + \cdots ;$$

consequently the coordinates of the image of the divisor $\mathfrak{d}_z$ under the general Abel-Jacobi mapping are

$$\begin{aligned}
w_k(\mathfrak{d}_z) &= \sum_{i=1}^{s}\sum_{j=1}^{\nu_i} w_k(z_{ij}) \\
&= \sum_{i=1}^{s}\sum_{j=1}^{\nu_i}\Big(w_k(a_i) + w_k'(a_i)z_{ij} + \frac{1}{2}w_k''(a_i)z_{ij}^2 + \cdots\Big) \\
&= w_k(\mathfrak{d}) + \sum_{i=1}^{s}\Big(w_k'(a_i)t_{i1}(z) + \frac{1}{2}w_k''(a_i)t_{i2}(z) + \cdots \\
&\qquad\qquad \cdots + \frac{1}{\nu_i!}w_k^{(\nu_i)}(a_i)t_{i\nu_i}(z) + \cdots\Big),
\end{aligned}$$

where the further terms in the last expansion are higher powers in the local coordinates t_{im}. The partial derivatives of the functions $w_k(\mathfrak{d}_z)$ with respect to the coordinates t_{im} in $M^{(r)}$ evaluated at the divisor $\mathfrak{d}_a$, at which $t_{im} = 0$, thus are precisely the entries in row k of the Brill-Noether matrix (11.9) in terms of the local coordinates in U_i, and that suffices to conclude the proof.

Corollary 12.6 *If M is a compact Riemann surface of genus $g > 0$ the rank of the differential of the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ at a divisor $\mathfrak{d} \in M^{(r)}$ is*

$$\operatorname{rank}_{\mathfrak{d}} dw \quad = \quad \operatorname{rank}\Omega(\mathfrak{d}) \quad = \quad r + 1 - \gamma(\zeta_{\mathfrak{d}}).$$

Proof: It follows from the preceding theorem that $\operatorname{rank}_{\mathfrak{d}} dw = \operatorname{rank}\Omega(\mathfrak{d})$, while $\operatorname{rank}\Omega(\mathfrak{d}) = r + 1 - \gamma(\zeta_{\mathfrak{d}})$ by the Riemann-Roch Theorem in the form of Theorem 11.3; that suffices for the proof.

Although there is some variation in the precise meaning of the term in the differential-geometric literature, it is convenient here to define a *critical point* of a holomorphic mapping $f : M \longrightarrow N$ between two connected complex manifolds to be any point $p \in M$ at which $\operatorname{rank}_p df < \min(\dim M, \dim N)$. If $\dim M = \dim N$ the critical points of the mapping f are precisely those points $p \in M$ at which the mapping f fails to be locally biholomorphic.

Corollary 12.7 *If M is a compact Riemann surface of genus $g > 0$ the critical points of the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ form a proper holomorphic subvariety $\mathrm{sp}M^{(r)} \subset M^{(r)}$ that consists precisely of the special positive divisors of degree r.*

Proof: A positive divisor $\mathfrak{d}$ of degree r on a compact Riemann surface of genus $g > 0$ is a special positive divisor if and only if $\gamma(\zeta_{\mathfrak{d}}) - 1 > \max(0, r - g)$, by definition (11.19), while the preceding corollary shows that $\gamma(\zeta_{\mathfrak{d}}) - 1 = r - \operatorname{rank}_{\mathfrak{d}} dw$; thus a positive divisor $\mathfrak{d}$ of degree r is special if and only if $\operatorname{rank}_{\mathfrak{d}} dw < \min(r, g)$, which is just the condition that $\mathfrak{d}$ is a critical point of the general

Abel-Jacobi mapping. The differential dw is an $r \times g$ matrix of holomorphic functions in any local coordinate neighborhood on the complex manifold $M^{(r)}$, so the points at which it has rank less than $\min(r, g)$ form the holomorphic subvariety $\mathrm{sp}M^{(r)} \subset M^{(r)}$ defined by the vanishing of some subdeterminants of that matrix. There are nonspecial positive divisors in any of the manifolds $M^{(r)}$ by Theorem 11.5; consequently $\mathrm{sp}M^{(r)}$ is a proper subvariety of the connected complex manifold $M^{(r)}$, and that suffices to conclude the proof.

12.4 The Subvariety of Special Positive Divisors

In view of the preceding theorem it is customary to call the subset $\mathrm{sp}M^{(r)} \subset M^{(r)}$ the *subvariety of special positive divisors* of degree r, since it is indeed a holomorphic subvariety of the complex manifold $M^{(r)}$. The set of general divisors thus is the complement of a proper holomorphic subvariety of $M^{(r)}$ so is a dense open subset of $M^{(r)}$; that observation perhaps explains the use of the terms special positive divisor and general positive divisor.

Theorem 12.8 (i) *If M is a compact Riemann surface of genus $g > 0$ the image $W_r = w(M^{(r)}) \subset J(M)$ of the complex manifold $M^{(r)}$ under the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ is an irreducible holomorphic subvariety of the Jacobi variety $J(M)$.*
(ii) *If $r \geq g$ then $W_r = J(M)$.*
(iii) *If $r < g$ then* $\dim\ W_r = r$.
(iv) *If $r \leq g$ the image* $\mathrm{sp}W_r = w(\mathrm{sp}M^{(r)})$ *of the subvariety* $\mathrm{sp}M^{(r)} \subset M^{(r)}$ *of special positive divisors is a proper holomorphic subvariety of W_r. The subvariety $W_r \subset J(M)$ is a complex submanifold of $J(M)$ outside* $\mathrm{sp}W_r$*; and the restriction of the general Abel-Jacobi mapping is a biholomorphic mapping*

$$w : \Big(M^{(r)} \sim \mathrm{sp}M^{(r)}\Big) \xrightarrow{\cong} \Big(W_r \sim \mathrm{sp}W_r\Big) \tag{12.24}$$

between these two complex manifolds.

Proof: (i) The image $W_r = w(M^{(r)})$ is a holomorphic subvariety of the Jacobi variety $J(M)$ by Remmert's Proper Mapping Theorem; and since W_r is the image of a connected complex manifold it is necessarily an irreducible holomorphic subvariety.
(ii) If $r \geq g$ the Abel-Jacobi mapping is of rank $g = \dim J(M)$ at any nonspecial positive divisor $\mathfrak{d} \in M^{(r)}$ by Corollary 12.7, so this mapping is a biholomorphic mapping between an open neighborhood of the point $\mathfrak{d}$ in $M^{(r)}$ and an open neighborhood of the image $w(\mathfrak{d})$ in $J(M)$; therefore $\dim W_r = g$, and consequently $W_r = J(M)$ since $J(M)$ is a connected g-dimensional complex manifold.
(iii) If $r < g$ the differential of the Abel-Jacobi mapping is of rank $r = \dim M^{(r)}$ at any general positive divisor $\mathfrak{d} \in (M^{(r)} \sim \mathrm{sp}M^{(r)})$ by Corollary 12.7, so this mapping is a biholomorphic mapping between an open neighborhood U of $\mathfrak{d}$ in

$M^{(r)}$ and its image $w(U) \subset J(M)$; thus $w(U)$is an open subset of W_r that is an r-dimensional submanifold of $J(M)$, so since W_r is irreducible $\dim W_r = r$.
(iv) The set $\mathrm{sp}M^{(r)}$ is a proper holomorphic subvariety of the connected complex manifold $M^{(r)}$ by Corollary 12.7; and by Remmert's Proper Mapping Theorem the image $\mathrm{sp}W_r = w(\mathrm{sp}M^{(r)})$ is a holomorphic subvariety of W_r and $\dim \mathrm{sp}W_r \le \dim \mathrm{sp}M^{(r)} < \dim M^{(r)} = \dim W_r$ hence $\mathrm{sp}W_r$ is a proper holomorphic subvariety of W_r. From Corollary 12.6 it follows that $\gamma(\zeta_{\mathfrak{d}}) = 1$ for a divisor $\mathfrak{d} \in (M^{(r)} \sim \mathrm{sp}M^{(r)})$, so there are no other divisors in $M^{(r)}$ that are linearly equivalent to $\mathfrak{d}$; therefore $w^{-1}\big(w(\mathfrak{d})\big) = \mathfrak{d}$ by Abel's Theorem, and consequently the restriction of the general Abel-Jacobi mapping to the complement $M^{(r)} \sim \mathrm{sp}M^{(r)}$ is a one-to-one holomorphic mapping between complex manifolds so must be a biholomorphic mapping. That concludes the proof.

The special case $r = g$ of part (ii) of the preceding theorem shows that the general Abel-Jacobi mapping $w : M^{(g)} \longrightarrow J(M)$ for a compact Riemann surface M of genus $g > 0$ is a surjective mapping from the g-fold symmetric product of the surface M to its Jacobi variety, and consequently that any point in the Jacobi variety can be represented as the image $w(\mathfrak{d})$ of a positive divisor $\mathfrak{d}$ of degree g on the surface M; this result is traditionally known as the *Jacobi Inversion Theorem.*

Corollary 12.9 *Let M be a compact Riemann surface of genus $g > 0$.*
(i) *If M has no special positive divisors of degree r the subvariety $W_r \subset J(M)$ is an r-dimensional complex submanifold of the Jacobi variety $J(M)$ of M and the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ is a biholomorphic mapping between the complex manifolds $M^{(r)}$ and W_r.*
(ii) *For $r = 1$ the Abel-Jacobi mapping $w : M \longrightarrow J(M)$ is a biholomorphic mapping between M and the complex submanifold $W_1 = w(M) \subset J(M)$.*
(iii) *If M is of genus $g = 1$ the Abel-Jacobi mapping $w : M \longrightarrow J(M)$ is a biholomorphic mapping between M and its Jacobi variety $J(M)$.*

Proof: (i) This is an immediate consequence of Theorem 12.8, the special case in which $\mathrm{sp}M^{(r)} = \mathrm{sp}W_r = \emptyset$.
(ii) This is a special case of (i), since by Corollary 11.6 there are no special positive divisors of degree 1 on a Riemann surface M of genus $g > 0$.
(iii) This is a special case of (ii), and that concludes the proof.

An incidental consequence of the preceding corollary is the observation that if $r > g$ there always exist special positive divisors of degree r on a compact Riemann surface of genus $g > 0$; this also could have been deduced from Corollary 11.9, but does not merit further discussion here since more refined results will be obtained later. The Abel-Jacobi imbedding $w : M \longrightarrow J(M)$ of a compact Riemann surface M of genus $g > 0$ as a submanifold $W_1 = w(M) \subset J(M)$ of its Jacobi variety is a very useful concrete representation of the Riemann surface M. The identification of a compact Riemann surface of genus $g = 1$ with the complex torus $J(M)$ is the classical way of handling compact Riemann surfaces of genus $g = 1$ through the theory of elliptic functions. For a fixed base point point $z_0 \in \tilde{M}$ the images $W_r = w_{z_0}(M^{(r)}) \subset J(M)$ of the manifolds of positive divisors when viewed as subvarieties of the Jacobi group are related to one another in a variety of ways through the group structure on $J(M)$. In terms of this group structure, for any subsets A, $B \subset J(M)$ set

$$A + B = \left\{ a + b \;\middle|\; a \in A,\ b \in B \right\}, \tag{12.25}$$

and following Henrik Martens set

$$A \ominus B = \left\{ t \in J(M) \;\middle|\; t + B \subset A \right\}. \tag{12.26}$$

It is convenient to insert here the following useful auxiliary result before examining these two operations.

Lemma 12.10 *If σ is a holomorphic line bundle over a compact Riemann surface M of genus $g > 0$ and if $c(\sigma) < g - 1$ and $\gamma(\sigma\zeta_a) > 0$ for all points $a \in M$ then $\gamma(\sigma) > 0$.*

Proof: If to the contrary $\gamma(\sigma) \le 0$ it follows from Lemma 2.6 that $\gamma(\sigma\zeta_a) = 1$ for all points $a \in M$ and $\gamma(\sigma) = 0$, so by the Riemann-Roch Theorem $\gamma(\kappa\sigma^{-1}\zeta_a^{-1}) = g - 1 - c(\sigma)$ for all $a \in M$ and $\gamma(\kappa\sigma^{-1}) = g - 1 - c(\sigma) > 0$. Thus the line bundle $\kappa\sigma^{-1}$ has nontrivial holomorphic cross-sections and in addition $\gamma(\kappa\sigma^{-1}\zeta_a^{-1}) = \gamma(\kappa\sigma^{-1})$ for all $a \in M$. However by Lemma 2.6 again $\gamma(\kappa\sigma^{-1}\zeta_a^{-1}) = \gamma(\kappa\sigma^{-1})$ if and only if all holomorphic cross-sections of $\kappa\sigma^{-1}$ vanish at the point $a \in M$, so this equality can hold for at most finitely many points of M; that contradiction to conclude the proof.

Theorem 12.11 *If M is a compact Riemann surface of genus $g > 0$ with a base point $z_0 \in \tilde{M}$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ the subvarieties $W_r = w_{z_0}(M^{(r)}) \subset J(M)$ of the Jacobi group for integers $r > 0$ satisfy*

(i) $W_r \subset W_{r+1}$ *and* $W_g = J(M)$

(ii) $W_r + W_s = W_{r+s}$

(iii) $W_r = W_1 + W_1 + \cdots + W_1$ *(r terms).*

(iv) $W_r \ominus W_s = \begin{cases} J(M) & \text{if } r \ge g, \\ W_{r-s} & \text{if } r < g \text{ and } r \ge s, \\ \emptyset & \text{if } r < g \text{ and } r < s. \end{cases}$

Proof: (i) The image under the Abel-Jacobi homomorphism of the point $a_0 = \pi(z_0) \subset M$ is the point of the Jacobi group $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ represented by the vector $w_{z_0}(z_0) = w(z_0, z_0) = 0 \subset \mathbb{C}^g$, so is the identity of the Jacobi group. Therefore $w_{z_0}(\mathfrak{d}) = w_{z_0}(\mathfrak{d}) + w_{z_0}(a_0) = w_{z_0}(\mathfrak{d} + a_0) \in W_{r+1} \subset J(M)$ for any divisor $\mathfrak{d} \in M^{(r)}$ so $W_r \subset W_{r+1}$. It follows from Theorem 12.8 (ii) that $W_g = J(M)$.

(ii) and (iii) If $t_1 = w_{z_0}(\mathfrak{d}_1) \in W_r$ and $t_2 = w_{z_0}(\mathfrak{d}_2) \in W_s$, for positive divisors $\mathfrak{d}_1 \in M^{(r)}$ and $\mathfrak{d}_2 \in M^{(s)}$, then $t_1 + t_2 = w_{z_0}(\mathfrak{d}_1 + \mathfrak{d}_2) \in W_{r+s}$ so

$W_r + W_s \subset W_{r+s}$. On the other hand if $t = w_{z_0}(\mathfrak{d}) \in W_{r+s}$ for a positive divisor $\mathfrak{d} \in M^{(r+s)}$ then it is possible to write $\mathfrak{d} = \mathfrak{d}_1 + \mathfrak{d}_2$ where $\mathfrak{d}_1 \in M^{(r)}$ and $\mathfrak{d}_2 \in M^{(s)}$ so $t = w_{z_0}(\mathfrak{d}_1 + \mathfrak{d}_2) \in W_r + W_s$ and consequently $W_{r+s} \subset W_r + W_s$. Altogether then $W_{r+s} = W_r + W_s$, and by iterating this result $W_1 + \cdots + W_1 = W_r$ for r copies of the subvariety W_1.

(iv) If $r \ge g$ then $W_r = J(M)$ by Theorem 12.8 (ii) so $t + W_s \subset J(M) = W_r$ for all $t \in J(M)$ and consequently from the definition (12.25) it follows that $W_r \ominus W_s = J(M)$. If $r < g$ and $r < s$ it follows from Theorem 12.8 that $\dim W_s = \min(s, g) > r = \dim W_r$ so no translate $t + W_s$ can be contained in W_r and consequently $W_r \ominus W_s = \emptyset$. The interesting case is that in which $s \le r < g$. If $t \in W_{r-s}$ then it follows from (ii) that $t + W_s \subset W_r$ and consequently that $t \in W_r \ominus W_s$, so $W_{r-s} \subset W_r \ominus W_s$. Any $t \in J(M)$ can be written as the image $t = w_{z_0}(\mathfrak{d})$ of some divisor $\mathfrak{d}$ with $\deg \mathfrak{d} = r - s$, since the group homomorphism w_{z_0} in the Abel-Jacobi diagram (12.16) is surjective and its kernel contains divisors of any degree. If $t \in W_r \ominus W_s$ and $\mathfrak{a} \in M^{(s)}$ then $w_{z_0}(\mathfrak{d} + \mathfrak{a}) = t + w_{z_0}(\mathfrak{a}) \in W_r$, so $w_{z_0}(\mathfrak{d} + \mathfrak{a}) = w_{z_0}(\mathfrak{b})$ for some divisor $\mathfrak{b} \in M^{(r)}$; and since $\deg(\mathfrak{d} + \mathfrak{a}) = \deg \mathfrak{b}$ it follows from Abel's Theorem that $(\mathfrak{d} + \mathfrak{a}) \sim \mathfrak{b}$. The holomorphic line bundles of linearly equivalent divisors are holomorphically equivalent so $\zeta_{\mathfrak{d}} \cdot \zeta_{\mathfrak{a}} = \zeta_{\mathfrak{b}}$; and since $\mathfrak{b}$ is a positive divisor its line bundle has nontrivial holomorphic cross-sections, so $\gamma(\zeta_{\mathfrak{d}} \cdot \zeta_{\mathfrak{a}}) > 0$ for all positive divisors $\mathfrak{a} \in M^{(s)}$. In particular $\gamma(\zeta_{\mathfrak{d}} \zeta_{\mathfrak{a}'} \zeta_a) > 0$ if $\mathfrak{a} = \mathfrak{a}' + a$ for any positive divisor $\mathfrak{a}' \in M^{(s-1)}$ and any point $a \in M$, so since $c(\zeta_{\mathfrak{d}} \zeta_{\mathfrak{a}'}) = r - 1 < g - 1$ it follows from Lemma 12.10 that $\gamma(\zeta_{\mathfrak{d}} \zeta_{\mathfrak{a}'}) > 0$; a repetition of this argument shows eventually that $\gamma(\zeta_{\mathfrak{d}}) > 0$ hence that $\mathfrak{d} \in M^{(r-s)}$, so $t \in W_{r-s}$ and consequently $W_r \ominus W_s \subset W_{r-s}$. Altogether then $W_r \ominus W_s = W_{r-s}$, and that suffices to conclude the proof.

It follows from (iii) of the preceding theorem that all of the subvarieties $W_r \subset J(M)$ are determined just by the submanifold W_1 alone. On the other hand it follows from (iv) of that theorem that $W_1 = W_{g-1} \ominus W_{g-2}$ so W_1 in turn is determined fully by the subvarieties W_{g-1} and W_{g-2}; that is of some interest since subvarieties of lower codimension are often easier to handle than subvarieties of higher codimension, and subvarieties of codimension 1 are usually the easiest to handle. A further significance of (iv) is that the holomorphic subvarieties $W_r \subset J(M)$ admit only trivial translations, in the sense that

$$t + W_r \subset W_r \quad \text{for } r < g \text{ if and only if } t = 0; \tag{12.27}$$

in that way these subvarieties differ significantly from subtori.

12.5 The Subvariety $W_1 \subset J(M)$

As the image $W_1 = w_{z_0}(M)$ of the 2-dimensional compact topological manifold M the subset $W_1 \subset J(M)$ can be viewed as a singular 2-cycle in the $2g$-dimensional manifold $J(M)$, so it represents a homology class $[W_1] \in H_2(J(M))$ in the complex torus $J(M)$. When the torus $J(M)$ is described as the quotient group $J(M) = \mathbb{C}^g/\Omega\ \mathbb{Z}^{2g}$ of the vector space $\mathbb{C}^g$ by the lattice subgroup generated by the column vectors $\omega^1, \ldots, \omega^{2g}$ of the period matrix Ω, a basis[1]for the homology group $H_2\big(J(M)\big)$ consists of the homology classes represented by the singular 2-cycles $\omega^{l,m}$ spanned by pairs of column vectors ω^l and ω^m, and $\omega^{l,m} = -\omega^{m,l}$ and $\omega^{l,l} = 0$.

Theorem 12.12 *If M is a compact Riemann surface of genus $g > 0$ with period matrix Ω and intersection matrix $P = \{p_{lm}\}$ in terms of any bases $\omega_i \in \Gamma(M,\ \mathcal{O}^*(1,0))$ and $\tau_j \in\ H_1(M)$, the submanifold $W_1 \subset\ J(M)$ represents the homology class*

$$[W_1] = \tfrac{1}{2} \sum_{l,m=1}^{2g} p_{lm}\ \omega^{l,m} \in H_2(J(M)). \tag{12.28}$$

Proof: If $\Pi = \{\pi_{ij}\}$ is the inverse period matrix to the period matrix Ω, the differential forms $\phi_i = \sum_{k=1}^{g} \big(\overline{\pi_{ki}}\omega_k(z) + \pi_{ki}\overline{\omega_k(z)}\big) \in \Gamma(M, \mathcal{E}^1_c)$ form a basis for the deRham group $\mathfrak{H}^1(M)$ dual to the homology basis $\tau_j \in H_1(M)$ since

$$\int_{\tau_j} \phi_i = \sum_{k=1}^{g} (\overline{\pi_{ki}}\omega_{kj} + \pi_{ki}\overline{\omega_{kj}}) = \delta^i_j$$

by (F.7). The integrals

$$t_i(z, z_0) = \int_{z_0}^{z} \phi_i = \sum_{k=1}^{g} \big(\overline{\pi_{ki}} w_k(z, z_0) + \pi_{ki}\overline{w_k(z, z_0)}\big)$$

for any base point $z_0 \in \tilde{M}$ are $\mathcal{C}^\infty$ functions of the variable $z \in \tilde{M}$ that describe a $\mathcal{C}^\infty$ mapping $\tilde{t}_{z_0} : \tilde{M} \longrightarrow \mathbb{R}^{2g}$; and $t_i(T_j z, z_0) = t_i(z, z_0) + \delta^i_j$ so the mapping $\tilde{t}_{z_0}$ induces a $\mathcal{C}^\infty$ mapping

$$t_{z_0} : M = \tilde{M}/\Gamma \longrightarrow T = \mathbb{R}^{2g}/Z^{2g} \tag{12.29}$$

of the Riemann surface M into the standard torus $T = (\mathbb{R}/\mathbb{Z})^{2g}$. Since $\tilde{t}_{z_0}(z) = \overline{{}^t\Pi}\tilde{w}_{z_0}(z) + {}^t\Pi\overline{\tilde{w}_{z_0}(z)}$ the mapping t_{z_0} is the composition of the Abel-Jacobi mapping $w_{z_0} : M \longrightarrow J(M)$ and the homeomorphism

$$\tilde{\Pi} : J(M) \longrightarrow T \tag{12.30}$$

of (F.15); thus the mapping t_{z_0} is a $\mathcal{C}^\infty$ imbedding of the surface M as a submanifold $t_{z_0}(M) \subset \mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Since $dt_k(z, z_0) = \phi_k(z)$ it follows that the differential form on M induced by the differentials dt_k on T under the mapping t_{z_0} is $t^*_{z_0}(dt_k) = \phi_k$; hence

$$\int_{t_0(M)} dt_k \wedge\ dt_l = \int_M \phi_k \wedge \phi_l = p_{kl} \tag{12.31}$$

[1]Some of the topological properties of complex tori are discussed in Appendix F.2.

in terms of the intersection matrix $P = \{p_{kl}\}$. The basis for the homology group $H_2(T)$ dual to the basis $dt_k \wedge dt_l$ of the deRham group $\mathfrak{H}^2(T)$ consists of the homology classes represented by the singular 2-cycles $\delta^{k,l}$ spanned by the pairs of basis vectors δ^k and δ^l, where the vectors δ^i are generators of the lattice subgroup $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$; consequently it follows from (12.31) that the homology class represented by the submanifold $t_{z_0}(M) \subset T$ is

$$[t_{z_0}(M)] = \frac{1}{2} \sum_{k,l=1}^{2g} p_{kl}\delta^{k,l} \in H_2(T). \tag{12.32}$$

The real linear mapping $\Omega : \mathbb{R}^{2g} \longrightarrow \mathbb{C}^g$ defined by the matrix Ω induces the homeomorphism $\Omega : T \longrightarrow J(M)$ inverse to the homeomorphism (12.30) as in (F.13). The mapping Ω therefore maps the subvariety $t_{z_0}(M) \subset T$ to the subvariety $W_1 = \Omega \cdot t_{z_0}(M) \subset J(M)$, so the homology class represented by W_1 is the corresponding image

$$[W_1] = [\Omega \cdot t_{z_0}(M)] = \Omega \cdot \tfrac{1}{2} \textstyle\sum_{k,l=1}^{2g} p_{kl}\delta^{k,l} = \tfrac{1}{2} \textstyle\sum_{k,l=1}^{2g} p_{kl} \cdot \Omega\delta^{k,l}$$

of the homology class (12.29) represented by the submanifold $t_{z_0}(M) \subset T$; since $\Omega\delta^{k,l}$ is the singular 2-cycle spanned by the pairs of basis vectors $\Omega\delta^k = \omega^k$ and $\Omega\delta^l = \omega^l$ that suffices to conclude the proof.

The preceding result can be extended to yield some information about the topological properties of other 1-dimensional holomorphic subvarieties of complex tori. If V is a one-dimensional irreducible holomorphic subvariety, possibly with singularities, in a complex torus T of dimension $h > 0$, the normalization[2] of V is a compact Riemann surface M with a holomorphic mapping $n : M \longrightarrow V$ that is a one-to-one locally biholomorphic mapping except possibly over finitely many points of V, the singularities of V and some points with finitely many inverse images under the mapping n. As the image of a 2-dimensional topological manifold the subset $V \subset T$ can be viewed as a singular 2-cycle so represents a homology class $[V] \in H_2(T)$. Just as for the Jacobi variety $J(M)$, if the torus T is described as the quotient $T = \mathbb{C}^h/\Lambda\ Z^{2h}$ of the vector space $\mathbb{C}^h$ by the lattice subgroup spanned by the columns $\lambda^1, \dots \lambda^{2h}$ of the period matrix Λ, the singular 2-cycles $\lambda^{l,m}$ spanned by pairs of column vectors λ^l and λ_m represent a basis for the homology group $H_2(T)$, and $\lambda^{l,m} = -\lambda^{m,l}$ and $\lambda^{l,l} = 0$.

Theorem 12.13 *If V is an irreducible one-dimensional holomorphic subvariety of an h-dimensional complex torus $T = \mathbb{C}^h/\Lambda\mathbb{Z}^{2h}$ and the normalization of V is a compact Riemann surface M of genus g, then for any basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ in terms of which Ω is the period matrix and P is the intersection matrix of the surface M, the normalization mapping $n : M \longrightarrow V$ is the composition $n = f \circ w$ of the Abel-Jacobi mapping*

$$w : M \longrightarrow J(M) = C^g/\Omega\mathbb{Z}^{2g}$$

[2]Further properties of holomorphic varieties, in particular the existence and nature of the normalization mapping for one-dimensional holomorphic varieties, are discussed in Appendix A.3.

of the Riemann surface M into its Jacobi variety $J(M)$ and a holomorphic mapping $f : J(M) \longrightarrow T$ between these two complex tori. If the mapping f is described by a Hurwitz relation (A, Q) from the period matrix Ω of the surface M to the period matrix Λ of the torus T then the homology class represented by the subvariety V is

$$[V] = \frac{1}{2} \sum_{j,k=1}^{2h} c_{jk} \lambda^{j,k} \in H_2(T) \tag{12.33}$$

where $\{c_{jk}\} = C = QP\,{}^tQ \in \mathbb{Z}^{2h \times 2h}$.

Proof: The normalization mapping $n : M \longrightarrow V$ is induced by a holomorphic mapping $n : M \longrightarrow T$ from the Riemann surface M into the torus T with image $n(M) = V$. By Theorem 3.7 this mapping can be factored as the composition $n = f \circ w$ of the Abel-Jacobi mapping $w : M \longrightarrow J(M)$ from the Riemann surface M to its Jacobi variety $J(M) = C^g/\Omega\mathbb{Z}^{2g}$ and a holomorphic mapping $f : J(M) \longrightarrow T$ from the Jacobi variety $J(M)$ to the complex torus T, where the mapping f is described up to a translation in T by a Hurwitz relation (A, Q) from Ω to Λ; the mapping f is induced by the affine mapping $\tilde{f}(z) = Az + a$ for some point $a \in \mathbb{C}^h$, and $A\Omega = \Lambda Q$. The mapping f induces a homomorphism $f_* : H_*(J(M)) \longrightarrow H_*(T)$ between the homology groups of these two tori that takes the homology class represented by the singular 1-cycle carried by the column ω^l of the period matrix Ω to the homology class carried by the singular cycle described by the linear combination $A\omega^l = A\Omega\delta^l = \Lambda\ Q\delta^l = \sum_{j=1}^{2h} \lambda^j q_{jl}$ of the singular 1-cycles carried by the columns of the period matrix Λ, and correspondingly takes the homology class represented by the singular 2-cycle $\omega^{l,m}$ spanned by the pair of column vectors ω^l and ω^m to the homology class represented by the linear combination $A\omega^{l,m} = \sum_{j,k=1}^{2h} \lambda^{j,k} q_{jl} q_{km}$ of the singular 2-cycles $\lambda^{j,k}$ spanned by the pairs of column vectors λ^j and λ^k. By the preceding theorem the homology class represented by the image $W_1 = w(M) \subset J(M)$ is $[W_1] = \frac{1}{2} \sum_{l,m=1}^{2g} p_{lm}\ \omega^{l,m} \in H_1(J(M)$; consequently the homology class represented by the subvariety $V = n(M) = f(w(M))$ is

$$[V] = f_*([W_1]) = f_* \left(\frac{1}{2} \sum_{l,m=1}^{2h} p_{lm} \omega^{l,m} \right) = \frac{1}{2} \sum_{l,m=1}^{2g} \sum_{j,k=1}^{2h} p_{lm} q_{jl} q_{km} \lambda^{j,k},$$

which is just (12.33) where $c_{jk} = \sum_{l,m=1}^{2g} q_{jl} p_{lm} q_{km}$, and that suffices for the proof.

Corollary 12.14 (Matsusaka's Theorem) *A complex torus $T = \mathbb{C}^g/\Lambda^{2g}$ described by a nonsingular period matrix Λ is biholomorphic to the Jacobi variety of a compact Riemann surface of genus g if and only if it contains an irreducible one-dimensional holomorphic subvariety V such that the normalization of V is a Riemann surface of genus g and the homology class in $H^2(T)$ represented by*

V is

$$[V] = \tfrac{1}{2}\sum_{j,k=1}^{2g} c_{jk}\lambda^{j,k} \in H_2(T) \tag{12.34}$$

where $\det C = \pm 1$. The subvariety V is necessarily a nonsingular submanifold of the complex torus T, and T is biholomorphic to the Jacobi variety $J(V)$ of the compact Riemann surface V.

Proof: If a complex torus T contains an irreducible one-dimensional holomorphic subvariety V with normalization a compact Riemann surface M of genus $g = \dim T$ such that V represents the homology class (12.34) then by the preceding theorem there is a holomorphic mapping $f : J(M) \longrightarrow T$ described by a Hurwitz relation (A, Q) from the period matrix Ω to the period matrix Λ such that the matrix C is given by $C = QP\,{}^tQ$. If $\det C = \pm 1$ then $\det Q = \pm 1$ as well, since $\det P = \pm 1$; thus $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ and the mapping f is biholomorphic by Theorem F.9 (ii). That mapping induces a biholomorphic mapping between M and V, and shows as well that V is imbedded as a submanifold of T and that $T = J(M) = J(V)$. Conversely if there is an analytic equivalence $f : J(M) \longrightarrow T$ between the complex torus T and the Jacobi variety $J(M)$ of a compact Riemann surface M of genus g then Theorem 12.12 exhibits the subvariety $V = f(W_1)$ as having the desired properties. That suffices to conclude the proof.

12.6 The Variety of Linear Equivalencence Classes of Divisors

The general Abel-Jacobi mapping (12.23) fails to be a locally biholomorphic mapping at the special positive divisors, since by Corollary 12.7 the special positive divisors are precisely the critical points of that mapping. To examine the local behavior of the general Abel-Jacobi mapping at such points, consider first a holomorphic line bundle λ of characteristic class $c(\lambda) = r > 0$ over a compact Riemann surface M of genus $g > 0$ and a collection $\{f_0, \ldots, f_\nu\}$ of linearly independent holomorphic cross-sections $f_i \in \Gamma(M, \mathcal{O}(\lambda))$; and introduce the mapping

$$\tilde{F} : \left(\mathbb{C}^{\nu+1} \sim 0\right) \longrightarrow M^{(r)} \tag{12.35}$$

that associates to any nonzero vector $t = (t_0, \ldots, t_\nu) \in \mathbb{C}^{\nu+1}$ the divisor

$$\tilde{F}(t) = \mathfrak{d}(t_0 f_0 + \cdots + t_\nu f_\nu) \in M^{(r)}. \tag{12.36}$$

Since two holomorphic cross-sections of the bundle λ have the same divisor if and only if they are constant multiples of one another, and the cross-sections f_i are linearly independent, it is evident that the divisor $\tilde{F}(t)$ depends only on the point $t \in \mathbb{P}^\nu$ represented by the vector $(t_0, \ldots, t_\nu) \in \mathbb{C}^{\nu+1}$ and that the naturally induced mapping

$$F : \mathbb{P}^\nu \longrightarrow M^{(r)} \tag{12.37}$$

is a one-to-one mapping from the complex projective space $\mathbb{P}^\nu$ into the complex manifold $M^{(r)}$.

Theorem 12.15 (i) *If M is a compact Riemann surface of genus $g > 0$ the mapping $F : \mathbb{P}^\nu \longrightarrow M^{(r)}$ determined by a collection $\{f_0, \ldots, f_\nu\}$ of linearly independent holomorphic cross-sections $f_i \in \Gamma(M, \mathcal{O}(\lambda))$ of a holomorphic line bundle λ over M with $c(\lambda) = r$ is a holomorphic mapping. The image of F is a complex submanifold $F(\mathbb{P}^\nu) \subset M^{(r)}$ consisting of linearly equivalent divisors, and F is a biholomorphic mapping between $\mathbb{P}^\nu$ and its image $F(\mathbb{P}^\nu)$.*
(ii) *If $\{f_0, \ldots, f_\nu\}$ is a basis for $\Gamma(M, \mathcal{O}(\lambda))$ the submanifold $F(\mathbb{P}^\nu) \subset M^{(r)}$ is a full linear equivalence class of divisors of degree r on M.*
(iii) *If the cross-sections $f_0, \ldots, f_\nu$ are holomorphic functions of additional parameters in a holomorphic variety V the mapping $F : \mathbb{P}^\nu \times V \longrightarrow M^{(r)}$ is also a holomorphic mapping.*

Proof: The image of a point $c \in \mathbb{P}^\nu$ represented by an arbitrary nonzero vector $(c_0, \ldots, c_\nu) \in \mathbb{C}^{\nu+1}$ can be written as a divisor

$$F(c) = \mathfrak{d}_0 = n_1 \cdot a_1 + \cdots + n_s \cdot a_s \in M^{(r)}$$

where a_i are distinct points of M. If U_i are disjoint open coordinate neighborhoods of the points a_i with local coordinates $z_i \in U_i$, then for any point $(t_0, \ldots, , t_\nu) \in C^{\nu+1}$ sufficiently near the point $(c_0, \ldots, c_\nu)$ the divisor of the cross-section $t_0 f_0 + \cdots + t_\nu f_\nu$ is near the divisor $\mathfrak{d}_0$ in $M^{(r)}$ so can be written

$$\mathfrak{d}(t_0 f_0 + \cdots + t_\nu f_\nu) = \sum_{i=1}^{s} \sum_{k=1}^{n_i} 1 \cdot \ z_{ik}$$

for some points $z_{i,1}, \ldots, z_{i,n_i} \in U_i$. As in the proof of Theorem 12.5, the values $\zeta_{ij} = z_{i1}^j + \cdots + z_{in_i}^j$ for $1 \le i \le s,\ 1 \le j \le n_i$ can be taken as local coordinates in the manifold $M^{(r)}$ in an open neighborhood of the point $\mathfrak{d}_0 \in M^{(r)}$. In terms of these coordinates the mapping F is described by r coordinate functions $\zeta_{ij}(t_0, \ldots, t_r)$, which by the Cauchy integral formula can be written

$$\zeta_{ij}(t_0, \ldots, t_r) = \frac{1}{2\pi i} \int_{\partial U_i} \frac{t_0 f_0'(z_i) + \cdots + t_\nu f_\nu'(z_i)}{t_0 f_0(z_i) + \cdots + t_\nu f_\nu(z_i)} z_i^j \, dz_i.$$

This integral clearly is a holomorphic function of the values $t_0, \ldots, t_\nu$ whenever the point $(t_0, \ldots, t_\nu)$ is sufficiently near the point $(c_0, \ldots, c_\nu)$ that the denominator is nonzero on the boundary ∂U_i, so the mapping F is holomorphic in an open neighborhood of the point $c \in \mathbb{P}^\nu$; and if the functions f_i are holomorphic functions of additional parameters in another holomorphic variety V the integral is also a holomorphic function of these other parameters as well, so the mapping $F : \mathbb{P}^\nu \times V \longrightarrow M^{(r)}$ also is holomorphic. For the mapping F itself, since $\mathbb{P}^\nu$ is a compact manifold it follows from Remmert's Proper Mapping Theorem that the image $F(\mathbb{P}^\nu) \subset M^{(r)}$ is an irreducible holomorphic subvariety of the complex manifold $M^{(r)}$ with $\dim F(\mathbb{P}^\nu) = \nu$; and it was already noted that the mapping F takes distinct points of $\mathbb{P}^\nu$ to distinct points of $M^{(r)}$. All the divisors in the image $F(\mathbb{P}^\nu) \subset M^{(r)}$ are divisors of holomorphic cross-sections of the same holomorphic line bundle, so all are linearly equivalent; and from Abel's Theorem, Theorem 5.10, it follows that the image of the subset $F(\mathbb{P}^\nu) \subset M^{(r)}$ under the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ is a single point $w_F = w\big(F(\mathbb{P}^\nu)\big) \subset J(M)$. If the cross-sections f_i are a basis for the space $\Gamma(M, \mathcal{O}(\lambda))$ the divisors $\mathfrak{d}(t_0 f_0 + \cdots + t_\nu f_\nu)$ for all nonzero vectors $(t_0, \ldots, t_\nu) \in \mathbb{C}^{\nu+1}$ are a full linear equivalence class of divisors so $F(\mathbb{P}^\nu) = w^{-1}(w_F)$; and since $\operatorname{rank}_{\mathfrak{d}_0} dw = r - \nu$ by Corollary 12.6 the ν-dimensional holomorphic subvariety $F(\mathbb{P}^\nu)$ is a ν-dimensional complex submanifold of $M^{(r)}$. The mapping $F : \mathbb{P}^\nu \longrightarrow F(\mathbb{P}^\nu)$ then is a one-to-one holomorphic mapping between two complex manifolds, so must be biholomorphic. Its restriction to a linear subspace $\mathbb{P}^\mu \subset \mathbb{P}^\nu$ is also a biholomorphic mapping between $\mathbb{P}^\mu$, with the image $F(\mathbb{P}^\mu) \subset F(\mathbb{P}^\nu)$; and since the mappings defined by an arbitrary set of linearly independent holomorphic cross-sections always can be obtained by restricting some basis of the space of cross-sections, which suffices to conclude the proof.

Corollary 12.16 *Under the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ of a compact Riemann surface M of genus $g > 0$ into its Jacobi variety the inverse image $w^{-1}\big(w(\mathfrak{d})\big) \subset M^{(r)}$ of any divisor $\mathfrak{d} \in M^{(r)}$ is a complex submanifold of $M^{(r)}$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$ of dimension $\nu = \gamma(\zeta_\mathfrak{d}) - 1$ and that consists of all those positive divisors of degree r linearly equivalent to $\mathfrak{d}$.*

Proof: If $\mathfrak{d} \in M^{(r)}$ and $f_0, \ldots, f_\nu \in \Gamma(M, \mathcal{O}(\zeta_\mathfrak{d}))$ is a basis for the space of holomorphic cross-sections of the line bundle $\zeta_\mathfrak{d}$, where $\nu = \gamma(\zeta_\mathfrak{d}) - 1$, then by the preceding theorem the image $F(\mathbb{P}^\nu) \subset M^{(r)}$ of the holomorphic mapping $F : \mathbb{P}^\nu \longrightarrow M^{(r)}$ described by these cross-sections is a holomorphic submanifold of $M^{(r)}$ that is biholomorphic to $\mathbb{P}^\nu$ and that consists of all the positive divisors linearly equivalent to $\mathfrak{d}$. By Abel's Theorem, Theorem 5.10, this set of divisors is also the inverse image $w^{-1}\big(w(\mathfrak{d})\big)$, and that suffices for the proof.

By definition (11.19) the special positive divisors in $M^{(r)}$ for a compact Riemann surface M of genus $g > 0$ are those divisors for which $\gamma(\zeta_\mathfrak{d}) - 1 > \max(0, r - g)$. It then follows from the preceding corollary that the special positive divisors of degree $r \le g$ are precisely those divisors $\mathfrak{d} \in M^{(r)}$ such that $\dim w^{-1}(w(\mathfrak{d})) > 0$. However if $r > g$ then for any divisor $\mathfrak{d} \in M^{(r)}$ the Riemann-Roch Theorem shows that $\gamma(\zeta_\mathfrak{d}) - 1 = \gamma(\kappa\zeta_\mathfrak{d}^{-1}) + r - g \ge r - g > 0$ and consequently that $\dim w^{-1}(w(\mathfrak{d})) \ge r - g > 0$; and $\mathfrak{d}$ is a special positive divisor if and only if $\dim w^{-1}\big(w(\mathfrak{d})\big) > r - g$. Thus the manifolds $M^{(r)}$ for $r > g$ always contain nontrivial complex projective spaces as submanifolds, as do the manifolds $M^{(r)}$ for $r \le g$ so long as there are special positive divisors of degree r on the surface M. It is perhaps worth noting that all projective subspaces of $M^{(r)}$ arise precisely in this way.

Corollary 12.17 *If M is a compact Riemann surface of genus $g > 0$ the image $F(\mathbb{P}^\nu) \subset M^{(r)}$ of any holomorphic mapping $F : \mathbb{P}^\nu \longrightarrow M^{(r)}$ from a complex projective into $M^{(r)}$ consists of linearly equivalent divisors in $M^{(r)}$.*

Proof: Since complex projective spaces are simply connected the composition $G = w \circ F : \mathbb{P}^\nu \longrightarrow J(M)$ of the mapping F followed by the general Abel-Jacobi mapping $w : M^{(r)} \longrightarrow J(M)$ can be lifted to a holomorphic mapping $\tilde{G} : \mathbb{P}^\nu \longrightarrow \mathbb{C}^g$ from the simply connected compact complex manifold $\mathbb{P}^\nu$ to the universal covering space $\mathbb{C}^g$ of the complex torus $J(M) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$; the image $\tilde{G}(\mathbb{P}^\nu)$ is a compact holomorphic submanifold of $\mathbb{C}^g$ so must be a single point since the coordinate functions in $\mathbb{C}^g$ are constant as a consequence of the maximum modulus theorem; therefore the image $(w \circ F)(\mathbb{P}^\nu)$ is a single point of the Jacobi variety $J(M)$. It then follows from Abel's Theorem, Theorem 5.10, that all the divisors in $F(\mathbb{P}^\nu)$ are linearly equivalent, which suffices for the proof.

Corollary 12.18 *If M is a compact Riemann surface of genus $g > 0$ and if there are no special positive divisors in $M^{(r)}$ for some $r < g$ then the complex manifold $M^{(r)}$ contains no rational curves, no holomorphic images of the projective space $\mathbb{P}^1$ under a nonconstant holomorphic mapping $F : \mathbb{P}^1 \longrightarrow M^{(r)}$.*

Proof: By the preceding Corollary 12.17 the image $F(\mathbb{P}^1) \subset M^{(r)}$ of a holomorphic mapping from the complex projective space $\mathbb{P}^1$ into $M^{(r)}$ consists of linearly equivalent divisors; and if the mapping F is nonconstant this set consists of more than one divisor. If $\mathfrak{d}$ is one of these divisors then by Corollary 12.16 the set of all divisors linearly equivalent to $\mathfrak{d}$ form a complex projective space $\mathbb{P}^\nu$ of dimension $\nu = \gamma(\zeta_\mathfrak{d}) - 1$. Since there are at least two such divisors it must be the case that $\nu > 0$, and consequently that $\gamma(\zeta_\mathfrak{d}) - 1 > 0$; and since $\deg \mathfrak{d} = r < g$ it is a consequence of the definition (11.19) that the divisor $\mathfrak{d}$ is a special positive divisor in $M^{(r)}$. That contradicts the assumption, and thereby concludes the proof.

12.7 The Subvarieties G_r^ν and W_r^ν

The divisors in the subvarieties $\mathrm{sp}M^{(r)} \subset M^{(r)}$ of special positive divisors can be grouped according to the extent to which they are special, that is, according to the extent to which $\gamma(\zeta_\mathfrak{d}) - 1$ exceeds the value $\max(0, \deg \mathfrak{d} - g)$. With a slight modification of the classical notation set

$$G_r^\nu = \left\{ \mathfrak{d} \in M^{(r)} \,\middle|\, \gamma(\zeta_\mathfrak{d}) - 1 \ge \nu \right\} \text{ for } r > 0 \text{ and } \nu \in \mathbb{Z}. \tag{12.38}$$

For Riemann surfaces of genus $g > 0$ with bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ it follows from Corollary 12.6 that (12.38) is equivalent to either of the alternative characterizations

$$G_r^\nu = \begin{cases} \Big\{\mathfrak{d} \in M^{(r)} \;\Big|\; \operatorname{rank}_{\mathfrak{d}} dw \le r - \nu\Big\}, \\ \\ \Big\{\mathfrak{d} \in M^{(r)} \;\Big|\; \operatorname{rank}\Omega(\mathfrak{d}) \le r - \nu\Big\}, \end{cases} \quad \text{for } r > 0 \text{ and } \nu \in \mathbb{Z}, \tag{12.39}$$

where $w : M^{(r)} \longrightarrow J(M)$ is the general Abel-Jacobi mapping and $\Omega(\mathfrak{d})$ is the Brill-Noether matrix at the divisor $\mathfrak{d}$. In the classical terminology a divisor $\mathfrak{d} \in G_r^\nu$ is called simply a g_r^ν.

Theorem 12.19 *If M is a compact Riemann surface of genus $g > 0$ the subsets $G_r^\nu \subset M^{(r)}$ for $r > 0$ and $\nu \in \mathbb{Z}$ are holomorphic subvarieties.*

Proof: In view of the first alternative characterization in (12.39), the subset $G_r^\nu \subset M^{(r)}$ consists of those points of the compact complex manifold $M^{(r)}$ at which all the $(r - \nu + 1) \times (r - \nu + 1)$ subdeterminants of the differential dw of the general Abel-Jacobi mapping vanish. The differential is a $g \times r$ matrix of holomorphic functions in a coordinate neighborhood of the complex manifold $M^{(r)}$, so the subdeterminants are well defined local holomorphic functions and their zeros consequently form a holomorphic subvariety of $M^{(r)}$. That suffices for the proof.

For convenience the following is a list of some useful general properties of the subvarieties G_r^ν.

Theorem 12.20 *If M is a compact Riemann surface of genus $g > 0$ the holomorphic subvarieties $G_r^\nu \subset M^{(r)}$ for $r > 0$ satisfy*

(i) $G_r^{\nu+1} \subset G_r^\nu$,
(ii) $G_r^\nu = M^{(r)}$ *for* $\nu \le \max(0, r - g)$,
(iii) $G_r^\nu = \left\{\begin{array}{lll} M^{(r)} & \textit{for} & \nu \le r - g \\ \emptyset & \textit{for} & \nu > r - g \end{array}\right\}$ *if* $r \ge 2g - 1$,
(iv) $G_r^{1+\max(0,r-g)} = \operatorname{sp} M^{(r)}$ *for any* $r > 0$.

Proof: It is clear from the definition (12.38) that the defining conditions for these subvarieties are increasingly restrictive as ν increases, which yields (i). Of course $\gamma(\zeta_{\mathfrak{d}}) - 1 \ge 0$ for every positive divisor $\mathfrak{d}$, and it follows from the Riemann-Roch Theorem that $\gamma(\zeta_{\mathfrak{d}}) - 1 = \gamma(\kappa\zeta_d^{-1}) + r - g \ge r - g$ for any divisor $\mathfrak{d} \in M^{(r)}$; thus $\gamma(\zeta_{\mathfrak{d}}) - 1 \ge \max(0, r - g)$ for every positive divisor $\mathfrak{d} \in M^{(r)}$, which yields (ii). If $r \ge 2g - 1$ it follows from the Riemann-Roch Theorem that $\gamma(\zeta_{\mathfrak{d}}) - 1 = \gamma(\kappa\zeta_d^{-1}) + r - g = r - g$ for any divisor $\mathfrak{d} \in M^{(r)}$, which yields (iii). If $\mathfrak{d} \in M^{(r)}$ then $\mathfrak{d} \in G_r^{1+\max(0,r-g)}$ precisely when $\gamma(\zeta_{\mathfrak{d}}) - 1 \ge 1 + \max(0, r - g)$ by (12.38), and that is just the condition that $\mathfrak{d}$ is a special positive divisor as defined in (11.19). That suffices to conclude the proof.

The characterization of special positive divisors in (iv) of the preceding theorem is perhaps most easily remembered in the form

$$\text{sp}M^{(r)} = \begin{cases} G_r^1 & \text{for} \quad 1 \le r \le g, \\ \\ G_r^{r-g+1} & \text{for} \quad r \ge g. \end{cases} \tag{12.40}$$

For Riemann surfaces of genus $g > 0$ it follows from Theorem 2.4 that $\gamma(\zeta_p) = 1$ and hence $G_1^1 = \emptyset$, while it follows from (ii) of the preceding theorem that $G_1^0 = M$; this observation together with (iii) of the preceding theorem show that the interesting range for the more detailed investigation of the subvarieties G_r^ν is $2 \le r \le 2g-2$, which is also the interesting range for the Riemann-Roch Theorem. The restriction of the Abel-Jacobi diagram (12.16) of Theorem 12.2 to the subvarieties $G_r^\nu \subset \Gamma(M, \mathcal{D})$ can be used to introduce corresponding subvarieties of the complex tori $J(M)$ and $P_r(M)$.

Theorem 12.21 *For any compact Riemann surface M of genus $g > 0$ the set $(G_r^\nu/\sim)$ of linear equivalence classes of divisors in the holomorphic subvariety $G_r^\nu \subset M^{(r)}$ can be given the structure of a holomorphic variety so that for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and for any base point $z_0 \in \tilde{M}$ with image $a_0 = \pi(z_0) \in M$ there is a commutative diagram of holomorphic mappings*

(12.41)

$$\begin{tikzcd} G_r^\nu \arrow[r, "\psi"] & (G_r^\nu/\sim) \arrow[rr, "\hat{\zeta}"] \arrow[dr, "\hat{w}_{z_0}"'] & & \hat{W}_r^\nu \subset P_r(M) \arrow[dl, "\hat{\phi}_{a_0}"] \\ & & W_r^\nu \subset J(M) & \end{tikzcd} \qquad \textit{for } r > 0.$$

The mappings $\hat{\zeta}$, $\hat{w}_{z_0}$ and $\hat{\phi}_{a_0}$ are biholomorphic, while the mapping ψ is surjective and the inverse image of any point of $(G_r^\nu/\sim)$ under the mapping ψ is a holomorphic subvariety of G_r^ν that is biholomorphic to a complex projective space $\mathbb{P}^\nu$.

Proof: The restriction of the Abel-Jacobi diagram (12.16) of Theorem 12.2 to the subset $G_r^\nu \subset \Gamma(M, \mathcal{D})$ yields the commutative diagram of mappings (12.41) in which $\hat{W}_r^\nu = \hat{\zeta}(G_r^\nu/\sim) \subset P_r(M)$ and $W_r^\nu = \hat{w}_{z_0}(G_r^\nu/\sim) \subset J(M)$ for the mappings $\hat{\zeta}$ and $\hat{w}_{z_0}$ as in (12.15). Since $w_{z_0} = \hat{w}_{z_0} \circ \psi$ the subset $W_r^\nu \subset J(M)$ can be described alternatively as the image $W_r^\nu = w_{z_0}(G_r^\nu)$ of the holomorphic variety G_r^ν under the proper holomorphic mapping $w_{z_0} : M^{(r)} \longrightarrow J(M)$; so by Remmert's Proper Mapping Theorem W_r^ν is a holomorphic subvariety of the Jacobi variety $J(M)$. All the divisors in G_r^ν are of degree r so it follows from Abel's Theorem, Corollary 5.10, that two divisors in G_r^ν have the same image under the mapping w_{z_0} if and only if they are linearly equivalent; consequently the mapping $\hat{w}_{z_0}$ is a one-to-one mapping. This mapping then can be used to identify the set $(G_r^\nu/\sim)$ with the holomorphic variety W_r^ν, and thereby to determine the structure of a holomorphic variety on $(G_r^\nu/\sim)$ for which the mapping

$\hat{w}_{z_0}$ is biholomorphic. The restriction of the mapping $\hat{\phi}_{a_0}$ in the Abel-Jacobi diagram (12.16) to the subset $P_r(M) \subset H^1(M, \mathcal{O}^*)$ is a biholomorphic mapping between the complex tori $P_r(M)$ and $J(M)$ by Theorem 12.2; and through this biholomorphic mapping the set $\hat{W}_r^\nu = \hat{\phi}_{a_0}^{-1}(W_r^\nu)$ receives the structure of a holomorphic subvariety of $P_r(M)$ that is biholomorphic to the variety W_r^ν. It then follows from the commutativity of the diagram (12.41) that the mapping $\hat{\zeta}$ also is a biholomorphic mapping. The inverse image under $\tilde{w}_{z_0} \circ \psi$ of any point of $J(M)$ is a complex submanifold of $M^{(r)}$ that is biholomorphic to the complex projective space $\mathbb{P}^\nu$ by Corollary 12.16, and it is contained in G_r^ν by definition of that subset; and since $\tilde{w}_{z_0}$ is a biholomorphic mapping it follows that the inverse image under ψ of any point of $(G_r^\nu/\sim)$ is a complex submanifold of G_r^ν that is biholomorphic to the complex projective space $\mathbb{P}^\nu$, which suffices to conclude the proof.

The commutative diagram (12.41) is called the *restricted Abel-Jacobi diagram.* Since the variety $\hat{W}_r^\nu \subset P_r(M)$ can be characterized as the set of line bundles $\zeta_{\mathfrak{d}}$ of the divisors $\mathfrak{d} \in G_r^\nu \subset M^{(r)}$ it is evident from the definition (12.38) that it can be described alternatively as

$$\hat{W}_r^\nu = \Big\{ \lambda \in P_r(M) \;\Big|\; \gamma(\lambda) - 1 \geq \nu \Big\} \subset P_r(M). \tag{12.42}$$

This holds initially for all ν but only for $r > 0$, for which the subvarieties G_r^ν are defined; but (12.42) can be used to define subsets $\hat{W}_r^\nu \subset P_r(M)$ for arbitrary integers ν and r, and then through the biholomorphic mapping $\hat{\phi}_{a_0}$ these in turn determine subsets $W_r^\nu \subset J(M)$ for arbitrary ν and r. For the extended range $r \leq 0$ the sets so defined are also holomorphic subvarieties, but the rather simple ones

$$\hat{W}_r^\nu = \begin{cases} P_r(M) & \text{for } r \leq 0 \text{ and } \nu < 0, \\ \emptyset & \text{for } r < 0 \text{ and } \nu \geq 0, \\ \emptyset & \text{for } r = 0 \text{ and } \nu > 0, \\ 1 & \text{for } r = 0 \text{ and } \nu = 0, \end{cases} \tag{12.43}$$

where $1 \in P_r(M)$ is the trivial line bundle, and

$$W_r^\nu = \begin{cases} J(M) & \text{for } r \leq 0 \text{ and } \nu < 0, \\ \emptyset & \text{for } r < 0 \text{ and } \nu \geq 0, \\ \emptyset & \text{for } r = 0 \text{ and } \nu > 0, \\ 0 & \text{for } r = 0 \text{ and } \nu = 0, \end{cases} \tag{12.44}$$

where $0 \in J(M)$ is the identity element of the Jacobi group $J(M)$. For many purposes it is more convenient to focus on the subvarieties $W_r^\nu \subset J(M)$ since all of them are contained in the same complex manifold $J(M)$, even though there is no direct characterization of them comparable to the characterization of the subvarieties $\hat{W}_r^\nu \subset P_r(M)$ in (12.42). For convenience a list of some useful general properties of these subvarieties is included here.

Theorem 12.22 *If M is a compact Riemann surface of genus $g > 0$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and any base point $z_0 \in \tilde{M}$ with image $a_0 = \pi(z_0) \in M$ the holomorphic subvarieties $W_r^\nu \subset J(M)$ of the Jacobi group $J(M)$ satisfy*

(i) $W_r^0 = W_r$ *for* $r > 0$,
(ii) $W_r^\nu \subset W_{r+1}^\nu$ *for all* r, ν,
(iii) $W_r^{\nu+1} \subset W_r^\nu$ *for all* r, ν,
(iv) $W_r^\nu = J(M)$ *for* $\nu \le \max(-1, r-g)$ *and all* r,
(v) $W_r^\nu = \left\{ \begin{array}{lll} J(M) & \textit{for} & \nu \le r-g \\ \emptyset & \textit{for} & \nu > r-g \end{array} \right\}$ *if* $r \ge 2g-1$.

Proof: From the restricted Abel-Jacobi diagram (12.41) it follows that $W_r^0 = \tilde{w}_{z_0}\big(\psi(G_r^0)\big) = w_{z_0}(G_r^0)$ for $r > 0$; and $G_r^0 = M^{(r)}$ by Theorem 12.20 (ii) so $W_r^0 = w_{z_0}(M^{(r)}) = W_r$, as the latter set was defined in Theorem 12.8, thus demonstrating (i). If $t \in W_r^\nu$ then $t = \hat{\phi}_{a_0}(\lambda)$ for some line bundle $\lambda \in \hat{W}_r^\nu$, as in the restricted Abel-Jacobi diagram (12.41), and $c(\lambda) = r$ while $\gamma(\lambda) - 1 \ge \nu$; then $c(\lambda\zeta_{a_0}) = r+1$ and $\gamma(\lambda\zeta_{a_0}) - 1 \ge \gamma(\lambda) - 1 \ge \nu$ by Lemma 2.6 so $\lambda\zeta_{a_0} \in \hat{W}_{r+1}^\nu$, and from the commutativity of the restricted Abel-Jacobi diagram (12.41) it follows that $\hat{\phi}_{a_0}(\zeta_{a_0}) = \hat{\phi}_{a_0}\big(\tilde{\zeta}(a_0)\big) = \tilde{w}_{z_0}(a_0) = 0 \in J(M)$ so $t = \hat{\phi}_{a_0}(\lambda\zeta_{a_0}) \in \hat{\phi}_{a_0}(\hat{W}_{r+1}^\nu) = W_{r+1}^\nu$, which demonstrates (ii). The defining conditions (12.42) clearly are more restrictive as ν increases, and that yields (iii). Of course $\gamma(\lambda) - 1 \ge -1$ for any line bundle λ, and it follows from the Riemann-Roch Theorem that $\gamma(\lambda) - 1 = \gamma(\kappa\lambda^{-1}) + r - g \ge r - g$ if $c(\lambda) = r$; thus $\gamma(\lambda) \ge \max(-1, r-g)$ for any line bundle $\lambda \in P_r(M)$, and that yields (iv). Finally if $r \ge 2g-1$ then $\gamma(\lambda) > 0$ so $\lambda = \zeta_{\mathfrak{d}}$ for a positive divisor $\mathfrak{d}$ and (v) then follows from Theorem 12.20 (iii), which concludes the proof.

It is perhaps worth pointing out, to avoid its being overlooked, that the result of part (iv) of the preceding theorem differs slightly from the corresponding result in part (ii) of Theorem 12.20, reflecting the facts that $\gamma(\zeta_{\mathfrak{d}}) \ge 1$ for all positive divisors $\mathfrak{d}$ while $\gamma(\lambda) = 0$ for line bundles λ that are not the line bundles of positive divisors. The point of including the rather trivial observation (i) is just to indicate explicitly that the subvarieties $W_r \subset J(M)$ introduced in Theorem 12.8 are included among the more extended class of subvarieties W_r^ν. In addition to the subvarieties $W_r^\nu \subset J(M)$ it is useful to consider explicitly their negatives

$$-W_r^\nu = \Big\{ -t \in J(M) \,\Big|\, t \in W_r^\nu \Big\}. \tag{12.45}$$

The holomorphic subvariety $-W_r^\nu \subset J(M)$ is biholomorphic to the holomorphic subvariety $W_r^\nu \subset J(M)$ under the biholomorphic mapping of the Jacobi group to itself that takes a point $t \in J(M)$ to its inverse $-t \in J(M)$. The negative varieties $-W_r^\nu$ can be identified with translates of positive varieties W_s^μ through a relation that is equivalent to the Riemann-Roch Theorem.

Theorem 12.23 (Riemann-Roch Theorem) *If M is a compact Riemann surface M of genus $g > 0$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and any base point $z_0 \in \tilde{M}$*

$$k - W^\nu_r = W^{\nu+g-1-r}_{2g-2-r} \ \text{ for all } r,\, \nu, \tag{12.46}$$

where $k = \hat{\phi}_{a_0}(\kappa) \in J(M)$ is the image of the canonical bundle κ in the Jacobi group.

Proof: If $\lambda \in \hat{W}^\nu_r$, so that $c(\lambda) = r$ and $\gamma(\lambda) - 1 \geq \nu$, then $c(\kappa\lambda^{-1}) = 2g-2-r$ and it follows from the Riemann-Roch Theorem that

$$\gamma(\kappa\lambda^{-1}) - 1 = \gamma(\lambda) - 1 + g - 1 - c(\lambda) \geq \nu + g - 1 - r$$

so that $\kappa\lambda^{-1} \in \hat{W}^{\nu+g-1-r}_{2g-2-r}$; and the same argument shows conversely that any line bundle $\lambda \in \hat{W}^{\nu+g-1-r}_{2g-2-r}$ can be written as the product $\kappa\lambda^{-1}$ for some line bundle $\lambda \in \hat{W}^\nu_r$. These observations are equivalent to (12.46) through the isomorphism $\hat{\phi}_{a_0}$ in the restricted Abel-Jacobi diagram (12.41), and that concludes the proof.

The simplest special cases of the preceding theorem are the identities

$$k - W^\nu_{g-1} = W^\nu_{g-1} \ \text{ for all } \nu, \tag{12.47}$$

where again $k = \hat{\phi}_{a_0}(\kappa) \in J(M)$ is the image of the canonical bundle κ in the Jacobi group. When expressed in terms of line bundles through the restricted Abel-Jacobi diagram (12.41) this is merely the observation that if $\lambda \in P_{g-1}(M)$ and $\gamma(\lambda) - 1 \geq \nu$ then $\gamma(\kappa\lambda^{-1}) - 1 \geq \nu$ as well, an immediate consequence of the Riemann-Roch Theorem. Another set of translation relations provides an extension of the result of Theorem 12.11 expressed in terms of the Martens difference operation (12.26).

Theorem 12.24 *If M is a compact Riemann surface M of genus $g > 0$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and any base point $z_0 \in \tilde{M}$ the holomorphic subvarieties $W_r \subset J(M)$ of the Jacobi group satisfy*

$$W_r \ominus (-W_s) = \begin{cases} J(M) & \text{if } r,\, s > 0 \text{ and } r \geq g, \\ \\ W^s_{r+s} & \text{if } r,\, s > 0 \text{ and } r < g. \end{cases} \tag{12.48}$$

Proof: Of course if $r \geq g$ then $W_r = J(M)$ and $t - W_s \subset J(M) = W_r$ for any $t \in J(M)$, so that $W_r \ominus (-W_s) = J(M)$. Any $t \in W_r \ominus (-W_s) \subset J(M)$ can be written as the image $t = \hat{\phi}_{a_0}(\lambda)$ of a holomorphic line bundle $\lambda \in P_{r+s}$, since the group homomorphism $\hat{\phi}_{a_0}$ in the Abel-Jacobi diagram (12.16) is surjective and its kernel contains line bundles of any characteristic class. The condition that $t \in W_r \ominus (-W_s)$, or equivalently that $t - W_s \subset W_r$, when expressed in terms of the associated line bundle λ is the condition that $\lambda\sigma^{-1} \in \hat{W}_r = \hat{W}^0_r$ for any line bundle $\sigma \in \hat{W}_s$; and if $r < g$ that is just the condition that $\gamma(\lambda\sigma^{-1}) - 1 \ge 0$ for any line bundle $\sigma \in \hat{W}_s$, or equivalently that $\gamma(\lambda\zeta_{a_1}^{-1}\cdots\zeta_{a_s}^{-1}) - 1 \ge 0$ for any points $a_i \in M$, since the line bundles $\sigma \in W_s$ can be written as products $\sigma = \zeta_{a_1}\cdots\zeta_{a_s}$ for points $a_i \in M$. Of course then $\gamma(\lambda) > 0$; if $\gamma(\lambda) = n > 0$ and $f_1, \ldots, f_n \in \Gamma(M, \mathcal{O}(\lambda))$ is a basis for the holomorphic cross-sections of the bundle λ, choose s points $a_i \in M$ such that $f_i(a_i) \ne 0$ for $1 \le i \le \min(s, n)$. If $n \le s$ then none of the cross-sections $f_1, \ldots, f_n$ can vanish at all of the points a_i and hence $\gamma(\lambda\zeta_{a_1}^{-1}\cdots\zeta_{a_s}^{-1}) = 0$, a contradiction; therefore $n > s$ so $\gamma(\lambda) > s$ or equivalently $\gamma(\lambda) - 1 \ge s$, hence $\lambda \in \hat{W}^s_{r+s}$ and $t \in W^s_{r+s}$, showing that $W_r \ominus (-W_s) \subset W^s_{r+s}$. Conversely if $t \in W^s_{r+s}$ and $t = \hat{\phi}_{a_0}(\lambda)$ for a line bundle $\lambda \in \hat{W}^s_{r+s}$ then $\gamma(\lambda) - 1 \ge s$. For any line bundle $\sigma = \zeta_{a_1}\cdots\zeta_{a_s} \in P_s(M)$ it follows from Lemma 2.6 that $\gamma(\lambda\sigma^{-1}) = \gamma(\lambda\zeta_{a_1}^{-1}\cdots\zeta_{a_s}^{-1}) \ge \gamma(\lambda) - s > 0$, hence $\lambda\sigma^{-1} \in \hat{W}^0_r = W_r$ so $t - \hat{\phi}_{a_0}(\sigma) \in W_r$; since the points $\hat{\phi}_{a_0}(\sigma) \subset J(M)$ form the subvariety $W_s \subset J(M)$ it follows that $t - W_s \subset W_r$ so $t \in W_r \ominus (-W_s)$ and consequently $W^s_{r+s} \subset W_r \ominus (-W_s)$. Altogether then $W^s_{r+s} = W_r \ominus (-W_s)$, which concludes the proof.

Corollary 12.25 *If M is a compact Riemann surface of genus $g > 0$ then*

$$G^\nu_r = \hat{W}^\nu_r = W^\nu_r = \emptyset \quad \begin{cases} \text{if } \nu \ge r & \text{for any } r > 0, \text{ or} \\ \text{if } \nu > \frac{r}{2} & \text{for } 0 < r \le 2g-2. \end{cases} \tag{12.49}$$

Proof: Since $g > 0$ it follows from Theorem 2.7 that $\gamma(\lambda) - 1 < c(\lambda)$ for every holomorphic line bundle λ over M with $c(\lambda) > 0$, so $\hat{W}^\nu_r = \emptyset$ whenever $\nu \ge r > 0$. Since the varieties $\hat{W}^\nu_r$ and W^ν_r are biholomorphic, as in the restricted Abel-Jacobi diagram (12.41), it is also the case that $W^\nu_r = \emptyset$ whenever $\nu \ge r > 0$; and since the general Abel-Jacobi mapping $w_{z_0} : G^\nu_r \longrightarrow W^\nu_r$ in the restricted Abel-Jacobi diagram (12.41) is surjective it follows that $G^\nu_r = \emptyset$ also whenever $\nu > r > 0$. Therefore $G^\nu_r = \hat{W}^\nu_r = W^\nu_r = \emptyset$ whenever $\nu \ge r > 0$. If $r < g$ and $r > \nu > \frac{r}{2} > 0$ then $\nu > r - \nu > 0$ and it follows from Theorem 12.8 (iii) that $\dim W_\nu = \nu > r - \nu = \dim W_{r-\nu}$; consequently there are no points $t \in J(M)$ for which $t - W_\nu \subset W_{r-\nu}$, so by the preceding theorem $W^\nu_r = W_{r-\nu} \ominus (-W_\nu) = \emptyset$, and of course the same is true for the varieties $\hat{W}^\nu_r$ and G^ν_r as before. If $g \le r \le 2g-2$ and $r > \nu > \frac{r}{2} > 0$ it follows from Theorem 12.23 that the varieties W^ν_r and W^μ_s are biholomorphic, where $\mu = \nu + g - 1 - r$ and $s = 2g - 2 - r$. Since $r \ge g$ it follows that $s < g$, since $r < 2g - 2$ it follows that $s > 0$, and since $2\nu > r$ it follows that $2\mu - s = 2\nu - r > 0$; thus altogether $s < g$ and $\mu > \frac{s}{2} > 0$. If $s \ge \mu$ the first condition in (12.49) is satisfied and consequently $W^\mu_s = G^\nu_r = \hat{W}^\nu_r = W^\nu_r = \emptyset$; and if $s < \mu$ it follows from what has already been demonstrated that $W^\mu_s = G^\nu_r = \hat{W}^\nu_r = W^\nu_r = \emptyset$. Finally if λ is a holomorphic line bundle for which $c(\lambda) = 2g - 2$ then it follows from the Riemann-Roch Theorem that $\gamma(\lambda) = \gamma(\kappa\lambda^{-1}) + c(\lambda) + 1 - g \le g$, since $c(\kappa\lambda^{-1}) = 0$ so $\gamma(\kappa\lambda^{-1}) \le 1$; consequently $W^\nu_{2g-2} = \emptyset$ whenever $\nu > (g-1)$, and correspondingly of course for W^ν_{2g-2} and $\hat{W}^\nu_{2g-2}$, which suffices to conclude the proof.

The stronger result in (12.49) holds just for indices r in the indicated range, the usual interesting range $0 < r \le 2g-1$; for if $c(\lambda) > 2g-2$ then $\gamma(\lambda) = c(\lambda) + 1 - g$ by the Riemann-Roch Theorem so

$$1 \ge \frac{\gamma(\lambda)}{c(\lambda)} = 1 - \frac{g-1}{c(\lambda)} > \frac{1}{2} \quad \text{if } c(\lambda) > 2g-2. \tag{12.50}$$

The results in (12.49) are the best possible for all Riemann surfaces, as will be demonstrated in the discussion of the Brill-Noether sequence in Chapter 11. Some further useful properties of the subvarieties of special positive divisors follow from an extension of Theorem 12.24, which in turn follows from an application of the following simple lemma.

Lemma 12.26 *For any subsets $A, B, C \subset J(M)$ of a complex torus $J(M)$*

$$(A \ominus B) \ominus C = A \ominus (B + C). \tag{12.51}$$

Proof: It is evident from the definition (12.26) of the Martens difference operator that $t \in (A \ominus B) \ominus C$ if and only if $t + C \subset (A \ominus B)$, that is if and only if $(t+C) + B \subset A$, which is just the condition that $t \in A \ominus (B+C)$. That suffices for the proof.

Theorem 12.27 *If M is a compact Riemann surface M of genus $g > 0$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and any base point $z_0 \in \tilde{M}$ with image $a_0 = \pi(z_0) \in M$ the holomorphic subvarieties $W^{\nu}_r \subset J(M)$ of the Jacobi group satisfy*

(i) $W^{\nu}_r \ominus W_s = W^{\nu}_{r-s}$ *if $r > \nu > 0$ and $g > r - \nu > s > 0$,*

(ii) $W^{\nu}_r \ominus (-W_s) = W^{\nu+s}_{r+s}$ *if $r > \nu > 0$, $s > 0$ and $r - \nu < g$.*

Proof: If $r > \nu > 0$ and $g > r - \nu > s > 0$ it follows from Theorem 12.24 that $W^{\nu}_r = W_{r-\nu} \ominus (-W_\nu)$ and further that

$$\begin{aligned}
W^{\nu}_r \ominus W_s &= \big(W_{r-\nu} \ominus (-W_\nu)\big) \ominus W_s \\
&= W_{r-\nu} \ominus \big((-W_\nu) + W_s\big) \quad \text{by Lemma 12.26} \\
&= W_{r-\nu} \ominus \big(W_s + (-W_\nu)\big) \quad \text{by rearranging the sum} \\
&= (W_{r-\nu} \ominus W_s) \ominus (-W_\nu) \quad \text{by Lemma 12.26} \\
&= W_{r-\nu-s} \ominus (-W_\nu) \quad \text{by Theorem 12.11 (iv)} \\
&= W^{\nu}_{r-s} \quad \text{by Theorem 12.24,}
\end{aligned}$$

which demonstrates (i). If $r > \nu > 0$, $s > 0$ and $r - \nu < g$ then

$$\begin{aligned}
W^{\nu}_r \ominus (-W_s) &= \big(W_{r-\nu} \ominus (-W_\nu)\big) \ominus (-W_s) \\
&= W_{r-\nu} \ominus \big((-W_\nu) + (-W_s)\big) \quad \text{by Lemma 12.26} \\
&= W_{r-\nu} \ominus (-W_{\nu+s}) \quad \text{by Theorem 12.11 (ii)} \\
&= W^{\nu+s}_{r+s} \quad \text{by Theorem 12.24,}
\end{aligned}$$

which demonstrates (ii) and concludes the proof.

There is another approach to the Abel-Jacobi mapping itself that is relevant to the discussion here. The Abel-Jacobi mapping $w_{z_0} : M \longrightarrow J(M)$ is defined as the mapping induced by the holomorphic mapping $\tilde{w}_{z_0} : \tilde{M} \longrightarrow \mathbb{C}^g$ that associates to a point $z \in \tilde{M}$ the point $\tilde{w}_{z_0}(z) = \{w_i(z, z_0)\} \in \mathbb{C}^g$, where $w_i(z, z_0) = \int_{z_0}^{z} \omega_i$ for a basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$; and as such it depends on the choice of the base point $z_0 \in \tilde{M}$. On the other hand there is also the mapping

$$w_2 : M \times M \longrightarrow J(M) \tag{12.52}$$

induced by the holomorphic mapping

$$\tilde{w}_2 : \tilde{M} \times \tilde{M} \longrightarrow \mathbb{C}^g \tag{12.53}$$

that associates to a point $(z_1, z_2) \in \tilde{M} \times \tilde{M}$ the point

$$\tilde{w}_2(z_1, z_2) = \{w_i(z_1, z_2)\} \in \mathbb{C}^g. \tag{12.54}$$

That the mapping (12.53) induces a mapping (12.52) follows just as for the ordinary Abel-Jacobi mapping; alternatively since $w_2(z_1, z_2) = w_{z_0}(z_1) - w_{z_0}(z_2)$ it follows that $w_2(a_1, a_2) = w_{z_0}(a_1) - w_{z_0}(a_2)$ for any points $a_1, a_2 \in M$ and any choice of a base point $z_0 \in \tilde{W}$. The mapping w_2 is more intrinsically defined than the general Abel-Jacobi mapping w_{z_0}, since the definition of w_2 does not depend on the choice of a base point $z_0 \in \tilde{M}$; and the image of the mapping w_2 is the intrinsically defined subset

$$W_1 - W_1 = W_1 + (-W_1) = \left\{ t_1 - t_2 \,\middle|\, t_1,\, t_2 \in W_1 \right\} \subset J(M), \tag{12.55}$$

which as the image of the proper holomorphic mapping (12.52) is a holomorphic subvariety of the Jacobi group $J(M)$ by Remmert's Proper Mapping Theorem.

Theorem 12.28 *If M is a compact Riemann surface of genus $g > 2$ and M has no special positive divisors of degree 2, the holomorphic mapping w_2 takes the diagonal subvariety*

$$\Delta = \left\{ (z, z) \,\middle|\, z \in M \right\} \subset M \times M \tag{12.56}$$

to the identity $0 \in J(M)$ and restricts to a biholomorphic mapping

$$w_2 : (M \times M) \sim \Delta \longrightarrow (W_1 - W_1) \sim 0. \tag{12.57}$$

Proof: It is clear from the definition that $\tilde{w}_2(z, z) = \{w_i(z,\, z)\} = 0 \in \mathbb{C}^g$ for any point $z \in \tilde{M}$ and consequently that $w_2(\Delta) = 0$. If $w_2(a_1, a_2) = w_2(b_1, b_2)$ for two points $(a_1, a_2),\ (b_1, b_2) \in M \times M$ then $w_{z_0}(a_1) - w_{z_0}(a_2) = w_{z_0}(b_1) - w_{z_0}(b_2)$ and consequently

$$w_{z_0}(a_1 + b_2) = w_{z_0}(b_1 + a_2);$$

and since there are no special divisors of degree 2 on the Riemann surface M it follows from Corollary 12.9 (i) that $a_1 + b_2 = b_1 + a_2$. Thus either $a_1 = a_2$ and $b_1 = b_2$, so the two points points (a_1, a_2), (b_1, b_2) lie on the diagonal Δ, or $a_1 = b_1$ and $a_2 = b_2$, so $(a_1, a_2) = (b_1, b_2)$; and therefore the restriction of the mapping w_2 to the complement of the diagonal is a one-to-one mapping onto $(W_1 - W_1) \sim 0$ and $w_2^{-1}(0) = \Delta$. If $a_1 \neq a_2$ and $z_{\alpha 1}$ and $z_{\alpha 2}$ are local coordinates centered at the points a_1 and a_2 respectively then $(z_{\alpha 1},\, z_{\alpha 2})$ is a local coordinate system in an open neighborhood of the point (a_1, a_2) in $M \times M$, in terms of which the mapping (12.57) has the local form

$$w_2 : (z_{\alpha 1},\, z_{\alpha 2}) \longrightarrow w(z_{\alpha 1},\, z_0) - w(z_{\alpha 2},\, z_0) \in \mathbb{C}^g;$$

the differential of this mapping then is the $2 \times g$ matrix

$$dw_2(z_{\alpha 1},\, z_{\alpha 2}) = \begin{pmatrix} \dfrac{\partial w(z_{\alpha 1},\, z_0)}{\partial z_{\alpha 1}} & -\dfrac{\partial w(z_{\alpha 2},\, z_0)}{\partial z_{\alpha 2}} \end{pmatrix},$$

and clearly rank $dw_2(z_{\alpha 1},\, z_{\alpha 2})$ = rank $\Omega(z_{\alpha 1} + z_{\alpha 2})$ in terms of the Brill-Noether matrix (11.10) of the divisor $z_{\alpha 1} + z_{\alpha 2}$. On the other hand

$$\text{rank } \Omega(z_{\alpha 1} + z_{\alpha 2}) = 3 - \gamma(\zeta_{z_{\alpha 1} + z_{\alpha 2}})$$

by the Riemann-Roch Theorem in the form of Theorem 11.3 while $\gamma(\zeta_{z_{\alpha 1} + z_{\alpha 2}}) = 1$ since the Riemann surface M has no special positive divisors of degree 2 by assumption, so rank $dw_2(z_{\alpha 1},\, z_{\alpha 2}) = 2$ and therefore the mapping w_2 is locally biholomorphic near the point $(a_1, a_2) \in M \times M$. That suffices for the proof.

Corollary 12.29 *If M is a compact Riemann surface M of genus $g > 0$ then for any bases $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and $\tau_j \in H_1(M)$ and any base point $z_0 \in \tilde{M}$ with image $a_0 = \pi(z_0) \in M$ the holomorphic subvarieties $W_r^\nu \subset J(M)$ of the Jacobi group satisfy*

$$W_r^\nu \ominus (W_1 - W_1) = W_r^{\nu+1} \quad \text{if } 0 < \nu < r \text{ and } 1 < r - \nu < g. \tag{12.58}$$

Proof: If $g > r - \nu > 1$ then

$$\begin{aligned} W_r^\nu \ominus (W_1 - W_1) &= (W_r^\nu \ominus W_1) \ominus (-W_1) \quad \text{by Lemma 12.26} \\ &= W_{r-1}^\nu \ominus (-W_1) \quad \text{by Theorem 12.27 (i)} \\ &= W_r^{\nu+1} \quad \text{by Theorem 12.27 (ii)}, \end{aligned}$$

and that suffices for the proof.

Corollary 12.30 *On a compact Riemann surface M of genus $g > 0$ the holomorphic subvariety $W_r^{\nu+1} \subset W_r^\nu$ is contained in the singular locus of the subvariety W_r^ν if $0 < \nu < r$ and $1 < r - \nu < g$.*

Proof: In an open neighborhood $U \subset J(M)$ of a point $t \in W_r^\nu$ the subvariety W_r^ν is the zero locus of a finite number of holomorphic functions $f_i \in \Gamma(U, \mathcal{O})$, which also generate the ideal of that subvariety at any point in U. It follows from the preceding corollary that if $t \in W_r^{\nu+1}$ then $t + W_1 - W_1 \subset W_r^\nu$, and consequently

$$f_i\big(t + \tilde{w}(z) - \tilde{w}(a)\big) = 0 \tag{12.59}$$

for any points $z, a \in M$ sufficiently close that $t + \tilde{w}(z) - \tilde{w}(a) \in U$. The differential of the function (12.59) with respect to the variable $z \in M$ therefore vanishes identically in z, so in particular for any point $a \in U$ and for $z = a$

$$0 = df_i\big(t + \tilde{w}(z) - \tilde{w}(a)\big)\Big|_{z=a} = \sum_{j=1}^{g} \partial_j f_i(t)\omega_j(a) \tag{12.60}$$

in terms of the holomorphic abelian differentials $\omega_j(a)$. These holomorphic abelian differentials are linearly independent, so (12.60) implies that $\partial_j f_i(t) = 0$ for all i, j; that implies that the point t lies in the singular locus of the subvariety W_r^ν defined by the functions f_i, which suffices for the proof.

Chapter 13

The General Cross-Ratio Function

13.1 The Product Cross-Ratio Function

PRELIMINARY VERSION Another approach to the Abel-Jacobi mapping is through an application of the cross-ratio function defined in (5.23). By Theorem 5.6 the intrinsic cross-ratio function for a compact Riemann surface M with the universal covering surface $\widetilde{M}$ is the meromorphic function $q(z, a; z^+, z^-)$ on the product $\widetilde{M}^4$ that as a function of the variable $z \in \widetilde{M}$ is meromorphic on $\widetilde{M}$, takes the value 1 at the point $z = a$, has simple zeros at the points Tz_+ and simple poles at the points Tz_- for all covering translations $T \in \Gamma$, and is a meromorphic relatively automorphic function for the flat factor of automorphy $\rho_{t(z^+,z^-)}$. In terms of a basis $\omega_i(z) \in \Gamma(M, \mathcal{O}^{((1,0)})$ and the generators $T_j \in \Gamma$ of the covering translation group of M this factor of automorphy has the explicit form

$$\rho_{z^+,z^-}(T) = \exp -2\pi \sum_{m,n=1}^{g} w_m(z^+, z^-) g_{mn} \overline{\omega_n(T)} \tag{13.1}$$

for any covering translation $T \in \Gamma$, where $w_m(z^+, z^-) = \int_{z^-}^{z^+} \omega_m$ are the integrals of the abelian differentials, $\omega_n(T)$ is the period of the abelian differential $\omega_n(z)$ on the covering translation T, P is the intersection matrix and $G = {}^t H^{-1}$ for the positive definite Hermitian matrix $H = i\Omega P\, {}^t\overline{\Omega}$ for the period matrix $\Omega = \{\omega_{ij}\} = \{\omega_i(T_j)\}$. For any two ordered sets of r points $A^+ = \{a_1^+, a_2^+, \ldots, a_r^+\}$ and $A^- = \{a_1^-, a_2^-, \ldots, a_r^-\}$ on $\widetilde{M}$ the *product cross-ratio function* of degree r is defined to be the meromorphic function

$$Q(z, a; A^+, A^-) = \prod_{\nu=1}^{r} q(z, a; a_\nu^+, a_\nu^-). \tag{13.2}$$

As a function of the variable $z \in \widetilde{M}$ this is a relatively automorphic function for the flat factor of automorphy

$$\rho_{A^+,A^-}(T) = \prod_{\nu=1}^{r} \rho_{a_\nu^+,a_\nu^-}(T) \tag{13.3}$$
$$= \exp -2\pi \sum_{m,n=1}^{g} \sum_{\nu=1}^{r} w_m(a_\nu^+, a_\nu^-) g_{mn} \overline{\omega_n(T)}.$$

The divisor of the relatively automorphic function $Q(z, a; A^+, A^-)$ of the variable $z \in \widetilde{M}$ is

$$\mathfrak{d}Q(z, a; A^+, A^-) = \mathfrak{d}^+ - \mathfrak{d}^- \ \text{ where } \mathfrak{d}^+ = \sum_{\nu=1}^{r} \pi(a_\nu^+) \ \ , \mathfrak{d}^+ = \sum_{\nu=1}^{r} \pi(a_\nu^-) \tag{13.4}$$

where $\pi : \tilde{M} \longrightarrow M$ is the universal covering projection.

Lemma 13.1 *There is a meromorphic function on the compact Riemann surface M with the divisor $\mathfrak{d}^+ - \mathfrak{d}^-$ if and only if there is a holomorphic abelian differential $\omega(z)$ on M with the period class $\omega(T)$ for which*

$$\rho_{A^+,A^-}(T) = \exp \omega(T); \tag{13.5}$$

if that condition is satisfied then for any integral $w(z) = \int_{z_0}^{z} \omega$

$$f(z) = Q(z, a; A^+, A^-) e^{-w(z)} \tag{13.6}$$

is the unique meromorphic function on M with the divisor $\mathfrak{d}^+ - \mathfrak{d}^-$, up to an arbitrary nonzero constant factor.

Proof: If there is a meromorphic function $f(z)$ on the Riemann surface M with the divisor $\mathfrak{d} = \mathfrak{d}^+ - \mathfrak{d}^-$ then the quotient $Q(z, a; A^+, A^-)/f(z)$ is a holomorphic and nowhere vanishing relatively automorphic function for the flat factor of automorphy ρ_{A^+,A^-}; consequently that factor of automorphy represents the trivial holomorphic line bundle, so as in Corollary 3.10 it has the form $\rho_{A^+,A^-}(T) = \exp \omega(T)$ where $\omega(T)$ is the period class of a holomorphic abelian differential $\omega(z)$. Conversely if $\rho_{A^+,A^-}(T) = \exp \omega(T)$ for the period class of the holomorphic abelian differential $\omega(z)$ and if $w(z) = \int_{z_0}^{z} \omega$ then $f(z) = Q(z, a; A^+, A^-) \exp -w(z)$ is a relatively automophic function for the factor of automorphy $\rho_{A^+,A^-}(T) \exp -\omega(T) = 1$ with the divisor $\mathfrak{d}^+ - \mathfrak{d}^-$; any meromorphic function on M with this divisor of course is a constant multiple of the function $f(z)$, and that suffices for the proof.

This simple lemma leads directly to the following rather more explicit formulation of Abel's Theorem than that given earlier in Corollary 5.10.

Theorem 13.2 *Let M be a compact Riemann surface of genus $g > 0$, let Ω be the period matrix of M in terms of a basis $\omega_i(z) \in \Gamma(M, \mathcal{O}^{((1,0)})$ and generators*

T_j of the covering translation group Γ, let $\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \mathbb{C}^g$ be the holomorphic mapping described by the integrals $w_i(z, z_0) = \int_{z_0}^{z} \omega_i$, and let $\pi : \widetilde{M} \longrightarrow M$ be the universal covering space of the surface M. Further let $\mathfrak{d}^+$ and $\mathfrak{d}^-$ be two positive divisors of degree r on M given by $\mathfrak{d}^+ = \sum_{\nu=1}^{r} \pi(a_\nu^+)$ and $\mathfrak{d}^- = \sum_{\nu=1}^{r} \pi(a_\nu^-)$ where $A^+ = \{a_1^+, a_2^+, \ldots, a_r^+\}$ and $A^- = \{a_1^-, a_2^-, \ldots, a_r^-\}$ are two ordered sets of points of $\widetilde{M}$.

(i) *The necessary and sufficient condition that there exists a meromorphic function on M with divisor $\mathfrak{d}^+ - \mathfrak{d}^-$ is that*

$$\sum_{j=1}^{r} \Big(\widetilde{w}_{z_0}(a_i^+) - \widetilde{w}_{z_0}(a_j^-) \Big) = \Omega\, n \quad \textit{for a vector } n \in \mathbb{Z}^{2g}. \tag{13.7}$$

(ii) *If the condition* (13.7) *is satisfied then the function*

$$f(z) = Q(z, a; A^+, A^-) \exp -2\pi i \sum_{k,l=1}^{g} \sum_{s=1}^{2g} w_k(z, z_0) g_{kl} \overline{\omega}_{ls} n_s, \tag{13.8}$$

where $G = {}^t H^{-1}$ for the matrix $H = i\Omega P {}^t\overline{\Omega}$ expressed in terms of the period matrix Ω and the intersection matrix P of the surface in terms of the given bases, is a meromorphic function on M with the divisor $\mathfrak{d} = \mathfrak{d}^+ - \mathfrak{d}^-$ and this function is unique up to a nonzero constant factor.

Proof: (i) The proof just amounts to interpreting the condition of the preceding Lemma 13.1. A holomorphic abelian differential $\omega(z)$ can be written as the sum $\omega(z) = \sum_{k=1}^{g} c_k \omega_k(z)$ in terms of the basis $\omega_k(z) \in \Gamma(M, \mathcal{O}^{(1,0)})$; and then $\exp \omega(T_j) = \exp \sum_{k=1}^{g} c_k \omega_{kj}$ in terms of the period matrix $\Omega = \{\omega_{kj}\}$ where $\omega_{kj} = \omega_k(\tau_j)$. To simplify the notation set $w_m(A^+) = \sum_{\nu=1}^{r} w_m(a_\nu^+, z_0)$, $w_m(A^-) = \sum_{\nu=1}^{r} w_m(a_\nu^+, z_0)$ and $w_m(A) = w_m(A^+) - w_m(A^-)$. By (13.3)

$$\rho_{A^+,A^-}(T_j) = \exp -2\pi \sum_{m,n=1}^{g} w_m(A) g_{mn} \overline{\omega}_{nj}. \tag{13.9}$$

Condition (13.5) that $\rho_{A^+,A^-}(T_j) = \exp \omega(T_j) = \exp \sum_{k=1}^{g} c_k \omega_{kj}$ can be written as the condition that

$$-2\pi \sum_{m,n=1}^{g} w_m(A) g_{mn} \overline{\omega}_{nj} = 2\pi i N_j + \sum_{k=1}^{g} c_k \omega_{kj} \tag{13.10}$$

for some integers N_j. The preceding equation can be viewed as a system of linear equations $v_j = 2\pi i N_j + \sum_{k=1}^{g} c_k \omega_{kj}$ in the unknowns N_j and c_k, where v_j denotes the left-hand side of (13.10); and in terms of the column vectors $v = \{v_j\}$, $N = \{N_j\} \in \mathbb{C}^{2g}$ and $c = \{c_k\} \in \mathbb{C}^g$ this equation can be written

$$v - 2\pi i N = {}^t\Omega c. \tag{13.11}$$

The inverse period matrix to Ω as defined in Theorem F.12 in Appendix F.1 is the $g \times 2g$ complex matrix Π for which

$$\Pi\,{}^t\Omega = 0, \quad \Pi\,{}^t\overline{\Omega} = I, \quad {}^t\Omega\overline{\Pi} + {}^t\overline{\Omega}\Pi = I. \tag{13.12}$$

In view of these properties of the inverse period matrix, if for some N there is a solution c to the linear equation (13.11) then $\Pi(v - 2\pi iN) = \Pi\,{}^t\Omega c = 0$; and conversely if $\Pi(v - 2\pi iN) = 0$ then

$$v - 2\pi iN = ({}^t\Omega\overline{\Pi} + {}^t\overline{\Omega}\Pi)(v - 2\pi iN) = {}^t\Omega\overline{\Pi}(v - 2\pi iN) = {}^t\Omega c.$$

Thus there is a solution c to (13.11) if and only if

$$\Pi(v - 2\pi iN) = 0, \tag{13.13}$$

and a solution is given explicitly by

$$c = \overline{\Pi}(v - 2\pi iN). \tag{13.14}$$

Condition (13.13) is just that $\sum_{k=1}^{g} \Pi_{kj}(v_j - 2\pi iN_j) = 0$ or, upon replacing v_j by its explicit value and using (13.10), that

$$-2\pi \sum_{m,n=1}^{g} \sum_{j=1}^{2g} w_m(A) g_{mn} \overline{\omega}_{nj} \Pi_{kj} - 2\pi i \sum_{j=1}^{2g} N_j \Pi_{kj} = 0 \tag{13.15}$$

for $1 \le k \le g$. If this condition is satisfied then a solution c_k is given by $c_k = \sum_{j=1}^{2g} \overline{\Pi}_{kj}(v_j - 2\pi iN_j)$ as in (13.14); upon replacing v_j by its explicit value and using (13.10) again this solution takes the form

$$c_k = -2\pi \sum_{m,n=1}^{g} \sum_{j=1}^{2g} w_m(A) g_{mn} \overline{\omega}_{nj} \overline{\Pi}_{kj} - 2\pi i \sum_{l=1}^{2g} N_j \overline{\Pi}_{kj} \tag{13.16}$$

for $1 \le k \le g$. However $\sum_{j=1}^{2g} \overline{\omega}_{nj} \Pi_{kj} = \delta_k^n$ by (13.12) so equation (13.15) reduces to

$$\sum_{m=1}^{g} w_m(A) g_{mk} = -i \sum_{j=1}^{2g} N_j \Pi_{kj} \tag{13.17}$$

for $1 \le k \le g$; and $\sum_{k=1}^{g} g_{mk} h_{rk} = \delta_r^m$ so multiplying the preceding equation by h_{rk} and adding the result for $1 \le k \le g$ yields the equation

$$w_r(A) = -i \sum_{j=1}^{2g} \sum_{k=1}^{g} N_j \Pi_{kj} h_{rk} \tag{13.18}$$

for $1 \le r \le g$. In this equation though $H\Pi = i\Omega P\,{}^t\overline{\Omega}\Pi = i\Omega P(I - {}^t\Omega\overline{\Pi}) = i\Omega P$ by (13.12) and Riemann's equality $\Omega P\,{}^t\Omega = 0$, so (13.18) can be rewritten

$$w_r(A) = \sum_{s,j=1}^{2g} \Omega_{rs} P_{sj} N_j \tag{13.19}$$

for $1 \le r \le g$. The intersection matrix P is an integral matrix of determinant $\det P = 1$, so as the entries N_j vary over all integral values in $\mathbb{Z}^{2g}$ so do the entries $n_s = \sum_{j=1}^{2g} P_{sj} N_j$; consequently the preceding equation (13.19) is just the assertion that

$$w_r(A) = w_r(A^+) - w_r(A^-) = \sum_{s=1}^{2g} \Omega_{rs} n_s \tag{13.20}$$

for some integers n_s, which is (13.7).
(ii) If the condition (i) is satisfied then as in the discussion preceding the statement of the theorem the function $f(z)$ of (13.6) is a meromorphic function with the divisor $\mathfrak{d}^+ - \mathfrak{d}^-$ where $w(z) = \sum_{k=1}^{g} c_k w_k(z)$ for the constants c_k of (13.16). Since $\overline{\Omega}\,{}^t\overline{\Pi} = 0$ by (13.12) equation (13.16 reduces to the simpler form

$$c_k = -2\pi i \sum_{l=1}^{2g} N_l \overline{\Pi}_{kl}. \tag{13.21}$$

Since condition (13.20) is expressed in terms of the constants $n_s = \sum_{l=1}^{2g} P_{sl} N_l$ it is natural to use those same constants in the expression for the function $f(z)$ hence to rewrite the preceding equation as

$$c_k = -2\pi i \sum_{l,s=1}^{2g} n_s P^{-1}_{ls} \overline{\Pi}_{kl}. \tag{13.22}$$

Equation (F.35) in Appendix F shows that $\overline{G} = i\Pi\,{}^tP^{-1}\,{}^t\overline{\Pi}$ where $G = {}^tH^{-1}$; hence

$${}^t\overline{G}\Omega = i\overline{\Pi}P^{-1}\,{}^t\Pi\overline{\Omega} = i\overline{\Pi}P^{-1}(I - {}^t\overline{\Pi}\Omega) = i\overline{\Pi}P^{-1}$$

from (13.12) and (F.35); substituting this into (13.22) shows that

$$c_k = -2\pi \sum_{k=1}^{g} \sum_{s=1}^{2g} g_{kl} \overline{\omega}_{ks} n_s, \tag{13.23}$$

and substituting these values of the coefficients c_k in the formula for the integral $w(z)$ yields (13.8) to conclude the proof.

To examine the product cross-ratio function further it is necessary to consider in somewhat more detail the complex manifolds $\widetilde{M}^{(r)}$ and various quotients of these manifolds. Since the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ is a normal subgroup the universal covering projection $\widetilde{\pi} : \widetilde{M} \longrightarrow M = \widetilde{M}/\Gamma$ can be decomposed as the composition $\widetilde{\pi} = \widehat{\pi} \circ \pi_a$ of the two mappings in the chain of covering projections

$$\widetilde{M} \xrightarrow{\pi_a} \widehat{M} = \widetilde{M}/[\Gamma, \Gamma] \xrightarrow{\widehat{\pi}} M = \widehat{M}/\Gamma_a \tag{13.24}$$

where $\Gamma_a = \Gamma/[\Gamma, \Gamma]$ is the abelianization of the group Γ. The group Γ can be generated by $2g$ generators with the single relation (D.4), as discussed in

Appendix D.1; consequently its abelianization Γ_a is a free abelian group on $2g$ generators. The subgroup $[\Gamma, \Gamma] \subset \Gamma$ is not of finite index, since the quotient $\Gamma_a = \Gamma/[\Gamma, \Gamma]$ is an infinite group, so $\widehat{M}$ is not a compact Riemann surface. The surface $\widehat{M}$ is not simply connected; indeed its fundamental group is isomorphic to $[\Gamma, \Gamma]$. The fundamental group of any noncompact connected surface is a free group[1]; so the group $[\Gamma, \Gamma]$ actually is a free group, a result which though interesting will not be used here. The Riemann surface $\widetilde{M}$ can be identified with the unit disc, through the general uniformization theorem. The Riemann surface $\widehat{M}$ however appears to be an example of a non-continuable[2] Riemann surface, a noncompact Riemann surface that cannot be realized as a proper subset of another Riemann surface; but that topic will not be pursued further here.

The holomorphic abelian differentials on M are represented by Γ-invariant holomorphic differential 1-forms ω_i on $\widetilde{M}$, and their integrals $w_i(z, z_0) = \int_{z_0}^{z} \omega_i$ are holomorphic functions on $\widetilde{M}$ such that

$$w_i(Tz, z_0) = w_i(z, z_0) + \omega_i(T) \quad \text{for all} \quad T \in \Gamma. \tag{13.25}$$

The set of period vectors $\omega(T) = \{\omega_i(T)\} \in \mathbb{C}^g$ for all $T \in \Gamma$ form the lattice subgroup $\mathcal{L}(\Omega) \subset \mathbb{C}^g$; and the set of integrals $w_i(z, z_0)$ describe a holomorphic mapping

$$\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \mathbb{C}^g \quad \text{where} \quad \widetilde{w}_{z_0}(z) = \{w_i(z, z_0)\} \in \mathbb{C}^g. \tag{13.26}$$

It follows from (13.25) that the mapping (13.26) commutes with the covering projections $\widetilde{\pi} : \widetilde{M} \longrightarrow M$ and $\pi : \mathbb{C}^g \longrightarrow J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$, so it induces the Abel-Jacobi mapping $w_{z_0} : M \longrightarrow J(M)$ as in the commutative diagram (3.4). Recall from the earlier discussion that the Abel-Jacobi mapping is a nonsingular biholomorphic mapping from the Riemann surface M to its image $W_1 = w_{z_0}(M) \subset J(M)$, which is an irreducible holomorphic submanifold of the complex torus $J(M)$. The holomorphic mapping (13.26) and the Abel-Jacobi mapping have the same local expression; so if the image of the mapping (13.26) is denoted by

$$\widetilde{W}_1 = \widetilde{w}_{z_0}(\widetilde{M}) \subset \mathbb{C}^g \tag{13.27}$$

then the mapping (13.26) is a nonsingular holomorphic mapping, hence is a locally biholomorphic mapping

$$\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \widetilde{W}_1. \tag{13.28}$$

This situation can be summarized in the commutative diagram of holomorphic

[1]See the discussion the book *Riemann Surfaces* by Lars Ahlfors and Leo Sario, section 44.
[2]See the discussion in the paper by S. Bochner, "Fortsetzung Riemannscher Flächen" *Math. Annalen* 98(1928), pp. 406-421.

mappings

$$
\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\widetilde{w}_{z_0}} & \widetilde{W}_1 & \xrightarrow[\subset]{\iota} & \mathbb{C}^g \\
\tilde{\pi}\big\downarrow & & \pi\big\downarrow & & \pi\big\downarrow \\
M = \widetilde{M}/\Gamma & \xrightarrow[\cong]{w_{z_0}} & W_1 = \widetilde{W}_1/\mathcal{L}(\Omega) & \xrightarrow[\subset]{\iota} & J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)
\end{array}
\tag{13.29}
$$

in which ι is the natural inclusion mapping. Although the subset $\widetilde{W}_1$ is defined as the image (13.27) it also can be characterized by

$$
\widetilde{W}_1 = \pi^{-1}(W_1) \quad \text{so} \quad \widetilde{W}_1 + \lambda = \widetilde{W}_1 \quad \text{for all } \lambda \in \mathcal{L}(\Omega). \tag{13.30}
$$

Indeed if $t \in \widetilde{W}_1 \subset \mathbb{C}^g$ then by definition $t = \widetilde{w}_{z_0}(z)$ for some point $z \in \widetilde{M}$. and if $\lambda \in \mathcal{L}(\Omega)$ then $\lambda = \omega(T)$ for some $T \in \Gamma$; it then follows from (13.25) that $w_{z_0}(Tz) = w_{z_0}(z) + \lambda = t + \lambda$, so $t + \lambda \in \widetilde{W}_1$. This also shows that the mapping $\widetilde{w}_{z_0}$ is a covering projection. Since $\widetilde{W}_1$ is the inverse image of the holomorphic submanifold W_1 by the holomorphic mapping π it follows that $\widetilde{W}_1$ is a holomorphic submanifold of $\mathbb{C}^g$.

The holomorphic mapping (13.28) is locally biholomorphic but it is not globally biholomorphic. Indeed if $\widetilde{w}_{z_0}(z_1) = \widetilde{w}_{z_0}(z_2)$ for two points $z_1, z_2 \in \widetilde{M}$ then by the commutativity of the diagram (13.29) the images $a_1 = \tilde{\pi}(z_1)$ and $a_2 = \tilde{\pi}(z_2)$ in M have the same image under the Abel-Jacobi mapping w_{z_0}; and since the mapping w_{z_0} is injective it follows that $a_1 = a_2$. Consequently $z_1 = Tz_2$ for some $T \in \Gamma$; and then $w_{z_0}(z_1) = w_{z_0}(Tz_2) = w_{z_0}(z_1) + \omega(T)$ so that $\omega(T) = 0$, which by Corollary 3.6 is equivalent to the condition that $T \in [\Gamma, \Gamma]$. The converse clearly holds, so

$$
\widetilde{w}_{z_0}(z_1) = \widetilde{w}_{z_0}(z_2) \quad \text{if and only if} \quad z_1 = Tz_2 \quad \text{where} \quad T \in [\Gamma, \Gamma]. \tag{13.31}
$$

That means that the mapping $\widetilde{w}_{z_0}$ in the diagram (13.29) is a covering projection, with the covering translation group $[\Gamma, \Gamma]$, and that this mapping can be factored through the quotient surface $\widehat{M} = \widetilde{M}/[\Gamma, \Gamma]$ so the diagram (13.29) can be factored into the commutative diagram of holomorphic mappings

(13.32)

$$
\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\widetilde{w}_{z_0}} & \widetilde{w}_{z_0}(\widetilde{M}) = \widetilde{W}_1 & \xrightarrow[\subset]{\iota} & \mathbb{C}^g \\
\pi_a\big\downarrow & & \Big\| & & \Big\| \\
\widehat{M} = \widetilde{M}/[\Gamma, \Gamma] & \xrightarrow[\cong]{\widehat{w}_{z_0}} & \widehat{w}_{z_0}(\widehat{M}) = \widetilde{W}_1 & \xrightarrow[\subset]{\iota} & \mathbb{C}^g \\
\hat{\pi}\big\downarrow & & \pi\big\downarrow & & \pi\big\downarrow \\
M = \widehat{M}/\Gamma_a & \xrightarrow[\cong]{w_{z_0}} & w_{z_0}(M) = W_1 = \widetilde{W}_1/\mathcal{L}(\Omega) & \xrightarrow[\subset]{\iota} & J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)
\end{array}
$$

where all the vertical arrows are covering projections, as also is the mapping $\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \widetilde{W}_1$. The holomorphic mapping $\widehat{w}_{z_0}$ clearly is surjective, it is injective as a consequence of (13.31), and it is locally biholomorphic since it has the same local expression as the Abel-Jacobi mapping w_{z_0}; hence it is a biholomorphic mapping, as indicated in the diagram. The image $\widehat{w}_{z_0}(\widehat{M}) = \widetilde{W}_1$ thus is an irreducible holomorphic submanifold of $\mathbb{C}^g$ that is biholomorphic to $\widetilde{M}$.

The holomorphic mapping $\widehat{w}_{z_0}$ is defined as the mapping induced by the mapping $\widetilde{w}_{z_0}$; but it also can be described somewhat independently. Indeed it follows from (13.25) that the holomorphic abelian integrals $w_i(z, z_0)$ are invariant under the covering translation group $[\Gamma, \Gamma]$ so they can be viewed as holomorphic functions $\widehat{w}_i(\widehat{z}, z_0)$ of points $\widehat{z}$ in the complex manifold $\widehat{M}$. Of course the holomorphic abelian differentials can be viewed as holomorphic differential forms on the Riemann surface $\widehat{M}$, which is not simply connected; but their integrals actually also are well defined global holomorphic functions $\widehat{w}_i(\widehat{z}, z_0)$ on the manifold $\widehat{M}$. In terms of these integrals the mapping $\widehat{w}_{z_0}$ can be viewed as the mapping defined by

$$\widehat{w}_{z_0}(\widehat{z}) = \{\widehat{w}_i(\widehat{z}, z_0)\} \in \mathbb{C}^g; \tag{13.33}$$

and

$$\widehat{w}_i(\widehat{T}\widehat{z}, z_0) = \widehat{w}_i(\widehat{z}, z_0) + \widehat{\omega}_i(\widehat{T}) \quad \text{for all} \quad \widehat{T} \in \Gamma_a \tag{13.34}$$

where $\widehat{\omega}_i(\widehat{T}) \in \mathbb{C}$ is the period $\omega_i(T)$ for any $T \in \Gamma$ representing $\widehat{T} \in \Gamma_a$. The set of period vectors $\widehat{\omega}(\widehat{T})$ for all $\widehat{T} \in \Gamma_a$ also form the lattice subgroup $\mathcal{L}(\Omega) \subset \mathbb{C}^g$.

The chain of covering projections (13.24) naturally induces a chain of covering projections

$$\widetilde{M}^r \xrightarrow{\pi_a^r} \widehat{M}^r \xrightarrow{\widehat{\pi}^r} M^r \tag{13.35}$$

between the cartesian products of these Riemann surfaces, with the composition

$$\widetilde{\pi}^r = \widehat{\pi}^r \circ \pi_a^r : \widetilde{M}^r \longrightarrow M^r. \tag{13.36}$$

The symmetric group $\mathfrak{S}_r$ of permutations of r points acts naturally on these products; and the quotients have the structures of complex manifolds of dimension r, as in Theorem 12.4 for the case of the surface M itself. The quotient $M^r/\mathfrak{S}_r$ was denoted by $M^{(r)}$ and was identified with the set of positive divisors of degree r on M in the discussion at the beginning of Chapter 12; the corresponding assertions and notation can be applied to surfaces $\widetilde{M}$ and $\widehat{M}$ as well. The holomorphic mappings in (13.35) commute with the action of the symmetric group $\mathfrak{S}_r$, so there results the corresponding chain of holomorphic mappings

$$\widetilde{M}^{(r)} \xrightarrow{\pi_a^{(r)}} \widehat{M}^{(r)} \xrightarrow{\widehat{\pi}^{(r)}} M^{(r)}. \tag{13.37}$$

with the composition

$$\widetilde{\pi}^{(r)} = \widehat{\pi}^{(r)} \circ \pi_a^{(r)} : \widetilde{M}^{(r)} \longrightarrow M^{(r)}. \tag{13.38}$$

Since $\pi_a^{(r)}(z_1'+\cdots+z_r') = \pi_a^{(r)}(z_1''+\cdots+z_r'')$ for two divisors in $\widetilde{M}^{(r)}$ if and only if $z_1'' = T_1 z_{i_1}', \ldots, z_r'' = T_r z_{i_r}'$ for some permutation $(i_1, \ldots, i_r)$ of the indices $(1, \ldots, r)$ and some mappings $T_i : \widetilde{M} \longrightarrow \widetilde{M}$ where $T_i \in \{\Gamma, \Gamma]$, it is evident that the mapping $\pi_a^{(r)} : \widetilde{M}^{(r)} \longrightarrow \widetilde{M}^{(r)}$ is a covering projection; and so is the mapping $\pi_a^{(r)} : \widetilde{M}^{(r)} \longrightarrow \widetilde{M}^{(r)}$, with the corresponding argument, and the composition (13.37).

The holomorphic mappings $\widetilde{w}_{z_0}$ and $\widehat{w}_{z_0}$ in the diagram (13.32) can be extended to the symmetric products, in analogy with the extension of the holomorphic mapping w_{z_0} to the Abel-Jacobi mapping (12.23); thus there is the holomorphic mapping

$$(13.39) \qquad \widetilde{w}_{z_0}^{(r)} : \widetilde{M}^{(r)} \longrightarrow \mathbb{C}^g$$

defined by

$$(13.40) \qquad \widetilde{w}_{z_0}^{(r)}(z_1 + \cdots + z_r) = \widetilde{w}_{z_0}(z_1) + \cdots + \widetilde{w}_{z_0}(z_r) \in \mathbb{C}^g$$

for any divisor $z_1 + \cdots + z_r \in \widetilde{M}^{(r)}$, where $\widetilde{w}_{z_0}(z)$ is the mapping (13.26), and this induces the corresponding holomorphic mapping $\widehat{w}_{z_0}^{(r)} : \widehat{M}^{(r)} \longrightarrow \mathbb{C}^g$ defined by the restricted abelian integrals (13.33) on $\widetilde{M}$. The image of the Abel-Jacobi mapping $w_{z_0}^{(r)} : M^{(r)} \longrightarrow J(M)$ is the irreducible holomorphic subvariety $W_r \subset J(M)$, as in Theorem 12.8; and if the image of the holomorphic mapping $\widetilde{w}_{z_0}^{(r)}$ is denoted correspondingly by $\widetilde{W}^{(r)}$ then it is also the image of the holomorphic mapping $\widetilde{w}_{z_0}^{(r)}$ and there results the commutative diagram of holomorphic mappings

$$(13.41) \qquad \begin{tikzcd}
\widetilde{M}^{(r)} \arrow[r, "\widetilde{w}_{z_0}^{(r)}"] \arrow[d, "\pi_a^{(r)}"'] & \widetilde{w}_{z_0}^{(r)}(\widetilde{M}^{(r)}) = \widetilde{W}_r \arrow[r, "\iota", "\subset"'] \arrow[d, equal] & \mathbb{C}^g \arrow[d, equal] \\
\widehat{M}^{(r)} \arrow[r, "\widehat{w}_{z_0}^{(r)}"] \arrow[d, "\widehat{\pi}^{(r)}"'] & \widehat{w}_{z_0}^{(r)}(\widehat{M}^{(r)}) = \widetilde{W}_r \arrow[r, "\iota", "\subset"'] \arrow[d, "\pi"'] & \mathbb{C}^g \arrow[d, "\pi"'] \\
M^{(r)} \arrow[r, "w_{z_0}^{(r)}"] & w_{z_0}^{(r)}(M^{(r)}) = W_r \arrow[r, "\iota", "\subset"'] & J(M).
\end{tikzcd}$$

The mappings in the vertical columns are covering projections; that was already noted for the first and third columns, and the second column is a restriction of the third column so is also a covering projection. It follows from (13.25) just as in the proof of (13.30) that

$$(13.42) \qquad \widetilde{W}_r = \widetilde{\pi}^{-1}(W_r) \quad \text{so} \quad \widetilde{W}_r + \lambda = \widetilde{W}_r \quad \text{for all } \lambda \in \mathcal{L}(\Omega);$$

indeed if $t \in \widetilde{W}_r$ then $t = \widetilde{w}_{z_0}^{(r)}(z_1 + \cdots + z_r)$ for some divisor $z_1 + \cdots + z_r \in \widetilde{M}^{(r)}$, and since any lattice vector $\lambda \in \mathcal{L}(\Omega)$ is the period $\lambda = \omega(T)$ for some covering translation $T \in \Gamma$ it follows that $t + \lambda = \widetilde{w}_{z_0}(Tz_1 + \cdots + z_r) \in \widetilde{W}_r$. As the inverse image of the holomorphic subvariety $W_r \subset J(M)$ under the holomorphic mapping $\widetilde{\pi}$ the subset $\widetilde{W}_r \subset \mathbb{C}^g$ is a holomorphic subariety; and as the image of a connected complex manifold under the holomorphic mapping $\widetilde{w}^{(r)}_{z_0}$ it is an irreducible holomorphic subvariety.

The diagram (13.41) for $r = 1$ reduces to the diagram (13.32) in which both w_{z_0} and $\widehat{w}_{z_0}$ are biholomorphic mappings which identify the Riemann surfaces M and $\widehat{M}$ with holomorphic submanifolds of $J(M)$ and $\mathbb{C}^g$ respectively; but for $r > 1$ the situation is a bit more complicated. Subsets $G^1_r \subset M^{(r)}$ for $1 < r \le g$ were defined in (12.38) and (12.39); they are holomorphic subvarieties by Theorem 12.19, and can be identified with the proper holomorphic subvarieties $\mathrm{sp}M^{(r)} \subset M^{(r)}$ of special positive divisors as in (12.40). Their images $W^1_r = w_{z_0}(G^1_r) \subset J(M)$ are holomorphic subvarieties of $J(M)$ by Remmert's Proper Mapping Theorem, as in Theorem 12.21. By Theorem 12.8 (iv) with the interpretation (12.40) the restriction

$$w^{(r)}_{z_0} : \Big(M^{(r)} \sim G^1_r\Big) \longrightarrow \Big(W_r \sim W^1_r\Big) \tag{13.43}$$

is a biholomorphic mapping. The inverse images $\widetilde{G}^1_r = (\widetilde{\pi}^{(r)})^{-1}(G^1_r) \subset \widetilde{M}^{(r)}$ and $\widetilde{W}^1_r = \pi^{-1}(W^1_r) \subset \widetilde{W}_r$ then are holomorphic subvarieties for which $\widetilde{W}^1_r = \widetilde{w}_{z_0}(\widetilde{G}^1_r)$, and correspondingly for $\widehat{G}^1_r$ and $\widehat{W}^1_r$; and the restriction of the holomorphic mapping $\widetilde{w}^{(r)}_{z_0}$ is a surjective holomorphic mapping

$$\widetilde{w}^{(r)}_{z_0} : \big(\widetilde{M}^{(r)} \sim \widetilde{G}^1_r\big) \longrightarrow \big(\widetilde{W}_r \sim \widetilde{W}^1_r\big). \tag{13.44}$$

The mappings $\widetilde{\pi}^{(r)} = \widehat{\pi}^{(r)} \circ \pi^{(r)}_a$ and π in (13.41) are covering mappings so it follows from (13.43) that the mapping (13.44) is a locally biholomorphic mapping.

Introduce an equivalence relation on the divisors in $\widetilde{M}^{(r)}$ by setting

$$\begin{aligned}&(\widetilde{z}_1 + \cdots + \widetilde{z}_r) \sim (T_1\widetilde{z}_1 + \cdots + T_r\widetilde{z}_r)\\ &\qquad \text{for any } T_i \in \Gamma \text{ for which}\\ &\qquad \omega(T_1) + \cdots + \omega(T_r) = \omega(T_1 \cdots T_r) = 0;\end{aligned} \tag{13.45}$$

and let $\overset{\bowtie}{M}{}^{(r)} = \widetilde{M}^{(r)}/\sim$ be the quotient of $\widetilde{M}^{(r)}$ by this equivalence relation. This is a weaker equivalence relation than that defined by the quotient mapping to $M^{(r)}$, in the sense that any divisors equivalent under the relation (13.45) have the same image in $M^{(r)}$; and since $\omega(T) = 0$ for all $T \in [\Gamma, \Gamma]$ it is a stronger equivalence relation than that defined by the quotient mapping to $\widehat{M}^{(r)}$, in the sense that any two divisors that have the same image in $\widehat{M}^{(r)}$ are equivalent under the relation (13.45). Consequently the covering projection $\widehat{\pi}^{(r)} : \widehat{M}^{(r)} \longrightarrow M^{(r)}$ can be factored into the composition $\widehat{\pi}^{(r)} = \overset{\bowtie}{\pi}{}^{(r)} \circ \widehat{\pi}^{(r)}_0$ of covering projections

$$\widehat{M}^{(r)} \xrightarrow{\widehat{\pi}^{(r)}_0} \overset{\bowtie}{M}{}^{(r)} \xrightarrow{\overset{\bowtie}{\pi}{}^{(r)}} M^{(r)}, \tag{13.46}$$

so the quotient space $\overset{\bowtie}{M}{}^{(r)}$ has the structure of a complex manifold for which the covering projections in (13.46) are holomorphic and locally biholomorphic mappings. If $\overset{\bowtie}{G}{}^1_r = (\overset{\bowtie}{\pi}{}^{(r)})^{(-1)}(G^1_r) \subset \overset{\bowtie}{M}{}^{(r)}$ then $\overset{\bowtie}{G}{}^1_r$ is a holomorphic subvariety of $\overset{\bowtie}{M}{}^{(r)}$ and $\overset{\bowtie}{G}{}^1_r = \widehat{\pi}{}^{(r)}_a(\widehat{G}^1_r)$.

Theorem 13.3 (i) *If two divisors $\mathfrak{d}', \mathfrak{d}'' \in \widetilde{M}^{(r)}$ are equivalent under the equivalence relation* (13.45) *then $\widetilde{w}^{(r)}_{z_0}(\mathfrak{d}') = \widetilde{w}^{(r)}_{z_0}(\mathfrak{d}'') \in \widetilde{W}_r$.*
(ii) *Conversely if $\widetilde{w}^{(r)}_{z_0}(\mathfrak{d}') = \widetilde{w}^{(r)}_{z_0}(\mathfrak{d}'') \in \widetilde{W}_r \sim \widetilde{G}^1_r$ then the divisors $\mathfrak{d}'$ and $\mathfrak{d}''$ are equivalent under the equivalence relation* (13.45)*.*

Proof: (i) If $\mathfrak{d}' = z'_1 + \cdots + z'_r$ and $\mathfrak{d}'' = z''_1 + \cdots + z''_r$ are equivalent divisors in $\widetilde{M}^{(r)}$ then after reordering the points z''_i if necessary there will be covering translations $T_i \in \Gamma$ such that $z''_i = T_i z'_i$ where $\sum_{i=1}^r \omega(T_i) = 0$. Then $\widetilde{w}^{(r)}_{z_0}(\mathfrak{d}'') = \sum_{i=1}^r \widetilde{w}^{(r)}_{z_0}(z''_i) = \sum_{i=1}^r \widetilde{w}^{(r)}_{z_0}(T_i z') = \sum_{i=1}^r \Big(\widetilde{w}^{(r)}_{z_0}(z') + \omega(T_i)\Big) = \widetilde{w}^{(r)}_{z_0}(\mathfrak{d}')$ as desired.
(ii) If $\widetilde{w}^{(r)}_{z_0}(\mathfrak{d}') = \widetilde{w}^{(r)}_{z_0}(\mathfrak{d}'')$ for two divisors $\mathfrak{d}', \mathfrak{d}'' \in \widetilde{M}^{(r)} \sim \widetilde{G}^1_r$ then from the commutativity of the diagram (13.41) it follows that

$$w^{(r)}_{z_0}\big(\widetilde{\pi}^{(r)}(\mathfrak{d}')\big) = w^{(r)}_{z_0}\big(\widetilde{\pi}^{(r)}(\mathfrak{d}'')\big) \in J(M).$$

Since the mapping (13.43) is injective it must be the case that $\widetilde{\pi}^{(r)}(\mathfrak{d}') = \widetilde{\pi}^{(r)}(\mathfrak{d}'')$; thus the divisors $\mathfrak{d}'$ and $\mathfrak{d}''$ in $\widetilde{M}^{(r)}$ represent the same divisor in $M^{(r)}$, so after reordering the points in these divisors as necessary there will be covering translations $T_i \in \Gamma$ such that $T_i z'_i = z''_i$ for each index i. Then $\widetilde{w}^{(r)}_{z_0}(\mathfrak{d}'') = \sum_{i=1}^r \widetilde{w}_{z_0}(T_i z'_i) = \sum_{i=1}^r (\widetilde{w}_{z_0}(z'_i) + \omega(T_i)) = \widetilde{w}^{(r)}_{z_0}(\mathfrak{d}') + \sum_{i=1}^r \omega(T_i)$, and since $\widetilde{w}^{(r)}_{z_0}(\widetilde{\mathfrak{d}}'') = \widetilde{w}^{(r)}_{z_0}(\widetilde{\mathfrak{d}}')$ by assumption it follows that $\sum_{i=1}^r \omega(T_i) = 0$ so the two divisors $\mathfrak{d}'$ and $\mathfrak{d}''$ are equivalent. That suffices for the proof.

Corollary 13.4 *The holomorphic mapping $\widetilde{w}^{(r)}_{z_0} : \widetilde{M}^{(r)} \longrightarrow \widetilde{W}_r$ for a compact Riemann surface M induces a biholomorphic mapping*

$$\overset{\bowtie}{w}{}^{(r)}_{z_0} : \overset{\bowtie}{M}{}^{(r)} \sim \overset{\bowtie}{G}{}^1_r \xrightarrow{\cong} \widetilde{W}_r \sim \widetilde{W}^1_r. \tag{13.47}$$

Proof: Part (i) of the preceding theorem shows that equivalent divisors $\mathfrak{d}', \mathfrak{d}'' \in \widetilde{M}$ have the same image under the holomorphic mapping $\widetilde{w}^{(r)}_{z_0} : \widetilde{M}^{(r)} \longrightarrow \widetilde{W}_r$, hence this mapping induces a holomorphic mapping $\overset{\bowtie}{w}{}^{(r)}_{z_0} : \overset{\bowtie}{M}{}^{(r)} \longrightarrow \widetilde{W}_r$; and part (ii) of the preceding theorem shows that the restriction of this mapping is a biholomorphic mapping which is the mapping (13.47), and that suffices for the proof.

From these observations it follows that the commutative diagram (13.41) can be extended to a further commutative diagram of holomorphic mappings (13.48), in which all the columns are covering projections, the first mappings in each row are surjective holomorphic mappings and the second mappings in each row are inclusion mappings.

$$
\begin{tikzcd}
\widetilde{M}^{(r)} \sim \widetilde{G}^1_r \arrow[r, "\widetilde{w}^{(r)}_{z_0}"] \arrow[d, "\pi^{(r)}_a"'] & \widetilde{W}_r \sim \widetilde{W}^1_r \arrow[r, "\iota", "\subset"'] \arrow[d, equal] & \mathbb{C}^g \arrow[d, equal] \\
\widehat{M}^{(r)} \sim \widehat{G}^1_r \arrow[r, "\widehat{w}^{(r)}_{z_0}"] \arrow[d, "\widehat{\pi}^{(r)}_0"'] & \widetilde{W}_r \sim \widetilde{W}^1_r \arrow[r, "\iota", "\subset"'] \arrow[d, equal] & \mathbb{C}^g \arrow[d, equal] \\
\overset{\bowtie}{M}{}^{(r)} \sim \overset{\bowtie}{G}{}^1_r \arrow[r, "\overset{\bowtie}{w}{}^{(r)}_{z_0}", "\cong"'] \arrow[d, "\overset{\bowtie}{\pi}{}^{(r)}"'] & \widetilde{W}_r \sim \widetilde{W}^1_r \arrow[r, "\iota", "\subset"'] \arrow[d, "\pi"'] & \mathbb{C}^g \arrow[d, "\pi"'] \\
M^{(r)} \sim G^1_r \arrow[r, "w^{(r)}_{z_0}", "\cong"'] & W_r \sim W^1_r \arrow[r, "\iota", "\subset"'] & J(M).
\end{tikzcd}
\tag{13.48}
$$

13.2 The General Cross-Ratio Function

The product cross-ratio function was defined in (13.2) as the meromorphic function

$$
Q(z; a; z_1^+, \dots, z_r^+; z_1^-, \dots, z_r^-) = \prod_{\nu=1}^{r} q(z, a; z_\nu^+, z_\nu^-) \tag{13.49}
$$

of the ordered set of variables

$$
(z; a; z_1^+, \dots, z_r^+; z_1^-, \dots, z_r^-) \in \widetilde{M} \times \widetilde{M} \times \widetilde{M}^r \times \widetilde{M}^r; \tag{13.50}
$$

but it is symmetric in the variables $(z_1^+, \dots, z_r^+)$ and $(z_1^-, \dots, z_r^-)$ and it is invariant under the mappings $\pi_a^{(r)} : \widetilde{M}^{(r)} \longrightarrow \widehat{M}^{(r)}$ and $\widehat{\pi}_0^{(r)} : \widehat{M}^{(r)} \longrightarrow \overset{\bowtie}{M}{}^{(r)}$.

Theorem 13.5 *The product cross-ratio function of degree r can be viewed as a meromorpic function $Q(z, a; \widetilde{\mathfrak{d}}^+, \widetilde{\mathfrak{d}}^-)$ of the variables*

$$
(z, a; \widetilde{\mathfrak{d}}^+, \widetilde{\mathfrak{d}}^-) \in \widetilde{M} \times \widetilde{M} \times \widetilde{M}^{(r)} \times \widetilde{M}^{(r)}, \tag{13.51}
$$

or as a meromorphic function $Q(z, a; \widehat{\mathfrak{d}}^+, \widehat{\mathfrak{d}}^-)$ of the variables

$$
(z, a; \widehat{\mathfrak{d}}^+, \widehat{\mathfrak{d}}^-) \in \widetilde{M} \times \widetilde{M} \times \widehat{M}^{(r)} \times \widehat{M}^{(r)}, \tag{13.52}
$$

or even as a meromorphic function $Q(z, a; \overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}_-)$ of the variables

$$
(z, a; \overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^-) \in \widetilde{M} \times \widetilde{M} \times \overset{\bowtie}{M}{}^{(r)} \times \overset{\bowtie}{M}{}^{(r)}. \tag{13.53}
$$

Proof: The intrinsic cross-ratio function $q(z,a;z^+,z^-)$ as a function of the variable $z \in \widetilde{M}$ is a meromorphic relatively automorphic function for the factor of automorphy $\rho_{z^+,z^-}(T)$ given explicitly in (13.1), and it has simple zeros at the points Tz^+ and simple poles at the points Tz^- for all $T \in \Gamma$. For any fixed points z_ν^+, z_ν^+ the quotient

$$g(z) = \frac{q(z,a;z_1^+,z_1^-)q(z,a;z_2^+,z_2^-)}{q(z,a;z_2^+,z_1^-)q(z,a;z_1^+,z_2^-)} \tag{13.54}$$

therefore is a nowhere vanishing holomorphic function in $\widetilde{M}$, since the zero divisor of the numerator is the same as the zero divisor of the denominator and correspondingly for the pole divisors. The function $g(z)$ is also a relatively automorphic function for the factor of automorphy

$$\rho(T) = \frac{\rho_{z_1^+,z_1^-}(T)\rho_{z_2^+,z_2^-}(T)}{\rho_{z_2^+,z_1^-}(T)\rho_{z_1^+,z_2^-}(T)}; \tag{13.55}$$

and from the explicit form (13.1) for the factor of automorphy $\rho_{z^+,z^-}(T)$ it is clear that $\rho(T) = 1$ for all $T \in \Gamma$. Therefore the function $g(z)$ really is a function on the compact Riemann surface M, so it is actually a constant in the variable z. Since $q(a,a;z_\nu^+,z_\nu^-) = 1$ for any z_ν^+, z_ν^-, it follows that $g(z) = 1$ for all $z \in \widetilde{M}$. That is the case for any values of the auxiliary parameters z_ν^+, z_ν^- so the function (13.54) is identically equal to 1 in all variables, or equivalently the product $q(z,a;z_1^+,z_1^-)q(z,a;z_2^+,z_2^-)$ is symmetric in the parameters z_1^+, z_2^+. This argument can be applied to any pair of points among those in $z_1^+, \dots, z_r^+$ or $z_1^-, \dots, z_r^-$, showing that the product cross-ratio function is symmetric in the parameters $z_1^+, \dots, z_r^+$ as well as in the parameters $z_1^-, \dots, z_r^-$ and consequently the product cross-ratio function can be viewed as a meromorphic function of the divisors $\mathfrak{d}^+ = z_1^+ + \cdots + z_r^+ \in \widetilde{M}^{(r)}$ and $\mathfrak{d}^- = z_1^- + \cdots + z_r^- \in \widetilde{M}^{(r)}$. In view of the symmetry $q(z,a;z^+,z^-) = q(z^+,z^-;z,a)$ of Theorem 5.28 (ii) the cross-ratio function is also a relatively automorphic function of the variable z^+, in the sense that

$$q(z,a;\,Tz^+,z^-) = \rho_{z,a}(T)q(z,a;z^+,z^-)$$

for any $T \in \Gamma$; consequently the product crosss-ratio function (13.2) as a function of the variables $\mathfrak{d}^+ = z_1^+ + \cdots + z_r^+ \in \widetilde{M}^{(r)}$ and $\mathfrak{d}^- = z_1^- + \cdots + z_r^- \in \widetilde{M}^{(r)}$ satisfies

$$\begin{aligned} &Q(z,a;T_1z_1^+ + \cdots + T_rz_r^+,\ z_1^- + \cdots + z_r^-) \\ &\qquad = \rho_{z,a}(T_1\cdots T_r)Q(z,a;z_1^+ \cdots + z_r^+,\ z_1^- + \cdots + z_r^-) \end{aligned} \tag{13.56}$$

for any covering translations $T_1, \dots, T_r \in \Gamma$. In particular

$$\begin{aligned} &Q(z,a;T_1z_1^+ + \cdots + T_rz_r^+,\ z_1^- + \cdots + z_r^-) \\ &\qquad = Q(z,a;z_1^+ \cdots + z_r^+,\ z_1^- + \cdots + z_r^-) \\ &\qquad \text{if } \ \omega(T_1\cdots T_r) = \omega(T_i) + \cdots + \omega(T_r) = 0 \end{aligned} \tag{13.57}$$

since $\rho_{z,a}(T_1 \cdots T_r) = 1$ if $\omega(T_1 \cdots T_r) = 0$; thus the product cross-ratio function $Q(z, a; z_1^+ \cdots + z_r^+,\ z_1^- + \cdots + z_r^i)$ as a function of the divisor $\mathfrak{d}^+ = z_1^+ \cdots + z_r^+ \in \widetilde{M}^{(r)}$ is invariant under the equivalence relation (13.45) on the manifold $\widetilde{M}^{(r)}$ and therefore it can be viewed as a meromorphic function of the variables $(z, a; \overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^-) \in \widetilde{M} \times \widetilde{M} \times \overset{\bowtie}{M}{}^{(r)} \times \overset{\bowtie}{M}{}^{(r)}$. The equivalence relation (13.45) is stronger than the equivalence relation defining the quotient $\widetilde{M}^{(r)}$, and if $r = 1$ it is just the equivalence relation defining $\widetilde{M}$; consequently the product cross-ratio function automatically can be viewed as a function of the variables $(\widetilde{z}, \widetilde{a}; \widehat{\mathfrak{d}}^+, \widehat{\mathfrak{d}}^-) \in \widetilde{M} \times \widetilde{M} \times \widehat{M}^{(r)} \times \widehat{M}^{(r)}$, and that suffices for the proof.

For the special case $r = g$ the subvariety $\widetilde{W}_g \subset \mathbb{C}^g$ is the entire space $\mathbb{C}^g$ and the subvariety $\widetilde{W}_g^1 \subset \widetilde{W} = \mathbb{C}^g$ is a holomorphic subvariety of dimension $g - 2$, since $\widetilde{W}_g^1$ is a covering space of the submanifold $W_g^1 \subset J(M)$ in the diagram (13.48) and $W_g^1 = k - W_{g-2}$ by Theorem 12.23. The mapping (13.47) then is a biholomorphic mapping

$$\overset{\bowtie}{w}{}^{(g)}_{z_0} : \left(\overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}^1_g\right) \xrightarrow{\cong} \left(\mathbb{C}^g \sim \widetilde{W}_g^1\right), \tag{13.58}$$

by Corollary 13.4 and as indicated in the diagram (13.48) ; and through this biholomorphic mapping the meromorphic function $Q(z, a; \overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^-)$ on $\widetilde{M} \times \widetilde{M} \times \left(\overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}^1_g\right) \times \left(\overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}^1_g\right)$ can be identified with a meromorphic function $\mathfrak{Q}(z, a; t^+, t^-)$ on the product manifold $\widetilde{M} \times \widetilde{M} \times \left(\mathbb{C}^g \sim \widetilde{W}_g^1\right) \times \left(\mathbb{C}^g \sim \widetilde{W}_g^1\right)$ for which

$$\mathfrak{Q}\big(z, a; \overset{\bowtie}{w}{}^{(g)}_{z_0}(\overset{\bowtie}{\mathfrak{d}}{}^+), \overset{\bowtie}{w}{}^{(g)}_{z_0}(\overset{\bowtie}{\mathfrak{d}}{}^-)\big) = Q(z, a; \overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^-) \tag{13.59}$$

for any $z, a \in \widetilde{M}$ and any divisors $\overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^- \in \overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}^1_g$. Since $\widetilde{W}_g^1$ is a holomorphic subvariety of codimension 2 in the complex manifold $\mathbb{C}^g$ it follows from the Theorem of Levi[3] that the function $\mathfrak{Q}(z, a; t^+, t^-)$ extends uniquely to a meromorphic function on the entire product manifold $\widetilde{M} \times \widetilde{M} \times \mathbb{C}^g \times \mathbb{C}^g$. This extension is called the *general cross-ratio function* of the Riemann surface M, and for this extension (13.59) holds for all divisors $\overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^- \in \overset{\bowtie}{M}{}^{(g)}$ by analytic continuation. When the product cross-ratio function is written out explicitly as a product of the intrinsic cross-ratio functions $q(z, a; z_j, a_j)$ of the variables $z, a; z_j, a_j \in \widetilde{M}$ equation (13.59) takes the form

$$\mathfrak{Q}\Big(z, a; \sum_{j=1}^{g} \widetilde{w}_{z_0}(z_j), \sum_{j=1}^{g} \widetilde{w}_{z_0}(a_j)\Big) = \prod_{j=1}^{g} q(z, a; z_j, a_j) \tag{13.60}$$

for all points $z, a, z_1, \ldots, z_g, a_1, \ldots, a_g \in \widetilde{M}$. Equations (13.59) and (13.60) thus are alternative characterizations of the general cross-ratio function. It should be

[3]See the discussion of extension properties of meromorphic functions on page 409 in Appendix A

observed that the mapping (13.47 and all the mappings in the diagram (13.48) are defined in terms of the choice of a base point $z_0 \in \widetilde{M}$; so the general cross-ratio function should be viewed as defined for a pointed Riemann surface M, one with a specified base point $z_0 \in \widetilde{M}$. A change of the base point amounts to a translation in the space $\mathbb{C}^g$, so the actual dependence on the choice of a base point is of limited significance. The basic properties of the general cross-ratio can be summarized as follows.

Theorem 13.6 (i) *On a compact Riemann surface $M = \widetilde{M}/\Gamma$ of genus $g > 0$, with the period matrix Ω for the choice of a basis $\omega_i \in \Gamma(M, \mathcal{O}^{(1,0)})$ and generators $T_j \in \Gamma$, the general cross-ratio function $\mathfrak{Q}(z, a; t^+, t^-)$ is a meromorphic function on the complex manifold $\widetilde{M} \times \widetilde{M} \times \mathbb{C}^g \times \mathbb{C}^g$ with the symmetries*

$$\mathfrak{Q}(z, a; t^+, t^-) = \mathfrak{Q}(a, z; t^+, t^-)^{-1} = \mathfrak{Q}(z, a; t^-, t^+)^{-1}. \tag{13.61}$$

and the normalizations

$$\mathfrak{Q}(a, a; t^+, t^-) = \mathfrak{Q}(z, a; t, t) = 1. \tag{13.62}$$

(ii) *For any $T \in \Gamma$*

$$\mathfrak{Q}(Tz, a; t^+, t^-) = \rho_{t^+,t^-}(T)\mathfrak{Q}(z, a; t^+, t^-) \tag{13.63}$$

where

$$\rho_{t^+,t^-}(T) = \exp -2\pi \sum_{m,n=1}^{g} \left(t^+_m - t^-_n\right) g_{mn}\overline{\omega_n(T)}. \tag{13.64}$$

(iii) *For any lattice vector $\lambda \in \mathcal{L}(\Omega)$*

$$\mathfrak{Q}(z, a;\ t^+ + \lambda, t^-) = \rho_{z,a}(\lambda)\ \mathfrak{Q}(z, a;\ t^+, t^-) \tag{13.65}$$

where

$$\rho_{z,a}(\lambda) = \exp -2\pi \sum_{m,n=1}^{g} w_m(z, a) g_{mn}\overline{\lambda}_n \tag{13.66}$$

for the abelian integrals $w_m(z, a) = \int_a^z \omega_m$.
(iv) *For any fixed points $a, b \in \widetilde{M}$ and $t_0 \in \mathbb{C}^g$ for which $\mathfrak{Q}(a, b; t, t_0)$ is a nontrivial meromorphic function of the variable $t \in \mathbb{C}^g$, neither identically 0 nor identically ∞, its zero locus is the holomorphic subvariety*

$$\widetilde{V}_a = \widetilde{w}_{z_0}(a) + \widetilde{W}_{g-1} \subset \mathbb{C}^g, \tag{13.67}$$

at which it has a simple zero, and its pole locus is the holomorphic subvariety

$$\widetilde{V}_a = \widetilde{w}_{z_0}(b) + \widetilde{W}_{g-1} \subset \mathbb{C}^g, \tag{13.68}$$

at which it has a simple pole.

Proof: (i) Since the cross-ratio function has the symmetries $q(z_1, z_2; z_3, z_4) = q(z_2, z_1; z_3, z_4)^{-1} = q(z_1, z_2; z_4, z_3)^{-1}$ by Theorem 5.6 (ii) it follows from (13.60) that the general cross-ratio has the symmetries (13.61); and since the cross-ratio function has the normalization $q(z_1, z_1; z_3, z_4) = q(z_1, z_2; z_3, z_3) = 1$ by Theorem 5.6 (i) it follows from (13.60) that the general cross-ratio has the normalization (13.62).
(ii) As noted in the discussion on page 342, the product cross-ratio function $Q(z, a; A^+, , A_-)$ for $A^+ = (a_1^+, \ldots, a_g^+)$ and $A^- = (a_1^-, \ldots, a_g^-)$ is a relatively automorphic function of the variable $z \in \widetilde{M}$ for the factor of automorphy $\rho_{A+,A^-}(T)$ of (13.3); therefore when the product cross-ratio function is viewed as a function of the divisors $\mathfrak{d}^+ = a_1^+ + \cdots + a_g^+$ and $\mathfrak{d}^- = a_1^- + \cdots + a_g^-$ as in Theorem 13.5

$$
\begin{aligned}
(13.69)\qquad Q(Tz, a\,; \mathfrak{d}^+, \mathfrak{d}^-) &= Q(z, a\,; \mathfrak{d}^+, \mathfrak{d}^-)\rho_{A^+,A^-}(T)\\
&= Q(z, a\,; \mathfrak{d}^+, \mathfrak{d}^-) \cdot \exp -2\pi \sum_{m,n=1}^{g} w_m(\mathfrak{d}^+, \mathfrak{d}^-)\, g_{mn}\, \overline{\omega_n(T)}
\end{aligned}
$$

where $w_m(\mathfrak{d}^+, \mathfrak{d}^-) = \sum_{\nu=1}^{g} w_m(a_\nu^+ - a_\nu^-)$. The image $t^+ = \widetilde{w}_{z_0}^{(g)}(\mathfrak{d}^+) \in \mathbb{C}^g$ is the vector with coefficients $t_m^+ = \sum_{\nu=1}^{g} w_m(a_\nu^+, z_0)$, and similarly for the vector $t^- \in \mathbb{C}^g$; thus $w_m(\mathfrak{d}^+, \mathfrak{d}^-) = w_m(\mathfrak{d}^+, z_0) - w_m(\mathfrak{d}^+, z_0) = t_m^+ - t_m^-$ so (13.69) can be written

$$
(13.70)\quad Q(Tz, a\,;\ \mathfrak{d}^+, \mathfrak{d}^-) = Q(z, a\,; \mathfrak{d}^+, \mathfrak{d}^-)\cdot\exp -2\pi \sum_{m,n=1}^{g} (t_m^+ - t_m^-)\, g_{mn}\, \overline{\omega_n(T)}.
$$

This equation is invariant under the equivalence relation (13.45) so through (13.59) yields the result of part (ii). (iii) For any $T \in \Gamma$ and any divisor $\overset{\bowtie}{\mathfrak{d}}{}^+ = z_1^+ + z_2^+ + \cdots + z_g^+ \in \overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}_g^1$ let $T\,\overset{\bowtie}{\mathfrak{d}}{}^+ = Tz_1^+ + z_2^+ + \cdots + z_g^+$; with this convention (13.56) takes the form

$$
(13.71)\qquad Q(z, a; T\,\overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^-) = \rho_{z,a}(T) Q(z, a; \overset{\bowtie}{\mathfrak{d}}{}^+, \overset{\bowtie}{\mathfrak{d}}{}^-)
$$

where the factor of automorphy $\rho_{z,a}(T)$ has the explicit form (13.1). Now if $\overset{\bowtie}{w}{}_{z_0}^{(g)}(\overset{\bowtie}{\mathfrak{d}}{}^\pm) = t^+$ then $\overset{\bowtie}{w}{}_{z_0}^{(g)}(T\,\overset{\bowtie}{\mathfrak{d}}{}^+) = \overset{\bowtie}{w}{}_{z_0}^{(g)}(\overset{\bowtie}{\mathfrak{d}}{}^+) + \lambda$ where $\lambda = \omega(T) \in \mathcal{L}(\Omega)$; so through the biholomorphic mapping (13.58) equation (13.71) takes the form

$$
(13.72)\qquad \mathfrak{Q}(z, a; t^+ + \lambda, t^-) = \rho_{z,a}(T)\mathfrak{Q}(z, a; t^+, t^-)
$$

where

$$
\rho_{z,a}(T) = \exp -2\pi \sum_{m,n=1}^{g} (w_m(z) - w_n(a))\, g_{mn} \overline{\lambda_n}.
$$

That identity extends across the subvariety $\overset{\bowtie}{G}{}_g^1$ by analytic continuation, thereby demonstrating (13.65). Since any lattice vector $\lambda \in \mathcal{L}(\Omega)$ can be written as

$\lambda = \omega(T)$ for some $T \in \Gamma$ that equation holds for all $\lambda \in \mathcal{L}(\Omega)$.
(iv) For any fixed points $a, b, a_1, , \ldots, a_g \in \widetilde{M}$ for which $a \neq Tb$ and $a \neq Ta_j$ and $b \neq Ta_j$ for any index j and any $T \in \Gamma$ the product cross-ratio function $Q(a, b; z_1, , \ldots, z_g; a_1, , \ldots, a_g) = \prod_{j=1}^{g} q(a, b; z_j, a_j)$ is a nontrivial meromorphic function of the variables $(z_1, \ldots, z_g) \in \widetilde{M}^g$ and is zero at those points $(z_1, \ldots, z_g) \in \widetilde{M}^g$ for which $z_j = Ta$ for some index j and some $T \in \Gamma$; consequently the zero locus of the product cross-ratio function viewed as a function of divisors $z_1 + \cdots + z_g \in \overset{\bowtie}{M}^{(g)}$ consists of divisors of the form $Ta + z_2 + \cdots + z_g$ for some $T \in \Gamma$ and for arbitrary points $z_j \in \widetilde{M}$. The set of those divisors that are contained in $\overset{\bowtie}{M}^{g)} \sim \overset{\bowtie}{G}{}^1_g$ is mapped through the biholomorphic mapping

$$\overset{\bowtie}{w}{}^{g)}_{z_0} : \overset{\bowtie}{M}^{g)} \sim \overset{\bowtie}{G}{}^1_g \longrightarrow \widetilde{W}_g \sim \widetilde{W}^1_g$$

in the diagram (13.48) to the subset

$$\left(\widetilde{w}_{z_0}(a) + \widetilde{W}_{g-1}\right) \cap \left(\mathbb{C}^g \sim \widetilde{W}^1_g\right) \subset \mathbb{C}^g;$$

so by (13.60) this is the zero locus of $\mathfrak{Q}(a, b; t, t_0)$ in the subset $\mathbb{C}^g \sim \widetilde{W}^1_g \subset \mathbb{C}^g$, where $t_0 \in \mathbb{C}^g$ is the image of the divisor $a_1 + \cdots + a_g$. The subvariety $\widetilde{W}^1_g \subset \mathbb{C}^g$ is the inverse image of the subvariety $W^1_g \subset J(M)$ under the covering projection $\pi : \mathbb{C}^g \longrightarrow J(M)$, so $\dim \widehat{W}^1_g = \dim W^1_g = g - 2$ since $W^1_g = k - W_{g-2}$ by the Riemann-Roch Theorem in the form of Theorem 12.23; consequently the zero locus of the general cross-ratio function $\mathfrak{Q}(a, b; t, t_0)$ as a function of the variable $t \in \mathbb{C}^g$ actually is the holomorphic subvariety $\widetilde{V}_a = \widetilde{w}^{(g)}_{z_0}(a) + \widetilde{W}_{g-1} \subset \mathbb{C}^g$. The composition $M^g \longrightarrow \mathbb{C}^g$ of the branched covering mapping $M^g \longrightarrow M^{(g)}$ and the other mappings in the diagram (13.48) is locally biholomorphic in a neighborhood of any point $(z_1, \ldots, z_g) \in \widetilde{M}^g$ for which $z_1, \ldots, z_g$ are distinct points of $\widetilde{M}$; and since each factor $q(z, a; z_j, a_j)$ has a simple zero at the point $z_j = z$ it follows that the general cross-ratio function $\mathfrak{Q}(a, b; t, t_0)$ vanishes to the first order on the subvariety $\widetilde{V}_a = \widetilde{w}^{(g)}_{z_0}(a) + \widetilde{W}_{g-1}$. In view of what was already proved in part (i) of this theorem the pole locus of the function $\mathfrak{Q}(a, b; t, t_0)$ as a function of the variable $t \in \mathbb{C}^g$ is the zero locus of the function $\mathfrak{Q}(b, a; t, t_0)$ as a function of the variable $t \in \mathbb{C}^g$, so it is a simple pole of the general cross-ratio function, and that concludes the proof.

Since the general cross-ratio function $\mathfrak{Q}(a, b; t, t_0)$ is a relatively automorphic function of the variable $t \in \mathbb{C}^g$ for the action of the lattice subgroup $\mathcal{L}(\Omega)$, by (iii) of the preceding theorem, it follows that its zero locus $\widetilde{V}_a$ is invariant under $\mathcal{L}(\Omega)$ so describes a holmorphic subvariety

$$V_a = \widetilde{V}_a / \mathcal{L}(\Omega) \subset J(M), \tag{13.73}$$

and similarly for the pole locus. The general cross-ratio function $\mathfrak{Q}(a, b; t, t_0)$ also is a relatively automorphic function of the variable $a \in \widetilde{M}$, by (ii) of the preceding theorem, so the zero locus V_a really depends only on the point of the Riemann surface M represented by the point $a \in \widetilde{M}$; so the subvariety V_a can be viewed alternatively as indexed by the point $a \in M$. By Corollary 5.7 the intrinsic cross-ratio function satisfies the product formula $q(a,b;z,z_0) = q(a,b;z,z_1)q(a,b;z_1,z_0)$ for any points $a,b,z,z_0,z_1 \in \widetilde{M}$; consequently the product cross-ratio function and therefore the general cross-ratio function as well satisfy the corresponding product formula

$$\mathfrak{Q}(a,b;t,t_0) = \mathfrak{Q}(a,b;t,t_1)\mathfrak{Q}(a,b;t_1,t_0). \tag{13.74}$$

Thus so long as the constant $\mathfrak{Q}(a,b;t_1,t_0)$ is nonzero the two general cross-ratio functions $\mathfrak{Q}(a,b;t,t_0)$ and $\mathfrak{Q}(a,b;t,t_1)$ of the variable $t \in \mathbb{C}^g$ have the same zero locus as well as the same pole locus; that indicates why in considering the zero locus or pole locus of the general cross-ratio $\mathfrak{Q}(a,b;t,t_0)$ the particular parameter value t_0 is essentially irrelevant.

Corollary 13.7 *For any fixed points $a,b \in M$ and $t_0 \in J(M)$ for which the general cross-ratio function $\mathfrak{Q}(a,b;t,t_0)$ is a nontrivial meromorphic function of the variable $t \in J(M)$ its zero locus V_a and pole locus V_b satisfy*

$$W_g^1 \subset V_a \cap V_b \subset J(M). \tag{13.75}$$

Proof: If $t = w_{z_0}(\mathfrak{d}) \in W_g^1 \subset J(M)$ for a divisor $\mathfrak{d} \in G_g^1 \subset M^{(g)}$ then $\gamma(\zeta_{\mathfrak{d}}) \geq 2$ so there are two linearly independent holomorphic cross-sections $h_1, h_2 \in \Gamma\big(M, \mathcal{O}(\zeta_{\mathfrak{d}})\big)$. The holomorphic cross-section $h = h_2(a)h_1 - h_1(a)h_2 \in \Gamma\big(M, \mathcal{O}(\zeta_{\mathfrak{d}})\big)$ vanishes at the point a so its divisor has the form $\mathfrak{d}(h) = a + a_1 + \cdots + a_{g-1}$ for some points $a_j \in M$; hence the image of this divisor under the Abel-Jacobi mapping is $w_{z_0}\big(\mathfrak{d}(h)\big) = w_{z_0}(a + a_1 + \cdots + a_{g-1}) \in w_{z_0}(a) + W_{g-1} = V_a$. Since $h \in \Gamma\big(M, \mathcal{O}(\zeta_{\mathfrak{d}})\big)$ the divisor $\mathfrak{d}(h)$ is linearly equivalent to the divisor $\mathfrak{d}$; so from Abel's Theorem, Theorem 5.10, it follows that $w_{z_0}\big(\mathfrak{d}(h)\big) = w_{z_0}(\mathfrak{d})$, hence $t = w_{z_0}(\mathfrak{d}) = w_{z_0}\big(\mathfrak{d}(h)\big) \subset V_a$. That is the case for any point $t \in W_g^1$, and therefore $W_g^1 \subset V_a$. The pole locus V_b of the general cross-ratio function $\mathfrak{Q}(a,b;t,t_0)$ is the zero locus of the general cross-ratio function $\mathfrak{Q}(b,a;t,t_0)$ so V_b^- also contains W_g^1, and that suffices for the proof.

The assertion of the preceding corollary that $W_g^1 \subset V_a = w_{z_0}(a) + W_{g-1}$ for all points $a \in M$ of course is equivalent to the assertion that that $W_g^1 - t \subset W_{g-1}$ for all $t \in W_1$; the converse is also true, and both assertions can be derived more directly as follows.

Lemma 13.8 *If $t \in J(M)$ is a point in the Jacobi variety $J(M)$ of a compact Riemann surface M of genus $g > 0$ then*

$$W_g^1 - t \subset W_{g-1} \quad \textit{if and only if} \quad t \in W_1. \tag{13.76}$$

Proof: Since $W_g^1 = k - W_{g-2}$ by the Riemann-Roch Theorem in the form of Theorem 12.23, where $k \in J(M)$ is the canonical point of the Jacobi variety, it follows that $W^1_g - t \subset W_{g-1}$ if and only if $k - t - W_{g-2} \subset W_{g-1}$ or equivalently if and only if

$$t + W_{g-2} \subset k - W_{g-1} = W_{g-1}, \tag{13.77}$$

where the last equality is the identity (12.47) for $\nu = 0$, a consequence of the Riemann-Roch theorem. In terms of the Martens differential operator on subvarieties of the Jacobian variety, as defined in (12.26), the preceding equation (13.77) is equivalent to $t \in W_{g-1} \ominus W_{g-2}$; and since $W_{g-1} \ominus W_{g-2} = W_1$ by Theorem 12.11 (iv), that suffices for the proof.

The set of common zeros of a collection of general cross-ratio functions also can be identified with a standard holomorphic subvariety of the Jacobi variety. More generally, for any fixed points $b \in \widetilde{M}, t_0 \in \mathbb{C}^g$ for which the general cross-ratio function $\mathfrak{Q}(z, b; t, t_0)$ is a nontrivial meromorphic relatively automorphic function of the variables $z \in \widetilde{M}, t \in \mathbb{C}$, let

$$\mathfrak{Q}^{(j)}(z, b; t, t_0) = \frac{\partial^j \mathfrak{Q}(z, b; t, t_0)}{\partial z^j} \tag{13.78}$$

be the derivatives of the function $\mathfrak{Q}(z, b; t, t_0)$ with respect to a local coordinate $z \in \widetilde{M}$; and for any divisor $\mathfrak{d} = \sum_{\nu=1}^{s} n_\nu a_\nu \in \widetilde{M}^{(r)}$ of degree $r = \sum_{\nu=1}^{s} n_\nu$ where $a_\nu \in \widetilde{M}$ represent distinct points on the Riemann surface M let

$$\begin{aligned} \widetilde{V}_{\mathfrak{d}} = \Big\{ t \in \mathbb{C}^g \,\Big|\, & \mathfrak{Q}(a_\nu, b; t, t_0) = \mathfrak{Q}^{(1)}(a_\nu, b; t, t_0) = \cdots \\ & \cdots = \mathfrak{Q}^{(n_\nu - 1)}(a_\nu, b; t, t_0) = 0 \ \text{ for } \ 1 \le \nu \le s \Big\} \subset \mathbb{C}^g. \end{aligned} \tag{13.79}$$

For the divisor $\mathfrak{d} = 1 \cdot a$ of degree 1 the subset $\widetilde{V}_{\mathfrak{d}}$ is just the subvariety $\widetilde{V}_a$ of (13.67); and for a divisor $\mathfrak{d} = \sum_{\nu=1}^{r} a_\nu$ where the points $a_\nu \in \widetilde{M}$ represent r distinct points on M the definition (13.79) takes the simpler form

$$\widetilde{V}_{\mathfrak{d}} = \Big\{ t \in \mathbb{C}^g \,\Big|\, \mathfrak{Q}(a_\nu, b; t, t_0) = 0 \ \text{ for } \ 1 \le \nu \le r \Big\} \subset \mathbb{C}^g. \tag{13.80}$$

The general cross-ratio function satisfies (13.65) for any arbitrariy lattice vector $\lambda \in \mathcal{L}(\Omega)$ and differentiating that equation repeatedly shows that

$$\begin{aligned} \mathfrak{Q}(a_\nu, b; t + \lambda, t_0) &= \rho_{z,a_\nu}(\lambda) \mathfrak{Q}(a_\nu, b; t, t_0) \\ \mathfrak{Q}^{(1)}(a_\nu, b; t + \lambda, t_0) &= \rho_{z,a_\nu}(\lambda) \mathfrak{Q}^{(1)}(a_\nu, b; t, t_0) + \\ &\qquad + \tfrac{\partial}{\partial z} \rho_{z,a_\nu}(\lambda) \mathfrak{Q}(a_\nu, b; t, t_0) \\ &\text{and so on.} \end{aligned} \tag{13.81}$$

In parallel with the discussion of the Brill-Noether matrix it follows that the zero locus $\widetilde{V}_{\mathfrak{d}}$ of these derivatives is invariant under translation through lattice vectors $\lambda \in \mathcal{L}(\Omega)$ so describes a holomorphic subvariety

$$V_{\mathfrak{d}} = \widetilde{V}_{\mathfrak{d}} / \mathcal{L}(\Omega) \subset J(M). \tag{13.82}$$

Theorem 13.9 (i) *For any divisor $\mathfrak{d} = \sum_{\nu=1}^{s} n_\nu a_\nu$ on $\widetilde{M}$ of degree r, where a_ν represent distinct points of M and $1 \le r \le g$, and for any points $b \in \widetilde{M},\ t_0 \in \mathbb{C}^g$ for which the general cross-ratio functions $\mathfrak{Q}(a_\nu, b; t, t_0)$ for all ν are nontrivial meromorphic functions of the variable $t \in \mathbb{C}^g$, the subvariety $V_\mathfrak{d} \subset J(M)$ has the form*

$$V_\mathfrak{d} = \Big(w_{z_0}(\mathfrak{d}) + W_{g-r}\Big) \cup W_g^1 \subset J(M); \tag{13.83}$$

it is the union of two irreducible holomorphic subvarieties of $J(M)$, one of dimension $g-r$ and the other of dimension $g-2$, and $w_{z_0}(\mathfrak{d}) + W_{g-r} \subset W_g^1$ if and only if $w_{z_0}(\mathfrak{d}) \in W_r^1$.
(ii) *If $n_\nu = 1$ for all ν the subvariety $V_\mathfrak{d}$ can be described alternatively as the intersection*

$$V_\mathfrak{d} = \bigcap_{\nu=1}^{r} \Big(w_{z_0}(a_\nu) + W_{g-1}\Big). \tag{13.84}$$

Proof: (i) The restriction of the general cross-ratio function to the complement of the subvariety $\widetilde{W}_g^1 \subset \mathbb{C}^g$ is identified with the restriction of the product cross-ratio function to the complement of the subvariety $\overset{\bowtie}{G}{}_g^1 \subset \overset{\bowtie}{M}{}^{(g)}$ through the biholomorphic mapping (13.58), under which a divisor $\mathfrak{d} \in \overset{\bowtie}{M}{}^{(g)}$ corresponds to the point $t = \overset{\bowtie}{w}{}_{z_0}^{(g)}(\mathfrak{d}) \in \mathbb{C}^g$. Therefore the restriction $\widetilde{V}_\mathfrak{d}\Big|\Big(\mathbb{C}^g \sim \widetilde{W}_g^1\Big)$ can be identified with the restriction $\widetilde{X}_\mathfrak{d}\Big|\Big(\overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}_g^1\Big)$, where the subvariety $\widetilde{X}_\mathfrak{d} \subset \overset{\bowtie}{M}{}^{(g)}$ is defined by the formulas analogous to (13.79) but involving the derivatives of the product cross-ratio function rather than of the general cross-ratio function. Since the product cross-ratio is defined as the product (13.2) it follows that the subvariety $\widetilde{X}_\mathfrak{d} \subset \overset{\bowtie}{M}{}^{(g)}$ can be described as the subset consisting of those divisors $z_1 + \cdots + z_g \in \overset{\bowtie}{M}{}^{(g)}$ such that $Q(z, b; z_1, \ldots, z_g; b_1, \ldots, b_g) = \prod_{k=1}^{g} q(z, b; z_k, b_k)$ as a function of the variable $z \in \widetilde{M}$ has zeros of order n_ν at the points $z = a_\nu$. Each factor $q(z, b; z_k, b_k)$ has a simple zero at the point $z = z_k$ and no other zeros, so the product has the desired zeros precisely when the divisor $z_1 + \cdots + z_g$ contains the divisor $\mathfrak{d}$, that is to say, precisely when $z_1 + \cdots + z_g = \mathfrak{d} + \mathfrak{d}'$ for some divisor $\mathfrak{d}' \in \overset{\bowtie}{M}{}^{(g-r)}$; and the image of such divisors under the holomorphic mapping $\overset{\bowtie}{w}{}_{z_0}^{(g)} : \overset{\bowtie}{M}{}^{(g)} \longrightarrow \mathbb{C}^g$ is the subset $\overset{\bowtie}{w}_{z_0}(\mathfrak{d}) + \widetilde{W}_{g-r} \subset \mathbb{C}^g$, which represents the subvariety $w_{z_0}(\mathfrak{d}) + W_{g-r} \subset J(M)$. By construction this subvariety is contained in $V_\mathfrak{d}$ and contains the complement $V_\mathfrak{d} \sim \big(V_\mathfrak{d} \cap W_g^1\big)$; but since $\mathfrak{Q}(z, b; t, t_0) = 0$ for all $z \in \widetilde{M}$ for any fixed point $t \in W_g^1$ as a consequence of Corollary 13.7, it follows that $W_g^1 \subset V_\mathfrak{d}$ so that altogether the subvariety $V_\mathfrak{d} \subset J(M)$ has the form (13.83). Here W_{g-r} is an irreducible holomorphic subvariety of dimension $g-r$ and as already observed $W_g^1 = k - W_{g-2}$ so W_g^1 is an irreducible holomorphic subvariety of dimension $g-2$. The inclusion $w_{z_0}(\mathfrak{d}) + W_{g-r} \subset W_g^1 = k - W_{g-2}$, or equivalently

$$k - w_{z_0}(\mathfrak{d}) - W_{g-r} \subset W_{g-2},$$

can be written in Marten's notation, discussed on page 322, as the inclusion $k - w_{z_0}(\mathfrak{d}) \in W_{g-2} \ominus -W_{g-r}$; and $W_{g-2} \ominus -W_{g-r} = W^{g-r}_{2g-r-2}$ by Theorem 12.24 while $W^{g-r}_{2g-r-2} = k - W^1_r$ by the Riemann-Roch Theorem in the form of Theorem 12.23. Therefore the inclusion amounts to the condidition that $w_{z_0}(\mathfrak{d}) \in W^1_r$.
(ii) If $n_\nu = 1$ the points a_ν are distinct, so as in (13.80) the subvariety $V_\mathfrak{d}$ is the locus of common zeros of the r meromorphic functions $\mathfrak{Q}(a_\nu, b; t, t_0)$; thus $V_\mathfrak{d}$ is the intersection of the zero loci of these two functions, and since the zero locus of the function $\mathfrak{Q}(a_\nu, b; t, t_0)$ is the subvariety $\widetilde{w}_{z_0}(a_\nu) + W_{g-1}$, that demonstrates (13.86), and thereby concludes the proof.

Some special cases of the preceding theorem are worth stating explicitly here for emphasis; the first is an extension of the result in Corollary 13.7.

Corollary 13.10 (i) *For any divisor $\mathfrak{d} = a_1 + a_2 \in M^{(2)}$, and for any points $b \in M, t_0 \in J(M)$ for which the general cross-ratio functions $\mathfrak{Q}(a_\nu, b; t, t_0)$ for $\nu = 1, 2$ are nontrivial meromorphic functions of the variable $t \in \mathbb{C}^g$,*

$$V_\mathfrak{d} = \Big(w_{z_0}(\mathfrak{d}) + W_{g-2}\Big) \cup W^1_g \subset J(M). \tag{13.85}$$

The subvarieties $w_{z_0}(\mathfrak{d}) + W_{g-2}$ and W^1_g are irreducible holomorphic subvarieties of dimension $g-2$ in $J(M)$, and are equal if and only if M is a hyperelliptic Riemann surface and $w_{z_0}(\mathfrak{d}) \in W^1_2$.
(ii) *If $a_1 \neq a_2$ in M then the subvariety $V_\mathfrak{d}$ can be described alternatively as*

$$V_\mathfrak{d} = \Big(w_{z_0}(a_1) + W_{g-1}\Big) \cap \Big(w_{z_0}(a_2) + W_{g-1}\Big). \tag{13.86}$$

Proof: This is just the special case $r = 2$ of the preceding Theorem 13.9, together with the observations first that the two subvarieties $w_{z_0}(\mathfrak{d}) + W_{g-2}$ and W^1_g are irreducible holomorphic subvarieties of the same dimension so an inclusion of one in the other is an equality of the two divisors, and second that there exist divisors $\mathfrak{d} \in W^1_2$ if and only if the Riemann surface M is hyperelliptic. That suffices for the proof.

Any intersection $\Big(t_1 + W_{g-1}\Big) \cap \Big(t_2 + W_{g-1}\Big)$ for points $t_1, t_2 \in J(M)$ is a holomorphic subvariety of the complex torus $J(M)$. It can be demonstrated[4] that this intersection is homologous to twice the homology class carried by W_{g-2}, as would be expected from (13.85). However in general this intersection is an irreducible subvariety of the torus $J(M)$; at least it follows from Lemma 13.8 that a translate of $W^1_g = k - W_{g-2}$ is contained in the subvariety W_{g-1} if and only it is the translate $W^1_g - t$ for a point $t \in W_{g-1}$, hence by the preceding lemma if and only if it is an irreducible component of the interssection (13.86).

Corollary 13.11 *For any divisor $\mathfrak{d} = \sum_{\nu=1}^{s} n_\nu a_\nu \in M^{(g-1)}$, of distinct points $a_\nu \in M$ and for any points $b \in M, t_0 \in J(M)$ for which the general cross-ratio*

[4] This is one case of Poincaré's fomula, as discussed on page 350 of Griffiths and Harris, *Principles of Algebraic Geometry*, and will be discussed later here as well.

functions $\mathfrak{Q}(a_\nu, b; t, t_0)$ *for* $1 \le \nu \le g$ *are nontrivial meromorphic functions of the variable* $t \in \mathbb{C}^g$*, the subvariety* $V_{\mathfrak{d}}$ *has the form*

$$V_{\mathfrak{d}} = \Big(w_{z_0}(\mathfrak{d}) + W_1\Big) \cup W_g^1 \subset J(M); \tag{13.87}$$

thus it is the union of an irreducible curve $w_{z_0}(\mathfrak{d}) + W_1$ *and the irreducible holomorphic subvariety* W_g^1 *of dimension* $g-2$*, and* $w_{z_0}(\mathfrak{d}) + W_1 \subset W_g^1$ *if and only if* $w_{z_0}(\mathfrak{d}) \in W_{g-1}^1$.
(ii) If $n_\nu = 1$ *for all* ν *the subvariety* $V_{\mathfrak{d}}$ *can be described alternatively as the intersection*

$$V_{\mathfrak{d}} = \bigcap_{\nu=1}^{g-1} \Big(w_{z_0}(a_\nu) + W_{g-1}\Big). \tag{13.88}$$

Proof: This is just the special case of the preceding Theorem 13.9 for which $r = g-1$, so no further proof is required.

The curve $w_{z_0}(\mathfrak{d})$ in the preceding corollary is the Riemann surface M imbedded in its Jacobi variety through the Abel-Jacobi mapping as usual; but the interest here is that the subvariety $M \subset J(M)$ is described as an intersection of $g-1$ translates of the hypersurface W_{g-1} outside of the exceptional subvariety W_g^1, which is contained in each of the translates of W_{g-1}.

Corollary 13.12 *If* $\mathfrak{d} = \sum_{\nu=1}^{g+1} a_\nu$ *is a divisor of degree* $g+1$ *consisting of distinct points on* M *then*

$$V_{\mathfrak{d}} = W_g^1 \subset J(M) \tag{13.89}$$

and

$$\bigcap_{\nu=1}^{g+1} \Big(w_{z_0}(a_\nu) + W_{g-1}\Big) = W_g^1. \tag{13.90}$$

Proof: In the proof of Theorem 13.9 the restriction $\widetilde{X}_{\mathfrak{d}}\Big|\Big(\overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}_g^1\Big)$ is the set of points in $\overset{\bowtie}{M}{}^{(g)} \sim \overset{\bowtie}{G}{}_g^1$ at which the product $\Pi_{\nu=1}^g(z, b; z_\nu, b_\nu)$ vanishes at points representing $g+1$ distinct points of M, and there are no such points. Hence $V_{\mathfrak{d}}$ consists just of the subvariety W_g^1; and since $V_{\mathfrak{d}}$ is still the intersection (13.90) that intersection too is just the subvariety W_g^1, which suffices for the proof.

13.3 The Role of the Classical Theta Function

The discussion in the first section of this chapter focused on the intrinsic cross-ratio function; but to relate it to more classical treatments it is necessary to consider instead the canonical cross-ratio function $\hat{q}(z_1, z_2; z_3, z_4)$, which is normalized in terms of a marking of the Riemann surface M. As discussed in Appendix D.1, a marking involves a choice of a base point $z_0 \in \widetilde{M}$ and a collection of paths $\tilde{\alpha}_j \subset \widetilde{M}$ from z_0 to $A_j z_0$ and $\tilde{\beta}_j \subset \widetilde{M}$ from z_0 to $B_j z_0$, where $A_j, B_j \in \Gamma$ are generators of the covering translation group Γ of the Riemann surface M. Translates of the paths $\tilde{\alpha}_j$ $\tilde{\beta}_j$ form the boundary of a fundamental domain $\Delta \subset \widetilde{M}$ for the action of the covering translation group Γ on $\widetilde{M}$; and the Riemann surface $M = \widetilde{M}/\Gamma$ itself can be recaptured by identifying appropriate boundary segments of Δ. The boundary $\partial\Delta$ indeed has the explicit form

(13.91)
$$\partial\Delta = \sum_{j=1}^{g} \Big(C_1 \cdots C_{j-1}\tilde{\alpha}_j + C_1 \cdots C_{j-1} A_j \tilde{\beta}_j - C_1 \cdots C_j B_j \tilde{\alpha}_j - C_1 \cdots C_j \tilde{\beta}_j \Big)$$

as in (D.2) in Appendix D.1, but written additively here. Associated to the marking is the canonical basis $\{\omega_i\}$ for the holomorphic abelian differentials, defined by having the periods $\omega_i(A_j) = \delta^i_j$. The period marix of the surface for this marking is the matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ where $Z = \{z_{ij}\}$ for the periods $z_{ij} = \omega_i(B_j)$; and the matrix Z is an element in the Siegel upper half-space $\mathfrak{H}_g$ of rank g, as discussed in Appendix F.3. The associated abelian integrals are defined by $w_i(z) = w_i(z, z_0) = \int_{z_0}^{z} \omega_i$, so that $w_i(z_0) = 0$ and $\int_{\alpha_j} \omega_i = \delta^i_j$ whlle $\int_{\beta_j} \omega_i = z_{ij}$. The Jacobi variety of the marked Riemann surface M of genus $g > 0$ is the quotient torus $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$ for the lattice subgroup $\mathcal{L}(\Omega) = \Omega\mathbb{Z}^{2g} = \{\, \mu + Z\nu \mid \mu, \nu \in \mathbb{Z}^g \,\}$.

Another invariant of the abelian differentials plays a significant role in the discussion of theta functions. In addition to the regular periods of the holomorphic abelian differentials there are the *quadratic periods*, defined in terms of the marking of the surface as the integrals

$$r_{ij} = \int_{\tilde{\alpha}_j} w_i(z)\omega_j(z) \ \text{ for } \ 1 \le i, j \le g. \tag{13.92}$$

The vector

$$\mathcal{R} = \{R_i\} \in \mathbb{C}^g \ \text{ where } \ R_i = \sum_{j=1}^{g} \big(r_{ij} + z_{ij} \big) - \tfrac{1}{2} z_{ii}, \tag{13.93}$$

defined in terms of the ordinary periods z_{ij} and the quadratic periods r_{ij} of the holomorphic abelian differentials, is called the *Riemann vector* of the marked Riemnann surface; the image of this vector in the Jacobi variety $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$ is a point

$$\mathfrak{R} = \mathcal{R} \mod \mathcal{L}(\Omega) \in J(M) \tag{13.94}$$

called the *Riemann point* of the surface M.

The canonical cross-ratio function $\hat{q}(z_1, z_2; z_3, z_4)$ is characterized in Theorem 5.16 as the meromorphic function of the variables $z_j \in \widetilde{M}$ that has the divisor and symmetries of the intrinsic cross-ratio function but is normalized by the alternate period conditions

$$\hat{q}(Tz_1, z_2; z_3, z_4) = \hat{\rho}_{z_3,z_4}(T)\hat{q}(z_1, z_2; z_3, z_4) \text{ where } \hat{\rho}_{z_3,z_4}(A_j) = 1 \quad \text{and} \quad \hat{\rho}_{z_3,z_4}(B_j) = \exp 2\pi i\, w_j(z_3, z_4) \tag{13.95}$$

in terms of the integrals $w_j(z_3, z_4) = \int_{z_4}^{z_3} \omega_j = w_j(z_3) - w_j(z_4)$ of the canonical abelian differentials ω_j. The associated canonical general cross-ratio function $\widetilde{\mathfrak{Q}}_c(a, b; t, t_0)$ is defined just as for the general cross-ratio function, but in terms of products of the canonical rather than the intrinsic cross-ratio functions. The basic results in the preceding part of this chapter carry over with just the corresponding change in the periods. In particular in the proof of Theorem 13.6 (iii) equation (13.71) holds for the canonical product cross-ratio function, with $\hat{\rho}_{z,a}(T)$ as in (13.95) in place of $\rho_{z,a}(T)$, so equation (13.72) takes the form

$$\widetilde{\mathfrak{Q}}_c(z, a;\ t^+ + \lambda,\ t^-) = \hat{\rho}_{z,a}(T)\ \mathfrak{Q}_c(z, a;\ t^+, t^-) \tag{13.96}$$

where $\lambda = \omega(T) \in \mathcal{L}(\Omega)$. If $T = \left(\prod_{j=1}^g A_j^{\mu_j}\right) \cdot \left(\prod_{j=1}^g B_j^{\nu_j}\right) \cdot C$ for some commutator $C \in [\Gamma, \Gamma]$ then $\hat{\rho}_{z,a}(T) = \hat{\rho}_{z,a}(\prod_{j=1}^g B_j^{\nu_j}) = \exp 2\pi i \sum_{j=1}^g \nu_j w_j(z, a)$ and also $\lambda = \omega(T) = \mu + Z\nu$ where $\mu = \{\mu_j\} \in \mathbb{Z}^g$ and $\nu = \{\nu_j\} \in \mathbb{Z}^g$. The preceding equation (13.96) then can be rewritten as the form of Theorem 13.6 (iii) for the canonical cross-ratio functions, thus as the condition that for any $\mu, \nu \in \mathbb{Z}^g$

$$\widetilde{\mathfrak{Q}}_c(z, a;\ t^+ + \mu + Z\nu,\ t^-) = \hat{\rho}_{z,a}(\mu + Z\nu)\ \mathfrak{Q}_c(z, a;\ t^+, t^-) \tag{13.97}$$

where

$$\hat{\rho}_{z,a}(\mu + Z\nu) = \exp 2\pi i \sum_{j=1}^g \nu_j w_j(z, a). \tag{13.98}$$

The result of part (ii) of Theorem 13.6 is changed correspondingly to hold for the canonical general cross-ratio function just by replacing the factor (13.64) by (13.95); otherwise that theorem holds as stated for the canonical general cross-ratio function.

The classical theta function associated to a period matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ for any matrix $Z \in \mathfrak{H}_g$ is defined by the series expansion

$$\Theta(t; Z) = \sum_{n \in \mathbb{Z}^g} \exp 2\pi i \Big(\tfrac{1}{2}\, {}^t n Z n + {}^t n\, t \Big), \tag{13.99}$$

as in (G.1) in Appendix G. It is a holomorphic even function in the variable $t \in \mathbb{C}^g$, in the sense that

$$\Theta(-t; Z) = \Theta(t; Z); \tag{13.100}$$

and it is a relatively automorphic function of the variable $t \in \mathbb{C}^g$ for the action of the lattice subgroup $\mathcal{L}(\Omega)$ with the factor of automorphy

$$\Xi(\mu + Z\nu, t) = \exp -2\pi i \Big(\tfrac{1}{2}\, {}^t\nu Z\, \nu + {}^t\nu t \Big) \tag{13.101}$$

for any $\mu, \nu \in \mathbb{Z}^g$, as in (G.7) in Appendix G. The zero locus of the classical theta function, called the *theta variety*, is the holomorphic subvariety

$$\widetilde{V}_\Theta = \left\{ s \in \mathbb{C}^g \;\middle|\; \Theta(s; Z) = 0 \right\} \subset \mathbb{C}^g \tag{13.102}$$

of dimension $g-1$ in $\mathbb{C}^g$. The theta variety is invariant under the lattice subgroup $\mathcal{L}(\Omega)$, since the theta function is a relatively automorphic function, so it determines a holomorphic subvariety $V_\Theta = \widetilde{V}_\Theta/\mathcal{L}(\Omega) \subset J(M)$, also called the theta variety. These subvarieties of course satisfy the symmetry conditions $\widetilde{V}_\Theta = -\widetilde{V}_\Theta$ and $V_\Theta = -V_\Theta$ as a consequence of (13.100).

If $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ is the period matrix of a compact Riemann surface the composition of the classical theta function $\Theta(t; Z)$ and the translate through a vector $s \in \mathbb{C}^g$ of the holomorphic mapping $\widetilde{w}_{z_0} : \widetilde{M} \longrightarrow \mathbb{C}^g$ of (13.26) defined by the abelian integrals $w_i(z) = w_i(z, z_0) = \int_{z_0}^{z} \omega_i$ is a well defined holomorphic function of the variable $z \in \widetilde{M}$ called the *Riemann theta function* and denoted by

$$\vartheta_s(z) = \Theta\big(s + \widetilde{w}_{z_0}(z); Z\big). \tag{13.103}$$

If $\Theta(s) \neq 0$ then $\vartheta_s(z_0) \neq 0$ so the function $\vartheta_s(z)$ does not vanish identically. However for some values $s \in \mathbb{C}^g$ the Riemann theta function $\vartheta_s(z)$ does vanish identically in the variable $z \in \widetilde{M}$; the functions $\vartheta_s(z)$ for such parameter values are the *trivial* Riemann theta functions. To examine this situation introduce the subset

$$\widetilde{X}_\Theta = \left\{ s \in \mathbb{C}^g \;\middle|\; \vartheta_s(z) = \Theta\big(s + \widetilde{w}_0(z); Z\big) = 0 \text{ for all } z \in \widetilde{M} \right\} \subset \widetilde{V}_\Theta. \tag{13.104}$$

This set too is invariant under the lattice subgroup $\mathcal{L}(\Omega)$ so it determines the corresonding subset $X_\Theta = \widetilde{X}_\Theta/\mathcal{L}(\Omega) \subset J(M)$. Another characterization of the latter set, familiar from the algebraic conditions on subvarieties of the Jacobi variety considered in Chapter 12, is that

$$X_\Theta = \left\{ s \in J(M) \;\middle|\; s + W_1 \subset V_\Theta \right\} = V_\Theta \ominus W_1. \tag{13.105}$$

Lemma 13.13 *For any marked Riemann surface M of genus $g > 0$ the subset $X_\Theta \subset J(M)$ is a holomorphic subvariety of $J(M)$ of dimension at most $g-2$.*

Proof: For each fixed point $z \in \widetilde{M}$ the equation $\Theta\big(s + \widetilde{w}_0(z); Z\big) = 0$ in the variable $s \in J(M)$ describes a holomorphic subvariety of $J(M)$; consequently X_Θ is the intersection of a collection of holomorphic subvarieties of $J(M)$ so is itself a holomorphic subvariety of $J(M)$. Since X_Θ is a proper subvariety of $J(M)$ necessarily $\dim X_\Theta \leq g-1$; so to prove the lemma it is enough just to show that $\dim X_\Theta \neq g-1$. For any irreducible component $X_0 \subset X_\Theta$ the sum

$$X_0 + W_1 = \left\{ s + t \;\middle|\; s \in X_0,\ t \in W_1 \right\} \subset J(M)$$

is an irreducible holomorphic subvariety of $J(M)$ by Remmert's proper mapping theorem[5], since it is the image of the compact irreducible holomorphic variety $X_0 \times W_1$ under the holomorphic mapping that takes a point $(s,t) \in X_0 \times W_1$ to the sum $s + t \in J(M)$. Since $X_0 + W_1 \subset V_\Theta$ by its definition it follows that $\dim(X_0 + W_1) \le g - 1$. If $\dim X_0 = g - 1$ then since X_0 is an irreducible holomorphic subvariety of dimension $g-1$ contained in the irreducible holomorphic subvariety $X_0 + W_1$ of dimension $g - 1$ it follows that $X_0 = X_0 + W_1$. By iterating this equation it follows further that $X_0 = X_0 + W_1 + W_1 + \cdots + W_1$; but since $W_1 + \cdots + W_1 = J(M)$ if there are g summands W_1 then $X_0 = J(M)$, an evident contradiction which serves to conclude the proof.

Lemma 13.14 *A nontrivial Riemann theta function ϑ_s for a marked compact Riemann surface M is a relatively automorphic function for the action of the covering translation group Γ on $\widetilde{M}$ with the factor of automorpny*

$$\xi_s(T,z) = \exp -2\pi i \left(\tfrac{1}{2}\, {}^t\nu Z\, \nu + {}^t\nu(s + \widetilde{w}_{z_0}(z))\right) \tag{13.106}$$

for any $T = \left(\prod_{j=1}^g A_j^{\mu_j}\right) \cdot \left(\prod_{j=1}^g B_j^{\nu_j}\right) \cdot C \in \Gamma$, where $A_j, B_j \in \Gamma$ are the generators of Γ associated to the marking and $C \in [\Gamma,\Gamma]$.

Proof: For the transformation $T \in \Gamma$ as in the statement of the lemma the canonical abelian integrals satisfy $\widetilde{w}_{z_0}(Tz) = \widetilde{w}_{z_0}(z) + \mu + Z\nu$ and consequently

$$\begin{aligned} \vartheta_s(Tz) &= \Theta\big(s + \widetilde{w}_{z_0}(Tz); Z\big) = \Theta\big(s + \widetilde{w}_{z_0}(z) + \mu + Z\nu; Z\big) \\ &= \Xi(\mu + Z\nu, s + \widetilde{w}_{z_0}(z))\Theta\big(s + \widetilde{w}_{z_0}(z); Z\big) \\ &= \exp -2\pi i\Big(\tfrac{1}{2}\, {}^t\nu Z\, \nu + {}^t\nu\big(s + \widetilde{w}_{z_0}(z)\big)\Big) \cdot \vartheta_s(z), \end{aligned}$$

and that suffices for the proof.

Since a nontrivial Riemann theta function ϑ_s is a relatively automorphic function for the action of the covering translation group Γ its divisor $\mathfrak{d}(\vartheta_s)$ on $\widetilde{M}$ is invariant under Γ so it can be viewed as a divisor $\mathfrak{d}_M(\vartheta_s)$ on the Riemann surface M itself.

Theorem 13.15 *The divisor $\mathfrak{d}_M(\vartheta_s)$ of a nontrivial Riemann theta function ϑ_s on a marked compact Riemann surface M of genus g is a positive divisor and* $\deg \mathfrak{d}_M(\vartheta_s) = g$.

Proof: The divisor $\mathfrak{d}_M(\vartheta_s)$ is a positive divisor since the Riemann theta function is holomorphic. The degree $\deg \mathfrak{d}_M(\vartheta_s)$ is the degree of that part of the divisor $\mathfrak{d}(\vartheta_s)$ that is contained in the fundamental domain $\Delta \subset \widetilde{M}$. By a suitable choice of the base point $z_0 \in \widetilde{M}$ it can be assumed that the divisor $\mathfrak{d}(\vartheta_s)$ is disjoint from the boundary $\partial\Delta$; hence by the Cauchy integral formula

$$\deg \mathfrak{d}_M(\vartheta_s) = \frac{1}{2\pi i}\int_{\partial\Delta} d\log\vartheta_s. \tag{13.107}$$

[5] Remmert's proper mapping theorem is discussed on page 423 of appendix A.3.

Since the boundary $\partial\Delta$ has the explicit form (13.91) it follows that

(13.108)
$$\begin{aligned}\int_{\partial\Delta} d\log\vartheta_s &= \sum_{j=1}^{g}\int_{C_1\cdots C_{j-1}\tilde\alpha_j - C_1\cdots C_jB_j\tilde\alpha_j + C_1\cdots C_{j-1}A_j\tilde\beta_j - C_1\cdots C_j\tilde\beta_j} d\log\vartheta_s(z)\\ &= \sum_{j=1}^{g}\int_{\tilde\alpha_j}\Big(d\log\vartheta_s(C_1\cdots C_{j-1}z) - d\log\vartheta_s(C_1\cdots C_jB_jz)\Big)\\ &\qquad + \sum_{j=1}^{g}\int_{\tilde\beta_j}\Big(d\log\vartheta_s(C_1\cdots C_{j-1}A_jz) - d\log\vartheta_s(C_1\cdots C_j)\Big)\end{aligned}$$

The factor of automorphy of the relatively automorphic function $\vartheta_s(z)$ has the form (13.106) by the preceding lemma, so in particular

(13.109)
$$\begin{aligned}&\vartheta_s(Cz) = \vartheta_s(A_jz) = \vartheta_s(z) \text{ for } A_j\in\Gamma,\ C\in[\Gamma.\Gamma] \text{ and}\\ &\vartheta_s(B_jz) = \exp -2\pi i\left(\tfrac12 z_{jj} + s_j + w_j(z,z_0)\right)\cdot\vartheta_s(z) \text{ for } B_j\in\Gamma\end{aligned}$$

where $Z = \{z_{ij}\}$; consequently

(13.110)
$$\begin{aligned}&d\log\vartheta_s(Cz) = d\log\vartheta_s(A_jz) = d\log\vartheta_s(z) \text{ and}\\ &d\log\vartheta_s(B_jz) = d\log\vartheta_s(z) - 2\pi i\omega_j(z).\end{aligned}$$

Therefore

$$\begin{aligned}&d\log\vartheta_s(C_1\cdots C_{j-1}z) - d\log\vartheta_s(C_1\cdots C_jB_jz)\\ &\qquad = d\log\vartheta_s(z) - d\log\vartheta_s(B_jz) = 2\pi i\omega_j(z)\end{aligned}$$

and

$$\begin{aligned}&d\log\vartheta_s(C_1\cdots C_{j-1}A_jz) - d\log\vartheta_s(C_1\cdots C_j)\\ &\qquad = d\log\vartheta_s(z) - d\log\vartheta_s(z) = 0.\end{aligned}$$

Upon substituting these observations into (13.108) it follows that

$$\deg\mathfrak{d}_M(\vartheta_s) = \frac{1}{2\pi i}\int_{\partial\Delta} d\log\vartheta_s = \sum_{j=1}^{g}\int_{\tilde\alpha_i}\omega_j(z) = g,$$

since $\int_{\tilde\alpha_i}\omega_i(z) = 1$, and that concludes the proof.

The image $w_{z_0}\big(\mathfrak{d}_M(\vartheta_s)\big)\in J(M)$ of the divisor $\mathfrak{d}_M(\vartheta_s)$ of the Riemann theta function under the Abel-Jacobi mapping $w_{z_0}: M\longrightarrow J(M)$ induced by the holomorphic abelian integrals as in (13.26) can be calculated quite explicitly by a straightforward modification of the Cauchy integral used in the proof of the preceding theorem, following Riemann.

Theorem 13.16 *For any marked Rieman surface M of genus $g > 0$ and any parameter $s \in \mathbb{C}^g \sim \widetilde{X}_\Theta$ the image $w_{z_0}\big(\mathfrak{d}_M(\vartheta_s)\big) \in J(M)$ of the divisor of ϑ_s under the Abel-Jacobi mapping is the point*

$$w_{z_0}\big(\mathfrak{d}_M(\vartheta_s)\big) = \mathfrak{R} - \mathfrak{s}, \tag{13.111}$$

where $\mathfrak{R} \in J(M)$ is the Riemann point of the Jacobi variety of M and $\mathfrak{s} \in J(M)$ is the image of the vector $s \in \mathbb{C}^g$ in $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$.

Proof: The divisor $\mathfrak{d}_M(\vartheta_s)$ of the Riemann theta function on M can be described as that part of the divisor $\mathfrak{d}(\vartheta_s)$ on $\widetilde{M}$ that is contained in the fundamental domain Δ, thus as a divisor

$$\mathfrak{d}_M(\vartheta_s) = \sum_{k=1}^{r} \nu_k\ a_k \quad \text{where } \nu_k > 0 \text{ and } a_k \in \Delta \subset \widetilde{M}. \tag{13.112}$$

By a suitable choice of the base point $z_0 \in \widetilde{M}$ it can be assumed that the divisor $\mathfrak{d}(\vartheta_s)$ is disjoint from the boundary $\partial\Delta$; hence by the Cauchy integral formula the image of the divisor (13.112) in $\mathbb{C}^g$ under the Abel-Jacobi mapping is the vector with the components

$$\widetilde{w}_i = \sum_{k=1}^{r} \nu_k w_i(a_k) = \frac{1}{2\pi i}\int_{\partial\Delta} w_i(z) d\log\vartheta_s(z). \tag{13.113}$$

Since the boundary $\partial\Delta$ has the explicit form (13.91) it follows that

(13.114)

$$\begin{aligned}
2\pi i\,\widetilde{w}_i &= \sum_{j=1}^{g} \int_{C_1\cdots C_{j-1}\tilde{\alpha}_j - C_1\cdots C_j B_j\tilde{\alpha}_j + C_1\cdots C_{j-1}A_j\tilde{\beta}_j - C_1\cdots C_j\tilde{\beta}_j} w_i(z)\, d\log\vartheta_s(z) \\
&= \sum_{j=1}^{g}\int_{\tilde{\alpha}_j}\Big(w_i(C_1\cdots C_{j-1}z) d\log\vartheta_s(C_1\cdots C_{j-1}z) \\
&\qquad\qquad - w_i(C_1\cdots C_jB_jz) d\log\vartheta_s(C_1\cdots C_jB_jz)\Big) \\
&\quad + \sum_{j=1}^{g}\int_{\tilde{\beta}_j}\Big(w_i(C_1\cdots C_{j-1}A_jz) d\log\vartheta_s(C_1\cdots C_{j-1}A_jz) \\
&\qquad\qquad - w_i(C_1\cdots C_jz) d\log\vartheta_s(C_1\cdots C_jz)\Big)
\end{aligned}$$

In view of (13.110) and the known periods of the holomorphic abelian differentials it follows that

$$\begin{aligned}
w_i(C_1\cdots C_{j-1}z) d\log\vartheta_s(C_1\cdots C_{j-1}z) &= w_i(z) d\log\vartheta_s(z) \\
w_i(C_1\cdots C_jB_jz) d\log\vartheta_s(C_1\cdots C_jB_jz)\Big) &= \Big(w_i(z) + z_{ij}\Big)\Big(d\log\vartheta_s(z) - 2\pi i\omega_j(z)\Big) \\
w_i(C_1\cdots C_{j-1}A_jz) d\log\vartheta_s(C_1\cdots C_{j-1}A_jz) &= \Big(w_i(z) + \delta^i_j\Big) d\log\vartheta_s(z) \\
w_i(C_1\cdots C_jz) d\log\vartheta_s(C_1\cdots C_jz) &= w_i(z) d\log\vartheta_s(z).
\end{aligned}$$

∎

Substituting these observations into (13.113) leads to the identity

(13.115)
$$\begin{aligned}2\pi i\,\widetilde{w}_i = &\sum_{j=1}^{g}\int_{\tilde\alpha_j}\Big(w_i(z)d\log\vartheta_s(z) - \big(w_i(z)+z_{ij}\big)\big(d\log\vartheta_s(z) - 2\pi i\omega_j(z)\big)\Big)\\ &\qquad + \sum_{j=1}^{g}\int_{\tilde\beta_j}\Big(\big(w_i(z)+\delta^i_j\big)d\log\vartheta_s(z) - w_i(z)d\log\vartheta_s(z)\Big)\\ = &\sum_{j=1}^{g}\int_{\tilde\alpha_j}\Big(2\pi i\,w_i(z)\omega_j(z) - z_{ij}\,d\log\vartheta_s(z) + 2\pi i\,z_{ij}\omega_j(z)\Big)\\ &\qquad + \sum_{j=1}^{g}\int_{\tilde\beta_j}\delta^i_j\,d\log\vartheta_s(z)\end{aligned}$$

Since $\vartheta_s(A_jz_0) = \vartheta_s(z_0)$ by (13.109) it follows that for any choice of a branch of $\log\vartheta_s(z)$ along the path $\tilde\alpha_j$

$$\int_{\tilde\alpha_j} d\log\vartheta_s(z) = \log\vartheta_s(A_jz_0) - \log\vartheta_s(z_0) = 2\pi i n_j \tag{13.116}$$

for some integer $n_j \in \mathbb{Z}$; for although the function $\vartheta_s(z)$ has the same value at the beginning and end points of the path $\tilde\alpha_j$ the analytic continuation of $\log\vartheta_s(z)$ along the path $\tilde\alpha_j$ may lead to a value that differs from $\vartheta_s(z_0)$ by some integral multiple of $2\pi i$. Similarly since $\vartheta_s(B_jz) = \exp -2\pi i\left(\frac{1}{2}z_{jj} + s_j + w_j(z)\right)\cdot\vartheta_s(z)$ by (13.109) while $w_j(z_0) = 0$ it follows that for any choice of a branch of $\log\vartheta_s(z)$ along the path $\tilde\beta_j$

$$\int_{\tilde\beta_j} d\log\vartheta_s(z) = \log\vartheta_s(B_jz_0) - \log\vartheta_s(z_0) = -2\pi i\Big(\tfrac{1}{2}z_{jj} + s_j + m_j\Big) \tag{13.117}$$

for some integer n_j. Substituting the auxiliary integrals (13.116), (13.117) and the quadratic period (13.92) into (13.115) shows that (13.115) takes the form

$$\widetilde{w}_i = \sum_{j=1}^{g}\Big(r_{ij} - z_{ij}n_j + z_{ij} - \delta^i_j\left(\tfrac{1}{2}z_{jj} + s_j + m_j\right)\Big). \tag{13.118}$$

Here $\sum_{j=1}^{g}(r_{ij} + z_{ij} - \delta^i_j\frac{1}{2}z_{jj}) = R_i$ are the components of the Riemann vector $\mathcal{R} \in \mathbb{C}^g$ defined in (13.93); and the image of this vector in the Jacobi variety $j(M)$ is the Riemann point $\mathfrak{R} \in J(M)$. The terms $\lambda_i = \sum_{j=1}^{g}(-z_{ij}n_j - \delta^i_jm_j)$ are the components of a vector $\lambda \in \mathcal{L}(\Omega)$ which maps to 0 in the Jacobi varietiy $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$. The vector $s = \{s_i\} \in \mathbb{C}^g$ represents a point $\mathfrak{s} \in J(M)$. Therefore (13.118) is just the equation $\widetilde{w}_i = R_i + \lambda_i - s_i$ among vectors in $\mathbb{C}^g$, which reduces to the equation (13.111) in the Jacobi variety; and that concludes the proof.

Corollary 13.17 *If $\mathfrak{s} \in J(M)$ is a point in the Jacobi variety $J(M) = \mathbb{C}^g/\mathcal{L}(\Omega)$ of a compact Riemann surface M of genus $g > 0$ and if $\mathfrak{s} \notin X_\Theta \cup (\mathfrak{R} - W_g^1)$ then $\Theta(\mathfrak{s}; Z) = 0$ if and only if $s \in \mathfrak{R} - W_{g-1}$.*

Proof: First suppose that $\Theta(s; Z) = 0$ for a point $s \in \mathbb{C}^g$ where $s \notin \widetilde{X}_\Theta$. The Riemann theta function $\vartheta_s(z) = \Theta\big(s + \widetilde{w}_{z_0}(z); Z\big)$ does not vanish identically in $z \in \widetilde{M}$ since $s \notin \widetilde{X}_\Theta$; so it follows from Theorems 13.15 and 13.16 that the divisor $\mathfrak{d}_M(\vartheta_s)$ is a positive divisor of degree g on M for which

$$w_{z_0}\big(\mathfrak{d}_M(\vartheta_s)\big) = \mathfrak{R} - \mathfrak{s}. \tag{13.119}$$

Now $\vartheta_s(z_0) = \Theta\big(s + \widetilde{w}_{z_0}(z_0); Z\big) = \Theta\big(s; Z\big) = 0$ since the abelian integrals are normalized by requiring that $\widetilde{w}_{z_0}(z_0) = 0$ and it is asssumed that $\Theta(s; Z) = 0$; thus the base point $z_0 \in \widetilde{M}$ represents one of the points in the divisor $\mathfrak{d}_M(\vartheta_s)$, so $\mathfrak{d}_M(\vartheta_s) = \mathfrak{d}'_M(\vartheta_s) + z_0$ where $\mathfrak{d}'_M(\vartheta_s)$ is a positive divisor of degree $g - 1$ for which $w_{z_0}\big(\mathfrak{d}_M(\vartheta_s)\big) = w_{z_0}\big(\mathfrak{d}'_M(\vartheta_s)\big)$. Substituting this in (13.119) shows that

$$\mathfrak{s} = \mathfrak{R} - w_{z_0}\big(\mathfrak{d}'_M(\vartheta_s)\big) \in \mathfrak{R} - W_{g-1}. \tag{13.120}$$

Conversely suppose that $\mathfrak{s} = \mathfrak{R} - w_{z_0}(\mathfrak{d}_0)$ where $\mathfrak{d}_0$ is a positive divisor of degree $g - 1$ on M and that $\mathfrak{s} \notin X_\Theta \cup (\mathfrak{R} - W_g^1)$, hence in particular that $w_{z_0}(\mathfrak{d}_0) = (\mathfrak{R} - \mathfrak{s}) \notin W_g^1$. If $\mathfrak{d} = \mathfrak{d}_0 + z_0$ then $\mathfrak{d}$ is a positive divisor of degree g on M, and since $w_{z_0}(\mathfrak{d}) = w_{z_0}(\mathfrak{d}_0)$ it is also the case that $w_{z_0}(\mathfrak{d}) \notin W_g^1$. Since $\mathfrak{s} \notin X_\Theta$ the function $\vartheta_s(z)$ does not vanish identically in the variable $z \in \widetilde{M}$, so by Theorems 13.15 and 13.16 again $\mathfrak{d}_M(\vartheta_s)$ is a positive divisor on M of degree g for which $\mathfrak{s} = \mathfrak{R} - w_{z_0}(\mathfrak{d}_M(\vartheta_s))$, hence

$$w_{z_0}\big(\mathfrak{d}_M(\vartheta_s)\big) = w_{z_0}(\mathfrak{d}). \tag{13.121}$$

Here $\deg \mathfrak{d}_M(\vartheta_s) = \deg \mathfrak{d} = g$ so by Abel's theorem (13.121) implies that $\mathfrak{d}_M(\vartheta_s)$ and $\mathfrak{d}$ are linearly equivalent divisors; and since $w_{z_0}(\mathfrak{d}) \notin W_g^1$ the two divisors actually must be equal. Consequently the base point z_0 also is one of the points of the divisor $\mathfrak{d}_M(\vartheta_s)$, so $\Theta(s; Z) = \Theta\big(s + \widetilde{w}_{z_0}(z_0); Z\big) = \vartheta_s(z_0) = 0$. That suffices for the proof,

Corollary 13.18 (Riemann Vanishing Theorem) *The function $\Theta(t; Z)$ on any compact Riemann surface of genus $g > 0$ vanishes to the first order on the subvariety $V_\Theta \subset J(M)$ and*

$$V_\Theta = \mathfrak{R} - W_{g-1} = -\mathfrak{R} + W_{g-1} \tag{13.122}$$

where $\mathfrak{R} \in J(M)$ is the Riemann point of $J(M)$.

Proof: The preceding Corollary 13.17 shows that the two holomorphic subvarieties V_θ and $\mathfrak{R} - W_{g-1}$ of pure dimension $g - 1$ in the Jacobi variety $J(M)$ coincide in the complement of the subvariety $X_\Theta \cup (\mathfrak{R} - W_g^1)$. The subvariety X_Θ is of dimension at most $g-2$ by Lemma 13.13, while $W_g^1 = k - W_{g-2}$ by the

Riemann-Roch Theorem in the form of Theorem 12.23 so is of pure dimension $g-2$; therefore necessarily $V_\theta = \mathfrak{R} - W_{g-1}$, since subvarieties of pure dimension $g-1$ are uniquely determined by their restrictions to the complements of subvarieties of dimension strictly less than $g-1$. Since $\Theta(t;Z)$ is an even function $V_\Theta = -V_\Theta$ as noted on page 365, so it is also the case that $V_\theta = -\mathfrak{R} + W_{g-1}$. Since the subvarietiy $V_\Theta \subset J(M)$ is an irreducible holomorphic subvariety then if the function $\Theta(t;Z)$ does not vanish to the first order it must vanish to an order $r > 1$ at each point of its zero locus V_Θ. That means then that in the proof of Corollary 13.17 the divisor $\mathfrak{d}_M(\vartheta_s)$ contains the multiple $r \cdot z_0$ so the divisor $\mathfrak{d}'_M(\vartheta_s)$ is of degree $g-r$ and consequently $\mathfrak{s} \in \mathfrak{R} - W_{g-s}$; that is impossible, since it would mean that the zero locus of $\Theta(t;Z)$ is of dimension $g-r$. That suffices for the proof.

The preceding corollary also yields some information about the Riemann point $\mathfrak{R}$ in the Jacobi variety $J(M)$ of a compact Riemann surface M, in the notation discussed in Chapter 12.

Corollary 13.19 *If z_0 is the base point for a marking of a compact Riemann surface of genus $g > 0$ then $\mathfrak{R}^2 \cdot \zeta_{z_0}^{2g-2} = \kappa$, the canonical bundle of M.*

Proof: Equation (13.122) implies that $2\mathfrak{R} - W_{g-1} = W_{g-1}$, or equivalently $2\mathfrak{R} \in W_{g-1} \ominus (-W_{g-1})$; and $W_{g-1} \ominus (-W_{g-1}) = W_{2g-2}^{g-1}$ by the Riemanan-Roch theorem in the form of Theorem 12.24. By the Canonical Bundle Theorem the canonical bundle κ is characterized by the condition that $c(\kappa) = 2g-2$ and $\gamma(\kappa) = g$; hence $W_{2g-2}^{g-1} = \kappa$ when points in the Jacobi variety are viewed as line bundles of characteristic class $2g-2$, which is the identification of a point $t \in J(M_)$ with the line bundle $t \cdot \zeta_{z_0}^{2g-2}$, which suffices for the proof.

The preceding corollary can be restated as the assertion that the Riemann point $\mathfrak{R} \subset J(M)$ is what is called a *theta characteristic*, a point of $J(M)$ such that $2\mathfrak{R} = k$ is the canonical point of $J(M)$; that determines the Riemann point up to a half period of the lattice $\mathcal{L}(\Omega)$. Mumford[6] gave a more precise characterization of the Riemann point. That is another topic, thoug; but to return to the canonical general cross-ratio function, the zero locus and pole locus of $\mathfrak{Q}_c(a,b;t,t_0)$ as a function of the variable $t \in \mathbb{C}^g$ are translates of the zero locus of the theta function $\Theta(t;Z)$, so these two functions must be closely related.

Theorem 13.20 *On a marked compact Riemann surface M of genus $g > 0$ the canonical general cross-ratio function $\mathfrak{Q}_c(a,b;t,t_0)$ for parameters a,b,t_0 for which it is a nontrivial meromorphic function of the variable $t \in \mathbb{C}^g$ and the classical theta function $\Theta(t;Z)$ are related by*

$$\mathfrak{Q}_c(a,b;t,t_0) = c\ \Theta\left(t - \mathcal{R} - \widetilde{w}_{z_0}(a);Z\right)\ \Theta\left(t - \mathcal{R} - \widetilde{w}_{z_0}(b);Z\right)^{-1} \tag{13.123}$$

for some nonzero constant c, where $\mathcal{R} \in \mathbb{C}^g$ is the Riemann point in $\mathbb{C}^g$.

[6]See the book *Tata Lectures on Theta*, vol. I, by David Mumford, (Birkhäuser), 1983.

Proof: By Theorem 13.6 (iv) for any fixed points $a, b \in \widetilde{M}$ and $t_0 \in \mathbb{C}^g$ for which $\mathfrak{Q}_c(a, b; t, t_0)$ is a nontrivial meromorphic function of the variable $t \in \mathbb{C}^g$ that function vanishes to first order on the subvariety $\widetilde{w}_{z_0}(a) + \widetilde{W}_{g-1} \subset \mathbb{C}^g$ and has a simple pole on the subvariety $w_{z_0}(b) + \widetilde{W}_{g-1} \subset \mathbb{C}^g$. By the Riemann Vanishing Theorem, Corollary 13.18, the function $\Theta(t; Z)$ vanishes to the first order on the subvariety $-\mathcal{R} + \widetilde{W}_{g-1} \subset J(M)$. Therefore $\Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(a); Z\big)$ has the same zero locus as $\mathfrak{Q}_c(a, b; t, t_0)$ and $\Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(b); Z\big)^{-1}$ has the same pole locus as $\mathfrak{Q}_c(a, b; t, t_0)$, hence the quotient

$$f(t) = \frac{\mathfrak{Q}_c(a, b; t, t_0)}{\Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(a); Z\big)\ \Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(b); Z\big)^{-1}} \tag{13.124}$$

is a holomorphic and nowhere vanishing function of the variable $t \in \mathbb{C}^g$. The functions $\mathfrak{Q}_c(a, b; t, t_0)$ and $\Theta\big(t - \widetilde{w}_{z_0}(a) - \mathcal{R}; Z\big)$ are relatively automorphic for the action of the lattice subgroup $\mathcal{L}(\Omega)$; indeed by (13.97)

$$\mathfrak{Q}_c(a, b; t + \mu + Z\nu, t_0) = \exp 2\pi i\, {}^t\nu \widetilde{w}_{z_0}(a, b) \cdot\ \mathfrak{Q}_c(a, b; t, t_0)$$

and by (13.98)

$$\begin{aligned}\Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(a) + \mu + Z\nu; Z\big) = \exp 2\pi i \left(\frac{1}{2}\, {}^t\nu Z\nu + {}^t\nu\big(t - \mathcal{R} - \widetilde{w}_{z_0}(a)\big)\right) \cdot \\ \cdot \Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(a); Z\big).\end{aligned}$$

Substituting these results into (13.124) shows that

$$f(t + \mu + Z\nu) = f(t)$$

so that $f(t) = c$ is a nonzero constant, and that suffices for the proof.

The descriptions of some standard holomorphic subvarieties of the Jacobi varieti $J(M)$ in terms of the general theta function can be rephrased in terms of the classical theta function by applying the preceding theorem relating these two functios.

Theorem 13.21 *If $\Theta(t; Z)$ is the classical theta function for the period matrix $\Omega = \big(I \quad Z\big)$ of a marked Riemann surface M of genus $g > 0$ then*

$$\Theta(t - \widetilde{W}_g^1; Z) = 0 \ \textit{ if and only if } \ t - \mathcal{R} \in \widetilde{W}_1 \subset \mathbb{C}^g \tag{13.125}$$

where $\mathcal{R} \in \mathbb{C}^g$ is the Riemann point.

Proof: The assumption is that $t - \widetilde{W}_g^1 \subset \widetilde{V}_\Theta$ where $\widetilde{V}_\Theta$ is the zero locus of the classical theta function $\Theta(t; Z)$. By Corollary 13.18 the zero locus is the subvariety $\widetilde{V}_\Theta = \mathcal{R} - \widetilde{W}_{g-1}$ when viewed as a holomorphic subvariety of $\mathbb{C}^g$; so the assumption is just that $t - \widetilde{W}_g^1 \subset \mathcal{R} - \widetilde{W}_{g-1}$ or equivalently that $\mathcal{R} - t + \widetilde{W}_g^1 \subset \widetilde{W}_{g-1}$; but by Lemma 13.8 that is just the condition that $t - \mathcal{R} \in \widetilde{W}_1$, when viewed as a subset of $\mathbb{C}^g$, so that suffices for the proof.

Theorem 13.22 *If $\Theta(t;Z)$ is the classical theta function for the period matrix $\Omega = \begin{pmatrix} I & Z \end{pmatrix}$ of a marked Riemann surface M of genus $g > 0$ and if the points $a_\nu \in \widetilde{M}$ for $1 \le a_\nu \le r \le g$ represent distinct points of M then*

$$\left\{ t \in \mathbb{C}^g \;\middle|\; \Theta\big(t - \mathcal{R} - \widetilde{w}_{z_0}(a_\nu; Z) = 0\big) \ \textit{for}\ 1 \le \nu \le r \right\} \tag{13.126}$$
$$= \Big(\widetilde{w}_0(a_1 + \cdots + a_r) + \widetilde{W}_{g-r}\Big) \cup \widetilde{W}^1_g \subset \mathbb{C}^g$$
$$= \bigcap_{\nu=1}^{r} \Big(\widetilde{w}_{z_0}(a_\nu) + \widetilde{W}_{g-1}\Big) \bigcup \widetilde{W}^1_g \subset \mathbb{C}^g$$

where $\mathcal{R} \in \mathbb{C}^g$ is the Riemann point.

Proof: Since $\mathfrak{Q}(a_\nu, b; t, t_0) = c\Theta\big(t - \mathcal{R} - \widetilde{w}(a_\nu); Z\big)\Theta\big(t - \mathcal{R} - \widetilde{w}(b); Z\big)^{-1}$ by Theorem 13.20 then if $\Theta\big(t-\mathcal{R}-\widetilde{w}_{z_0}(a_\nu); Z\big) = 0$ it follows that $\mathfrak{Q}(a_\nu, b; t, t_0) = 0$, assuming that $b \in \widetilde{M}$ represents a point of M that is distinct from the points a_ν so that the zero locus and the pole locus of the function $\mathfrak{Q}(a_\nu, b; t, t_0)$ are distinct; so the assumption of the theorem implies that the point t is contained in the zero locus $\widetilde{V}_{\mathfrak{d}}$ of the divisor $\mathfrak{d} = a_1 + \cdots + a_r$, as defined in (13.80). The conclusion of the present theorem then follows from Theorem 13.9.

It is evident from the preceding discussion that the subvariety $W^1_g = k - W_{g-2} \subset J(M)$ plays a surprisingly central role in Jacobi varieties. The Riemann surface M imbedded in the Jacobi variety as the submanifold $W_1 \subset J(M)$ can be described in terms of the subvariety W^1_g as

$$W_1 = \left\{ t \in J(M) \;\middle|\; \Theta(t + \mathfrak{R} - W^1_g; Z) = 0 \right\}; \tag{13.127}$$

the Riemann point $\mathfrak{R}$ is not needed to describe the submanifold W_1 just up to translation in $J(M)$. By Theorem 13.22 the submanifold W^1_g can be characterized as one of the two irreducible components $\widetilde{w}_{z_0}(a_1 + a_2)$ and $k - w_{g-2} = w^1_g$ of the intersection $\Theta(\widetilde{w}_{z_0}(a_1); Z) = \Theta(t + \widetilde{w}_{z_0}(a_2); Z) = 0$. The subvariety W^1_g is also the exceptional locus of the Abel-Jacobi mapping $\widetilde{w}_{z_0} : M^{(g)} \longrightarrow J(M)$.

Chapter 14

Pseudouniformizing Mappings

REVISED IN PART

A Riemann surface has the pseudogroup structure of a complex manifold; but it may possess in addition finer pseudogroup structures, leading to what can be called the *pseudouniformization* of Riemann surfaces. The pseudogroups of interest here are the pseudogroup $\mathcal{F}_1$ of affine transformations and the pseudogroup $\mathcal{F}_2$ of linear fractional or projective transformations; explicitly

$$T \in \mathcal{F}_1 \quad \Leftrightarrow \quad T(z) = az + b \quad \text{where } a \neq 0 \quad \text{for } z \in \mathbb{C} \tag{14.1}$$

$$T \in \mathcal{F}_2 \quad \Leftrightarrow \quad T(z) = \frac{az+b}{cd+d} \quad \text{where} \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \quad \text{for } z \in \mathbb{P}^1. \tag{14.2}$$

There are pseudogroup structures on Riemann surfaces associated to each of the pseudogroups $\mathcal{F}_\nu$, *affine structures* associated to the pseudogroup $\mathcal{F}_1$ and *projective structures* associated to the pseudogroup $\mathcal{F}_2$. The pseudogroup $\mathcal{F}_1$ consists of actions of the affine Lie group

$$G_1 = \mathrm{A}(1,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| \, a, b \in \mathbb{C}, \;\; a \neq 0 \right\} \tag{14.3}$$

on the complex manifold $V_1 = \mathbb{C}$ while the pseudogroup $\mathcal{F}_2$ consists of actions of the projective Lie group

$$G_2 = \mathrm{Pl}(2,\mathbb{C}) = \mathrm{Gl}(2,\mathbb{C})/\mathbb{C}^* = \mathrm{Sl}(2,\mathbb{C})/\pm\mathrm{I} \tag{14.4}$$

on the complex manifold $V_2 = \mathbb{P}^1$. The mapping that associates to each matrix in $G_1 = \mathrm{A}(1,\mathbb{C})$ the projective linear transformation represented by that matrix is an injective homomorphism from the affine Lie group G_1 to the projective Lie group G_2 which determines an injective homomorphism from the pseudogroup $\mathcal{F}_1$ into the pseudogroup $\mathcal{F}_2$.

The pseudogroups $\mathcal{F}_\nu$ are particularly interesting in that they can be described as sets of solutions of systems of differential equations; indeed it will be demonstrated next that the pseudogroup $\mathcal{F}_\nu$ consists of local biholomorphic mappings that are solutions of the differential equation $D_\nu f = 0$, where

$$(D_1 f)(z) = \frac{f''(z)}{f'(z)}, \tag{14.5}$$

$$(D_2 f)(z) = \frac{2f'(z)f'''(z) - 3f''(z)^2}{2f'(z)^2}. \tag{14.6}$$

The differential operator D_2, customarily called the *Schwarzian differential operator*, can be written in either of the alternative forms

(14.7)
$$(D_2 f)(z) = \begin{cases} -2\dfrac{h''(z)}{h(z)} & \text{where } h(z) = f'(z)^{-1/2}, \text{ or} \\ k''(z) - \frac{1}{2}k'(z)^2 & \text{where } k(z) = \log f'(z), \text{ or} \\ \frac{d}{dz}(D_1 f)(z) - \frac{1}{2}(D_1 f)(z)^2, & \end{cases}$$

as can be seen by a straightforward calculation. In addition if $u_1(z)$ and $u_2(z)$ are two linearly independent holomorphic solutions of the differential equation $u(z)'' + g(z)u(z) = 0$ for some holomorphic function $g(z)$ and if $f(z) = u_2(z)/u_1(z)$ then

$$(D_2 f)(z) = g(z) \tag{14.8}$$

by another straightforward calculation, accomplished most easily by using the second expression for the differential operator D_2 in (14.7); and

$$\frac{d}{dz}\Big(u_1(z)u_2'(z) - u_1'(z)u_2(z)\Big) = u_1(z)u_2''(z) - u_1''(z)u_2(z) = 0, \tag{14.9}$$

so the Wronskian of these two functions is a nonzero constant c and $f'(z) = cu_1(z)^{-2} \neq 0$.

Theorem 14.1 *The differential operators D_ν have the following properties:*
(i) *if $h = f \circ g$ is the composition of the holomorphic local homeomorphisms f and g, so that $h(z) = f(w)$ where $w = g(z)$, then*

$$(D_\nu h)(z) = (D_\nu f)(w) \cdot g'(z)^\nu + (D_\nu g)(z); \tag{14.10}$$

(ii) *$D_\nu f = 0$ for a holomorphic local homeomorphism f if and only if $f \in \mathcal{F}_\nu$;*
(iii) *for any holomorphic function g in an open neighborhood of a point $a \in \mathbb{C}$ there exist holomorphic functions f_ν in an open subneighborhood of a such that $f_\nu'(a) \neq 0$ and $D_\nu f_\nu = g$; and if $\tilde{f}_\nu$ is another holomorphic function for which $\tilde{f}_\nu'(a) \neq 0$ and $D_\nu \tilde{f}_\nu = g$ then $\tilde{f}_\nu = T \circ f_\nu$ for some $T \in \mathcal{F}_\nu$.*

Proof: (i) It follows immediately from the chain rule for differentiation that

$$\begin{aligned} h' &= f'g', \\ h'' &= f''g'^2 + f'g'', \\ h''' &= f'''g'^3 + 3f''g'g'' + f'g''', \end{aligned}$$

where the variables are omitted to simplify the notation; but it should be understood that f is viewed as a function of the variable w and g and h are viewed as functions of the variable z. A straightforward calculation using these forms of the chain rule shows that

$$(D_1h) = \frac{f''g'^2 + f'g''}{f'g'} = (D_1f) \cdot g' + (D_1g)$$

and

$$\begin{aligned} (D_2h) &= \frac{2f'g'(f'''g'^3 + 3f''g'g'' + f'g''') - 3(f''g'^2 + f'g'')^2}{2f'^2g'^2} \\ &= (D_2f) \cdot g'^2 + (D_2g). \end{aligned}$$

(ii) It is obvious that $D_1f = 0$ if and only if $f'' = 0$, hence if and only if $f(z) = az + b$ or equivalently $f \in \mathcal{F}_1$. It is equally obvious from (14.7) that $D_2f = 0$ if and only if $h'' = 0$, or equivalently $h(z) = cz+d$, where $h = (f')^{-1/2}$. If $h(z) = cz + d$ then $f'(z) = (cz + d)^{-2}$ hence $f(z) = \frac{-1}{c(cz+d)} + c_1 = \frac{az+b}{cz+d}$ for some constants c_1, a and b. Conversely if $f(z) = \frac{az+d}{cz+d}$ then $f'(z) = \frac{ad-bc}{(cz+d)^2}$ and $h = (f')^{-1/2} = (c_1z + d_1)$ for some constants c_1 and d_1. Thus $D_2f = 0$ if and only if $f \in \mathcal{F}_2$.
(iii) If $g(z)$ is holomorphic in an open neighborhood of a point $a \in \mathbb{C}$ its indefinite integral $F(z) = \int^z g(\zeta)d\zeta$ is a holomorphic function near that point, as is the further indefinite integral $f_1(z) = \int^z \exp F(\zeta)d\zeta$; and $f_1'(a) = \exp F(a) \neq 0$ while $D_1f_1(z) = f_1''(z)/f_1'(z) = F'(z)\exp F(z)/\exp F(z) = g(z)$. Next it is a familiar result from the local theory of holomorphic differential equations[1] that if $g(z)$ is a holomorphic function in an open neighborhood of a point $a \in \mathbb{C}$ then there exist two linearly independent holomorphic functions $u_1(z)$ and $u_2(z)$ near that point that are solutions of the differential equation $u''(z) + g(z)u(z) = 0$; it is always possible to choose these solutions so that $u_1(a) \neq 0$, and it then follows from (14.8) that the quotient $f_2(z) = u_2(z)/u_1(z)$ is a holomorphic function which satisfies $D_2f_2 = g$ and $f_2'(a) \neq 0$. If $D_\nu f_\nu = D_\nu\tilde{f}_\nu = g$ for holomorphic functions f_ν and $\tilde{f}_\nu$ such that $f_\nu'(a)\tilde{f}_\nu'(a) \neq 0$ then the restriction of each of these functions to a sufficiently small open neighborhood of the point a is a biholomorphic mapping, and consequently $\tilde{f}_\nu = h_\nu \circ f_\nu$ for the biholomorphic mapping $h_\nu = \tilde{f}_\nu \circ f_\nu^{-1}$ from an open neighborhood of $f_\nu(a)$ to an open neighborhood of $\tilde{f}_\nu(a)$; it then follows from (i) that $g = D_\nu\tilde{f}_\nu =$

[1]These results about holomorphic differential equations are discussed among other places in C. Carathéodory, *Theory of Functions of a Complex Variable*, vol. II (Chelsea, 1960), where the role of the Schwarzian differential operator is examined in some detail.

$(D_\nu h_\nu)(f'_\nu)^\nu + (D_\nu f_\nu) = (D_\nu h_\nu)(f'_\nu)^\nu + g$ hence that $D_\nu h_\nu = 0$ and consequently $h_\nu \in \mathcal{F}_\nu$ by (ii). That suffices to conclude the proof.

The result (14.10) of the preceding theorem can be written more succinctly as

$$D_\nu(f \circ g) = D_\nu f \cdot (g')^\nu + D_\nu g. \tag{14.11}$$

It is clear from this that the composition of any two solutions of the differential equation $D_\nu f = 0$ is again a solution of that differential equation, hence that the set of solutions is a pseudogroup of locally biholomorphic mappings, even without determining explicitly what the set of solutions is. The differential operators D_ν are chosen particularly so that this identity is satisfied; these operators of course can be simplified by eliminating the denominators in (14.5) and (14.6), which does not change the set of solutions of the differential equations $D_\nu f = 0$ but does not lead to quite as convenient a formula as (14.11). It is worth noting that the derivatives $f'(a)$ at any particular point a of the solutions f of the differential equations $D_\nu f = 0$ near that point can have quite arbitrary values; that can be viewed as the assertion that changes of coordinates in the pseudogroups $\mathcal{F}_\nu$ can have quite arbitrary actions on the tangent space at any point, which is often phrased as the assertion that these particular pseudogroups of transformations are *tangentially transitive*.

It can be shown that up to a natural equivalence these are the only tangentially transitive pseudogroups in one variable that can be described as the sets of solutions of systems of differential equations; that is one reason for focusing attention on these particular pseudogroups. Although the proof of that assertion would lead far too far afield, it may be worth including here a brief digression[2] on the classification of pseudogroups defined by systems of ordinary or partial differential equations in order to clarify the role of the particular differential operators D_1 and D_2. First, though, it is worth pointing that the pseudogroup of holomorphic mappings in one complex variable can be defined as the pseudogroup of $\mathcal{C}^1$- mappings $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that are solutions of the Cauchy-Riemann differential equation $\overline{\partial} f = 0$, which really amounts to the condition that the differential of the mapping f lies in the subgroup $\mathrm{Gl}(1,\mathbb{C}) \subset \mathrm{Gl}(2,\mathbb{R})$. This is perhaps the simplest interesting example of a non tangentially transitive pseudogroup that can be defined by a system of partial differential equations, and one for which the defining differential equations are linear. There are very few pseudogroups that can be defined by linear systems of partial differential equations, and all impose some restrictions on the differentials of the mappings[3]. For further examples it is necessary to consider pseudogroups defined by systems of nonlinear differential equations, which is not as difficult as might be expected.

[2] This topic is discussed among other places in R. C. Gunning, *On Uniformization of Complex Manifolds: the Role of Connections*, Mathematical Notes number 22 (Princeton University Press, 1978), and in S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, 1964.

[3] This is discussed in S. Bochner and R. C. Gunning, "Infinite linear pseudogroups of transformations", Annals of Mathematics **75** (1962), 93-104.

The set of germs at the origin of holomorphic functions of a complex variable that vanish but have nonvanishing first derivative at the origin form a group $\mathcal{G}$ under composition; and the Taylor expansions of these functions up to any finite order k with the induced group structure form a Lie group $\mathcal{G}_k$, the *group of k-jets* of holomorphic mappings in one variable. For instance the 3-jets of two germs of holomorphic functions $f(z) = \sum_{n=1}^{\infty} x_n \cdot z^n$ and $g(z) = \sum_{n=1}^{\infty} y_n \cdot z^n$ are described by the points (x_1, x_2, x_3) and (y_1, y_2, y_3) in the complex manifold $\mathbb{C}^* \times \mathbb{C}^2$, and the group structure defined by the composition of functions is easily seen to be explicitly

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 y_1, \;\; x_1 y_2 + x_2 y_1^2, \;\; x_1 y_3 + 2x_2 y_1 y_2 + x_3 y_1^3),$$

thus identifying the complex manifold underlying the Lie group $\mathcal{G}_3$ with the three-dimensional complex manifold $\mathbb{C}^* \times \mathbb{C}^2$. For any subgroup $G \subset \mathcal{G}_k$, described as the set of points $x \in \mathcal{G}_k$ such that $P_i(x) = 0$ for some polynomials P_i, the set of holomorphic local homeomorphisms with k-jets in G is closed under composition and is the pseudogroup defined by the system of differential equations $P_i(f, f'', \ldots, f^{(k)}) = 0$.

The first step in classifying the pseudogroups defined by systems of differential equations is the classification of subgroups of $\mathcal{G}_k$, a simple algebraic problem that can be handled most readily by examining the Lie algebra of the group $\mathcal{G}_k$. Of course conjugate subgroups yield pseudogroups that are equivalent under changes of coordinates in $\mathbb{C}$, so it is only the classification of subgroups up to inner automorphism that is relevant. However quite different nonconjugate subgroups $G \subset \mathcal{G}_k$ may lead to equivalent pseudogroups of transformations. For instance it is not necessarily the case that the set of points of $\mathcal{G}_k$ consisting of the values $(f(x), f''(x), \ldots, f^{(k)}(x))$ for mappings defined by that group actually span the entire group G; they may lie in a proper subgroup $G' \subset G$, and then the subgroups G and G' describe the same pseudogroup of transformations. Hence the next step is to examine the explicit pseudogroups defined by various subgroups of $\mathcal{G}_k$. That also can be handled quite readily by using the Lie algebras, at least for the tangentially transitive pseudogroups. In this way it is not difficult to show that the only tangentially transitive pseudogroups described by differential equations are, first, the pseudogroup described by the subgroup $x_2 = 0$ in $\mathcal{G}_2$, and second, the pseudogroup described by the subgroup $x_1 x_3 - x_2^2 = 0$ in $\mathcal{G}_3$; these are just the pseudogroups described by the differential operators D_1 and D_2, which are expressed in terms of the derivatives $f^{(n)} = x_n/n!$ rather than in terms of coefficients in the power series expansions. Higher order differential operators lead to no additional pseudogroups. The first pseudogroup of course also is described by the subgroup $x_2 = x_3 = 0$ in $\mathcal{G}_3$, a simple example of distinct subgroups determining the same pseudogroup. The situation in higher dimensions is quite similar. There are up to equivalence only three nontrivial tangentially transitive pseudogroups defined by systems of partial differential equations in dimensions $n > 2$: the pseudogroup of affine mappings, the pseudogroup of mappings having constant Jacobian determinants, and the pseudogroup of projective transformations in n dimensions. All of these pseudogroups actually are defined by systems of second-order partial differential equations.

To examine the role of the differential operators (14.5) and (14.6) further, consider a Riemann surface M having a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ with local coordinates z_α and holomorphic coordinate transition functions $f_{\alpha\beta}$, so that $z_\alpha = f_{\alpha\beta}(z_\beta)$ in an intersection $U_\alpha \cap U_\beta$. Since $f_{\alpha\gamma} = f_{\alpha\beta} \circ f_{\beta\gamma}$ in any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ it follows from (14.11) that $D_\nu f_{\alpha\gamma} = D_\nu f_{\alpha\beta} \cdot (f'_{\beta\gamma})^\nu + D_\nu f_{\beta\gamma}$, or equivalently that

$$D_\nu f_{\alpha\gamma} = \kappa^\nu_{\gamma\beta} \cdot D_\nu f_{\alpha\beta} + D_\nu f_{\beta\gamma} \tag{14.12}$$

where $f'_{\beta\gamma} = dz_\beta/dz_\gamma = \kappa_{\gamma\beta}$ are the coordinate transition functions of the canonical bundle κ as defined in (2.23); that is just the condition that the functions $D_\nu f_{\alpha\beta}$ describe a cocycle in $Z^1(\mathfrak{U}, \mathcal{O}(\kappa^\nu))$, as in (1.45). For $\nu = 1$ the sheaf $\mathcal{O}(\kappa)$ is identified as usual with the sheaf $\mathcal{O}^{(1,0)}$ of germs of abelian differentials on M by associating to a germ $f_\alpha \in \mathcal{O}(\kappa)$ the germ of the holomorphic differential $f_\alpha dz_\alpha \in \mathcal{O}^{(1,0)}$. Correspondingly for $\nu = 2$ to each germ $f_\alpha \in \mathcal{O}(\kappa^2)$ can be associated the invariantly defined expression $f_\alpha \cdot dz_\alpha^2$, called the germ of a *quadratic differential* on M, and in this way the sheaf $\mathcal{O}(\kappa^2)$ is identified with the sheaf $\mathcal{O}^{2(1,0)}$ of germs of quadratic differentials on M; the associated vector space $\Gamma(M, \mathcal{O}^{2(1,0)})$ is called the space of *quadratic differentials* on the Riemann surface M. In these terms (14.12) can be written somewhat more intrinsically in the form

$$D_\nu f_{\alpha\gamma} \cdot dz_\gamma^\nu = D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu + D_\nu f_{\beta\gamma} \cdot dz_\gamma^\nu \tag{14.13}$$

involving abelian differentials if $\nu = 1$ and quadratic differentials if $\nu = 2$; this is just the condition that the expressions $D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu$ describe a cocycle in $Z^1(M, \mathcal{O}^{\nu(1,0)})$.

A holomorphic change of coordinates on the manifold M is effected by biholomorphic mappings $w_\alpha = f_\alpha(z_\alpha)$ in the coordinate neighborhoods U_α, after a refinement of the coordinate covering if necessary; and the new coordinates have the coordinate transition functions $w_\alpha = \tilde{f}_{\alpha\beta}(w_\beta)$ where $\tilde{f}_{\alpha\beta} = f_\alpha \circ f_{\alpha\beta} \circ f_\beta^{-1}$, or equivalently where $\tilde{f}_{\alpha\beta} \circ f_\beta = f_\alpha \circ f_{\alpha\beta}$ in an intersection $U_\alpha \cap U_\beta$. Applying the differential operator D_ν to this last identity and using (14.11) lead to the result that

$$\begin{aligned} D_\nu(\tilde{f}_{\alpha\beta} \circ f_\beta) &= D_\nu \tilde{f}_{\alpha\beta} \cdot (f'_\beta)^\nu + D_\nu f_\beta \\ &= D_\nu(f_\alpha \circ f_{\alpha\beta}) \;=\; D_\nu f_\alpha \cdot (f'_{\alpha\beta})^\nu + D_\nu f_{\alpha\beta}, \end{aligned}$$

so that

$$D_\nu f_{\alpha\beta} - D_\nu \tilde{f}_{\alpha\beta} \cdot (f'_\beta)^\nu = D_\nu f_\beta - D_\nu f_\alpha \cdot (f'_{\alpha\beta})^\nu. \tag{14.14}$$

Since $f'_\beta = dw_\beta/dz_\beta$ and $f'_{\alpha\beta} = dz_\alpha/dz_\beta$ this can be written more intrinsically as

$$D_\nu \tilde{f}_{\alpha\beta} \cdot dw_\beta^\nu - D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu = D_\nu f_\alpha \cdot dz_\alpha^\nu - D_\nu f_\beta \cdot dz_\beta^\nu \tag{14.15}$$

in terms of abelian differentials if $\nu = 1$ or quadratic differentials if $\nu = 2$, and in this form it is just the condition (1.42)that the cocycles $D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu$ and $D_\nu \tilde{f}_{\alpha\beta} \cdot dw_\beta^\nu$ differ by the coboundary of the cochain $D_\nu f_\alpha \cdot dz_\alpha^\nu \in C^0(\mathfrak{U}, \mathcal{O}^{\nu(1,0)})$ so are cohomologous. Therefore the cohomology class

$$D_\nu(M) \in H^1(M, \mathcal{O}^{\nu(1,0)}) \tag{14.16}$$

represented by the cocycle $D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu$ is intrinsically defined, so is independent of the choice of coordinate covering of the Riemann surface. The cohomology class $D_\nu(M)$ is trivial if and only if, after a refinement of the coordinate covering if necessary, the cocycle $D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu$ is the coboundary of a cochain in $C^0(\mathfrak{U}, \mathcal{O}^{\nu(1,0)})$; such a cochain is called an $\mathcal{F}_\nu$ *connection* on the surface M, an *affine connection* if $\nu = 1$ and a *projective connection* if $\nu = 2$, using a terminology modeled on the traditional terminology in differential geometry. Thus an $\mathcal{F}_\nu$ connection on M is a collection of local holomorphic ν-differentials $p_\alpha \cdot dz_\alpha^\nu$ in the coordinate neighborhoods U_α such that

$$D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu = p_\beta \cdot dz_\beta^\nu - p_\alpha \cdot dz_\alpha^\nu \tag{14.17}$$

in each intersection $U_\alpha \cap U_\beta$, as in (1.42).

Theorem 14.2 *A Riemann surface M admits an $\mathcal{F}_\nu$ structure if and only if $D_\nu(M) = 0$ in $H^1(M, \mathcal{O}^{\nu(1,0)})$. If the surface admits an $\mathcal{F}_\nu$ structure then the set of all $\mathcal{F}_\nu$ structures is in canonical one-to-one correspondence with the set of $\mathcal{F}_\nu$ connections on the surface M, a linear space of dimension $\gamma(\kappa^\nu)$.*

Proof: Choose a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the Riemann surface M with local coordinates z_α and coordinate transition functions $f_{\alpha\beta}$. If the surface admits an $\mathcal{F}_\nu$ structure then after a refinement of the coordinate covering if necessary there is a change of coordinates $w_\alpha = f_\alpha(z_\alpha)$ such that the coordinate transition functions $w_\alpha = \tilde{f}_{\alpha\beta}(w_\beta)$ belong to the pseudogroup $\mathcal{F}_\nu$. It then follows from (14.15) that $0 = D_\nu \tilde{f}_{\alpha\beta} \cdot dw_\beta^\nu = D_\nu f_{\alpha\beta} \cdot dz_\beta^\nu + D_\nu f_\alpha \cdot dz_\alpha^\nu - D_\nu f_\beta \cdot dz_\beta^\nu$, so $D_\nu f_\alpha \cdot dz_\alpha^\nu$ is an $\mathcal{F}_\nu$ connection on the surface M and therefore $D_\nu(M) = 0$. Conversely if $D_\nu(M) = 0$ then there is an $\mathcal{F}_\nu$ connection on the surface M; thus after a refinement of the coordinate covering if necessary there are holomorphic abelian or quadratic differentials $p_\alpha \cdot dz_\alpha^\nu$ in the coordinate neighborhoods U_α satisfying (14.17). It follows from Lemma 14.1 (iii) that, after a further refinement of the coordinate covering if necessary, there is a holomorphic change of coordinates $w_\alpha = f_\alpha(z_\alpha)$ for which $D_\nu f_\alpha = p_\alpha$; the coordinate transition functions $\tilde{f}_{\alpha\beta}$ for the coordinates w_α satisfy (14.15), and since $D_\nu f_\alpha = p_\alpha$ while the functions p_α satisfy (14.17) it follows that $D_\nu \tilde{f}_{\alpha\beta} = 0$ so the coordinates w_α provide an $\mathcal{F}_\nu$ structure on M.

If there is an $\mathcal{F}_\nu$ structure on M then the most general coordinate covering defining the same $\mathcal{F}_\nu$ structure is of the form $h_\alpha \circ f_\alpha$ where $h_\alpha \in \mathcal{F}_\nu$, after another refinement of the coordinate covering if necessary; and since $D_\nu h_\alpha = 0$ it follows from (14.11) that $D_\nu(h_\alpha \circ f_\alpha) = D_\nu f_\alpha = p_\alpha$, so all these $\mathcal{F}_\nu$ coordinate coverings yield the same $\mathcal{F}_\nu$ connection on M. On the other hand it further follows from

Lemma 14.1 (iii) that all solutions f_α of the differential equation $D_\nu f_\alpha = p_\alpha$ are of the form $h_\alpha \circ f_\alpha$, where f_α is any one solution and h_α are biholomorphic mappings such that $D_\nu h_\alpha = 0$, after a further refinement of the coordinate covering if necessary; thus any such solutions yield the same $\mathcal{F}_\nu$ structure on M. Altogether, if $D_\nu(M) = 0$ then the set of possible $\mathcal{F}_\nu$ structures on M is in natural one-to-one correspondence with the set of $\mathcal{F}_\nu$ connections on the surface M. If there is an $\mathcal{F}_\nu$ connection on M then any two such connections differ by a cocycle, that is, by an element of $\Gamma(M, \mathcal{O}(\kappa^\nu))$; thus the set of all $\mathcal{F}_\nu$ connections form a linear space of dimension $\gamma(\kappa^\nu)$, and that suffices to conclude the proof of the theorem.

Corollary 14.3 *If $p_\alpha \cdot dz_\alpha^\nu$ is an $\mathcal{F}_\nu$ connection on a Riemann surface M in terms of local coordinates z_α, the associated $\mathcal{F}_\nu$ structure is described by local coordinates $w_\alpha = f_\alpha(z_\alpha)$ for any local solutions of thke differential equation $D_\nu f_\alpha = p_\alpha$.*

Proof: This was demonstrated in the proof of the preceding theorem, and is included explicitly here just as a convenience for later reference.

Corollary 14.4 (i) *A compact Riemann surface M admits an affine structure if and only it has genus $g = 1$; and if $g = 1$ the set of affine structures naturally form a one-dimensional linear space.*
(ii) *Any compact Riemann surface M admits a projective structure; if $g = 0$ the structure is unique, if $g = 1$ the set of projective structures naturally form a one-dimensional linear space, and if $g > 1$ the set of projective structures naturally form a $(3g-3)$-dimensional linear space.*

Proof: (i) If a compact Riemann surface M admits an affine structure, described by local coordinates z_α for some coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the Riemann surface M, then $z_\alpha = a_{\alpha\beta} z_\beta + b_{\alpha\beta}$ for some constants $a_{\alpha\beta}$ and $b_{\alpha\beta}$ in any intersection $U_\alpha \cap U_\beta$ of coordinate neighborhoods. The canonical bundle κ consequently is described by constant coordinate transition functions $\kappa_{\alpha\beta} = dz_\beta / dz_\alpha = a_{\beta\alpha}$, so it is a flat bundle and therefore $c(\kappa) = 0$ by Corollary 3.9; and since $c(\kappa) = 2g - 2$ by the Canonical Bundle Theorem, Theorem 2.24, it follows that $g = 1$. Conversely any compact Riemann surface of genus $g = 1$ is biholomorphic to its Jacobi variety under the Abel-Jacobi mapping, as in Corollary 12.9 (iii), so is a complex torus and hence has a natural affine structure. By Theorem 14.2 the set of all affine structures is in one-to-one correspondence with the set of affine connections on M, a linear space of dimension $\gamma(\kappa)$, and $\gamma(\kappa) = 1$ since the canonical bundle is trivial.
(ii) By Theorem 14.2 a Riemann surface M admits a projective structure if and only if the cohomology class $D_2(M) \in H^1(M, \mathcal{O}(\kappa^2))$ is zero. If M is a compact Riemann surface of genus g the Serre Duality Theorem in the form of Corollary 1.18 shows that $\dim H^1(M, \mathcal{O}(\kappa^2)) = \dim \Gamma(M, \mathcal{O}^{(1,0)}(\kappa^{-2})) = \dim \Gamma(M, \mathcal{O}(\kappa^{-1}))$. If $g > 1$ the canonical bundle κ has Chern class $c(\kappa) = 2g - 2 > 0$ and consequently $\Gamma(M, \mathcal{O}(\kappa^{-1})) = 0$ so $D_2(M) = 0$ and the surface

does admit a projective structure; furthermore by Theorem 14.2 the set of all projective structures is in one-to-one correspondence with the set of projective connections, a linear space of dimension $\gamma(\kappa^2)$, and $\gamma(\kappa^2) = 3g - 3$ when $g > 1$ by the Riemann-Roch Theorem. If $g = 1$ then by the first part of this theorem the surface has an affine structure, which is a special case of a projective structure; again the set of projective structures is in one-to-one correspondence with the set of projective connections, which is a linear space of dimension $\gamma(\kappa^2)$, and $\gamma(\kappa^2) = 1$ since the canonical bundle is trivial. Finally if $g = 0$ the surface is just the Riemann sphere $\mathbb{P}^1$, which has the natural projective structure described by its standard coordinate covering; yet again the set of all projective structures is in one-to-one correspondence with the set of projective connections on M, which is a linear space of dimension $\gamma(\kappa)$, and $\gamma(\kappa) = 0$ since $c(\kappa) = -2$ so this projective structure is unique. That suffices to conclude the proof.

Theorem 14.2 holds for noncompact Riemann surfaces as well. If M is a noncompact Riemann surface it is a standard result[4] that $H^1(M, \mathcal{O}^{\nu(1,0)}) = 0$ for any ν, since M is a Stein manifold and $\mathcal{O}^{\nu(1,0)}$ is a coherent analytic sheaf over M; hence M admits both affine and projective structures, and the vector space $\Gamma(M, \mathcal{O}(\kappa^\nu))$ is infinite dimensional so there are a vast number of distinct affine and projective structures on M. That topic will not be pursued further here, since the discussion generally has been limited to compact Riemann surfaces, although the next few general definitions do not require compactness. If a Riemann surface M has an $\mathcal{F}_\nu$ structure and if $\mathfrak{U} = \{U_\alpha\}$ is a coordinate covering describing that structure, with local coordinates z_α and coordinate transition functions $f_{\alpha\beta}$ consisting of the operation of group elements $\hat{f}_{\alpha\beta} \in G_\nu$, these group elements can be viewed as constant mappings $\hat{f}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G_\nu$ to the Lie group G_ν and they satisfy $\hat{f}_{\alpha\gamma} = \hat{f}_{\alpha\beta} \cdot \hat{f}_{\beta\gamma}$ in any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ so they describe a *flat G_ν bundle* over M, a flat fibre bundle with group the Lie group G_ν and with fibre the complex manifold V_ν on which the group G_ν acts, the complex plane $V_1 = \mathbb{C}$ or the Riemann sphere $V_2 = \mathbb{P}^1$. Any other coordinate covering describing the same $\mathcal{F}_\nu$ structure on M, after refining the coordinate covering if necessary, is of the form $w_\alpha = h_\alpha(z_\alpha)$ for mappings $h_\alpha \in \mathcal{F}_\nu$ consisting of the operations of group elements $\hat{h}_\alpha \in G_\nu$; and the coordinate transition functions for this covering are $h_\alpha \circ f_{\alpha\beta} \circ h_\beta^{-1} \in \mathcal{F}_\nu$, which consist of the actions of the group elements $\hat{h}_\alpha \circ \hat{f}_{\alpha\beta} \circ \hat{h}_\beta^{-1} \in G_\nu$ so describe an equivalent flat G_ν bundle over M. In this way there is associated to any $\mathcal{F}_\nu$ structure on an arbitrary Riemann surface M a unique flat G_ν bundle over M, called the *holonomy bundle* of the $\mathcal{F}_\nu$ structure of M. The holonomy bundles are in some ways simpler than the $\mathcal{F}_\nu$ structures to which they are associated, so they are convenient tools to use in examining these structures. Some caution is required, though, since generally there is not a one-to-one correspondence between $\mathcal{F}_\nu$ structures on M and flat G_ν bundles over M; not all flat G_ν bundles are holonomy bundles of $\mathcal{F}_\nu$ structures on M, and a flat G_ν bundle over M may

[4]For this property of noncompact Riemann surfaces see for instance O. Forster, *Lectures on Riemann Surfaces*, Graduate Texts in Mathematics, No. 81 (Springer-Verlag, 1981).

be the holonomy bundle of a variety of different $\mathcal{F}_\nu$ structures on M.

Corollary 14.5 *The holonomy bundles of $\mathcal{F}_\nu$ structures on a Riemann surface M are the flat G_ν bundles over M that admit locally biholomorphic cross-sections.*

Proof: The local coordinates z_α describing an $\mathcal{F}_\nu$ structure on M are locally biholomorphic mappings $z_\alpha : U_\alpha \longrightarrow V_\nu$ such that $z_\alpha = f_{\alpha\beta}(z_\beta)$ in each intersection $U_\alpha \cap U_\beta$ of coordinate neighborhoods on M, where $f_{\alpha\beta}$ are the coordinate transition functions of the holonomy bundle of this structure; thus the local coordinates z_α can be viewed as a locally biholomorphic cross-section of the holonomy bundle of the $\mathcal{F}_\nu$ structure on M. Conversely if a flat G_ν bundle over a Riemann surface M has a locally biholomorphic cross-section z_α then the functions z_α can be taken as local coordinates on M, and clearly they describe an $\mathcal{F}_\nu$ structure with the given G_ν bundle as holonomy bundle. That suffices for the proof.

It is possible more generally to begin with a two-dimensional topological manifold M, rather than with a Riemann surface, and to consider $\mathcal{F}_\nu$ structures on M. Any $\mathcal{F}_\nu$ structure is in particular a complex structure; but of course there are $\mathcal{F}_\nu$ structures on M associated to the various different complex structures on M. Each $\mathcal{F}_\nu$ structure though has an associated holonomy bundle; and it is evident that the analogue of Corollary 14.5 is the assertion that the holonomy bundles of $\mathcal{F}_\nu$ structures on a topological manifold M are precisely those flat G_ν bundles over M that admit locally homeomorphic cross-sections. The problem of determining whether a particular flat G_ν bundle has a locally homeomorphic cross-section is a purely topological problem, but one of some difficulty as will become apparent in the further discussion.

As for the case of flat vector bundles, discussed in Part **????**, flat $\mathcal{F}_\nu$ bundles over a Riemann surface or over just a two-dimensional topological manifold M can be described by flat factors of automorphy for the action of the covering translation group Γ on the universal covering space $\tilde{M}$ of the surface M. Indeed a flat $\mathcal{F}_\nu$ bundle $\mathcal{B}$ over M lifts to a Γ-invariant flat $\mathcal{F}_\nu$ bundle $\tilde{\mathcal{B}}$ over $\tilde{M}$, and $\mathcal{B}$ can be identified with the quotient $\mathcal{B} = \tilde{\mathcal{B}}/\Gamma$ of the bundle $\tilde{\mathcal{B}}$ under the natural induced action of Γ. Since $\tilde{M}$ is simply connected the usual monodromy argument shows that the flat bundle $\tilde{\mathcal{B}}$ is equivalent to a product bundle $\tilde{M} \times V_\nu$ over M. The action of a covering translation $T \in \Gamma$ on the product bundle $\tilde{M} \times V_\nu$ must have the form

$$T \cdot (z, v) = \big(Tz,\ \rho(T) \cdot v\big) \tag{14.18}$$

for some group homomorphism $\rho \in \operatorname{Hom}(\Gamma, \mathcal{F}_\nu)$; and this homomorphism is the flat factor of automorphy describing the bundle $\mathcal{B}$. Clearly any homomorphism $\rho \in \operatorname{Hom}(\Gamma, \mathcal{F}_\nu)$ can be taken as the flat factor of automorphy describing a flat $\mathcal{F}_\nu$ bundle over M. On the other hand the bundle $\tilde{\mathcal{B}}$ is equivalent to a product bundle in various ways, for it is always possible to apply an automorphism $A : \tilde{M} \times V_\nu \longrightarrow \tilde{M} \times V_\nu$ of flat product bundles; if this automorphism is given explicitly by $A(z,v) = (z, \alpha\cdot v)$ for some $\alpha \in \mathcal{F}_\nu$ then it follows from (14.18) that $ATA^{-1}(z,v) = (Tz,\ \alpha\rho(T)\alpha^{-1}\cdot v)$, so this automorphism changes the factor of automorphy describing the bundle $\mathcal{B}$ to the conjugate homomorphism $\alpha\rho\alpha^{-1}$. It is evident from this that just as in the case of flat vector bundles *two flat factors of automorphy describe the same flat $\mathcal{F}_\nu$ bundle if and only if they are conjugate homomorphisms.*

This global description of flat $\mathcal{F}_\nu$ bundles can be used to derive a very convenient global description of $\mathcal{F}_\nu$ structures on a Riemann surface. The holonomy bundle of an $\mathcal{F}_\nu$ structure can be described by a flat factor of automorphy $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_\nu)$, which is called simply the *holonomy* of the $\mathcal{F}_\nu$ structure; it is of course determined only up to conjugation. Cross-sections of the holonomy bundle correspond to relatively automorphic functions for the flat factor of automorphy or holonomy $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_\nu)$, hence to mappings $\phi : \tilde{M} \longrightarrow V_\nu$ such that

$$\phi(Tz) = \rho(T)\cdot\phi(z) \tag{14.19}$$

for all $T \in \Gamma$. It follows from Corollary 14.5 that a group homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_\nu)$ is the holonomy of an $\mathcal{F}_\nu$ structure on M if and only if there is a locally biholomorphic mapping ϕ satisfying (14.19); this mapping, called the *development mapping* of the $\mathcal{F}_\nu$ structure, provides local coordinates on M describing the $\mathcal{F}_\nu$ structure. If $A \in \mathcal{F}_\nu$ then the composition $A\circ\phi : \tilde{M} \longrightarrow V_\nu$ is a locally biholomorphic mapping such that $(A\circ\phi)(Tz) = A\rho(T)A^{-1}\cdot(A\circ\phi)(z)$, so it is the development mapping for the same $\mathcal{F}_\nu$ structure but for a conjugate holonomy. It follows from Theorem 14.2 that if there are other $\mathcal{F}_\nu$ structures on M then they are in one-to-one correspondence with the $\mathcal{F}_\nu$ connections on M, where the correspondence is as described explicitly in Corollary **??**. The local coordinates provided by the development mapping ϕ are an $\mathcal{F}_\nu$ coordinate covering though, so their coordinate transition functions $f_{\alpha\beta}$ are solutions of the differential equation $D_\nu f_{\alpha\beta} = 0$; consequently it is evident from the defining equation (14.17) for an $\mathcal{F}_\nu$ connection that such a connection is just an $\mathcal{F}_\nu$ differential $\pi(z) = p_\alpha(z_\alpha)\cdot dz_\alpha^\nu$ on the surface M in terms of this coordinate covering, an abelian differential if $\nu = 1$ or a quadratic differential if $\nu = 2$. The local coordinates w_α describing the $\mathcal{F}_\nu$ structure associated to this connection $\pi(z)$ then are given by $w_\alpha = f_\alpha(z_\alpha)$ for any solutions of the differential equation $D_\nu f_\alpha(z_\alpha) = p_\alpha(z_\alpha)$ by Corollary **??**. Of course this new $\mathcal{F}_\nu$ structure also can be described by a development mapping $\psi : \tilde{M} \longrightarrow V_\nu$ for its holonomy $\sigma \in \mathrm{Hom}(\Gamma, \mathcal{F}_\nu)$, so it can be assumed that the local coordinates w_α are given by this development mapping and hence that $w_\alpha = \psi(z_\alpha)$; therefore the development mapping is a global solution of the differential equation $D_\nu\psi(z) = \pi(z)$ on the universal covering space $\tilde{M}$, and the holonomy σ is determined from this since $\psi(Tz) = \sigma(T)\cdot\psi(z)$ for all $T \in \Gamma$.

The use of holonomy and the development mapping to study pseudogroup structures can be illustrated nicely by examining in some detail the simplest cases. The situation for the Riemann sphere is rather trivial, since as in Corollary 14.4 it has a unique projective structure and no affine structures. A Riemann surface M of genus $g = 1$ has both affine and projective structures, as in Corollary 14.4. Indeed when M is identified with its Jacobi variety it appears as the quotient of the complex plane $\mathbb{C}$ by the lattice subgroup $\mathcal{L}$ generated by two complex numbers ω_1 and ω_2 that are linearly independent over the real numbers, and that provides a natural affine structure on M. The complex plane is the universal covering surface $\tilde{M}$ of M, and the lattice subgroup $\mathcal{L}$ is the covering translation group, generated by the translations $T_j : z \longrightarrow z + \omega_j$ for $i = 1, 2$. The development mapping $\phi : \tilde{M} = \mathbb{C} \longrightarrow V_1 = \mathbb{C}$ for this affine structure is just the identity mapping, and the holonomy $\rho : \Gamma = \mathcal{L} \longrightarrow \mathcal{F}_1$ is the trivial homomorphism $\rho(T_j) = T_j$. Other affine structures on M are described by affine connections; in terms of the coordinates on the torus, affine connections are just holomorphic abelian differentials, which are of the form $\pi(z) = a \cdot dz$ for arbitrary complex constants $a \in \mathbb{C}$. The development mapping ψ for the affine structure described by such a differential is a global solution of the differential equation $D_1\psi(z) = \psi''(z)/\psi'(z) = \frac{d}{dz} \log \psi'(z) = a$ on the universal covering space $\mathbb{C}$. If $a = 0$ the identity mapping is a solution, so as might be expected the trivial affine connection corresponds to the initial affine structure. If $a \neq 0$ the function $\psi(z) = e^{az}$ is a solution; in this case the development mapping satisfies

$$\psi(T_j z) = e^{a(z+\omega_j)} = e^{a\omega_j}\psi(z), \tag{14.20}$$

so the holonomy of this affine structure is the homomorphism $\rho : \mathcal{L} \longrightarrow \mathcal{F}_1$ for which

$$\rho(T_j) = \begin{pmatrix} e^{a\omega_j} & 0 \\ 0 & 1 \end{pmatrix}. \tag{14.21}$$

Of course any other solution ψ_1 of the differential equation $D_1\psi_1 = a$ is of the form $\psi_1 = A \circ \psi$ for some $A \in \mathcal{F}_1$ by Lemma 14.1 (iii); that solution leads to an equivalent affine structure on M, with the conjugate holonomy $\rho_1 = A\rho A^{-1}$. The initial affine structure is also a projective structure and other projective structures are described by projective connections; in terms of the coordinates on the torus, projective connections are just holomorphic quadratic differentials, which are of the form $\pi(z) = a \cdot dz^2$ for arbitrary complex constants $a \in \mathbb{C}$. The development mapping ψ for the projective structure described by such a differential is a global solution of the differential equation $D_2\psi(z) = a$ on the universal covering space $\mathbb{C}$. If $a = 0$ the identity mapping is a solution, so again as might be expected the trivial projective connection corresponds to the initial projective structure. If $a \neq 0$ and the differential operator D_2 is taken in the first form of (14.7) then solving the differential equation $D_2\psi = a$ amounts first to solving the differential equation $-2h''(z)/h(z) = a$ and then to solving the differential equation $\psi'(z) = h(z)^{-2}$. If $h(z) = e^{bz}$ then $-2h''(z)/h(z) = -2b^2$, so it is possible to take $h(z) = e^{bz}$ where $-2b^2 = a$; note that there are two choices of the parameter b that provide solutions. For either choice of the parameter b the function ψ is a solution of the differential equation $\psi'(z) = h(z)^{-2} = e^{-2bz}$, and it is possible to take $\psi(z) = -\frac{1}{2b}e^{-2bz}$. The development mapping satisfies

$$\psi(T_j z) = -\frac{1}{2b}e^{-2b(z+\omega_j)} = e^{-2b\omega_j}\psi(z) \tag{14.22}$$

so the holonomy of this projective structure is the group homomorphism $\rho : \mathcal{L} \longrightarrow \mathcal{F}_2$ for which

$$\rho(T_j) = \begin{pmatrix} e^{-2b\omega_j} & 0 \\ 0 & 1 \end{pmatrix}. \tag{14.23}$$

This of course is an affine mapping; so *all projective structures on M are equivalent to affine structures.* In addition when $a \neq 0$ there are two choices of the parameter b that correspond to the parameter a describing the projective structure; as in the earlier discussion of affine structures these two choices of the parameter b determine different affine structures, but the two different affine structures are equivalent when viewed as projective structures.

It is worth examining a bit more closely the holonomy bundles of affine and projective structures on surfaces of genus $g = 1$ as special flat affine and projective bundles; and for that purpose it is convenient to consider marked surfaces. Suppose therefore that M is a compact Riemann surface of genus $g = 1$ and that $T_1, T_2 \in \Gamma$ are generators of the covering translation group Γ. A flat affine bundle over M can be described by a conjugacy class of homomorphisms $\rho \in \text{Hom}(\Gamma, \mathcal{F}_1)$ of the covering translation group Γ. A homomorphism ρ in turn is described completely by the two matrices

$$\rho(T_j) = \begin{pmatrix} a_j & b_j \\ 0 & 1 \end{pmatrix} \in \mathcal{F}_1 \ \text{ for } \ j = 1, 2, \tag{14.24}$$

which can be any two matrices of this form that commute. Finally these two matrices are described fully by the parameters $(a_1, a_2; b_1, b_2) \in \mathbb{C}^4$ where $a_1 a_2 \neq 0$, and the condition that the two matrices commute is easily seen to be that $a_1 b_2 + b_1 = a_2 b_1 + b_2$. Thus the set $\text{Hom}(\Gamma, \mathcal{F}_1)$ of all these homomorphisms can be identified with the subset $V \subset \mathbb{C}^4$ defined by

$$V = \Big\{ (a_1, a_2; b_1, b_2) \Bigm| a_1 a_2 \neq 0, \ \ (a_1 - 1) b_2 = (a_2 - 1) b_1 \Big\} \tag{14.25}$$

for a chosen marking of the surface M; this is an algebraic subvariety of the complex manifold $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$. It is easy to see from this explicit description that the subvariety V actually is a submanifold except for the single singular point $P_0 = (1, 1; 0, 0)$ at which all the partial derivatives of the defining equation vanish. Since

$$\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & cb + d(1-a) \\ 0 & 1 \end{pmatrix}$$

conjugation in $\text{Hom}(\Gamma, \mathcal{F}_1)$ amounts to the action of the group $\mathbb{C}^* \times \mathbb{C}$ on the subvariety V given by

$$(c, d) \cdot (a_1, a_2; b_1, b_2) = \big(a_1, a_2; cb_1 + d(1 - a_1), cb_2 + d(1 - a_2)\big) \tag{14.26}$$

where $(c, d) \in \mathbb{C}^* \times \mathbb{C}$ and $(a_1, a_2; b_1, b_2) \in V$. The set of orbits of this group action has a somewhat complicated structure. First it is clear that the only point of V left fixed by the action of all (c, d) is the singular point $P_0 = (1, 1; 0, 0) \in V$; this point describes the identity homomorphism and represents the trivial affine bundle over M. Next the orbit of a point $(1, 1; b_1, b_2)$ where $(b_1, b_2) \neq (0, 0)$ consists of all points $(1, 1; cb_1, cb_2)$ for arbitrary $c \in \mathbb{C}^*$, so is naturally isomorphic to $\mathbb{C}^*$; and the set of orbits of this form can be put into one-to-one correspondence with the points of the Riemann sphere $\mathbb{P}^1$ by associating to the orbit of a point $(1, 1; b_1, b_2)$ the point of $\mathbb{P}^1$ represented by (b_1, b_2). The points of $\mathbb{P}^1$ thus parametrize the flat affine bundles described by flat factors of automorphy consisting of pure translations. Finally for any point $(a_1, a_2; b_1, b_2)$ for which $(a_1, a_2) \neq (1, 1)$ the system of linear equations

$$cb_1 + (a_1 - 1)d = 0, \qquad cb_2 + (a_2 - 1)d = 0$$

in the variables c, d has rank 1, since the determinant of this system of equations is zero in view of (**??**); consequently this system of equations has a nontrivial solution, showing that the orbit of the point $(a_1, a_2; b_1, b_2)$ contains a point of the form $(a_1, a_2; 0, 0)$, which clearly is unique. The orbit of the point $(a_1, a_2; 0, 0)$ when $(a_1, a_2) \neq (1, 1)$ consists of all points $\big(a_1, a_2; d(1 - a_1), d(1 - a_2)\big)$ for arbitrary $d \in \mathbb{C}$, so is naturally isomorphic to $\mathbb{C}$. The set of orbits of this form, parametrizing the set of nontrivial flat affine bundles that can be described by factors of automorphy consisting entirely of pure multiplications, that is, of transformations of the form $z \longrightarrow cz$, can be put into one-to-one correspondence with the complement of the point $(1, 1)$ in the product $(\mathbb{C}^*)^2$ by associating to the orbit of a point $(a_1, a_2; b_1, b_2)$ for which $(a_1, a_2) \neq (1, 1)$ the point $(a_1, a_2) \in (\mathbb{C}^*)^2 \sim (1, 1)$. Altogether then the set of all orbits under the group action (**??**), parametrizing the set of all flat affine bundles over the marked surface M, has the form

$$V/\mathcal{F}_1 \cong P_0 \cup \mathbb{P}^1 \cup \Big((\mathbb{C}^*)^2 \sim (1, 1)\Big); \tag{14.27}$$

each of the three components is a complex manifold, and they are of dimensions 0, 1, and 2.

To examine more closely the way in which these separate manifolds are linked consider the surjective projection

$$p : V \longrightarrow (\mathbb{C}^*)^2 \quad \text{defined by} \quad p(a_1, a_2; b_1, b_2) = (a_1, a_2). \tag{14.28}$$

The inverse image $p^{-1}(a_1, a_2)$ of a point $(a_1, a_2) \in (\mathbb{C}^*)^2$ consists of those points $(b_1, b_2) \in \mathbb{C}^2$ such that $(a_1 - 1)b_2 = (a_2 - 1)b_1$, so is a one-dimensional linear subspace of $\mathbb{C}^2$ if $(a_1, a_2) \neq (1, 1)$ but is $\mathbb{C}^2$ if $(a_1, a_2) = (1, 1)$. This inverse image is preserved by the group action (**??**), so the projection p induces a surjective projection

$$p_* : V/\mathcal{F}_1 \longrightarrow (\mathbb{C}^*)^2. \tag{14.29}$$

The inverse image $p_*^{-1}(a_1, a_2)$ of a point $(a_1, a_2) \in (\mathbb{C}^*)^2$ is a single point so long as $(a_1, a_2) \neq (1, 1)$ but is $P_0 \cup \mathbb{P}^1$ if $(a_1, a_2) = (1, 1)$. Thus when the singular point P_0 is excluded the mapping p_* exhibits the remainder of set of orbits $V/\mathcal{F}_1$ as the result of "blowing up" the point $(1,1) \in (\mathbb{C}^*)^2$ to a projective space $\mathbb{P}^1$; indeed the quotient set aside from the point P_0 is the algebraic subvariety of the product $(\mathbb{C}^*)^2 \times \mathbb{P}^1$ defined by (**??**) when the values (b_1, b_2) are viewed as projective coordinates of a point in $\mathbb{P}^1$. Alternatively when the subvariety $\mathbb{P}^1$ is excluded the mapping p_* identifies the remainder of the set of orbits $V/\mathcal{F}_1$ with $\mathbb{C}^2$, where the origin represents the trivial affine bundle over M viewed in this way as a special case of the affine bundle described by a factor of automorphy consisting entirely of pure multiplications. This is a nice example of a general phenomenon. Often the set of general mathematical structures of some sort naturally has the structure of a noncompact complex variety, which can be completed to a compact variety in which it appears as the complement of a proper holomorphic subvariety consisting of special structures; but there can be different compactifications, in which the complementary subvarieties are the moduli of different sorts of special structures.

To interpret the moduli space (14.27) in terms of fibre bundles over M, the projection p of (14.28) can be identified with the surjective mapping

$$p : \mathrm{Hom}(\Gamma, \mathcal{F}_1) \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{C}^*) \tag{14.30}$$

that associates to a homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_1)$ the composition $p(\rho) = \pi \circ \rho$ with the homomorphism π of (**??**), since any homomorphism $\phi \in \mathrm{Hom}(\Gamma, \mathbb{C}^*)$ is determined by the values $\phi(T_j) = a_j$ and these can be any points $(a_1, a_2) \in (\mathbb{C}^*)^2$. The image $p(\rho)$ of a homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_1)$ depends only on its conjugacy class, hence on the flat affine bundle described by that homomorphism; the mapping (14.28) therefore induces a surjective mapping from flat affine bundles over M to flat line bundles over M, which can be identified with the mapping p_* of (14.29). The image of a flat affine bundle is called the *subordinate line bundle* to the affine bundle. The preceding discussion shows that there is a unique flat affine bundle having a nontrivial flat line bundle as a subordinate line bundle; but the set of flat affine bundles having the identity line bundle as a subordinate line bundle is the set $P_0 \bigcup \mathbb{P}^1$ consisting of the identity bundle, represented by the point P_0, together with the set of flat affine bundles that can be described by nontrivial flat factors of automorphy consisting entirely of translations, which set is naturally parametrized by the complex manifold $\mathbb{P}^1$. Not all flat affine bundles are holonomy bundles of affine structures on a topological surface of genus $g = 1$ however.

Theorem 14.6 *All flat affine bundles over a compact topological surface of genus $g = 1$ are the holonomy bundles of affine structures on that surface except for those bundles represented by factors of automorphy $\rho \in Hom(\Gamma, \mathcal{F}_1)$ that are conjugate to homomorphisms that either*
(i) *consist of pure translations through vectors all of which are parallel, or*
(ii) *consist of pure multiplications by complex numbers of absolute value* 1.
A flat affine bundle that is the holonomy bundle of an affine structure on the surface determines that affine structure completely.

Proof: The identity bundle, represented by the singular point $P_0 \in V/\mathcal{F}_1$ and a special case of the exceptional bundles of type (ii) in the statement of the theorem, obviously is not the holonomy bundle of any affine structure. A topological surface with an affine structure of course has the associated complex structure, so is naturally a Riemann surface; and a marked Riemann surface M of genus $g = 1$ has a unique affine structure with holonomy parametrized by a point of the subset $\mathbb{P}^1 \subset V/\mathcal{F}_1$. That is the affine structure arising from the identification of M with its Jacobi variety and hence from the representation of M as the quotient of the complex plane by the lattice subgroup generated by two complex numbers $b_1, b_2 \in \mathbb{C}$ that are linearly independent over the real numbers; and by (14.21) all other affine structures on M are parametrized by points of the set $(\mathbb{C}^*)^2 \sim (1,1)$. Thus all flat affine bundles parametrized by points of $\mathbb{P}^1 \subset V/\mathcal{F}_1$ are the holonomy bundles of affine structures except for those for which b_1, b_2 are linearly dependent over the real numbers; and in that exceptional case the homomorphisms $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_1)$ are translations through vectors that are integral linear combinations of b_1 and b_2 and hence that are parallel vectors in the complex plane. It is evident that such a holonomy describes the flat affine structure uniquely. A Riemann surface of genus $g = 1$ represented as the quotient of $\mathbb{C}$ by the lattice subgroup generated by $b_1, b_2 \in \mathbb{C}$ also has an affine structure with the development mapping $\psi(z) = e^{az}$ for any nonzero complex number a; the holonomy of this affine structure clearly is parametrized by the point $(a_1, a_2) \in \left(\mathbb{C}^*\right)^2 \sim (1,1) \in V/\mathcal{F}_1$ where $a_1 = e^{ab_1}, a_2 = e^{ab_2}$. In this case too it is easy to see that the holonomy determines the affine structure uniquely. Indeed if the affine structures determined by two affine connections adz and $a'dz$ have the same holonomy then $e^{a'b_j} = e^{ab_j}$ so that $a'b_j = ab_j + 2\pi i n_j$ for some integers n_j for $j = 1, 2$; but then $a' - a = 2\pi i n_j / b_j$ for $j = 1, 2$ so that $n_1 b_2 = n_2 b_1$, which is impossible since the complex numbers b_1, b_2 are linearly independent over the real numbers. Any pair of parameter values a_1, a_2 can be written in the form $a_j = e^{b_j}$, and if b_1, b_2 are linearly independent over the real numbers then the associated Riemann surface has the affine structure with that holonomy. If b_1, b_2 are linearly dependent over the real numbers, say for instance if $b_2 = rb_1$ for some real number r, and if b_1 is not purely imaginary, then b_1 and $b_2 + 2\pi i$ are linearly independent over the real numbers so a_1, a_2 are the parameter values for the holonomy of an affine structure of this surface. If b_1 and b_2 are both purely imaginary though no combinations $b_j + 2\pi i n_j$ are ever linearly independent over the real numbers so the associated bundle cannot be the holonomy bundle of an affine structure on any surface; in this exceptional case the homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_1)$ consists of pure multiplications of absolute value 1. That suffices to conclude the proof of the theorem.

To turn next to flat projective bundles over a marked Riemann surface M of genus $g = 1$, any such bundle can be described by a conjugacy class of homomorphisms $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_2)$ of the covering translation group Γ of M. Any such homomorphism in turn can be described by matrices $A_j \in \mathrm{Sl}(2, \mathbb{C})$ representing the projective transformations $\rho(T_j) \in \mathcal{F}_2$ for $j = 1, 2$; these matrices are determined uniquely only up to a factor of ± 1, and can be any two matrices representing projective linear transformations that commute, hence any two special linear matrices A_1, A_2 such that $A_1A_2 = \epsilon A_2A_1$ where $\epsilon = \pm 1$. Note that if $\epsilon = 1$ then the homomorphism ρ actually lifts to a homomorphism in $\mathrm{Hom}(\Gamma, \mathrm{Sl}(2,\mathbb{C}))$, while that is not the case if $\epsilon = -1$.

Theorem 14.7 *A flat projective bundle over a compact topological surface of genus $g = 1$ is equivalent to a flat affine bundle if and only a factor of automorphy $\rho \in Hom(\Gamma, \mathcal{F}_2)$ describing that bundle lifts to a homomorphism $\rho \in Hom(\Gamma, Sl(2,\mathbb{C}))$. A flat projective bundle is the holonomy bundle of a projective structure on the surface M if and only it is equivalent to a flat affine bundle that is the holonomy bundle of an affine structure on that surface.*

Proof: Suppose that $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_2)$ is a factor of automorphy describing a flat projective bundle over a topological surface M of genus $g = 1$, where the homomorphism ρ is described by matrices A_1, $A_2 \in \mathrm{Sl}(2,\mathbb{C})$ such that $A_1A_2 = \epsilon A_2A_1$. If v is an eigenvector of the matrix A_1 with eigenvalue a then $A_1(A_2v) = \epsilon A_2A_1v = \epsilon A_2av = \epsilon a(A_2v)$, so A_2v is also an eigenvector of the matrix A_1 but with eigenvalue ϵa. There are three cases to consider.
(i) First if A_1 has but a single eigenvector v then A_2v must be a multiple of that eigenvector and $\epsilon = 1$; and v is also an eigenvector of A_2. After conjugation it can be assumed that $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $A_1 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ where $ad = 1$; and since A_1 has but a single eigenvector necessarily $b \neq 0$ and $a = d = \pm 1$. Of course since A_1 can be multiplied by ± 1 it can be assumed that $a = d = 1$, so that A_1 actually is a pure translation $A_1 = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}$ by the vector $b_1 \neq 0$. Similarly the matrix A_2 can be taken to be a pure translation $A_2 = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$ by a vector b_2. An easy calculation shows that a pair of such matrices is conjugate in $\mathcal{F}_2$ to another pair if and only if they are conjugate in $\mathcal{F}_1$. Thus a flat projective bundle of this special type is equivalent to a unique flat affine bundle represented by a homomorphism $\rho \in \mathrm{Hom}(\Gamma, \mathcal{F}_1)$ consisting of pure translations, a flat affine bundle parametrized by a point in the projective line $\mathbb{P}^1$.
(ii) Next suppose that A_1 has two distinct eigenvectors v_1 and v_2, and that A_2v_j is a multiple of v_j for $j = 1, 2$ so v_1 and v_2 are also eigenvectors for A_2. After conjugation it can be assumed that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so the matrices A_1 and A_2 are diagonal matrices $A_j = \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}$ where $a_jb_j = 1$; and since they clearly commute $\epsilon = 1$. Another easy calculation shows that a pair of diagonal matrices with diagonal entries a_j, b_j for $j = 1, 2$ is conjugate in $\mathcal{F}_2$ to another pair of diagonal matrices with diagonal entries a'_j, b'_j if and only if either $a'_j = a_j$ and $b'_j = b_j$ or $a'_j = b_j$ and $b'_j = a_j$. Thus a flat projective bundle of this special type is equivalent to a flat affine bundle consisting of pure multiplications, that is, described by affine transformations of the form $\rho(T_j)z = c_jz$, so it is either the identity bundle or one of the bundles parametrized by points of $(\mathbb{C}^*)^2 \sim (1, 1)$. Furthermore flat affine bundles generated by pure

multiplications by c_j and c'_j for $j = 1, 2$ represent the same flat projective bundle if and only either $c'_j = c_j$ or $c'_j = 1/c_j$.
(iii) Finally suppose that A_1 has two distinct eigenvectors v_1 and v_2, and that $A_2 v_1$ is a multiple of v_2 while $A_2 v_2$ is a multiple of v_1; after conjugation it can be assumed that $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so that $A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & b_2 \\ c_2 & 0 \end{pmatrix}$ where $a_1 d_1 = -b_2 c_2 = 1$. If $\epsilon = 1$ another easy calculation shows that $a_1 = d_1 = \pm 1$, so after a change of sign $A_1 = \mathrm{I}$, the identity matrix; in that case the arguments iin (i) and (ii) can be applied to the matrix A_2 rather than A_1, so the bundle falls under either case (i) or case (ii). On the other hand if $\epsilon = -1$ then $d_1 = -a_1$ so that $a_1 = \pm i$, and after a change of sign $A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Another easy calculation shows that matrix pairs of this form are conjugate in $\mathcal{F}_2$ if and only if the quotients b_2/c_2 are the same; thus a flat projective bundle of this special type can be reduced to the form $\rho(T_1)z = -z, \rho(T_2)z = c/z$ for some nonzero constant c when $\epsilon = -1$, and is not equivalent to a flat affine bundle. Since all projective structures on M are equivalent to affine structures, having holonomy (14.23), that suffices to conclude the proof of the theorem.

Suppose that M is a Riemann surface with an $\mathcal{F}_\nu$ structure described by a coordinate covering $\{U_\alpha, z_\alpha\}$ with coordinate transition functions $z_\alpha = f_{\alpha\beta}(z_\beta)$ in the intersections $U_\alpha \cap U_\beta$, where $f_{\alpha\beta} \in \mathcal{F}_\nu$. The mappings $f_{\alpha\beta}$ can be viewed as abstract mappings, either as affine mappings $f_{\alpha\beta} : \mathbb{C} \longrightarrow \mathbb{C}$ for $\nu = 1$ or as projective mappings $f_{\alpha\beta} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ for $\nu = 2$; and for any triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ these mappings satisfy the compatibility condition $f_{\alpha\beta} f_{\beta\gamma} = f_{\alpha\gamma}$, so they define coordinate bundles over M, indeed define flat coordinate bundles since the mappings are constant in each intersection. For an equivalent $\mathcal{F}_\nu$ structure, given by a change of coordinates $w_\alpha = f_\alpha(z_\alpha)$ after passing to a refinement of the coordinate covering if necessary, the corresponding mappings are $f_\alpha f_{\alpha\beta} f_\beta^{-1}$, which determine equivalent flat coordinate bundles over M. Thus to each $\mathcal{F}_\nu$ structure on M there is associated in this way a flat fibre bundle over M, with group $\mathrm{A}(1, \mathbb{C})$ and fibre $\mathbb{C}$ in the case of an affine structure and with group $\mathrm{Pl}(1, \mathbb{C})$ and fibre $\mathbb{P}^1$ in the case of a projective structure. This bundle is called the *monodromy bundle* associated to the $\mathcal{F}_\nu$ structure on M. The local coordinates z_α can be viewed as holomorphic cross-sections of the monodromy bundle. Actually these cross-sections have the special property that they are holomorphic local homeomorphisms from M into the fibre $\mathbb{C}$ or $\mathbb{P}^1$; indeed it is evident that *a flat fibre bundle of one or the other of these two types is the monodromy bundle of an $\mathcal{F}_\nu$ structure on the Riemann surface M for the appropriate value of ν if and only if it admits a holomorphic cross-section that is a local homeomorphism from M to the fibre.*

Any fibre bundle over M can be represented by a factor of automorphy for the action of the covering translation group Γ on the universal covering space $\tilde{M}$. Although that was discussed only for holomorphic line bundles in Addendum **??**, it is true, and even easier to prove, for the monodromy bundles of $\mathcal{F}_\nu$

structures over M. Indeed any flat fibre bundle over M naturally lifts to a flat fibre bundle over the universal covering space $\tilde{M}$ by using the same coordinate transition functions in each lift to $\tilde{M}$ of an intersection $U_\alpha \cap U_\beta$ of coordinate neighborhoods in M, since it can be assumed by passing to a suitable refinement that the coordinate covering is chosen so that these intersections are simply connected; and the quotient of the lifted bundle over $\tilde{M}$ by the natural action of the covering translation group Γ is isomorphic to the initial bundle over M. A familiar argument shows that the associated flat principal bundle over $\tilde{M}$ has a locally constant cross-section. Indeed the identity group element viewed as a constant cross-section over a base coordinate neighborhood U_{α_0} can be extended to a cross-section over any intersecting coordinate neighborhood through the appropriate constant coordinate transition function, and that process can be continued; the resulting cross-section of the principal bundle over the simply connected manifold $\tilde{M}$ is well defined and single valued as a consequence of the familiar monodromy principle, since it admits a continuation along any path in $\tilde{M}$. Such a cross-section then can be used to reduce the lifted bundle over $\tilde{M}$ to an equivalent flat bundle that is trivial, a product of the manifold $\tilde{M}$ and the fibre; of course the trivialization is unique only up to a flat isomorphism of the product bundle. This trivialization transforms the action of the covering translation group on the lifted bundle to an action on the product bundle such that the quotient again is the initial bundle over M; and this action is described by a flat factor of automorphy for the covering translation group Γ, where equivalent trivializations lead to equivalent factors of automorphy.

For a flat affine bundle over M, for example, the lifted bundle is equivalent to the product bundle $\tilde{M} \times \mathbb{C}$, and the action of a covering translation $T \in \Gamma$ on the product takes a point $(z, t) \in \tilde{M} \times \mathbb{C}$ to the point $T(z, t) = (Tz, \rho(T)t)$ where Tz is the image of the point $z \in \tilde{M}$ under the covering translation T and $\rho(T) \in \mathrm{A}(1, \mathbb{C})$ is an affine transformation depending on the element T and acting on the fibre $\mathbb{C}$. This exhibits Γ as a group of transformations acting on $\tilde{M} \times \mathbb{C}$, so $\rho(ST) + \rho(S)\rho(T)$ for any two covering translation mappings $S, T, \in \Gamma$. The group homomorphism $\rho : \Gamma \longrightarrow \mathbb{A}$, which is the flat factor of automorphy describing the flat affine bundle over M, is called the *monodromy representation* of that bundle. Equivalent trivializations of the lifted bundle arise from flat bundle isomorphisms $(z, t) \longrightarrow (z, \sigma t)$ for some affine transformations $\sigma \in \mathrm{A}(1, \mathbb{C})$, which clearly transform the monodromy representation to the conjugate $\tilde{\rho}(T) = \sigma\rho(T)\sigma^{-1}$. If the initial flat affine bundle is the monodromy bundle of an affine structure on the Riemann surface M the local coordinate mappings describe a holomorphic cross-section of the monodromy bundle which is a holomorphic local homeomorhism into the fibre $\mathbb{C}$. This cross-section lifts to a Γ invariant cross-section of the lifted bundle; and when that bundle is trivialized it becomes a holomorphic local homeomorphism $f : \tilde{M} \longrightarrow \mathbb{C}$ that is invariant under the action of Γ on the product bundle, hence a holomorphic local homeomorhism such that $f(Tz) = \rho(T) \cdot f(z)$ for every point $z \in \tilde{M}$ and every covering translation $T \in \Gamma$. This mapping is called the *development mapping* of the affine structure on M. The same affine structure is described by the composite development mapping $\sigma \circ f$ with the monodromy representation

$\sigma\rho\sigma^{-1}$ for any affine transformation $\sigma \in \mathrm{A}(1,\mathbb{C})$. The situation is quite the same for flat projective structures, which are described by development mappings $f : \tilde{M} \longrightarrow \mathbb{P}^1$ which are holomorphic local homeomorphisms such that $f(Tz) = \rho(T)f(z)$ for the monodromy representation $\rho : \Gamma \longrightarrow \mathrm{Pl}(1,\mathbb{C})$; equivalent projective structures are described by the equivalent development mappings $\sigma \circ f$ for the conjugate homomorphisms $\sigma\rho\sigma^{-1}$ for any element $\sigma \in \mathrm{Pl}(1,\mathbb{C})$.

The only compact Riemann surfaces that admit an affine structure are those of genus $g = 1$; and since that case can be worked out quite simply and completely it may serve as a useful model for the general theory of such structures on compact Riemann surfaces. Consider therefore a compact Riemann surface M of genus $g = 1$, which can be represented by the quotient $M = \mathbb{C}/\mathcal{L}$ where $\mathcal{L} \subset \mathbb{C}$ is a lattice subgroup that can be taken as generated by the complex numbers 1 and ω where $\Im\omega > 0$. The complex coordinate z the plane provides an affine structure on the quotient, for which the coordinate transformations are either the identity or translation by a vector in the lattice subgroup $\mathcal{L}$. The covering translation group is just the lattice group $\mathcal{L}$ itself, generated by the translations $T_1 : z \longrightarrow z + 1$ and $T_2 : z \longrightarrow z + \omega$; and the universal covering space $\mathbb{C}$ is already a trivial affine structure, so the monodromy is the identity mapping from the lattice $\mathcal{L}$ to itself and the development mapping is the identity mapping.

The monodromy of any flat affine bundle over Riemann surface M of genus 1 is described by an element of the quotient space $V/\mathbb{A}$ where $V = \mathrm{Hom}(\Gamma, \mathbb{A})$ and the affine group $\mathbb{A}$ acts on V by conjugation. If the affine group is viewed as the subgroup $\mathbb{A} \subset \mathrm{Gl}(2,\mathbb{C})$ consisting of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and each such matrix is described by the pair of complex numbers (a, b) for which $a \neq 0$ then an element $\rho \in V$ can be described by the two matrices $\rho(T_i)$ or equivalently by the four complex numbers $(a_1, a_2; b_1, b_2)$ where $a_1a_2 \neq 0$. These four numbers describe an element of V precisely when the matrices $\rho(T_i)$ commute, which is easily seen to be the condition that $a_1b_2 + b_1 = a_2b_1 + b_2$; thus there results the natural identification

$$V \cong \Big\{ (a_1, a_2; b_1, b_2) \Big| \, a_1a_2 \neq 0, \;\; (a_1 - 1)b_2 = (a_2 - 1)b_1 \Big\}. \tag{14.31}$$

This describes the set V as a three-dimensional complex analytic or even algebraic subvariety of the product manifold $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}$, defined by the vanishing of this single polynomial equation. It is easy to see from this explicit description that this subvariety V actually is a submanifold except for the single singular point $(1, 1; 0, 0)$, at which all the partial derivatives of the defining equation vanish. Under conjugation

$$STS^{-1} = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c} & -\frac{d}{c} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & cb + d(1-a) \\ 0 & 1 \end{pmatrix}$$

where $S = \begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. Thus conjugation by the affine transformation described by the parameters (c, d) has the effect

$$(c,d)(a_1,a_2;b_1,b_2) = (a_1,a_2;cb_1 + d(1-a_1), cb_2 + d(1-a_2)). \tag{14.32}$$

This action actually is rather nontrivial, and the space of orbits has a somewhat complicated structure. First it is easy to see that the only point left fixed by conjugation by all elements of the affine group $\mathbb{A}$ is the singular point $P_0 = (1,1;0,0)$; it thus represents one special flat affine bundle over M, the identity bundle. Next the orbit of a point $(1,1;b_1,b_2)$ where $(b_1,b_2) \neq (0,0)$ consists of all points $(1,1;cb_1,cb_2)$ for arbitrary $c \in \mathbb{C}^*$; thus each orbit is naturally isomorphic to $\mathbb{C}^*$, and the set of orbits can be put into one-to-one correspondence with the points of the Riemann sphere $\mathbb{P}^1$. The flat affine bundles represented by these orbits are those defined by pure translations. Finally for any point $a_1,a_2;b_1,b_2)$ where $(a_1,a_2) \neq (1,1)$ the system of linear equations

$$\begin{aligned} cb_1 + (a_1-1)d &= 0 \\ cb_2 + (a_2-1)d &= 0 \end{aligned}$$

in the variables c, d has rank 1, since the determinant of this system of equations is just the polynomial that characterizes the commutativity condition; thus any orbit contains a point of the form $(a_1,a_2;0,0)$, indeed obviously a unique point of this form, so the space of orbits can be put into one-to-one correspondence with the complement of the point $(1,1)$ in the produce $\mathbb{C}^* \times \mathbb{C}^*$. Altogether therefore

$$V/\mathbb{A} \cong P_0 \cup \mathbb{P}^1 \cup \{\mathbb{C}^* \times \mathbb{C}^* - (1,1)\}. \tag{14.33}$$

The variety V can be mapped onto the product space $\mathbb{C}^* \times \mathbb{C}^*$ by the natural projection mapping $\pi : (a_1,a_2;b_1,b_2) \longrightarrow (a_1,a_2)$; the fibre of this mapping over any point is naturally the product $\mathbb{C} \times \mathbb{C}$, consisting of the points (b_1,b_2). The action of the group $\mathbb{A}$ of analytic mappings of this space to itself commutes with the projection π, so acts on the fibres separately. It is transitive on all the fibres except those over the point $(1,1)$, while its action on the fibre over that point identifies the quotient with the union of a point and the projective line $\mathbb{P}^1$; so the quotient variety $V/\mathbb{A}$ appears as a rather singular blowing up of the product $\mathbb{C}^* \times \mathbb{C}^*$ at the point $(1,1)$). Although it is possible to say more, this is probably quite enough for present purposes.

When $a = 0$ the mapping $f(z)$ is just the identity mapping, yielding the initial affine structure on M. When $a \neq 0$ the development mapping $f : \mathbb{C} \longrightarrow \mathbb{C}$ exhibits the universal covering spce $\mathbb{C}$ of the surface M as the universal covering space of the punctured plane $\mathbb{C}^*$. The development mapping clearly satisfies

$$f(T_1 z) = e^{a(z+1)} = e^a f(z),$$

$$f(T_2(z) = e^{a(z+\omega)} = e^{a\omega} f(z),$$

so the flat affine bundle associated to the affine structure is described by the representation

$$\rho(T_1) = e^a, \quad \rho(T_2) = e^{a\omega};$$

it thus corresponds to the point $(a_1, a_2; b_1, b_2) = (e^a, e^{a\omega}; 00) \in V$. The case $a = 0$ thus yields a point on the moduli space $V/\mathbb{A}$ corresponding to the singular point P_0. Since $\Im\omega \neq 0$ it is evident that $(a_1, a_2) \neq (1, 1)$, so there are no affine structures corresponding to the points $(1, 1; b_1, b_2) \in V/\mathbb{A}$, which means that the representations in that orbit do not correspond to any affine structures on a torus. Finally the points in $V/\mathbb{A}$ for which $(a_1, a_2) \neq (1, 1)$ in the moduli space for flat affine bundles that actually arise from affine structures are those for which $a_1 = e^a, a_2 = e^{a\omega}$ for some complex number ω with $\Im\Omega \neq 0$. That condition on ω is equivalent to the condition that the points a_1, a_2 are not real multiples of one another, that is, are not collinear with the origin in the complex plane; so the representations corresponding to such bundles also can never be represented by an affine structure. Thus the subset of the moduli space of flat affine bundles that arise from affine structures is the complement of a proper subset of the moduli space $V/\mathbb{A}$ as described. Note further that

$$\omega = \frac{\log a_1 + 2\pi i n_1}{\log a_2 + 2\pi i n_2}$$

for some integers n_1, n_2, for any given determination of the logarithms of the constants a_1, a_2; thus the bundle determines the Riemann surface itself at least locally, in the obvious sense.

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX■

[CHECK THIS SECTION]

The situation for affine or projective structures on Riemann surfaces is simplified by the observation that the monodromy group really determines the structure. The situation for affine structures is particularly simple.

Theorem 14.8 *A flat affine structure on a compact Riemann surface is determined uniquely by the flat affine bundle associated to the structure.*

Proof: Consider a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of the Riemann surface M for which there are two sets of affine coordinates $\{z_\alpha\}$ and $\{w_\alpha\}$ that determine the same flat affine bundle over M. Thus in each intersection $U_\alpha \cap U_\beta$ the coordinate functions satisfy

$$z_\alpha = a_{\alpha\beta} z_\beta + b_{\alpha\beta} \quad \text{and} \quad w_\alpha = a_{\alpha\beta} w_\beta + b_{\alpha\beta}.$$

The two sets of local coordinates are also satisfy $w_\alpha = f_\alpha(z_\alpha)$ for some holomorphic functions f_α, and for each intersection $U_\alpha \cap U_\beta$ these functions must therefore be such that

$$a_{\alpha\beta} f_\beta(z_\beta) + b_{\alpha\beta} = f_\alpha(a_{\alpha\beta} z_\beta + b_{\alpha\beta})$$

where $w_\beta = f_\beta(z_\beta)$; differentiating this equation with respect to the variable z_β shows that

$$a_{\alpha\beta} f'_\beta(z_\beta) = f'_\alpha(a_{\alpha\beta} z_\beta + b_{\alpha\beta}) a_{\alpha\beta}.$$

The derivatives $g_\alpha = f'_\alpha(z_\alpha)$ thus are holomorphic functions in the coordinate neighborhoods U_α such that $g_\alpha = g_\beta$ in the intersection $U_\alpha \cap U_\beta$, hence they describe a global holomorphic function on the entire compact Riemann surface M, which must be a constant c; and therefore $w_\alpha = cz_\alpha + d_\alpha$ for some constants d_α, so that the two systems of local coordinates are related by a affine transformations so determine the same affine structure. That suffices to conclude the proof of the theorem.

xxxxxxxxxxxxxxxxxxxxxxxxxxxx

Compact Riemann surfaces of genus $g > 1$ admit branched affine structures though; indeed for any choice of a base point $a_0 \in M$ there is a unique normalized basic affine structure on M, an affine structure on $M \sim a_0$ branched at the point a_0 and with unitary submonodromy group. To be more explicit, choose a coordinate covering U_α of the surface M with local coordinates z_α; and assume that the branch point $a_0 \in M$ is contained in a single coordinate neighborhood U_0. If $p_\alpha dz_\alpha$ is the meromophic affine connection describing the normalized branched affine structure, and $f_\alpha(z_\alpha)$ are holomorphic functions in the coordinate neighborhoods such that $D_1 f_\alpha dz_\alpha = p_\alpha dz_\alpha$ in each coordinate neighborhood then $w_\alpha = f_\alpha(z_\alpha)$ are local coordinates in U_α for all coordinate neighborhoods other than U_0 while the local coordinate w_0 in the coordinate neighborhood U_0 is such that $f_0(z_0) = w_0^{2g-1}$. The functions $f_\alpha(z_\alpha)$ satisfy

$$
\begin{aligned}
f_\alpha(z_\alpha) &= \phi_{\alpha\beta}\big(f_\beta(z_\beta)\big) \\
&= a_{\alpha\beta} f_\beta(z_\beta) + b_{\alpha\beta}
\end{aligned}
$$

in each intersection $U_\alpha \cap U_\beta$, for the affine transformations $\phi_{\alpha\beta}$ defining the monodromy bundle of the branched affine structure.

The branched affine structure lifts to a branched affine structure on the universal covering space $\tilde{M}$, branched at all the points covering a_0; this is as usual accomplished merely by using the local coordinates w_α in all the lifts of the coordinate neighborhoods U_α, which can be assumed to be simply connected. The resulting structure, and its monodromy bundle, are invariant under the covering translation group. The monodromy bundle can be trivialized over $\tilde{M}$, by choosing a flat cross-section ϕ_α; the functions $\phi_\alpha^{-1} \circ f_\alpha$ are then equal in the intersections of the lifted coordinate neighborhoods on $\tilde{M}$, but are transformed by affine transformations under the action of the covering translation group. Thus there results a holomorphic function f on the universal covering space $\tilde{M}$ such that the restriction of this function to the inverse image of any coordinate neighborhood $U_\alpha \subset M$ is the composition $f = \phi_\alpha^{-1} \circ f_\alpha$ for the affine mapping ϕ_α, and

$$f(Tz) = \phi(T) \circ f(z)$$

for any covering translation $T \in \Gamma$ and for some affine mapping $\phi(T)$. It is evident that

$$\phi : \Gamma \longrightarrow \mathrm{UA}(1, \mathbb{C})$$

is a homomorphism from the covering translation group Γ to the group of unitary affine transformations, called the *monodromy mapping*, and its image is a subgroup of the unitary affine group called the *monodromy group* of the branched affine structure. The holomorphic mapping

$$f : \tilde{M} \longrightarrow \mathbb{C}$$

is a local homeomorphism except at the points lying over the branch point $a_0 \in M$, where it is a branched covering of degree $2g-2$. The mapping f is the *development mapping* of the branched affine structure and the homomorphism ϕ is its *monodromy representation*. It is evident that the development mapping is an isometric mapping from $\tilde{M}$ to the complex plane, since the local form of the metric is precisely that induced by the euclidean metric in the complex plane by the affine coordinates.

Theorem 14.9 *If M is a compact Riemann surface of genus $g > 0$ with the normalized basic branched affine structure with a branch point $a_0 \in M$ and if the image of the development mapping f is the open subset $U \subset \mathbb{C}$ then the monodromy group acts as a properly discontinuous group of transformations of the set U; the quotient $U/\phi(\Gamma) = N$ is a compact Riemann surface, and the development mapping f induces a branched covering mapping $f : M \longrightarrow N$.*

Proof: Consider then a point $\tilde{p} \in \tilde{M}$, which may or may not be a branch point of the branched affine structure. The points $T\tilde{p}$ are precisely the points of $\tilde{M}$ that cover the same image $p = \pi(\tilde{p}) \in M$; and $f(T\tilde{p}) = \phi(T)f(\tilde{p})$ for all covering translations $T \in \Gamma$. Since the development mapping is an isometry, the distance $|\phi(T)f(\tilde{p}) - f(\tilde{p})|$ is the length of a geodesic in $\tilde{M}$ from $\tilde{p}$ to $T\tilde{p}$. Each such geodesic projects to a geodesic on the initial Riemann surface M from the point p to itself but representing a nontrivial class in the fundamental group of the surface M at that point. Since an open neighborhood of p is simply connected, each such geodesic must leave and re-enter that neighborhood, so that its length is bounded below by ϵ, the distance from the point p to the boundary of that neighborhood. Consequently if $z = f(\tilde{p})$ then $\phi(T)z - z| \ge \epsilon$ for every $T \in \Gamma$ other than the identity element. That is just the condition that the group $\phi(\Gamma)$ is properly discontinuous. It is well known that that is a sufficient condition for the quotient $N = \mathbb{C}/\phi(\Gamma)$ to have the natural structure of a Riemann surface itself. The development mapping induces a surjective holomorphic mapping $f : M \longrightarrow N$, and since M is compact necessarily N also is compact. That suffices for a proof of the theorem.

Although the only compact Riemann surfaces that admit affine structures are those of genus $g = 1$, by Corollary 14.4, nonetheless any compact Riemann surface admits branched affine structures arising from meromorphic affine connections on the surface. A meromorphic affine connection expressed in terms of a coordinate covering $\mathfrak{U} = \{U_\alpha\}$ of a Riemann surface M, with local coordinates z_α and coordinate transition functions $f_{\alpha\beta}$, is a collection of local meromorphic differential forms $p_\alpha \cdot dz_\alpha$ in the coordinate neighborhoods $\{U_\alpha\}$ of a coordinate covering $\mathfrak{U}$ satisfying (14.17). That is just the condition that the one-cocycle $D_1 f_{\alpha\beta} \cdot dz_\beta \in Z^1(\mathfrak{U}, \mathcal{M}^{(1,0)})$ is the coboundary of the zero-cochain $-p_\alpha \cdot dz_\alpha \in C^0(\mathfrak{U}, \mathcal{M}^{(1,0)})$; and since $H^1(M, \mathcal{M}^{(1,0)}) = 0$ by Corollary **??** it follows that any compact Riemann surface actually has meromorphic affine connections. For any meromorphic affine connection the difference $p_\alpha \cdot dz_\alpha - p_\beta \cdot dz_\beta$ is a holomorphic differential form in the intersection $U_\alpha \cap U_\beta$, so the local differential forms $p_\alpha \cdot dz_\alpha$ and $p_\beta \cdot dz_\beta$ have the same differential principal part in the intersection $U_\alpha \cap U_\beta$; thus any meromorphic affine connection has a well defined differential principal part on the entire Riemann surface M, naturally called the *principal part* of that affine connection. In particular the *residue* $\mathrm{res}_a(p_\alpha \cdot dz_\alpha)$ of a meromorphic affine connection at a point $a \in M$ is well defined, and of course is just the residue of the principal part of that affine connection.

Theorem 14.10 *If $p_\alpha \cdot dz_\alpha$ is a meromorphic affine connection on a compact Riemann surface M of genus $g > 0$ then*

$$\sum_{a \in M} res_a(p_\alpha \cdot dz_\alpha) = 2g - 2.$$

Proof: Choose a coordinate covering $\{U_\alpha\}$ of the surface M such that each pole a_i of the affine connection $p_\alpha \cdot dz_\alpha$ is contained in a single coordinate neighborhood $U_{\alpha_i} \subset M$. Choose $\mathcal{C}^\infty$ functions $r_i(z_{\alpha_i})$ in each of these coordinate neighborhoods U_{α_i} such that $r_i(z_{\alpha_i})$ is identically zero in an open neighborhood of the pole a_i and $r_i(z_{\alpha_i})$ is identically equal to 1 in any nontrivial intersection $U_\alpha \cap U_{\alpha_i}$; and extend these local functions to a global $\mathcal{C}^\infty$ function $r(z)$ on the entire surface M by setting $r(z) = 1$ outside the neighborhoods U_{α_i}. The product $q_\alpha \cdot dz_\alpha = r\, p_\alpha \cdot dz_\alpha$ is equal to $p_\alpha dz_\alpha$ in any intersection of coordinate neighborhoods, so the local $\mathcal{C}^\infty$ differential forms $q_\alpha \cdot dz_\alpha$ form a $\mathcal{C}^\infty$ affine connection in the sense that they satisfy

$$D_1 f_{\alpha\beta} \cdot dz_\beta = -q_\alpha \cdot dz_\alpha + q_\beta \cdot dz_\beta$$

in any intersection $U_\alpha \cap U_\beta$ of coordinate neighborhoods. Since $D_1 f_{\alpha\beta} \cdot dz_\beta$ is holomorphic $\overline{\partial}(q_\alpha \cdot dz_\alpha) = \overline{\partial}(q_\beta \cdot dz_\beta)$ in $U_\alpha \cap U_\beta$, so these local differentials form a global differential form of degree 2 on the entire Riemann surface M; and $\overline{\partial}(q_\alpha \cdot dz_\alpha) = 0$ outside the coordinate neighborhoods U_{α_i}, since q_α coincides with p_α and hence is holomorphic there, so it follows from Stokes's theorem that

$$\begin{aligned}
\frac{1}{2\pi i} \int_M \overline{\partial}(q_\alpha \cdot dz_\alpha) &= \frac{1}{2\pi i} \sum_i \int_{U_{\alpha_i}} d(q_{\alpha_i} \cdot dz_{\alpha_i}) \\
&= \frac{1}{2\pi i} \sum_i \int_{\partial U_{\alpha_i}} q_{\alpha_i} \cdot dz_{\alpha_i} = \frac{1}{2\pi i} \sum_i \int_{\partial U_{\alpha_i}} p_{\alpha_i} \cdot dz_{\alpha_i} \\
&= \sum_i \mathrm{res}_{a_i}(p_{\alpha_i} \cdot dz_{\alpha_i})
\end{aligned}$$

since $q_{\alpha_i} = p_{\alpha_i}$ on the boundary ∂U_{α_i}. On the other hand if $\tilde{q}_\alpha \cdot dz_\alpha$ is any other $\mathcal{C}^\infty$ affine connection on M then the differences $\phi_\alpha = \tilde{q}_\alpha \cdot dz_\alpha - q_\alpha \cdot dz_\alpha$ satisfy $\phi_\alpha = \phi_\beta$ in any intersection $U_\alpha \cap U_\beta$ so form a global $\mathcal{C}^\infty$ differential form ϕ of type $(1,0)$ on the compact Riemann surface M; by Stokes's Theorem again

$$\int_M \overline{\partial}(\tilde{q}_\alpha \cdot dz_\alpha) - \int_M \overline{\partial}(q_\alpha \cdot dz_\alpha) = \int_M \overline{\partial}\phi = \int_M d\phi = 0,$$

and consequently from this and the preceding equation it follows that

$$\sum_i \operatorname{res}_{a_i}(p_{\alpha_i} \cdot dz_{\alpha_i}) = \frac{1}{2\pi i}\int_M \overline{\partial}(\tilde{q}_\alpha \cdot dz_\alpha) \tag{14.34}$$

for any $\mathcal{C}^\infty$ affine connection $\tilde{q}_\alpha \cdot dz_\alpha$ on M. Now there are $\mathcal{C}^\infty$ functions $r_\alpha > 0$ in the coordinate neighborhoods U_α such that $r_\alpha = |\kappa_{\alpha\beta}|^2\ r_\beta$ in each intersection $U_\alpha \cap U_\beta$, and

$$-\partial \log r_\alpha + \partial \log r_\beta = -d\log \kappa_{\alpha\beta} = d\log f'_{\alpha\beta} = \frac{f''_{\alpha\beta}}{f'_{\alpha\beta}} dz_\beta = D_1 f_{\alpha\beta} \cdot dz_\beta.$$

Therefore $\tilde{q}_\alpha dz_\alpha = \partial \log r_\alpha$ is a $\mathcal{C}^\infty$ affine connection on M, and it follows from Lemma **??** that

$$\frac{1}{2\pi i}\int_M \overline{\partial}(\tilde{q}_\alpha dz_\alpha) = \frac{1}{2\pi i}\int_M \overline{\partial}\partial \log r_\alpha = c(\kappa) = 2g-2$$

since $c(\kappa) = 2g-2$ by the Canonical Bundle Theorem, Theorem 2.24. Combining thhis identity with (14.34) yields the desired result, which concludes the proof of the theorem.

The preceding actually is the only restriction on the singularities of meromorphic affine connections on a compact Riemann surface.

Corollary 14.11 *If $\mathfrak{p}$ is a differential principal part with total residue $2g-2$ on a compact Riemann surface M of genus g there is a meromorphic affine connection on M with the principal part $\mathfrak{p}$.*

Proof: There exists at least one meromorphic affine connection on any compact Riemann surface M, as already observed. It is evident from the defining equation (14.17) of an affine connection that the difference between any two meromorphic affine connections is a meromorphic abelian differential, and that the sum of a meromorphic affine connection and a meromorphic abelian differential is again a meromorphic affine connection. Since by Theorem 4.4 (ii) there exists a meromorphic abelian differential having any chosen principal part of total residue zero, the desired result is immediate. That suffices for a proof of the corollary.

If $p_\alpha dz_\alpha$ is a meromorphic affine connection on a compact Riemann surface M with the principal part $\mathfrak{p}$ then it is a holomorphic affine connection on the dense open subset $M^* \subset M$ complementary to the poles of the principal part $\mathfrak{p}$, so it describes an affine structure on M^*; but that structure does not extend to the poles.

Lemma 14.12 *If $p(z)$ is a meromorphic function in a contractible open neighborhood U of the origin in the complex plane, and if $p(z)$ has the principal part*

$$\mathfrak{p}(z) = \sum_{n>0} c_n z^{-n}$$

at the origin and is holomorphic at all other points of U, then in an open neighborhod of any point of U other than the origin itself there exist solutions $f(z)$ of the differential equation $D_1 f(z) = p(z)$, and for any solution

$$f'(z) = z^{c_1} h(z) \exp\left(-\sum_{n>1} \frac{c_n}{1-n} z^{1-n} \right) \tag{14.35}$$

where $h(z)$ is holomorphic and nowhere vanishing in all of U; if $f(z)$ is one solution then all other solutions are of the form $T(f(z))$ for affine mappings $T \in A(1, \mathbb{C})$.

Proof: Write $p(z) = h_1(z) + \mathfrak{p}(z)$ where $h_1(z)$ is holomorphic in U, choose an indefinite integral $h_2(z) = \int h_1(z)dz$ in U, and set $h(z) = \exp h_2(z)$; thus $h(z)$ is a holomorphic and nowhere vanishing function in all of U. The differential equation can be written

$$\begin{aligned} \frac{d}{dz} \log f'(z) &= D_1 f(z) = h_1(z) + \mathfrak{p}(z) \\ &= \frac{d}{dz} \left(h_2(z) + c_1 \log z + \sum_{n>1} \frac{c_n}{1-n} z^{1-n} \right), \end{aligned}$$

and consequently near any point of U except the origin

$$\log f'(z) = h_2(z) + c_1 \log z + \sum_{n>1} \frac{c_n}{1-n} z^{1-n}$$

is a solution of the differential equation for any local choice of a branch of the logarithim; thus $f'(z)$ has the asserted form. If $D_1 f(z) = D_1 g(z) = p(z)$ then it follows from Lemma 14.1 (iii) that $g(z) = T(f(z))$ for some affine transformation T so the derivatives of all solutions have the form (14.35). That suffices to conclude the proof of the lemma.

The derivative $f'(z)$ of the solution of the differential equation $D_1 f(z) = p(z)$ of the preceding lemma is single-valued near the origin only when the principal part has an integral residue at the origin, and in that case $f'(z)$ has an essential singularity unless the singularity at the origin is a simple pole; even then the solution $f(z)$ itself may have a logarithmic branch point at the origin if $n < 0$. Thus the simplest case is that in which the function $p(z)$ has a simple pole at the origin with residue a positive integer n; and in that case there is a holomorphic solution $f(z)$ that has a zero of order $n+1$ at the origin, while all other solutions are of the form $T(f(z))$ for an arbitrary affine mapping $T \in \mathcal{F}_1$. Meromorphic affine connections having as their singularities only simple poles with positive integral residues are called *regular* meromorphic affine connections; and these are the meromorphic affine connections that describe branched affine structures on M. It follows from Theorem 14.10 and Corollary 14.11 that on any compact Riemann surface M of genus $g > 1$ there exist regular meromorphic affine connections, although with at most $2g - 2$ poles.

To describe the branched affine structures that arise from these connections, suppose that $p_\alpha \cdot dz_\alpha$ is a regular meromorphic affine connection having the principal part

$$\mathfrak{p} = \sum_i \frac{\nu_i}{z - a_i}$$

with distinct poles $a_i \in M$, and choose a coordinate cover $\{U_\alpha\}$ of the surface M so that each pole a_i is contained in a single coordinate neighborhood U_{α_i}. There are holomorphic solutions of the differential equation $D_1 f_\alpha = p_\alpha$ in each coordinate neighborhood U_α; if these neighborhoods are sufficiently small the mappings $f_\alpha : U_\alpha \longrightarrow V_\alpha$ are biholomorphic mappings from the coordinate neighborhoods U_α to open subsets $V_\alpha \subset \mathbb{C}$ provided that U_α does not contain any of the poles of the affine connection, but these mappings exhibit U_α as an $(n+1)$-sheeted branched covering of the subset $V_\alpha \subset \mathbb{C}$ branched at a pole of the affine connection having residue n. It is still the case that the solutions f_α are uniquely determined up to affine transformations and that $f_\alpha = f_{\alpha\beta}(f_\beta$ for some affine transformation $f_{\alpha\beta} \in \mathcal{F}_1$ in each intersection $U_\alpha \cap U_\beta$; so these mappings describe a unique affine structure on the complement of the poles of the affine connection $p_\alpha \cdot dz_\alpha$. Such a collection of local mappings f_α describes a *branched affine structure* on the Riemann surface M, branched at the poles of the affinie connection $p_\alpha \cdot dz_\alpha$.

The affine transformations $f_{\alpha\beta}$ associated to the intersections $U_\alpha \cap U_\beta$ still describe a flat affine bundle over M, which is the *holonomy bundle* of the branched affine structure on M. When these coordinate transformations are written out more explicitly as $f_\alpha(z_\alpha) = a_{\alpha\beta} f_\beta(z_\beta) + b_{\alpha\beta}$ the coefficients $a_{\alpha\beta}$ satisfy $a_{\alpha\beta}a_{\beta\gamma}a_{\gamma\alpha} = 1$ in each triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ so define a flat line bundle α over M called the *subordinate* line bundle to the holonomy bundle; it is evident that an affine change of the local functions f_α yields an equivalent flat line bundle, so the subordinate line bundle is uniquely determined by the holonomy bundle. The differentials $\phi_\alpha(z_\alpha) = df_\alpha(z_\alpha)$ of the functions $f_\alpha(z_\alpha$ are holomorphic differential forms in the coordinate neighborhoods U_α such that $\phi_\alpha(z_\alpha) = a_{\alpha\beta}\phi_\beta(z_\beta)$ in any intersection $U_\alpha \cap U_\beta$; thus these differential forms describe a holomorphic Prym differential on the Riemann surface M, called the *associated Prym differential* to the branched affine structure. It is evident that the divisors of these Prym differentials are the divisors $\mathfrak{d}(\phi_\alpha) = \sum_i n_i \cdot a_i$ of degree $2g-2$, where the principal part of the meromorphic affine connection is $\mathfrak{p} = \sum_i c_i(z_i - a_i)^{-n_i}$. Since the coefficients $f'_\alpha(z_\alpha)$ are holomorphic functions such that $f'_\alpha(z_\alpha) = a_{\alpha\beta}\kappa_{\alpha\beta}f'_\beta(z_\beta)$ and since the divisor of these functions is the divisor $\mathfrak{d} = \sum_i n_i \cdot a_i$ it follows that $\zeta_{\mathfrak{d}} = \alpha\kappa$, which determines the subordinate line bundle α explicitly in terms of the principal part of the meromorphic affine connection.

oooooooooooooooooooo

Lemma 14.13 *The submonodromy line bundles of the branched affine structures associated to all the basic meromorphic affine connections on a compact Riemann surface M of genus $g > 0$ are all analytically equivalent flat line bundles, and all flat line bundles arise from some branched affine structure.*

Proof: It was already observed that if $p_\alpha dz_\alpha$ and $\tilde{p}_\alpha dz_\alpha$ are two basic meromorphic affine connections on M then they differ by a holomorphic abelian differential, so that $\tilde{p}_\alpha dz_\alpha = p_\alpha dz_\alpha + dw_\alpha$ for some holomorphic abelian integral w_α on M; so if $D_1 f_\alpha = p_\alpha$ and $D_1\tilde{f}_\alpha = \tilde{p}_\alpha$ then

$$\begin{aligned}\tilde{p}_\alpha dz_\alpha &= D_1\tilde{f}_\alpha dz_\alpha \\ &= \quad d\log\tilde{f}'_\alpha + dw_\alpha\end{aligned}$$

so that $\tilde{f}'_\alpha(z_\alpha) = f'_\alpha(z_\alpha)\exp w_\alpha(z_\alpha)$. Now for the submonodromy bundles since $d\tilde{f}_\alpha(z_\alpha) = \tilde{a}_{\alpha\beta} d\tilde{f}_\beta(z_\beta)$ it follows that

$$\begin{aligned}d\tilde{f}_\alpha &= \exp w_\alpha(z_\alpha) df_\alpha(z_\alpha) = \exp w_\alpha(z_\alpha) a_{\alpha\beta} df_\beta(z_\beta) \\ &= \quad \tilde{a}_{\alpha\beta} d\tilde{f}_\beta(z_\beta) \\ &= \quad \tilde{a}_{\alpha\beta}\exp w_\beta(z_\beta) df_\beta(z_\beta)\end{aligned}$$

and consequently that $\tilde{a}_{\alpha\beta} = a_{\alpha\beta}\exp\big(w_\alpha(z_\alpha) - w_\beta(z_\beta)\big)$ so that these two bundles are indeed analytically equivalent. Conversely any analytic equivalence of these two bundles must be actually holomorphic of course, so the affine structure is unbranched; and since there is always the standard affine structure associated to the representation of the surface as the quotient of the complex plane by a lattice subgroup, for which the submonodromy bundle is the identity bundle, that is the normalized coordinate structure of an elliptic curve. In that case of course the structure actually holomorphic of course, so the affine structure is unbranched; and since there is always the standard affine structure associated to the representation of the surface as the quotient of the complex plane by a lattice subgroup, for which the submonodromy bundle is the identity bundle, that is the normalized coordinate structure of an elliptic curve. In that case of course the structure does not depend on the choice of a base point. In general the coordinatization is an affine structure on the complement of the base point, with the standard branched coordinate system in an open neighborhood of the base point.

The branched affine structure is a regular affine structure on the complement $M \sim a_0$ of the point a_0; on the complement straight line segments clearly are well defined. Furthermore straight line segments from the base point a_0 in the coordinate neighborhood U_0 also correspond to straight lines in any intersecting coordinate neighborhood, since any straight line segment beginning at the origin $w_0 = 0$ is a straight line segment in terms of the power w_0^{2g-1}, so is a straight line in any other coordinate neighborhood. It is evident from this that a straight line segment in one coordinate neighborhood extends naturally to a straight line throughout the compact manifold M; the extension is uniquely determined except at the base point a_0, where there are $2g-1$ choices of the extension of the line corresponding to the $2g-1$ roots of $w_0^{1/(2g-1)}$. For the normalized affine structure $w_\alpha = a_{\alpha\beta} w_\beta$ in any intersection $U_\alpha \cap U_\beta$ of coordinate neighborhoods not containing the base point so a_0 it follows that $|w_\alpha| = |w_\beta|$; consequently that the ordinary Euclidean distance between two points is well defined invariant on the complement $M \sim a_0$. Alternatively the distance can be defined in terms of the complex Riemannian metric $g = dw_\alpha\, d\overline{w}_\alpha$ in the coordinate neighborhood U_α, since this metric is independent of the local coordinate system. In an intersection $U_0 \cap U_\alpha$, where the local coordinates are related by $w_\alpha = a_{\alpha 0} w_0^{2g-1} + b_{\alpha 0}$, this metric induces in terms of the local coordinate in U_0 the metric form $g = (2g-1)^2 |w_0|^{4(g-1)} dw_0\, d\overline{w}_0$; this is a riemannian metric with a conical singularity at the point a_0. It is indeed a straightforward calculation to show that the neighborhood U_0 with this metric is isometric to a cone in three space with the metric induced by the euclidean metric in the ambient space; thus the riemannian metric on M does define a distance function on M, although with a conical singularity at the single branch point.

Part III

Appendices

Appendix A

Manifolds and Varieties

A.1 Holomorphic Functions

This appendix contains a survey of some general properties of complex manifolds and holomorphic varieties, an acquaintance with which is presupposed in the present book. The emphasis is on those properties that are relevant to the study of Riemann surfaces. The discussion here is rather abbreviated and generally does not include complete proofs; references for more detailed treatments of particular topics will be included along the way. Any investigation of Riemann surfaces of course presupposes familiarity with the standard properties of holomorphic functions of a single variable; but many topics also involve some properties of holomorphic functions of several variables, and since these properties may not be quite so familiar the appendix will begin with a survey of some of the results that are used in this book[1].

A complex-valued function $f(z)$ defined in an open subset U of the n-dimensional complex vector space $\mathbb{C}^n$ is *holomorphic* in U if in an open neighborhood of any point $a \in U$ it has a convergent power series expansion

$$f(z_1, \ldots, z_n) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1\ldots i_n}(z_1 - a_1)^{i_1} \cdots (z_n - a_n)^{i_n}. \tag{A.1}$$

The set of holomorphic functions in U form a ring $\mathcal{O}_U$ under pointwise addition and multiplication of functions; the units or invertible elements in this ring are the nowhere vanishing holomorphic functions, which form a multiplicative group $\mathcal{O}^*_U$. The series (A.1) is absolutely convergent in an open neighborhood of the point $(a_1, \ldots, a_n)$ so it can be rearranged as a convergent series in any

[1]For more extensive treatments of the general properties of holomorphic functions of several variables see for instance R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables*, (Wadsworth and Brooks/Cole, 1990), (references to which for short will be given in the form G-IIIC12 for Theorem/Corollary/Definition 12, section C, volume III), or L. and B. Kaup, *Holomorphic Functions of Several Variables*, (deGruyter, 1983), or S. Krantz, *Function Theory of Several Complex Variables*, (Wadsworth and Brooks/Cole, 1992).

one of the variables when the remaining variables are held constant; thus a holomorphic function of several is holomorphic in each variable separately. A basic and nontrivial result is *Hartogs's Theorem*[2] that conversely any function that is holomorphic in each variable separately is holomorphic in all variables, without any additional hypothesis of continuity or measurability in all variables; thus holomorphic functions of several complex variables can be characterized by separate conditions in each complex variable $z_j = x_j + i\, y_j$ or each pair of real variables (x_j, y_j). For instance a function of several complex variables that is continuously differentiable in each pair of variables (x_j, y_j) and that satisfies the *Cauchy-Riemann equations* in each pair of variables (x_j, y_j) is a holomorphic function of all variables. It is convenient to write the Cauchy-Riemann equations in terms of the linear partial differential operators

$$\text{(A.2)} \qquad \frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \frac{1}{i}\frac{\partial f}{\partial y_j}\right), \quad \frac{\partial f}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \frac{1}{i}\frac{\partial f}{\partial y_j}\right)$$

where $z_j = x_j + iy_j$; in these terms if f is a continuously differentiable function in an open subset $U \subset \mathbb{C}^n$, or just a function that is continuously differentiable in each pair of variables (x_j, y_j), then

$$\text{(A.3)} \qquad f \text{ is holomorphic if and only if } \frac{\partial f}{\partial \overline{z}_j} = 0 \text{ for } 1 \le j \le n.$$

If f is holomorphic then $\partial f/\partial z_j$ is just the ordinary complex derivative of the holomorphic function $f(z_j)$ of the complex variable z_j when the remaining variables are held constant.

The zero locus of a holomorphic function of a single complex variable is a discrete of points, but the situation is rather more complicated for holomorphic functions of several complex variables. A *holomorphic subvariety* of an open subset $U \subset \mathbb{C}^n$ is a subset $V \subset U$ with the property that for each point $a \in U$ there are an open neighborhood U_a of that point and finitely many holomorphic functions f_{ai} in U_a, not all of which vanish identically, such that

$$V \cap U_a = \left\{ z \in U_a \;\middle|\; f_{a1}(z) = f_{a2}(z) = \cdots = 0 \right\}.$$

It is not required that a holomorphic subvariety $V \subset U$ be the set of common zeros of a collection of functions defined and holomorphic in all of U; the notion of a holomorphic subvariety is essentially local in nature. It is evident from this definition that a holomorphic subvariety of U is a closed subset of U. If $V \subset U$ is a holomorphic subvariety of a connected open subset $U \subset \mathbb{C}^n$ then the complement $U \sim V$ is a connected dense open subset [3]. The analogue for functions of several complex variables of the Riemann Removable Singularities Theorem[4] for functions of a single complex variable is the theorem that if f is a bounded holomorphic function in the complement $U \sim V$ of a holomorphic

[2] Theorem G-IB6
[3] Corollary G-IA9 and Corollary G-ID3
[4] Theorem G-ID2

subvariety V of a connected open subset $U \subset \mathbb{C}^n$ then f has a unique extension to a holomorphic function on the entire set. There is actually a stronger removable singularities theorem, a special case of a number of extension theorems that arise only for functions of more than one variable. This theorem[5] asserts that if V is a holomorphic subvariety of an open subset $D \subset \mathbb{C}^n$ and if $\dim V \le n-2$ then any holomorphic function in $D \sim V$ extends uniquely to a holomorphic function in D. If f and g are two holomorphic functions in a connected open subset $U \subset \mathbb{C}^n$ and if they agree on a subset of U that is not a holomorphic subvariety of U, such as an open subset of U, then clearly they must agree at all points of U.

If f and g are two holomorphic functions in a connected open subset $U \subset \mathbb{C}^n$ and if the function g does not vanish identically then its zero locus is a holomorphic subvariety $V_g \subset U$ and the quotient $m = f/g$ is a well defined complex-valued function on the connected dense open subset $U \sim V_g \subset U$. A complex-valued function m that is defined in the complement of a holomorphic subvariety $V_m \subset U$ of an open subset $U \subset \mathbb{C}^n$ and that can be represented in an open neighborhood of each point of U as such a quotient of holomorphic functions is called a *meromorphic function* in U. Clearly the set of meromorphic functions in a connected open subset $U \subset \mathbb{C}^n$ form a field under pointwise addition and multiplication of functions; this field is denoted by $\mathcal{M}_U$. If the open subset U is not connected $\mathcal{M}_U$ is not a field, since meromorphic functions that vanish in a connected component of U but not in all of U are nontrivial but do not have multiplicative inverses. Of course any holomorphic function in an open subset $U \subset \mathbb{C}^n$ is also meromorphic, so $\mathcal{O}_U \subset \mathcal{M}_U$; and it follows from the Riemann Removable Singularities Theorem that a bounded meromorphic function in U actually is holomorphic in U. It is evident that if m is meromorphic in an open subset $U \subset \mathbb{C}^n$ then it is a meromorphic function in each variable separately in U when the remaining variables are held constant, except when all such points lie in the holomorphic subvariety V_m where m is not necessarily well defined. An analogue of Hartogs's Theorem for meromorphic functions is Rothstein's Theorem[6] that conversely a complex valued function in the complement of a holomorphic subvariety V of an open subset $U \subset \mathbb{C}^n$ that is a meromorphic function in each variable separately in $U \sim V$ is a meromorphic function in U. There is also an analogue for meromorphic functioins of the extension theorem for holomorphic functions. The Theorem of Levi[7] asserts that if V is holomorphic subvariety of an open subset $D \subset \mathbb{C}^n$ and if $\dim V \le n-2$ then any meromorphic function in $D \sim V$ extends uniquely to a meromorphic function in D.

A *holomorphic mapping* from an open subset $U \subset \mathbb{C}^n$ into $\mathbb{C}^m$ is a mapping that sends a point $z = (z_1, \ldots, z_n) \in U$ to the point $w = (w_1, \ldots, w_m) \in \mathbb{C}^m$

[5] Theorem G-IIK1. The theorem requires the notion of the dimension of a holomorhic subvariety, which will be taken up later in Section A.3; but it is more convenient to include the statement here in the discussion of functions.

[6] See the paper by W. Rothstein "Ein neuer Beweis des Hartogsschen Hauptsatzes und eine Ausdehnung auf meromorphe Funktionen", Math. Zeit., vol. 53 (1950), pp. 84 - 95.

[7] Theorem G-IIO6

where $w_j = f_j(z_1, \ldots, z_n)$ for some holomorphic functions $f_j \in \mathcal{O}_U$. A holomorphic function f in an open subset $U \subset \mathbb{C}^n$ can be viewed as a holomorphic mapping $f : U \longrightarrow \mathbb{C}$. It is familiar that a holomorphic mapping $f : U \longrightarrow \mathbb{C}$ defined in an open subset $U \subset \mathbb{C}$ is an open mapping; but trivial examples show that is not the case for holomorphic mappings from open subsets of $\mathbb{C}^n$ into $\mathbb{C}^n$ for $n > 1$. However if $F : U \longrightarrow V$ is a one-to-one holomorphic mapping from an open subset $U \subset \mathbb{C}^n$ onto a subset $V \subset \mathbb{C}^n$ then V is necessarily an open subset of $\mathbb{C}^n$ and the mapping F is an open mapping with a holomorphic inverse[8]. A holomorphic mapping $F : U \longrightarrow V$ between two open subsets $U, V \subset \mathbb{C}^n$ that has a holomorphic inverse mapping is said to be *biholomorphic*. A holomorphic mapping $F : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ for which $\det\{\partial f_j/\partial z_k\} \neq 0$ at a point $a \in \mathbb{C}^n$ describes a biholomorphic mapping from an open neighborhood of the point $a \in \mathbb{C}^n$ to an open neighborhood of the image point $F(a) \in \mathbb{C}^n$.

Differential forms play a more useful role in several complex variables than in one variable. A complex-valued differential form ϕ in an open subset $U \subset \mathbb{C}^n$ can be written either in terms of the differentials dx_j, dy_j of the real coordinates in $\mathbb{C}^n$ or in terms of the complex linear combinations

$$dz_j = dx_j + i\,dy_j, \quad d\overline{z}_j = dx_j - i\,dy_j \tag{A.4}$$

of these differentials; a differential form of degree r that can be written

$$\phi = \sum_{j,k} f_{j_1 \ldots j_p, k_1 \ldots k_q} dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \tag{A.5}$$

in terms of the complex differentials dz_j and $d\overline{z}_k$ it is said to be of *type* (p, q) and *degree* $r = p + q$. The vector space of complex-valued $\mathcal{C}^\infty$ differential forms of degree r in U is denoted by $\mathcal{E}^r_U$, and the vector space of complex-valued $\mathcal{C}^\infty$ differential forms of type (p, q) in U is denoted by $\mathcal{E}^{(p,q)}_U$, so there is the direct sum decomposition

$$\mathcal{E}^r_U = \bigoplus_{p+q=r} \mathcal{E}^{(p,q)}_U. \tag{A.6}$$

The exterior derivative of a differentiable function f in U is the differential 1-form

$$df = \sum_{j=1}^n \Big(\frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \Big). \tag{A.7}$$

A straightforward calculation shows that when written in terms of the complex differentials dz_j and $d\overline{z}_j$ the exterior derivative takes the form

$$df = \sum_{j=1}^n \Big(\frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \Big)$$

[8] Corollary G-IIE10

in terms of the differential operators (A.2). The separate differential forms

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j \tag{A.8}$$

are the components of the differential df of type $(1, 0)$ and type $(1, 0)$ respectively; thus the exterior derivative of a $\mathcal{C}^\infty$ function $f \in \mathcal{E}_U$ can be written as the sum

$$df = \partial f + \overline{\partial} f \tag{A.9}$$

of a differential form $\partial f \in \mathcal{E}_U^{(1,0)}$ of type $(1, 0)$ and a differential form $\overline{\partial} f \in \mathcal{E}_U^{(0,1)}$ of type $(0, 1)$. If f is holomorphic then $df = \partial f$ since $\overline{\partial} f = 0$; and conversely if f is a continuously differentiable function such that $df = \partial f$ then $\overline{\partial} f = 0$ so the Cauchy-Riemann equations show that f is holomorphic. Under a biholomorphic mapping $w_k = f_k(z_j)$ between open subsets of $\mathbb{C}^n$

$$dw_k = \sum_{j=1}^{n} \frac{\partial w_k}{\partial z_j} dz_j \quad \text{and} \quad d\overline{w}_k = \sum_{j=1}^{n} \overline{\frac{\partial w_k}{\partial z_j}} d\overline{z}_j;$$

it is evident from this that the type of a differential form is unchanged under biholomorphic changes of coordinates in $\mathbb{C}^n$, so to that extent the decomposition (A.6) is intrinsic. The exterior derivative of the differential form (A.5) is the differential form

$$\begin{aligned} d\phi &= \sum_{j,k} df_{j_1 \ldots j_p, k_1 \ldots k_q} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \\ &= \sum_{j,k} \partial f_{j_1 \ldots j_p, k_1 \ldots k_q} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q} \\ &\qquad + \sum_{j,k} \overline{\partial} f_{j_1 \ldots j_p, k_1 \ldots k_q} \wedge dz_{j_1} \wedge \cdots \wedge dz_{j_p} \wedge d\overline{z}_{k_1} \wedge \cdots \wedge d\overline{z}_{k_q}; \end{aligned} \tag{A.10}$$

thus if $\phi \in \mathcal{E}^{(p,q)}$ then $d\phi = \partial\phi + \overline{\partial}\phi$ where $\partial\phi \in \mathcal{E}_U^{(p+1,q)}$ and $\overline{\partial} \in \phi\mathcal{E}_U^{(p,q+1)}$, so there is the direct sum decomposition $d = \partial \oplus \overline{\partial}$ of exterior differentiation of arbitrary differential forms in terms of linear differential operators

$$\partial : \mathcal{E}_U^{(p,q)} \longrightarrow \mathcal{E}_U^{(p+1,q)} \quad \text{and} \quad \overline{\partial} : \mathcal{E}_U^{(p,q)} \longrightarrow \mathcal{E}_U^{(p,q+1)}. \tag{A.11}$$

In particular if ϕ is a differential form of type $(p, 0)$ then $d\phi = 0$ if and only if both $\partial\phi = 0$ and $\overline{\partial}\phi = 0$. When the differential form ϕ is written

$$\phi = \sum_{j} f_{j_1 \ldots j_p} dz_{j_1} \wedge \cdots \wedge dz_{j_p} \tag{A.12}$$

the condition that $\overline{\partial}\phi = 0$ clearly is equivalent to the condition that the coefficients $f_{j_1 \ldots j_p}$ are holomorphic functions; a differential form of type $(p, 0)$

satisfying this condition is called a *holomorphic differential form* of type $(p, 0)$, and the space of such differential forms is denoted by $\mathcal{O}^{(p,0)}$. Exterior differentiation satisfies $d\,d = 0$; the kernel of the linear operator d is the subspace of *closed* differential forms in U, the image of d is the subspace of *exact* differential forms in U, and every exact form is closed since $d\,d = 0$. When exterior differentiation is written as the sum $d = \partial + \overline{\partial}$ the identity $d\,d = 0$ is equivalent to the identities

$$\partial\,\partial = \partial\,\overline{\partial} + \overline{\partial}\,\partial = \overline{\partial}\,\overline{\partial} = 0, \tag{A.13}$$

so $\partial\,d = \partial\,\overline{\partial} = d\,\overline{\partial}$. It is familiar that any closed differential form is at least locally exact. If ϕ is a holomorphic differential form of type $(p, 0)$ that is closed then it too is locally the exterior derivative $\phi = d\psi$ of a differential form ψ of degree $p-1$, indeed clearly a differential form ψ of type $(p-1, 0)$; and since $\phi = \partial\psi + \overline{\partial}\psi$ it is evident that ψ must be a holomorphic differential form of type $(p-1, 0)$. Thus if a holomorphic differential form ϕ of type $(p, 0)$ is closed then locally it is the exterior derivative of a holomorphic differential form ψ of type $(p-1, 0)$.

A.2 Manifolds

A *manifold* or *topological manifold* of dimension n is a second countable Hausdorff topological space M such that each point of M has an open neighborhood homeomorphic to an open subset of the n-dimensional Euclidean space $\mathbb{R}^n$. A *coordinate covering* $\{U_\alpha, x_\alpha\}$ of the manifold M is a covering of M by open subsets $U_\alpha \subset M$, for each of which there is a homeomorphism $x_\alpha : U_\alpha \longrightarrow W_\alpha$ between U_α and an open subset $W_\alpha \subset \mathbb{R}^n$. The subsets U_α are called the *coordinate neighborhoods*, and the mappings x_α are called the *coordinate mappings* or the *local coordinates* of the coordinate covering. In the intersections $U_\alpha \cap U_\beta$ of coordinate neighborhoods there are two homeomorphisms to subsets of $\mathbb{R}^n$, the restrictions of x_α and of x_β; the compositions

$$f_{\alpha\beta} = x_\alpha \circ\, x_\beta^{-1} : x_\beta(U_\alpha \cap U_\beta) \to x_\alpha(U_\alpha \cap U_\beta) \tag{A.14}$$

are homeomorphisms called the *coordinate transition mappings* of the coordinate covering, and the two local coordinates in an intersection $U_\alpha \cap U_\beta$ are related by $x_\alpha = f_{\alpha\beta}(x_\beta)$. The manifold M is determined completely by the open subsets $W_\alpha = x_\alpha(U_\alpha) \subset \mathbb{R}^n$ and the coordinate transition mappings $f_{\alpha\beta}$ of a coordinate covering, since M can be recovered from the disjoint union of the sets W_α by identifying points $x_\alpha \in W_\alpha$ and $x_\beta \in W_\beta$ whenever $x_\alpha = f_{\alpha\beta}(x_\beta)$. If $\{U_\alpha, x_\alpha\}$ and $\{V_\beta, y_\beta\}$ are two coordinate coverings of the manifold M their union also is a coordinate covering of M, consisting of the total collection of coordinate neighborhoods and local coordinates from the two separate coordinate coverings. The set of coordinate transition mappings for the union is properly larger than the union of the sets of coordinate transition mappings for the two separate coverings, though, since it must include the coordinate transition mappings relating the local coordinates x_α and y_β in the intersections $U_\alpha \cap V_\beta$ of coordinate neighborhoods from the two separate coverings.

Coordinate coverings with special properties can be used to describe additional structures on a topological manifold. A collection $\mathcal{G}$ of homeomorphisms between open subsets of $\mathbb{R}^n$ that is determined by local conditions, that includes with any homeomorphism its restrictions to open subsets and its inverse, and that includes with any two homeomorphisms their composition wherever it is defined, is called a *pseudogroup*[9]. One example is the pseudogroup $\mathcal{G}_1$ of all continuously differentiable or $\mathcal{C}^1$ homeomorphisms between open subsets of $\mathbb{R}^n$; a second example is the pseudogroup $\mathcal{G}_2$ of all infinitely differentiable or $\mathcal{C}^\infty$ homeomorphisms between open subsets of $\mathbb{R}^n$; a third example is the pseudogroup $\mathcal{G}_3$ of all holomorphic homeomorphisms between open subsets of $\mathbb{C}^m$, when the real vector space is of dimension $n = 2m$ and is identified with the complex vector space $\mathbb{C}^m$; a fourth example is the pseudogroup $\mathcal{G}_4$ of nonsingular complex linear mappings between open subsets of $\mathbb{C}^m$. This last example is actually a group, since the composition of any two nonsingular complex linear mappings is again a nonsingular complex linear mapping; in the previous examples only those mappings with suitably overlapping ranges and domains can be composed, hence the terminology pseudogroup rather than group. These four examples are increasingly restrictive, in the obvious sense that $\mathcal{G}_4 \subset \mathcal{G}_3 \subset \mathcal{G}_2 \subset \mathcal{G}_1$. A coordinate covering $\{U_\alpha, x_\alpha\}$ is called a $\mathcal{G}$ *coordinate covering* if all of its coordinate transition mappings $f_{\alpha\beta}$ belong to the pseudogroup $\mathcal{G}$. Two $\mathcal{G}$ coordinate coverings are called *equivalent* if their union is again a $\mathcal{G}$ coordinate covering; this is an equivalence relation in the usual sense, as a simple consequence of the definition of a pseudogroup, and is actually a nontrivial equivalence relation, since there are more coordinate transition mappings in the union of two coordinate coverings than just the union of the two sets of coordinate transition mappings. An equivalence class of $\mathcal{G}$ coordinate coverings is called a $\mathcal{G}$ *structure* on the manifold M, and a manifold M with a fixed $\mathcal{G}$ structure is called a $\mathcal{G}$ *manifold*. Thus for the four examples of pseudogroups just considered there are *continuously differentiable* or $\mathcal{C}^1$ manifolds, *infinitely differentiable* or $\mathcal{C}^\infty$ manifolds, *complex analytic* manifolds, usually called just *complex* manifolds, and *flat complex linear* manifolds. A complex manifold also is a $\mathcal{C}^\infty$ manifold, since any complex analytic coordinate covering is also a $\mathcal{C}^\infty$ coordinate covering and any two equivalent complex analytic coordinate coverings are equivalent as $\mathcal{C}^\infty$ coordinate coverings; thus a complex manifold can be viewed as a $\mathcal{C}^\infty$ manifold by ignoring some of the structure, or alternatively a complex structure is an additional structure that can be imposed on an underlying $\mathcal{C}^\infty$ manifold. Similar considerations of course apply to any pseudogroups $\mathcal{G}'' \subset \mathcal{G}'$.

Complex manifolds[10] are of particular interest in the present book. As a

---

[9] There is an extensive literature devoted to pseudogroups and pseudogroup structures following the initial treatment by E. Cartan, which can be found in his *Oeuvres Complétes*, partie II, vol. 2. (Gauthier-Villars, 1953). See for instance the discussion in S. Sternberg, *Lectures on Differential Geometry*, (Prentice-Hall, 1964).

[10] A more detailed discussion of complex manifolds can be found in K. Kodaira and J. Morrow, *Complex Manifolds*, (Holt, Rhinehart and Winston, 1971) or R. O. Wells, *Differential*

matter of convention, a complex manifold of topological dimension $n = 2m$ customarily is referred to as a complex manifold of dimension m, viewing the complex dimension rather than the real dimension as the more significant index. A complex-valued function f defined in an open subset U of a complex manifold M is *holomorphic* if for each intersection $U \cap U_\alpha$ of the set U with a coordinate neighborhood of a holomorphic coordinate covering $\{U_\alpha, x_\alpha\}$ of M the composition $f \circ x_\alpha^{-1} : x_\alpha(U \cap U_\alpha) \longrightarrow \mathbb{C}$ is a holomorphic function in the open subset $x_\alpha(U \cap U_\alpha) \subset \mathbb{C}^n$. This condition clearly is independent of the choice of a complex coordinate covering representing the complex structure of M, so depends only on the complex structure of M. The same notation is used for functions on complex manifolds as for functions on open subsets of $\mathbb{C}^n$; thus the ring of holomorphic functions in the subset $U \subset M$ is denoted by $\mathcal{O}_U$, the multiplicative group of nowhere vanishing holomorphic functions is denoted by $\mathcal{O}_U^*$, and if U is connected the field of meromorphic functions is denoted by $\mathcal{M}_U$ and the multiplicative group of not identically vanishing meromorphic functions is denoted by $\mathcal{M}_U^*$. The ring $\mathcal{C}_U$ of continuous complex-valued functions on a topological manifold, the ring $\mathcal{E}_U$ of $\mathcal{C}^\infty$ complex-valued functions on a $\mathcal{C}^\infty$ manifold, and the ring $\mathcal{F}_U$ of locally constant complex-valued functions on a flat manifold are defined correspondingly. For a connected open subset $U \subset M$ of a complex manifold M there are the natural inclusions $\mathcal{O}_U \subset \mathcal{M}_U$ and $\mathcal{O}_U^* \subset \mathcal{M}_U^*$; and $\mathcal{F}_U \subset \mathcal{O}_U \subset \mathcal{E}_U \subset \mathcal{C}_U$, but of course $\mathcal{M}_U$ is not a subset of $\mathcal{E}_U$ or $\mathcal{C}_U$.

In a coordinate neighborhood U_α of a complex manifold M with local coordinates $z_{\alpha j} = x_{\alpha j} + i\, y_{\alpha j}$ it follows readily from (A.4) that

$$\text{(A.15)}\quad \left(\frac{i}{2}\right)^n dz_{\alpha 1} \wedge d\overline{z}_{\alpha 1} \wedge \cdots \wedge dz_{\alpha n} \wedge d\overline{z}_{\alpha n} = dx_{\alpha 1} \wedge dy_{\alpha 1} \wedge \cdots \wedge dx_{\alpha n} \wedge dy_{\alpha n},$$

so this differential form can be used as an element of volume in the coordinate neighborhood $U_\alpha \subset M$; in particular in a coordinate neighborhood U_α on a Riemann surface M with local coordinate $z_\alpha = x_\alpha + i\, y_\alpha$ the differential form $\frac{i}{2} dz_\alpha \wedge d\overline{z}_\alpha = dx_\alpha \wedge dy_\alpha$ can be taken as an element of area. For another local coordinate $z_\beta = x_\beta + i\, y_\beta$

$$\text{(A.16)}\quad \begin{aligned} dx_\alpha \wedge dy_\alpha = \frac{i}{2}\, dz_\alpha \wedge d\overline{z}_\alpha &= \left|\frac{dz_\alpha}{dz_\beta}\right|^2 \frac{i}{2}\, dz_\beta \wedge d\overline{z}_\beta \\ &= \left|\frac{dz_\alpha}{dz_\beta}\right|^2 dx_\beta \wedge dy_\beta, \end{aligned}$$

so this element of area remains positive under any complex analytic change of coordinates on the Riemann surface; equivalently the Jacobian determinant of a complex analytic change of coordinates is everywhere positive. The analogous result holds for n-dimensional complex manifolds as well, so *complex manifolds are orientable topological spaces.* In this book the positive orientation of a

Analysis on Complex Manifolds, (Prentice Hall, 1973).

complex manifold is taken to be that for which (A.15) is the positive volume element on the manifold; in particular the orientation of a Riemann surface is that for which (A.16) is the positive element of area.

A continuous mapping $F : M \longrightarrow N$ between complex manifolds M and N of dimensions m and n with coordinate coverings $\{U_\alpha, x_\alpha\}$ and $\{V_\beta, y_\beta\}$ respectively is *holomorphic* if for any point $p \in U_\alpha \subset M$ for which $F(p) \in V_\beta \subset N$ the composition $y_\beta \circ F \circ x_\alpha^{-1}$ is a holomorphic mapping from an open neighorhood of the point $x_\alpha(p) \in \mathbb{C}^m$ into the space $\mathbb{C}^n$. Two complex manifolds M and N are said to be *analytically equivalent* or *biholomorphic* if there is a homeomorphism $F : M \longrightarrow N$ such that both F and F^{-1} are holomorphic mappings; the mapping F is called an *analytic equivalence* or a *biholomorphic mapping*. Any one-to-one holomorphic mapping between two complex manifolds of the same dimension is a biholomorphic mapping since as noted in the preceding section a one-to-one holomorphic mapping from an open subset of $\mathbb{C}^n$ into $\mathbb{C}^n$ is a biholomorphic mapping. For the most part it is only the analytic equivalence classes or biholomorphic equivalence classes of complex manifolds that are of primary interest.

Riemann surfaces are defined as one-dimensional connected complex manifolds, and are the main topic of this book; however various complex manifolds of higher dimension, such as complex projective spaces and complex tori, arise naturally in the discussion of Riemann surfaces. The n-dimensional complex projective space $\mathbb{P}^n$ is defined to be the set of equivalence classes of nonzero points $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$, where the equivalence relation is defined by $(z_0, z_1, \ldots, z_n) \sim (t\,z_0, t\,z_1, \ldots, t\,z_n)$ for any nonzero complex number $t \in \mathbb{C}^*$; alternatively $\mathbb{P}^n$ can be defined to be the set of one-dimensional linear subspaces of $\mathbb{C}^{n+1}$, since any such subspace is an equivalence class as just defined. The space $\mathbb{P}^n$ is topologized with the natural quotient topology, so the open subsets of $\mathbb{P}^n$ are the equivalence classes of points in open subsets of $\mathbb{C}^{n+1}$. The equivalence class containing a point $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ is denoted by $[z_0, z_1, \ldots, z_n] \in \mathbb{P}^n$, and the point $(z_0, z_1, \ldots, z_n)$ is called the set of *homogeneous coordinates* for the point of $\mathbb{P}^n$ that it represents. In the open subset $U_i \subset \mathbb{P}^n$ consisting of points with homogeneous coordinates $(z_0, z_1, \ldots, z_n)$ for which $z_i \neq 0$, where $0 \leq i \leq n$, any point is represented by unique homogeneous coordinates of the form $(z^i_0, \ldots, z^i_{i-1}, 1, z^i_{i+1}, \ldots, z^i_n)$, which are called the *inhomogeneous coordinates* of that point; these provide local coordinates in U_i, identifying that subset of $\mathbb{P}^n$ with the complex vector space $\mathbb{C}^n$. Points in the intersection $U_i \cap U_j$ for $i \neq j$ then are described by two sets of inhomogeneous coordinates which are related by $(z^i_0, \ldots, z^i_{i-1}, 1, z^i_{i+1}, \ldots, z^i_n) = t(z^j_0, \ldots, z^j_{j-1}, 1, z^j_{j+1}, \ldots, z^j_n)$, where clearly $t = z^i_j$ so that

$$z^i_k = z^i_j z^j_k \quad \text{for} \quad k \neq i, j; \tag{A.17}$$

that is a nonsingular linear, hence holomorphic, change of coordinates, so the inhomogeneous coordinates describe on the space $\mathbb{P}^n$ the structure of a complex manifold of dimension n. The unit sphere $S^{2n-1} \subset \mathbb{C}^{n+1}$ consists of points $Z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ for which $\sum_{i=0}^n |z_i|^2 = 1$, and is of course a compact subset of $\mathbb{C}^{n+1}$. The natural mapping $\mathbb{C}^{n+1} \longrightarrow \mathbb{P}^n$ restricts to a continuous mapping $S^{2n-1} \longrightarrow \mathbb{P}^n$ with image all of $\mathbb{P}^n$, and consequently $\mathbb{P}^n$ is a compact complex manifold. The inverse image of a point $w \in \mathbb{P}^n$ is the circle $\{t\,Z \mid |t| = 1\}$; it is not difficult to see that the mapping $\mathbb{C}^{n+1} \longrightarrow \mathbb{P}^n$ is a fibration over the projective space $\mathbb{P}^n$ with fibres the unit circle, and that determines the topology of $\mathbb{P}^n$.

A subset $V \subset M$ of a complex manifold such as $\mathbb{C}^n$ is a **submanifold** if in an open neighborhood U_a of any point $a \in M$ there are local coordinates in M such that $U_a \cap V$ is a linear subspace in terms of these coordinates. It is easy to see that a submanifold has the natural structure of a complex manifold; the dimension of the submanifold is the dimension of that manifold. If a subset V of an open neighborhood U_a of a point $a \in \mathbb{C}^n$ is the set of common zeros of $k \le n$ holomorphic functions $f_1, \dots f_k$ in U_a for which the $n \times k$ matrix $\{\partial_i f_k(a)\}$ is of rank $k \le n$ then V is a submanifold of dimension $n-k$ near k; indeed if $f_{k+1}, \dots, f_n$ are any holomorphic functions in U_a such that the $n \times n$ matrix $\{\partial_i f_j(a)\}$ has rank n then these functions can be taken as local coordinates near a and in terms of these coordinates the subset V is the linear subset as the set of zeros of the coordinates $f_1, \dots, f_k$. Similarly if $f_1, \dots f_n$ are $n \ge k$ holomorphic functions in an open neighborhood U_a of a point $a \in \mathbb{C}^k$ for which the $k \times n$ matrix $\{\partial_i f_k(a)\}$ is of rank $k \le n$ then the image $f(U_a) \subset \mathbb{C}^k$ of a subneighborhood of the point a under the mapping $F : U_a \longrightarrow \mathbb{C}^n$ defined by $F(z) = (f_1(z), \dots f_n(z)$ is a submanifold of an open neighborhood of $F(a)$ of dimension k; for if $\mathbb{C}^k$ is viewed as the subspace consisting of the first k variables $z_1, \dots, z_k$ in the space $\mathbb{C}^n$ with the coordinates $z_1, \dots, z_n$ then the functions $f_1, \dots, f_k, z_{k+1}, \dots z_n$ are local coordinates in $\mathbb{C}^n$ for which C^k is the linear subspace defined as the set of zeros of the coordinates $z_{k+1}, \dots, z_n$, while the mapping $F : \mathbb{C}^n \longrightarrow C^n$ for which $F(z_1, \dots, z_n) = (f_1(z), \dots, f_k(z), z_{k+1}, \dots, z_n)$ is a locally biholomorphic mapping which takes the linear subspace $z_{k+1} = \cdots = z_n = 0$ to the image of the mapping F.

A.3 Holomorphic Varieties

Almost any consideration of complex manifolds eventually leads to more general entities as well. A *holomorphic subvariety* of an open subset $U \subset \mathbb{C}^n$ is a subset $V \subset U$ with the property that for each point $a \in U$ there exist an open neighborhood U_a and finitely many holomorphic functions f_{ai} in U_a such that

$$V \cap U_a = \{z \in U_a \mid f_{ai}(z) = 0\}.$$

If a holomorphic subvariety is defined locally by a finite number of holomorphic functions having a nonsingular Jacobian determinant at each point then that subvariety has the natural structure of a complex manifold; thus a complex submanifold of an open subset $U \subset \mathbb{C}^n$ is a special case of a holomorphic subvariety of U. A holomorphic subvariety of U always is a closed subset of U, as an immediate consequence of this definition. It is not required that the subvariety be the set of common zeros of a collection of functions defined and holomorphic in all of U; the notion of a holomorphic subvariety is essentially local in nature. For this reason it is of course possible to consider a homorphic subvariety of an arbitrary complex manifold. More generally if $V_1, V_2 \subset U$ are holomorphic subvarieties of a complex manifold U and $V_1 \subset V_2$ then V_1 is called a *holomorphic subvariety* of V_2. A function f on a holomorphic subvariety $V \subset U$ in a complex manifold U is *holomorphic* on V if in an open neighborhood of each point of V it is the restriction to V of a holomorphic function in an open neighborhood of that point in the manifold U; and a function f on V is *meromorphic* if it can be represented in an open neighborhood of each point of V as a quotient of holomorphic functions on V. On a complex manifold any bounded meromorphic function actually is holomorphic; but that is not the case for meromorphic functions on holomorphic subvarieties, as is illustrated in the examples in the discussion of singular points on page 419. The bounded meromorphic functions on a holomorphic subvariety are known as *weakly holomorphic* functions; and a holomorphic subvariety for which all weakly holomorphic functions are actually holomorphic is known as a *normal* holomorphic variety.

A mapping $F : V_1 \longrightarrow V_2$ between two holomorphic subvarieties $V_1 \subset U_1$ and $V_2 \subset U_2$ of complex manifolds U_1 and U_2 is *holomorphic* if for any holomorphic function f in an open neighborhood of a point $a \in V_2$ the composition $f \circ F$ is a holomorphic function in an open neighborhod of the point $F^{-1}(a) \in V_1$; this is readily seen to be equivalent to the condition that in an open neighborhood of each point $a \in V_1$ the mapping F is the restriction to V_1 of a holomorphic mapping of an open neighborhood of a in U_1 into U_2. Two holomorphic subvarieties are *analytically equivalent* or *biholomorphic* if there are holomorphic mappings $F : V_1 \longrightarrow V_2$ and $G : V_2 \longrightarrow V_1$ that are inverse to one another; and a *holomorphic variety* is a biholomorphic equivalence class of holomorphic subvarieties. A holomorphic variety thus is an abstract version of a holomorphic subvariety, independent of a particular representation as a subvariety of a complex manifold; a complex manifold is a special case of a holomorphic variety. A holomorphic variety V is *reducible* if it can be written as a nontrivial union of holomorphic varieties, and otherwise is *irreducible*; in particular a complex manifold is an irreducible holomorphic variety if and only if it is connected. A holomorphic variety V is *locally reducible* at a point $a \in V$ if the restriction of V to any sufficiently small open neighborhood of the point a is reducible, and otherwise is *locally irreducible* at that point. A complex manifold is locally irreducible at each of its points; but a holomorphic varietiy may be locally reducible at some of its points. Any holomorphic variety V can be written uniquely as a union of irreducible subvarieties, called its *irreducible components*; and somewhat less trivially, an open neighborhood of any point of a holomorphic variety can be written uniquely as a finite union of locally irreducible varieties at that point.

Holomorphic varieties are generalizations of complex manifolds, but actually are complex manifolds at most points; for an arbitrary holomorphic variety V is a complex manifold outside a proper holomorphic subvariety $\mathfrak{S}(V) \subset V$ called the *singular locus* of V and consisting of precisely those points at which V fails to be a complex manifold. An irreducible holomorphic variety V is a connected complex manifold outside its singular locus $\mathfrak{S}(V)$; the dimension of the manifold

$V \sim \mathfrak{S}(V)$ is considered to be the *dimension* of the holomorphic variety V and is denoted by $\dim V$. The dimension of a reducible holomorphic variety is defined to be the largest of the dimensions of its irreducible components. If all irreducible components have the same dimension n the variety is said to be of *pure dimension* n. For some purposes it is more useful to consider the local dimension of a holomorphic variety V at a point $p \in V$, the dimension of arbitrarily small open neighborhods of that point, rather than the global dimension of the variety V; the local dimension is denoted by $\dim_p V$, and may vary from point to point unless the variety V is irreducible.

A few more detailed properties[11] of the dimension of a holomorphic variety are also needed. If V_1 is a holomorphic subvariety of a holomorphic variety V_2 then $\dim V_1 \le \dim V_2$, and this is a strict inequality unless V_1 and V_2 have a common irreducible component of the common dimension; in particular $\dim \mathfrak{S}(V) < \dim V$ for the singular locus $\mathfrak{S}(V)$ of an irreducible holomorphic subvariety V. If f is a nontrivial holomorphic function on an irreducible holomorphic variety V of dimension n then the zero locus of the function f is a holomorphic subvariety of pure dimension $n-1$ in V. Consequently if $f_1, \ldots, f_k$ are holomorphic functions on an irreducible holomorphic variety V of dimension n then the locus of common zeros of these functions is a holomorphic subvariety $W \subset V$ for which $\dim W \ge n-k$. It is not generally true that conversely a holomorphic subvariety W of dimension $n-k$ of an irreducible holomorphic variety V of dimension n can be defined as the set of common zeros of precisely k holomorphic functions on V; the minimal number of functions required to describe such a holomorphc subvariety even locally can exceed k. However a holomorphic subvariety W of dimension $n-1$ in a complex manifold V of dimension n always is locally the set of zeros of a single holomorphic function. In general if W_1, W_2 are holomorphic subvarieties of a complex manifold of dimension n and W is an irreducible component of the intersection $W_1 \cap W_2$ then $\dim W \ge \dim W_1 + \dim W_2 - n$.

The singular locus $\mathfrak{S}(V)$ of a one-dimensional holomorphic variety V is a discrete set of points, called the *singular points* of V, and the complement $V \sim \mathfrak{S}(V)$ has the natural structure of a union of Riemann surfaces. It can be shown that if $a \in \mathfrak{S}(V)$ is a singular point of the one-dimensional holomorphic variety V and if V_i are the local irreducible components of V in a neighborhood of the point a then to each separate irreducible component V_i there can be associated a Riemann surface $\hat{V}_i$ and a holomorphic mapping $f_i : \hat{V}_i \longrightarrow V_i$ such that $f_i^{-1}(a)$ is a single point of $\hat{V}_i$ and the restriction $f_i : \hat{V}_i \tilde{f}^{-1}(a) \longrightarrow V\tilde{a}$ is an analytic equivalence of Riemann surfaces. This construction can be carried out at each singular point, yielding a union of Riemann surfaces $\hat{V}$ called the *normalization* of the variety V or the *nonsingular model* of the variety V. The local normalization mappings lead to a global normalization mapping $f : \hat{V} \longrightarrow V$ that is a biholomorphic mapping between $V \sim \mathfrak{S}(V)$ and $f^{-1}(V \sim \mathfrak{S}(V)) \subset \hat{V}$; both $V \sim (V \sim \mathfrak{S}(V)) = \mathfrak{S}(V) \subset V$ and $\hat{V} \sim f^{-1}(V \sim \mathfrak{S}(V)) \subset \hat{V}$ are

[11] These properties are discussed and proved for instance in R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables* (Wadsworth & Brooks/Cole, 1990), vol. II.

discrete sets of points. For example, if $V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$, so that V is the union of the two irreducible components consisting of the two coordinate axes in $\mathbb{C}^2$, then the singular locus $\mathfrak{S}(V)$ consists of the origin itself; and $\hat{V}$ is the disjoint union of two copies of $\mathbb{C}^1$ corresponding to the two irreducible components of V. In this case the singularity arises as the intersection of two separate manifolds. For another example, if $V = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 - z_2^3 = 0\}$ then the singular locus $\mathfrak{S}(V)$ again consists of the origin itself, $\hat{V}$ is a copy of $\mathbb{C}^1$, and the mapping $f : \hat{V} \longrightarrow V$ is given explicitly by $z \longrightarrow (z^3, z^2)$. In this case the holomorphic subvariety V is globally irreducible, is locally irreducible at each point, and has a definite singularity at the origin, a point at which V fails to be a submanifold even topologically. The singularities of a one-dimensional subvariety are just those points at which there are locally weakly holomorphic functions (bounded meromorphic functions) that fail to be holomorphic. In the first of the two preceding examples the function $f(z_1, z_2) = (bz_1 + az_2)(z_1 + z_2)^{-1}$ is a meromorphic function on V that takes the value a on the component $z_1 = 0$ and takes the value b on the component $z_2 = 0$, so is not continuous hence not holomorphic; in the second example the function $f(z_1, z_2) = z_1/z_2$ is a meromorphic function that is bounded, since $z_1/z_2 = z$ is the value of the normalization mapping, but that is not holomorphic. There is an extensive literature dealing with the classification of the singular points of one-dimensional holomorphic subvarieties, for the most part in the context of algebraic geometry when these subvarieties are viewed as algebraic curves.[12]

There are considerably more complicated results for subvarieties of higher dimensions, where the singular loci can be proper holomorphic subvarieties of various dimensions and the singularities can be resolved only by much more complicated mappings; nothing further about the resolution of singularities of higher dimensional varieties is needed in the discussion in the body of the book, but some familiarity with a few general properties of the singularities of holomorphic varieties is required[13]. For any point $p \in V$ of a holomorphic subvariety $V \subset U$ of an open subset $U \subset \mathbb{C}^n$ the *ideal* $\mathrm{id}_p V \in \mathcal{O}_p$ of the subvariety V at a point $p \in V$ is the ideal in the local ring $\mathcal{O}_p$ of germs of holomorphic functions of n complex variables at the point p consisting of the germs of those holomorphic functions that vanish on V near the point p. It can be shown that this ideal always is finitely generated, so has a *basis* consisting of finitely many germs in $\mathcal{O}_p$. It is not necessarily the case that a collection of holomorphic functions in U having V as their set of common zeros generate the ideal of that subvariety at any point $p \in V$. However if V is a holomorphic subvariety of dimension $n - 1$ any holomorphic function f in U that vanishes to the first order at the regular points of V does generate the ideal of the subvari-

[12] A discussion of the singularities of algebraic curves in $\mathbb{C}^2$ from a geometric point of view can be found in E. Brieskorn and H. Knörrer, *Plane Algebraic Curves*, (Birkhäuser, 1986); a discussion of the classical results can be found in R. J. Walker *Algebraic Curves*, (Princeton University Press, 1950).

[13] For the proofs of these assertion and a more detailed discussion see for instance R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables* (Wadsworth & Brooks/Cole, 1990), vol. II.

ety V at each of its points If p is a regular point of the subvariety V, a point at which V is a submanifold of dimension r, then the set of differentials at p of a basis for the ideal $\mathrm{id}_p V$ has dimension $n - r$; the locus of zeros of these differentials, which are just the linear approximations at p of the holomorphic functions in the basis, is an r-dimensional linear subspace of $\mathbb{C}^n$ that can be identified with the complex tangent space $T_p(V)$ to the manifold V at the point p. However if p is a singular point of the subvariety V the differentials of this basis at p span a linear subspace of dimension strictly less than r; indeed it may be the case that all the differentials vanish at the point p. The common zero locus of these differentials still form a linear subspace of $\mathbb{C}^n$, defined to be the *complex tangent space* $T_p(V)$ of the holomorphic subvariety V at the point p. Thus for any r-dimensional subvariety $V \subset U \subset \mathbb{C}^n$ it is always the case that $r \le \dim T_p(V) \le n$; and p is a singular point precisely when $r < \dim T_p(V)$. The dimension of the tangent space is called the *tangential dimension* of the subvariety V at the point p and is denoted by $\mathrm{tdim}_p V$; it can be characterized alternatively as the least dimension of a complex submanifold of an open neighborhood of p in $\mathbb{C}^n$ containing the subvariety V in that neighborhood, so sometimes is called the *imbedding dimension* of the subvariety V at the point p. The point p is a regular point of the subvariety V, a point at which V is a submanifold, precisely when $\mathrm{tdim}_p V = \dim_p V$; equivalently $p \in \mathfrak{S}(V)$ precisely when $\mathrm{tdim}_p V > \dim_p V$.

The tangential dimension is a measure of the singularity of the subvariety V at the point p, the greater the tangential dimension the worse the singularity. An additional measure of the singularity of a point $p \in V$ for a proper holomorphic subvariety V of an open subset of $\mathbb{C}^n$ is the *multiplicity* of the subvariety V at the point p, defined as the least integer μ such that

$$\text{(A.18)} \qquad \left. \frac{\partial^k f}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right|_p = 0 \quad \text{whenever} \quad k = k_1 + \cdots + k_n < \mu$$

for all functions f in a basis for the ideal $\mathrm{id}_p V$ of the holomorphic subvariety V near p; the multiplicity of the subvariety V at the point p is denoted by $\mathrm{mult}_p V$. If $p \notin V$ then by this definition $\mathrm{mult}_p V = 0$. On the other hand $\mathrm{mult}_p V \ge 1$ at all points $p \in V$, and $\mathrm{mult}_p V = 1$ if and only if in an open neighborhood of the point p the subvariety V is contained in a proper complex submanifold of $\mathbb{C}^n$, since that is just the condition that there is a nontrivial holomorphic function near p that vanishes on the subvariety V but has a nonzero differential at the point p. The points $p \in V$ at which $\mathrm{mult}_p V > 1$ are singular points $p \in V \in \mathbb{C}^n$ at which $\mathrm{tdim}_p V = n$. The multiplicity consequently distinguishes between singularities at which the tangential dimension is maximal, that is, singularities at which the differentials of all functions in the ideal of the subvariety vanish; it is in this sense a finer measure of the nature of these somewhat extreme singularities. It should be noted that if V is not a proper subvariety but actually coincides with $\mathbb{C}^n$ then the local defining basis consists just of the function 0 and the multiplicity as defined by (A.18) would be infinite; that is the reason for restricting this invariant to proper subvarieties of $\mathbb{C}^n$.

It is useful to note that both the tangential dimension and the multiplicity of a holomorphic subvariety are monotonic, in the sense that for holomorphic subvarieties V and W of an open subset of $\mathbb{C}^n$

$$\text{(A.19)} \qquad \text{if } \ p \in V \subset W \ \text{ then } \ \begin{cases} \operatorname{tdim}_p V \le \operatorname{tdim}_p W, \\ \\ \operatorname{mult}_p V \le \operatorname{mult}_p W. \end{cases}$$

To see that this is the case, suppose that $f_1, \ldots, f_r$ is a basis for the ideal $\mathrm{id}_p V$ of the holomorphic subvariety V at the point $p \in V$ and that g is a holomorphic function in an open neighborhood of the point p in $\mathbb{C}^n$ that is part of a defining basis for the ideal $\mathrm{id}_p W$ of the subvariety W at p. Since the function g vanishes on the subvariety W it must also vanish on V, so its germ is in the ideal $\mathrm{id}_p V$ generated by the germs of the functions f_i and consequently $g = \sum_{i=1}^r h_i f_i$ for some holomorphic functions h_i in an open neighborhood of p. For any vector $t \in T_p(V)$ it then follows that

$$d_p g(t) = \sum_{i=1}^r h_i(p) d_p f_i(t) = 0;$$

therefore $t \in T_p(W)$, so that $T_p(V) \subset T_p(W)$ and consequently $\operatorname{tdim}_p(V) \le \operatorname{tdim}_p(W)$. Furthermore

$$\left. \frac{\partial^k g}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right|_p = \sum_{i=1}^r h_i(p) \left. \frac{\partial^k f_i}{\partial z_1^{k_1} \cdots \partial z_n^{k_n}} \right|_p + \text{lower derivatives of } f_i,$$

and in view of the definition (A.18) it is evident that $\operatorname{mult}_p V \le \operatorname{mult}_p W$

A refinement of the notion of the tangent space of a holomorphic subvariety, providing a more precise description of the singularities of holomorphic subvarieties, is the *tangent cone*[14] of a holomorphic subvariety, which can be defined in a number of equivalent ways. Geometrically the tangent cone $C_p(V)$ of a holomorphic subvariety V of an open subset $U \subset \mathbb{C}^n$ at a point $p \in V$ is defined to be the set of all vectors $v \in \mathbb{C}^n$ such that there exist a sequence of points $p_i \in V$ tending to the point $p \in V$ and a sequence of complex numbers $c_i \in \mathbb{C}$ such that $c_i(p_i - p) \longrightarrow v$; thus the tangent cone $C_p(V)$ is the set of limits of the secant lines joining points of V to the point p. Although far from obvious, the tangent cone $C_p(V)$ can be described as the set of all tangent vectors to smooth curves through V at the point p, that is, as the set of derivatives $v = \phi'(0)$ of $\mathcal{C}^1$ mappings $\phi : (-\epsilon, \epsilon) \longrightarrow V$ from an open neighborhood of the origin in the real line into the subvariety V such that $\phi(0) = p$. On the other hand the tangent cone can be defined algebraically as the zero locus of the initial polynomials $f_i^*(z)$ of a basis f_i for the ideal $\mathrm{id}_p V \subset \mathcal{O}_p$ of the subvariety V at the point p; here the *initial polynomial* $f^*(z)$ at the point p of a holomorphic function f in

[14]For a further discussion of tangent cones, and proofs of the results described here, see for instance H. Whitney, *Complex Analytic Varieties* (Addison Wesley, 1972), particularly Chapter 7.

an open neighborhood of the point p is the homogeneous polynomial consisting of the terms of lowest degree in the power series expansion of the function f in terms of local coordinates z_i centered at the point p. That too requires proof and is not trivial. It can be shown also that $\mathrm{mult}_p V = \min_i \deg f_i^*$ for a proper holomorphic subvariety in $\mathbb{C}^n$; so the multiplicity of the subvariety V at a point $p \in V$ is one of the properties of the tangent cone $C_p(V)$. It is evident from any of these equivalent definitions that the tangent cone is a cone at the origin in $\mathbb{C}^n$ in the usual sense, namely that if $v \in C_p(V) \subset \mathbb{C}^n$ then $cv \in C_p(V)$ for every complex number $c \in \mathbb{C}$; consequently the tangent cone determines a well defined subset $\mathbb{P}C_p(V) \subset \mathbb{P}^{n-1}$ in the complex projective space of dimension $g-1$, the *projective tangent cone* of the subvariety V at the point $p \in V$. It is frequently more convenient to describe the projective tangent cone rather than the tangent cone itself; the tangent cone then can be described as the set of all vectors in $\mathbb{C}^n$ that represent points in the subset $\mathbb{P}C_p(V) \subset \mathbb{P}^{n-1}$. The tangent cone $C_p(V)$ is a holomorphic subvariety of $\mathbb{C}^n$, and the projective tangent cone is a holomorphic hence an algebraic subvariety of $\mathbb{P}^{n-1}$. It can be shown that $C_p(V) \subset T_p(V)$ and that $\dim C_p(V) = \dim_p V$. Moreover the tangent cone also is monotonic, in the sense that if $p \in V \subset W$ for some holomorphic subvarieties V and W in an open neighborhood of the point $p \in \mathbb{C}^n$ then $C_p(V) \subset C_p(W)$. The tangent cone $C_p(V)$ may be reducible even though the holomorphic subvariety V is irreducible at the point $p \in V$.

There are other possible notions of the tangent cone to a holomorphic subvariety at a point, although the preceding is the commonly used notion and is almost inevitably what is meant by the term "tangent cone". One alternative notion that is useful for some purposes is the *extended tangent cone* $C_p^*(V)$ of a holomorphic subvariety V of an open neighborhood of a point p in $\mathbb{C}^n$, defined as the set of all vectors $v \in \mathbb{C}^n$ that are the limits of tangent vectors to the regular part of the variety V at points approaching p; the extended tangent cone coincides with the tangent space at any regular point of V, and is a natural extension of the tangent space of the regular part of V to the singular points[15]. The extended tangent cone is a holomorphic cone containing the usual tangent cone, so that $C_p(V) \subset C_p^*(V)$ at any point p of a holomorphic subvariety V; but this may be a strict inclusion, and indeed the dimension of the extended tangent cone may exceed the dimension of the subvariety V at the point p.

In addition to the preceding properties of the singularities of holomorphic varieties, some acquaintance with the properties[16] of some special classes of holomorphic mappings between holomorphic varieties also will be required at some points in the discussion in the body of the book. Particularly important

[15] The extended tangent cone is the cone C_4 in Whitney's terminology, while the usual tangent cone is C_3.

[16] For the proofs and further discussion of these topics see for instance R. C. Gunning, *Introduction to Holomorphic Functions of Several Variables*, Vol. II, (Wadsworth and Brooks/Cole, 1990), especially Sections L and N. Remmert's proper mapping theorem was proved in the paper by R.Remmert, "Holomorphe und meromorphe Abbildungen komplexer Räume", *Math. Ann.*, vol 133(1957), pp. 328-370; that result and the local properties of holomorphic mappings have been discussed extensively in the literature.

are *proper holomorphic mappings*, those holomorphic mappings such that the inverse image of any compact set is compact, and *finite holomorphic mappings*, those holomorphic mappings such that the inverse image of any point is a finite set of points. The simplest finite proper holomorphic mappings are *finite branched holomorphic coverings*, holomorphic mappings $F : V \longrightarrow W$ between two holomorphic varieties V and W with the properties that

(i) F is a finite, proper, surjective holomorphic mapping;
(ii) there are dense open subsets $V_0 \subset V$ and $W_0 \subset W$ such that $V_0 = F^{-1}(W_0)$ and the restriction $F|V_0 : V_0 \longrightarrow W_0$ is a locally biholomorphic covering mapping;
(iii) the complement $W - W_0$ is a holomorphic subvariety of W.

Any holomorphic mapping between one-dimensional holomorphic varieties is locally a finite branched holomorphic covering, as is quite familiar; finite branched holomorphic coverings are those holomorphic mappings between holomorphic varieties of arbitrary dimensions that are most like holomorphic mappings between one-dimensional holomorphic varieties. The *local parametrization theorem* asserts that any irreducible holomorphic variety of dimension n can be represented locally as a finite branched holomorphic covering of an open subset of $\mathbb{C}^n$; that provides a particularly convenient local representation for the study of holomorphic varieties. A holomorphic mapping $F : V \longrightarrow W$ between two holomorphic varieties V and W is said to be *finite* if $F^{-1}(p)$ is a finite subset of V for each point $p \in W$; it can be shown that a holomorphic mapping $F : V \longrightarrow W$ is finite if and only if for each irreducible component V_i of V the restriction $F|V_i : V_i \longrightarrow F(V_i)$ is locally a finite branched holomorphic covering. More general proper holomorphic mappings arise quite frequently. One of their most important properties is given in *Remmert's proper mapping theorem*, which asserts that if $F : V \longrightarrow W$ is a proper holomorphic mapping between holomorphic varieties V and W then the image $F(V)$ is a holomorphic subvariety of W; and if V is irreducible then so is its image, and

$$\dim F(V) = \sup_{p\in V} \big(\dim V - \dim_p F^{-1}(F(p))\big). \tag{A.20}$$

For a finite proper holomorphic mapping $\dim_p F^{-1}(F(p)) = 0$ for all points $p \in V$ so the preceding formula reduces to

$$\dim F(V) = \dim V. \tag{A.21}$$

In general the fibres $F^{-1}(q)$ over points $q \in W$ need not be irreducible, and their dimensions may vary from point to point. However there is at least some regularity to the behavior of the dimension, as a consequence of *Remmert's semi-continuity theorem*, which asserts that if $F : V \longrightarrow W$ is a holomorphic mapping between holomorphic varieties V and W, not necessarily a proper holomorphic mapping, then for any integer ν the subset $\big\{\, p \in V \;\big|\; \dim_p F^{-1}(F(p)) \ge \nu \,\big\}$ is a holomorphic subvariety of V; this is an extension of the condition that $\dim_p F^{-1}(F(p))$ is an upper semi-continuous function of the point $p \in V$. Images of holomorphic varieties under holomorphic mappings that are not proper also may be holomorphic subvarieties. The *local mapping theorem* asserts that if $F : V \longrightarrow W$ is a holomorphic mapping between two holomorphic varieties V and W, not necessarily a proper holomorphic mapping, and if $\dim_p F^{-1}(F(p)) = \nu$ is independent of the point $p \in V$, then each point $p \in V$ has arbitrarily small open neighborhoods V_p such that $F(V_p)$ is a holomorphic subvariety of an open neighborhood of $F(p)$ in W and $\dim_{F(p)} F(V_p) = \dim_p V - \nu$. If the subvariety V is irreducible at the point $p \in V$ then the converse also holds: if there are arbitrily small open neighborhoods V_p of the point $p \in V$ such that $F(V_p)$ is a holomorphic subvariety of an open neighborhood of the point $f(p) \in W$ then $\dim_q F^{-1}(F(q))$ is a constant independent of the point $q \in V$ in some open neighborhood of the point $p \in V$. Thus the fact that the fibres of a holomorphic mapping have constant dimension really is almost equivalent to the condition that the image of the mapping is locally a holomorphic subvariety. These results will be used at various points in the discussion in the body of this book.

Appendix B

Vector Bundles

B.1 Definitions

A *complex vector bundle* of *rank* r over a topological space M is a topological space λ with a continuous mapping $\pi : \lambda \longrightarrow M$ such that (i) in an open neighborhood U of each point $p \in M$ there is a commutative diagram

$$\begin{array}{ccc} \lambda \supset \pi^{-1}(U) & \xrightarrow{\lambda_U} & U \times \mathbb{C}^r \\ {\scriptstyle\pi}\big\downarrow & & {\scriptstyle\pi_1}\big\downarrow \\ M \supset U & = & U \end{array} \tag{B.1}$$

where λ_U is a homeomorphism from $\pi^{-1}(U)$ to the product $U \times \mathbb{C}^r$ and π_1 is the projection of the product to its first factor; and (ii) in an intersection $U \cap V$ of two such neighborhoods of p there is a continuous mapping

$$\lambda_{VU} : U \cap V \longrightarrow \mathrm{Gl}(r, \mathbb{C}) \tag{B.2}$$

such that the composite mapping $\lambda_V \circ \lambda_U^{-1} : (U \cap V) \times \mathbb{C}^r \longrightarrow (U \cap V) \times \mathbb{C}^r$ has the form

$$(\lambda_V \circ \lambda_U^{-1})(p, t) = \big(p,\, \lambda_{VU}(p)t\big) \tag{B.3}$$

for any point $(p, t) \in (U \cap V) \times \mathbb{C}^r$. The space M is called the *base space* of the vector bundle λ, the mapping π is called the *projection*, the mappings λ_U are called the *coordinate mappings* or *local coordinates*, the linear transformations $\lambda_{VU}(p)$ are called the *coordinate transition functions*, and the inverse image $\lambda_p = \pi^{-1}(p)$ of a point $p \in M$ is called the *fibre* over the point p. The local product structure provided by the homeomorphism λ_U describes a point in the open subset $\pi^{-1}(U) \subset \lambda$ by a pair $(p, t_U) \in U \times \mathbb{C}^r$, where the vector $t_U \in \mathbb{C}^r$ is the *fibre coordinate* of that point in terms of the local product structure over U; the vector t_U will be viewed as a column vector of length r when explicit formulas are required, and the coordinate transition functions then will be viewed as $r \times r$

complex matrices. If $p \in U \cap V$ the fibre coordinates of points in $\pi^{-1}(p)$ in terms of the local product structures over U and V are related by

$$t_V = \lambda_{VU}(p)\, t_U. \tag{B.4}$$

The simplest example of a complex vector bundle of rank r over a topological space M is the *product bundle* or *trivial bundle*, the product $\lambda = M \times \mathbb{C}^r$ where π is the natural projection to the first factor; for this bundle all the coordinate transition functions can be taken to be the identity mapping $\lambda_{VU}(p) = I$ since the coordinate mappings can be taken to be the identity mapping. A complex vector bundle of rank 1 also is called a *complex line bundle*; for a complex line bundle the coordinate transition functions are merely nowhere vanishing functions in the intersections $U \cap V$. If the base space M is a topological manifold of dimension n and the subsets $U \subset M$ are coordinate neighborhoods in M that are identified with subsets of $\mathbb{R}^n$ then the local coordinate mappings λ_U impose on the space λ the structure of a topological manifold for which $\dim \lambda = n + r$; in addition, since the homeomorphisms $\lambda_V \circ \lambda_U^{-1}$ belong to the pseudogroup $\mathcal{CL}$ consisting of local homeomorphisms between products $\mathbb{R}^n \times \mathbb{C}^r$ that are complex linear mappings on $\mathbb{C}^r$, the manifold λ is a $\mathcal{CL}$ manifold. If the base space M is a $\mathcal{C}^\infty$ manifold and the coordinate transition functions $\lambda_{UV}(p)$ are $\mathcal{C}^\infty$ functions the manifold λ is a $\mathcal{C}^\infty$ manifold and the bundle is said to be a $\mathcal{C}^\infty$ *vector bundle.* If the base space M is a complex manifold and the coordinate transition functions $\lambda_{UV}(p)$ are holomorphic functions the manifold λ is a complex manifold and the bundle is said to be a *holomorphic vector bundle.* If the coordinate transition functions $\lambda_{VU}(p)$ are locally constant functions the bundle is called a *flat vector bundle.* A holomorphic vector bundle also has the weaker structure of a $\mathcal{C}^\infty$ vector bundle, and a flat vector bundle also has the weaker structure of a holomorphic vector bundle.

A *cross-section* of a complex vector bundle λ over a topological space M is a continuous mapping $f : M \longrightarrow \lambda$ such that $\pi \circ f(p) = p$ for each point $p \in M$. The composition of a cross-section f and the coordinate mapping λ_U over an open subset $U \subset M$ has the form

$$(\lambda_U \circ f)(p) = (p, f_U(p)) \in M \times \mathbb{C}^r \tag{B.5}$$

for any point $p \in U$, where $f_U : U \longrightarrow \mathbb{C}^r$ is a continuous mapping called the *local form* of the cross-section over U; thus $f_U(p)$ is the fibre coordinate of the point $f(p) \in \lambda$ in terms of the local product structure over U. It is clear that a cross-section f is described completely by its local form over subsets $U \subset M$, and that the local forms satisfy

$$f_V(p) = \lambda_{VU}(p)\, f_U(p) \quad \text{for} \quad p \in U \cap V \tag{B.6}$$

since the fibre coordinates satisfy (B.4). A cross-section f of a $\mathcal{C}^\infty$ complex vector bundle λ is a $\mathcal{C}^\infty$ *cross-section* if the mapping $f : M \longrightarrow \lambda$ is a $\mathcal{C}^\infty$ mapping, or equivalently if the local forms $f_U : U \longrightarrow \mathbb{C}^r$ are $\mathcal{C}^\infty$ mappings; a cross-section f of a holomorphic vector bundle λ is a *holomorphic cross-section* if the mapping $f : M \longrightarrow \lambda$ is a holomorphic mapping, or equivalently if the local forms $f_U : U \longrightarrow \mathbb{C}^r$ are holomorphic mappings; and a cross-section f of a flat vector bundle λ is a *flat cross-section* if the local forms $f_U : U \longrightarrow \mathbb{C}^r$ are locally constant mappings. Cross-sections can be added and multiplied by complex constants, by using the structure of a complex vector space in the fibre; thus the set of cross-sections has the natural structure of a complex vector space. Clearly linear combinations of $\mathcal{C}^\infty$ cross-sections of a $\mathcal{C}^\infty$ vector bundle again are $\mathcal{C}^\infty$ cross-sections, and correspondingly for holomorphic or flat cross-sections; so the set of all continuous, $\mathcal{C}^\infty$, holomorphic or flat cross-sections of a complex vector bundle having the appropriate regularity are also complex vector spaces. The vector space of continuous cross-sections of a vector bundle λ is denoted by $\Gamma(M, \mathcal{C}(\lambda))$, the vector space of $\mathcal{C}^\infty$ cross-sections of a $\mathcal{C}^\infty$ vector bundle λ over a $\mathcal{C}^\infty$ manifold M is denoted by $\Gamma(M, \mathcal{E}(\lambda))$, the vector space of holomorphic cross-sections of a holomorphic vector bundle λ over a complex manifold M is denoted by $\Gamma(M, \mathcal{O}(\lambda))$, and the vector space of flat cross-sections of a flat vector bundle λ over a topological manifold M is denoted by $\Gamma(M, \mathcal{F}(\lambda))$.

If λ^i for $1 \le i \le n$ are vector bundles over a topological space M described by coordinate transition functions λ^i_{UV} for $1 \le i \le n$ their *direct sum* $\lambda^1 \oplus \cdots \oplus \lambda^n$ is the vector bundle with the coordinate transition functions $\lambda^1_{UV} \oplus \cdots \oplus \lambda^n_{UV}$ and their *tensor product* $\lambda^1 \otimes \cdots \otimes \lambda^n$ is the vector bundle with the coordinate transition functions $\lambda^1_{UV} \otimes \cdots \otimes \lambda^n_{UV}$, where

$$\text{(B.7)} \qquad \lambda^1_{UV} \oplus \cdots \oplus \lambda^n_{UV} = \begin{pmatrix} \lambda^1_{UV} & 0 & \cdots & 0 \\ 0 & \lambda^2_{UV} & \cdots & 0 \\ & \cdots & & \cdots \\ 0 & 0 & \cdots & \lambda^n_{UV} \end{pmatrix}$$

and $\lambda^1_{UV} \otimes \cdots \otimes \lambda^n_{UV}$ is the linear transformation on tensors $v_{U,i_1 i_2 \cdots i_n}$ defined by

$$\text{(B.8)} \qquad v_{U,i_1 i_2 \cdots i_n} = \sum_{j_1, \cdots j_n} \lambda^1_{UV,i_1 j_1}\, \lambda^2_{UV,i_2 j_2} \cdots \lambda^n_{UV,i_n j_n}\, v_{V,j_1 j_2 \cdots j_n}.$$

It is evident that

$$\text{(B.9)} \qquad \operatorname{rank}(\lambda^1 \oplus \cdots \oplus \lambda^n) = \operatorname{rank}\lambda^1 + \cdots + \operatorname{rank}\lambda^n \quad \text{and}$$

$$\text{(B.10)} \qquad \operatorname{rank}(\lambda^1 \otimes \cdots \otimes \lambda^n) = (\operatorname{rank}\lambda^1) \cdots (\operatorname{rank}\lambda^n).$$

If $\lambda^1 = \cdots = \lambda^n = \lambda$ the tensor product is denoted by $\lambda^{\otimes n}$. The tensor product $\lambda^1 \otimes \lambda^2$ of two vector bundles can be viewed alternatively as a vector bundle in which the fibres as well as the coordinate transition functions are matrices; for in this case (B.8) becomes

$$v_{U,i_1 i_2} = \sum_{j_1, j_2} \lambda^1_{UV,i_1 j_1}\, v_{V,j_1 j_2}\, \lambda^2_{UV,i_2 j_2},$$

and when the values $v_{U,i_1 i_2}$ are interpreted as entries in a matrix v_U this is the matrix identity

$$\text{(B.11)} \qquad v_U = \lambda^1_{UV}\, v_V\ {}^t\lambda^2_{UV}$$

where ${}^t\lambda^2_{UV}$ is the transpose of the matrix λ^2_{UV}. A tensor product of line bundles λ^1 and λ^2 is a line bundle, and the notation usually is simplified by setting $\lambda^1 \otimes \lambda^2 = \lambda^1\lambda^2$; correspondingly the notation for the tensor product of a line bundle σ and a vector bundle λ usually also is simplified by setting $\sigma \otimes \lambda = \sigma\lambda$. The tensor product of n copies of a line bundle λ with itself usually is denoted by λ^n.

If $\lambda = \{\lambda_{UV}\}$ is a vector bundle of rank r over a topological space M then for any group homomorphism $\theta : \mathrm{Gl}(r, \mathbb{C}) \longrightarrow \mathrm{Gl}(s, \mathbb{C})$ the mappings $\theta(\lambda_{UV})$ can be taken as the coordinate transition functions describing a vector bundle $\theta(\lambda)$ of rank s over M. For example, associated to any vector bundle λ of rank r over M is its *determinant bundle* $\det\lambda$, the line bundle over M described by the coordinate transition functions $\det\lambda_{UV}$, and its *dual bundle* $\lambda^* = {}^t\lambda^{-1}$, the vector bundle of rank r described by the coordinate transition functions $\lambda^*_{UV} = {}^t\lambda^{-1}_{UV}$. Similarly to any vector bundle λ of rank r over M can be associated its *adjoint bundle* $\mathrm{Ad}\,\lambda$, the vector bundle of rank r^2 described by the coordinate transition functions $\mathrm{Ad}(\lambda_{UV})$ where $\mathrm{Ad} : \mathrm{Gl}(r, \mathbb{C}) \longrightarrow \mathrm{Gl}(r^2, \mathbb{C})$ is the adjoint representation, the mapping that associates to a matrix $A \in \mathrm{Gl}(r, \mathbb{C})$ the linear transformation on the vector space $\mathbb{C}^{r\times r}$ of $r \times r$ complex matrices defined by $\mathrm{Ad}(A)Z = A\,Z\,A^{-1}$. The linear subspace $\mathbb{C}^{r\times r}_0 \subset \mathbb{C}^{r\times r}$ consisting of matrices of trace zero is preserved under the adjoint representation, and the restriction of the adjoint representation to this subspace $\mathbb{C}^{r\times r}_0$ is another group homomorphism $\mathrm{Ad}_0 : \mathrm{Gl}(r, \mathbb{C}) \longrightarrow \mathrm{Gl}(r^2 - 1, \mathbb{C})$ that can be used to associate to the vector bundle λ its *restricted adjoint bundle* $\mathrm{Ad}_0\,\lambda$ of rank $r^2 - 1$, defined by the coordinate transition functions $\mathrm{Ad}_0(\lambda_{UV})$. The changes of coordinates in the fibres of the bundle $\mathrm{Ad}\,\lambda$ are given by $Z_U = \mathrm{Ad}(\lambda_{UV})Z_V = \lambda_{UV} Z_V \lambda_{UV}$ where Z_U, $Z_V \in \mathbb{C}^{r\times r}$, so in view of (B.11)there is the natural identification

$$\mathrm{Ad}\,\lambda = \lambda \otimes \lambda^* \tag{B.12}$$

that is quite commonly used.

If λ and σ are vector bundles of ranks r and s over the same space M, with projections $\pi_\lambda : \lambda \longrightarrow M$ and $\pi_\sigma : \sigma \longrightarrow M$, a *bundle homomorphism* $\phi : \sigma \longrightarrow \lambda$ is a continuous mapping between the topological spaces σ and λ such that (i) the diagram

$$\begin{tikzcd} \sigma \arrow[r, "\phi"] \arrow[d, "\pi_\sigma"'] & \lambda \arrow[d, "\pi_\lambda"'] \\ U \arrow[r, equal] & U \end{tikzcd} \tag{B.13}$$

is commutative, so that $\phi(\sigma_p) \subset \lambda_p$ for the fibres over any point $p \in M$; and (ii) the restriction $\phi\big|\sigma_p : \sigma_p \longrightarrow \lambda_p$ of the mapping ϕ to the fibre σ_p is a linear mapping for each point $p \in M$. In terms of the fibre coordinates t_U for the bundle λ and s_U for the bundle σ over an open subset $U \subset M$ the composite mapping $\lambda_U \circ \phi \circ \sigma_U^{-1} : U \times \mathbb{C}^s \longrightarrow U \times \mathbb{C}^r$ has the form

$$(\lambda_U \circ \phi \circ \sigma_U^{-1})(p, s_U) = (p, t_U) = (p, \phi_U(p)\, s_U) \tag{B.14}$$

where $\phi_U(p) : \mathbb{C}^s \longrightarrow \mathbb{C}^r$ is a linear mapping for each point $p \in U$ that is a continuous function of the point $p \in U$, called the *local form* of the homomorphism ϕ. If $p \in U \cap V$

$$\begin{aligned}(p, \phi_V(p)\, s_V) &= \big(\lambda_V \circ \phi \circ \sigma_V^{-1}\big)(p, s_V)\\ &= \big(\lambda_V \circ \lambda_U^{-1}\big) \circ \big(\lambda_U \circ \phi \circ \sigma_U^{-1}\big) \circ \big(\sigma_U \circ \sigma_V^{-1}\big)(p,\ s_V)\\ &= \big(\lambda_V \circ \lambda_U^{-1}\big) \circ \big(\lambda_U \circ \phi \circ \sigma_U^{-1}\big)(p,\ \sigma_{UV}(p)\, s_V)\\ &= \big(\lambda_V \circ \lambda_U^{-1}\big)(p,\ \phi_U(p) \cdot \sigma_{UV}(p)\, s_V)\\ &= \big(p,\ \lambda_{VU}(p) \cdot \phi_U(p) \cdot \sigma_{UV}(p)\, s_V\big);\end{aligned}$$

consequently

$$\phi_V(p) = \lambda_{VU}(p) \cdot \phi_U(p) \cdot \sigma_{UV}(p) \quad \text{if} \quad p \in U \cap V. \tag{B.15}$$

The homomorphism ϕ is a $\mathcal{C}^\infty$ *homomorphism* if λ and σ are $\mathcal{C}^\infty$ bundles and the mapping $\phi : \sigma \longrightarrow \lambda$ is a $\mathcal{C}^\infty$ mapping, or equivalently if the local forms $\phi_U(p)$ are $\mathcal{C}^\infty$ functions; the homomorphism is a *holomorphic homomorphism* if λ and σ are holomorphic bundles and the mapping $\phi : \sigma \longrightarrow \lambda$ is a holomorphic mapping, or equivalently if the local forms $\phi_U(p)$ are holomorphic functions; and the homomorphism ϕ is a *flat homomorphism* if σ and λ are flat bundles and the local forms $\phi_U(p)$ are locally constant functions. It is evident from (B.15) that if $\phi = \{\phi_U\}$ and $\psi = \{\psi_U\}$ are two homomorphisms from a vector bundle σ to a vector bundle λ over M then $a\phi + b\psi = \{a\phi_U + b\psi_U\}$ is also a homomorphism from σ to λ for any complex constants $a, b \in \mathbb{C}$; the set of homomorphisms from σ to λ thus naturally form a complex vector space, denoted by $\mathrm{Hom}(\sigma, \lambda)$. If the bundles λ and σ are $\mathcal{C}^\infty$ the set of $\mathcal{C}^\infty$ homomorphisms form a vector subspace $\mathrm{Hom}_{\mathcal{E}}(\sigma, \lambda) \subset \mathrm{Hom}(\sigma, \lambda)$, as do the further subspaces $\mathrm{Hom}_{\mathcal{O}}(\sigma, \lambda)$ of holomorphic homomorphisms between holomorphic vector bundles and $\mathrm{Hom}_{\mathcal{F}}(\sigma, \lambda)$ of flat homomorphisms between flat vector bundles. When $\sigma = \lambda$ vector bundle homomorphisms also are called *endomorphisms* of that bundle. Since the composition of two endomorphisms is again an endomorphism it is evident that the set of endomorphisms $\mathrm{End}(\lambda) = \mathrm{Hom}(\lambda, \lambda)$ has the natural structure of a complex algebra; of course the same is true for special classes of endomorphisms such as $\mathrm{End}_{\mathcal{E}}(\lambda)$, $\mathrm{End}_{\mathcal{O}}(\lambda)$ and $\mathrm{End}_{\mathcal{F}}(\lambda)$.

If $\phi : \sigma \longrightarrow \lambda$ is a bundle homomorphism, the rank of the linear mapping $\phi|\sigma_p : \sigma_p \longrightarrow \lambda_p$ is called the *rank* of the homomorphism ϕ at the point $p \in M$ and is denoted by $\mathrm{rank}_p(\phi)$; of course $\mathrm{rank}_p(\phi) = \mathrm{rank}\phi_U(p)$ in terms of the local form ϕ_U of the homomorphism ϕ for any coordinate neighborhood U containing the point p. The maximal rank of a homomorphism ϕ at all the points of M is called simply the *rank* of the homomorphism ϕ and is denoted by $\mathrm{rank}\,\phi$; thus $\mathrm{rank}\,\phi = \sup_{p\in M} \mathrm{rank}_p\phi$. The rank of a homomorphism ϕ can vary from point to point on the set M, except in the case of a flat homomorphism of flat vector bundles over a connected topological space. The condition that $\mathrm{rank}_p\phi \le n$ amounts to the vanishing of all $(n+1) \times (n+1)$ subdeterminants of the matrix $\phi_U(p)$ at the point $p \in U$; so for a holomorphic homomorphism ϕ between two holomorphic vector bundles over a complex manifold M the set of points $p \in M$ at which $\operatorname{rank}_p\phi \le t$ is either the entire complex manifold M or a holomorphic subvariety of M, and the set of points $p \in M$ at which $\operatorname{rank}_p\phi < \operatorname{rank}\phi$ is a proper holomorphic subvariety of M. A homomorphism ϕ is said to be of *constant rank* if $\operatorname{rank}_p\phi = \operatorname{rank}\phi$ at all points $p \in M$. A bundle homomorphism $\phi : \sigma \longrightarrow \lambda$ is *injective* (*surjective*) if its restriction $\phi\big|\sigma_p : \sigma_p \longrightarrow \lambda_p$ is an injective linear mapping (a surjective linear mapping) over each point $p \in M$; it is an *isomorphism* if it is both injective and surjective, or equivalently if it has an inverse vector bundle homomorphism $\psi : \lambda \longrightarrow \sigma$. Isomorphic bundles of course have the same rank; and a homomorphism between two vector bundles λ and σ for which $\operatorname{rank}\lambda = \operatorname{rank}\sigma = r$ is an isomorphism if and only if the homomorphism is of constant rank r.

For many purposes it is not necessary to consider the local product structures of a vector bundle over a space M for all open subsets of M, but suffices to consider only those for a single open covering of M. If λ is a vector bundle of rank r over a topological space M then for any sufficiently fine open covering $\mathfrak{U} = \{U_\alpha\}$ of M there will be coordinate mappings

$$\lambda_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^r$$

for the bundle λ; and in an intersection $U_\alpha \cap U_\beta$ as in (B.3) there are the coordinate transition functions $\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Gl}(r, \mathbb{C})$ for which

$$(\lambda_\alpha \circ \lambda_\beta^{-1})(p, t) = (p, \lambda_{\alpha\beta}(p)t).$$

The collection $\{U_\alpha, \lambda_{\alpha\beta}\}$ of the open subsets $U_\alpha \subset M$ and coordinate transition functions $\lambda_{\alpha\beta}$ is called a *coordinate bundle* describing the vector bundle λ. The fibre coordinates t_α and t_β of a point of λ lying over a point $p \in U_\alpha \cap U_\beta \subset M$ are related by

$$t_\alpha = \lambda_{\alpha\beta}(p)\, t_\beta \quad \text{for} \quad p \in U_\alpha \cap U_\beta \tag{B.16}$$

as in (B.4). It is clear that the coordinate transition functions $\lambda_{\alpha\beta}$ satisfy the compatibility conditions

$$\begin{aligned} \lambda_{\alpha\alpha}(p) &= \mathrm{I} \quad \text{if} \quad p \in U_\alpha, \\ \lambda_{\alpha\beta}(p) \cdot \lambda_{\beta\alpha}(p) &= \mathrm{I} \quad \text{if} \quad p \in U_\alpha \cap U_\beta, \\ \lambda_{\alpha\beta}(p) \cdot \lambda_{\beta\gamma}(p) \cdot \lambda_{\gamma\alpha}(p) &= \mathrm{I} \quad \text{if} \quad p \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned} \tag{B.17}$$

The sets U_α, U_β, U_γ in (B.17) are not necessarily distinct; the second condition follows from the first and third upon setting $\gamma = \alpha$, but is included separately in (B.17) for emphasis. Any collection of open subsets $U_\alpha \subset M$ covering M and of continuous mappings

$$\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{Gl}(r, \mathbb{C}) \tag{B.18}$$

satisfying the compatibility conditions (B.17) form a coordinate bundle describing a vector bundle over M. Indeed it is a straightforward matter to verify that, in terms of the equivalence relation on pairs $(p_\alpha, t_\alpha) \in U_\alpha \times \mathbb{C}^r$ defined by $(p_\alpha, t_\alpha) \sim (p_\beta, t_\beta)$ whenever $p_\alpha = p_\beta = p \in M$ and $t_\alpha = \lambda_{\alpha\beta}(p)t_\beta$, which is an equivalence relation in the usual sense as an immediate consequence of (B.17), the quotient of the disjoint union of the products $U_\alpha \times \mathbb{C}^r$ by this equivalence relation is a complex vector bundle described by the coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$. This is a particularly common and useful way of describing complex vector bundles; $\mathcal{C}^\infty$, holomorphic or flat complex vector bundles can be described in this way for mappings (B.18) that are $\mathcal{C}^\infty$, holomorphic, or locally constant. It should be noted that for there to be a description of a given vector bundle λ by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ the covering $\mathfrak{U} = \{U_\alpha\}$ of the space M must be sufficiently fine that the sets $\pi^{-1}(U_\alpha)$ have the necessary product structure. However if the sets U_α have the property that any vector bundle of rank r over U_α is a product bundle then any vector bundle of rank r over M can be described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ for this covering; and the corresponding assertion of course holds for $\mathcal{C}^\infty$, holomorphic or flat vector bundles.

Two coordinate bundles $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$ that describe the same complex vector bundle over M are called *equivalent* coordinate bundles. If the fibre coordinates are t_α over U_α and t_k over V_k equivalence means that in addition to the relations (B.16) between the fibre coordinates t_α and t_β over intersections $U_\alpha \cap U_\beta$ and the corresponding relations $t_k = \sigma_{kl}t_l$ between the fibre coordinates t_k and t_l over $V_k \cap V_l$ there are further relations of the form

$$\text{(B.19)} \qquad t_\alpha = \mu_{\alpha k}(p)t_k \quad \text{and} \quad t_k = \mu_{k\alpha}(p)t_\alpha \quad \text{for} \quad p \in U_\alpha \cap V_k$$

between the fibre coordinates t_α over U_α and t_k over V_k for some continuous mappings

$$\text{(B.20)} \qquad \mu_{\alpha k},\ \mu_{k\alpha} : U_\alpha \cap V_k \longrightarrow \mathrm{Gl}(r, \mathbb{C}).$$

Consequently in addition to the compatibility conditions (B.17) for the coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ and the corresponding conditions for the coordinate bundle $\{V_k, \sigma_{kl}\}$ there are the further compatibility conditions

$$\text{(B.21)} \qquad \begin{aligned} \lambda_{\alpha\beta}(p)\mu_{\beta m}(p)\mu_{m\alpha}(p) &= I \quad \text{for} \quad p \in U_\alpha \cap U_\beta \cap V_m, \\ \mu_{\alpha l}(p)\sigma_{lm}(p)\mu_{m\alpha}(p) &= I \quad \text{for} \quad p \in U_\alpha \cap V_l \cap V_m. \end{aligned}$$

The corresponding conditions for other orders of the products of the coordinate transition functions follow automatically from these relations; and for the special cases in which $\beta = \alpha$ or $l = k$ it follows that

$$\text{(B.22)} \qquad \mu_{\alpha k}(p)\mu_{k\alpha}(p) = I.$$

Conversely two coordinate bundles $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$ of the same rank over M are equivalent coordinate bundles if there are mappings (B.20) satisfying (B.21), since in that case the collection of all the sets U_α and V_k and of all the mappings $\lambda_{\alpha\beta}$, σ_{kl}, $\mu_{\alpha k}\,\mu_{k\alpha}$ form a coordinate bundle over M describing a vector bundle over M that is also described by the two separate coordinate bundles $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$. When it is useful or necessary to specify an additional regularity condition for the vector bundle the coordinate bundles are said to be $\mathcal{C}^\infty$ *equivalent* or *holomorphic equivalent* or *flat equivalent* coordinate bundles; for the equivalence of bundles with these further regularity conditions the mappings $\lambda_{\alpha\beta}$, σ_{kl}, $\mu_{\alpha k}$ also must satisfy the appropriate regularity conditions. A somewhat simpler and more useful condition for the equivalence of two coordinate bundles arises from the observation that if $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$ are equivalent coordinate bundles and if $p \in U_\alpha \cap U_\beta \cap V_k \cap V_l$ then it follows from (B.21) that $\mu_{k\alpha}(p)\lambda_{\alpha\beta}(p)\mu_{\beta l}(p) = \mu_{k\alpha}(p) \cdot \mu_{\alpha k}(p)\mu_{k\beta}(p) \cdot \mu_{\beta k}(p) = \mu_{k\beta}(p)\mu_{\beta l}(p) = \sigma_{kl}(p)$ and consequently

$$\sigma_{kl}(p) = \mu_{k\alpha}(p)\lambda_{\alpha\beta}(p)\mu_{\beta l}(p) \quad \text{for} \quad p \in U_\alpha \cap U_\beta \cap V_k \cap V_l. \tag{B.23}$$

Conversely if $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{V_k, \sigma_{kl}\}$ are two coordinate bundles of the same rank over M and there are mappings (B.20) satisfying (B.22) and (B.23) then these two coordinate bundles are equivalent; indeed when $k = l$ the equations (B.23) reduce to the first equations in (B.21) while when $\alpha = \beta$ they reduce to the second equations in (B.21), and consequently the two coordinate bundles are equivalent. In particular a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ describes a trivial bundle if and only if it is equivalent to the coordinate bundle described by a single coordinate neighborhood $V_k = M$; and in that case condition (B.23) takes the form

$$\lambda_{\alpha\beta}(p) = \mu_\alpha(p)\mu_\beta(p)^{-1}. \tag{B.24}$$

On the other hand for two coordinate bundles defined in terms of the same covering of M, so for two coordinate bundles $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{U_\alpha, \sigma_{\alpha\beta}\}$, condition (B.23) for the case that $V_k = U_\alpha$ and $V_l = U_\beta$ takes the form

$$\sigma_{\alpha\beta}(p) = \mu_\alpha(p)\lambda_{\alpha\beta}(p)\mu_\beta(p)^{-1} \quad \text{for} \quad p \in U_\alpha \cap U_\beta \tag{B.25}$$

where

$$\mu_\alpha = \mu_{\alpha\alpha} : U_\alpha \longrightarrow \mathrm{Gl}(r, \mathbb{C}); \tag{B.26}$$

thus this condition must be satisfied if $\{U_\alpha, \lambda_{\alpha\beta}\}$ and $\{U_\alpha, \sigma_{\alpha\beta}\}$ are equivalent coordinate bundles. Conversely if this condition is satisfied then for any point $p \in U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$\begin{aligned} \sigma_{\alpha\beta}(p) &= \mu_\alpha(p)\lambda_{\alpha\beta}(p)\mu_\beta(p)^{-1} \\ &= \mu_\alpha(p)\lambda_{\alpha\gamma}(p) \cdot \lambda_{\gamma\delta}(p) \cdot \lambda_{\delta\beta}(p)\mu_\beta(p)^{-1} \\ &= \mu_{\alpha\gamma}(p)\lambda_{\gamma\delta}(p)\mu_{\delta\beta}(p) \end{aligned}$$

where $\mu_{\alpha\gamma}(p) = \mu_\alpha(p)\lambda_{\alpha\gamma}(p)$ and $\mu_{\delta\beta}(p) = \lambda_{\delta\beta}(p)\mu_\beta(p)^{-1}$, and since this is just (B.23) it follows that the two coordinate bundles are equivalent. Consequently

(B.25) is a necessary and sufficient condition for the equivalence of the two coordinate bundles.

If a vector bundle λ over M is described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ and if $f \in \Gamma(M, \mathcal{C}(\lambda))$ then the mappings $f_\alpha = f_{U_\alpha}$ of the local form of the cross-section f, as defined by (B.5) for the open subsets U_α, describe that cross-section completely. It follows from (B.16) that these mappings satisfy

$$f_\alpha(p) = \lambda_{\alpha\beta}(p) f_\beta(p) \tag{B.27}$$

for any point $p \in U_\alpha \cap U_\beta$; conversely any collection of mappings $f_\alpha : U_\alpha \longrightarrow \mathbb{C}^r$ satisfying (B.27) describe a cross-section $f \in \Gamma(M, \mathcal{C}(\lambda))$. If vector bundles σ and λ are described by coordinate bundles $\{U_\alpha, \sigma_{\alpha\beta}\}$ and $\{U_\alpha, \lambda_{\alpha\beta}\}$ in terms of a the same covering $\{U_\alpha\}$ of M, the local form of a homomorphism $\phi : \sigma \longrightarrow \lambda$ between these two vector bundles for the subsets U_α consist of linear mappings $\phi_\alpha(p) = \phi_{U_\alpha}$ defined for points $p \in U_\alpha$; and as in (B.15) these linear mappings satisfy

$$\phi_\alpha(p) = \lambda_{\alpha\beta}(p)\phi_\beta(p)\sigma_{\beta\alpha}(p) \quad \text{for} \quad p \in U_\alpha \cap U_\beta. \tag{B.28}$$

Conversely any collection of linear mappings ϕ_α satisfying these conditions describes a vector bundle homomorphism $\phi : \sigma \longrightarrow \lambda$.

B.2 Basic Properties

Vector bundles of rank $r > 1$ are more complicated than line bundles in many ways, so it may be useful to discuss their basic properties in a bit more detail here. A cross-section $\phi = \{\phi_\alpha\}$ of a vector bundle λ described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ over a topological space M satisfies $\phi_\alpha = \lambda_{\alpha\beta}\, \phi_\beta$ over any intersection $U_\alpha \cap U_\beta$, as in (B.27); and that can be viewed as the special case of (B.28) in which $\sigma_{\alpha\beta}(p) = 1$ for all points $p \in U_\alpha \cap U_\beta$, so ϕ can be identified with a bundle homomorphism $\phi : 1 \longrightarrow \lambda$ from the trivial line bundle 1 to the bundle λ. Conversely any such homomorphism can be viewed as a cross-section of the bundle λ, so for instance there is the natural identification

$$\mathrm{Hom}_{\mathcal{O}}(1, \lambda) = \Gamma(M, \mathcal{O}(\lambda)) \tag{B.29}$$

and correspondingly for the other regularity classes of bundles. Similarly a collection of s cross-sections of the bundle λ can be viewed as a homomorphism $\phi : \mathrm{I}_s \longrightarrow \lambda$ from the trivial vector bundle of rank s to the bundle λ and conversely, so that there is the further natural identification

$$\mathrm{Hom}_{\mathcal{O}}(\mathrm{I}_s, \lambda) = \Gamma(M, \mathcal{O}(\lambda))^s. \tag{B.30}$$

More generally for any bundle homomorphism $\phi : \sigma \longrightarrow \lambda$ equation (B.28) can be rewritten

$$\phi_\alpha(p) = \lambda_{\alpha\beta}(p)\, \phi_\beta(p)\, \sigma_{\alpha\beta}(p)^{-1}; \tag{B.31}$$

and in view of (B.11) this can be interpreted as the condition that the mappings ϕ_α form a cross-section of the vector bundle $\lambda \otimes \sigma^*$ where σ^* is the dual vector bundle to σ; thus there is also the natural identification

$$\text{Hom}_{\mathcal{O}}(\sigma, \lambda) = \Gamma(M, \mathcal{O}(\lambda \otimes \sigma^*)), \tag{B.32}$$

and correspondingly for the other regularity classes. In particular if $\sigma = \lambda$ it follows from (B.31) and (B.12) that

$$\text{End}_{\mathcal{O}}(\lambda) = \text{Hom}_{\mathcal{O}}(\lambda, \lambda) = \Gamma(M, \mathcal{O}(\lambda \otimes \lambda^*)) = \Gamma(M, \mathcal{O}(\text{Ad}\lambda)). \tag{B.33}$$

Incidentally it follows from (B.31) that

$${}^t\phi_\alpha(p) = {}^t\sigma_{\alpha\beta}^{-1}\, {}^t\phi_\beta(p)\, {}^t\lambda_{\alpha\beta}(p) = \sigma^*_{\alpha\beta}\, {}^t\phi_\beta(p)\lambda^*_{\beta\alpha}(p), \tag{B.34}$$

hence that ${}^t\phi \in \text{Hom}(\lambda^*, \sigma^*)$; thus taking the transpose of the coordinate functions of a bundle homomorphism yields the natural isomorphism

$$\text{Hom}(\sigma, \lambda) \cong \text{Hom}(\lambda^*, \sigma^*), \tag{B.35}$$

so from (B.33) it follows that

$$\text{End}_{\mathcal{O}}(\lambda) = \text{Hom}_{\mathcal{O}}(\lambda, \lambda) \cong \text{Hom}_{\mathcal{O}}(\lambda^*, \lambda^*) = \text{End}_{\mathcal{O}}(\lambda^*). \tag{B.36}$$

If $\phi = \{\phi_\alpha(p)\} \in \text{End}(\lambda)$ is an endomorphism then

$$\det \phi_\alpha(p) = \det\Big(\lambda_{\alpha\beta}(p)\phi_\beta(p)\lambda_{\alpha\beta}(p)^{-1}\Big) = \det \phi_\beta(p)$$

in any intersection $U_\alpha \cap U_\beta$, so these local functions describe a global function on M that is called the *determinant* of the endomorphism ϕ and is denoted by $\det \phi$. Similarly the traces $\operatorname{tr} \phi_\alpha$ describe a global function on M that is called the *trace* of the endomorphism ϕ and is denoted by $\operatorname{tr} \phi$. If M is a compact complex manifold and λ is a holomorphic vector bundle over M it follows from the maximum modulus theorem that both $\det \phi$ and $\operatorname{tr} \phi$ are complex constants; and if $\det \phi \neq 0$ the endomorphism is an *automorphism* of the bundle λ, an isomorphism from the holomorphic vector bundle λ to itself.

A subset $\sigma \subset \lambda$ of a vector bundle λ of rank r over a topological space M is called a *subbundle* if it has the natural structure of a vector bundle over M under the restriction of the projection $\pi : \lambda \longrightarrow M$. Thus if the bundle λ is locally the product $\lambda|U = U \times \mathbb{C}^r$ then a subbundle $\sigma \subset \lambda$ is locally the product $\sigma|U = U \times \mathbb{C}^s$ for a subspace $\mathbb{C}^s \subset \mathbb{C}^r$; and after a suitable linear change of coordinates in the fibres, it can be assumed that the subspace $\mathbb{C}^s$ consists of the first s elements of the column vectors comprising the fibre $\mathbb{C}^r$. If the bundle λ is described by a coordinate bundle $\{U_\alpha, \lambda_{\alpha\beta}\}$ the coordinate transition functions $\lambda_{\alpha\beta}(p)$ must map the fibre $\mathbb{C}^s \subset \mathbb{C}^r$ to itself, and consequently

$$\lambda_{\alpha\beta}(p) = \begin{pmatrix} \sigma_{\alpha\beta}(p) & \sigma_{\alpha\beta}(p)x_{\alpha\beta}(p) \\ 0 & \tau_{\alpha\beta}(p) \end{pmatrix} \tag{B.37}$$

where $\sigma_{\alpha\beta}(p) \in \mathrm{Gl}(s, \mathbb{C})$ are the coordinate transition functions describing the subbundle $\sigma \subset \lambda$ and $\tau_{\alpha\beta}(p) \in \mathrm{Gl}(r-s, \mathbb{C})$ are the coordinate transition functions describing a vector bundle of rank $r-s$ over M that is called the *quotient bundle* and is denoted by $\tau = \lambda/\sigma$. The remaining entries of the matrix $\lambda_{\alpha\beta}(p)$ of course can be written $\sigma_{\alpha\beta}(p)x_{\alpha\beta}(p)$ for some $s \times (r-s)$ matrices $x_{\alpha\beta}(p)$, since the matrices $\sigma_{\alpha\beta}(p)$ are nonsingular. Conversely whenever the coordinate transition functions for a coordinate bundle λ can be put into the form (B.37) where $0 < \text{rank } \sigma_{\alpha\beta} = s < r$ then the subset of λ consisting of the first s elements of the column vectors comprising the fibre $\mathbb{C}^r$ form a subbundle $\sigma \subset \lambda$. Clearly $\operatorname{rank} \lambda = \operatorname{rank} \sigma + \operatorname{rank} \tau$ when $\sigma \subset \lambda$ and $\tau = \lambda/\sigma$.

A vector bundle λ is said to be *reducible* if it contains a nontrivial subbundle, and otherwise is said to be *irreducible*; thus λ is reducible precisely when its coordinate transition functions can be put into the form (B.37) nontrivially. On the other hand a vector bundle λ is said to be *decomposable* if it is a nontrivial direct sum $\lambda = \sigma \oplus \tau$ of two other vector bundles, and otherwise is said to be *indecomposable*; thus λ is decomposable precisely when its coordinate transition functions can be put into the form (B.37) in a nontrivial way and $x_{\alpha\beta} = 0$. For example the tensor product $\lambda^{\otimes 2}$ of a vector bundle with itself is the direct sum of the subbundle of symmetric tensors and the subbundle of skew-symmetric tensors, hence $\lambda^{\otimes 2}$ is decomposable. Reducibility and decomposability depend of course upon the regularity category being considered; for instance a reducible holomorphic vector bundle also is reducible when viewed as a $\mathcal{C}^\infty$ vector bundle, but the converse is not necessarily true since there may be a $\mathcal{C}^\infty$ equivalence of coordinate bundles exhibiting the reducibility of the vector bundle but not a holomorphic equivalence. Of course any decomposable bundle is reducible, or equivalently any irreducible bundle is indecomposable; but the converse is not always true.

A collection of vector bundles and bundle homomorphisms

$$0 \longrightarrow \sigma \xrightarrow{\phi} \lambda \xrightarrow{\psi} \tau \longrightarrow 0 \tag{B.38}$$

is called a *short exact sequence* of vector bundles if its restriction to the fibres over any point is a short exact sequence of vector spaces and linear mappings. For example if the coordinate bundle of λ has the form (B.37), so that σ is a subbundle and $\tau = \lambda/\sigma$ is the quotient bundle, there is a short exact sequence of vector bundles (B.38) in which ϕ and ψ are the bundle homomorphisms described by the local forms

$$\phi_\alpha(p) = \begin{pmatrix} \mathrm{I}_s \\ 0 \end{pmatrix}, \quad \psi_\alpha(p) = \begin{pmatrix} 0 & \mathrm{I}_t, \end{pmatrix} \tag{B.39}$$

where I_s is the $s \times s$ identity matrix, I_t is the $t \times t$ identity matrix, $r = \operatorname{rank} \lambda$, $s = \operatorname{rank} \sigma$, $t = \operatorname{rank} \tau$ and $r = s + t$. The homomorphism ϕ is the *inclusion mapping* of the subbundle $\sigma \subset \lambda$, and the homomorphism ψ is the *projection mapping* to the quotient bundle τ. On the other hand for any short exact sequence of vector bundles (B.38) the homomorphisms ϕ and ψ both must be of maximal rank at each point, so by passing to equivalent coordinate bundles they can be represented by coordinate functions of the form (B.39); and in that case the bundle λ is described by a coordinate bundle of the form (B.37), so σ is a subbundle of λ and $\tau = \lambda/\sigma$ is the quotient bundle τ. Thus the existence of a short exact sequence (B.38) is equivalent to the condition that σ is a subbundle of λ with quotient bundle $\tau = \lambda/\sigma$. Note that the dual of the exact sequence (B.38) is the exact sequence

$$0 \longrightarrow \tau^* \xrightarrow{{}^t\psi} \lambda^* \xrightarrow{{}^t\phi} \sigma^* \longrightarrow 0 \tag{B.40}$$

exhibiting τ^* as a subbundle of λ^* with $\sigma^* = \lambda^*/\tau^*$ as quotient bundle, as follows from the observation that

$$\lambda^*_{\alpha\beta} = \begin{pmatrix} \sigma^*_{\alpha\beta} & 0 \\ -\tau^*_{\alpha\beta}\,{}^t x_{\alpha\beta} & \tau^*_{\alpha\beta} \end{pmatrix}. \tag{B.41}$$

This is an alternate form of the description (B.37) of a subbundle and quotient bundle.

The short exact sequence of vector bundles (B.38) can be viewed not just as expressing the reducibility of the vector bundle λ but also as describing λ as an *extension* of the subbundle σ by the bundle τ, thus as a new vector bundle formed by combining the bundles σ and τ. Two extensions λ_1, λ_2 of the bundle σ by the bundle τ are called *equivalent* if there is a bundle homomorphism $\phi : \lambda_1 \longrightarrow \lambda_2$ such that

$$\begin{tikzcd} 0 \arrow[r] & \sigma \arrow[r] \arrow[d, "1"'] & \lambda_1 \arrow[r] \arrow[d, "\phi"'] & \tau \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & \sigma \arrow[r] & \lambda_2 \arrow[r] & \tau \arrow[r] & 0 \end{tikzcd} \tag{B.42}$$

is a commutative diagram of short exact sequences, where 1 denotes the identity homomorphism. It is easy to see from this diagram that ϕ is an isomorphism, and that equivalence in this sense is an equivalence relation in the usual sense. In particular when $\lambda = \sigma \oplus \tau$ the extension is said to be the *trivial extension* of vector bundles. The set of equivalence classes of extensions of a continuous vector bundle σ by a continuous vector bundle τ is denoted by $\mathrm{Ext}_{\mathcal{C}}(\sigma, \tau)$; correspondingly the sets of equivalence classes of extensions of $\mathcal{C}^\infty$, holomorphic, or flat vector bundes are denoted by $\mathrm{Ext}_{\mathcal{E}}(\sigma, \tau)$, $\mathrm{Ext}_{\mathcal{O}}(\sigma, \tau)$ or $\mathrm{Ext}_{\mathcal{F}}(\sigma, \tau)$. These sets have the natural structures of complex vector spaces arising from the following explicit descriptions.

Theorem B.1 *For any vector bundles σ and τ on a topological space M there is a canonical identification*

$$\mathrm{Ext}_{\mathcal{C}}(\sigma, \tau) = H^1(M, \mathcal{C}(\sigma \otimes \tau^*)).$$

If M is a $\mathcal{C}^\infty$ manifold and the bundles are $\mathcal{C}^\infty$ bundles there is in addition the canonical identification

$$\mathrm{Ext}_{\mathcal{E}}(\sigma, \tau) = H^1(M, \mathcal{E}(\sigma \otimes \tau^*));$$

if the manifold M and the bundles are holomorphic there is the further canonical identification
$$\operatorname{Ext}_{\mathcal{O}}(\sigma,\tau) = H^1(M, \mathcal{O}(\sigma\otimes\tau^*));$$
and if the bundles are flat there is the canonical identification
$$\operatorname{Ext}_{\mathcal{F}}(\sigma,\tau) = H^1(M, \mathcal{F}(\sigma\otimes\tau^*)).$$

Proof: The short exact sequence of vector bundles (B.38) expresses the condition that the vector bundle λ can be described by a coordinate bundle of the form (B.37), in which $\sigma_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are coordinate bundles describing the vector bundles σ and τ and the extension itself is described by the matrices $x_{\alpha\beta}$. As before let $r = \operatorname{rank}\lambda$, $s = \operatorname{rank}\sigma$, and $t = \operatorname{rank}\tau$, where $r = s+t$. In order that the matrices (B.37) satisfy the consistency conditions (B.17) to be a coordinate bundle the matrices $x_{\alpha\beta}$ must be such that $x_{\alpha\alpha} = 0$ and
$$\begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta}x_{\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix}\begin{pmatrix} \sigma_{\beta\gamma} & \sigma_{\beta\gamma}x_{\beta\gamma} \\ 0 & \tau_{\beta\gamma} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha\gamma} & \sigma_{\alpha\gamma}x_{\alpha\gamma} \\ 0 & \tau_{\alpha\gamma} \end{pmatrix}$$
in any intersection $U_\alpha \cap U_\beta \cap U_\gamma$, which is easily seen to be just the condition that $\sigma_{\alpha\beta}\sigma_{\beta\gamma}x_{\beta\gamma} + \sigma_{\alpha\beta}x_{\alpha\beta}\tau_{\beta\gamma} = \sigma_{\alpha\gamma}x_{\alpha\gamma}$, or alternatively that

(B.43)
$$x_{\alpha\gamma} = \sigma_{\gamma\beta}x_{\alpha\beta}\tau_{\beta\gamma} + x_{\beta\gamma};$$

and that is just the condition that the matrices $x_{\alpha\beta}$ describe a one-cocycle

(B.44)
$$x_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{C}(\sigma\otimes\tau^*))$$

of the covering $\mathfrak{U} = \{U_\alpha\}$ with coefficients in the sheaf of germs of continuous cross-sections of the vector bundle $\sigma\otimes\tau^*$, the condition for the vector bundle $\sigma\otimes\tau^*$ paralleling the corresponding condition for line bundles as in (1.45. The same considerations of course apply to extensions of more restrictive regularity classes of vector bundles; for instance extensions of a holomorphic vector bundle σ by a holomorphic vector bundle τ are holomorphic vector bundles λ described by cocycles in the group $Z^1(\mathfrak{U}, \mathcal{O}(\sigma\otimes\tau^*))$. The extensions λ_1, λ_2 described by two cocycles $x_{1\alpha\beta}, x_{2\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{C}(\sigma\otimes\tau^*))$ are equivalent if and only if there is a bundle homomorphism $\phi : \lambda_1 \longrightarrow \lambda_2$ leading to a commutative diagram of exact sequences of the form (B.42). The homomorphism ϕ can be described in a suitable refinement of the covering of M by coordinate functions ϕ_α, which must be of the form

(B.45)
$$\phi_\alpha = \begin{pmatrix} \mathrm{I}_s & f_\alpha \\ 0 & \mathrm{I}_t \end{pmatrix}$$

for some matrices f_α, since ϕ induces the identity mapping on the subbundle σ and the quotient bundle τ. The condition that these coordinate functions describe a bundle homomorphism $\phi : \lambda_1 \longrightarrow \lambda_2$ is (B.28), which is equivalent to
$$\begin{pmatrix} \mathrm{I}_s & f_\alpha \\ 0 & \mathrm{I}_t \end{pmatrix}\begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta}\, x_{1\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta}\, x_{2\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix}\begin{pmatrix} \mathrm{I}_s & f_\beta \\ 0 & \mathrm{I}_t \end{pmatrix}$$

in any intersection $U_\alpha \cap U_\beta$; and that amounts to the condition that the matrices $x_{\alpha\beta}$ satisfy $\sigma_{\alpha\beta}x_{1\alpha\beta} + f_\alpha\tau_{\alpha\beta} = \sigma_{\alpha\beta}f_\beta + \sigma_{\alpha\beta}x_{2\alpha\beta}$ or equivalently

$$x_{1\alpha\beta} - x_{2\alpha\beta} = f_\beta - \sigma_{\beta\alpha}f_\alpha\tau_{\alpha\beta}, \tag{B.46}$$

which is the condition that the cocycle $x_{2\alpha\beta} - x_{1\alpha\beta}$ is the coboundary of the cochain $f_\alpha \in C^0(\mathfrak{U}, \mathcal{C}(\sigma\otimes\tau^*))$, again the condition for the vector bundle $\sigma\otimes\tau^*$ paralleling the corresponding condition for line bundles as in (1.43). Thus the set of equivalence classes of extensions $\mathrm{Ext}(\sigma,\tau)$ is in one-to-one correspondence with the cohomology classes in $H^1(M, \mathcal{C}(\sigma\otimes\tau^*))$ represented by the cocycles $x_{\alpha\beta} \in Z^1(\mathfrak{U}, \mathcal{C}(\sigma\otimes\tau^*))$, and similary for extensions of the more restrictive regularity classes. That suffices to conclude the proof.

Corollary B.2 *For any vector bundles σ,τ on a topological manifold M*

$$\mathrm{Ext}_{\mathcal{C}}(\sigma,\tau) = 0;$$

and if the bundles and the manifold are $\mathcal{C}^\infty$ then

$$\mathrm{Ext}_{\mathcal{E}}(\sigma,\tau) = 0.$$

Proof: Since the sheaves $\mathcal{C}(\sigma\otimes\tau^*)$ and $\mathcal{E}(\sigma\otimes\tau^*)$ are fine sheaves

$$H^1(M, \mathcal{C}(\sigma\otimes\tau^*)) = H^1(M, \mathcal{E}(\sigma\otimes\tau^*)) = 0,$$

as in the discussion of the cohomology groups of fine sheaves on page 456 of Appendix C.2; the corollary is an immediate consequence of this observation and the preceding theorem.

For emphasis, and for convenience of reference, the preceding corollary can be restated equivalently as follows.

Corollary B.3 *Reducibility and decomposability are equivalent properties for continuous or $\mathcal{C}^\infty$ vector bundles.*

Proof: The preceding corollary shows that any reducible continuous or $\mathcal{C}^\infty$ vector bundle is decomposable, while as noted earlier the converse always holds; that suffices for the proof.

It should be noted particularly though that distinct extension classes in $\mathrm{Ext}_{\mathcal{O}}(\sigma,\tau)$ can lead to analytically equivalent vector bundles. The simplest instance of this, which arises sufficiently often to merit a separate statement for purposes of reference, is the following.

Lemma B.4 *Two nontrivial extension classes $x, y \in \mathrm{Ext}_{\mathcal{O}}(\sigma,\tau)$ describe analytically equivalent holomorphic vector bundles whenever $y = cx$ for some nonzero complex constant $c \in \mathbb{C}$.*

Proof: This is an immediate consequence of the identity

$$\begin{pmatrix} c\mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix}\begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta}\, x_{\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix}\begin{pmatrix} c^{-1}\mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta}\, c\, x_{\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix},$$

which suffices for a proof.

It is not the case that a reducible holomorphic or flat vector bundle is necessarily decomposable, as will become evident as the discussion continues. It is particularly useful to have available some simple tests to see whether a particular extension of holomorphic vector bundles is trivial or not. Note that if the short exact sequence (B.38) is the trivial extension, so that $\lambda = \sigma \oplus \tau$, there is also the bundle homomorphism $\theta : \tau \longrightarrow \lambda$ described by the coordinate functions

$$\theta_\alpha(p) = \begin{pmatrix} 0 \\ \mathrm{I}_t \end{pmatrix}; \tag{B.47}$$

and the composition $\psi\theta : \tau \longrightarrow \tau$ is the identity homomorphism. A short exact sequence (B.38) is said to *split* if there is a homomorphism $\theta : \tau \longrightarrow \lambda$ such that $\psi\theta = \mathrm{I}$ is the identity homomorphism; thus if (B.38) is a trivial extension then the short exact sequence splits.

Theorem B.5 *A short exact sequence of holomorphic vector bundles*

$$0 \longrightarrow \sigma \xrightarrow{\phi} \lambda \xrightarrow{\psi} \tau \longrightarrow 0 \tag{B.48}$$

splits if and only if $\lambda = \sigma \oplus \tau$.

Proof: It has been noted already that for the trivial extension $\lambda = \sigma \oplus \tau$ the short exact sequence (B.48) splits. Conversely suppose that the short exact sequence (B.48) splits, so that there is a bundle homomorphism $\theta : \tau \longrightarrow \lambda$ for which the composition $\psi\theta : \tau \longrightarrow \tau$ is the identity homomorphism, and let $r = \operatorname{rank}\lambda$, $s = \operatorname{rank}\sigma$, and $t = \operatorname{rank}\tau$ so that $r = s + t$. When the vector bundle λ is described by a coordinate bundle $\lambda_{\alpha\beta}$ of the form (B.37) the coordinate functions of the bundle homomorphisms ϕ and ψ have the form (B.39); and since $\psi\theta = \mathrm{I}_t$ the coordinate functions of the homomorphism θ must have the form

$$\theta_\alpha = \begin{pmatrix} \theta'_\alpha \\ \mathrm{I}_t \end{pmatrix}$$

where θ'_α is an $s \times t$ matrix. These matrices satisfy

$$\begin{pmatrix} \theta'_\alpha \\ \mathrm{I}_t \end{pmatrix}\tau_{\alpha\beta} = \begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta} x_{\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix}\begin{pmatrix} \theta'_\beta \\ \mathrm{I}_t \end{pmatrix},$$

so $\theta'_\alpha\tau_{\alpha\beta} = \sigma_{\alpha\beta}\theta'_\beta + \sigma_{\alpha\beta}x_{\alpha\beta}$ or equivalently

$$x_{\alpha\beta} = \sigma_{\beta\alpha}\theta'_\alpha\tau_{\alpha\beta} - \theta'_\beta. \tag{B.49}$$

That is just the condition that the cocycle $x_{\alpha\beta} \in Z^2(M, \mathcal{O}(\sigma \otimes \tau^*))$ is the coboundary of the cochain $\theta'_\alpha \in C^1(M, \mathcal{O}(\sigma \otimes \tau^*))$ as in (B.46), hence that this cocycle represents the trivial cohomology class in $H^2(M, \mathcal{O}(\sigma \otimes \tau^*))$; and by Theorem B.1 that is just the condition that the extension is trivial, hence that $\lambda = \sigma \oplus \tau$. That concludes the proof.

There is another sometimes useful interpretation of the cohomology class $x \in H^1(M, \mathcal{O}(\sigma \otimes \tau^*))$ describing an extension of holomorphic vector bundles. Associated to the short exact sequence of holomorphic vector bundles (B.38) describing this extension is the exact sequence

$$0 \longrightarrow \mathcal{O}(\sigma) \xrightarrow{\phi} \mathcal{O}(\lambda) \xrightarrow{\psi} \mathcal{O}(\tau) \longrightarrow 0 \tag{B.50}$$

of sheaves of germs of holomorphic cross-sections of these bundles, since over a sufficiently small coordinate neighborhood the bundle λ is the direct sum of the bundles σ and τ. The exact cohomology sequence associated to this exact sequence of sheaves includes the segment

(B.51)

$$0 \longrightarrow \Gamma(M, \mathcal{O}(\sigma)) \xrightarrow{\phi} \Gamma(M, \mathcal{O}(\lambda)) \xrightarrow{\psi} \Gamma(M, \mathcal{O}(\tau)) \xrightarrow{\delta} H^1(M, \mathcal{O}(\sigma)).$$

Theorem B.6 *If λ is a holomorphic vector bundle over a complex manifold M and λ is the extension of a vector bundle σ by a vector bundle τ described by a cohomology class $x \in \mathrm{Ext}(\sigma, \tau) = H^1(M, \mathcal{O}(\sigma \otimes \tau^*))$ then multiplication by this cohomology class x yields a homomorphism*

$$x : \Gamma(M, \mathcal{O}(\tau)) \longrightarrow H^1(M, \mathcal{O}(\sigma)) \tag{B.52}$$

that is precisely the coboundary mapping δ in the exact cohomology sequence (B.51)*; so if $K \subset \Gamma(M, \mathcal{O}(\tau))$ is the kernel of the homomorphism* (B.52) *then*

$$\gamma(\lambda) = \gamma(\sigma) + \dim K. \tag{B.53}$$

Proof: Suppose that the vector bundle λ is described by a coordinate bundle of the form (B.37) for a covering $\mathfrak{U}$ of the surface M. For any holomorphic cross-section $f_{2\alpha} \in \Gamma(M, \mathcal{O}(\tau))$ of the vector bundle τ it follows from the cocycle condition (B.43) that

$$x_{\alpha\gamma} f_{2\gamma} = \sigma_{\gamma\beta} x_{\alpha\beta} f_{2\beta} + x_{\beta\gamma} f_{2\gamma},$$

which is just the condition that the products $x_{\alpha\beta} f_{2\beta}$ describe a cocycle in $Z^1(\mathfrak{U}, \mathcal{O}(\sigma))$; thus multiplication by the matrices $x_{\alpha\beta}$ determines a homomorphism (B.52). A cross-section $f_{2\alpha} \in \Gamma(M, \mathcal{O}(\tau))$ is in the kernel K of this homomorphism if and only if the cocycle $x_{\alpha\beta} f_{2\beta}$ is a coboundary, so if and only if after a refinement of the covering if necessary there will be holomorphic functions $f_{1\alpha}$ in the open sets of the covering $\mathfrak{U}$ such that

$$\sigma_{\beta\alpha} f_{1\alpha} - f_{1\beta} = x_{\alpha\beta} f_{2\beta}$$

as in (B.46). This condition, together with the condition that the functions $f_{2\alpha}$ are a cross-section of the bundle τ, are easily seen to amount to the condition that

$$\begin{pmatrix} f_{1\alpha} \\ f_{2\alpha} \end{pmatrix} = \begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha\beta}x_{\alpha\beta} \\ 0 & \tau_{\alpha\beta} \end{pmatrix} \begin{pmatrix} f_{1\beta} \\ f_{2\beta} \end{pmatrix},$$

which in turn is just the condition that

$$f_\alpha = \begin{pmatrix} f_{1\alpha} \\ f_{2\alpha} \end{pmatrix} \in \Gamma(U_\alpha, \mathcal{O}(\lambda)).$$

Thus a cross-section $f_{2\alpha} \in \Gamma(M, \mathcal{O}(\tau))$ is in the kernel K if and only if it is the image of a cross-section $f_\alpha \in \Gamma(U_\alpha, \mathcal{O}(\lambda))$ under the inclusion mapping ϕ; that is precisely the condition satisfied by the coboundary mapping in the exact cohomology sequence (B.51), which identifies the homomorphism (B.52) with that coboundary mapping. The sequence (B.51) yields the exact sequence

$$0 \longrightarrow \Gamma(M, \mathcal{O}(\sigma)) \xrightarrow{\phi} \Gamma(M, \mathcal{O}(\lambda)) \xrightarrow{\psi} K \longrightarrow 0,$$

from which it follows immediately that $\gamma(\lambda) = \gamma(\sigma) + \dim K$. That suffices to conclude the proof.

For some purposes the particular extensions involved in building up a vector bundle from bundles of smaller ranks are not relevant; and for these purposes it is convenient to introduce another construction. For any complex manifold M let $V(M)$ be the free abelian group generated by all holomorphic vector bundles over M, and let $V_0(M) \subset V(M)$ be the subgroup generated by the expressions $\lambda - \sigma - \tau$ whenever λ, σ, τ are holomorphic vector bundles for which there is a short exact sequence

$$0 \longrightarrow \sigma \longrightarrow \lambda \longrightarrow \tau \longrightarrow 0. \tag{B.54}$$

The quotient group $V(M)/V_0(M) = K(M)$ is called the *Grothendieck group* of holomorphic vector bundles of the manifold M. For any exact sequence of holomorphic vector bundles (B.54) it is evident that $\det \lambda = (\det \sigma)(\det \tau)$; thus if the operation of taking the determinant line bundle of a holomorphic vector bundle is extended to a homomorphism $\det : V(M) \longrightarrow H^1(M, \mathcal{O}^*)$ by setting

$$\det(\lambda_1 + \cdots + \lambda_n) = (\det \lambda_1) \cdots (\det \lambda_n)$$

then this homomorphism is trivial on the subgroup $V_0(M) \subset V(M)$ and consequently induces a homomorphism

$$\det : K(M) \longrightarrow H^1(M, \mathcal{O}^*).$$

It is thus possible to define the determinant line bundle of an arbitrary element in the Grothendieck group $K(M)$. At least some other constructions for vector bundles also can be extended to the Grothendieck group; but the further discussion discussion of this topic will be deferred.

If $\phi : \sigma \longrightarrow \lambda$ is a homomorphism of vector bundles over a manifold M the *kernel* of ϕ is the union of the kernels of the linear mappings $\phi_p : \sigma_p \longrightarrow \lambda_p$ on the fibres of these bundles over all the points $p \in M$. The kernel of ϕ is a well defined subset of the vector bundle σ and is a linear subspace of each fibre of σ; but if the rank of the homomorphism ϕ is not constant the dimensions of these linear subspaces may vary with the point $p \in M$, so the kernel cannot be a subbundle of σ. The *image* of ϕ correspondingly is the subset $\phi(\sigma) \subset \lambda$, which is a well defined subset of the vector bundle λ and is a linear subspace of each fibre of λ; but again if the rank of the homomorphism ϕ is not constant then this subset too may not be a subbundle of λ.

Lemma B.7 *If $F : U \longrightarrow \mathbb{C}^{r\times s}$ is a continuous, $\mathcal{C}^\infty$, holomorphic, or locally constant mapping from an open neighborhood $U \subset \mathbb{C}^n$ of the origin in the space $\mathbb{C}^n$ to the space of $r \times s$ complex matrices, and if* $\operatorname{rank} F(z) = t$ *at all points $z \in U$, then in an open subneighborhood $V \subset U$ of the origin there are mappings $A : V \longrightarrow \mathrm{Gl}(r,\mathbb{C})$ and $B : V \longrightarrow \mathrm{Gl}(s,\mathbb{C})$ such that*

$$A(z)F(z)B(z) = \begin{pmatrix} \mathrm{I}_t & 0 \\ 0 & 0 \end{pmatrix}$$

at all points $z \in V$, where I_t is the $t \times t$ identity matrix; and these mappings have the same regularity properties as the mapping F.

Proof: Multiplying the matrix $F(z)$ on the right by a matrix $B(z)$ has the effect of replacing the columns of the matrix $F(z)$ by linear combinations of those columns with coefficients from the matrix B. By multiplying on the right by a nonsingular constant matrix it can be arranged that the first t columns of the matrix $F(z)$ are of rank t at the origin; and they remain of rank t at all points of a sufficiently small open neighborhood V of the origin. By then multiplying on the right by another nonsingular matrix, which has the effect of subtracting the appropriate linear combinations of the first t columns from the last $s - t$ columns, it can be arranged that the last $s - t$ columns of the matrix $F(z)$ vanish; the coefficients of these linear combinations are determined explicitly by Cramer's rule, so are continuous, $\mathcal{C}^\infty$, holomorphic or locally constant according to the regularity of the entries of the matrix $F(z)$. Multiplying the matrix $F(z)$ on the left by a matrix $A(z)$ has the corresponding effect on the rows of $F(z)$, so it can be arranged similarly that the last $r - t$ rows of the matrix $F(z)$ also vanish. The leading $t \times t$ block of the resulting matrix, consisting of the only nonzero terms in this matrix, then is of rank t throughout V; so by multiplying on the left or right by another nonsingular matrix that block can be reduced to the identity matrix of rank t as asserted, which suffices to conclude the proof.

Theorem B.8 *If $\phi : \sigma \longrightarrow \lambda$ is a homomorphism of constant rank t between two vector bundles over a manifold M there is a commutative diagram of short* *exact sequences of vector bundles and bundle homomorphisms of the form*

(B.55)
$$\begin{tikzcd}
0 \arrow[r] & \lambda_1 \arrow[r, "\rho_1"] & \lambda \arrow[r, "\rho_2"] & \lambda_2 \arrow[r] & 0 \\
0 & \sigma_1 \arrow[l] \arrow[u, "\theta"'] & \sigma \arrow[l, "\rho_1^*"'] \arrow[u, "\phi"'] & \sigma_2 \arrow[l, "\rho_2^*"'] & 0 \arrow[l]
\end{tikzcd}$$

in which $\operatorname{rank}\lambda_1 = \operatorname{rank}\sigma_1 = t$ *and the bundle homomorphism* θ *is an isomorphism. The bundles and bundle homomorphisms in this diagram are* $\mathcal{C}^\infty$*, holomorphic, or flat if the initial bundles have those regularity properties.*

Proof: If the bundles σ and λ are described by coordinate bundles $\{U_\alpha, \sigma_{\alpha\beta}\}$ and $\{U_\alpha, \lambda_{\alpha\beta}\}$ the local form of the homomorphism ϕ is described by the matrix functions $\phi_\alpha : U_\alpha \longrightarrow \mathbb{C}^{r\times s}$ such that $\phi_\alpha(p)\sigma_{\alpha\beta}(p) = \lambda_{\alpha\beta}(p)\phi_\beta(p)$ at all points $p \in U_\alpha \cap U_\beta$, as in (B.28), and the matrices ϕ_α are all of constant rank t. It follows from the preceding lemma that after passing to a refinement of the covering if necessary there are mappings $A_\alpha : U_\alpha \longrightarrow \mathrm{Gl}(r,\mathbb{C})$ and $B_\alpha : U_\alpha \longrightarrow \mathrm{Gl}(s,\mathbb{C})$ such that $A_\alpha(p)\phi_\alpha(p)B_\alpha(p) = \psi_\alpha(p)$ for all points $p \in U_\alpha$ where

$$\psi_\alpha(p) = \begin{pmatrix} \mathrm{I}_t & 0 \\ 0 & 0 \end{pmatrix},$$

in which I_t is the identity matrix of rank t; and the mappings A_α and B_α have the same regularity as the mapping ϕ_α and the bundles λ and σ. Then

$$A_\alpha(p)\phi_\alpha(p)B_\alpha(p)B_\alpha(p)^{-1}\sigma_{\alpha\beta}(p)B_\beta(p) = A_\alpha(p)\lambda_{\alpha\beta}(p)A_\beta(p)^{-1}A_\beta(p)\phi_\beta(p)B_\beta(p)$$

or equivalently

$$\psi_\alpha(p)\tilde{\sigma}_{\alpha\beta}(p) = \tilde{\lambda}_{\alpha\beta}(p)\psi_\beta(p)$$

for all points $p \in U_\alpha \cap U_\beta$, where $\tilde{\sigma}_{\alpha\beta}(p) = B_\alpha(p)^{-1}\sigma_{\alpha\beta}(p)B_\beta(p)$ and $\tilde{\lambda}_{\alpha\beta}(p) = A_\alpha(p)\lambda_{\alpha\beta}(p)A_\beta(p)^{-1}$; thus the vector bundles σ and λ can be described by the coordinate bundles $\tilde{\sigma}_{\alpha\beta}$ and $\tilde{\lambda}_{\alpha\beta}$, and the homomorphism ϕ by the coordinate functions ψ_α. When the coordinate bundles $\tilde{\sigma}_{\alpha\beta}$ and $\tilde{\lambda}_{\alpha\beta}$ are decomposed into matrix blocks corresponding to the decomposition of the matrices ψ_α then

$$\begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{11\alpha\beta} & \sigma_{12\alpha\beta} \\ \sigma_{21\alpha\beta} & \sigma_{22\alpha\beta} \end{pmatrix} = \begin{pmatrix} \lambda_{11\alpha\beta} & \lambda_{12\alpha\beta} \\ \lambda_{21\alpha\beta} & \lambda_{22\alpha\beta} \end{pmatrix}\begin{pmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix},$$

and consequently

(B.56)
$$\sigma_{11\alpha\beta} = \lambda_{11\alpha\beta} \quad \text{and} \quad \sigma_{12\alpha\beta} = \lambda_{21\alpha\beta}0,$$

so the coordinate transition functions for the two bundles have the form

$$\tilde{\sigma}_{\alpha\beta} = \begin{pmatrix} \sigma_{11\alpha\beta} & 0 \\ \sigma_{21\alpha\beta} & \sigma_{22\alpha\beta} \end{pmatrix}, \quad \tilde{\lambda}_{\alpha\beta} = \begin{pmatrix} \lambda_{11\alpha\beta} & \lambda_{12\alpha\beta} \\ 0 & \lambda_{22\alpha\beta} \end{pmatrix}.$$

Thus the vector bundles σ and λ are reducible, and there are short exact sequences

$$0 \longrightarrow \sigma_2 \longrightarrow \sigma \longrightarrow \sigma_1 \longrightarrow 0$$

$$0 \longrightarrow \lambda_1 \longrightarrow \lambda \longrightarrow \lambda_2 \longrightarrow 0$$

in which the vector bundles σ_i are described by the coordinate bundles $\sigma_{ii\alpha\beta}$ and the vector bundles λ_i are described by the coordinate bundles $\lambda_{ii\alpha\beta}$. Moreover the bundle homomorphism $\theta : \sigma_1 \longrightarrow \lambda_1$ defined by the identity mapping is an isomorphism and is just the homomorphism ϕ applied to the quotient bundle $\sigma_1 \subset \sigma$ with image contained in the subbundle $\lambda_1 \subset \lambda$; that yields the commutative diagram of the theorem, and suffices for the proof.

The mapping $\theta : \sigma_1 \longrightarrow \lambda_1$ in the preceding theorem is a homomorphism between two vector bundles of the same rank, and consequently its determinant is a well defined cross-section $\det\theta \in \Gamma(M, \mathcal{O}\big((\det\lambda_1)(\det\sigma_1)^{-1}\big)$; this cross-section is called the *determinant* of the initial bundle homomorphism $\phi : \sigma \longrightarrow \lambda$, and as such is denoted by $\det\phi$. It is worth pointing out explicitly that if $\operatorname{rank}\sigma = \operatorname{rank}\phi$ in the commutative diagram (B.55) then $\sigma_2 = 0$ and the second line reduces to the assertion that $\sigma_1 \cong \sigma$; correspondingly if $\operatorname{rank}\lambda = \operatorname{rank}\phi$ then $\lambda_2 = 0$ and the first line reduces to the assertion that $\lambda_1 \cong \lambda$. Of course if $\operatorname{rank}\sigma = \operatorname{rank}\lambda = \operatorname{rank}\phi$ the theorem is rather vacuous. The theorem is most useful in the following form.

Corollary B.9 *If $\phi : \sigma \longrightarrow \lambda$ is a homomorphism of constant rank t between vector bundles σ λ over a manifold M, where* rank $\sigma = s$ *and* rank $\lambda = r$, *there is an exact sequence of vector bundles*

$$0 \longrightarrow \sigma_2 \xrightarrow{\rho_2^*} \sigma \xrightarrow{\phi} \lambda \xrightarrow{\rho_2} \lambda_2 \longrightarrow 0 \tag{B.57}$$

over M, where rank $\sigma_2 = s - t$ *and* rank $\lambda_2 = r - t$. *The bundles and bundle homomorphisms are $\mathcal{C}^\infty$, holomorphic, or flat if the initial bundles and bundle homomorphisms have those regularity properties.*

Proof: This follows from the preceding theorem by a chase through the diagram (B.55). From the top short exact sequence it follows that rank $\lambda_2 =$ rank $\lambda -$ rank $\lambda_1 = r - t$, and from the bottom exact sequence it follows that rank $\sigma_2 =$ rank $\sigma -$ rank $\sigma_1 = s - t$. From these two exact sequences it also follows that ρ_2^* is injective and ρ_2 is surjective. From the commutativity of (B.55) it follows that $\phi \cdot \rho_2^* = \rho_1 \cdot \theta \cdot \rho_1^* \cdot \rho_2^* = 0$ since $\rho_1^* \cdot \rho_2^* = 0$. If $s \in \sigma$ and $\phi(s) = 0$ then from the commutativity of (B.55) again $0 = \phi(s) = \rho_1 \cdot \theta \cdot \rho_1^*(s)$; since θ and ρ_1 are injective necessarily $\rho_1^*(s) = 0$ and hence $s = \rho_2^*(s_2)$ for some $s_2 \in \sigma_2$, so (B.57) is exact at the bundle σ. From the commutativity of (B.55) yet again $\rho_2 \cdot \phi = \rho_2 \cdot \rho_1 \cdot \theta \cdot \rho_1^* = 0$ since $\rho_2 \cdot \rho_1 = 0$. Finally if $t \in \lambda$ and $\rho_2(t) = 0$ then $t = \rho_1(t_1)$ for some $t_1 \in \lambda_1$; since θ and ρ_1^* are surjective necessarily $t_1 = \theta \cdot \rho_1^*(s)$ for some $s \in \sigma$ and $t = \rho_1 \cdot \theta \cdot \rho_1^*(s) = \phi(s)$, so (B.57) is exact at the bundle λ, and that concludes the proof.

Corollary B.10 *The kernel and image of a continuous, $\mathcal{C}^\infty$, holomorphic or flat vector bundle homomorphism of constant rank are both subbundles of the same regularity class.*

Proof: In the exact sequence (B.57) of the preceding corollary the kernel of the homomorphism $\phi : \sigma \longrightarrow \lambda$ is the vector bundle σ_2, so the kernel of a bundle homomorphism of constant rank is a subbundle; and the image of ϕ is the kernel of ρ_2, so it is a subbundle by the first part of the proof of the present corollary, and that concludes the proof.

Appendix C

Sheaves

C.1 General Properties

Sheaves[1] were introduced into complex analysis in the early 1950's, in part to provide a tool for passing systematically from local to global results and in part to handle more readily some of the rather complicated semi-local properties of holomorphic functions of several variables. A *sheaf* of abelian groups over a topological space M is a topological space $\mathcal{S}$ with a mapping $\pi : \mathcal{S} \longrightarrow M$ such that: (i) π is a surjective local homeomorphism; (ii) for each point $p \in M$ the inverse image $\pi^{-1}(p) \subset \mathcal{S}$ has the structure of an abelian group; and (iii) the group operations are continuous in the topology of $\mathcal{S}$. To clarify condition (iii), the product $\mathcal{S} \times \mathcal{S}$ of a sheaf $\mathcal{S}$ with itself can be given the product topology, and the subset $\mathcal{S} \times_\pi \mathcal{S}$ consisting of those points (s_1, s_2) such that $\pi(s_1) = \pi(s_2)$ inherits a topology as a subset of $\mathcal{S} \times \mathcal{S}$; the mapping that takes a point $(s_1, s_2) \in \mathcal{S} \times_\pi \mathcal{S}$ to the point $s_1 - s_2 \in \mathcal{S}$ is a well defined mapping $\mathcal{S} \times_\pi \mathcal{S} \longrightarrow \mathcal{S}$ between two topological spaces, and (iii) is just the condition that this mapping is continuous. There are of course analogous definitions for sheaves of rings or of other algebraic structures. It is convenient to speak of a sheaf without specifying the algebraic structure when the particular structure is not relevant, and to speak of a sheaf of abelian groups or of another special algebraic structure when that structure is of particular significance. The space M is called the *base space* of the sheaf, the mapping π is called the *projection*, and the subset $\pi^{-1}(p) = \mathcal{S}_p$ is called the *stalk* over the point $p \in M$. The simplest example of a sheaf of groups is a *product sheaf* or *trivial sheaf* over M, the Cartesian product $\mathcal{S} = M \times G$ of the space M and a discrete group G with the product topology

[1]The role of sheaves in complex analysis is discussed in G-III and in the books by H. Grauert and R. Remmert, *Theorie der Steinsche Räumen*, (Springer, 1977), by H. Grauert and K. Fritzsche, *Einführung in die Funktionentheorie mehrerer Veränderlcher*, (Springer, 1974), by L. and B. Kaup, *Holomorphic Functions of Several Variables*, (deGruyter, 1983), by S. G. Krantz, *Function Theory of Several Complex Variables*, (Wadsworth & Brooks/Cole, 1992), and by R. Narasimhan, *Introduction to the Theory of Analytic Spaces*, (Springer, 1966), among other places. A definitive treatment of sheaves in general is in the book by R. Godement, *Topologie algèbrique et thèorie des faisceaux*, (Hermann, 1958).

and the projection mapping $\pi : M \times G \longrightarrow M$ to the first factor; the product sheaf with the group G usually is denoted just by G, and when G is the zero group it is called the *zero sheaf* and is denoted by 0. If $N \subset M$ the restriction $\mathcal{S}\big|N$ of a sheaf $\mathcal{S}$ of groups over M to the subset N is clearly a sheaf of groups over N; in particular the restriction of $\mathcal{S}$ to a point $p \in M$ is just the stalk $\mathcal{S}\big|p = \mathcal{S}_p$ of the sheaf $\mathcal{S}$ over p. A *section* of a sheaf $\mathcal{S}$ over a subset $U \subset M$ of its base M is a continuous mapping $s : U \longrightarrow \mathcal{S}$ such that the composition $\pi \circ s : U \longrightarrow U$ is the identity mapping; the set of all sections of $\mathcal{S}$ over U is denoted by $\Gamma(U, \mathcal{S})$. By condition (i) in the definition of a sheaf it follows that for any point $s \in \mathcal{S}$ there is an open neighborhood V of s in $\mathcal{S}$ such that the restriction $\pi|V$ of the projection to that set is a homeomorphism between $V \subset \mathcal{S}$ and an open subset $U \subset M$; the inverse of the restriction $\pi|V$ then is a section of the sheaf $\mathcal{S}$ over U, so there is a section of the sheaf $\mathcal{S}$ through any point $s \in \mathcal{S}$ and the images of sections over the open subsets of M form a basis for the topology of $\mathcal{S}$. Any two sections through a point $s \in \mathcal{S}$ coincide locally with the inverse of the projection mapping π, so any two sections of the sheaf $\mathcal{S}$ that agree at a point $p \in M$ necessarily agree in a full open neighborhood of p in M. By condition (iii) in the definition of a sheaf it follows that for any sections $s_1, s_2 \in \Gamma(U, \mathcal{S})$ the mapping that associates to a point $p \in U$ the difference $s_1(p) - s_2(p) \in \mathcal{S}_p$ also is a section; thus the set $\Gamma(U, \mathcal{S})$ of sections of $\mathcal{S}$ over any subset $U \subset M$ has the natural structure of an abelian group, and the corresponding result holds for sheaves of other algebraic structures.

A sheaf over a topological space M is described fully by the collection of its sections over the open subsets of M; indeed that is one of the standard ways in which to construct a sheaf. To make this more precise, a *presheaf* $\{\mathcal{S}_U, \rho_{V,U}\}$ of abelian groups over a topological space M is a collection (i) of abelian groups $\mathcal{S}_U$ indexed by the open subsets $U \subset M$, with $\mathcal{S}_\emptyset = 0$, and (ii) of group homomorphisms $\rho_{V,U} : \mathcal{S}_U \longrightarrow \mathcal{S}_V$ indexed by pairs $V \subset U$ of open subsets of M such that $\rho_{U,U}$ is the identity mapping and $\rho_{W,V}\,\rho_{V,U} = \rho_{W,U}$ whenever $W \subset V \subset U$. There are analogous definitions for presheaves of other algebraic structures; and as in the case of sheaves it is convenient to speak of a presheaf without specifying the algebraic structure when the particular structure is not relevant, and to speak of a presheaf of abelian groups or of another special algebraic structure when that structure is of particular significance. The set of sections $\mathcal{S}_U = \Gamma(U, \mathcal{S})$ of a sheaf $\mathcal{S}$ over the open subsets $U \subset M$ with the natural restrictions $\rho_{V,U}$ of sections over a set U to a subset $V \subset U$ clearly form a presheaf, which is called the *associated presheaf* of the sheaf. Conversely to any presheaf $\{\mathcal{S}_U, \rho_{V,U}\}$ over M there is an *associated sheaf* constructed as follows. For any point $p \in M$ let $\mathcal{U}_p$ be the collection of all open subsets $U \subset M$ that contain p and let $\mathcal{S}_p^*$ be the disjoint union of the sets $\mathcal{S}_U$ for all $U \in \mathcal{U}_p$. Two elements $s_U \in \mathcal{S}_U$ and $s_V \in \mathcal{S}_V$ are considered to be equivalent if there is a subset $W \subset M$ such that $p \in W \subset U \cap V$ and $\rho_{W,U}(s_U) = \rho_{W,V}(s_V)$; it is easy to see that this is an equivalence relation in the usual sense. The set $\mathcal{S}_p$ of equivalence classes of elements in $\mathcal{S}_p^*$ is a well defined group, known as the *direct limit* of the partially ordered collection of groups $\mathcal{S}_U$. Let $\mathcal{S} = \bigcup_{p \in M} \mathcal{S}_p$ be the union of these groups and $\pi : \mathcal{S} \longrightarrow M$ be the mapping for which $\pi(\mathcal{S}_p) = p$;

and introduce on $\mathcal{S}$ the topology for which the images in $\mathcal{S}$ of the elements $s_U \in \mathcal{S}_U$ for all the open subsets $U \subset M$ are a basis for the open subsets of $\mathcal{S}$. It is straightforward to verify that $\mathcal{S}$ is a sheaf over M with the projection π and with the same algebraic structure as that of the presheaf. It may be the case that the sheaf associated to a nontrivial presheaf is the zero sheaf, as for instance when all the homomorphisms $\rho_{V,U}$ are the zero mappings; so some conditions must be imposed on presheaves to ensure that they determine interesting sheaves. A presheaf $\{\mathcal{S}_U, \rho_{VU}\}$ over M is called a *complete presheaf* provided that whenever an open subset $U \subset M$ is covered by open subsets $U_\alpha \subset M$ (i) if $\rho_{U_\alpha,U}(s) = 0$ for an element $s \in \mathcal{S}_U$ and all subsets U_α then $s = 0$; and (ii) if $\rho_{U_\alpha\cap U_\beta,U_\alpha}(s_\alpha) = \rho_{U_\alpha\cap U_\beta,U_\beta}(s_\beta)$ for some elements $s_\alpha \in \mathcal{S}_{U_\alpha}$ and all pairs of subsets $U_\alpha,\ U_\beta$ then there is an element $s \in \mathcal{S}_U$ such that $s_\alpha = \rho_{U_\alpha,U}(s)$ for all U_α. It is evident that the presheaf of sections of a sheaf is a complete presheaf; and it is straightforward to verify that a complete presheaf can be identified naturally with the presheaf of sections of its associated sheaf.

As examples of particular interest here, the collection of rings $\mathcal{O}_U$ of holomorphic functions in the open subsets $U \subset \mathbb{C}^n$ clearly form a complete presheaf of rings over $\mathbb{C}^n$; the associated sheaf, denoted by $\mathcal{O}$, is called the *sheaf of germs of holomorphic functions* over $\mathbb{C}^n$, and there is the natural identification $\mathcal{O}_U \cong \Gamma(U, \mathcal{O})$ for any open subset $U \subset \mathbb{C}^n$. Similarly the collection of fields $\mathcal{M}_U$ of meromorphic functions is a complete presheaf of fields over $\mathbb{C}^n$; the associated sheaf, denoted by $\mathcal{M}$, is called the *sheaf of germs of meromorphic functions* over $\mathbb{C}^n$, and there is the natural identification $\mathcal{M}_U \cong \Gamma(U, \mathcal{M})$ for any open subset $U \subset \mathbb{C}^n$. The sheaf $\mathcal{C}$ of germs of continuous functions and the sheaf $\mathcal{E}$ of germs of $\mathcal{C}^\infty$ functions are sheaves of rings defined correspondingly, and the sheaves $\mathcal{E}^{(p,q)}$ of germs of $\mathcal{C}^\infty$ complex valued differential forms of type (p, q) are sheaves of abelian groups. All of these sheaves are defined purely locally, so can be considered as sheaves over arbitrary complex manifolds as well as over subsets of $\mathbb{C}^n$.

A *subsheaf* of a sheaf $\mathcal{S}$ of abelian groups over a topological space M is an open subset $\mathcal{R} \subset \mathcal{S}$ such that $\mathcal{R}_p = \mathcal{R} \cap \mathcal{S}_p$ is a subgroup of $\mathcal{S}_p$ for each point $p \in M$; a subsheaf of a sheaf of abelian groups over M clearly is itself a sheaf of abelian groups over M, and subsheaves of sheaves of other algebraic structures are defined correspondingly. If $\mathcal{R}$ is a subsheaf of a sheaf $\mathcal{S}$ of abelian groups over M the quotient groups $\mathcal{S}_p/\mathcal{R}_p$ are well defined for each point $p \in M$ and the union $\mathcal{T} = \bigcup_{p\in M} \mathcal{S}_p/\mathcal{R}_p$ with the natural quotient topology is another sheaf of abelian groups over M called the *quotient sheaf* and denoted by $\mathcal{S}/\mathcal{R}$. A *homomorphism* between two sheaves $\mathcal{R}$ and $\mathcal{S}$ of abelian groups over the same base space M is a continuous mapping $\phi : \mathcal{R} \longrightarrow \mathcal{S}$ that commutes with the projections of the two sheaves, so that $\phi(\mathcal{R}_p) \subset \mathcal{S}_p$ for each point $p \in M$, and that restricts to group homomorphisms $\phi|\mathcal{R}_p : \mathcal{R}_p \longrightarrow \mathcal{S}_p$ between the stalks of the two sheaves at all points $p \in M$. It is a straightforward matter to verify that a homomorphism between the two sheaves is always an open mapping. The *kernel* of a homomorphism $\phi : \mathcal{R} \longrightarrow \mathcal{S}$ between two sheaves of abelian groups is the union of the kernels of the homomorphisms $\phi : \mathcal{R}_p \longrightarrow \mathcal{S}_p$ between the stalks of the sheaves, and easily is seen to be a subsheaf of the sheaf $\mathcal{R}$. The *image* of the homomorphism ϕ is the union of the images of the homomorphisms $\phi : \mathcal{R}_p \longrightarrow \mathcal{S}_p$ between the stalks of the sheaves, and also easily is seen to be a subsheaf of the sheaf $\mathcal{S}$ since a sheaf homomorphism is an open mapping. An *isomorphism* between two sheaves of abelian groups is a homomorphism with an inverse that is also a homomorphism; a homomorphism $\phi : \mathcal{R} \longrightarrow \mathcal{S}$ is an isomorphism if and only if it is *injective*, has trivial kernel, and is *surjective*, has the full sheaf $\mathcal{S}$ as its image. The inclusion mapping $\iota : \mathcal{R} \longrightarrow \mathcal{S}$ of a subsheaf $\mathcal{R} \subset \mathcal{S}$ of abelian groups into the sheaf $\mathcal{S}$ is injective, and the natural homomorphism from the sheaf $\mathcal{S}$ to the quotient sheaf $\mathcal{S}/\mathcal{R}$ is surjective. A sequence

$$\xrightarrow{\phi_{n-2}} \mathcal{S}_{n-1} \xrightarrow{\phi_{n-1}} \mathcal{S}_n \xrightarrow{\phi_n} \mathcal{S}_{n+1} \xrightarrow{\phi_{n+1}}$$

of sheaves of abelian groups and homomorphisms is an *exact sequence* if for each n the image of ϕ_{n-1} is precisely the kernel of ϕ_n; a *short exact sequence* of sheaves of abelian groups is an exact sequence of the form

$$0 \longrightarrow \mathcal{R} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathcal{T} \longrightarrow 0 \tag{C.1}$$

in which 0 are zero sheaves. That (C.1) is an exact sequence means that ϕ is injective, that its image $\phi(\mathcal{R}) \subset \mathcal{S}$ is the kernel of the homomorphism ψ, and that ψ is surjective; or equivalently it just means that ϕ is an imbedding of $\mathcal{R}$ as a subsheaf of $\mathcal{S}$ and ψ identifies the quotient sheaf $\mathcal{S}/\mathcal{R}$ with the image sheaf $\mathcal{T}$. It is easy to see that if (C.1) is a short exact sequence of sheaves of abelian groups over a topological space M then the induced sequence of sections

$$0 \longrightarrow \Gamma(M, \mathcal{R}) \xrightarrow{\phi} \Gamma(M, \mathcal{S}) \xrightarrow{\psi} \Gamma(M, \mathcal{T}) \tag{C.2}$$

is exact; but the homomorphism ψ on sections is not necessarily surjective. For example if e is the mapping that sends the germ of a holomorphic function $f(z)$ to the germ of the nowhere vanishing homomorphic function $\exp 2\pi i\, f(z)$ there is the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0$$

over $\mathbb{C}^n$; for any open subset $M \subset \mathbb{C}^n$ the induced sequence of sections

$$0 \longrightarrow \Gamma(M, \mathbb{Z}) \longrightarrow \Gamma(M, \mathcal{O}) \xrightarrow{e} \Gamma(M, \mathcal{O}^*)$$

is exact, but the mapping e on sections is not necessarily surjective when M is not simply connected. A measure of the extent to which such a sequence of sections fails to be exact is provided by the cohomology theory of sheaves.

C.2 Sheaf Cohomology

Although there are more general approaches to the cohomology theory of sheaves, for present purposes it is most convenient to consider skew-symmetric Čech cohomology. For any covering $\mathfrak{U}$ of a topological space M by open subsets $U_\alpha \subset M$ let $\mathfrak{U}^p$ be the collection of all ordered $(p+1)$-tuples $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p})$ of sets of $\mathfrak{U}$ with nonempty intersection $|\sigma| = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \neq \emptyset$. A (skew-symmetric) *p-cochain* s of the covering $\mathfrak{U}$ with coefficients in a sheaf of abelian groups $\mathcal{S}$ over M is a mapping that associates to each $(p+1)$-tuple $\sigma = (U_{\alpha_0}, U_{\alpha_1}, \ldots, U_{\alpha_p}) \in \mathfrak{U}^p$ a section

$$s_{\alpha_0,\alpha_1\ldots,\alpha_p} \in \Gamma(|\sigma|, \mathcal{S}) \tag{C.3}$$

over the intersection $|\sigma|$ such that for any permutation $\pi \in \mathfrak{S}_{p+1}$ of the indices $(0, 1, \ldots, p)$

$$s_{\alpha_{\pi 0},\alpha_{\pi 1},\ldots,\alpha_{\pi p}} = (\operatorname{sgn} \pi) \cdot s_{\alpha_0,\alpha_1,\ldots,\alpha_p} \tag{C.4}$$

where $\operatorname{sgn}\pi$ is the sign of the permutation π. For example a 0-cochain s associates to each set U_{α_0} a section $s_{\alpha_0} \in \Gamma(U_{\alpha_0}, \mathcal{S})$; and a 1-cochain s associates to each ordered pair of sets $(U_{\alpha_0}, U_{\alpha_1})$ with a nonempty intersection $U_{\alpha_0} \cap U_{\alpha_1} \neq \emptyset$ a section $s_{\alpha_0,\alpha_1} \in \Gamma(U_{\alpha_0} \cap U_{\alpha_1}, \mathcal{S})$ such that $s_{\alpha_1,\alpha_0} = -s_{\alpha_0,\alpha_1}$, so that in particular $s_{\alpha_0,\alpha_0} = 0$. The set of all p-cochains is denoted by $C^p(\mathfrak{U}, \mathcal{S})$ and clearly is an abelian group under addition. The *coboundary homomorphism* δ is the group homomorphism

$$\delta : C^p(\mathfrak{U}, \mathcal{S}) \longrightarrow C^{p+1}(\mathfrak{U}, \mathcal{S})$$

for any $p \geq 0$ taking a cochain $s \in C^p(\mathfrak{U}, \mathcal{S})$ to the cochain $\delta s \in C^{p+1}(\mathfrak{U}, \mathcal{S})$ that associates to each $(p+2)$-tuple $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_{p+1}}) \in \mathfrak{U}^{p+1}$ the section

$$(\delta s)_{\alpha_0,\ldots,\alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \rho_{|\sigma|}(s_{\alpha_0,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_{p+1}}) \tag{C.5}$$

where $\rho_{|\sigma|}(s)$ is the restriction of the section s to the intersection $|\sigma|$. A straightforward calculation[2] shows that the coboundary of a skew-symmetric cochain satisfies the skew-symmetry condition (C.4) and that $\delta\,\delta = 0$. The kernel of the coboundary homomorphism is called the subgroup of *p-cocycles* and is denoted by $Z^p(\mathfrak{U}, \mathcal{S}) \subset C^p(\mathfrak{U}, \mathcal{S})$; the image $\delta C^{p-1}(\mathfrak{U}, \mathcal{S}) \subset C^p(\mathfrak{U}, \mathcal{S})$ is called the subgroup of *p-coboundaries*. Since $\delta\,\delta = 0$ it follows that $\delta C^{p-1}(\mathfrak{U}, \mathcal{S}) \subset Z^p(\mathfrak{U}, \mathcal{S})$ for $p > 0$; the group

$$H^p(\mathfrak{U}, \mathcal{S}) = \begin{cases} \dfrac{Z^p(\mathfrak{U}, \mathcal{S})}{\delta C^{p-1}(\mathfrak{U}, \mathcal{S})} & \text{for} \quad p > 0, \\[2ex] Z^0(\mathfrak{U}.\mathcal{S}) & \text{for} \quad p = 0 \end{cases} \tag{C.6}$$

[2]For details see G-IIIE.

is called the p-th (skew-symmetric) *Cech cohomology group* of the covering $\mathfrak{U}$ with coefficients in the sheaf $\mathcal{S}$. For example if $s \in C^0(\mathfrak{U}, \mathcal{S})$ then

$$(\delta s)_{\alpha_0,\alpha_1}(a) = s_{\alpha_1}(a) - s_{\alpha_0}(a) \quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1}, \tag{C.7}$$

and clearly $\delta s_{\alpha_1,\alpha_0} = -\delta s_{\alpha_0,\alpha_1}$. The cochain s is a cocycle if and only if

$$s_{\alpha_0}(a) = s_{\alpha_1}(a) \quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1} \tag{C.8}$$

so that the local sections s_{α_0} are the restrictions to the various sets U_α of a section $s \in \Gamma(M, \mathcal{S})$ over all of M; thus there is the natural identification

$$H^0(\mathfrak{U}, \mathcal{S}) = \Gamma(M, \mathcal{S}). \tag{C.9}$$

If $s \in C^1(\mathfrak{U}, \mathcal{S})$ then

$$\begin{aligned}(\delta s)_{\alpha_0,\alpha_1,\alpha_2}(a) &= s_{\alpha_1,\alpha_2}(a) - s_{\alpha_0,\alpha_2}(a) + s_{\alpha_0,\alpha_1}(a) \\ &\quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2},\end{aligned} \tag{C.10}$$

which is easily seen to satisfy the skew-symmetry condition (C.4); the cochain is a cocycle $s \in Z^1(\mathfrak{U}, \mathcal{S})$ if and only if

$$s_{\alpha_0,\alpha_1}(a) + s_{\alpha_1,\alpha_2}(a) + s_{\alpha_2,\alpha_0}(a) = 0 \quad \text{for} \quad a \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}. \tag{C.11}$$

A special case of particular interest in this book is that of the sheaf $\mathcal{S} = \mathcal{O}^*$ of germs of nowhere vanishing holomorphic functions on a complex manifold M, a sheaf of multiplicative abelian groups. A cocycle $s \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ is a collection of nowhere vanishing holomorphic functions $s_{\alpha_0,\alpha_1}(z)$ in the intersections $U_{\alpha_0} \cap U_{\alpha_1}$ such that

$$\begin{aligned} s_{\alpha_0,\alpha_0}(z) &= 1 && \text{for} \quad z \in U_{\alpha_0} \\ s_{\alpha_0,\alpha_1}(z) &= s_{\alpha_1,\alpha_0}(z)^{-1} && \text{for} \quad z \in U_{\alpha_0} \cap U_{\alpha_1} \\ s_{\alpha_0,\alpha_1}(z) s_{\alpha_1,\alpha_2}(z) s_{\alpha_2\alpha_0}(z) &= 1 && \text{for} \quad z \in U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}; \end{aligned} \tag{C.12}$$

thus if $s \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ then the cross-sections s_{α_0,α_1} satisfy (B.17) so that $\{U_\alpha, s_{\alpha_0\alpha_1}\}$ is a coordinate line bundle describing a holomorphic line bundle over M. The cocycle $s \in Z^1(\mathfrak{U}, \mathcal{O}^*)$ is a coboundary if and only if

$$s_{\alpha_0,\alpha_1}(z) = t_{\alpha_1}(z) t_{\alpha_0}(z)^{-1} \;\; \text{for} \;\; z \in U_{\alpha_0} \cap U_{\alpha_1} \tag{C.13}$$

where $t_{\alpha_0}(z)$ are nowhere vanishing holomorphic functions in the open subsets U_α, hence if and only if the cross-sections s_{α_0,α_1} satisfy (B.24) so that the coordinate line bundle $\{U_\alpha, s_{\alpha_0\alpha_1}\}$ describes the trivial holomorphic line bundle over M. Thus the cohomology group $H^1(\mathfrak{U}, \mathcal{O}^*)$ can be identified with the set of those holomorphic line bundles over M that can be described by coordinate line bundles in terms of the covering $\mathfrak{U}$.

There remains the question of the relations between the cohomology groups of different coverings of the space M. A covering $\mathfrak{V}$ of M is called a *refinement* of the covering $\mathfrak{U}$ if there is a mapping $\mu : \mathfrak{V} \longrightarrow \mathfrak{U}$ which associates to each set $V_\alpha \in \mathfrak{V}$ a set $\mu(V_\alpha) = U_{\mu(\alpha)} \in \mathfrak{U}$ such that $V_\alpha \subset U_{\mu(\alpha)}$. The mapping μ, called a *refining mapping*, induces group homomorphisms $\mu : C^p(\mathfrak{U}, \mathcal{S}) \longrightarrow C^p(\mathfrak{V}, \mathcal{S})$ for any sheaf $\mathcal{S}$ of abelian groups over M; for a cochain $s \in C^p(\mathfrak{U}, \mathcal{S})$ the image $\mu s \in C^p(\mathfrak{V}, \mathcal{S})$ is the cochain that associates to each $\sigma = (V_{\alpha_0}, \ldots, V_{\alpha_p}) \in \mathfrak{V}^p$ the section

$$(\mu s)_{\alpha_0,\ldots,\alpha_p} = \rho_{|\sigma|}(s_{\mu(\alpha_0),\ldots,\mu(\alpha_p)}) \tag{C.14}$$

where $\rho_{|\sigma|}(s)$ is the restriction of the section s to the intersection $|\sigma|$. This homomorphism clearly commutes with the coboundary homomorphism δ and consequently induces group homomorphisms

$$\mu^* : H^p(\mathfrak{U}, \mathcal{S}) \longrightarrow H^p(\mathfrak{V}, \mathcal{S}). \tag{C.15}$$

Of course if $\mathfrak{V}$ is a refinement of the covering $\mathfrak{U}$ there may be a number of different refining mappings; but a straightforward calculation[3] shows that the induced homomorphisms μ^* of the cohomology groups are independent of the choice of a refining mapping. In the disjoint union of the cohomology groups $H^p(\mathfrak{U}, \mathcal{S})$ for all coverings $\mathfrak{U}$ two cohomology classes $s \in H^p(\mathfrak{U}, \mathcal{S})$ and $t \in H^p(\mathfrak{V}, \mathcal{S})$ are considered to be equivalent if there is a common refinement $\mathfrak{W}$ of the coverings $\mathfrak{U}$ and $\mathfrak{V}$, with refining mappings $\mu_{\mathfrak{U}} : \mathfrak{W} \longrightarrow \mathfrak{U}$ and $\mu_{\mathfrak{V}} : \mathfrak{W} \longrightarrow \mathfrak{V}$, such that $\mu_{\mathfrak{U}}^*(s) = \mu_{\mathfrak{V}}^*(t)$; this easily is seen to be an equivalence relation in the usual sense. The set of equivalence classes is a well defined abelian group, the direct limit of the directed set of groups indexed by coverings $\mathfrak{U}$ of M, called the (skew-symmetric) *Čech cohomology group* of the space M with coefficients in the sheaf $\mathcal{S}$ and denoted by $H^p(M, \mathcal{S})$. For any covering $\mathfrak{U}$ there is then the natural homomorphism

$$\iota_{\mathfrak{U}}^* : H^p(\mathfrak{U}, \mathcal{S}) \longrightarrow H^p(M, \mathcal{S}) \tag{C.16}$$

that takes a cohomology class in $H^p(\mathfrak{U}, \mathcal{S})$ to its equivalence class in $H^p(M, \mathcal{S})$; and for any refining mapping $\mu : \mathfrak{V} \longrightarrow \mathfrak{U}$ these homomorphisms commute in the sense that $\iota_{\mathfrak{U}}^* = \iota_{\mathfrak{V}}^* \circ \mu^*$. For example, since $H^0(\mathfrak{U}, \mathcal{S}) = \Gamma(M, \mathcal{S})$ for any covering $\mathfrak{U}$ it follows that

$$H^0(M, \mathcal{S}) = \Gamma(M, \mathcal{S}), \tag{C.17}$$

so $H^0(\mathfrak{U}, \mathcal{S}) \cong H^0(M, \mathcal{S})$ for any covering $\mathfrak{U}$ of M. The cohomology groups $H^p(\mathfrak{U}, \mathcal{S})$ and $H^p(M, \mathcal{S})$ generally are not isomorphic for $p > 0$, although they are for some special covers of suitably regular topological spaces.

To any short exact sequence (C.1) of sheaves of abelian groups over a topological space M there corresponds the exact sequence of sections (C.2). Since

[3]See Theorem G-IIIE8

the cochain groups $C^p(\mathfrak{U}, \mathcal{S})$ are just the direct sums of groups of sections over various subsets of M there are corresponding exact sequences

$$0 \longrightarrow C^p(\mathfrak{U}, \mathcal{R}) \xrightarrow{\phi} C^p(\mathfrak{U}, \mathcal{S}) \xrightarrow{\psi} C^p(\mathfrak{U}, \mathcal{T}) \tag{C.18}$$

of cochain groups. The homomorphisms ϕ and ψ commute with the coboundary operators and consequently induce homomorphisms

$$\phi^* : H^p(\mathfrak{U}, \mathcal{R}) \longrightarrow H^p(\mathfrak{U}, \mathcal{S}),$$
$$\psi^* : H^p(\mathfrak{U}, \mathcal{S}) \longrightarrow H^p(\mathfrak{U}, \mathcal{T}).$$

If M is a paracompact Hausdorff space, a Hausdorff space such that every open covering has a locally finite refinement, these homomorphisms can be combined and lead to the *exact cohomology sequence*[4]

$$\begin{aligned}
&0 \longrightarrow \Gamma(M, \mathcal{R}) \xrightarrow{\phi^*} \Gamma(M, \mathcal{S}) \xrightarrow{\psi^*} \Gamma(M, \mathcal{T}) \xrightarrow{\delta^*} \\
&\xrightarrow{\delta^*} H^1(M, \mathcal{R}) \xrightarrow{\phi^*} H^1(M, \mathcal{S}) \xrightarrow{\psi^*} H^1(M, \mathcal{T}) \xrightarrow{\delta^*} \cdots \\
&\qquad\qquad \cdots \\
&\cdots \xrightarrow{\delta^*} H^p(M, \mathcal{R}) \xrightarrow{\phi^*} H^p(M, \mathcal{S}) \xrightarrow{\psi^*} H^p(M, \mathcal{T}) \xrightarrow{\delta^*} \\
&\xrightarrow{\delta^*} H^{p+1}(M, \mathcal{R}) \xrightarrow{\phi^*} H^{p+1}(M, \mathcal{S}) \xrightarrow{\psi^*} H^{p+1}(M, \mathcal{T}) \xrightarrow{\delta^*} \cdots \\
&\qquad\qquad \cdots
\end{aligned} \tag{C.19}$$

for suitable connecting homomorphisms δ^*. To define these connecting homomorphisms and demonstrate the exactness of the sequence (C.19) extend the exact sequences (C.18) to the short exact sequences

$$0 \longrightarrow C^p(\mathfrak{U}, \mathcal{R}) \xrightarrow{\phi} C^p(\mathfrak{U}, \mathcal{S}) \xrightarrow{\psi} \overline{C}^p(\mathfrak{U}, \mathcal{T}) \longrightarrow 0 \tag{C.20}$$

in which $\overline{C}^p(\mathfrak{U}, \mathcal{T}) \subset C^p(\mathfrak{U}, \mathcal{T})$ is the image of the homomorphism ψ. These short exact sequences are mapped to one another by the coboundary homomorphism δ, leading to the following commutative diagram of abelian groups and

[4] See Theorem G-IIID2.

homomorphisms.

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{p-1}(\mathfrak{U},\mathcal{R}) & \xrightarrow{\phi} & C^{p-1}(\mathfrak{U},\mathcal{S}) & \xrightarrow{\psi} & \overline{C}^{p-1}(\mathfrak{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta_{p-1}\downarrow & & \delta_{p-1}\downarrow & & \delta_{p-1}\downarrow & & \\
0 & \longrightarrow & C^{p}(\mathfrak{U},\mathcal{R}) & \xrightarrow{\phi} & C^{p}(\mathfrak{U},\mathcal{S}) & \xrightarrow{\psi} & \overline{C}^{p}(\mathfrak{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta_{p}\downarrow & & \delta_{p}\downarrow & & \delta_{p}\downarrow & & \\
0 & \longrightarrow & C^{p+1}(\mathfrak{U},\mathcal{R}) & \xrightarrow{\phi} & C^{p+1}(\mathfrak{U},\mathcal{S}) & \xrightarrow{\psi} & \overline{C}^{p+1}(\mathfrak{U},\mathcal{T}) & \longrightarrow & 0 \\
 & & \delta_{p+1}\downarrow & & \delta_{p+1}\downarrow & & \delta_{p+1}\downarrow & &
\end{array}
$$

Each row is exact by (C.20) while the sheaf cohomology groups measure the inexactness of the columns, in the sense that $H^p(\mathfrak{U},\mathcal{R}) = \ker\delta_p/\operatorname{im}\delta_{p-1}$ and similarly for the cohomology groups $H^p(\mathfrak{U},\mathcal{S})$ and $\overline{H}^p(\mathfrak{U},\mathcal{T})$, where the latter are defined in terms of the cochain groups $\overline{C}^p(\mathfrak{U},\mathcal{T})$. A simple diagram chase shows that under the induced homomorphisms on the cohomology groups the sequences

$$H^p(\mathfrak{U},\mathcal{R}) \xrightarrow{\phi^*} H^p(\mathfrak{U},\mathcal{S}) \xrightarrow{\psi^*} \overline{H}^p(\mathfrak{U},\mathcal{T})$$

are exact sequences. For any $t \in \overline{C}^p(\mathfrak{U},\mathcal{T})$ for which $\delta t = 0$ select an element $s \in C^p(\mathfrak{U},\mathcal{S})$ for which $t = \psi(s)$. Since $\psi(\delta s) = \delta\psi(s) = \delta t = 0$ it follows from the exactness of the next row that there is an element $r \in C^{p+1}(\mathfrak{U},\mathcal{R})$ such that $\phi(r) = \delta s$; and $\phi(\delta r) = \delta\phi(r) = \delta\,\delta s = 0$, so since ϕ is an inclusion it follows that $\delta r = 0$. The homomorphism

$$\delta^* : \overline{Z}^p(\mathfrak{U},\mathcal{T}) \longrightarrow Z^{p+1}(\mathfrak{U},\mathcal{R})$$

is defined by $\delta^*(t) = r$. Further diagram chases show first that the cohomology class of r is independent of the choice of s, next that cohomologous elements t lead to cohomologous elements $\delta^* t$, and finally that there results a long exact cohomology sequence of the form

$$\cdots \longrightarrow H^p(\mathfrak{U},\mathcal{R}) \xrightarrow{\phi^*} H^p(\mathfrak{U},\mathcal{S}) \xrightarrow{\psi^*} \overline{H}^p(\mathfrak{U},\mathcal{T}) \xrightarrow{\delta^*} H^{p+1}(\mathfrak{U},\mathcal{S}) \longrightarrow \cdots .$$

There is a corresponding exact cohomology sequence for any refinement $\mathfrak{V}$ of the covering $\mathfrak{U}$, and it is easy to see that the homomorphisms induced by the refining mapping commute with the homomorphisms in these exact sequences; it follows readily from this that there results the exact sequence

$$\cdots H^p(M,\mathcal{R}) \xrightarrow{\phi^*} H^p(M,\mathcal{S}) \xrightarrow{\psi^*} \overline{H}^p(M,\mathcal{T}) \xrightarrow{\delta^*} H^{p+1}(M,\mathcal{S}) \cdots .$$

Finally it is a straightforward matter to show that if M is a paracompact Hausdorff space then $\overline{H}^p(M,\mathcal{T}) \cong H^p(M,\mathcal{T})$, since for any locally finite covering $\mathfrak{U}$

it is possible to choose a locally finite refinement $\mathfrak{V}$ in which the sets V_α are sufficiently small that sections of the sheaf $\mathcal{T}$ over intersections of these sets are the images of sections of the sheaf $\mathcal{S}$. That demonstrates the exactness of the sequence (C.19).

Various auxiliary sheaves often are used to calculate cohomology groups explicitly. A sheaf $\mathcal{S}$ of abelian groups over a topological space M is a *fine sheaf* if for any locally finite open covering $\mathfrak{U} = \{U_\alpha\}$ of M there are sheaf homomorphisms $\epsilon_\alpha : \mathcal{S} \longrightarrow \mathcal{S}$ such that (i) $\epsilon_\alpha(s) = 0$ if $s \in \mathcal{S}_a$ for a point $a \in M \sim U_\alpha$, and (ii) $\sum_\alpha \epsilon_\alpha(s) = s$ for any $s \in \mathcal{S}$; the latter sum is finite since the covering $\mathfrak{U}$ is locally finite so by (i) only finitely many entries in the sum are nonzero. The collection of homomorphisms $\{\epsilon_\alpha\}$ is called a *partition of unity* for the sheaf $\mathcal{S}$ subordinate to the covering $\mathfrak{U}$. For example the sheaves $\mathcal{C}(\lambda)$ and $\mathcal{E}(\lambda)$ of continuous or $\mathcal{C}^\infty$ cross-sections of a holomorphic line bundle λ over a Riemann surface M are fine sheaves since it is a standard result of analysis that for any locally finite open covering $\mathfrak{U} = \{U_\alpha\}$ of M there are $\mathcal{C}^\infty$ real-valued functions ϵ_α on M such that $\epsilon_\alpha(a) \ge 0$ at each point $a \in M$, that the support of the function ϵ_α is contained in U_α, and that $\sum_\alpha \epsilon_\alpha(a) = 1$ at each point $a \in M$; and multiplication of the sheaves $\mathcal{C}(\lambda)$ or $\mathcal{E}(\lambda)$ by such functions ϵ_α is a partition of unity for these sheaves. The basic property of fine sheaves is that $H^p(M, \mathcal{S}) = 0$ for all $p > 0$ if $\mathcal{S}$ is a fine sheaf over a paracompact Hausdorff space. To see this it suffices to show that $H^p(\mathfrak{U}, \mathcal{S}) = 0$ for a locally finite covering $\mathfrak{U}$ of M. The first step in doing so is to demonstrate that if $s \in Z^p(\mathfrak{U}, \mathcal{S})$ is a cocycle for $p > 0$ and if $s(a) = 0$ whenever $a \in M \sim U_\beta$ for some set U_β of the covering $\mathfrak{U}$ then the cocycle s is cohomologous to zero. Indeed for such a cocycle s consider the $(p-1)$-cochain $s^\beta \in C^{p-1}(\mathfrak{U}, \mathcal{S})$ that associates to a p-tuple $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_{p-1}}) \in \mathfrak{U}^{p-1}$ the cross-section over $|\sigma|$ defined by

$$s^\beta_{\alpha_0,\ldots,\alpha_{p-1}}(a) = \begin{cases} s_{\beta,\alpha_0,\ldots,\alpha_{p-1}}(a) & \text{if } a \in |\sigma| \cap U_\beta, \\ 0 & \text{if } a \in |\sigma| \sim |\sigma| \cap U_\beta, \end{cases}$$

noting that this is a well defined cross-section since the cocycle s vanishes outside U_β. Since s is a cocycle it follows that for any $(p+2)$-tuple $\tau = (U_\beta, U_{\alpha_0}, \ldots, U_{\alpha_p}) \in \mathfrak{U}^{p+1}$ and any point $a \in |\tau|$

$$\begin{aligned} 0 &= (\delta\, s)_{\beta,\alpha_0,\ldots,\alpha_p}(a) \\ &= s_{\alpha_0,\ldots,\alpha_p}(a) - \sum_{j=0}^{p} (-1)^j s_{\beta,\alpha_0,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots\alpha_p}(a) \\ &= s_{\alpha_0,\ldots,\alpha_p}(a) - \sum_{j=0}^{p} (-1)^j s^\beta_{\alpha_0,\ldots,\alpha_{j-1},\alpha_{j+1},\ldots\alpha_p}(a); \end{aligned}$$

this identity holds trivially if $\alpha \notin U_\beta$, since the cocycle s vanishes outside U_β, so it actually holds for all points $a \in |\sigma|$ where $\sigma = (U_{\alpha_0}, \ldots, U_{\alpha_p})$ and consequently $s = \delta\, s^\beta$, showing that the cocycle s is cohomologous to zero. Next choose a partition of unity ϵ_β for the sheaf $\mathcal{S}$ subordinate to the covering $\mathfrak{U}$. The sheaf mappings $\epsilon_\beta : \mathcal{S} \longrightarrow \mathcal{S}$ then induce homomorphisms $\epsilon_\beta : \Gamma(U,\mathcal{S}) \longrightarrow \Gamma(U,\mathcal{S})$ between the sections of the sheaf $\mathcal{S}$ over any open subset $U \subset M$; and since the cochain groups $C^p(\mathfrak{U},\mathcal{S})$ consist of sections over subsets of M these sheaf mappings also induce homomorphisms $\epsilon_\beta : C^p(U,\mathcal{S}) \longrightarrow C^p(U,\mathcal{S})$. Then for any cocycle $s \in Z^p(U,\mathcal{S})$ the image $\epsilon_\beta(s) \in Z^p(U,\mathcal{S})$ vanishes outside the set U_β so there are cochains $t^\beta \in C^{p-1}(U,\mathcal{S})$ such that $\delta t^\beta = \epsilon_\beta(s)$; and then $\delta \sum_\beta t^\beta = \sum_\beta \epsilon_\beta(s) = s$, and consequently the cocycle s is cohomologous to zero as desired.

If for a sheaf $\mathcal{S}$ of abelian groups over a paracompact Hausdorff space M there is an exact sequence of the form

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \xrightarrow{d_0} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_2 \xrightarrow{d_2} \cdots \tag{C.21}$$

in which all the sheaves $\mathcal{S}_j$ are fine sheaves, a sequence called a *fine resolution* of the sheaf $\mathcal{S}$, the cohomology groups of $\mathcal{S}$ can be calculated in terms of the groups of cross-sections of these auxiliary fine sheaves. Explicitly the cohomology groups of $\mathcal{S}$ are isomorphic to the cohomology groups of the not necessarily exact sequence

$$0 \longrightarrow \Gamma(M,\mathcal{S}) \longrightarrow \Gamma(M,\mathcal{S}_0) \xrightarrow{d_0^*} \Gamma(M,\mathcal{S}_1) \xrightarrow{d_1^*} \Gamma(M,\mathcal{S}_2) \xrightarrow{d_2^*} \cdots \tag{C.22}$$

in the sense that

$$H^q(M,\mathcal{S}) \cong \frac{\ker d_q^*}{\operatorname{im} d_{q-1}^*} \tag{C.23}$$

If $\mathcal{K}_j \subset \mathcal{S}_j$ is the kernel of the homomorphism d_j the initial segment of the long exact sequence (C.21) is equivalent to the short exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}_0 \xrightarrow{d_0} \mathcal{K}_1 \longrightarrow 0;$$

and since $\mathcal{S}_0$ has trivial cohomology groups in strictly positive dimensions it follows from the exact cohomology sequence associated to this short exact sequence of sheaves that

$$H^1(M,\mathcal{S}) \cong \frac{\Gamma(M,\mathcal{K}_1)}{d_0^*\,\Gamma(M,\mathcal{S}_0)}$$

and

$$H^q(M,\mathcal{S}) \cong H^{q-1}(M,\mathcal{K}_1) \quad \text{for} \quad q > 1.$$

The remainder of the long exact sequence (C.21) is equivalent to the collection of short exact sequences

$$0 \longrightarrow \mathcal{K}_j \longrightarrow \mathcal{S}_j \xrightarrow{d_j} \mathcal{K}_{j+1} \longrightarrow 0$$

for $j > 0$; and since the sheaves $\mathcal{S}_j$ also have trivial cohomology groups in strictly positive dimensions it follows from the exact cohomology sequence associated to these short exact sequences of sheaves that

$$H^1(M,\mathcal{K}_j) \cong \frac{\Gamma(M,\mathcal{K}_{j+1})}{d_j^*\,\Gamma(M,\mathcal{S}_j)}$$

and

$$H^q(M, \mathcal{K}_{j+1}) \cong H^{q+1}(M, \mathcal{K}_j) \quad \text{for} \quad j, q > 0.$$

From these sets of isomorphisms it follows that

$$H^q(M, \mathcal{S}) \cong H^{q-1}(M, \mathcal{K}_1) \cong H^{q-2}(M, \mathcal{K}_2) \cong \cdots \cong H^1(M, \mathcal{K}_{q-1})$$

and hence that

$$H^q(M, \mathcal{S}) \cong \frac{\Gamma(M, \mathcal{K}_q)}{d^*_{q-1}\,\Gamma(M, \mathcal{S}_{q-1})} \quad \text{for} \quad q > 0,$$

where of course $\Gamma(M, \mathcal{K}_q)$ is just the kernel of the homomorphism

$$d^*_q : \Gamma(M, \mathcal{S}_q \longrightarrow \Gamma(M, \mathcal{S}_{q+1})$$

and consequently this suffices to demonstrate (C.23).

This result can be used to calculate the cohomology group ts of sheaves in another way. A covering $\mathfrak{U}$ of a topological space M is called a *Leray covering* for a sheaf of abelian groups $\mathcal{S}$ if $H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}, \mathcal{S}) = 0$ for all indices $p \geq 0$, $q \geq 1$ and all $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p} \in \mathfrak{U}^p$. In these terms the *Theorem of Leray*[5] asserts that if $\mathfrak{U}$ is a Leray covering of a paracompact Hausdorff space M for a sheaf of abelian groups $\mathcal{S}$ then the natural homomorphisms (C.16) are isomorphisms; thus for paracompact Hausdorff spaces the cohomology groups $H^p(M, \mathcal{S})$ can be calculated in terms of any Leray covering of the space M for the sheaf $\mathcal{S}$. To demonstrate this result construct a fine resolution of the sheaf $\mathcal{S}$ over M as follows. For any open subset $U \subset M$ let $\Gamma^*(U, \mathcal{S})$ be the group of not necessarily continuous cross-sections of the sheaf $\mathcal{S}$ over U, the group of quite arbitrary mappings $f : U \longrightarrow S$ such that $\pi f(p) = p$ for all points $p \in U$. The set of such groups form a complete presheaf over M, and the associated sheaf $\mathcal{S}^*$ is a fine sheaf since for any locally finite covering $\mathfrak{U} = \{U_\alpha\}$ of M and any subsets $K_\alpha \subset U_\alpha$ that are pairwise disjoint and also cover M the mappings $\rho_\alpha : \mathcal{S}^* \longrightarrow \mathcal{S}^*$ for which $\rho_\alpha(s) = s$ if $s \in K_\alpha$ and $\rho_\alpha(s) = 0$ otherwise form a partition of unity for the sheaf $\mathcal{S}^*$ for the covering $\mathfrak{U}$. The same argument shows that the restriction of the sheaf $\mathcal{S}^*$ to any open subset of M is a fine sheaf over that subset. The inclusion $\Gamma(U, \mathcal{S}) \subset \Gamma^*(U, \mathcal{S})$ of continuous cross-sections into the group of not necessarily continuous cross-sections is a homomorphism of presheaves which leads to an imbedding $\iota : \mathcal{S} \longrightarrow \mathcal{S}^*$. For the fine resolution of $\mathcal{S}$ take $\mathcal{S}_0 = \mathcal{S}^*$, $\mathcal{S}_1 = (\mathcal{S}_0/\mathcal{S})^*$, and so on. This is a fine resolution over the entire space M or over any open subset of M; so the cohomology of M or of any open subset of M with coefficients in the sheaf $\mathcal{S}$ can be calculated from this fine resolution. In particular since $\mathfrak{U}$ is assumed to be a Leray covering $H^1(|\sigma|, \mathcal{S}) = 0$ for any intersection $|\sigma| = U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ of sets U_α of $\mathfrak{U}$, so the sequence of sections

$$0 \longrightarrow \Gamma(|\sigma|, \mathcal{S}) \longrightarrow \Gamma(|\sigma|, \mathcal{S}_0) \xrightarrow{d^*_0} \Gamma(|\sigma|, \mathcal{S}_1) \xrightarrow{d^*_1} \cdots$$

[5] See Theorem G-IIIE5.

over $|\sigma|$ is exact. The cochain groups are finite direct sums of these sequences of sections, so there is also the exact sequence

$$0 \longrightarrow C^q(|\sigma|, \mathcal{S}) \longrightarrow C^q(|\sigma|, \mathcal{S}_0) \xrightarrow{d_0^*} C^q(|\sigma|, \mathcal{S}_1) \xrightarrow{d_1^*} \cdots .$$

The coboundary homomorphisms commute with the homomorphisms of these exact sequences, so there results the following commutative diagram of abelian groups and group homomorphisms.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(M, \mathcal{S}) & \longrightarrow & \Gamma(M, \mathcal{S}_0) & \xrightarrow{d_0^*} & \Gamma(M, \mathcal{S}_1) & \xrightarrow{d_1^*} & \Gamma(M, \mathcal{S}_2) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^0(\mathfrak{U}, \mathcal{S}) & \longrightarrow & C^0(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{d_0^*} & C^0(\mathfrak{U}, \mathcal{S}_1) & \xrightarrow{d_1^*} & C^0(\mathfrak{U}, \mathcal{S}_2) \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
0 & \longrightarrow & C^1(\mathfrak{U}, \mathcal{S}) & \longrightarrow & C^1(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{d_0^*} & C^1(\mathfrak{U}, \mathcal{S}_1) & \xrightarrow{d_1^*} & C^1(\mathfrak{U}, \mathcal{S}_2) \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow \\
0 & \longrightarrow & C^2(\mathfrak{U}, \mathcal{S}) & \longrightarrow & C^2(\mathfrak{U}, \mathcal{S}_0) & \xrightarrow{d_0^*} & C^2(\mathfrak{U}, \mathcal{S}_1) & \xrightarrow{d_1^*} & C^2(\mathfrak{U}, \mathcal{S}_2) \\
 & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow & & \delta\downarrow
\end{array}$$

All rows except the first are exact, while the cohomology of M with coefficients in the sheaf $\mathcal{S}$ measures the extent to which the first row fails to be exact as in (C.23). All columns except the first are exact, since the sheaves $\mathcal{S}_j$ are all fine sheaves, while the cohomology of the covering $\mathfrak{U}$ with coefficients in the sheaf $\mathcal{S}$ measures the extent to which the first column fails to be exact. A straightforward diagram chase then yields the desired result.

Appendix D

Topology of Surfaces

D.1 Homotopy

This appendix contains a brief survey of some of the basic topological properties of surfaces, an acquaintance with which is presupposed in the discussion in this book.[1] The choice of topics and the order of presentation are not those of a standard introduction to the topology of surfaces, in particular are not those most appropriate for a rigorous development of the subject with complete proofs. Instead those results that are of primary interest in the study of compact Riemann surfaces will be discussed from a rather intuitive and geometric point of view; proofs can be found in the references noted.

A *surface* is a connected two-dimensional topological manifold; only orientable surfaces will be considered here. A *path* on a surface M is a continuous image of the closed unit interval $[0, 1]$ oriented in the direction of increasing parameter values; the beginning point is the image of 0 and the end point is the image of 1. The path is *closed* if its beginning and end points coincide, and is *simple* if distinct real numbers in $[0, 1]$ have distinct images, except possibly for the beginning and end points; if the beginning and end point do coincide the path is a *simple closed* path. Two closed paths $\sigma : [0, 1] \longrightarrow M$ and $\tau : [0, 1] \longrightarrow M$ beginning and ending at p are *homotopic* if there is a continuous mapping $F : [0, 1]^2 \longrightarrow M$ of the unit square $[0, 1]^2 = \big\{ (t_1, t_2) \,\big|\, 0 \le t_i \le 1 \big\}$ into M such that $F(t_1, 0) = \sigma(t_1),\ F(t_1, 1) = \tau(t_1),\ F(0, t_2) = F(1, t_2) = p$; this is readily seen to be an equivalence relation in the usual sense. The *fundamental group* $\pi_1(M, p_0)$ of a surface M at a point $p_0 \in M$ is the set of homotopy classes of closed paths in M beginning and ending at the point p_0. The product $\sigma \cdot \tau$ of two paths σ and τ beginning and ending at p_0 is the path that arises by traversing first σ and then τ. If σ is homotopic to σ' and τ is homotopic to τ' then $\sigma\tau$ is homotopic to $\sigma'\tau'$, so this defines a group structure on $\pi_1(M, p_0)$. The

[1] These properties are treated in detail in H. Seifert and W. Threlfall, *Lehrbuch der Topologie* (Teubner, 1934), [English translation *A Textbook of Topology* (Academic Press, 1980)], in W. S. Massey, *A basic course in algebraic topology* (Springer-Verlag, 1991), and in W. Fulton, *Algebraic Topology* (Springer, 1995), among other places.

identity element in the group $\pi_1(M, p_0)$ is the homotopy class of the constant mapping of the unit interval to the point p, while the inverse of an element is the homotopy class of that path traversed in the reverse direction. To any choice of a path σ from a point $p_0 \in M$ to another point $q_0 \in M$ there corresponds the natural isomorphism $\sigma^* : \pi_1(M, p_0) \longrightarrow \pi_1(M, q_0)$ that associates to the homotopy class of a closed path τ beginning and ending at the point p_0 the homotopy class of the closed path $\sigma^{-1}\tau\sigma$ beginning and ending at the point q_0, since if $\tau' \sim \tau$ then $\sigma^{-1}\tau'\sigma \sim \sigma^{-1}\tau\sigma$. The isomorphism σ^* depends only on the homotopy class of the path σ, any two such isomorphisms differ by an inner automorphism of the fundamental group, and any inner automorphism of the fundamental group can be realized in this way. In the subsequent discussion the homotopy classes represented by a path σ also will be denoted by σ to avoid complicating the notation; it should be quite clear from context what is meant in any particular case. A topological space M is *simply connected* if its fundamental group is trivial.

The fundamental group is closely related to properties of covering spaces, which play an important role in the analytic study of Riemann surfaces. A *covering space* over a surface M is a surface N together with a continuous mapping $\pi : N \longrightarrow M$, called the *covering projection*, such that each point of M has an open neighborhood U for which the inverse image $\pi^{-1}(U)$ consists of a collection of disjoint open subsets of N each of which is homeomorphic to U under the covering projection. The *universal covering space* $\widetilde{M}$ over M is the unique simply connected covering space over M. There is a properly discontinuous group Γ of homeomorphisms acting without fixed points on the universal covering space $\widetilde{M}$, the *covering translation group*, such that the quotient space $\widetilde{M}/\Gamma$ is homeomorphic to the surface M and the natural quotient mapping $\tilde{\pi} : \widetilde{M} \longrightarrow \widetilde{M}/\Gamma = M$ is the covering projection mapping; that the group Γ is properly discontinuous means that for each point $z \in \widetilde{M}$ there is an open neighborhood U of z in $\widetilde{M}$ such that $S(U) \cap T(U) = \emptyset$ whenever S, T are distinct elements of Γ. For any contractible open subset $U \subset M$ the elements of the group Γ permute the connected components of $\pi^{-1}(U)$ transitively. For any subgroup $\Gamma_N \subset \Gamma$ the quotient space $N = \widetilde{M}/\Gamma_N$ is a surface and the natural mappings $\tilde{\pi}$ and π in the diagram

$$\begin{array}{ccccc}
\widetilde{M} & \xrightarrow{\tilde{\pi}} & \widetilde{M}/\Gamma_N & \xrightarrow{\pi} & \widetilde{M}/\Gamma \\
 & & \| & & \| \\
 & & N & \xrightarrow{\pi} & M
\end{array}$$

are covering projections; and conversely any for any covering projection $\pi : N \longrightarrow M$ there corresponds a subgroup $\Gamma_N \subset \Gamma$ such that the covering projection π is that arising from the action of the group Γ_N on the universal covering space $\widetilde{M}$. The covering projection $\pi : N \longrightarrow M$ is a *regular* covering if the subgroup Γ_N is a normal subgroup of Γ; in that case the quotient group Γ/Γ_N acts as a group of covering transformations of the space N with quotient M.

The covering translation group Γ is isomorphic to the fundamental group of the surface M, and the isomorphism can be made canonical by the choice of a

Figure D.1: A marking of a compact oriented surface M.

base point $z_0 \in \widetilde{M}$. A surface M together with of a base point in its universal covering space $\widetilde{M}$ is called a *pointed surface*[2]. For a pointed surface M with base point $z_0 \in \widetilde{M}$ let $\pi : \widetilde{M} \longrightarrow M$ be the universal covering projection and $\pi_{z_0} : \Gamma \longrightarrow \pi_1(M, p(z_0))$ be the mapping that associates to a covering translation $T \in \Gamma$ the homotopy class in $\pi_1(M, \pi(z_0))$ of the image $\tau = \pi(\tilde{\tau})$ in M of any path $\tilde{\tau} \subset \widetilde{M}$ from z_0 to Tz_0; since $\widetilde{M}$ is simply connected the homotopy class of the path τ is independent of the choice of the path $\tilde{\tau}$. If $\tilde{\sigma} \subset \widetilde{M}$ is a path from the base point z_0 to the point Sz_0 for another covering translation $S \in \Gamma$ then the path $\tilde{\sigma} \cdot S\tilde{\tau}$ extends from z_0 to the point STz_0 so $\pi_{z_0}(ST) = \pi\big(\tilde{\sigma} \cdot (S\tilde{\tau})\big) = \pi(\tilde{\sigma}) \cdot \pi(\tilde{\tau}) = \pi_{z_0}(S) \cdot \pi_{z_0}(T)$, showing that π_{z_0} is a group homomorphism; that it is an isomorphism is a simple consequence of the simple connectivity of $\widetilde{M}$. Changing the base point $z_0 \in \widetilde{M}$ to Az_0 for a covering translation $A \in \Gamma$ has the effect of changing the isomorphism π_{z_0} by an inner automorphism of the group Γ, and any inner automorphism of the group Γ can be realized in this way.

A compact orientable surface can be represented as a sphere with g handles, where the integer g is called the *genus* of the surface; thus a surface of genus $g = 0$ is just a sphere, a surface of genus $g = 1$ is a torus, a surface of genus $g = 2$ is a sphere with two handles, and so on. A surface of genus $g > 0$ can be represented as a sphere with g handles in a number of different ways though. A *marking* of a compact oriented surface M of genus $g > 0$ with universal covering space $\widetilde{M}$ and covering projection $\pi : \widetilde{M} \longrightarrow M$ is the choice of a base

[2]It is more customary to define a pointed surface as a surface M together with the choice of a base point in M itself. Of course the choice of a base point $z_0 \in \widetilde{M}$ yields automatically the choice of the base point $\pi(z_0) \in M$ where $\pi : \widetilde{M} \longrightarrow M$ is the covering projection; but for a canonical isomorphism between the fundamental group and the covering translation group it is necessary to choose a base point in the universal covering space.

point $z_0 \in \widetilde{M}$, of a representation of M as a sphere with g handles, ordered as the first handle, the second handle, and so on, and of $2g$ simple closed paths $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ in $\widetilde{M}$ beginning and ending at the point $p_0 = \pi(z_0) \in M$, disjoint except for the common point p_0 and such that the paths α_i and β_i encircle the i-th handle as sketched in Figure D.1; the surface M together with a marking is a *marked surface*. Any orientation-preserving homeomorphism of the surface M that preserves the base point $p_0 \in M$ transforms a marking of the surface to another marking; two markings related in this way are called *equivalent markings* of the surface, and it is really the equivalence classes of markings of a surface that are of primary interest. When the $2g$ paths α_i and β_i are removed from the surface M the result is a contractible open subset $D \subset M$. The boundary of D can be traversed from the interior of D in the positive sense of the orientation it inherits from the orientation of the surface, beginning at the point 1 in Figure D.1 and proceeding first along the path α_1 in the direction of its orientation back to the point 2, then along the path β_1 in the direction of its orientation back to the point 3, then along the path α_1 but in the reverse direction to its orientation back to the point 4, then along the path β_1 but again in the reverse direction to its orientation back to the point 5, then along the path α_2 in the direction of its orientation back to the point 6, and so on; the traverse ends by proceeding along the path β_g in the reverse direction to its orientation back to the initial point 1.

If $\tilde{\alpha}_i \subset \widetilde{M}$ is the lifting of the path $\alpha_i \subset M$ to a path in the universal covering space $\widetilde{M}$ beginning at the base point $z_0 \in \widetilde{M}$ the end point of the path $\tilde{\alpha}_i$ is the point $A_i z_0 \in \widetilde{M}$ for a uniquely determined covering translation $A_i \in \Gamma$, that element of the group Γ for which $\pi_{z_0}(A_i) \in \pi_1(M, p_0)$ is the homotopy class of the path α_i under the canonical isomorphism from the covering translation group Γ to the fundamental group of the surface M. Correspondingly if $\tilde{\beta}_i \subset \widetilde{M}$ is the lifting of the path $\beta_i \subset M$ to a path beginning at z_0 it will end at the point $B_i z_0$ for a covering translation $B_0 \in \Gamma$ for which $\pi_{z_0}(B_i) \in \pi_1(M, p_0)$ is the homotopy class of the path β_i. To simplify the formulas in the subsequent discussion it is convenient to introduce the commutators

$$C_i = [A_i, B_i] = A_i\, B_i\, A_i^{-1} B_i^{-1} \in \Gamma \tag{D.1}$$

of these covering translations. The inverse image $\pi^{-1}(D) \subset \widetilde{M}$ of the open subset $D \subset M$ consists of a collection of disjoint open subsets of the universal covering space $\widetilde{M}$ that are homeomorphic to D under the covering projection $\pi : \widetilde{M} \longrightarrow M$ and that are permuted by the action of the covering translation group Γ; the boundaries of the disjoint connected components of $\pi^{-1}(D)$ consist of the images under various covering translations in Γ of the paths $\tilde{\alpha}_i$ and $\tilde{\beta}_i$. The set Δ sketched in Figure D.2 is that connected component of $\pi^{-1}(D)$ with the base point z_0 and the paths $\tilde{\alpha}_1$ and $\tilde{\beta}_g$ on its boundary; it is called the *fundamental domain* for the action of the covering translation group Γ. The translates $T\Delta$ for all covering translations $T \in \Gamma$ are disjoint subsets of $\widetilde{M}$ that cover the entire space $\widetilde{M}$ except for points over the removed paths α_i and β_i. The boundary of Δ can be traversed from the interior of Δ by lifting the traverse

Figure D.2: The fundamental domain $\Delta \subset \widetilde{M}$.

of the boundary of D, beginning at the point 1 and proceeding first along the path $\tilde{\alpha}_1$ covering α_1 to the point 2, then along the path $A_1\tilde{\beta}_1$ covering β_1 to the point 3, then in the reverse direction along the path $C_1B_1\tilde{\alpha}_1$ covering α_1 to the point 4, then in the reverse direction along the path $C_1\tilde{\beta}_1$ covering β_1 to the point 5, then along the path $c_1\tilde{\alpha}_1$ covering α_1 to the point 6, and so on, ending back at the point 1. The vertices of Δ are the images of the base point z_0 under the indicated elements of the group Γ, and the particular lifts of the paths $\tilde{\alpha}_i$ and $\tilde{\beta}_i$ that form the boundary of Δ are determined by their beginning points as in Figure D.2. A neighborhood of the base point $z_0 \in \widetilde{M}$ is mapped homeomorphically to a neighborhood of the point $p_0 \in M$ by the projection π; consequently $2g$ translates of the fundamental domain Δ meet at the vertex z_0 in a manner reflecting the configuration of paths emerging from the point $p_0 \in M$ as sketched in Figure D.1, and similarly of course at all the points Γz_0. The surface M itself can be constructed from the fundamental domain Δ by identifying the sides $\tilde{\alpha}_1$ and $C_1B_1\tilde{\alpha}_1$ and the other pairs of sides correspondingly; this is the traditional "scissors and paste" description of a compact surface, probably most familiar for surfaces of genus $g = 1$ described by identifying the opposite sides of a parallelogram in the traditional treatment of elliptic functions.

Any closed path τ on the marked surface M beginning at the point p_0 can be deformed homotopically to a closed path lying entirely on the boundary of D, so is homotopic to the product of the paths α_i and β_i and their inverses in some order; thus the fundamental group $\pi_1(M, p_0)$ is generated by the group elements α_i and β_i. It is evident from Figure D.2 that the boundary of the fundamental domain Δ can be described as the product

$$\text{(D.2)} \qquad \partial\overline{\Delta} = \prod_{i=1}^{g}\Big((C_1\cdots C_{i-1}\tilde{\alpha}_i)\cdot(C_1\cdots C_{i-1}A_i\tilde{\beta}_i)\cdot (C_1\cdots C_iB_i\tilde{\alpha}_i)^{-1}\cdot(C_1\cdots C_i\tilde{\beta}_i)^{-1}\Big),$$

where the product is taken in increasing order of the index i and a product $C_1\cdots C_{i-1}$ is interpreted as being the identity element when $i=1$; and that this product is homotopic to the identity element, so that the homotopy classes α_i and β_i in $\pi_1(M,p_0)$ are subject to the relation

$$\text{(D.3)} \qquad \mathrm{I} = \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\cdots\beta_{g-1}^{-1}\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}.$$

All the relations between these generators are consequences of this single relation, so the fundamental group $\pi_1(M,p_0)$ has the presentation as the quotient of the free group F on the symbols α_i and β_i by the normal subgroup K generated by the relation (D.3). The fundamental group is isomorphic to the covering translation group; so the covering translation group Γ can be described correspondingly as the quotient of the free group F on the symbols A_i and B_i by the normal subgroup K generated by the single word

$$\text{(D.4)} \qquad C = C_1\cdots C_g = A_1B_1A_1^{-1}B_1^{-1}\cdots A_gB_gA_g^{-1}B_g^{-1}$$

in the commutator subgroup $[F,F]\subset F$. Conversely whenever α_i, β_i are generators of the fundamental group $\pi_1(M,p_0)$ of a surface M of genus g, subject only to the relation (D.3), these generators can be taken to arise from the paths of a geometric marking of the surface[3]; for many purposes this associated presentation of the fundamental group or covering translation group of the surface is the most significant aspect of a marking of the surface.

There are situations in which it is necessary to consider the noncompact surfaces that arise by removing from a surface M a set of n points $q_1,\ldots,q_n$. By expanding the holes made by removing the points $q_1,\ldots,q_n$ to small discs about these points and expanding the whole made by removing the point q_n any closed path in the complement $D\sim(q_1\cup\cdots\cup q_n)$ can be deformed homotopically to a path on the boundary of the set D together with paths $\gamma_1,\ldots,\gamma_{n-1}$ from the point p_0 out to small circles around the points $q_1,\ldots,q_{n-1}$ and then back to the point p_0; that can be visualized by considering the domain D and its pairs of boundary paths as sketched in Figure D.3. The collection of the paths

[3]That any presentation of the fundamental group with the single relation (D.3) can be realized by a geometric marking of the surface is a consequence of the result of J. Nielsen that an automorphism of the covering translation group can be realized by a homeomorphism of the surface. This result can be found in the paper by J. Nielsen, "Untersuchungen zur Topologie der geschlossenen zweiseitige Flächen" I, *Acta Math.* **50** (1927), pp. 189-358; an English translation is in J. Nielsen, *Collected Papers* I, Birkhäuser, 1986, pp. 223-341. See also the discussion in the book by W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory*, Interscience, 1966, page 176.

Figure D.3: The set resulting from the removal of the points q_i from D can be shrunk to the union of the paths $\alpha_i, \beta_i, \gamma_i$.

$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{n-1}$. amounts to the set consisting of $2g+n-1$ circles with a single point in common, a set called a *bouquet of circles*; its fundamental group, and therefore the fundamental group of the complement $D \sim (q_1 \cup \cdots \cup q_n)$, is the free group generated by the paths $\alpha_i, \beta_i, \gamma_j$, for $1 \leq i \leq g$ and $1 \leq j \leq n-1$, a free group on $2g+n-1$ generators.

D.2 Homology

To examine the homology groups of a compact Riemann surface M of genus $g > 0$ it is convenient to assume that the surface is triangulated[4]. The group C_i of i-dimensional *chains* of the triangulated surface M is the free abelian group generated by the i-dimensional oriented simplices of the triangulation. The boundary of a 2-dimensional simplex is the 1-dimensional chain consisting of the sum of the three 1-dimensional simplices forming the boundary of the triangle, oriented so that the boundary is traversed in the natural orientation; and the boundary of a 1-dimensional simplex is the 0-dimensional chain consisting of the end point of the simplex minus the beginning point of the simplex, as sketched in Figure D.4. The mapping that associates to any chain the sum of the boundaries of the simplices comprising that chain is a group homomorphism $\partial_i : C_i \longrightarrow C_{i-1}$ for $i = 1, 2$; it is apparent from Figure D.4 that $\partial_1 \partial_2 = 0$. The kernel of the homomorphism ∂_i for $i = 1, 2$ is the subgroup $Z_i \subset C_i$ of i-dimensional *cycles*,

[4]That every compact surface can be triangulated is a classical result; the definition and general properties of triangulations are discussed in the general references cited on page 461.

Figure D.4: Boundary cycles of simplices: $\partial_2\tau_1^2 = \tau_1^1 + \tau_2^1 + \tau_3^1, \quad \partial_1\tau_1^1 = \tau_2^0 - \tau_1^0$

and the image of the homomorphism ∂_{i+1} for $i = 0, 1$ is the subgroup $B_i \subset C_i$ of i-dimensional *boundaries*. Clearly $B_1 \subset Z_1$ since $\partial_1\partial_2 = 0$; the 1-dimensional *homology group* of the surface M is defiined to be the quotient group $H_1(M) = Z_1/B_1$. There are no boundaries $B_2 \subset C_2$ so the 2-dimensional homology group is defined as the quotient group $H_2(M) = Z_2(M)$; and every cochain in C_0 can be viewed as a cocycle so the 0-dimensional homology group is defined as the quotient group $H_0(M) = C_0/B_0$. A basic result is that the homology groups are independent of the choice of the triangulation. Customarily two cycles that differ from one another by a boundary are called *homologous*, so that alternatively the homology group can be viewed as the group of homology classes of cycles on M. The rank of the group $H_i(M)$ is called the i-th *Betti number* of the surface M and is denoted by b_i.

All 2-cycles of the surface M are integral multiples of the *fundamental cycle*, the sum of all of the 2-dimensional simplices of the triangulation; the fundamental cycle usually is denoted simply by M; consequently

$$H_2(M) \cong \mathbb{Z}, \tag{D.5}$$

or equivalently $b_2 = 1$. On a marked surface, with the marking described by a base point $z_0 \in \widetilde{M}$ and paths α_i and β_i as in Figure D.2, any 1-cycle is homologous to a sum of the cycles α_i and β_i, and no nontrivial combination of these cycles is homologous to zero; consequently

$$H_1(M) \cong \mathbb{Z}^{2g}, \tag{D.6}$$

or equivalently $b_1 = 2g$. A basic result is that the homology group $H_1(M)$ is the abelianization of the fundamental group $\pi_1(M, p_0)$; the homology group is a simpler invariant than the fundamental group since it ignores the information carried by the commutator subgroup of the fundamental group. The homology classes represented by the paths α_i, β_i also are denoted by α_i, β_i, to avoid complicating the notation; it should be quite clear from context what is meant in any particular case. If $p_0 = \pi(z_0) \in M$, the composition of the homomorphism $\pi_{z_0} : \Gamma \longrightarrow \pi_1(M, p_0)$ and the mapping from the fundamental group to the first homology group yields the natural identification

$$H_1(M) \cong \frac{\Gamma}{[\Gamma, \Gamma]} \tag{D.7}$$

describing the homology of M in terms of the covering translation group. Since the mapping π_{z_0} changes by an inner automorphism when the base point is changed, it follows that this identification is canonical, independent of the choice of the base point. Finally any 0-dimensional chain is a cycle, and two such cycles are homologous precisely when the sums of the multiplicities of the points involved coincide, so that

$$H_0(M) \cong \mathbb{Z}, \tag{D.8}$$

or equivalently $b_0 = 1$.

The alternating sum of the ranks of the homology groups of the surface M, the expression

$$\chi(M) = \sum_{i=0}^{3}(-1)^i \text{rank } H_i(M) = b_0 - b_1 + b_2, \tag{D.9}$$

is called the *Euler characteristic* of the surface M; in view of the formulas for the Betti numbers just derived, it follows that

$$\chi(M) = 2 - 2g \tag{D.10}$$

where g is the genus of the surface. Since the homology groups arise from the exact sequences

$$0 \longrightarrow Z_2 \longrightarrow C_2 \xrightarrow{\partial} B_1 \longrightarrow 0,$$

$$0 \longrightarrow Z_1 \longrightarrow C_1 \xrightarrow{\partial} B_0 \longrightarrow 0,$$

$$0 \longrightarrow Z_0 \longrightarrow C_0 \xrightarrow{\partial} 0,$$

it follows that the Euler characteristic can be expressed alternatively as

$$\begin{aligned}
\chi(M) &= \text{rank } H_2(M) - \text{rank } H_1(M) + \text{rank } H_0(M) \\
&= \text{rank } Z_2 - \big(\text{rank } Z_1 - \text{rank } B_1\big) + \big(\text{rank } Z_0 - \text{rank } B_0\big) \\
&= \big(\text{rank } Z_2 + \text{rank } B_1\big) - \big(\text{rank } Z_1 + \text{rank } B_0\big) + \text{rank } Z_0 \\
&= \text{rank } C_2 - \text{rank } C_1 + \text{rank } C_0.
\end{aligned}$$

Here rank $C_i = n_i$ is just the total number of i-dimensional simplices in the triangulation, so there results the *Euler formula*

$$\chi(M) = n_0 - n_1 + n_2; \tag{D.11}$$

this expresses the Euler characteristic directly in terms of the number of simplices in any triangulation of the surface.

For some purposes it is more convenient to consider the singular homology groups rather than the homology groups associated to a triangulation of the surface. A singular simplex of a surface M is a continuous mapping of a standard simplex, either a point, a line segment, or a triangle, into the surface M; and the singular chain complex of M is the free abelian group generated by the singular simplices of M. The boundary of a singular simplex is the element of the singular chain complex consisting of the singular simplices that are formed by restricting the mapping of a standard simplex into M to the boundary of that simplex. Again the boundary of a boundary is zero, so it is possible to define the singular homology groups of a surface M as the homology groups of this chain complex. It is a standard result that the homology groups formed from the singular complex of a surface are isomorphic to the homology groups formed from a triangulation of the surface. The singular homology groups are clearly invariantly defined, so this shows that the homology groups defined in terms of a triangulation really are independent of the choice of the triangulation.

Just as important as the homology groups, and in some ways even more convenient, are the dual cohomology groups. If C_i is the group or $\mathbb{Z}$-module of i-dimensional chains in a triangulation of the surface M and $\mathcal{R}$ is any $\mathbb{Z}$-module then $C^i(\mathcal{R}) = \mathrm{Hom}(C_i, \mathcal{R})$ is the group of i-dimensional *cochains* of that triangulation with coefficients in the $\mathbb{Z}$-module $\mathcal{R}$. Of primary interest here are the cases in which $\mathcal{R} = \mathbb{Z}, \mathbb{R}$, or $\mathbb{C}$, and it will be assumed henceforth that $\mathcal{R}$ is one of these modules. The boundary homomorphisms $\partial : C_{i+1} \longrightarrow C_i$ naturally lead to dual coboundary homomorphisms $\delta : C^i(\mathcal{R}) \longrightarrow C^{i+1}(\mathcal{R})$, where $\delta(\phi)(c) = \phi(\partial c)$ for any cochain $\phi \in C^i(\mathcal{R}) = \mathrm{Hom}(C_i, \mathcal{R})$ and any chain $c \in C_{i+1}$; and $\delta\delta = 0$ since $\partial\partial = 0$. The kernel of the homomorphism δ is the subgroup $Z^i(\mathcal{R}) \subset C^i(\mathcal{R})$ of i-dimensional *cocycles* with coefficients in $\mathcal{R}$, and the image of that homomorphism is the subgroup $B^i(\mathcal{R}) = \delta C^{i-1}(\mathcal{R})$ of i-dimensional *coboundaries* with coefficients in $\mathcal{R}$; clearly $B^i(\mathcal{R}) \subset Z^i(\mathcal{R})$ since $\delta\delta = 0$. The quotient group $H^i(M, \mathcal{R}) = Z^i(\mathcal{R})/B^i(\mathcal{R})$ is the i-th *cohomology group* of the surface M with coefficients $\mathcal{R}$. In the case of surfaces the situation is particularly simple, for $H^i(M, \mathcal{R}) \cong \mathrm{Hom}(H_i(M), \mathcal{R})$ since the homology groups are free abelian groups; in particular

$$H^i(M, \mathbb{Z}) \cong \mathrm{Hom}(H_i(M), \mathbb{Z}), \tag{D.12}$$

$$H^i(M, \mathbb{R}) = H^i(M, \mathbb{Z}) \otimes \mathbb{R}, \quad H^i(M, \mathbb{C}) = H^i(M, \mathbb{Z}) \otimes \mathbb{C}$$

for $i = 0, 1, 2$. Furthermore there is the canonical identification

$$H^1(M, \mathbb{Z}) = \mathrm{Hom}(\Gamma, \mathbb{Z}), \tag{D.13}$$

since homomorphisms from Γ to the abelian group $\mathbb{Z}$ are necessarily trivial on the commutator subgroup $[\Gamma, \Gamma] \subset \Gamma$ and $\Gamma/[\Gamma, \Gamma] \cong H_1(M)$.

For any $\mathbb{Z}$-module $\mathcal{R}$ the cohomology groups $H^i(M, \mathcal{R})$ can be identified with the sheaf cohomology groups of M with coefficients in the constant sheaf $\mathcal{R}$. Any finite open covering of a compact Riemann surface M has a refinement $\mathfrak{U}$ consisting of nonempty open sets U_i such that any 4 of the sets U_i have an empty intersection[5]. Associated to this covering of M is the two-dimensional simplicial complex in which the vertices are the sets U_i, the one-simplices are pairs of distinct intersecting sets $U_i \cap U_j \neq \emptyset$, and the two-simplices are triples

[5]See for instance Hurewicz and Wallman, *Dimension Theory*, Princeton University Press, 1948.

of distinct intersecting sets $U_i \cap U_j \cap U_k \neq \emptyset$; this simplicial complex can be viewed as a simplicial approximation to the topological space M, associating to each set U_i a point in that set, associating to each intersection $U_i \cap U_j$ a segment connecting the points associated to these two separate sets, and associating to the intersection $U_i \cap U_j \cap U_k$ the triangle formed by the segments associated to the separate pairs of intersecting sets. The sheaf cochain groups of the covering $\mathfrak{U}$ with coefficients in the constant sheaf $\mathcal{R}$ can be identified with the ordinary cochains $\mathcal{C}^i$ of this simplicial complex, and the coboundary operators then clearly coincide so the two sets of cochain groups lead to the same cohomology groups.

The cohomology groups of M with real coefficients can be expressed in terms of differential forms by deRham's Theorem[6] For an arbitrary $\mathcal{C}^\infty$ manifold M let $\mathcal{E}^p$ be the sheaf of germs of $\mathcal{C}^\infty$ complex-valued differential forms of degree p on M and let $d : \mathcal{E}^p \longrightarrow \mathcal{E}^{p+1}$ be the operator of exterior differentiation, which satisfies $dd = 0$. The kernel of the operator d is the subsheaf $\mathcal{E}^p_c \subset \mathcal{E}^p$ of *closed differential forms* of degree p, and the local form of deRham's Theorem is the assertion that the sequence

$$0 \longrightarrow \mathcal{E}^{p-1}_c \longrightarrow \mathcal{E}^{p-1} \xrightarrow{d} \mathcal{E}^p_c \longrightarrow 0 \tag{D.14}$$

is an exact sequence of sheaves for any degree $p > 0$. Of course $\mathcal{E}^0_c = \mathbb{C}$, the subsheaf of $\mathcal{E}^0$ consisting of germs of constant functions; and $\mathcal{E}^n_c = \mathcal{E}^n$ and $\mathcal{E}^p = 0$ for $p > n$ for a manifold M of dimension n. For surfaces $\mathcal{E}^2_c = \mathcal{E}^2$ and $\mathcal{E}^p = 0$ for $p > 2$, so that there are just the two exact sequences

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1_c \longrightarrow 0 \tag{D.15}$$

and

$$0 \longrightarrow \mathcal{E}^1_c \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \longrightarrow 0. \tag{D.16}$$

The sequences of cross-sections of these two exact sequence of sheaves are not necessarily exact at the right end; the extent to which it fails to be exact is measured by the *deRham groups* of the manifold M, the quotient groups

$$\mathfrak{H}^1(M) = \frac{\Gamma(M, \mathcal{E}^1_c)}{d\Gamma(M, \mathcal{E}^0)} \text{ and } \mathfrak{H}^2(M) = \frac{\Gamma(M, \mathcal{E}^2)}{d\Gamma(M, \mathcal{E}^1)} \tag{D.17}$$

where as usual $\Gamma(M, \mathcal{E}^p)$ denotes the space of sections of the sheaf $\mathcal{E}^p$. The sheaves $\mathcal{E}^p$ are fine sheaves, so that $H^q(M, \mathcal{E}^{p-1}) = 0$ whenever $p, q > 0$, and from the exact cohomology sequences associated to the exact sequences of sheaves (D.15) (D.16) there result the global form of deRham's Theorem, the isomorphisms

$$\mathfrak{H}^p(M) \cong H^p(M, \mathbb{C}) \text{ for } p = 1, 2. \tag{D.18}$$

[6] A general discussion of differential forms on differentiable manifolds, and proofs of the basic properties that will be used here, can be found in M. Spivak, *Calculus on Manifolds*, (Benjamin, 1965) as well as in W. Fulton, *Algebraic Topology* (Springer, 1995), among other places. The general properties of sheaves that arise in this discussion are reviewed in Appendix C.

The cohomology class in $H^p(M,\mathbb{C})$ that is associated to the element in the deRham group $\mathfrak{H}^p(M)$ represented by a closed differential form $\phi \in \Gamma(M,\mathcal{E}^p_c)$ under the deRham isomorphism (D.18) is the *period class* of the differential form ϕ. This abstract cohomological interpretation of the period class has a more geometric form; the classical statement of deRham's Theorem is the assertion that *the mapping that associates to a differential form $\phi \in \Gamma(M,\mathcal{E}^p_c)$ the linear functional on the homology group $H_p(M)$ defined by integration of the differential form ϕ along representative cycles is an isomorphism between the deRham group $\mathfrak{H}^p(M)$ and the cohomology group $H^p(M,\mathbb{C})$* in each dimension p. In particular the exact differential forms, those in $d\Gamma(M,\mathcal{E}^{p-1})$, are precisely the differential forms having zero integrals along all the cycles of M; and any linear functional on the cycles can be represented as the integral of a suitable closed differential form. Two closed differential forms ϕ,ψ that differ by an exact differential form are said to be *cohomologous*, and that is indicated by writing $\phi \sim \psi$; so the deRham isomorphism can be rephrased as the assertion that the space of cohomology classes of closed differential forms of degree p is a vector space that is naturally dual to the homology group $H_p(M)$ by integration. The deRham isomorphism for real cohomology is described correspondingly. The subgroup of cohomology classes of closed differential forms having integral periods on all the cycles of M form a lattice subgroup of the deRham group that is naturally isomorphic to the integral cohomology group $H^p(M,\mathbb{Z})$.

The exterior product of any two closed differential forms $\phi,\psi \in \Gamma(M,\mathcal{E}^1_c)$ is a closed differential form $\phi\wedge\psi \in \Gamma(M,\mathcal{E}^2)$; and if $\phi \sim \phi'$ and $\psi \sim \psi'$ then clearly $\phi\wedge\psi \sim \phi'\wedge\psi'$, so this yields a well defined skew-symmetric bilinear mapping

$$\mathfrak{H}^1(M)\times\mathfrak{H}^1(M) \longrightarrow \mathfrak{H}^2(M),$$

the *cup product* mapping. Under the deRham isomorphism through the period classes of these differential forms this induces the skew-symmetric bilinear mapping

$$H^1(M,\mathbb{C})\times H^1(M,\mathbb{C}) \longrightarrow H^2(M,\mathbb{C}),$$

that associates to cohomology classes $\phi,\psi \in H^1(M,\mathbb{C})$ a cohomology class $\phi\cup\psi$ called the *cup product* of these cohomology classes. Since the manifold M is two-dimensional the composition of the exterior product mapping in the deRham group and the isomorphism $\mathfrak{H}^2(M)\cong\mathbb{C}$ that associates to the deRham class represented by a differential form $\phi \in \Gamma(M,\mathcal{E}^2)$ the value $\int_M \phi$ is the skew-symmetric bilinear mapping

$$\mathfrak{H}^1(M)\times\mathfrak{H}^1(M) \longrightarrow \mathbb{C}$$

that associates to the deRham classes represented by any two differential forms $\phi,\psi \in \Gamma(M,\mathcal{E}^1_c)$ the complex number

$$(\phi,\psi) = \int_M \phi\wedge\psi;$$

this is called the *intersection form* on the surface M. In terms of a basis $\tau_j \in H_1(M)$ for the homology of M and the dual basis $\phi_i \in \Gamma(M,\mathcal{E}^1_c)$ for the deRham group $\mathfrak{H}^1(M)$, characterized by the period conditions $\int_{\tau_j}\phi_i = \delta^i_j$ for $1 \le i,j \le 2g$, the intersection form is described by the *intersection matrix* $P = \{p_{ij}\}$, the $2g \times 2g$ skew-symmetric integral matrix with entries

$$p_{ij} = (\phi_i, \phi_j) = \int_M \phi_i \wedge \phi_j. \tag{D.19}$$

For the basis associated to a marking of the surface the intersection matrix has the following normal form.

Theorem D.1 *If M is a compact oriented surface of genus $g > 0$ with a marking described by covering translations $A_j, B_j \in \Gamma$ and if $\phi_i \in \Gamma(M, \mathcal{E}^1_c)$ is the dual basis for the first deRham group of M then in terms of this basis the intersection matrix is the $2g \times 2g$ basic skew-symmetric matrix*

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \tag{D.20}$$

where I is the $g \times g$ identity matrix and 0 is the $g \times g$ zero matrix.

Proof: When the differential forms ϕ_i are viewed as Γ-invariant differential forms on the universal covering space $\widetilde{M}$ their integrals $f_i(z) = \int_{z_0}^{z}\phi_i(z)$ are functions on the universal covering space $\widetilde{M}$ such that $f_i(A_j z) = f_i(z) + \delta^i_j$, $f_i(B_j z) = f_i(z) + \delta^i_{j+g}$, and $f_i(C_j z) = f_i(z)$ for the commutator $C_j = [A_j, B_j]$. By Stokes's Theorem

$$p_{jk} = \int_M \phi_j \wedge \phi_k = \int_\Delta d(f_j\phi_k) = \int_{\partial\Delta} f_j\phi_k$$

in terms of the fundamental polygon Δ. The boundary $\partial\Delta$ is described explicitly in equation (D.2), and it follows that

$$\begin{aligned}
p_{jk} &= \sum_{i=1}^{g} \int_{C_1\cdots C_{i-1}\tilde{\alpha}_i - C_1\cdots C_i B_i\tilde{\alpha}_i} f_j(z)\phi_k(z) \\
&\qquad + \sum_{i=1}^{g} \int_{C_1\cdots C_{i-1}A_i\tilde{\beta}_i - C_1\cdots C_i\tilde{\beta}_i} f_j(z)\phi_k(z) \\
&= \sum_{i=1}^{g} \int_{\tilde{\alpha}_i} \Big(f_j(z)\phi_k(z) - \big(f_j(z) + \delta^j_{i+g}\big)\phi_k(z) \Big) \\
&\qquad + \sum_{i=1}^{g} \int_{\tilde{\beta}_i} \Big(\big(f_j(z) + \delta^j_i\big)\phi_k(z) - f_j(z)\phi_k(z) \Big) \\
&= -\sum_{i=1}^{g} \delta^j_{i+g} \int_{\tilde{\alpha}_i} \phi_k(z) + \sum_{i=1}^{g} \delta^j_i \int_{\tilde{\beta}_i} \phi_k(z) \\
&= \sum_{i=1}^{g} \Big(-\delta^j_{i+g}\delta^k_i + \delta^j_i\delta^k_{i+g} \Big) \\
&= \delta^{j+g}_k - \delta^j_{k+g},
\end{aligned}$$

which is the result asserted and thereby concludes the proof.

Corollary D.2 *Any two intersection matrices P and $\tilde{P}$ for a compact oriented surface of genus $g > 0$ are related by $\tilde{P} = QP\,{}^t\!Q$ for some matrix $Q \in \mathrm{Gl}(2g, \mathbb{Z})$, and consequently* $\det P = 1$ *for any intersection matrix P.*

Proof: If P is the intersection matrix of the surface M in terms of a basis ϕ_i of the deRham group the intersection matrix $\tilde{P}$ in terms of another basis $\tilde{\phi}_i = \sum_{j=1}^{2g} q_{jl}\phi_j$ has the form

$$\begin{aligned}
\tilde{p}_{ij} &= \int_M \tilde{\phi}_i \wedge \tilde{\phi}_j = \sum_{k,l=1}^{2g} \int_M q_{ik}\phi_k \wedge q_{jl}\phi_l \\
&= = \sum_{k,l=1}^{2g} \int_M q_{ik}p_{kl}q_{jl},
\end{aligned}$$

or in matrix terms $\tilde{P} = QP\,{}^t\!Q$. Since one intersection matrix is the basic skew-symmetric matrix J by the preceding theorem it follows that any other intersection matrix is of the form $P\mathrm{J}\,{}^t\!Q$ for some invertible matrix Q and consequently $\det P = \det \mathrm{J} = 1$. That suffices for the proof.

Appendix E

Cohomology of Groups

E.1 Definitions and Basic Properties

Various analytical and geometrical constructions that arise in the study of compact Riemann surfaces involve the action of the covering translation group on the universal covering space of the surface. The universal covering space is both topologically and analytically trivial, in natural senses, so structures on the quotient space to a considerable extent are determined by the structure of the covering translation group; in particular the cohomology of the covering translation group reflects significant properties of the geometry of the quotient space. Since the cohomology of groups possibly is not so familiar and the notation that will be adopted here is not altogether standard, among other things in that groups will be viewed as acting on the right rather than on the left, a brief survey of the notation and of some of the basic properties of the cohomology of groups will be included in this appendix.[1]

A multiplicative group Γ acts as a group of operators on the right on an additive abelian group V if there is a mapping $V \times \Gamma \longrightarrow V$ that associates to any elements $v \in V$ and $T \in \Gamma$ an element $v|T \in V$ such that:

(i) for each $T \in \Gamma$ the mapping $v \longrightarrow v|T$ is an automorphism of the group V;
(ii) if $I \in \Gamma$ is the identity then $v|I = v$ for all $v \in V$;
(iii) $v|(T_1T_2) = (v|T_1)|T_2$ for all $v \in V$ and $T_1, T_2 \in \Gamma$.

If Γ acts as a group of operators on the right on two additive abelian groups V_1, V_2, a Γ-homomorphism $\phi : V_1 \longrightarrow V_2$ is a homomorphism of abelian groups such that $\phi(v|T) = \phi(v)|T$ for all $v \in V_1$ and all $T \in \Gamma$.

For any multiplicative group Γ and for any integer $n \geq 0$ let $X_n(\Gamma)$ be the additive free abelian group generated by the symbols $(T_0, T_1, \ldots, T_n)$ for arbitrary $T_i \in \Gamma$, but where $(T_0, T_1, \ldots, T_n) = 0$ if $T_i = T_{i-1}$ for any index i.

[1] A more detailed treatment of this material can be found in S. MacLane, *Homology*, (Springer, 1994), to which reference is made for the proofs that are not included here.

The group Γ acts as a group of operators on the right on the abelian group $X_n(\Gamma)$ by setting

$$(T_0, T_1, \ldots, T_n)|T = (T_0T, T_1T, \ldots, T_nT) \tag{E.1}$$

for the free generators of $X_n(\Gamma)$. For any index $n > 0$ introduce the group homomorphism

$$\partial : X_n(\Gamma) \longrightarrow X_{n-1}(\Gamma)$$

defined on the free generators of X_n by

$$\partial(T_0, T_1, \ldots, T_n) = \sum_{i=0}^{n}(-1)^i(T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n); \tag{E.2}$$

it is a straightforward exercise to verify that this is compatible with the condition that $(T_0, T_1, \ldots, T_n) = 0$ if $T_i = T_{i-1}$. These homomorphisms clearly commute with the operation of Γ on $X_n(\Gamma)$, so are also Γ-homomorphisms; and it is another straightforward exercise to verify that $\partial\partial = 0$. On the other hand it is somewhat less straightforward to see that

$$X_0(\Gamma) \xleftarrow{\partial} X_1(\Gamma) \xleftarrow{\partial} X_2(\Gamma) \xleftarrow{\partial} \cdots \tag{E.3}$$

is an exact sequence of Γ-homomorphisms. To demonstrate that, introduce the group homomorphisms $\sigma : X_n(\Gamma) \longrightarrow X_{n+1}(\Gamma)$ defined on the free generators of X_n by $\sigma(T_0, T_1, \ldots, T_n) = (I, T_0, T_1, \ldots, T_n)$ where $I \in \Gamma$ is the identity element. One more straightforward calculation shows that $\partial\sigma + \sigma\partial = I$ is the identity homomorphism on $X_n(\Gamma)$ for $n > 0$; hence if $f \in X_n(\Gamma)$ for $n > 0$ and if $\partial f = 0$ then $f = (\partial\sigma + \sigma\partial)f = \partial(\sigma f)$, so the sequence (E.3) is exact. The cohomology groups of Γ with coefficients in an abelian group V on which Γ acts on the right are defined to be the cohomology groups of the sequence of Γ-homomorphisms

(E.4)

$$\mathrm{Hom}_\Gamma(X_0(\Gamma), V) \xrightarrow{\delta} \mathrm{Hom}_\Gamma(X_1(\Gamma), V) \xrightarrow{\delta} \mathrm{Hom}_\Gamma(X_2(\Gamma), V) \xrightarrow{\delta} \cdots,$$

where Hom_Γ denotes the group of Γ-homomorphisms and $\delta(f) = f \circ \partial$. In more detail, the group $C_0^n(\Gamma, V) = \mathrm{Hom}_\Gamma(X_n(\Gamma), V)$, called the *group of homogeneous n-cochains* of Γ with coefficients in V, can be described alternatively as

(E.5)

$$C_0^n(\Gamma, V) = \left\{ f : \Gamma^{n+1} \longrightarrow V \;\middle|\; \begin{array}{l} f(T_0T,\ T_1T, \ldots, T_nT) = \\ \qquad\qquad f(T_0,\ T_1, \ldots, T_n)|T, \quad \text{and} \\ \\ f(T_0,\ T_1, \ldots, T_n) = 0 \\ \qquad\qquad \text{if} \quad T_i = T_{i-1} \quad \text{for any} \quad i, \end{array} \right\}$$

since a homomorphism $f \in \mathrm{Hom}(X_n(\Gamma), V)$ is determined by its values on the free generators of $X_n(\Gamma)$. The *coboundary homomorphism*

$$\delta : C_0^n(\Gamma, V) \longrightarrow C_0^{n+1}(\Gamma, V)$$

takes an n-cochain $f \in C_0^n(\Gamma, V)$ to the $(n+1)$-cochain $\delta f \in C_0^{n+1}(\Gamma, V)$ defined by

$$\begin{aligned}\delta f(T_0,\, T_1, \ldots, T_{n+1}) &= f\partial(T_0,\, T_1, \ldots, T_{n+1}) \\ &= \sum_{i=0}^{n+1}(-1)^i f(T_0, \ldots, T_{i-1}, T_{i+1} \ldots, T_{n+1}).\end{aligned} \tag{E.6}$$

A cochain f is a *cocycle* if $\delta f = 0$ and is a *coboundary* if $f = \delta g$ for some cochain g; the cocycles are the kernels of the coboundary homomorphisms and form subgroups $Z_0^n(\Gamma, V) \subset C_0^n(\Gamma, V)$ for all $n \geq 0$, while the coboundaries are the images of the coboundary homomorphisms and form subgroups $B_0^n(\Gamma, V) \subset C_0^n(\Gamma, V)$ for all $n > 0$, where the latter definition is extended by setting $B_0^0(\Gamma, V) = 0$. Since $\partial\partial = 0$ every coboundary is a cocycle, or equivalently $B_0^n(\Gamma, V) \subset Z_0^n(\Gamma, V)$; the quotient groups are the *cohomology groups*

$$H^n(\Gamma, V) = \frac{Z_0^n(\Gamma, V)}{B_0^n(\Gamma, V)} \tag{E.7}$$

of the group Γ with coefficients in V for all indices $n \geq 0$.

The cohomology groups can be shown to satisfy the expected naturality properties, although the details will not be included here; in particular any Γ-homomorphism $\phi : V_1 \longrightarrow V_2$ naturally induces homomorphisms

$$\phi^* : H^n(\Gamma, V_1) \longrightarrow H^n(\Gamma, V_2), \tag{E.8}$$

and the compositions of Γ-homomorphisms induce the corresponding compositions of homomorphisms of the cohomology groups. As for any cohomology theory, a critical property is that to any short exact sequence of Γ-homomorphisms

$$0 \longrightarrow V_1 \xrightarrow{\phi} V_2 \xrightarrow{\psi} V_3 \longrightarrow 0$$

there is associated a long exact sequence of cohomology groups

$$\cdots \longrightarrow H^n(\Gamma, V_1) \xrightarrow{\phi} H^n(\Gamma, V_2) \xrightarrow{\psi} H^n(\Gamma, V_3) \xrightarrow{\delta} H^{n+1}(\Gamma, V_1) \longrightarrow \cdots .$$

The proof of the exactness of the cohomology sequence in general will not be included, since it parallels quite closely the proof of the exactness of the corresponding cohomology sequence in sheaf cohomology; but the proof of the exactness in those special cases in which explicit forms of the connecting homomorphism δ are needed will be included, and the proof of the general case can be constructed by following the pattern of the proof of the exactness in those special cases. The naturality properties, the exactness of the cohomology sequence, and the identification of the cohomology groups in a few standard cases can be shown to characterize the cohomology groups intrinsically.

In calculations it is often more convenient to use the group $C^n(\Gamma, V)$ of *inhomogeneous cochains* defined by

$$C^n(\Gamma, V) = \left\{ v : \Gamma^n \longrightarrow V \;\middle|\; \begin{array}{l} v(T_1, \ldots, T_n) = 0 \\ \quad\text{if} \quad T_i = I \quad \text{for any} \quad i \end{array} \right\}. \tag{E.9}$$

To a homogeneous cochain $f \in C_0^n(\Gamma, V)$ there can be associated its inhomogeneous form v defined by

$$v(T_1, \ldots, T_n) = f(I,\ T_n,\ T_{n-1}T_n,\ T_{n-2}T_{n-1}T_n, \ldots, T_1T_2\cdots T_n) \tag{E.10}$$

for any $T_i \in \Gamma$. If $S_i = T_{n-i+1}T_{n-i+2}\cdots T_n$ for $1 \le i \le n$, so that $S_1 = T_n,\ S_2 = T_{n-1}T_n,\ S_3 = T_{n-2}T_{n-1}T_n, \ldots, S_n = T_1T_2\cdots T_n$, then conversely $T_n = S_1$ and $T_{n-i+1} = S_iS_{i-1}^{-1}$ for $2 \le i \le n$; and the homogeneous cochain can be recaptured from its inhomogeneous form by

$$f(I, S_1, S_2, \ldots, S_n) = v(S_nS_{n-1}^{-1},\ S_{n-1}S_{n-2}^{-1}, \ldots, S_2S_1^{-1},\ S_1) \tag{E.11}$$

since the homogeneous cochain f clearly is determined fully just by the values $f(I,\ S_1,\ S_2, \ldots, S_n)$. The condition that $f(I,\ S_1,\ S_2, \ldots, S_n) = 0$ if $S_1 = I$ or $S_i = S_{i-1}$ for any index i in the range $2 \le i \le n$ corresponds to the condition that $v(T_1, \ldots, T_n) = 0$ if $T_i = I$ for any index i in the range $1 \le i \le n$. The mapping that associates to a homogeneous cochain its inhomogeneous form thus is an isomorphism from the group $C_0^n(\Gamma, V)$ of homogeneous cochains to the group $C^n(\Gamma, V)$ of inhomogeneous cochains. In particular to a homogeneous cochain $f \in C_0^0(\Gamma, V)$ there is associated the inhomogeneous form $v = f(I)$, so that $C^0(\Gamma, V) = V$; and the homogeneous form is determined by its inhomogeneous form since $f(T) = f(I)|T = v|T$. If $v \in C^{n-1}(\Gamma, V)$ is the inhomogeneous form of a cochain $f \in C_0^{n-1}(\Gamma, V)$ and if $w \in C^n(\Gamma, V)$ is the inhomogeneous form of the coboundary $\delta f \in C_0^n(\Gamma, V)$ then for any elements $S_i, T_i \in \Gamma$

$$w(T_1, \ldots, T_n) = (\delta f)(I,\ T_n,\ T_{n-1}T_n, \ldots, T_1T_2\cdots T_n)$$

$$\begin{aligned}
&= (\delta f)(I,\ S_1,\ S_2, \ldots, S_n)\\
&= f(S_1,\ S_2, \ldots S_n) + \sum_{i=1}^{n}(-1)^i f(I,\ S_1, \ldots, S_{i-1},\ S_{i+1} \ldots S_n)\\
&= f(I,\ S_2S_1^{-1}, \ldots S_nS_1^{-1})|S_1 + \sum_{i=1}^{n}(-1)^i f(I,\ S_1, \ldots, S_{i-1},\ S_{i+1} \ldots S_n)\\
&= v(S_nS_{n-1}^{-1},\ S_{n-1}S_{n-2}^{-1}, \ldots, S_2S_1^{-1})|S_1\\
&\qquad -v(S_nS_{n-1}^{-1},\ S_{n-1}S_{n-2}^{-1}, \ldots, S_3S_2^{-1},\ S_2)\\
&\qquad +\sum_{i=2}^{n-1}(-1)^i v(S_nS_{n-1}^{-1}, \ldots, S_{i+2}S_{i+1}^{-1},\ S_{i+1}S_{i-1}^{-1},\ S_{i-1}S_{i-2}^{-1}, \ldots, S_1)\\
&\qquad +(-1)^n v(S_{n-1}S_{n-2}^{-1},\ S_{n-2}S_{n-3}^{-1}, \ldots, S_2S_1^{-1},\ S_1)\\
&= v(T_1,\ T_2, \ldots, T_{n-1})|T_n\\
&\qquad -v(T_1,\ T_2, \ldots, T_{n-2},\ T_{n-1}T_n)\\
&\qquad +\sum_{i=2}^{n-1}(-1)^i v(T_1, \ldots, T_{n-i-1},\ T_{n-i}T_{n-i+1},\ T_{n-i+2}, \ldots, T_n)\\
&\qquad +(-1)^n v(T_2, T_3, \ldots, T_n).
\end{aligned}$$

This expresses the coboundary operator in terms of the inhomogeneous cocycles; but it is perhaps clearer to change the index of summation and to rewrite the formula as

$$\begin{aligned}(\delta v)(T_1,\ldots,T_n) &= \\ &= v(T_1,\, T_2,\ldots,T_{n-1})|T_n + (-1)^n v(T_2,\, T_3,\ldots,T_n) \\ &\quad + \sum_{i=1}^{n-1}(-1)^{n+i} v(T_1,\ldots,T_{i-1},\, T_iT_{i+1},\, T_{i+2},\ldots T_n)\end{aligned} \tag{E.12}$$

for any inhomogeneous cochain $v \in C^{n-1}(\Gamma, V)$. For the initial cases

(E.13)

$$\begin{aligned} v \in C^0(\Gamma,V): &\quad (\delta v)(T_1) &=&\; v|T_1 - v, \\ v \in C^1(\Gamma,V): &\quad (\delta v)(T_1,T_2) &=&\; v(T_1)|T_2 + v(T_2) - v(T_1T_2) \\ v \in C^2(\Gamma,V): &\quad (\delta v)(T_1,T_2,T_3) &=&\; v(T_1,T_2)|T_3 - v(T_2,T_3) \\ &&&\quad + v(T_1T_2,\, T_3) - v(T_1,\, T_2T_3). \end{aligned}$$

The *inhomogeneous cocycles* are the cochains $v \in C^n(\Gamma, V)$ such that $\delta v = 0$ and form subgroups $Z^n(\Gamma, V) \subset C^n(\Gamma, V)$, while the *inhomogeneous coboundaries* are the cochains $v \in \delta C^{n-1}(\Gamma, V)$ and form subgroups $B^n(\Gamma, V) \subset C^n(\Gamma, V)$ where $B^0(\Gamma, V) = 0$. The cohomology groups are isomorphic to the quotients

$$H^n(\Gamma,V) \cong \frac{Z^n(\Gamma,V)}{B^n(\Gamma,V)} \quad \text{for } n > 0 \tag{E.14}$$

while

(E.15)

$$H^0(\Gamma,V) \cong Z^0(\Gamma,V) = V^\Gamma$$

where

$$V^\Gamma = \left\{ v \in V \;\middle|\; v|T = v \quad \text{for all } T \in \Gamma \right\} \tag{E.16}$$

is the subgroup of Γ-invariant elements of V. Then for $n = 1$ the group of inhomogeneous 1-cocycles is

$$Z^1(\Gamma,V) = \left\{ v : \Gamma \longrightarrow V \;\middle|\; \begin{array}{l} v(T_1)|T_2 = v(T_1T_2) - v(T_2) \\ \\ v(I) = 0 \end{array} \right\} \tag{E.17}$$

while the subgroup of inhomogeneous 1-coboundaries is

$$B^1(\Gamma,V) = \left\{ v : \Gamma \longrightarrow V \;\middle|\; v(T) = w|T - w \quad \text{for some } w \in V \right\}, \tag{E.18}$$

and for $n = 2$

(E.19)

$$Z^2(\Gamma, V) = \left\{ v : \Gamma \times \Gamma \longrightarrow V \;\middle|\; \begin{array}{l} v(T_1, T_2)|T_3 = \\ \qquad v(T_1, T_2T_3) - v(T_1T_2, T_3) + v(T_2, T_3) \\ \\ v(I, T) = v(T, I) = 0 \end{array} \right\}$$

while

(E.20)

$$B^2(\Gamma, V) = \left\{ v : \Gamma \times \Gamma \longrightarrow V \;\middle|\; \begin{array}{l} v(T_1, T_2) = w(T_1T_2) - w(T_2) - w(T_1)|T_2) \\ \\ \text{where } w : \Gamma \longrightarrow V \text{ and } w(I) = 0. \end{array} \right\}$$

E.2 Example: Trivial Group Action

A particularly simple case is that in which a multiplicative group Γ acts trivially on an additive abelian group V, so that $v|T = v$ for all $T \in \Gamma$ and all $v \in V$, or equivalently $V^\Gamma = V$; by (E.15) then

(E.21) $$H^0(\Gamma, V) \cong V \quad \text{if } \Gamma \text{ acts trivially on } V.$$

Next by (E.17) the inhomogeneous 1-cocycles are mappings $v : \Gamma \longrightarrow V$ such that $v(I) = 0$ and $v(T_1T_2) = v(T_1) + v(T_2)$, so $Z^1(\Gamma, V) = \mathrm{Hom}(\Gamma, V)$; by (E.18) the inhomogeneous 1-coboundaries are trivial, and consequently

(E.22) $$H^1(\Gamma, V) \cong \mathrm{Hom}(\Gamma, V) \quad \text{if } \Gamma \text{ acts trivially on } V.$$

The second cohomology group is equally interesting and possibly less familiar. By (E.19) the group of inhomogeneous two-cocycles consists of those mappings $v : \Gamma \times \Gamma \longrightarrow V$ such that $v(I, T) = v(T, I) = 0$ and

(E.23) $$v(R, S) - v(R, ST) + v(RS, T) - v(S, T) = 0$$

for any $R, S, T \in \Gamma$; and by (E.20) the subgroup $B^2(\Gamma, V)$ of inhomogeneous two-coboundaries consists of those two-cocycles of the form

(E.24) $$v(S, T) = w(S) + w(T) - w(ST)$$

for a mapping $w : \Gamma \longrightarrow V$ such that $w(I) = 0$. While there are natural direct interpretations of the quotient cohomology group, what is quite useful for present purposes is a rather more indirect interpretation of the second cohomology group in terms of a presentation of the group Γ, following H. Hopf[2] and beginning with the following preliminary observation.

[2] H. Hopf, "Fundamentalgruppe und zweite Bettische Gruppe," *Commentarii Mathematici Helvetici* **14**(1941), pp. 257-309. See the historical discussion in MacLane's *Homology*, p.137.

Lemma E.1 *If Γ is a finitely generated free group acting trivially on the right on an abelian group V then $H^2(\Gamma, V) = 0$.*

Proof: By (E.24) it is only necessary to show that for any inhomogeneous cocycle $v \in Z^2(\Gamma, V)$ there is a mapping $w : \Gamma \longrightarrow V$ such that $w(I) = 0$ and

$$w(ST) = w(S) + w(T) - v(S,T) \tag{E.25}$$

for all $S, T \in \Gamma$. If $v \in Z^2(\Gamma, V)$ it follows from (E.23) for $R = S^{-1} = T$ that $v(T, T^{-1}) = v(T^{-1}, T)$. Now choose arbitrary values $w(T_i) \in V$ for a set of free generators T_i of the group Γ, and set $w(T_i^{-1}) = v(T_i, T_i^{-1}) - w(T_i) = v(T_i^{-1}, T_i) - w(T_i)$. Since any element of the free group Γ can be written uniquely as a product of the symbols T_i and T_i^{-1}, equation (E.25) can be used to define $w(S)$ for any element $S \in \Gamma$ if it is demonstrated that $w(T_iT_i^{-1}) = w(T_i^{-1}T_i) = 0$ for each free generator T_i and that the value assigned to $w(RST)$ for any elements $R, S, T \in \Gamma$ is independent of the way in which this triple product is associated. The first follows readily from the way in which $w(T_i^{-1})$ is defined, while the second is a consequence of the cocycle condition (E.23) and can be verified by a straightforward calculation. That suffices for the proof.

The preceding lemma is also true for a finitely generated free group Γ acting trivially on a multiplicative abelian group, such as the group $\mathbb{C}^*$ or any finite subgroup of $\mathbb{C}^*$, rather than on an additive abelian group V; for the argument used only the commutativity of the coefficient group V. For present purposes the principal application of the preceding lemma is to the following general result.

Theorem E.2 (Hopf's Theorem) *If a group Γ acts trivially on the right on an abelian group V and if Γ has a presentation $\Gamma = F/K$, where F is a finitely generated free group and $K \subset F$ is a normal subgroup, then*

$$H^2(\Gamma, V) \cong \frac{\mathrm{Hom}\Big(K/[K,F], V\Big)}{i\Big(\mathrm{Hom}(F,V)\Big)} \tag{E.26}$$

where $[K,F] \subset K$ is the normal subgroup of F generated by commutators $[S,T]$ for $S \in K$ and $T \in F$ and

$$i : \mathrm{Hom}(F,V) \longrightarrow \mathrm{Hom}\Big(K/[K,F], V\Big) \tag{E.27}$$

is the restriction of a homomorphism in $\mathrm{Hom}\,(F,V)$ *to the subgroup K.*

Proof: If $v \in Z^2(\Gamma, V)$ and $p : F \longrightarrow \Gamma$ is the natural quotient mapping then $v_p(S,T) = v(p(S), p(T)) \in Z^2(F, V)$. Since $H^2(F, V) = 0$ by the preceding lemma there is an inhomogeneous 1-cochain $w \in C^1(F, V)$ such that $v_p = \delta w$, hence such that $v_p(S,T) = w(S) + w(T) - w(ST)$ for all $S, T \in F$. If $S \in K$ then $v_p(S,T) = v(p(S), p(T)) = v(I, p(T)) = 0$ and hence $w(ST) = w(S) + w(T)$, and the same of course is true if $T \in K$. One consequence of this observation is that $w|K \in \operatorname{Hom}(K,V)$. Another consequence is that if $S \in K$ and $T \in F$ then $T^{-1}ST \in K$ and $w(S)+w(T) = w(ST) = w(T \cdot T^{-1}ST) = w(T)+w(T^{-1}ST)$; therefore $w(T^{-1}ST) = w(S)$, and consequently $w\big|[K,F] = 0$ so the cochain w restricts to a homomorphism $w|K \in \operatorname{Hom}(K/[K,F],V)$. Since any two cochains that have the same coboundary $v \in Z^2(\Gamma,V)$ differ by an element of $Z^1(F,V) = \operatorname{Hom}(F,V)$, there results a well defined homomorphism

$$p^* : Z^2(\Gamma,V) \longrightarrow \frac{\operatorname{Hom}(K/[K,F],V)}{i(\operatorname{Hom}(F,V))}. \tag{E.28}$$

The kernel of this homomorphism consists of those cocycles $v \in Z^2(\Gamma,V)$ such that $v_p = \delta w$ for a cochain $w \in C^1(F,V)$ which, after modification by the addition of a cocycle in $Z^1(F,V) = \operatorname{Hom}(F,V)$, can be supposed to satisfy $w|K = 0$; but then $w \in C^1(\Gamma,V)$ so $v = \delta w \in B^2(\Gamma,V)$ and hence the kernel of the homomorphism p^* is the subgroup $B^2(\Gamma,V) \subset Z^2(\Gamma,V)$. To conclude the proof it remains only to show that the homomorphism p^* is surjective. Any element $w \in \operatorname{Hom}(K,V)$ can be extended to a mapping $w : F \longrightarrow V$ by choosing a coset decomposition $F = \cup_i K\,T_i$, choosing arbitrary values $w(T_i) \in V$, and setting $w(ST_i) = w(S) + w(T_i)$ for all $S \in K$. If $R \in K$ and $T \in F$ then $T = ST_i$ for some $S \in K$ and $w(RT) = w(RST_i) = w(RS) + w(T_i) = w(R) + w(S) + w(T_i) = w(R) + w(T)$. If $w\big|[K,F] = 0$ as well then for any $S \in K$ necessarily $w(T_iS) = w(S \cdot S^{-1}T_iST_i^{-1} \cdot T_i) = w(S[S^{-1},T_i]T_i) = w(S[S^{-1},T_i]) + w(T_i) = w(S) + w(T_i)$; and as in the preceding argument it is also the case that $w(RT) = w(R) + w(T)$ whenever $R \in F$ and $T \in K$. The expression $v(S,T) = w(S) + w(T) - w(ST)$ is a cocycle $v \in Z^2(F,V)$. If $R \in K$ then $v(RS,T) = w(RS)+w(T)-w(RST) = w(R)+w(S)+w(T)-w(R)-w(ST) = v(S,T)$, so that $v(S,T)$ depends only on the coset of S modulo K; and the same argument shows that $v(S,T)$ also depends only on the coset of T modulo K. That shows that actually $v \in Z^2(\Gamma,V)$ as desired, hence concludes the proof.

In the special cases in which the group Γ has a presentation $\Gamma = F/K$ where F is a free group and $K \subset F$ is actually a subgroup $K \subset [F,F]$ of the commutator subgroup of F, Hopf's Theorem can be restated in a simpler and more explicit form; this is the special case that is of interest for surface groups.

Corollary E.3 *If a group Γ acts trivially on the right on an abelian group V and if Γ has a presentation $\Gamma = F/K$, where F is a finitely generated free group and $K \subset F$ is a normal subgroup such that $K \subset [F,F]$, the natural quotient mapping $p : F \longrightarrow \Gamma$ induces an isomorphism*

$$p^* : H^2(\Gamma,V) \xrightarrow{\cong} \operatorname{Hom}(K/[K,F],V); \tag{E.29}$$

this isomorphism takes the cohomology class represented by an inhomogeneous cocycle $v \in Z^2(\Gamma,V)$ to the homomorphism in $\operatorname{Hom}(K/[K,F],V)$ that is the restriction to $K \subset [F,F]$ of the mapping $w : [F,F] \longrightarrow V$ for which

(E.30)

$$w([S,T]) = v\Big([p(S),p(T)],\, p(T)p(S)\Big) - v\Big(p(S),p(T)\Big) + v\Big(p(T),p(S)\Big)$$

for any $S,T \in F$ *and*

$$w(C_1C_2) = w(C_1) + w(C_2) - v\Big(p(C_1),p(C_2)\Big) \tag{E.31}$$

for any commutators $C_1, C_2 \in [F,F]$.

Proof: If $K \subset [F,F]$ then $w|K = 0$ for any homomorphism $w \in \mathrm{Hom}(F,V)$, so $i(\mathrm{Hom}(F,V)) = 0$ and the isomorphism (E.26) of the preceding theorem takes the simpler form (E.29). In the proof of the preceding theorem the homomorphism $w \in \mathrm{Hom}(K/[K,F],V)$ associated to an inhomogeneous cocycle $v \in Z^2(\Gamma,V)$ is the restriction to $K \subset [F,F]$ of any inhomogeneous cochain $w \in C^1(F,V)$ such that $\delta w = v_p$ for the cocycle $v_p \in Z^2(\Gamma,V)$ defined by $v_p(S,T) = v\big(p(S),p(T)\big)$ for all $S,T \in F$; explicitly

$$v_p(S,T) = w(S) + w(T) - w(ST),$$

which is (E.31) in the special cases in which $S,T \in [F,F]$. Since $w(\mathrm{I}) = 0$ it follows from this for $S = T^{-1}$ that

$$w(T^{-1}) = -w(T) + v_p(T^{-1},T) = -w(T) + v_p(T,T^{-1})$$

for any $T \in F$. It also follows that

$$\begin{aligned}
w([S,T]) &= w\big(ST(TS)^{-1}\big)\\
&= \quad w(ST) + w\big((TS)^{-1}\big) - v_p\big(ST,(TS)^{-1}\big)\\
&= \quad w(ST) - w(TS) + v_p\big(TS,(TS)^{-1}\big) - v_p\big(ST,(TS)^{-1}\big)\\
&= \quad -v_p(S,T) + v_p(T,S) + v_p\big(TS,(TS)^{-1}\big) - v_p\big(ST,(TS)^{-1}\big);
\end{aligned}$$

but upon replacing R by $[S,T]$, S by TS, and T by $(TS)^{-1}$ the cocycle condition (E.23) takes the form

$$0 = v_p\big([S,T],TS\big) - v_p\big(TS,(TS)^{-1}\big) + v_p\big(ST,(TS)^{-1}\big) - v_p\big([S,T],\mathrm{I}\big),$$

so since $v_p\big([S,T],\mathrm{I}\big) = v\big(p([S,T]),\mathrm{I}\big) = 0$ then

$$w\big([S,T]\big) = -v_p(S,T) + v_p(T,S) + v_p\big([S,T],TS\big),$$

which is (E.30). That suffices to conclude the proof.

E.3 Example: Surface Groups

When a Riemann surface is represented as the quotient of its universal covering space $\widetilde{M}$ by the covering translation group Γ, the group Γ acts on the right on the complex vector space $V^p = \Gamma(\widetilde{M}, \mathcal{E}^p)$ of $\mathcal{C}^\infty$ differential forms of degree p on $\widetilde{M}$ by $(\phi|T)(z) = \phi(Tz)$. The differential forms on $\widetilde{M}$ that are invariant under this action of the group Γ are precisely the differential forms on the quotient space M, so in view of (E.15)

$$H^0(\Gamma, V^p) = \Gamma(M, \mathcal{E}^p) \quad \text{for} \quad p = 0, 1, 2. \tag{E.32}$$

To see that

$$H^q(\Gamma, V^p) = 0 \quad \text{for} \quad p = 0, 1, 2 \quad \text{and} \quad q > 0, \tag{E.33}$$

a homogeneous q-cocycle $w(T_0,\, T_1, \ldots, T_q) \in Z_0^q(\Gamma, V^p)$ can be viewed as a $\mathcal{C}^\infty$ differential form $w(T_0,\, T_1, \ldots, T_q; z)$ of degree p on the manifold $\widetilde{M}$ indexed by the elements $T_0, \ldots, T_q \in \Gamma$. For any simply-connected open subset $U \subset M$ the complete inverse image $\pi^{-1}(U) \subset \widetilde{M}$ is the set $\Gamma\tilde{U} = \{T\tilde{U} \mid T \in \Gamma\}$, where $\tilde{U}$ is a connected component of $\pi^{-1}(U)$ and $T_1\tilde{U} \cap T_2\tilde{U} = \emptyset$ whenever $T_1 \neq T_2$. For any C^∞ function $r(z)$ on M with support contained in U, viewed as a Γ-invariant C^∞ function on $\widetilde{M}$ with support contained in $\pi^{-1}(U) = \Gamma\tilde{U}$, the product $r(z)w(T_0, T_1, \ldots, T_q; z)$ also is a $\mathcal{C}^\infty$ differential form on $\widetilde{M}$, so is a homogeneous q-cocycle, and its support is contained in $\Gamma\tilde{U}$. The homogeneous $(q-1)$-cochain in $\Gamma\tilde{U}$ defined by

$$\begin{aligned} v(T_0, T_1, \ldots, T_{q-1}; z) &= r(z)w(I, T_0, T_1, \ldots, T_{q-1}; z) \quad \text{for} \quad z \in U, \\ v(T_0, T_1, \ldots, T_{q-1}; Tz) &= v(T_0T, T_1T, \ldots T_{q-1}T; z) \quad \text{for} \quad z \in U,\ T \neq \mathrm{I}, \end{aligned}$$

can be extended to all of $\widetilde{M}$ by setting it equal to zero outside $\Gamma\tilde{U}$. It is a straightforward calculation to verify that the coboundary of the cochain $v(T_0, T_1, \ldots, T_{q-1}; z)$ is the cocycle $r(z)w(T_0, T_1, \ldots, T_q; z)$, so this cocycle is cohomologous to zero. Since any cocycle can be written as a sum of such cocycles for functions $r_i(z)$ forming a $\mathcal{C}^\infty$ partition of unity on M it follows that any cocycle in $Z^q(\Gamma, V^p)$ is cohomologous to zero as asserted.

From the exact sequence of sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}_c^1 \longrightarrow 0 \tag{E.34}$$

on the universal covering space $\widetilde{M}$, where $\mathcal{E}_c^1$ is the sheaf of closed $\mathcal{C}^\infty$ differential forms of degree 1 and d is exterior differentiation, there follows the exact sheaf cohomology sequence beginning

$$0 \longrightarrow \mathbb{C} \longrightarrow \Gamma(\widetilde{M}, \mathcal{E}^0) \xrightarrow{d} \Gamma(\widetilde{M}, \mathcal{E}_c^1) \xrightarrow{\delta} H^1(\widetilde{M}, \mathbb{C});$$

since M is contractible $H^1(\widetilde{M},\mathbb{C})=0$, so this reduces to the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow V^0 \xrightarrow{d} V_c^1 \longrightarrow 0, \tag{E.35}$$

where $V^0=\Gamma(\widetilde{M},\mathcal{E}^0)$ as before and $V_c^1=\Gamma(\widetilde{M},\mathcal{E}_c^1)$. This is an exact sequence of right Γ-modules, so there results the exact cohomology sequence beginning

(E.36)

$$0 \longrightarrow H^0(\Gamma,\mathbb{C}) \longrightarrow H^0(\Gamma,V^0) \xrightarrow{d} H^0(\Gamma,V_c^1) \xrightarrow{\delta} H^1(\Gamma,\mathbb{C}) \longrightarrow H^1(\Gamma,V^0);$$

and $H^1(\Gamma,V^0)=0$ by (E.33). The coboundary mapping δ in this exact sequence can be described explicitly by a diagram chase through the cochain complex associated to the exact sequence of Γ homomorphisms (E.35), the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(\Gamma,\mathbb{C}) & \longrightarrow & C^0(\Gamma,V^0) & \xrightarrow{d} & C^0(\Gamma,V_c^1) & \longrightarrow & 0\\
 & & \delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \\
0 & \longrightarrow & C^1(\Gamma,\mathbb{C}) & \longrightarrow & C^1(\Gamma,V^0) & \xrightarrow{d} & C^1(\Gamma,V_c^1) & \longrightarrow & 0\\
 & & \delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \\
0 & \longrightarrow & C^2(\Gamma,\mathbb{C}) & \longrightarrow & C^2(\Gamma,V^0) & \xrightarrow{d} & C^2(\Gamma,V_c^1) & \longrightarrow & 0\\
 & & \delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & &
\end{array} \tag{E.37}$$

An inhomogeneous cocycle $\phi\in C^0(\Gamma,V_c^1)$ representing a cohomology class in $H^0(\Gamma,V_c^1)$ is just a closed differential form on the universal covering space $\widetilde{M}$ that is invariant under the covering translation group as in (E.15). This differential form can be written as the exterior derivative $\phi=df$ of a $\mathcal{C}^\infty$ function f on $\widetilde{M}$, and this function in turn is a cochain $f\in C^0(\Gamma,V^0)$ that maps to ϕ under the Γ-homomorphism d in the first line of the commutative diagram (E.37). The coboundary of this cochain is a 1-cochain $\delta f\in C^1(\Gamma,V^0)$, which actually is a cocycle contained in the cochain group $C^1(\Gamma,\mathbb{C})$; so by (E.22) it can be viewed as a homomorphism $p_1(\phi)\in\mathrm{Hom}(\Gamma,\mathbb{C})$. By (E.13) this homomorphism is given explicitly by

$$p_1(\phi)(T)=\delta f(T)=f(Tz)-f(z)=\int_z^{Tz}\phi \tag{E.38}$$

so it is just the usual period class of the closed differential form ϕ; thus the exact cohomology sequence (E.36) reduces to the deRham isomorphism

$$p_1:\frac{\Gamma(M,\mathcal{E}_c^1)}{d\,\Gamma(M,\mathcal{E}^0)} \xrightarrow{\cong} H^1(\Gamma,\mathbb{C})=\mathrm{Hom}(\Gamma,\mathbb{C}) \tag{E.39}$$

where p_1 is the usual period mapping (E.38).

For a possibly more interesting and less familiar result, the next segment of the exact cohomology sequence associated to the exact sequence (E.35) is the exact sequence

$$H^1(\Gamma, V^0) \longrightarrow H^1(\Gamma, V_c^1) \xrightarrow{\delta} H^2(\Gamma, \mathbb{C}) \longrightarrow H^2(\Gamma, V^0);$$

and since $H^1(\Gamma, V^0) = H^2(\Gamma, V^0) = 0$ by (E.33) this reduces to the isomorphism

$$\delta : H^1(\Gamma, V_c^1) \xrightarrow{\cong} H^2(\Gamma, \mathbb{C}). \tag{E.40}$$

The coboundary mapping giving this isomorphism can be described explicitly by another diagram chase through the commutative diagram (E.37). A cocycle in $C^1(\Gamma, V_c^1)$ representing a cohomology class $\theta \in H^1(M, V_c^1)$ is a collection of $\mathcal{C}^\infty$ closed differential 1-forms $\theta(T, z)$ on the universal covering space $\widetilde{M}$ such that $\theta(\mathrm{I}, z) = 0$ and that $\theta(T_1T_2, z) = \theta(T_1, T_2z) + \theta(T_2, z)$, the cocycle condition (E.13). Each of these differential forms can be written as the exterior derivative $\theta(T, z) = df(T, z)$ of a $\mathcal{C}^\infty$ function $f(T, z)$ on $\widetilde{M}$, and this collection of functions is a cochain $f \in C^1(\Gamma, V^0)$ that maps to the cochain θ under the Γ-homomorphism d in the second line of the commutative diagram (E.37). The coboundary of this cochain is a cochain $\delta f \in C^2(\Gamma, V^0)$, which actually is a cocycle contained in the cochain group $C^2(\Gamma, \mathbb{C})$. By (E.13) this cocycle is given explicitly by

$$\delta f(S, T) = f(S, Tz) + f(T, z) - f(ST, z). \tag{E.41}$$

The next segment of the exact cohomology sequence arising from the exact sequence of sheaves (E.34) is

$$H^1(\widetilde{M}, \mathcal{E}^0) \xrightarrow{d} H^1(\widetilde{M}, \mathcal{E}_c^1) \xrightarrow{\delta} H^2(\widetilde{M}, \mathbb{C}) \longrightarrow H^2(\widetilde{M}, \mathcal{E}^0);$$

and $H^1(\widetilde{M}, \mathcal{E}^0) = H^2(\widetilde{M}, \mathcal{E}^0) = 0$ since $\mathcal{E}^0$ is a fine sheaf while $H^2(\widetilde{M}, \mathbb{C}) = 0$ since the universal covering space $\widetilde{M}$ is contractible, so this exact sequence reduces to the identity

$$H^1(\widetilde{M}, \mathcal{E}_c^1) = 0. \tag{E.42}$$

The exact cohomology sequence arising from the exact sequence of sheaves

$$0 \longrightarrow \mathcal{E}_c^1 \longrightarrow \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \longrightarrow 0 \tag{E.43}$$

on the universal covering space $\widetilde{M}$ includes the segment

$$0 \longrightarrow \Gamma(\widetilde{M}, \mathcal{E}_c^1) \longrightarrow \Gamma(\widetilde{M}, \mathcal{E}^1) \xrightarrow{d} \Gamma(\widetilde{M}, \mathcal{E}^2) \xrightarrow{\delta} H^1(\widetilde{M}, \mathcal{E}_c^1),$$

in which $H^1(\widetilde{M}, \mathcal{E}_c^1) = 0$ by (E.42); so this amounts to the exact sequence of Γ-homomorphisms

$$0 \longrightarrow V_c^1 \longrightarrow V^1 \xrightarrow{d} V^2 \longrightarrow 0, \tag{E.44}$$

from which there follows the exact cohomology sequence containing the segment

$$0 \longrightarrow H^0(\Gamma, V_c^1) \longrightarrow H^0(\Gamma, V^1) \xrightarrow{d} H^0(\Gamma, V^2) \xrightarrow{\delta} H^1(\Gamma, V_c^1) \longrightarrow H^1(\Gamma, V^1).$$

Since $H^1(\Gamma, V^1) = 0$ by (E.33) while $H^0(\Gamma, V^q) = \Gamma(M, \mathcal{E}^q)$ by (E.32) this reduces to the isomorphism

$$\text{(E.45)} \qquad \delta : \frac{\Gamma(M, \mathcal{E}^2)}{d\,\Gamma(M, \mathcal{E}^1)} \xrightarrow{\cong} H^1(\Gamma, V_c^1).$$

The coboundary mapping giving this isomorphism can be described explicitly by a diagram chase through the cochain complex associated to the exact sequence of Γ-homomorphisms (E.44), the commutative diagram

$$\text{(E.46)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & C^0(\Gamma, V_c^1) & \longrightarrow & C^0(\Gamma, V^1) & \xrightarrow{d} & C^0(\Gamma, V^2) & \longrightarrow & 0 \\ & & \delta\big\downarrow & & \delta\big\downarrow & & \delta\big\downarrow & & \\ 0 & \longrightarrow & C^1(\Gamma, V_c^1) & \longrightarrow & C^1(\Gamma, V^1) & \xrightarrow{d} & C^1(\Gamma, V^2) & \longrightarrow & 0. \end{array}$$

An inhomogeneous cocycle $\phi \in C^0(\Gamma, V^2)$ representing a cohomology class in $H^0(\Gamma, V^2)$ is a $\mathcal{C}^\infty$ differential 2-form on the universal covering space $\widetilde{M}$ that is invariant under the covering translation group, as in (E.15). Since ϕ is automatically closed it can be written as the exterior derivative $\phi = d\psi$ of a $\mathcal{C}^\infty$ differential 1-form ψ on $\widetilde{M}$, and this differential form in turn is a cochain $\psi \in C^0(\Gamma, V^1)$ that maps to ϕ under the Γ-homomorphism d in the first line of the commutative diagram (E.46). The coboundary of this cochain is a 1-cochain $\delta\psi \in C^1(\Gamma, V^1)$, which actually is a cocycle contained in the cochain group $C^1(\Gamma, V_c^1)$ and represents the cohomology class that is the image of the class ϕ under the isomorphism (E.45); by (E.13) this cocycle is explicitly

$$\text{(E.47)} \qquad \delta(\phi)(T) = \delta\psi(T) = \psi(Tz) - \psi(z).$$

Combining the isomorphisms (E.40) and (E.45) yields the isomorphism

$$\text{(E.48)} \qquad p_2 : \frac{\Gamma(M, \mathcal{E}^2)}{d\,\Gamma(M, \mathcal{E}^1)} \xrightarrow{\cong} H^2(\Gamma, \mathbb{C}).$$

Combining the explicit descriptions of the isomorphisms (E.40) and (E.45) given in (E.41) and (E.47) shows that the isomorphism p_2 associates to the differential 2-form ϕ on the surface M the cohomology class in $H^2(\Gamma, \mathbb{C})$ represented by the cocycle $p_2(\phi) \in Z^2(\Gamma, \mathbb{C})$ for which

$$\text{(E.49)} \quad \begin{aligned} p_2(\phi)(S, T) &= f(S, Tz) + f(T, z) - f(ST, z) \\ &\qquad \text{where} \quad \phi = d\psi \ \text{ and } \ \psi(Tz) - \psi(z) = df(T, z). \end{aligned}$$

This too can be viewed as a period isomorphism, extending (E.39) to the next higher dimension.

The more familiar period mapping of course is the isomorphism

$$p : \frac{\Gamma(M,\mathcal{E}^2)}{d\,\Gamma(M,\mathcal{E}^1)} \xrightarrow{\cong} \mathbb{C} \tag{E.50}$$

that associates to any $\mathcal{C}^\infty$ differential form $\phi \in \Gamma(M,\mathcal{E}^2)$ its period $p(\phi) = \int_M \phi$; that this mapping is an isomorphism is just the classical deRham theorem. Combining the two isomorphisms (E.48) and (E.50) leads to an isomorphism

$$p \cdot p_2^{-1} : H^2(\Gamma,\mathbb{C}) \xrightarrow{\cong} \mathbb{C}, \tag{E.51}$$

which can be described quite explicitly in terms of a marking of the surface M. As discussed in Appendix D.1, a marking of a compact Riemann surface M of genus $g > 0$ is a representation of M as a sphere with g handles together with the choice of a base point $z_0 \in \widetilde{M}$ in the universal covering space of M and a collection of $2g$ simple closed paths $\alpha_i, \beta_i \subset M$ as in Figure D.1. When the paths α_i and β_i are lifted to simple paths $\tilde\alpha_i, \tilde\beta_i \subset \widetilde{M}$ beginning at the base point $z_0 \in \widetilde{M}$ their end points are $A_i z_0$ and $B_i z_0$, where $A_i, B_i \in \Gamma$ are covering translations corresponding to the homotopy classes of the paths $\alpha_i \beta_i$; and the surface M can be recaptured from the fundamental domain $\Delta \subset \widetilde{M}$ bounded by pairs of translates of the paths $\tilde\alpha_i, \tilde\beta_i$ by identifying the boundary paths as in Figure D.2. The covering translations A_i, B_i are generators of the group Γ and are subject to the single relation $C_1 \cdot C_2 \cdots C_g = \mathrm{I}$ for the commutators $C_i = [A_i, B_i]$. Alternatively the group Γ has a presentation as the quotient $\Gamma = F/K$ of the free group F on $2g$ generators $\tilde A_i, \tilde B_i$, representing the generators A_i, B_i of Γ, modulo the normal subgroup $K \subset F$ generated by the element $\tilde C = \tilde C_1 \cdot \tilde C_2 \cdots \tilde C_g$ for the commutators $\tilde C_i = [\tilde A_i, \tilde B_i]$.

Theorem E.4 *In terms of the presentation of the covering translation group Γ of a compact Riemann surface M of genus $g > 0$ derived from a marking of M, the image under the isomorphism $p \cdot p_2^{-1} : H^2(\Gamma,\mathbb{C}) \longrightarrow \mathbb{C}$ of the cohomology class $v \in H^2(\Gamma,\mathbb{C})$ represented by a cocycle $v(S,T) \in Z^2(\Gamma,\mathbb{C})$ is the complex number*

$$\begin{aligned} p \cdot p_2^{-1}(v) \quad = \quad & \sum_{i=1}^{g} \Big(v(C_1 \cdots C_{i-1},\, C_i) \\ & -v(C_i,\, B_i A_i) + v(A_i,\, B_i) - v(B_i,\, A_i) \Big). \end{aligned} \tag{E.52}$$

Proof: It follows from the isomorphism (E.48) in the explicit form (E.49) that, after replacing the cocycle $v(S,T)$ by a cohomologous cocycle if necessary, it can be assumed that $v(S,T) = f(S,Tz) + f(T,z) - f(ST,z)$ for some $\mathcal{C}^\infty$ functions $f(T,z)$ on $\widetilde{M}$ indexed by covering translations $T \in \Gamma$, where $df(T,z) = \psi(Tz) - \psi(z)$ and $d\psi = \phi$ is a differential form $\phi \in \Gamma(M,\mathcal{E}^2)$; thus $v = p_2(\phi)$ and consequently $p \cdot p_2^{-1}(v) = p(\phi) = \int_M \phi$. The functions $f(T,z)$ can be modified by a suitable additive constant so that $f(T, z_0) = 0$ for each $T \in \Gamma$, which amounts to replacing the cocycle $v(S,T)$ by yet another cohomologous cocycle; and then $v(S,T) = f(S, Tz_0)$. By Stokes's Theorem for the region Δ in Figure D.2 it follows that

$$\begin{aligned}
p \cdot p_2^{-1}(v) \quad &= \quad \int_M \phi \quad = \quad \int_\Delta d\psi \quad = \quad \int_{\partial\Delta} \psi \\
&= \sum_{i=1}^{g} \int_{C_1 \cdots C_{i-1}\tilde{\alpha}_i - C_1 \cdots C_i B_i \tilde{\alpha}_i} \psi(z) \\
&\qquad + \sum_{i=1}^{g} \int_{C_1 \cdots C_{i-1} A_i \tilde{\beta}_i - C_1 \cdots C_i \tilde{\beta}_i} \psi(z) \\
&= \sum_{i=1}^{g} \int_{\tilde{\alpha}_i} \big(\psi(C_1 \cdots C_{i-1} z) - \psi(C_1 \cdots C_i B_i z)\big) \\
&\qquad + \sum_{i=1}^{g} \int_{\tilde{\beta}_i} \big(\psi(C_1 \cdots C_{i-1} A_i z) - \psi(C_1 \cdots C_i z)\big) \\
&= \sum_{i=1}^{g} \int_{\tilde{\alpha}_i} \big(df(C_1 \cdots C_{i-1},\, z) - df(C_1 \cdots C_i B_i,\, z)\big) \\
&\qquad + \sum_{i=1}^{g} \int_{\tilde{\beta}_i} \big(df(C_1 \cdots C_{i-1} A_i,\, z) - df(C_1 \cdots C_i,\, z)\big) \\
&= \sum_{i=1}^{g} \big(f(C_1 \cdots C_{i-1},\, A_i z_0) - f(C_1 \cdots C_i B_i,\, A_i z_0)\big) \\
&\qquad + \sum_{i=1}^{g} \big(f(C_1 \cdots C_{i-1} A_i,\, B_i z_0) - f(C_1 \cdots C_i,\, B_i z_0)\big) \\
&= \sum_{i=1}^{g} \big(v(C_1 \cdots C_{i-1},\, A_i) - v(C_1 \cdots C_i B_i,\, A_i)\big) \\
&\qquad + \sum_{i=1}^{g} \big(v(C_1 \cdots C_{i-1} A_i,\, B_i) - v(C_1 \cdots C_i,\, B_i)\big).
\end{aligned}$$

By using the cocycle condition

$$v(T_1 T_2,\, T_3) = v(T_1,\, T_2 T_3) - v(T_1,\, T_2) + v(T_2, T_3)$$

following from (E.13) and noting that

$$\begin{aligned}
v(C_1 \ldots C_{i-1} \cdot C_i,\, B_i A_i) \quad &= \quad v(C_1 \ldots C_{i-1},\, A_i B_i) \\
&\qquad - v(C_1 \ldots C_{i-1},\, C_i) + v(C_i,\, B_i A_i)
\end{aligned}$$

this can be rewritten

$$\begin{aligned} p\cdot p_2^{-1}(v) &= \sum_{i=1}^{g}\Big(v(C_1\cdots C_{i-1},\,A_iB_i)-v(C_1\cdots C_i,\,B_iA_i)\\ &\qquad +v(A_i,\,B_i)-v(B_i,\,A_i)\Big)\\ &= \sum_{i=1}^{g}\Big(v(C_1\cdots C_{i-1},\,C_i)\\ &\qquad -v(C_i,\,B_iA_i)+v(A_i,\,B_i)-v(B_i,\,A_i)\Big), \end{aligned}$$

and that suffices to conclude the proof.

The explicit form (E.52) of the isomorphism $p{\cdot}p_2^{-1}$ can be interpreted alternatively in terms of Hopf's Theorem in the simplified form given in Corollary E.3. A cohomology class $v\in H^2(\Gamma,\mathbb{C})$ is determined uniquely by its image $p^*(v)$ under the isomorphism

$$p^*: H^2(\Gamma,\mathbb{C})\longrightarrow \mathrm{Hom}(K/[K,F],\mathbb{C}) \tag{E.53}$$

of (E.29); and since the group K is generated by the single element $\tilde C\in K$ the image homomorphism $p^*(v)$ in turn is determined uniquely by its value $p^*(v)(\tilde C)\in\mathbb{C}$ on this generator.

Corollary E.5 *In terms of the presentation of the covering translation group Γ of a compact Riemann surface M of genus $g>0$ derived from a marking of M, for which $\Gamma\cong F/K$ where $K\subset F$ is the normal subgroup of the free group F generated by a single commutator $\tilde C\in F$, the image under the isomorphism p^* of the cohomology class $v\in H^2(\Gamma,\mathbb{C})$ represented by a cocycle $v(S,T)\in Z^2(\Gamma,\mathbb{C})$ is the homomorphism $p^*(v)\in\mathrm{Hom}(K/[K,F],\mathbb{C})$ characterized by*

$$p^*(v)(\tilde C) = -p\cdot p_2^{-1}(v) \tag{E.54}$$

where $p\cdot p_2^{-1}(v)$ has the explicit form as in the preceding theorem.

Proof: If $v(S,T)\in Z^2(\Gamma,\mathbb{C})$ is a cocycle representing the cohomology class $v\in H^2(\Gamma,\mathbb{C})$ then by Corollary E.3 the image homomorphism $p^*(v)$ is the restriction to $K\subset[F,F]$ of the mapping $w:[F,F]\longrightarrow\mathbb{C}$ determined by the cocycle $v(S,T)$ through the two conditions (E.30) and (E.31). From (E.30) it follows that

$$w(\tilde C_i)=w([\tilde A_i,\tilde B_i])=v(C_i,B_iA_i)-v(A_i,B_i)+v(B_i,A_i).$$

and from (E.31) it follows by induction on g that

$$w(\tilde C_1\cdots\tilde C_g)=\sum_{i=1}^{g}\Big(w(\tilde C_i)-v(C_1\cdots C_{i-1},\;C_i)\Big)$$

with the understanding that $v(C_1 \cdots C_{i-1},\ C_i) = 0$ if $i = 1$. Combining these two observations shows that

$$\begin{aligned} p^*(v)(\tilde{C}) &= w(\tilde{C}) = w(\tilde{C}_1\tilde{C}_2 \cdots \tilde{C}_g) \\ &= \sum_{i=1}^{g} \Big(v(C_i,\ B_iA_i) - v(A_i,\ B_i) + v(B_i,\ A_i) \\ &\qquad\qquad - v(C_1 \cdots C_{i-1},\ C_i) \Big), \end{aligned}$$

so in view of (E.52) it follows that $p^*(\tilde{C}) = -p \cdot p_2^{-1}$, which suffices for the proof.

The negative sign in (E.54) is yet another instance of conventional choices made in interpreting abstract cohomology groups in concrete terms, such as the convention that associates to a divisor $\mathfrak{d}$ the holomorphic line bundle $\zeta_{\mathfrak{d}} = \delta(-\mathfrak{d})$ as discussed on page 6. Some useful properties of the cohomology of surface groups follow from these various results about the period classes of closed differential forms on a compact Riemann surface.

Theorem E.6 *If M is a compact Riemann surface of genus $g > 0$ with the covering translation group Γ, and if $v \in H^2(\Gamma, \mathbb{C})$ is a cohomology class such that $p \cdot p_2^{-1}(v) \in \mathbb{Z}$, then the cohomology class v can be represented by an integral cocycle $v(S, T) \in Z^2(\Gamma, \mathbb{Z})$.*

Proof: Choose a marking of the surface M, in terms of which the covering translation group Γ can be presented as the quotient $\Gamma = F/K$ of a free grop F modulo the normal subgroup $K \subset F$ generated by a single commutator $\tilde{C} \in K \subset [F, F]$ as before. For any cohomology class $v \in H^2(\Gamma, \mathbb{C})$ the image $p^*(v) \in \mathrm{Hom}(K/[K, F], \mathbb{Z})$ under the isomorphism (E.53) is the homomorphism that is characterized by $p^*(v)(\tilde{C}) = -p \cdot p_2^{-1}(v)$ as in Theorem E.5; therefore if $p \cdot p_2^{-1}(v) \in \mathbb{Z}$ then $p^*(v)(\tilde{C}) \in \mathbb{Z}$, and since K is generated by the single commutator $\tilde{C}$ it follows further that $p^*(v) \in \mathrm{Hom}(K/[K, F], \mathbb{Z})$. Corollary E.3 for the case that $V = \mathbb{Z}$ is the isomorphism $p^* : H^2(\Gamma, \mathbb{Z}) \longrightarrow \mathrm{Hom}(K/[K, F], \mathbb{Z})$, and consequently $p^*(v)$ is the image of an integral cohomology class so the cohomology class v can be represented by an integral cocycle, which suffices to conclude the proof.

A special case of a general construction in the cohomology of groups plays a role in the study of surface groups. To any two inhomogeneous 1-cocycles $v_i \in Z^1(\Gamma, \mathbb{C}) = \mathrm{Hom}(\Gamma, \mathbb{C})$ of the group Γ acting trivially on the complex numbers $\mathbb{C}$ there can be associated the 2-cocycle $v_1 \cup v_2 \in C^2(\Gamma, \mathbb{C})$ defined by

$$(v_1 \cup v_2)(T_1, T_2) = v_1(T_1) \cdot v_2(T_2) \quad \text{for all} \quad T_1, T_2 \in \Gamma; \tag{E.55}$$

that it is a cocycle can be demonstrated by noting by (E.13) that

$$\begin{aligned}\delta v(T_1,T_2,T_3) &= v(T_1,T_2)-v(T_2,T_3)+v(T_1T_2,T_3)-v(T_1,T_2T_3)\\ &= v_1(T_1)v_2(T_2)-v_1(T_2)v_2(T_3)+\Big(v_1(T_1)+v_1(T_2)\Big)v_2(T_3)\\ &\qquad -v_1(T_1)\Big(v_2(T_2)+v_2(T_3)\Big)\\ &= 0.\end{aligned}$$

The cohomology class of this cocycle is called the *cup product* of the cohomology classes $v_i \in H^1(\Gamma,\mathbb{C})$ and also is denoted by $v_1 \cup v_2$. This operation is a reflection in the cohomology of groups of the exterior product of differential forms, in the following sense.

Theorem E.7 *If $\phi_i \in \Gamma(M,\mathcal{E}^1_c)$ are closed differential 1-forms on a compact Riemann surface M of genus $g>0$ the period period class $p_2(\phi_1\wedge\phi_2)\in H^2(\Gamma,\mathbb{C})$ of their exterior product can be expressed in terms of the period classes $p_1(\phi_i)\in H^1(\Gamma,\mathbb{C})$ of theses 1-forms by*

$$p_2(\phi_1\wedge\phi_2)=p_1(\phi_1)\cup p_1(\phi_2). \tag{E.56}$$

Proof: If $\phi_i(z)=df_i(z)$ for some functions $f_i\in\Gamma(\widetilde{M},\mathcal{E}^0)$ then as in (E.38) the period classes of these differential forms are represented by the cocycles $v_i(T)=f_i(Tz)-f_i(z)$ for any covering translation $T\in\Gamma$. The product form $\phi(z)=\phi_1(z)\wedge\phi_2(z)$ can be written as the derivative $\phi(z)=d\psi(z)$ of the differential form $\psi(z)=f_1(z)\phi_2(z)$ on $\widetilde{M}$; and $\psi(Tz)-\psi(z)=v_1(T)\cdot\phi_2(z)=df(T,z)$ for the function $f(T,z)=v_1(T)\cdot f_2(z)$. It then follows from (E.49) that the period class of the differential form ϕ is represented by the cocycle

$$\begin{aligned}v(S,T) &= v_1(S)f_2(Tz)+v_1(T)\cdot f_2(z)-v_1(ST)\cdot f_2(z)\\ &= v_1(S)\cdot\big(v_1(T)+f_2(z)\big)+v_1(T)\cdot f_2(z)-v_1(ST)\cdot f_2(z)\\ &= v_1(S)\cdot v_1(T),\end{aligned}$$

and that suffices to conclude the proof.

The factors of automorphy describing holomorphic line bundles over compact Riemann surfaces can be interpreted in terms of the cohomology of groups. The exact sequence of sheaves

$$0\longrightarrow\mathbb{Z}\xrightarrow{\iota}\mathcal{O}\xrightarrow{e}\mathcal{O}^*\longrightarrow 0 \tag{E.57}$$

over the universal covering space $\widetilde{M}$ of a compact Riemann surface M of genus $g>0$ as in (1.38), in which $e(f)=\exp 2\pi i f$ for any $f\in\mathcal{O}$, leads to an exact cohomology sequence beginning

$$0\longrightarrow\Gamma(\widetilde{M},\mathbb{Z})\xrightarrow{\iota}\Gamma(\widetilde{M},\mathcal{O})\xrightarrow{e}\Gamma(\widetilde{M},\mathcal{O}^*)\longrightarrow 0, \tag{E.58}$$

since $H^1(\widetilde{M},\mathbb{Z})=0$ for the simply connected surface $\widetilde{M}$; and there is the natural identification $\Gamma(\widetilde{M},\mathbb{Z})\cong\mathbb{Z}$. When the covering translation group Γ of the surface M acts on the right on these groups of cross-sections by setting $(f|T)(z) = f(Tz)$ the exact sequence (E.58) can be viewed as an exact sequence of Γ-homomorphisms; and it leads to an exact group cohomology sequence containing the segment

(E.59)

$$H^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{O})\big) \xrightarrow{e} H^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{O}^*)\big) \xrightarrow{\delta} H^2\big(\Gamma, \mathbb{Z})\big) \xrightarrow{\iota} H^2\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{O})\big).$$

An inhomogeneous 1-cocycle $\lambda \in Z^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{O}^*)\big)$ is a collection of holomorphic and nowhere vanishing functions $\lambda(T, z)$ on $\widetilde{M}$ such that $\lambda(\mathrm{I}, z) = 1$ and $\lambda(ST, z) = \lambda(S, Tz)\lambda(T, z)$, the multiplicative form of the cocycle condition (E.17); hence it is a holomorphic factor of automorphy for the action of the covering translation group Γ. A 1-coboundary is a 1-cocyle $\lambda(T, z)$ of the form $\lambda(T, z) = h(Tz)/h(z)$ for a holomorphic nowhere vanishing function $h(z)$ on $\widetilde{M}$, the multiplicative form of (E.18); hence it is a holomorphically trivial holomorphic factor of automorphy. Therefore the cohomology group $H^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{O}^*)\big)$ can be identified with the group of holomorphic equivalence classes of holomorphic factors of automorphy for the covering translation group of the surface M, which by Theorem 3.11 in turn can be identified with the group of holomorphic equivalence classes of holomorphic line bundles over M. From the usual chase through the diagram of cochain groups associated to the exact sequence of Γ-homomorphisms (E.58), the diagram analogous to (E.37), it follows that the coboundary mapping

$$\delta : H^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{O}^*)\big) \longrightarrow H^2(\Gamma, \mathbb{Z}) \tag{E.60}$$

associates to the cohomology class represented by a factor of automorphy $\lambda(T, z)$ the cohomology class $\delta(\lambda) \in H^2(\Gamma, \mathbb{Z})$ represented by the cocycle $\delta(\lambda)(S, T) \in Z^2(\Gamma, \mathbb{Z})$ given explicitly by

$$\delta(\lambda)(S, T) = f(S, Tz) + f(T, z) - f(ST, z) \tag{E.61}$$

where $\lambda(T, z) = \exp 2\pi i f(T, z)$; this cohomology class is called the *characteristic class* of the factor of automorphy $\lambda(T, z)$. Parallel constructions can be carried out for the sheaf $\mathcal{C}$ of germs of continuous functions and the sheaf $\mathcal{E}$ of germs of $\mathcal{C}^\infty$ functions on $\widetilde{M}$; so the cohomology group $H^1(\Gamma, \Gamma(\widetilde{M}, \mathcal{C}^*))$ can be identified and the group of equivalence classes of continuous factors of automorphy while the group $H^1(\Gamma, \Gamma(\widetilde{M}, \mathcal{E}^*))$ can be identified with the group of equivalence classes of $\mathcal{C}^\infty$ factors of automorphy. In these cases the analogues of the exact sequence (E.59) reduce to the isomorphisms

$$\begin{aligned} \delta : H^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{C}^*)\big) \quad &\xrightarrow{\cong} \quad H^2(\Gamma, \mathbb{Z}), \\ \delta : H^1\big(\Gamma, \Gamma(\widetilde{M}, \mathcal{E}^*)\big) \quad &\xrightarrow{\cong} \quad H^2(\Gamma, \mathbb{Z}), \end{aligned}$$

since $H^i(\Gamma, \Gamma(\widetilde{M}, \mathcal{E})) = 0$ for $i > 0$ by (E.33) and $H^i(\Gamma, \Gamma(\widetilde{M}, \mathcal{C})) = 0$ for $i > 0$ by the corresponding argument. Thus the characteristic class of a holomorphic factor of automorphy provides a complete description of the continuous or $\mathcal{C}^\infty$ equivalence class of that factor of automorphy.

Theorem E.8 *If $\lambda(T,z)$ is a holomorphic factor of automorphy for the covering translation group Γ of a compact Riemann surface M of genus $g>0$ then the image $p\cdot p_2^{-1}\big(\delta(\lambda)\big)\in\mathbb{Z}$ of the characteristic class $\delta(\lambda)\in H^2(\Gamma,\mathbb{Z})$ of the factor of automorphy $\lambda(T,z)$ is equal to the characteristic class of the holomorphic line bundle represented by that factor of automorphy.*

Proof: A meromorphic relatively automorphic function $f(z)$ for the factor of automorphy $\lambda(T,z)\in H^1(\Gamma,\Gamma(\widetilde{M},\mathcal{O}^*))$ corresponds to a meromorphic cross-section of the holomorphic line bundle λ represented by that factor of automorphy, as in Theorem 3.11; so by definition (1.14) the characteristic class of the line bundle λ is the integer $\deg\mathfrak{d}(f)$, where $\mathfrak{d}(f)$ is the divisor of the function $f(z)$ on the Riemann surface M. If M is identified with the quotient $M=\widetilde{M}/\Gamma$ of its universal covering space $\widetilde{M}$ by the group Γ of covering translations and if $\Delta\subset\widetilde{M}$ is a fundamental domain for the action of Γ on $\widetilde{M}$ as in the discussion of marked surfaces in Appendix D.1, where Δ is chosen so that there are no zeros or poles of the function $f(z)$ on its boundary $\partial\Delta$, then

$$\deg\mathfrak{d}(f)=\frac{1}{2\pi i}\int_{\partial\Delta} d\,\log f(z) \tag{E.62}$$

by the residue theorem. Now the factor of automorphy can be written $\lambda(T,z)=\exp 2\pi i f(T,z)$ for some holomorphic functions $f(T,z)$ on $\widetilde{M}$; and its characteristic is the cohomology class represented by the 2-cocycle

$$\delta(\lambda)(S,T)=f(S,Tz)+f(T,z)-f(ST,z)\in\mathbb{Z}^2(\Gamma,\mathbb{Z}).$$

If $\psi(z)$ is any $\mathcal{C}^\infty$ differential form on $\widetilde{M}$ such that $\psi(Tz)-\psi(z)=d\,f(T,z)$ for all $T\in\Gamma$ it follows from (E.49) that the cocycle $\delta(\lambda)(S,T)$ represents the period class $p_2(\phi)$ of the differenteial form $\phi=d\,\psi$, and consequently that

$$p\cdot p_2^{-1}\big(\delta(\lambda)\big)=p(\phi)=\int_M\phi. \tag{E.63}$$

The absolute value $|f(z)|^2$ of the relatively automorphic function $f(z)$ is a well defined positive $\mathcal{C}^\infty$ function on $\widetilde{M}$ except at the zeros and poles of the meromorphic function $f(z)$, and $|f(Tz)|^2=|\lambda(T,z)|^2\cdot|f(z)|^2$ for all covering translations $T\in\Gamma$. The function $|f(z)|^2$ can be modified in small open neighborhoods of the zeros or poles of the meromorphic function $f(z)$ to yield a strictly positive $\mathcal{C}^\infty$ function $r(z)$ on $\widetilde{M}$ that is equal to $|f(z)|^2$ except near these zeros or poles, in particular that is equal to $|f(z)|^2$ on the boundary $\partial\Delta$, and that satisfies $r(Tz)=|\lambda(T,z)|^2 r(z)$ for all covering translations $T\in\Gamma$. The $\mathcal{C}^\infty$ differential form $\psi(z)=\frac{1}{2\pi i}\partial\log r(z)$ then satisfies

$$\begin{aligned}\psi(Tz) &= \frac{1}{2\pi i}\partial\log r(Tz) = \frac{1}{2\pi i}\partial\Big(\log r(z)+\log\lambda(T,z)+\log\overline{\lambda(T,z)}\Big)\\ &= \psi(z)+d\log\lambda(T,z),\end{aligned}$$

since $\partial \log \lambda(T,z) = d \log \lambda(T,z)$ and $\partial \log \overline{\lambda(T,z)} = 0$; consequently if $\phi(z) = d\psi(z)$ it follows from (E.62) and (E.63) that

$$\begin{aligned}
p \cdot p_2^{-1}\big(\delta(\lambda)\big) &= \int_M \phi \quad = \quad \int_\Delta d\psi \quad = \quad \int_{\partial\Delta} \psi \\
&= \frac{1}{2\pi i}\int_{\partial\Delta} \partial \log r(z) \quad = \quad \frac{1}{2\pi i}\int_{\partial\Delta} d\log |f(z)|^2 \\
&= \frac{1}{2\pi i}\int_{\partial\Delta} d\log f(z) \quad = \quad \deg \mathfrak{d}(f),
\end{aligned}$$

since $r(z) = |f(z)|^2$ on $\partial\Delta$, and that suffices to conclude the proof.

Appendix F

Complex Tori

F.1 Period Matrices

A *lattice subgroup* $\mathcal{L} \subset \mathbb{C}^g$ in the space of g complex variables is an additive subgroup generated by $2g$ vectors in $\mathbb{C}^g$ that are linearly independent over the real numbers. These $2g$ vectors viewed as column vectors of length g can be taken as the columns of a $g \times 2g$ complex matrix Ω, and $\mathcal{L} = \Omega\mathbb{Z}^{2g} \subset \mathbb{C}^g$ also is called the *lattice subgroup described by the period matrix* Ω and is denoted by $\mathcal{L} = \mathcal{L}(\Omega)$. A complex $g \times 2g$ matrix is called a *period matrix*, and a period matrix with columns that are linearly independent over the real numbers is called a *nonsingular period matrix*. To a $g \times 2g$ period matrix Ω there can be associated the $2g \times 2g$ matrix $\binom{\Omega}{\overline{\Omega}}$, called the *full period matrix* associated to the period matrix Ω.

Lemma F.1 *A $g \times 2g$ period matrix Ω is a nonsingular period matrix if and only if its associated $2g \times 2g$ full period matrix is an invertible square matrix.*

Proof: If the column vectors of the period matrix Ω are linearly dependent over the real numbers there is a nontrivial real column vector $x \in \mathbb{R}^{2g}$ such that $\Omega \cdot x = 0$. Since the vector x is real $\overline{\Omega} \cdot x = 0$ as well, so the $2g \times 2g$ complex matrix $\binom{\Omega}{\overline{\Omega}}$ is singular. Conversely if the square matrix $\binom{\Omega}{\overline{\Omega}}$ is singular there is a nontrivial complex column vector $z = x + iy \in \mathbb{C}^{2g}$ such that $\Omega \cdot z = \overline{\Omega} \cdot z = 0$; then $\Omega \cdot \overline{z} = \overline{\overline{\Omega} \cdot z} = 0$ as well, so $\Omega \cdot x = \Omega \cdot y = 0$, and since not both $x = 0$ and $y = 0$ the columns of Ω must be linearly dependent over the real numbers. That suffices to conclude the proof.

Theorem F.2 *Two nonsingular $g \times 2g$ period matrices Ω_1 and Ω_2 describe the same lattice subgroup $\mathcal{L}(\Omega_1) = \mathcal{L}(\Omega_2)$ if and only if $\Omega_1 = \Omega_2\, Q^{-1}$ for some matrix $Q \in \mathrm{Gl}(2g, \mathbb{Z})$.*

Proof: The group $\mathrm{Gl}(2g, \mathbb{Z})$ of $2g \times 2g$ integral matrices with integral inverses can be characterized as the set of $2g \times 2g$ complex matrices Q such that $Q\mathbb{Z}^{2g} =$

$\mathbb{Z}^{2g}$, since a complex matrix Q clearly has this property if and only if it has integral entries and an integral inverse. If $\Omega_2 = \Omega_1 Q$ where $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ then $\mathcal{L}(\Omega_2) = \Omega_2 \mathbb{Z}^{2g} = \Omega_1 Q \mathbb{Z}^{2g} = \Omega_1 \mathbb{Z}^{2g} = \mathcal{L}(\Omega_1)$. Conversely if $\mathcal{L}(\Omega_2) = \mathcal{L}(\Omega_1)$ then $\Omega_2 \mathbb{Z}^{2g} = \Omega_1 \mathbb{Z}^{2g}$, and by complex conjugation $\overline{\Omega}_2 \mathbb{Z}^{2g} = \overline{\Omega}_1 \mathbb{Z}^{2g}$ as well so the associated full period matrices satisfy

$$\begin{pmatrix} \Omega_2 \\ \overline{\Omega}_2 \end{pmatrix} \mathbb{Z}^{2g} = \begin{pmatrix} \Omega_1 \\ \overline{\Omega}_1 \end{pmatrix} \mathbb{Z}^{2g}.$$

The full period matrices are nonsingular by Lemma F.1 so the matrix

$$Q = \begin{pmatrix} \Omega_1 \\ \overline{\Omega}_1 \end{pmatrix}^{-1} \begin{pmatrix} \Omega_2 \\ \overline{\Omega}_2 \end{pmatrix}$$

is well defined; this matrix satisfies $Q \mathbb{Z}^{2g} = \mathbb{Z}^{2g}$ so $Q \in \mathrm{Gl}(2g, \mathbb{Z})$, and since $\Omega_2 = \Omega_1 Q$ that suffices to conclude the proof.

The linear mapping $A : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ described by a nonsingular complex matrix $A \in \mathrm{Gl}(g, \mathbb{C})$ takes a lattice subgroup $\mathcal{L} \subset \mathbb{C}^g$ to the lattice subgroup $A\mathcal{L} \subset \mathbb{C}^g$; two lattice subgroups related in this way are called *linearly equivalent* lattice subgroups.

Corollary F.3 *Lattice subgroups $\mathcal{L}(\Omega_1)$ and $\mathcal{L}(\Omega_2)$ in $\mathbb{C}^g$ are linearly equivalent if and only if $\Omega_1 = A\,\Omega_2\,Q^{-1}$ for matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$.*

Proof: If $\mathcal{L}(\Omega_1) = A\mathcal{L}(\Omega_2) = \mathcal{L}(A\Omega_2)$ for some matrix $A \in \mathrm{Gl}(g, \mathbb{C})$ then $\Omega_1 = A\,\Omega_2\,Q^{-1}$ for some matrix $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ by the preceding theorem. Conversely if $\Omega_1 = A\,\Omega_2\,Q^{-1}$ for some matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ then $\mathcal{L}(\Omega_1) = \Omega_1 \mathbb{Z}^{2g} = A\,\Omega_2\,Q^{-1}\mathbb{Z}^{2g} = A\,\Omega_2\,\mathbb{Z}^{2g} = A\,\mathcal{L}(\Omega_2)$. That suffices for the proof.

Two $g \times 2g$ period matrices Ω_1 and Ω_2 are called *equivalent period matrices* if $\Omega_1 = A\,\Omega_2\,Q^{-1}$ for matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$, and the equivalence of these two period matrices is denoted by $\Omega_1 \simeq \Omega_2$. If it is only the case that $Q \in \mathrm{Gl}(2g, \mathbb{Q})$, that Q is a nonsingular rational matrix, the two matrices are called *weakly equivalent period matrices*, and the weak equivalence of these two period matrices is denoted by $\Omega_1 \sim \Omega_2$. It is quite evident that both are equivalence relations in the customary sense, and that equivalent period matrices are weakly equivalent period matrices. To summarize, *the equivalence of period matrices is defined by*

$$\text{(F.1)} \qquad \Omega \simeq A\,\Omega Q^{-1} \quad \text{for any} \quad A \in \mathrm{Gl}(g, \mathbb{C}),\ Q \in \mathrm{Gl}(2g, \mathbb{Z}),$$

and *the weak equivalence of period matrices is defined by*

$$\text{(F.2)} \qquad \Omega \sim A\,\Omega Q^{-1} \quad \text{for any} \quad A \in \mathrm{Gl}(g, \mathbb{C}),\ Q \in \mathrm{Gl}(2g, \mathbb{Q}).$$

In these terms the preceding corollary can be restated as follows.

Corollary F.4 *Two lattice subgroups $\mathcal{L}(\Omega_1)$ and $\mathcal{L}(\Omega_2)$ in $\mathbb{C}^g$ are linearly equivalent if and only if the period matrices Ω_1 and Ω_2 are equivalent period matrices.*

Proof: This is equivalent to the preceding Corollary in view of the definition of equivalent period matrices, so no further proof is necessary.

Perhaps it should be repeated for emphasis that equivalence and weak equivalence of period matrices are defined for arbitrary period matrices, not necessarily just for nonsingular period matrices; but these equivalences preserve nonsingularity.

Corollary F.5 *A period matrix weakly equivalent (or equivalent) to a nonsingular period matrix is itself a nonsingular period matrix.*

Proof: It is course sufficient to demonstrate this corollary just for weakly equivalent period matrices. If Ω_1 and Ω_2 are weakly equivalent period matrices then by definition $\Omega_2 = A\,\Omega_1\,Q^{-1}$ for some nonsingular square matrices A and Q, so the associated full period matrices satisfy

$$\begin{pmatrix} \Omega_2 \\ \overline{\Omega}_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \overline{\Omega}_1 \end{pmatrix} Q^{-1}.$$

Since $\begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}$ and Q are nonsingular matrices it follows that if one of the two full period matrices is nonsingular so is the other; the desired result is then a consequence of Lemma F.1, and that suffices for the proof.

A *complex torus* of dimension g is the quotient $\mathbb{C}^g/\mathcal{L}$ of the additive group $\mathbb{C}^g$ by a lattice subgroup $\mathcal{L} \subset \mathbb{C}^g$. As a quotient group a complex torus has the natural structure of an abelian group. The natural quotient mapping $\pi : \mathbb{C}^g \longrightarrow \mathbb{C}^g/\mathcal{L}$ is the universal covering projection, and the complex torus $\mathbb{C}^g/\mathcal{L}$ inherits from its universal covering space $\mathbb{C}^g$ a natural complex structure; with this complex structure the complex torus is a compact complex abelian Lie group. For many purposes though the primary interest is in just the complex manifold structure of a complex torus rather than its full complex Lie group structure.

Theorem F.6 *A holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ between two complex tori is induced by an affine mapping $\tilde{f}(z) = Az + a$ between their universal covering spaces, where $A \in \mathbb{C}^{g_2 \times g_1}$ and $a \in \mathbb{C}^{g_2}$. An affine mapping $f(z) = Az + a$ induces a holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ between the complex tori for lattice subgroups $\mathcal{L}_1 \subset \mathbb{C}^{g_1}$ and $\mathcal{L}_2 \subset \mathbb{C}^{g_2}$ if and only if $A\mathcal{L}_1 \subset \mathcal{L}_2$; and this mapping is a group homomorphism if and only if $a \in \mathcal{L}_2$.*

Proof: A holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ lifts to a holomorphic mapping $\tilde{f} : \mathbb{C}^{g_1} \longrightarrow \mathbb{C}^{g_2}$ between the universal covering spaces of these two complex manifolds. A holomorphic mapping $\tilde{f} : \mathbb{C}^{g_1} \longrightarrow \mathbb{C}^{g_2}$ induces a holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ between the two quotient groups if and only if it takes points in $\mathbb{C}^{g_1}$ that differ by a lattice vector in $\mathcal{L}_1$ to points in $\mathbb{C}^{g_2}$ that differ by a lattice vector in $\mathcal{L}_2$, hence if and only if for any point $z \in \mathbb{C}^{g_1}$ and any lattice vector $\lambda_1 \in \mathcal{L}_1$ there is a lattice vector $\lambda_2 \in \mathcal{L}_2$ such that

$$\tilde{f}(z + \lambda_1) = \tilde{f}(z) + \lambda_2. \tag{F.3}$$

Since lattice subgroups are discrete the lattice vector λ_2 must be independent of the point z; so for any lattice vector λ_1 there must be a lattice vector λ_2 such that (F.3) holds as an identity in the variable $z \in \mathbb{C}^{g_1}$. The partial derivative $\partial\tilde{f}/\partial z_j$ then is a holomorphic mapping from $\mathbb{C}^{g_1}$ to $\mathbb{C}^{g_2}$ that is invariant under the lattice subgroup $\mathcal{L}_1$, so it is bounded in $\mathbb{C}^{g_1}$ and hence constant by the maximum modulus theorem for vector-valued holomorphic mappings; therefore the mapping $\tilde{f}$ must be of the form $\tilde{f}(z) = Az + a$ for some complex matrix A and complex vector a. For such a mapping (F.3) reduces to the condition that for any lattice vector $\lambda_1 \in \mathcal{L}_1$ there is a lattice vector $\lambda_2 \in \mathcal{L}_2$ such that $A\lambda_1 = \lambda_2$, hence to the condition that $A\mathcal{L}_1 \subset \mathcal{L}_2$. A holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ is a group homomorphism if and only if for any points $z_1, z_2 \in \mathbb{C}^{g_1}$ the image of their sum is the sum of their images in the torus $C^{g_2}/\mathcal{L}_2$; for the mapping $f(z) = Az+a$ that is the condition that $A(z_1+z_2)+a = (Az_1 + a) + (Az_2 + a) - l_2$ for some lattice vector $l_2 \in \mathcal{L}_2$, and since the lattice is discrete this must be an identity in the variables z_i so it is just the condition that $a = l_2 \in \mathcal{L}_2$. That suffices to conclude the proof.

Corollary F.7 *A holomorphic mapping between complex tori is the composition of a group homomorphism from one torus to the other and a translation in the image torus.*

Proof: A holomorphic mapping $f : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$ between two complex tori is induced by an affine mapping $\tilde{f}(z) = Az + a$ between their universal covering spaces for a matrix $A \in \mathbb{C}^{g_2\times g_1}$ such that $A\mathcal{L}_1 \subset \mathcal{L}_2$, by the preceding theorem. The mapping $\tilde{f}$ can be written as the composition $\tilde{f} = \tilde{g} \cdot \tilde{h}$ where $\tilde{g}(z) = z + a$ and $\tilde{h}(z) = Az$. The mapping $\tilde{g}(z)$ induces a translation g in the torus $\mathbb{C}^{g_2}/\mathcal{L}_2$, and by the preceding theorem again the mapping $\tilde{h}(z)$ induces a group homomorphism $h : \mathbb{C}^{g_1}/\mathcal{L}_1 \longrightarrow \mathbb{C}^{g_2}/\mathcal{L}_2$; since $f = g \cdot h$ that suffices for the proof.

The complex torus for the lattice subgroup $\mathcal{L}(\Omega)$ described by a nonsingular period matrix Ω is denoted by $J(\Omega)$, so that $J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega)$. A *Hurwitz relation* (A, Q) from a period matrix $\Omega_1 \in \mathbb{C}^{g_1\times 2g_1}$ to a period matrix $\Omega_2 \in \mathbb{C}^{g_2\times 2g_2}$ is defined to be a pair of matrices $A \in \mathbb{C}^{g_2\times g_1}$ and $Q \in \mathbb{Z}^{2g_2\times 2g_1}$ such that

$$A\Omega_1 = \Omega_2 Q, \tag{F.4}$$

whether the period matrices are nonsingular period matrices or not. A Hurwitz relation is not a symmetric relation, but rather involves a definite ordering of the period matrices Ω_1 and Ω_2. It is clear from (F.4) that if (A_1, Q_1) and (A_2, Q_2) are Hurwitz relations from Ω_1 to Ω_2 then so is the linear combination $n_1(A_1, Q_1) + n_2(A_2, Q_2) = (n_1A_1 + n_2A_2,\ n_1Q_1 + n_2Q_2)$ for any integers $n_1, n_2 \in \mathbb{Z}$; thus the set of Hurwitz relations from Ω_1 to Ω_2 form a $\mathbb{Z}$-module. In the special case that $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ the Hurwitz relation (F.4) when written $\Omega_2 = A\Omega_1 Q^{-1}$ amounts to the equivalence (F.1) of the period matrices Ω_1 and Ω_2.

Lemma F.8 *If (A, Q) is a Hurwitz relation from a nonsingular period matrix Ω_1 to a nonsingular period matrix Ω_2 then* $\operatorname{rank} Q = 2 \operatorname{rank} A$ *and either one of the matrices A or Q determines the other matrix uniquely.*

Proof: It is evident that the Hurwitz relation (F.4) is equivalent to the relation

$$\text{(F.5)} \qquad \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \overline{\Omega}_1 \end{pmatrix} = \begin{pmatrix} \Omega_2 \\ \overline{\Omega}_2 \end{pmatrix} Q$$

between the associated full period matrices. If the period matrices Ω_1 and Ω_2 are nonsingular period matrices the full period matrices are nonsingular square matrices by Lemma F.1; hence it follows from (F.5) that $\operatorname{rank} Q = 2 \operatorname{rank} A$ and that either one of the matrices A or Q determines the other uniquely. That suffices for the proof.

Theorem F.9 (i) *Holomorphic mappings $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ from the complex torus described by a nonsingular $g_1 \times 2g_1$ period matrix Ω_1 to the complex torus described by a nonsingular $g_2 \times 2g_2$ period matrix Ω_2 are in one-to-one correspondence with triples (A, Q, a_0) where (A, Q) is a Hurwitz relation from Ω_1 to Ω_2 and $a_0 \in J(\Omega_2)$; the holomorphic mapping f corresponding to (A, Q, a_0) is that induced by the affine mapping $\tilde{f}(z) = Az + a$ from $\mathbb{C}^{g_1}$ to $\mathbb{C}^{g_2}$ for any point $a \in \mathbb{C}^{g_2}$ representing the point $a_0 \in J(\Omega_2)$.*
(ii) *A holomorphic mapping $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ between two complex tori of the same dimension g corresponding to a Hurwitz relation (A, Q) from Ω_1 to Ω_2 is a biholomorphic mapping if and only if $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$.*

Proof: (i) By Theorem F.6 a holomorphic mapping $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ is induced by an affine mapping $\tilde{f}(z) = Az+a$; and an affine mapping $\tilde{f}(z) = Az+a$ induces a holomorphic mapping $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ if and only if $A\mathcal{L}(\Omega_1) \subset \mathcal{L}(\Omega_2)$, or equivalently if and only if $A\,\Omega_1 = \Omega_2\, Q$ for some integral matrix $Q \in \mathbb{Z}^{2g_2 \times 2g_1}$, which is just the condition that (A, Q) is a Hurwitz relation from Ω_1 to Ω_2. Two affine mappings $f_1(z) = A_1z+a_1$ and $f_z(z) = A_1z+a_1$ induce the same mapping $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ precisely when $(A_1-A_2)z+(a_1-a_2) = z+\lambda_2$ for all points $z \in \mathbb{C}^g$ and some lattice vector $\lambda_2 \in \mathcal{L}(\Omega_2)$, hence precisely when $A_1 = A_2$ and $a_1 - a_2 = \lambda_2$; that is the condition that the Hurwitz relations are the same and that a_1 and a_2 represent the same point of $J(\Omega_2)$.
(ii) Let $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ be a holomorphic mapping between two complex tori of dimension g corresponding to a triple (A, Q, a) where (A, Q) is a Hurwitz relation from Ω_1 to Ω_2. If $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ then

$A^{-1} \in \mathrm{Gl}(g,\mathbb{C})$ and $Q^{-1} \in \mathrm{Gl}(2g,\mathbb{Z})$, and it follows from the Hurwitz relation $A\Omega_1 = \Omega_2 Q$ that $A^{-1}\Omega_2 = \Omega_1 Q^{-1}$; thus (A^{-1}, Q^{-1}) is a Hurwitz relation from Ω_2 to Ω_1, and hence the affine mapping $\tilde{h}(z) = A^{-1}z - A^{-1}a$ describes a holomorphic mapping $h : J(\Omega_2) \longrightarrow J(\Omega_1)$. Since the affine mapping $\tilde{f}\tilde{h}$ is the identity mapping it follows that the induced mapping fh also is the identity mapping, and consequently that f itself is a biholomorphic mapping. Conversely if $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ is a biholomorphic mapping the inverse biholomorphic mapping $h = f^{-1}$ must be induced by an affine mapping $\tilde{h}(z) = Bz + b$. Since the mappings f and h are inverse to one another, for any point $z \in \mathbb{C}^g$ that point and the point $\tilde{f}(\tilde{h}(z)) = A(Bz+b)+a \in \mathbb{C}^g$ must represent the same point in the torus $J(\Omega_1)$; consequently there must be a lattice vector $\lambda_1 \in \mathcal{L}(\Omega_1)$ such that $ABz + Ab + a = z + \lambda_1$. This holds identically in z by continuity, hence $AB = \mathrm{I}$ so $A \in \mathrm{Gl}(g,\mathbb{C})$; and it then follows from Lemma F.8 that $Q \in \mathrm{Gl}(2g,\mathbb{Z})$. That suffices to conclude the proof.

The image of any holomorphic mapping between two complex tori is a holomorphic subvariety of the image manifold by Remmert's Proper Mapping Theorem[1]. The holomorphic mapping $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ corresponding to a Hurwitz relation (A, Q) from Ω_1 to Ω_2 is induced by the affine mapping $\tilde{f}(z) = Az+a$ for some point $a \in J(\Omega_2)$, so its image is a connected complex submanifold of $J(\Omega_2)$ of dimension equal to the rank of the matrix A. If rank $A = \dim J(\Omega_2)$ the induced mapping f is surjective, with image the full complex torus $J(\Omega_2)$. If rank $A = \dim J(\Omega_1) = \dim J(\Omega_2)$ the mapping f is a surjective and locally biholomorphic mapping between these two complex tori; such a mapping is called an *isogeny* from the complex torus $J(\Omega_1)$ to the complex torus $J(\Omega_2)$. When a Hurwitz relation (A, Q) determines an isogeny the matrix A is nonsingular, and then Q also is nonsingular matrix by Lemma F.8; and since $A\Omega_1 = \Omega_2 Q$ it follows that $A^{-1}\Omega_2 = \Omega_1 Q^{-1}$. Although Q^{-1} is not necessarily an integral matrix it is at least a rational matrix, so qQ^{-1} will be integral for some integer q; then $(q\,A^{-1},\ q\,Q^{-1})$ is a Hurwitz relation from the period matrix Ω_2 to the period matrix Ω_1. Thus if there is an isogeny from the complex torus $J(\Omega_1)$ to the complex torus $J(\Omega_2)$ there also is an isogeny from the complex torus $J(\Omega_2)$ to the complex torus $J(\Omega_1)$. Two complex tori are *isogenous* if there is an isogeny from one to another; this clearly is an equivalence relation between complex tori. Of course a biholomorphic mapping is a special case of an isogeny, so biholomorphic complex tori are isogenous.

Theorem F.10 *An isogeny $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ that is a group homomorphism induces a group isomorphism $J(\Omega_1)/K \cong J(\Omega_2)$ where $K \subset J(\Omega_1)$ is a finite subgroup; the mapping f exhibits the torus $J(\Omega_1)$ as a finite unbranched covering space of the torus $J(\Omega_2)$.*

Proof: By Theorem F.6 an isogeny $f : J(\Omega_1) \longrightarrow J(\Omega_2)$ is induced by an affine mapping $\tilde{f}(z) = Az + a$ between the universal covering spaces of the complex tori, and A must be a nonsingular matrix such that $A\mathcal{L}(\Omega_1) \subset \mathcal{L}(\Omega_2)$;

[1]For a discussion of Remmert's Proper Mapping Theorem see page 423 in Appendix A.3.

this isogeny is a group homomorphism if and only if $a \in \mathcal{L}(\Omega_2)$, in which case the mapping f also is induced by the linear mapping $\tilde{f}(z) = Az$. The kernel $K \subset J(\Omega_1)$ of the homomorphism f consists of those points of $J(\Omega_1)$ represented by vectors $\lambda \in \mathbb{C}^g$ such that $A\lambda \in \mathcal{L}(\Omega_2)$; consequently $K = \mathcal{K}/\mathcal{L}(\Omega_1)$ where $\mathcal{L}(\Omega_1) \subset \mathcal{K} = A^{-1}\mathcal{L}(\Omega_2) \subset \mathbb{C}^g$, and since A^{-1} is a linear isomorphism $\mathcal{K}$ is a lattice subgroup of $\mathbb{C}^g$. As the quotient of two lattice subgroups the group K is a finite group. The representation

$$J(\Omega_2) = \frac{\mathbb{C}^g}{\mathcal{L}(\Omega_2)} \cong \frac{A^{-1}\mathbb{C}^g}{A^{-1}\mathcal{L}(\Omega_2)} = \frac{\mathbb{C}^g}{\mathcal{K}}$$

exhibits $\mathbb{C}^g$ as the universal covering space of the complex torus $J(\Omega_2)$ with covering translation group $\mathcal{K}$. The subgroup $\mathcal{L}(\Omega_1) \subset \mathcal{K}$ of the covering translation group then corresponds to the sequence of covering projections

$$\begin{tikzcd}
\mathbb{C}^g \arrow[r] & \dfrac{\mathbb{C}^g}{\mathcal{L}(\Omega_1)} \arrow[r] \arrow[d, equal] & \dfrac{\mathbb{C}^g}{\mathcal{K}} \arrow[d, equal] \\
 & J(\Omega_1) \arrow[r] & J(\Omega_2);
\end{tikzcd}$$

and since the groups are abelian $\mathcal{L}(\Omega_1)$ is a normal subgroup of $\mathcal{K}$ so $J(\Omega_2) = J(\Omega_1)/K$, which suffices to conclude the proof.

Theorem F.11 *The complex tori $J(\Omega_1)$ and $J(\Omega_2)$ described by nonsingular period matrices Ω_1 and Ω_2 are biholomorphic if and only if the period matrices Ω_1 and Ω_2 are equivalent, and are isogenous if and only if the period matrices Ω_1 and Ω_2 are weakly equivalent.*

Proof: By Theorem F.9 the tori $J(\Omega_1)$ and $J(\Omega_2)$ are biholomorphic if and only if they are of the same dimension g and there are matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ such that $A\Omega_1 = \Omega_2 Q$; and that is precisely the condition (F.1) that the two period matrices are equivalent. The two tori are isogenous if and only they are of the same dimension g and there are matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathbb{Z}^{2g\times 2g}$ such that $A\Omega_1 = \Omega_2 Q$; by Lemma F.8 the matrix Q has rank $2g$ so that $Q \in \mathrm{Gl}(2g, \mathbb{Q})$, and that is precisely the condition (F.2) that the two period matrices are weakly equivalent. That suffices for the proof.

A useful alternative description of the complex torus $J(\Omega)$ involves a matrix Π closely related to the period matrix Ω.

Theorem F.12 *If Ω is a nonsingular period matrix*

$$\overline{{}^t\begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}^{-1}} = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$$

where Π also is a nonsingular period matrix.

Proof: The full period matrix associated to the period matrix Ω is a nonsingular $2g \times 2g$ matrix, so its inverse transpose conjugate exists and can be written in the form $\binom{\Pi_1}{\Pi_2}$ for some $g \times 2g$ period matrices Π_1 and Π_2. Since

$$\begin{pmatrix} \mathrm{I} & 0 \\ 0 & \mathrm{I} \end{pmatrix} = \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \overline{{}^t\begin{pmatrix} \Pi_1 \\ \Pi_2 \end{pmatrix}} = \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} ({}^t\overline{\Pi}_1 \quad {}^t\overline{\Pi}_2) = \begin{pmatrix} \Omega\, {}^t\overline{\Pi}_1 & \Omega\, {}^t\overline{\Pi}_2 \\ \overline{\Omega}\, {}^t\overline{\Pi}_1 & \overline{\Omega}\, {}^t\overline{\Pi}_2 \end{pmatrix}$$

it follows that

$$\Omega\, {}^t\overline{\Pi}_1 = \overline{\Omega}\, {}^t\overline{\Pi}_2 = \mathrm{I} \quad \text{and} \quad \Omega\, {}^t\overline{\Pi}_2 = \overline{\Omega}\, {}^t\overline{\Pi}_1 = 0.$$

By conjugation $\Omega\, {}^t\Pi_2 = \overline{\overline{\Omega}\, {}^t\overline{\Pi}_2} = \mathrm{I}$ and $\overline{\Omega}\, {}^t\Pi_2 = \overline{\Omega\, {}^t\overline{\Pi}_2} = 0$ as well, so

$$\Omega\, ({}^t\overline{\Pi}_1 - {}^t\Pi_2) = \overline{\Omega}\, ({}^t\overline{\Pi}_1 - {}^t\Pi_2) = 0;$$

and since the full period matrix is nonsingular it follows that $\Pi_2 = \overline{\Pi}_1$. The inverse of the full period matrix of course is also nonsingular, so Π itself is a nonsingular period matrix, and that suffices to conclude the proof.

The matrix Π of the preceding lemma is called the *inverse period matrix* to Ω. That Π is the inverse period matrix to Ω when viewed as the identity

$$\mathrm{I} = \overline{\begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}} \cdot {}^t\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = \begin{pmatrix} \overline{\Omega} \\ \Omega \end{pmatrix} \cdot ({}^t\Pi \;\; {}^t\overline{\Pi}) = \begin{pmatrix} \overline{\Omega}\, {}^t\Pi & \overline{\Omega}\, {}^t\overline{\Pi} \\ \Omega\, {}^t\Pi & \Omega\, {}^t\overline{\Pi} \end{pmatrix}$$

is equivalent to the conditions that

$$\Omega\, {}^t\Pi = 0 \quad \text{and} \quad \Omega\, {}^t\overline{\Pi} = \mathrm{I}; \tag{F.6}$$

and that Π is the inverse period matrix to Ω when viewed as the identity

$$\mathrm{I} = \overline{{}^t\begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = ({}^t\overline{\Omega} \;\; {}^t\Omega) \cdot \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = {}^t\overline{\Omega}\, \Pi + {}^t\Omega\, \overline{\Pi}$$

is equivalent to the condition that

$${}^t\Omega\, \overline{\Pi} + {}^t\overline{\Omega}\, \Pi = \mathrm{I}. \tag{F.7}$$

Clearly if Π is the inverse period matrix to Ω then Ω is the inverse period matrix to Π. Furthermore if $\Omega_1 \simeq \Omega_2$ so that $\Omega_2 = A\Omega_1 Q^{-1}$ where $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ then

$$\begin{pmatrix} \Omega_2 \\ \overline{\Omega}_2 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \overline{\Omega}_1 \end{pmatrix} Q^{-1},$$

and the complex conjugate of the inverse transpose of this equation is the equation

$$\begin{pmatrix} \Pi_2 \\ \overline{\Pi}_2 \end{pmatrix} = \begin{pmatrix} {}^t\overline{A}^{-1} & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} \Pi_1 \\ \overline{\Pi}_1 \end{pmatrix} {}^tQ,$$

showing that $\Pi_2 = {}^t\overline{A}^{-1}\Pi_1\,{}^tQ$ and consequently that $\Pi_1 \simeq \Pi_2$. Similarly of course if $\Omega_1 \sim \Omega_2$ then $\Pi_1 \sim \Pi_2$ by the same formula. The inverse period matrix to a nonsingular period matrix can be defined more intrinsically in terms of the real bilinear form

$$< z, w >= 2\Re(\,{}^tz\,\overline{w}) = 2\Re\Big(\sum_{i=1}^{g} z_i\overline{w}_i\Big) \tag{F.8}$$

for any column vectors $z = \{z_i\}$, $w = \{w_i\} \in \mathbb{C}^g$ when $\mathbb{C}^g$ is viewed as the real linear vector space $\mathbb{R}^g$; here $\Re(z)$ denotes the real part of the complex number z. For most purposes it is sufficient to describe the lattice subgroup $\mathcal{L}(\Pi) = \Pi\mathbb{Z}^{2g} \subset \mathbb{C}^g$ rather than the inverse period matrix Π itself.

Theorem F.13 *If $\Omega \in \mathbb{C}^{g\times 2g}$ is a nonsingular period matrix the lattice subgroup $\mathcal{L}(\Pi)$ described by the inverse period matrix Π is the dual lattice subgroup to $\mathcal{L}(\Omega)$ in terms of the real bilinear form* (F.8) *in the sense that*

$$\mathcal{L}(\Pi) = \Big\{\pi \in \mathbb{C}^g \;\Big|\; < \omega, \pi >\in \mathbb{Z} \quad \textit{for all} \quad \omega \in \mathcal{L}(\Omega)\Big\}.$$

Proof: The lattice subgroup $\mathcal{L}(\Omega) = \Omega\mathbb{Z}^{2g} \subset \mathbb{C}^g$ is generated over the integers by the column vectors $\omega_i \in \mathbb{C}^g$ of the matrix Ω, where $\omega_i = \{\omega_{ki} \mid 1 \le k \le g\}$ in terms of the entries ω_{ki} of the matrix Ω. The dual lattice then is generated over the integers by the $2g$ column vectors $\pi_j \in \mathbb{C}^g$ defined by the conditions that $< \omega_i, \pi_j >= \delta^i_j$ for $1 \le i, j \le 2g$. If $\pi_j = \{\pi_{kj} \mid 1 \le k \le g\} \in \mathbb{C}^g$ and Π is the matrix $\Pi = \{\pi_{kj}\} \in \mathbb{C}^{g\times 2g}$ this duality condition is just that $\delta^i_j = 2\Re(\,{}^t\omega_i\,\overline{\pi}_j) = {}^t\omega_i\overline{\pi}_j + {}^t\overline{\omega}_i\pi_j = \sum_{k=1}^{g}\big(\omega_{ki}\overline{\pi}_{kj} + \overline{\omega}_{ki}\pi_{kj}\big)$, or in matrix terms $\mathrm{I} = {}^t\Omega\overline{\Pi} + {}^t\overline{\Omega}\Pi$; and by (F.7) that is just the condition that Π is the inverse period matrix to Ω, which suffices to conclude the proof.

For another use of the inverse period matrix, if Ω is a nonsingular period matrix the columns of the $2g \times 2g$ matrix $\big({}^t\Omega \;\, {}^t\overline{\Omega}\big)$ are linearly independent vectors, so there is a direct sum decomposition

$$\mathbb{C}^{2g} = {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g \tag{F.9}$$

of the complex vector space $\mathbb{C}^{2g}$ into two complementary linear subspaces, one spanned by the columns of the matrix ${}^t\Omega$ and the other spanned by the columns of the matrix ${}^t\overline{\Omega}$. It follows from (F.7) that any point $t \in \mathbb{C}^{2g}$ can be written

$$t = {}^t\Omega\,\overline{\Pi}t + {}^t\overline{\Omega}\,\Pi t, \tag{F.10}$$

which is an explicit formula for splitting a vector $t \in \mathbb{C}^{2g}$ into its components in the direct sum decomposition (F.9); indeed by (F.6) with the matrices Ω and Π interchanged the square matrices ${}^t\Omega\overline{\Pi}$ and ${}^t\overline{\Omega}\Pi$ are the natural projection operators

$$\begin{aligned} {}^t\Omega\overline{\Pi} :\; {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g \quad &\longrightarrow \quad {}^t\Omega\mathbb{C}^g \\ {}^t\overline{\Omega}\Pi :\; {}^t\Omega\mathbb{C}^g \oplus {}^t\overline{\Omega}\mathbb{C}^g \quad &\longrightarrow \quad {}^t\overline{\Omega}\mathbb{C}^g \end{aligned} \tag{F.11}$$

in the direct sum decomposition (F.9), since ${}^t\Omega\overline{\Pi}\cdot{}^t\Omega\overline{\Pi} = {}^t\Omega\overline{\Pi}$ while both ${}^t\Omega\overline{\Pi}\cdot{}^t\Omega t = {}^t\Omega t$ and ${}^t\Omega\overline{\Pi}\cdot{}^t\overline{\Omega} t = 0$ for any $t \in \mathbb{C}^g$, and correspondingly for the complex conjugates. A particularly useful application of the inverse period matrix suggested by these observations is the following.

Theorem F.14 *If Ω is a nonsingular $g \times 2g$ period matrix there is the exact sequence of abelian groups*

$$0 \longrightarrow \mathbb{Z}^{2g} + {}^t\Omega\mathbb{C}^g \xrightarrow{\iota} \mathbb{C}^{2g} \xrightarrow{\Pi} \frac{\mathbb{C}^g}{\Pi\mathbb{Z}^{2g}} \longrightarrow 0$$

where ι is the natural inclusion homomorphism and Π is the linear mapping defined by the inverse period matrix to Ω.

Proof: The complex linear mapping $\Pi : \mathbb{C}^{2g} \longrightarrow \mathbb{C}^g$ defined by the inverse period matrix Π is surjective and has as its kernel the linear subspace ${}^t\Omega\mathbb{C}^g$, as is evident from (F.6); and since this linear mapping takes the subgroup $\mathbb{Z}^{2g} \subset \mathbb{C}^{2g}$ to the lattice subgroup $\Pi\mathbb{Z}^{2g} \subset \mathbb{C}^g$ that suffices for the proof.

Corollary F.15 *The complex torus $J(\Omega)$ defined by a nonsingular period matrix Ω can be described alternatively as the quotient group*

$$J(\Omega) = \frac{\mathbb{C}^{2g}}{\mathbb{Z}^{2g} + {}^t\Pi\mathbb{C}^g}$$

where Π is the inverse period matrix to Ω.

Proof: This follows immediately from the exact sequence of the preceding theorem, when the roles of the period matrices Ω and Π are interchanged, since the complex torus $J(\Omega)$ is the quotient $J(\Omega) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$, and that suffices for the proof.

The inverse period matrix also can be used to provide alternative characterizations of Hurwitz relations (F.4) between nonsingular period matrices in terms of either the matrix A or the matrix Q.

Theorem F.16 *Let Ω_1 be a $g_1 \times 2g_1$ nonsingular period matrix and Ω_2 be a $g_2 \times 2g_2$ nonsingular period matrix.*
(i) *A matrix $A \in \mathbb{C}^{g_2 \times g_1}$ is part of a Hurwitz relation (A, Q) from the period matrix Ω_1 to the period matrix Ω_2 if and only if*

$$2\Re({}^t\overline{\Pi}_2 A \Omega_1) \in \mathbb{Z}^{2g_2 \times 2g_1}$$

where Π_2 is the inverse period matrix to Ω_2 and $\Re(Z)$ denotes the real part of the complex matrix Z; and $Q = 2\Re({}^t\overline{\Pi}_2 A \Omega_1)$.
(ii) *A matrix $Q \in \mathbb{Z}^{2g_2 \times 2g_1}$ is part of a Hurwitz relation (A, Q) from the period matrix Ω_1 to the period matrix Ω_2 if and only if*

$$\Omega_2 Q\, {}^t\Pi_1 = 0 \in \mathbb{C}^{g_2 \times g_1}$$

where Π_1 is the inverse period matrix to Ω_1; and $A = \Omega_2 Q\, {}^t\overline{\Pi}_1$.

Proof: A matrix $A \in \mathbb{C}^{g_2\times g_1}$ is part of a Hurwitz relation from the period matrix Ω_1 to the period matrix Ω_2 if and only if there is a matrix $Q \in \mathbb{Z}^{2g_2\times 2g_1}$ satisfying (F.5); and that equation can be rewritten equivalently as

$$\begin{aligned} Q &= {}^t\!\begin{pmatrix} \overline{\Pi}_2 \\ \Pi_2 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \begin{pmatrix} \Omega_1 \\ \overline{\Omega}_1 \end{pmatrix} = \begin{pmatrix} {}^t\overline{\Pi}_2 & {}^t\Pi_2 \end{pmatrix} \begin{pmatrix} A\Omega_1 \\ \overline{A\Omega_1} \end{pmatrix} \\ &= {}^t\overline{\Pi}_2 A\Omega_1 + {}^t\Pi_2 \overline{A\Omega_1} \\ &= 2\Re\Big({}^t\overline{\Pi}_2 A\Omega_1 \Big). \end{aligned}$$

Similarly a matrix $Q \in \mathbb{Z}^{2g_2\times 2g_1}$ is part of a Hurwitz relation from the period matrix Ω_1 to the period matrix Ω_2 if and only if there is a matrix $A \in \mathbb{C}^{g_2\times g_1}$ satisfying (F.5); and that equation can be rewritten equivalently as

$$\begin{aligned} \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} &= \begin{pmatrix} \Omega_2 \\ \overline{\Omega}_2 \end{pmatrix} Q\, {}^t\!\begin{pmatrix} \overline{\Pi}_1 \\ \Pi_1 \end{pmatrix} = \begin{pmatrix} \Omega_2 Q \\ \overline{\Omega}_2 Q \end{pmatrix} \begin{pmatrix} {}^t\overline{\Pi}_1 & {}^t\Pi_1 \end{pmatrix} \\ &= \begin{pmatrix} \Omega_2 Q\, {}^t\overline{\Pi}_1 & \Omega_2 Q\, {}^t\Pi_1 \\ \overline{\Omega}_2 Q\, {}^t\overline{\Pi}_1 & \overline{\Omega}_2 Q\, {}^t\Pi_1 \end{pmatrix}. \end{aligned}$$

That suffices to conclude the proof.

Corollary F.17 *Two nonsingular period matrices Ω_1, Ω_2 of the same rank g are equivalent if and only if either of the following two equivalent conditions hold:*
(i) *there is a matrix $A \in \mathrm{Gl}(g, \mathbb{C})$ such that $2\Re({}^t\overline{\Pi}_2 A\Omega_1) \in \mathrm{Gl}(2g, \mathbb{Z})$ where Π_2 is the inverse period matrix to Ω_2;*
(ii) *there is a matrix $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ such that $\Omega_2 Q\, {}^t\Pi_1 = 0$ where Π_1 is the inverse period matrix to Ω_1.*

Proof: Two nonsingular period matrices Ω_1, Ω_2 of the same rank g are equivalent if and only if there is a Hurwitz relation (A, Q) from the period matrix Ω_1 to the period matrix Ω_2 where the matrices A and Q are invertible. By Lemma F.8 it is enough just to show that one of the two matrices A or Q is invertible. As in the proof of the preceding theorem, for a given matrix A the matrix Q in the Hurwitz relation is $Q = 2\Re({}^t\overline{\Pi}_2 A\Omega_1)$ while for a given matrix Q there is such a matrix A if and only if $\Omega_2 Q\, {}^t\Pi_1 = 0$, and that suffices to conclude the proof.

Corollary F.18 *Two nonsingular period matrices Ω_1, Ω_2 of the same rank g are weakly equivalent if and only if either of the following two equivalent conditions holds:*
(i) *there is a matrix $A \in \mathrm{Gl}(g, \mathbb{C})$ such that $2\Re({}^t\overline{\Pi}_2 A\, \Omega_1) \in \mathrm{Gl}(2g, \mathbb{Q})$ where Π_2 is the inverse period matrix to Ω_2;*
(ii) *there is a matrix $Q \in \mathrm{Gl}(2g, \mathbb{Q})$ such that $\Omega_2 Q\, {}^t\Pi_1 = 0$ where Π_1 is the inverse period matrix to Ω_1.*

Proof: Two nonsingular period matrices Ω_1, Ω_2 of the same rank g are weakly equivalent if and only if there is the analogue of a Hurwitz relation (A, Q) from the period matrix Ω_1 to the period matrix Ω_2 in which the matrices A and Q are invertible but Q is only rational rather than integral; with this modification the proof is as that of the preceding corollary, and that suffices for the proof.

F.2 Topological Properties

Topologically a complex torus of dimension g is a product of $2g$ circles. To make this more explicit, if $\Omega = (\omega^1, \ldots, \omega^{2g})$ is a nonsingular period matrix its column vectors $\omega^i \in \mathbb{C}^g$ are linearly independent over the real numbers. The real linear mapping that takes a vector $t \in \mathbb{R}^{2g}$ to the vector $z = \Omega t \in \mathbb{C}^g$ is an isomorphism

$$\Omega : \mathbb{R}^{2g} \longrightarrow \mathbb{C}^g \tag{F.12}$$

of real vector spaces that maps the standard basis column vector $\delta^j = \{\delta^j_k\}$ in $\mathbb{R}^{2g}$ to the column vector $\omega^j \in \mathbb{C}^g$ for $1 \le j \le 2g$ and consequently maps the lattice subgroup $\mathbb{Z}^{2g} \subset \mathbb{R}^{2g}$ to the lattice subgroup $\mathcal{L}(\Omega) = \Omega\mathbb{Z}^{2g} \subset \mathbb{C}^g$; it therefore determines a one-to-one mapping

$$\Omega : \mathbb{R}^{2g}/\mathbb{Z}^{2g} \longrightarrow \mathbb{C}^g/\Omega\mathbb{Z}^{2g} \tag{F.13}$$

that identifies the real torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ with the complex torus $J(\Omega) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ topologically. If Π is the inverse period matrix to Ω as introduced in Theorem F.12 then the real linear mapping that takes a vector $z \in \mathbb{C}^g$ to the vector

$$\tilde{\Pi}(z) = {}^t\overline{\Pi}z + {}^t\Pi\overline{z} \in \mathbb{R}^{2g} \tag{F.14}$$

is the real linear mapping inverse to (F.12) since $\Omega\tilde{\Pi}(z) = \Omega\,{}^t\overline{\Pi}z + \Omega\,{}^t\Pi\overline{z} = z$ by (F.6) and conversely $\tilde{\Pi}(\Omega t) = {}^t\overline{\Pi}\Omega t + {}^t\Pi\overline{\Omega}t = t$ by (F.7); the mapping (F.14) consequently determines the one-to-one mapping

$$\tilde{\Pi} : \mathbb{C}^g/\Omega\mathbb{Z}^{2g} \longrightarrow \mathbb{R}^{2g}/\mathbb{Z}^{2g} \tag{F.15}$$

that is the inverse mapping to (F.13). The real torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g} = (\mathbb{R}/\mathbb{Z})^{2g}$ is the product of $2g$ circles $\mathbb{R}/\mathbb{Z}$, and consequently the complex torus $J(\Omega) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$ is topologically the product of $2g$ circles as well.

The first homology group of a circle is the free abelian group $\mathbb{Z}$, so it follows from the Künneth formula[2] for the homology groups of product spaces that the homology groups of a torus T are finitely generated free abelian groups. Consequently the cohomology and homology groups of T are dual to one another, and both can be described fully by considering only the homology and cohomology

[2]For the general results about homology and cohomology groups of topological spaces see for instance the book by E. H. Spanier, *Algebraic Topology*, McGraw-Hill 1966.

groups with real or complex coefficients; and the complex cohomology group can be identified with the tensor product of the real cohomology group with $\mathbb{C}$. Thus for the purposes at hand it is enough to describe the homology and cohomology groups of T in terms of the complex deRham groups $\mathfrak{H}^p(T)$ of T, the quotients of the groups of closed complex-valued differential forms by the subgroups of exact complex-valued differential forms on the differentiable manifold T, effectively reducing the topological considerations to rather straightforward analytic considerations.

A complex-valued differential p-form on the torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ can be viewed as a complex-valued differential p-form on $\mathbb{R}^{2g}$ that is invariant under translations by vectors in $\mathbb{Z}^{2g}$; and a complex-valued differential p-form on T that is invariant under all translations of the torus T can be viewed correspondingly as a constant complex-valued differential p-form on $\mathbb{R}^{2g}$. Such a differential form can be written in terms of the real coordinates $t_1, \ldots, t_{2g}$ on $\mathbb{R}^{2g}$ as

$$\phi = \sum_{1 \le i_1 < \cdots < i_p \le 2g} c_{i_1 \ldots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p} \tag{F.16}$$

for arbitrary complex constants $c_{i_1 \ldots i_p}$; alternatively when the coefficients $c_{i_1 \ldots i_p}$ are extended to all values of the indices $i_1, \ldots, i_p$ to be skew-symmetric in these indices then

$$\phi = \frac{1}{p!} \sum_{i_1, \ldots i_p = 1}^{2g} c_{i_1 \ldots i_p} dt_{i_1} \wedge \cdots \wedge dt_{i_p}. \tag{F.17}$$

One of the many applications of the theory of harmonic differential forms[3] is that on a compact Lie group such as a torus any closed differential form is cohomologous to a unique group invariant differential form, and consequently that the deRham group $\mathfrak{H}^p(T)$ of the torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ is isomorphic to the space of constant complex-valued differential p-forms on $\mathbb{R}^{2g}$. Since the deRham group is isomorphic to the complex cohomology group it follows that

$$\dim H^p(T, \mathbb{C}) = \dim \mathfrak{H}^p(T) = \binom{2g}{p}. \tag{F.18}$$

The exterior product of differential forms determines an exterior product structure on the cohomology group $H^p(T, \mathbb{C})$, exhibiting it as the complex exterior algebra generated by the first cohomology group $H^1(T, \mathbb{C})$.

The cohomology group $H^p(T, C)$ is dual to the homology group $H_p(T, \mathbb{C})$, so $\dim H_p(T, \mathbb{C}) = \binom{2g}{p}$ as a consequence of (F.18). To describe the homology

[3] Invariant integrals on groups were introduced by E. Cartan, and the applications of harmonic differential forms to show that the harmonic differential forms on compact Lie groups are the invariant differentials forms of Cartan and consequently that harmonic differentials can be used to describe the deRham groups was due to W. V. D. Hodge, described in detail in his book *Harmonic Integrals*, Cambridge Univ. Press, 1941. There are many other derivations of the same result; for the case of complex tori a short proof of a quite different sort can be found in the book by C. Birkenhake and H. Lange, *Complex Abelian Varieties*, Springer-Verlag 2004.

group $H_p(T,\mathbb{C})$ of the torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ more concretely, for any p distinct integers $j_1,\ldots,j_p$ in the range $1 \le j_i \le 2g$ let $[\delta^{j_1,\cdots,j_p}]$ be the singular p-cycle

$$[\delta^{j_1,\cdots,j_p}] : [0,1]^p \longrightarrow T \tag{F.19}$$

that is the composition of the linear mapping from $[0,1]^p$ to the vector space $\mathbb{R}^{2g}$ defined by

$$\delta^{j_1,\ldots,j_p}(s_1,\ldots,s_p) = \sum_{k=1}^{p} s_k \delta^{j_k} \in \mathbb{R}^{2g} \tag{F.20}$$

for any point $(s_1,\ldots,s_p)$ where $0 \le s_i \le 1$, followed by the natural projection

$$t \in \mathbb{R}^{2g} \longrightarrow [t] \in T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}; \tag{F.21}$$

with the natural orientation provided by the parameter space $\mathbb{R}^p$ these singular cycles are skew-symmetric in the indices $j_1,\ldots,j_p$. For $p=1$ the singular 1-cycle $[\delta^j]$ can be viewed as being spanned by the column vector δ^j itself; the singular p-cycle $[\delta^{j_1,\cdots,j_p}]$ can be viewed as being spanned by the p column vectors $\delta^{j_1},\ldots,\delta^{j_p}$ so sometimes it is denoted also by $[\delta^{j_1}]\wedge\ldots\wedge[\delta^{j_p}]$. In terms of the coordinates $(t_1,\ldots,t_{2g})$ on $\mathbb{R}^{2g}$ the image of the mapping (F.20) is described parametrically by $t_l = \sum_{k=1}^{p}\delta_l^{j_k}s_k$; so the differential form dt_l on $\mathbb{R}^{2g}$ induces the differential form $dt_l = \sum_{k=1}^{p}\delta_l^{j_k}ds_k$ in terms of the parameters $(s_1,\ldots,s_p)$ of the singular p-cycle $[\delta^{j_1,\cdots,j_p}]$, and more generally

$$\begin{aligned} dt_{l_1}\wedge\cdots\wedge dt_{l_p} &= \sum_{k_1,\ldots,k_p=1}^{p} \delta_{l_1}^{j_{k_1}} ds_{k_1}\wedge\cdots\wedge\delta_{l_p}^{j_{k_p}} ds_{k_p} \\ &= \sum_{k_1,\ldots,k_p=1}^{p} \delta_{l_1}^{j_{k_1}}\cdots\delta_{l_p}^{j_{k_p}} ds_{k_1}\wedge\cdots\wedge ds_{k_p}. \end{aligned}$$

This differential form is clearly 0 unless the set of indices $(l_1,\ldots,l_p)$ is a permutation of the set of indices $(j_1,\ldots,j_p)$; and if $(l_1,\ldots,l_p) = \pi(j_{k_1},\ldots,j_{k_p})$ for a permutation $\pi \in \mathfrak{S}^p$ then

$$dt_{l_1}\wedge\cdots\wedge dt_{l_p} = \operatorname{sgn}(\pi) ds_1\wedge\cdots\wedge ds_p$$

where $\operatorname{sgn}(\pi)$ is the sign of the permutation π. It thus follows that

$$\int_{[\delta^{j_1\cdots j_p}]} dt_{l_1}\wedge\cdots\wedge dt_{l_p} = \delta_{l_1,\ldots,l_p}^{j_1,\ldots,j_p} \tag{F.22}$$

where $\delta_{l_1,\ldots,l_p}^{j_1,\ldots,j_p}$ is 0 unless $(j_1,\ldots,j_p)$ is a permutation of $(l_1,\ldots,l_p)$ and then is the sign of that permutation. As a consequence it is clear that the differential forms $dt_{i_1}\wedge\cdots\wedge dt_{i_p}$ are dual to the cycles $\delta^{j_1\cdots j_p}$, so since these differential forms are a basis for the deRham group $\mathfrak{H}^p(T)$ and hence represent a basis for the cohomology group $H^p(T,\mathbb{C})$ it follows that the homology classes represented by the p-cycles $[\delta^{j_1,\dots,j_p}]$ for $1 \le j_1 < \cdots < j_p \le 2g$ are a basis for the homology group $H_p(T,\mathbb{C})$; this establishes explicitly the duality between the homology and cohomology groups of the torus.

The homeomorphism (F.13) carries the skew-symmetric singular p-cycle $[\delta^{j_1,\dots,j_p}]$ in the torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ to the skew-symmetric singular p-cycle

$$\Omega[\delta^{j_1,\dots,j_p}] = [\Omega\delta^{j_1,\dots,j_p}] = [\omega^{j_1,\dots,j_p}] \tag{F.23}$$

in the torus $J(\Omega)$ spanned by the columns $\omega^j = \Omega\delta^j$ of the period matrix Ω, in analogy with (F.19), (F.20), (F.21), and the homology classes represented by the skew-symmetric singular p-cycles $[\omega^{j_1,\dots,j_p}]$ are a basis for the homology $H_p\big(J(\Omega),\mathbb{C}\big)$ of the complex torus $J(\Omega) = \mathbb{C}^g/\Omega\mathbb{Z}^{2g}$. The dual basis for the deRham group $\mathfrak{H}^p\big(J(M)\big)$ then consists of the differential forms $\phi_{j_1} \wedge \cdots \wedge \phi_{j_p}$ on the torus $J(\Omega)$ induced by the differential forms $= dt_{j_1} \wedge \cdots \wedge dt_{j_p}$ on the real torus $T = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$, so by (F.14)

$$\phi_j(z) = \sum_{k=1}^{g} (\overline{\pi_{kj}} dz_k + \pi_{kj} \overline{dz_k}) \tag{F.24}$$

for 1-forms and the exterior products have the corresponding forms.

For another period matrix $\Lambda \in \mathbb{C}^{h\times 2h}$ describing a complex torus $J(\Lambda)$ of dimension h, a basis for the homology $H_p\big(J(\lambda),\mathbb{C}\big)$ is represented by the skew-symmetric singular p-cycles $[\lambda^{j_1,\dots,j_p}]$ spanned by the columns $\lambda^j = \Lambda\delta^j$ of the matrix Λ, and a dual basis for the deRham group $\mathfrak{H}^p\big(j(\Lambda)\big)$ consists of the differential forms $\psi_{j_1} \wedge \cdots \wedge \psi_{j_p}$ for $1 \le j_1 < \cdots < j_p \le 2h$ where in analogy with (F.24)

$$\psi_j(w) = \sum_{k=1}^{h} (\overline{\sigma}_{kj} dw_k + \sigma_{kj} \overline{dw}_k) \tag{F.25}$$

in terms of the complex coordinates $w_1, \dots, w_h$ in $\mathbb{C}^h$ and $\Sigma = \{\sigma_{kj}\}$ is the inverse period matrix to the period matrix Λ; the exterior products have the corresponding forms. Both (F.24) and (F.25) can be rewritten conveniently in matrix notation as

$$\phi(z) = \overline{\Pi}\, dz + \Pi\, \overline{dz}. \tag{F.26}$$

and

$$\psi(w) = \overline{\Sigma}\, dw + \Sigma\, \overline{dw}. \tag{F.27}$$

where $\phi = \{\phi_j\}$, $\psi = \{\psi_j\}$, $dz = \{dz_j\}$ and $dw = \{dw_j\}$ are viewed as column vectors of differential forms; and in view of (F.6) the inverse of (F.26) is

$$dz = \Omega\, \phi(z) \qquad \overline{dz} = \overline{\Omega}\, \phi(z). \tag{F.28}$$

If $f : J(\Omega) \longrightarrow J(\Lambda)$ is a holomorphic mapping between these complex tori described by a Hurwitz relation (A, Q) from the period matrix Ω to the period matrix Λ, so that the mapping f is induced by the linear mapping $A : \mathbb{C}^g \longrightarrow \mathbb{C}^h$ aside from a translation in the vector space $\mathbb{C}^h$, then under this mapping the differential forms dw on $J(\Lambda)$ induce the differential forms $f^*(dw) = Adz$ on $J(\Omega)$ in matrix notation; and consequently the differential forms ψ on $J(\Lambda)$ induce the differential forms

$$\begin{aligned} f^*\big(\psi(w)\big) &= f^*(\overline{\Sigma}dw + \Sigma\overline{dw}) = \overline{\Sigma}f^*(dw) + \Sigma f^*(\overline{dw}) \\ &= \overline{\Sigma}A\,dz + \Sigma\overline{A\,dz} = \overline{\Sigma}A\Omega\phi + \Sigma\overline{A\Omega}\phi = \overline{\Sigma}\Lambda Q\phi + \Sigma\overline{\Lambda Q}\phi = Q\phi \end{aligned}$$

since $A\Omega = \Lambda Q$ and $\overline{\Sigma}\Lambda + \Sigma\overline{\Lambda} = \mathrm{I}$, or more explicitly

$$f^*\big(\psi_j(w)\big) = \sum_{k=1}^{g} q_{jk}\phi_k(z) \quad \text{for } 1 \le j \le h. \tag{F.29}$$

The induced differential p-forms then are just the wedge products of the induced differential 1-forms. Dually the mapping f takes the singular 1-cycle $[\omega^l]$ spanned by column l of the period matrix Ω for $1 \le l \le 2g$ to a linear combination $f_*([\omega^l]) = \sum_{m=1}^{2g} c_{lm}[\lambda^m]$ of the singular 1-cycles spanned by the columns of the period matrix Λ, where

$$\begin{aligned} q_{jk} &= \sum_{l=1}^{2h} q_{jl}\delta_k^l = \sum_{l=1}^{2h} \int_{[\omega^k]} q_{jl}\phi_l(z) = \int_{[\omega^k]} f^*\big(\psi_j(w)\big) \\ &= \int_{f_*([\omega^k])} \psi_j(w) = \sum_{m=1}^{2h} \int_{c_{km}[\lambda^m]} \psi_j(w) = c_{kj} \end{aligned}$$

and consequently

$$f_*([\omega^k]) = \sum_{m=1}^{2g} q_{mk}[\lambda^m]; \tag{F.30}$$

thus the matrix Q describes the effect of the mapping f described by the Hurwitz relation (A, Q) on the first homology groups of the two tori, the group homomorphism $f_* : H_1(J(\Omega_1)) \longrightarrow H_1(J(\Omega_2))$ induced by the mapping f, and that extends to the wedge products of the 1-cycles correspondingly. When the mapping f is the biholomorphic mapping corresponding to a Hurwitz relation (A, Q) that is an equivalence of the period matrices defining the complex tori, the induced homomorphism f_* is the isomorphism described by the matrix $Q \in \mathrm{Gl}(2g, \mathbb{Z})$. When the mapping f is merely an isogeny corresponding to a Hurwitz relation (A, Q) that is a weak equivalence of the period matrices defining the complex tori, the mapping exhibits the torus $J(\Omega_1)$ as an unbranched covering of the torus $J(\Omega_2)$; the induced homomorphism f_* described by the matrix Q determines this covering topologically, since the fundamental groups of complex tori are abelian so coincide with the first homology groups.

F.3 Riemann Matrices

A $g \times 2g$ period matrix Ω is called a *Riemann matrix* if there is a skew-symmetric integral matrix P such that:
(i) $\Omega\, P\, {}^t\Omega = 0$, and
(ii) the matrix $H = i\,\Omega\, P\, {}^t\overline{\Omega}$ is positive definite Hermitian;
the matrix P is called a *principal matrix* for the Riemann matrix Ω. If P is a principal matrix for the Riemann matrix Ω then so is any positive scalar multiple $r\,P$ that is also an integral matrix; among these multiples there is a unique one having relatively prime integral entries, called a *primitive principal matrix* for the Riemann matrix Ω. There are Riemann matrices admitting principal matrices not all of which are scalar multiples of one another, hence admitting a number of distinct primitive principal matrices; these are called *singular* Riemann matrices. The choice of a principal matrix up to arbitrary positive scalar multiples, or equivalently the choice of a primitive principal matrix, is called a *polarization* of the Riemann matrix Ω; and the pair consisting of a Riemann matrix Ω and its polarization is called a *polarized Riemann matrix.* A polarized Riemann matrix is denoted either by $(\Omega,\, P)$, where P is a primitive principal matrix for the Riemann matrix Ω, or $(\Omega,\, \{P\})$, where P is any matrix a multiple of which is a principal matrix for the Riemann matrix Ω. A polarized Riemann matrix (Ω, J) where J is the basic skew-symmetric matrix

$$\mathrm{J} = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} \tag{F.31}$$

is called a *principally polarized* Riemann matrix. Condition (i) in the definition of a Riemann matrix, often called *Riemann's equality*, can be rewritten in terms of the associated full period matrix as

$$i \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} P\, \overline{{}^t\begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix}} = i \begin{pmatrix} \Omega\, P\, {}^t\overline{\Omega} & \Omega\, P\, {}^t\Omega \\ \overline{\Omega}\, P\, {}^t\overline{\Omega} & \overline{\Omega}\, P\, {}^t\Omega \end{pmatrix} = \begin{pmatrix} H & 0 \\ 0 & -\overline{H} \end{pmatrix} \tag{F.32}$$

where $H = i\,\Omega\, P\, {}^t\overline{\Omega}$; and condition (ii) in the definition of a Riemann matrix, often called *Riemann's inequality*, is that the Hermitian matrix H is positive definite. An immediate consequence of this expanded form of the Riemann matrix conditions is the following auxiliary observation.

Theorem F.19 *A Riemann matrix Ω is a nonsingular period matrix; and if P is a principal matrix for the Riemann matrix Ω then P is a nonsingular matrix and* $\det P > 0$*.*

Proof: Since the matrix H in (F.32) is positive definite the right-hand side of that equation is a nonsingular matrix; consequently both the full period matrix $\binom{\Omega}{\overline{\Omega}}$ and the principal matrix P must be nonsingular. Taking the determinant in the identity (F.32) among $2g \times 2g$ square matrices yields the result that

$$i^{2g} \left| \det \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \right|^2 \det P = (-1)^g\, |\det H|^2,$$

hence that $\det P > 0$, and that suffices for the proof.

Two period matrices Ω and $\tilde{\Omega}$ were defined to be equivalent in (F.1) if there are matrices $A \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$ such that $\tilde{\Omega} = A\Omega\, Q^{-1}$. If (Ω, P) is a polarized Riemann matrix, so that $\Omega P\, {}^t\Omega = 0$ and $i\, \Omega P\, {}^t\overline{\Omega}$ is positive definite Hermitian, and if $\tilde{P} = Q\, P\, {}^tQ$, so $\tilde{P}$ is a skew-symmetric matrix with relatively prime integral entries, then

$$\tilde{\Omega}\tilde{P}\, {}^t\tilde{\Omega} = A\Omega Q^{-1} \cdot QP{}^tQ \cdot {}^tQ^{-1}\, {}^t\Omega\, {}^tA = A\Omega P\, {}^t\Omega\, {}^tA = 0$$

and the matrix

$$i\, \tilde{\Omega}\tilde{P}\, {}^t\overline{\tilde{\Omega}} = i\, A\Omega Q^{-1} \cdot QP{}^tQ \cdot {}^tQ^{-1}\, {}^t\overline{\Omega}\, {}^t\overline{A} = iA\Omega P\, {}^t\overline{\Omega}\, {}^t\overline{A}$$

is positive definite Hermitian, so $(\tilde{\Omega},\ \tilde{P})$ also is a polarized Riemann matrix. The two polarized Riemann matrices (Ω, P) and $(\tilde{\Omega}, \tilde{P})$ are called *equivalent polarized Riemann matrices*, and the equivalence of these two polarized Riemann matrices is denoted by $(\Omega, P) \simeq (\tilde{\Omega}, \tilde{P})$. If the period matrices Ω and $\tilde{\Omega}$ are just weakly equivalent, so that it is only the case that $Q \in \mathrm{Gl}(2g, \mathbb{Q})$, then the matrix $QP\, {}^tQ$ is only a rational matrix; but the other conditions for a polarized Riemann matrix are satisfied, so if r is a positive rational number such that $r\, QP\, {}^tQ$ is an integral matrix with relatively prime entries then the pair $(A\, \Omega\, Q^{-1},\ r\, QP\, {}^tQ)$ is a polarized Riemann matrix. These two polarized Riemann matrices are called *weakly equivalent polarized Riemann matrices*, and the weak equivalence of these two polarized Riemann matrices is denoted by $(\Omega, P) \sim (\tilde{\Omega}, \tilde{P})$ or $(\Omega, \{P\}) \sim (\tilde{\Omega}, \{\tilde{P}\})$. Of course $(A\, \Omega\, Q^{-1},\ \{r\, QP\, {}^tQ\}) = (A\, \Omega\, Q^{-1},\ \{\, QP\, {}^tQ\})$, so this is the more convenient notation when considering the weak equivalence of polarized Riemann matrices. Both evidently are equivalence relations in the customary sense. In summary, *the equivalence of polarized Riemann matrices is defined by*

(F.33)

$$(\Omega, P) \simeq (A\, \Omega\, Q^{-1},\ QP\, {}^tQ) \quad \text{or} \quad (\Omega, \{P\}) \simeq (A\, \Omega\, Q^{-1},\ \{QP\, {}^tQ\})$$

$$\text{whenever}\ \ A \in \mathrm{Gl}(g, \mathbb{C})\ \ \text{and}\ \ Q \in \mathrm{Gl}(2g, \mathbb{Z});$$

and *the weak equivalence of polarized Riemann matrices is defined by*

(F.34)

$$(\Omega, P) \sim (A\, \Omega\, Q^{-1},\ r\, QP\, {}^tQ) \quad \text{or} \quad (\Omega, \{P\}) \sim (A\, \Omega\, Q^{-1},\ \{\, QP\, {}^tQ\})$$

$$\text{whenever}\ \ A \in \mathrm{Gl}(g, \mathbb{C})\ \ \text{and}\ \ Q \in \mathrm{Gl}(2g, \mathbb{Q})$$

where r is the unique positive rational number such that $r\, QP\, {}^tQ$ is an integral matrix with relatively prime entries. An important case of this equivalence is the following.

Theorem F.20 *If* (Ω, P) *is a polarized Riemann matrix,* r *is a positive integer such that* $N = r^{-1}P^{-1}$ *is an integral matrix with relatively prime entries and* Π *is the inverse period matrix to* Ω*, then* $(\Pi, {}^tN)$ *is a polarized Riemann matrix that is weakly equivalent to* (Ω, P)*; and if in addition* $\det P = 1$ *then* $r = 1$ *and the weak equivalence is actually an equivalence.*

Proof: From Riemann's equality $\Omega P\,{}^t\Omega = 0$ and Theorem F.16 (ii) it follows that P is part of the Hurwitz relation (A, P) from the period matrix Π to the period matrix Ω where $A = \Omega P\,{}^t\overline{\Omega}$. Thus $A\Pi = \Omega P$, and since the principal matrix P is nonsingular by Theorem F.19, and consequently the matrix A also is nonsingular by Lemma F.8, it follows that $A \in \mathrm{Gl}(g, \mathbb{C})$ and $P \in \mathrm{Gl}(2g, \mathbb{Q})$ so this Hurwitz relation exhibits the weak equivalence of the period matrices Ω and Π. In addition $rP\,{}^tN\,{}^tP = P$, so the polarized Riemann matrices $(\Pi, {}^tN)$ and (Ω, P) are weakly equivalent. Of course if $\det P = 1$ then $r = 1$ and $P \in \mathrm{Gl}(2g, \mathbb{Z})$ so $(\Pi, {}^tN)$ and (Ω, P) are equivalent polarized Riemann matrices, and that suffices to conclude the proof.

Corollary F.21 *If* (Ω, P) *is a polarized Riemann matrix and* Π *is the inverse period matrix to* Ω *then* $(\Omega P\,{}^t\overline{\Omega}, P)$ *is a Hurwitz relation from the period matrix* Π *to the period matrix* Ω *describing an isogeny from the complex torus* $J(\Pi)$ *to the complex torus* $J(\Omega)$*; and if* $\det P = 1$ *this isogeny is a biholomorphic mapping.*

Proof: In the proof of the preceding theorem the weak equivalence of the polarized Riemann matrices $(\Pi, {}^tN)$ and (Ω, P) was exhibited by the Hurwitz relation $(A, P) = (\Omega P\,{}^t\overline{\Omega}, P)$ from the period matrix Π to the period matrix Ω, and this Hurwitz relation exhibits an isogeny from the complex torus $J(\Pi)$ to the complex torus $J(\Omega)$. If $\det P = 1$ the weak equivalence is an equivalence and the isogeny is a biholomorphic mapping, and that suffices for the proof.

For some purposes it is convenient to have a more explicit statement of the preceding observations. If (Ω, P) is a polarized Riemann matrix, (F.32) is an identity between invertible matrices and its inverse transpose is the equation

$$-i\ {}^t\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}^{-1} P^*\ \overline{\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}^{-1}} = \begin{pmatrix}G & 0\\ 0 & -\overline{G}\end{pmatrix}$$

or equivalently in terms of the inverse period matrix Π

$$-i\ \overline{\begin{pmatrix}\Pi\\ \overline{\Pi}\end{pmatrix}}\ P^*\ {}^t\begin{pmatrix}\Pi\\ \overline{\Pi}\end{pmatrix} = \begin{pmatrix}G & 0\\ 0 & -\overline{G}\end{pmatrix}$$

where $P^* = {}^tP^{-1}$ and $G = {}^tH^{-1} = \overline{H^{-1}}$. This equation is equivalent to the identities

$$\text{(F.35)} \qquad \Pi P^*\,{}^t\Pi = 0 \quad i\,\Pi P^*\,{}^t\overline{\Pi} = \overline{G},$$

which if rP^* is an integral matrix is the pair of conditions (i) and (ii) showing that rP^* is a principal matrix for the Riemann matrix Π since the matrix G is positive definite Hermitian.

Principally polarized Riemann matrices are of particular interest in the study of compact Riemann surfaces, so it is worth examining that special case in somewhat more detail.

Theorem F.22 (i) *For any principally polarized Riemann matrix* $(\Omega,\ \mathrm{J})$ *there is a unique nonsingular complex matrix* A *such that* $A\,\Omega = (\mathrm{I}\quad Z)$*, where* I *is the identity matrix.*
(ii) *A period matrix of the form* $(\mathrm{I}\quad Z)$*, where* I *is the identity matrix, is a Riemann matrix with the principal matrix* J *if and only if the matrix block* Z *is a complex symmetric matrix with positive definite imaginary part.*

Proof: If (Ω, J) is a principally polarized Riemann matrix and the $g\times 2g$ matrix Ω is decomposed into $g\times g$ square blocks $\Omega = (\Omega_1\quad \Omega_2)$ Riemann's equality is that

$$\text{(F.36)}\qquad 0 = (\Omega_1\quad \Omega_2)\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}\begin{pmatrix} {}^t\Omega_1 \\ {}^t\Omega_2 \end{pmatrix} = \Omega_1\,{}^t\Omega_2 - \Omega_2\,{}^t\Omega_1$$

and Riemann's inequality is that the $g\times g$ matrix

$$\text{(F.37)}\qquad H = i(\Omega_1\quad \Omega_2)\begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix}\begin{pmatrix} {}^t\overline{\Omega}_1 \\ {}^t\overline{\Omega}_2 \end{pmatrix} = i\left(\Omega_1\,{}^t\overline{\Omega}_2 - \Omega_2\,{}^t\overline{\Omega}_1\right)$$

is positive definite Hermitian. If the square matrix Ω_1 is singular there is a nontrivial row vector $c\in\mathbb{C}^g$ such that $c\,\Omega_1 = 0$, and then $c\,H\,\overline{{}^tc} = ic\,\Omega_1\cdot {}^t\overline{(c\Omega_2)} - ic\,\Omega_2\cdot {}^t\overline{(c\Omega_1)} = 0$ which contradicts the condition that the matrix H is positive definite Hermitian; therefore the matrix Ω_1 is nonsingular, and if $A = \Omega_1^{-1}$ it follows that $A\Omega = (\mathrm{I}\quad Z)$ for a $g\times g$ square complex matrix Z. The principally polarized Riemann matrix $\big((\mathrm{I}\quad Z), \mathrm{J}\big)$ thus is equivalent to $(\Omega,\ \mathrm{J})$, and it must satisfy the analogues of (F.36) and (F.37). From the analogue to (F.36) it follows that $0 = {}^tZ - Z$, so the matrix Z is symmetric; and from the analogue to (F.37) it follows that the matrix $H = i({}^t\overline{Z} - Z) = i({}^t\overline{Z} - Z) = 2\Im(Z)$ is positive definite, where $\Im(Z)$ is the imaginary part of the matrix Z. That suffices to conclude the proof.

The set of complex symmetric $g\times g$ matrices $Z = X + iY$ such that the imaginary part $\Im(Z) = Y$ is positive definite is called the *Siegel upper half-space of rank* g and is denoted by $\mathfrak{H}_g$. In the special case $g = 1$ the space $\mathfrak{H}_1$ is just the ordinary upper half-plane; in general $\mathfrak{H}_g$ is an open convex subspace of the vector space $\mathbb{C}^g$. A principally polarized Riemann matrix (Ω, J) for which $\Omega = (\mathrm{I}\quad Z)$ for $Z\in\mathfrak{H}_g$ is called a *normalized principally polarized Riemann matrix.* In these terms one of the consequences of the preceding theorem is that any principally polarized Riemann matrix is equivalent to a normalized principally polarized Riemann matrix. However distinct normalized principally polarized Riemann matrices still can be equivalent principally polarized Riemann matrices. The description of this situation involves the group $\mathrm{Sp}(2g, \mathbb{Z})$ of integral symplectic matrices of rank $2g$, the group consisting of those $2g \times 2g$ integral matrices Q such that $Q\mathrm{J}\,{}^t\!Q = \mathrm{J}$ for the basic skew-symmetric matrix $J = \big({0 \atop -\mathrm{I}} {\mathrm{I} \atop 0}$, as discussed in more detail in Appendix H.

Theorem F.23 *Normalized principally polarized Riemann matrices* $\big((\mathrm{I} \quad Z), \mathrm{J}\big)$ *and* $\big((\mathrm{I} \quad \tilde{Z}), \mathrm{J}\big)$ *are equivalent polarized Riemann matrices if and only if*

$$\tilde{Z} = (A + ZC)^{-1}(B + ZD) \tag{F.38}$$

for a symplectic matrix

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}).$$

Proof: By definition (F.33) two normalized principally polarized Riemann matrices $\big(\Omega, \mathrm{J}\big)$ and $\big(\tilde{\Omega}, \mathrm{J}\big)$ are equivalent polarized Riemann matrices if and only if $\tilde{\Omega} = E\Omega Q$ and $Q^{-1}P\,{}^t\!Q^{-1} = \mathrm{J}$ for some matrices $E \in \mathrm{Gl}(g, \mathbb{C})$ and $Q \in \mathrm{Gl}(2g, \mathbb{Z})$. It must therefore be the case that $Q \in \mathrm{Sp}(2g, \mathbb{Z})$, and when Q is decomposed into $g \times g$ matrix blocks

$$\begin{aligned} (\mathrm{I} \quad \tilde{Z}) \quad &= \quad E\,(\mathrm{I} \quad Z)\,Q \quad = \quad E\,(\mathrm{I} \quad Z)\begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \quad E\,(A + ZC \quad B + ZD); \end{aligned}$$

thus $E = (A + ZC)^{-1}$ and $\tilde{Z} = (A + ZC)^{-1}(B + ZD)$, which suffices for the proof.

For $g = 1$ equation (F.38) is just the familiar action of the classical modular group $\mathrm{Sl}(2, \mathbb{Z})$ as a group of biholomorphic mappings of the upper half-plane $\mathfrak{H}_1$ to itself, and the quotient space $\mathcal{A}_1 = \mathfrak{H}_1/\mathrm{Sl}(2, \mathbb{Z})$ is the familiar space of moduli of complex tori, that is, is a space of parameters for biholomorphic equivalence classes of complex tori of dimension 1. By Theorem F.22 normalized principally polarized Riemann matrices are of the form $\big((\mathrm{I} \quad Z), \mathrm{J}\big)$ for arbitrary matrices $Z \in \mathfrak{H}_g$, so (F.38) describes an action of the symplectic modular group as a group of biholomorphic mappings of the Siegel upper half-space $\mathfrak{H}_g$ of rank g to itself. The quotient space $\mathcal{A}_g = \mathfrak{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$ then is a well defined topological space with the natural structure of a holomorphic variety of dimension $g(g-1)/2$, the Siegel moduli space[4]. The holomorphic variety $\mathcal{A}_g$ can be considered as the space of moduli or of parameters for the set of equivalence classes of principally polarized Riemann matrices; in view of Theorem F.11 the variety $\mathcal{A}_g$ also can be viewed as the space of moduli or of parameters for the set of biholomorphic equivalence classes of complex tori $J(\Omega)$ described by period matrices Ω that are Riemann matrices with principal matrix J.

[4]The Siegel upper half-space and its quotient under the symplectic modular group were investigated extensively by C. L. Siegel; see for instance his papers " Einführung in die Theorie der Modulfunktionen n-ten Grades", Math. Ann., vol 116 (1939), pp. 617 - 657; "Zur Theorie der Modulfunktionen n-ten Grades", Comm. Pure and Appl. Math., vol. 8 (1955), pp. 677-681, and the discussion in his book *Topics in Complex Function Theory*, vol. III, (Wiley, 1989).

F.4 Form Matrices

An alternative characterization and interpretation of polarized Riemann matrices is quite useful for some purposes.

Theorem F.24 *A $g \times 2g$ nonsingular period matrix Ω is a Riemann matrix if and only if there is a positive definite Hermitian matrix G such that the matrix $N = 2\,\Im({}^t\Omega\, G\, \overline{\Omega})$ is a rational matrix, where $\Im(Z)$ denotes the imaginary part of the complex matrix Z.*

Proof: If Ω is a Riemann matrix then Ω is a nonsingular period matrix by the preceding theorem; thus (F.32) is an identity among invertible matrices, and its inverse readily is seen to be the identity

$$\text{(F.39)} \qquad \begin{aligned} P^{-1} &= i\, \overline{{}^t\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}} \begin{pmatrix} H^{-1} & 0 \\ 0 & -\overline{H}^{-1} \end{pmatrix} \begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix} \\ &= i\left({}^t\overline{\Omega}\, H^{-1}\,\Omega - {}^t\Omega\, \overline{H}^{-1}\, \overline{\Omega}\right) \\ &= 2\Im({}^t\Omega \overline{H}^{-1} \overline{\Omega}) = 2\,\Im({}^t\Omega\, {}^tH^{-1}\, \overline{\Omega}), \end{aligned}$$

so $P^{-1} = 2\Im({}^t\Omega\ {}^tH^{-1}\,\overline{\Omega})$ is a rational matrix where ${}^tH^{-1}$ is positive definite Hermitian. Conversely if Ω is a nonsingular period matrix and G is a positive definite Hermitian matrix such that $N = 2\Im\big({}^t\Omega\, G\,\overline{\Omega}\big)$ is a rational matrix then the inverse of equation (F.39) is (F.32) in which $H = {}^tG^{-1}$ and $P = N^{-1}$; the matrix H is positive definite Hermitian and P is rational, so a suitable positive multiple of P is integral and hence is a principal matrix for Ω, and consequently Ω is a Riemann matrix. That suffices to conclude the proof.

If Ω is a Riemann matrix a positive definite Hermitian matrix G such that $N = 2\Im({}^t\Omega G \overline{\Omega})$ is an integral matrix is called a *form matrix* for the Riemann matrix Ω, and the matrix N is called the associated *characteristic matrix.* If G is a form matrix for the Riemann matrix Ω then so is any positive scalar multiple rG for which the associated characteristic matrix is an integral matrix; among these multiples there is a unique one for which the associated characteristic matrix is integral with relatively prime entries, called a *primitive form matrix* for the Riemann matrix Ω.

Corollary F.25 (i) *If Ω is a Riemann matrix with principal matrix P and $H = i\,\Omega P\, {}^t\overline{\Omega}$ then $G = r^{-1}\, {}^tH^{-1}$ is a form matrix for the Riemann matrix Ω with characteristic matrix $N = r^{-1}P^{-1}$ for any positive number r such that N is integral.*
(ii) *If Ω is a Riemann matrix with form matrix G and associated characteristic matrix N then $P = r^{-1}N^{-1}$ is a principal matrix for the Riemann matrix Ω for any positive number r such that P is integral, and $i\Omega P\, {}^t\overline{\Omega} = r^{-1}\, {}^tG^{-1}$.*

Proof: In the proof of the preceding theorem it was demonstrated that (F.32) is equivalent to the condition that $N = 2\Im({}^t\Omega, G\overline{\Omega})$ where $N = P^{-1}$ and $G = {}^tH^{-1}$, from which the corollary follows. That suffices for the proof.

It follows that a polarization of a Riemann matrix Ω also can be described as the choice of a form matrix for Ω up to arbitrary positive scalar multiples, or equivalently the choice of a primitive form matrix for Ω; in view of this a polarized Riemann matrix also can be denoted either by $[\Omega, G]$, where G is a primitive form matrix for the Riemann matrix Ω, or by $[\Omega, \{G\}]$, where G is any positive definite Hermitian matrix a multiple of which is a form matrix for the Riemann matrix Ω. Polarizations (Ω, P) and $[\Omega, G]$ related as in the preceding corollary are considered as describing the same polarization of a Riemann matrix, so

$$(\Omega, \{P\}) = [\Omega, \{G\}] \tag{F.40}$$

$$\text{where } \ {}^t G^{-1} = i\,\Omega P\, {}^t\overline{\Omega} \ \text{ or equivalently } \ P^{-1} = 2\Im({}^t\Omega G \overline{\Omega}).$$

In view of this the characteristic matrix N of the form matrix G often is called the characteristic matrix of the polarized Riemann matrix (Ω, P). This alternative description of a polarized Riemann matrix is more convenient for some purposes in that it exhibits the polarization of a Riemann matrix Ω as a natural property of the complex torus $J(\Omega)$ determined by the period matrix Ω. Indeed a positive definite $g \times g$ Hermitian matrix G can be viewed as describing a constant, or equivalently a translation-invariant, Hermitian metric[5] of the form $\sum_{j,k=1}^{g} g_{jk} dw_j d\overline{w}_k$ on the torus $J(\Omega)$, the complex form of a Riemannian metric expressed in terms of the complex coordinates w_j on the complex manifold $J(\Omega)$. Associated to this Hermitian metric is the differential form

$$\phi = \frac{1}{i} \sum_{j,k=1}^{g} g_{jk} dw_j \wedge d\overline{w}_k$$

of type $(1,1)$ on the torus $J(\Omega)$. Since the matrix G is Hermitian it follows that $\overline{\phi} = \phi$, hence that ϕ is a real differential form; and since the coefficients of this differential form are constant it is a closed differential form. A Hermitian metric with the property that the associated differential form of type $(1,1)$ is closed is called a *Kähler metric*.

Theorem F.26 *A nonsingular period matrix Ω is a Riemann matrix if and only if the complex torus $J(\Omega)$ admits a translation-invariant Kähler metric such that the associated differential form has integral periods on all the two-cycles of the torus; the coefficient matrix of the metric is a form matrix for the Riemann matrix Ω, and the associated characteristic matrix describes the periods of this differential form.*

Proof: The integral of the differential form ϕ associated to the Kähler metric described by a positive definite Hermitian matrix G on the 2-cycle $[\omega^{lm}]$ defined

[5] The basic properties of Hermitian and Kähler metrics are discussed in most texts on differential geometry that deal with complex as well as real manifolds; see for instance R. O. Wells, *Differential Geometry on Complex Manifolds*, (Prentice-Hall, 1973).

as on page 536 is

$$\begin{aligned}
n_{lm} = \int^{\omega_{l,m}} \phi = \frac{1}{i}\sum_{j,k=1}^{g}\int_{s=0}^{1}\int_{t=0}^{1} g_{jk} dw_j(s,t)\wedge \overline{dw_k(s,t)} \\
&= \frac{1}{i}\sum_{j,k=1}^{g} g_{jk}\int_{s=0}^{1}\int_{t=0}^{1}(\omega_{jl}ds+\omega_{jm}dt)\wedge(\overline{\omega}_{kl}ds+\overline{\omega}_{km}dt) \\
&= \frac{1}{i}\sum_{j,k=1}^{g} g_{jk}\Big(\omega_{jl}\overline{\omega}_{km}-\omega_{jm}\overline{\omega}_{kl}\Big)\int_{s=0}^{1}\int_{t=0}^{1} ds\wedge dt \\
&= \frac{1}{i}\sum_{j,k=1}^{g} g_{jk}\Big(\omega_{jl}\overline{\omega}_{km}-\omega_{jm}\overline{\omega}_{kl}\Big).
\end{aligned}$$

When these periods are viewed as forming a $2g\times 2g$ matrix $N=\{n_{lm}\}$ the preceding equation can be rewritten as the matrix identity

$$\begin{aligned}
N &= \frac{1}{i}\Big({}^t\Omega\, G\overline{\Omega} - {}^t\overline{\Omega}\,{}^tG\Omega\Big) = \frac{1}{i}\Big({}^t\Omega\, G\overline{\Omega} - {}^t\overline{\Omega}\,\overline{G}\Omega\Big) \\
&= 2\Im({}^t\Omega\, G\,\overline{\Omega}).
\end{aligned}$$

Consequently the condition that Ω is a Riemann matrix, expressed in terms of the form matrix G, is just that the differential form ϕ associated to the matrix G has integral periods on the basic cycles of the torus $J(\Omega)$; and these periods form the characteristic matrix associated to the form matrix G, which concludes the proof.

Since the integrated average over the torus $J(\Omega)$ of any differentiable Kähler metric is a translation-invariant Kähler metric such that the closed differential form of type $(1,1)$ associated to the averaged metric and that associated to the initial metric have the same periods, the period matrix Ω of any complex torus $J(\Omega)$ that admits a differentiable Kähler metric with integral periods is a Riemann matrix; the coefficient matrix of the averaged Kähler metric is a form matrix for Ω. A Kähler metric with integral periods is called a *Hodge metric.* Although the topic will not be pursued further here, at least it should be mentioned that the existence of a Hodge metric on a complex torus is equivalent to the condition that the torus is an algebraic variety; that is traditionally approached through the study of theta functions on the torus.[6] Alternatively and more generally, it was demonstrated by K. Kodaira[7] that a compact complex manifold is an algebraic variety if and only if it admits a Hodge metric.

The notions of equivalence and weak equivalence of polarized Riemann matrices can be expressed alternatively in terms of the form matrix as well.If $(\Omega, P)\simeq$

[6]See for instance the discussion in F. Conforto, *Abelsche Funktionen und algebraische Geometrie*, (Springer, 1956); D. Mumford, *Abelian Varieties*, (Oxford, 1970); or A. I. Markushevich, *Introduction to the Classical Theory of Abelian Functions*, Translations of Mathematical Monographs, vol. 96, (American Mathematical Society, 1992).

[7]K. Kodaira, *On Kähler varieties of restricted type*, Annals of Math. 60 (1954), pages 28-48.

$(\tilde\Omega, \tilde P)$ and if the equivalence is described by matrices $A = \{a_{ik}\} \in \mathrm{Gl}(g,\mathbb{C})$ and $Q = \{q_{ik}\} \in \mathrm{Gl}(2g,\mathbb{Z})$ as in (F.33) then the positive definite matrices $H = i\Omega\, P\, {}^t\overline{\Omega}$ and $\tilde H = i\,\tilde\Omega\,\tilde P\,{}^t\overline{\tilde\Omega}$ are related by $\tilde H = i\, A\Omega Q\cdot Q^{-1} P\, {}^tQ^{-1}\cdot {}^tQ\, {}^t\overline{\Omega}\, {}^t\overline{A} = A\, H\, {}^t\overline{A}$. The form matrix describing the same polarization of the Riemann matrix Ω is $G = r^{-1}\, {}^tH^{-1}$ as in Corollary F.25, where r is the unique positive rational number such that the matrix $N = r^{-1}\, P^{-1}$ is an integral matrix with relative prime entries; and since $\tilde P^{-1} = {}^tQ\, P^{-1} Q$ where $Q \in \mathrm{Gl}(2g,\mathbb{Z})$ it follows that $\tilde N = r^{-1}\, {}^t\tilde P^{-1}$ is also an integral matrix with relatively prime entries, so that $\tilde G = r^{-1}\, {}^t\tilde H^{-1}$ is the form matrix describing the same polarization of the Riemann matrix $\tilde\Omega$. These two form matrices thus are related by $\tilde G = {}^tA^{-1}\, G\, \overline{A^{-1}}$, and the associated characteristic matrices $N = r^{-1}\, P^{-1}$ and $\tilde N = r^{-1}\, \tilde P^{-1}$ are related by $\tilde N = {}^tQ\, N\, Q$. In summary then, *the equivalence of polarized Riemann matrices described in terms of form matrices is defined by*

(F.41)

$$[\Omega, G] \simeq [A\,\Omega\, Q,\ {}^tA^{-1} G\, \overline{A}^{-1}] \ \text{ or } \ [\Omega, \{G\}] \simeq [A\,\Omega\, Q,\ \{{}^tA^{-1} G\, \overline{A}^{-1}\}]$$
$$\text{whenever } \ A \in \mathrm{Gl}(g,\mathbb{C}) \ \text{ and } \ Q \in \mathrm{Gl}(2g,\mathbb{Z}),$$

and the associated characteristic matrices are related by

(F.42) $$\tilde N = {}^tQ\, N\, Q.$$

For weak equivalence the matrix Q is only a nonsingular rational matrix and the form and characteristic matrices must be multiplied by that positive rational number for which the characteristic matrix is integral with relatively prime entries. This complication can be avoided by considering only the alternative notation for polarized Riemann matrices in terms of form matrices, so that *the weak equivalence of polarized Riemann matrices described in terms of families of form matrices is defined by*

(F.43)

$$[\Omega,\ \{G\}] \ \sim \ [A\,\Omega\, Q,\ \{{}^tA^{-1}\, G\, \overline{A}^{-1}\}]$$
$$\text{whenever } \ A \in \mathrm{Gl}(g,\mathbb{C}), \ \text{ and } \ Q \in \mathrm{Gl}(2g,\mathbb{Q}).$$

The equivalence of polarized Riemann matrices has an interesting interpretation in terms of the associated complex tori. If (Ω, P) and $(\tilde\Omega, \tilde P) = (A\Omega Q, Q^{-1}\, P\, {}^tQ^{-1})$ are equivalent polarized Riemann matrices with form matrices G and $\tilde G = {}^tA^{-1}\, G\overline{A}^{-1}$ then $A\Omega = \tilde\Omega Q^{-1}$ so (A, Q^{-1}) is a Hurwitz relation from the period matrix Ω to the period matrix $\tilde\Omega$; and since $Q \in \mathrm{Gl}(2g,\mathbb{Z})$ the linear mapping $A : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ induces a biholomorphic mapping $A : J(\Omega) \longrightarrow J(\tilde\Omega)$ between the complex tori described by these period matrices, as in Theorem F.9. This biholomorphic mapping, viewed as a nonsingular linear change of coordinates $\tilde w_j = \sum_{k=1}^g a_{jk} w_k$, transforms the translation-invariant Hermitian metric $\sum_{jk=1}^{g} \tilde{g}_{jk} d\tilde{w}_j d\overline{\tilde{w}}_k$ on the torus $J(\tilde{\Omega})$ described by the form matrix $\tilde{G}$ to the metric

$$\text{(F.44)} \qquad \sum_{jk=1}^{g} \tilde{g}_{jk} d\tilde{w}_j d\overline{\tilde{w}}_k = \sum_{jklm=1}^{g} \tilde{g}_{jk} a_{jl} \overline{a}_{km} dw_l d\overline{w}_m = \sum_{lm=1}^{g} g_{lm} dw_l d\overline{w}_m,$$

the translation-invariant Hermitian metric on the torus $J(\Omega)$ described by the form matrix $G = {}^t\!A\tilde{G}\overline{A}$. Thus the equivalence of polarized Riemann matrices amounts to the existence of a biholomorphic mapping between the complex tori described by these period matrices that transforms the translation-invariant metrics describing the polarizations into one another. To phrase this in another way, a *polarized complex torus* is a complex torus together with a family of translation-invariant Hermitian metrics, all of which are scalar multiples of one another and some of which have integral periods; and in these terms *the polarized Riemann matrices in an equivalence class describe the same polarized complex torus*, with the various polarized Riemann matrices in the equivalence class merely being descriptions of the same polarized torus in terms of other linear coordinate systems on the torus. This provides an intrinsic interpretation of a polarized Riemann matrix.

If the polarized Riemann matrices (Ω, P) and $(\tilde{\Omega}, \tilde{P}) = (A\Omega Q,\, Q^{-1}P\,{}^tQ^{-1})$ are just weakly equivalent the linear mapping $A : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ is nonsingular and transforms the translation-invariant Hermitian metric on the torus $J(\tilde{\Omega})$ described by the form matrix $\tilde{G}$ locally to the translation-invariant Hermitian metric on the torus $J(\Omega)$ described by the form matrix G as in equation (F.44); but the induced mapping on complex tori is not well defined globally. However for any positive rational number r for which $r\,Q^{-1}$ is an integral matrix the pair $(r\,A,\, rQ^{-1})$ is a Hurwitz relation from the period matrix Ω to the period matrix $\tilde{\Omega}$, so the linear mapping $r\,A : \mathbb{C}^g \longrightarrow \mathbb{C}^g$ defines an isogeny $r\,A : J(\Omega) \longrightarrow J(\tilde{\Omega})$ and this isogeny transforms the family of translation-invariant Hermitian metrics $\{\tilde{G}\}$ defining the polarization of the torus $J(\tilde{\Omega})$ to the family of translation-invariant Hermitian metrics $\{G\}$ defining the polarization of the torus $J(\Omega)$; that the metrics are transformed locally to one another follows from (F.44), and since the metrics are constant the same transformation arises at all points of $J(\Omega)$ that have the same image in $J(\tilde{\Omega})$ under this isogeny. Two polarized complex tori are *polarized-isogenous* if there is an isogeny between the tori that transforms the families of translation-invariant Hermitian metrics defining the polarizations to one another; and in these terms *all the polarized Riemann matrices in a weak equivalence class describe polarized-isogenous polarized complex tori*, with the various polarized Riemann matrices in the weak equivalence class merely being descriptions of the polarizations of isogenous tori in various linear coordinate systems.

Appendix G

Theta Series

The classical theta function in one variable plays a significant role in the study of elliptic functions, which are just meromorphic functions on compact Riemann surfaces of genus $g = 1$; the theta function in several variables plays an equally significant role in the study of meromorphic functions on complex tori of higher dimensions. Although a detailed discussion of function theory on complex tori in higher dimensions would lead far too far afield[1], at least some of the basic properties of theta functions in several variables required in the discussion in this book will be reviewed here. The classical theta function is defined by the series

$$\theta(t; z) = \sum_{n\in\mathbb{Z}} \exp 2\pi i \left(\tfrac{1}{2} z n^2 + tn\right), \tag{G.1}$$

where $t \in \mathbb{C}$ and $z = x + iy \in \mathbb{C}$ with $y > 0$. It is no doubt quite familiar that this is a nontrivial entire function of the complex variable t and a holomorphic function of the complex variable z in the upper half-plane. The theta function in g variables is defined by the analogous series

$$\Theta(t; Z) = \sum_{n\in\mathbb{Z}^g} \exp 2\pi i \left(\tfrac{1}{2}\, {}^t n Z n + {}^t n t\right) \tag{G.2}$$

for a complex vector $t \in \mathbb{C}^g$ and a $g \times g$ complex matrix $Z \in \mathfrak{H}_g$, where $\mathfrak{H}_g$ is the Siegel upper half-space of rank g consisting of complex symmetric $g \times g$ matrices $Z = X + iY$ with positive definite imaginary part Y, as discussed on page 543. Since Y is positive definite ${}^t n Y n \geq \epsilon ||n||^2$ for some $\epsilon > 0$ and all $n \in \mathbb{Z}^g$, so

$$|\, {}^t n(Yn - 2\,i\,t)| \geq \epsilon ||n||^2 - 2|\, {}^t n t| \geq \epsilon ||n||^2 - 2||n|| \cdot ||t|| \geq \frac{\epsilon}{2} ||n||^2$$

[1] For a general discussion of complex tori from an analytic point of view see for instance F. Conforto, *Abelsche Funktionen und algebraische Geometrie*, (Springer, 1956), A. I. Markushevich *Introduction to the Classical Theory of Abelian Functions*, (American Mathematical Society, 1992), or C. Birkenhake and H. Lange, *Complex Abelian Varieties*, (Springer-Verlag, 1992); and from an algebraic as well as an analytic point of view see D. Mumford *Abelian Varieties*, (Oxford, 1970), or more extensively his *Tata Lectures on Theta*, volumes I, II, III, (Birkhäuser 1983, 1984, 1991).

whenever $||n|| \ge \frac{4}{\epsilon}||t||$, where as usual $||t||^2 = \sum_i |t_i|^2$; consequently

$$\begin{aligned} \left| \exp 2\pi i\, {}^t\!n\left(\tfrac{1}{2}Zn + t\right)\right| &= \left| \exp 2\pi i\, {}^t\!n\left(\tfrac{1}{2}iYn + t\right)\right| \\ &= \left| \exp -\pi\, {}^t\!n\,(Yn - 2\,i\,t)\right| \le \exp -\tfrac{1}{2}\pi\epsilon||n||^2 \end{aligned}$$

whenever $||n|| \ge \frac{4}{\epsilon}||t||$, so the series (G.2) is locally uniformly convergent for $(t, Z) \in \mathbb{C}^g \times \mathfrak{H}_g$ and hence it represents a holomorphic function on the product manifold $\mathbb{C}^g \times \mathfrak{H}_g$. For any fixed point $Z \in \mathfrak{H}_g$ this function is nontrivial in the variable $t \in \mathbb{C}^g$, since (G.2) is a Fourier series with nonzero coefficients.

The parameter of summation $n \in \mathbb{Z}^g$ in the series (G.2) can be replaced by $-n$, so that

$$\text{(G.3)} \qquad \Theta(t; Z) = \sum_{n\in\mathbb{Z}^g} \exp 2\pi i\left(\tfrac{1}{2}\,{}^t\!nZn - {}^t\!nt\right) = \Theta(-t; Z).$$

Thus the theta function is an even function of the variable $t \in \mathbb{C}^g$; and it can be described by either of the series expansions (G.2) or (G.3), so the sign of the term ${}^t\!nt$ in the series expansion can be chosen arbitrarily as convenient. It is clear from (G.2) or(G.3) that

$$\text{(G.4)} \qquad \Theta(t + \mu; Z) = \Theta(t; Z) \quad \text{for} \quad \mu \in \mathbb{Z}^g,$$

since $\exp 2\pi\, i\, {}^t\!n\mu = 1$. On the other for any $\nu \in \mathbb{Z}^g$ the parameter of summation $n \in \mathbb{Z}^g$ in (G.2) can be replaced by $n + \nu$, and since Z is a symmetric matrix it follows that

$$\begin{aligned} \Theta(t; Z) &= \sum_{n\in\mathbb{Z}^g} \exp 2\pi i\, {}^t(n+\nu)\left(\tfrac{1}{2}Z(n+\nu) + t\right) \\ &= \sum_{n\in\mathbb{Z}^g} \exp 2\pi i\Big({}^t\!n\left(\tfrac{1}{2}Zn + Z\nu + t\right) + {}^t\nu\left(\tfrac{1}{2}Z\nu + t\right)\Big) \\ &= \Theta(t + Z\nu; Z)\cdot \exp 2\pi i\left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu\, t\right); \end{aligned}$$

consequently

$$\text{(G.5)} \qquad \Theta(t + Z\nu; Z) = \Theta(t; Z)\cdot \exp -2\pi i\left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu\, t\right) \ \text{ for } \ \nu \in \mathbb{Z}^g.$$

Equations (G.4) and (G.5) determine the behaviour of the function $\Theta(t; Z)$ when the variable $t \in \mathbb{C}^g$ is translated by any vector λ in the lattice subgroup $\mathcal{L}(\Omega) = \Omega\mathbb{Z}^{2g} \subset \mathbb{C}^g$ spanned by the columns of the $g \times 2g$ period matrix $\Omega = (\mathrm{I} \quad Z)$. These equations taken together can be written as the condition that

$$\text{(G.6)} \qquad \Theta(t + \lambda; Z) = \Xi_Z(\lambda, t)\cdot\Theta(t; Z) \quad \text{for all} \quad \lambda = \mu + Z\nu \in \mathcal{L}(\Omega)$$

where

$$\text{(G.7)} \qquad \Xi_Z(\mu + Z\nu, t) = \exp -2\pi i\left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu\, t\right) \quad \text{for all} \quad \mu, \nu \in \mathbb{Z}^g.$$

That is just the condition that the theta function $\Theta(t; Z)$ associated to a point $Z \in \mathfrak{H}_g$, viewed as a function of the complex variable $t \in \mathbb{C}^g$, is a holomorphic relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda, t)$ for the action of the lattice subgroup $\mathcal{L}(\Omega)$ on the space $\mathbb{C}^g$; this factor of automorphy consequently is called the *theta factor of automorphy*. Of course in view of (G.3) the function $\Theta(t; Z)$ can be viewed as a relatively automorphic function for the larger group that arises by adjoining to the lattice group $\mathcal{L}(\Omega)$ the additional mapping $\iota : t \longrightarrow -t$, and extending the factor of automorphy (G.7) to the larger group by setting $\Xi_Z(\iota, t) = 1$; however it is usually more convenient to view the theta function as a symmetric relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda; t)$ for the lattice subgroup $\mathcal{L}(\Omega)$ itself. Slightly more generally, it is also possible to replace the parameter of summation $n \in \mathbb{Z}^g$ in the series (G.2) by $Q\,n$ for any matrix $Q \in \mathrm{Gl}(g, \mathbb{Z})$; it then follows from a straightforward calculation that

$$\Theta(t; Z) = \Theta({}^tQt;\, {}^tQZQ) \quad \text{for any} \quad Q \in \mathrm{Gl}(g, \mathbb{Z}), \tag{G.8}$$

hence $\Theta({}^tQt; Z) = \Theta(t; Z)$ for all matrices Q in the subgroup

$$\mathcal{F}(Z) = \Big\{\, Q \in \mathrm{Gl}(g, \mathbb{Z}) \;\Big|\; {}^tQ\,Z\,Q = Z \,\Big\}, \tag{G.9}$$

which for some matrices $Z \in \mathfrak{H}_g$ can be properly larger than the subgroup consisting merely of $\pm\mathrm{I}$. The theta factor of automorphy $\Xi_Z(\lambda, t)$ describes a holomorphic line bundle Ξ_Z over the complex torus $J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega)$, called the *theta bundle* over $J(\Omega)$; and the theta function $\Theta(t; Z)$ describes a holomorphic cross-section of that line bundle. The vector space of holomorphic cross-sections of this line bundle is denoted by $\Gamma(J(\Omega), \mathcal{O}(\Xi_Z))$, paralleling the notation for the space of holomorphic cross-sections of a holomorphic line bundle over a compact Riemann surface; and correspondingly the dimension of this vector space is denoted by $\gamma(\Xi_Z) = \dim \Gamma(J(\Omega), \mathcal{O}(\Xi_Z))$.

Theorem G.1 *The space of holomorphic cross-sections of the line bundle Ξ_Z over the complex torus $J(\Omega)$ associated to the period matrix $\Omega = \begin{pmatrix} \mathrm{I} & Z \end{pmatrix}$ for any matrix $Z \in \mathfrak{H}_g$ is one-dimensional, that is, $\gamma(\Xi_Z) = 1$.*

Proof: The theta function $\Theta(t; Z)$ for a matrix $Z \in \mathfrak{H}_g$ describes a nontrivial holomorphic cross-section of the bundle Ξ_Z so $\gamma(\Xi_Z) \geq 1$. On the other hand if $f(t) \in \Gamma(J(\Omega), \mathcal{O}(\Xi_Z))$ then $f(t)$ satisfies (G.4) so that $f(t + \mu) = f(t)$ for all $\mu \in \mathbb{Z}^g$; hence $f(t)$ must have a holomorphic Fourier expansion

$$f(t) = \sum_{n \in \mathbb{Z}^g} a_n \exp 2\pi i\, {}^tn \cdot t \tag{G.10}$$

for some constants a_n. Since $f(t)$ also satisfies (G.5), that result for the special case that $\nu = \delta_j$, where δ_j are the basis vectors in $\mathbb{R}^g$, shows that

$$\begin{aligned} f(t + Z\delta_j) &= f(t) \cdot \exp -2\pi i \left(t_j + \tfrac{1}{2} z_{jj}\right) \\ &= \left(\sum_{n \in \mathbb{Z}^g} a_n \exp 2\pi i \, {}^t n \cdot t \right) \cdot \exp -2\pi i \left(t_j + \tfrac{1}{2} z_{jj}\right) \\ &= \sum_{n \in \mathbb{Z}^g} a_n \exp 2\pi i \left({}^t(n - \delta_j) \cdot t - \tfrac{1}{2} z_{jj}\right) \\ &= \sum_{n \in \mathbb{Z}^g} a_{n+\delta_j} \exp 2\pi i \left({}^t n \cdot t - \tfrac{1}{2} z_{jj}\right), \end{aligned}$$

where the last equality follows by replacing the parameter of summation n in the preceding line by $n + \delta_j$. However a direct substitution in (G.10) shows that

$$f(t + Z\delta_j) = \sum_{n \in \mathbb{Z}^g} a_n \exp 2\pi i \, {}^t n \cdot (t + Z\delta_j).$$

Comparing the Fourier coefficients in the two preceding expansions of $f(t + Z\delta_j)$ shows that

$$a_{n+\delta_j} \cdot \exp -\pi i z_{jj} = a_n \exp 2\pi i \, {}^t n Z \delta_j \quad \text{for all } n;$$

and since this recurrence relation determines all the coefficients a_n in terms of a_0 it follows that the space of holomorphic cross-sections of the bundle Ξ_Z has dimension at most one, which suffices to conclude the proof.

It follows from the preceding result that the theta function in several variables viewed as a holomorphic function of the variable $t \in \mathbb{C}^g$ can be described uniquely up to a constant factor as a holomorphic relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda, t)$ for the action of the lattice subgroup $\mathcal{L}(\Omega)$ on the vector space $\mathbb{C}^g$, where $\Omega = \begin{pmatrix} \mathrm{I} & Z \end{pmatrix}$. The explicit formulas for the theta function and its factor of automorphy are for period matrices that are principally polarized Riemann matrices in the normal form $\Omega = \begin{pmatrix} \mathrm{I} & Z \end{pmatrix}$ where $Z \in \mathfrak{H}_g$. If Λ is a nonsingular period matrix describing a complex torus $J(\Lambda)$ of dimension h for which there is a holomorphic mapping

$$\phi : J(\Lambda) \longrightarrow J(\Omega), \tag{G.11}$$

the theta function associated to the torus $J(\Omega)$ induces a holomorphic function associated to the torus $J(\Lambda)$ that can be viewed as a generalized theta function. Explicitly, if the mapping ϕ is described by a Hurwitz relation (A, N) from the period matrix Λ to the period matrix Ω as in Theorem F.9, for a complex matrix $A \in \mathbb{C}^{g \times h}$ and an integral matrix $N \in \mathbb{Z}^{2g \times 2h}$, then $A \cdot \Lambda = \Omega \cdot N$ and the holomorphic function $f(t) = \Theta(At;\, Z)$ of the variable $t \in \mathbb{C}^h$ satisfies

$$\begin{aligned} f(t + \Lambda\nu) &= \Theta(At + A\Lambda\nu; Z) = \Theta(At + \Omega \cdot N\nu; Z) \\ &= \Xi_Z(\Omega \cdot N\nu, At) \cdot \Theta(At; z) = \Xi_Z(\Omega N\nu, At) \cdot f(t) \end{aligned}$$

for $\nu \in \mathbb{Z}^{2h}$. If $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ where $N_1, N_2 \in \mathbb{Z}^{g\times 2h}$, so that $\Omega\cdot N\nu = N_1\nu + ZN_2\nu$, the factor of automorphy $\Xi_Z(\Omega\cdot N\nu, At)$ can be written out explicitly as in (G.7) and the preceding equation takes the form

$$\text{(G.12)}\qquad f(t+\Lambda\nu) = f(t)\exp -2\pi i\left(\tfrac{1}{2}\,{}^t\nu\,{}^tN_2ZN_2\nu + {}^t\nu\,{}^tN_2At\right)\quad \text{for } \nu\in\mathbb{Z}^{2h}.$$

Of course the function $f(t)$ may vanish identically if the image of the mapping (G.11) is contained in the zero locus of the theta function $\Theta(t;Z)$. On the other hand if the mapping (G.11) is a biholomorphic mapping then the function $f(t)$ is a nontrivial relatively automorphic function for the induced factor of automorphy, and is determined uniquely up to a constant factor by (G.12).

The zero locus of the theta function $\Theta(t)$ is a proper holomorphic subvariety $\widetilde{V}_\Theta \subset \mathbb{C}^g$ of dimension $g-1$ that is invariant under translation by any vector in the lattice subgroup $\mathcal{L}(\Omega)\subset\mathbb{C}^g$ spanned by the columns of the period matrix Ω, as a consequence of (G.6). This subvariety thus represents a holomorphic subvariety $V_\Theta = \widetilde{V}_\Theta/\mathcal{L}(\Omega)$ of dimension $g-1$ in the quotient torus $J(\Omega) = \mathbb{C}^g/\mathcal{L}(\Omega)$, sometimes called the *theta locus* in $J(\Omega)$. Since the theta function is an even function by (G.3) it follows that

$$\text{(G.13)}\qquad V_\Theta = -V_\Theta \subset J(\Omega);$$

more generally $V_\Theta = {}^tQV_\Theta$ for any matrix $Q\in\mathcal{F}(Z)$ where $\mathcal{F}(Z)\subset \mathrm{Gl}(g,\mathbb{Z})$ is the subgroup defined by (G.9), as an immediate consequence of (G.8). A particularly interesting finite set of points on the complex torus $J(\Omega)$, actually a finite subgroup of that complex torus, is the set of *half-periods*, the set of 2^{2g} points $\delta_i\in J(\Omega)$ such that $2\delta_i\in J(M)$ is the point of the torus represented by the origin in $\mathbb{C}^g$; alternatively this is the subset $\frac{1}{2}\Omega\mathbb{Z}^{2g}/\Omega\mathbb{Z}^{2g}\subset J(\Omega)$. The subset of *real half-periods* for the period matrix $\Omega = \begin{pmatrix} \mathrm{I} & Z\end{pmatrix}$ is the subset $\frac{1}{2}\mathbb{Z}^g/\Omega\mathbb{Z}^{2g}\subset J(\Omega)$ consisting of 2^g of the half-periods. Frequently points of $\mathbb{C}^{2g}$ representing the half-periods of the torus $\mathbb{C}^{2g}/\mathcal{L}(\Omega)$ also are called half-periods; that is essentially an equivalent definition, so the identification of the terms should not cause any confusion.

For any integer $r>0$ it follows from (G.6) that the function $f(t) = \Theta(t;Z)^r$ is a holomorphic relatively automorphic function for the factor of automorphy $\Xi_Z(\lambda,t)^r$ in terms of the period matrix $\Omega = \begin{pmatrix} \mathrm{I} & Z\end{pmatrix}$, the r-the power of the theta factor of automorphy for that period matrix, so that for any $\mu,\nu\in\mathbb{Z}^g$

$$\text{(G.14)}\qquad f(t+\mu+Z\nu;Z) = f(t)\cdot\exp -2\pi i\, r\,\left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu t\right).$$

The holomorphic relatively automorphic functions for this factor of automorphy are called *theta functions of order* r for the period matrix $\Omega = \begin{pmatrix} \mathrm{I} & Z\end{pmatrix}$, and describe holomorphic cross-sections of the r-th power Ξ_Z^r of the theta bundle over M; as before, the dimension of the vector space of theta functions of order r is denoted by $\gamma(\Xi_Z^r)$.

Theorem G.2 *The vector space of theta functions of order* $r>0$ *for the period matrix* $\Omega = \begin{pmatrix} \mathrm{I} & Z\end{pmatrix}$ *has dimension* r^g*, that is,* $\gamma(\Xi_Z^r) = r^g$.

Proof: The holomorphic relatively automorphic functions for the theta factor of automorphy $\Xi_Z(\mu + Z\nu)^r$ are holomorphic functions $f(t)$ of the variables $t \in \mathbb{C}^g$ satisfying (G.14). As in the proof of Theorerm G.1, for $\nu = 0$ it follows that the function $f(t)$ is invariant under translation through integer vectors, so it has a complex Fourier expansion

$$f(t) = \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \, {}^t n \cdot t \tag{G.15}$$

for some $c_n \in \mathbb{C}$. Then for $\mu = 0$ it follows from (G.14) that

$$\begin{aligned} f(t + Z\nu) &= f(t) \cdot \exp -2\pi i r \left(\tfrac{1}{2}\, {}^t\nu Z \nu + {}^t\nu t\right) \\ &= \left(\sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \, {}^t n \cdot t\right) \cdot \exp -2\pi i r \left(\tfrac{1}{2}\, {}^t\nu Z \nu + {}^t\nu t\right) \\ &= \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \left({}^t(n - r\nu) \cdot t - \tfrac{r}{2}\, {}^t\nu Z \nu\right) \\ &= \sum_{n \in \mathbb{Z}^g} c_{n+r\nu} \exp 2\pi i \left({}^t n \cdot t - \tfrac{r}{2}\, {}^t\nu Z \nu\right) \end{aligned} \tag{G.16}$$

where the last equality follows by replacing the parameter of summation n in the preceding line by $n + r\nu$; on the other hand replacing t in (G.15) by $t + Z\nu$ shows that

$$f(t + Z\nu) = \sum_{n \in \mathbb{Z}^g} c_n \exp 2\pi i \, {}^t n \cdot (t + Z\nu). \tag{G.17}$$

Upon comparing the Fourier coefficients in the Fourier expansions (G.16) and (G.17) it follows that

$$c_{n+r\nu} = c_n \exp 2\pi i \left({}^t n Z \nu + \tfrac{r}{2}\, {}^t\nu Z \nu\right) \quad \text{for all } n \in \mathbb{Z}^g. \tag{G.18}$$

Since every vector $n \in \mathbb{Z}^g$ can be written uniquely as $n = \delta + r\nu$ for some $\delta = (\delta_1, \ldots, \delta_g) \in \mathbb{Z}^g$ for which $0 \le \delta_i < r$ and some $\nu \in \mathbb{Z}^g$ it follows that all the coefficients c_n of the Fourier expansion (G.15) are determined by the r^g coefficients c_δ through the recurrence relation (G.18), where these coefficients c_δ can be chosen arbitrarily; hence $\gamma(\Xi_Z^r) = r^g$, which suffices for the proof.

The special case of second order theta functions is particularly interesting. The preceding theorem shows that the space of second-order theta functions has dimension 2^g. For notational convenience identify $\mathbb{Z}_2^g$ with the set of g-tuples of integers $\delta = (\delta_1, \ldots, \delta_g)$ where $\delta_i = 0$ or 1, or equivalently, with the set of real half-periods for the period matrix Ω; in these terms, every vector $n \in \mathbb{Z}^g$ can be written uniquely as $n = \delta + r\nu$ for some $\delta \in \mathbb{Z}_2^g$ and some $\nu \in \mathbb{Z}^g$. Then by using the recurrence relation (G.18) for the Fourier coefficients of second-order theta functions and the symmetry ${}^tZ = Z$, any second-order theta function $f(t)$

can be written

$$
\begin{aligned}
\text{(G.19)} \qquad f(t) &= \sum_{\substack{\delta\in\mathbb{Z}_2^g\\ \nu\in\mathbb{Z}^g}} c_{\delta+2\nu} \exp 2\pi i\, {}^t(\delta+2\nu)t \\
&= \sum_{\substack{\delta\in\mathbb{Z}_2^g\\ \nu\in\mathbb{Z}^g}} c_\delta \exp 2\pi i \left({}^t\delta Z\nu + {}^t\nu Z\nu + {}^t(\delta+2\nu)t\right) \\
&= \sum_{\delta\in\mathbb{Z}_2^g} c_\delta \exp 2\pi i\, {}^t\delta t \cdot \sum_{\nu\in\mathbb{Z}^g} \exp 2\pi i \left({}^t\nu Z\nu + {}^t\nu(2t+Z\delta)\right) \\
&= \sum_{\delta\in\mathbb{Z}_2^g} c_\delta \exp 2\pi i\, {}^t\delta t \cdot \Theta(2t+Z\delta; 2Z)
\end{aligned}
$$

in view of (G.2), or equivalently

$$
\text{(G.20)} \qquad f(t) = \sum_{\delta\in\mathbb{Z}_2^g} c_\delta\ \Theta_{2,\delta}(t;Z)
$$

where

$$
\text{(G.21)} \qquad \Theta_{2,\delta}(t;Z) = \exp 2\pi i\, {}^t\delta t \cdot \Theta(2t+Z\delta; 2Z).
$$

It follows from (G.6) and (G.7) that for any $\mu,\nu\in\mathbb{Z}^g$

$$
\begin{aligned}
\Theta_{2,\delta}(t+\mu+Z\nu;Z) &= \exp 2\pi i\, {}^t\delta(t+\mu+Z\nu)\cdot\Theta(2t+2\mu+2Z\nu+Z\delta;2Z) \\
&= \exp 2\pi i\, {}^t\delta(t+\mu+Z\nu)\cdot \\
&\qquad \cdot \exp -2\pi i \left({}^t\nu Z\nu + {}^t\nu(2t+Z\delta)\right)\cdot\Theta(2t+Z\delta;2Z) \\
&= \exp 2\pi i\, {}^t\delta t\cdot \exp -2\pi i\left({}^t\nu Z\nu + 2\, {}^t\nu t\right)\cdot\Theta(2t+Z\delta;2Z) \\
&= \exp -2\pi i\cdot ({}^t\nu Z\nu + 2\, {}^t\nu t)\Theta_{2,\delta}(t;Z),
\end{aligned}
$$

so in view of (G.14) each of the functions $\Theta_{2,\delta}(t;Z)$ is a second-order theta function and (G.20) is an expansion of an arbitrary second-order theta function as a linear combination of the 2^gfunctions $\Theta_\delta(t;Z)$, which consequently are a basis for the vector space of second-order theta functions. The Fourier expansions of these basic second-order theta functions also can be read directly from (G.19), since it is clear from the second line of that formula that

$$
\text{(G.22)} \qquad \Theta_{2,\delta}(t;Z) = \sum_{\nu\in\mathbb{Z}^g} \exp 2\pi i\, {}^t\nu Z(\nu+\delta)\cdot \exp 2\pi i\, {}^t(\delta+2\nu)t.
$$

Thus the basic second-order theta function $\Theta_{2,\delta}(t;Z)$ has nonzero Fourier coeffcients only for indices $\delta+2\nu\equiv\delta \pmod 2$, and all these Fourier coefficients are nonzero; in particular for $\nu=0$ the Fourier expansion includes the term $1\cdot\exp 2\pi i\, {}^t\delta t$, so the functions $\Theta_{2,\delta}(t;Z)$ are in a natural sense the standard basis for the set of second-order theta functions.

For any integral vectors $\delta, \mu, \nu \in \mathbb{Z}^g$ it follows from (G.6) and (G.7) that

$$\begin{aligned}\Theta(t + \tfrac{1}{2}\delta + \mu + Z\nu; Z) &= \exp -2\pi i \left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu(t + \tfrac{1}{2}\delta)\right) \cdot \Theta(t + \tfrac{1}{2}\delta; Z)\\ &= (-1)^{\,{}^t\nu\cdot\delta} \exp -2\pi i \left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu t\right) \cdot \Theta(t + \tfrac{1}{2}\delta; Z)\end{aligned}$$

and consequently that

$$\text{(G.23)} \qquad \Theta(t + \tfrac{1}{2}\delta + \mu + Z\nu; Z)^2 = \exp -4\pi i \left(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu t\right) \cdot \Theta(t + \tfrac{1}{2}\delta; Z)^2;$$

so in view of (G.14) the squares $\Theta(t + \frac{1}{2}\delta; Z)^2$ for all parameter values $\delta \in \mathbb{Z}_2^g$ are 2^g second-order theta functions.

Theorem G.3 *The second-order theta functions functions $\Theta(t + \frac{1}{2}\delta; Z)^2$ for real half-periods $\delta \in \mathbb{Z}_2^g$ can be written in terms of the basic second-order theta functions $\Theta_{2,\delta}(t; Z)$ as*

$$\text{(G.24)} \qquad \Theta(t + \tfrac{1}{2}\delta; Z)^2 = \textstyle\sum_{\epsilon\in\mathbb{Z}_2^g} (-1)^{\,{}^t\epsilon\cdot\delta} e^{\pi i\,{}^t\epsilon Z\epsilon}\Theta(Z\epsilon; 2Z) \cdot \Theta_{2,\epsilon}(t; Z);$$

consequently $\Theta(Z\epsilon; 2Z) \neq 0$ for at least some $\epsilon \in \mathbb{Z}_2^g$, and the dimension of the space of second-order theta functions spanned by the squares $\Theta(t + \frac{1}{2}\delta; Z)^2$ is equal to the number of real half-periods $\epsilon \in \mathbb{Z}_2^g$ such that $\Theta(Z\epsilon; 2Z) \neq 0$.

Proof: By the definition (G.2) of the theta function

$$\Theta(t + \tfrac{1}{2}\delta; Z) = \textstyle\sum_{n\in\mathbb{Z}^g} (-1)^{\,{}^tn\cdot\delta} \exp 2\pi i \left(\tfrac{1}{2}\,{}^tnZn + {}^tnt\right)$$

so

$$\Theta(t + \tfrac{1}{2}\delta; Z)^2 = \textstyle\sum_{m,n\in\mathbb{Z}^g} (-1)^{\,{}^t(m+n)\cdot\delta} \exp 2\pi i \left(\tfrac{1}{2}\,{}^tnZn + \tfrac{1}{2}\,{}^tmZm + {}^t(m+n)t\right),$$

and setting $m = \nu - n$ this can be rewritten

$$\begin{aligned}\Theta(t + \tfrac{1}{2}\delta; Z)^2 &= \sum_{\nu,n\in\mathbb{Z}^g} (-1)^{\,{}^t\nu\cdot\delta} \exp 2\pi i \left(\tfrac{1}{2}\,{}^tnZn + \tfrac{1}{2}\,{}^t(\nu - n)Z(\nu - n) + {}^t\nu t\right)\\ &= \sum_{\nu\in\mathbb{Z}^g} (-1)^{\,{}^t\nu\cdot\delta} \exp 2\pi i \left(\tfrac{1}{2}{}^t\nu Z\nu + {}^t\nu t\right) \cdot \sum_{n\in\mathbb{Z}^g} \exp 2\pi i \left({}^tnZn - {}^tnZ\nu\right)\\ &= \sum_{\nu\in\mathbb{Z}^g} (-1)^{\,{}^t\nu\cdot\delta} \exp 2\pi i \left(\tfrac{1}{2}{}^t\nu Z\nu + {}^t\nu t\right) \cdot \Theta(Z\nu; 2Z);\end{aligned}$$

$\blacksquare$

thus the Fourier series expansion of $\Theta(t + \frac{1}{2}\delta; Z)^2$ is

$$\text{(G.25)} \qquad \Theta(t + \tfrac{1}{2}\delta; Z)^2 = \textstyle\sum_{\nu\in\mathbb{Z}^g} (-1)^{\,{}^t\nu\cdot\delta}\ a_\nu\ e^{2\pi i\,{}^t\nu t}$$

where

$$\text{(G.26)} \qquad a_\nu = \exp \pi i \left({}^t\nu Z\nu\right) \cdot \Theta(Z\nu; 2Z).$$

This function can be written as a linear combination

$$\Theta(t+\tfrac{1}{2}\delta;Z)^2 = \textstyle\sum_{\epsilon\in\mathbb{Z}_2^g} b_\epsilon \Theta_{2,\epsilon}(t;Z)$$

of the basis $\Theta_{2,\epsilon}(t;Z)$ for the space of second-order theta functions, and comparing the Fourier coefficients of $\exp 2\pi i\,{}^t\epsilon t$ for these second-order theta functions shows that

$$b_\epsilon = (-1)^{\,{}^t\epsilon\cdot\delta} a_\epsilon,$$

which yields (G.24). The final conclusion of the theorem is an immediate consequence, since the squares $\Theta(t+\frac{1}{2}\delta;Z)^2$ are nontrivial second-order theta functions, and that suffices for the proof.

Corollary G.4 *The translates $\Theta(t+\frac{1}{2}\delta;Z)$ of the theta function satisfy the quadratic equations*

$$\text{(G.27)}\qquad \sum_{\delta\in\mathbb{Z}_2^g} (-1)^{\,{}^t\gamma\cdot\delta}\Theta(t+\tfrac{1}{2}\delta;Z)^2 = 0$$

for all $\gamma\in\mathbb{Z}_2^g$ such that $\Theta(Z\gamma;2Z)=0$.

Proof: It is clear that for any $\epsilon,\gamma\in\mathbb{Z}_2^g$

$$\sum_{\delta\in\mathbb{Z}_2^g} (-1)^{\,{}^t(\epsilon-\gamma)\cdot\delta} = \begin{cases} 2^g & \text{if} \quad \gamma=\epsilon, \\ 0 & \text{if} \quad \gamma\neq\epsilon \end{cases}$$

From this and (G.24) it follows that

$$\begin{aligned}\sum_{\delta\in\mathbb{Z}_2^g} (-1)^{\,{}^t(\epsilon-\gamma)\cot\delta}\Theta(2+\tfrac{1}{2}\delta;Z)^2 &= \sum_{\delta,\epsilon\in\mathbb{Z}_2^g} (-1)^{\,{}^t(\epsilon-\gamma)\cdot\delta} e^{\pi i\,{}^t\epsilon Z\epsilon}\Theta(Z\epsilon;2Z)\cdot\Theta_{2,\epsilon}(t;Z) \\ &= 2^g e^{\pi i\,{}^t\gamma Z\gamma}\Theta(Z\gamma;2Z)\cdot\Theta_{2,\gamma}(t;Z). \qquad \blacksquare\end{aligned}$$

In particular then

$$\sum_{\delta\in\mathbb{Z}_2^g} (-1)^{\,{}^t(\epsilon-\gamma)\cot\delta}\Theta(2+\tfrac{1}{2}\delta;Z)^2 = 0$$

whenever $\gamma\in\mathbb{Z}_2^g$ is a real half-period for which $\Theta(Z\gamma;2Z)=0$, and that suffices for the proof.

More generally the complex tori $J(\Omega)$ are homogeneous spaces under arbitrary translations, and these translations have a fairly simple effect on theta functions. Indeed if $\Theta(t;s,Z)=\Theta(s+t;Z)$ is the translate of the theta function $\Theta(t;z)$ through s it follows from (G.4), (G.5) and (G.7) that

$$\text{(G.28)}\qquad \begin{aligned}\Theta(t+\mu+Z\nu;s,Z) &= \Theta(s+t+\mu+Z\nu;Z) \\ &= \Theta(s+t;Z)\cdot\exp -2\pi i\big(\tfrac{1}{2}\,{}^t\nu Z\nu + {}^t\nu(t+s)\big) \\ &= \exp -2\pi i\,{}^t\nu\, s\cdot\Xi_Z(\mu+Z\nu,t)\cdot\Theta(t;s,Z);\end{aligned}$$

thus $\Theta(t;s,Z)$ is a relatively automorphic function for the product of the flat factor of automorphy

$$\sigma_s(\mu+Z\nu)=\exp -2\pi i\,{}^t\nu\, s \tag{G.29}$$

and the theta factor of automorphy $\Xi_Z(\mu+Z\nu,t)$. It is clear from its definition that the flat factor of automorphy $\sigma_s(\mu+Z\nu)$ is trivial if and only if $s\in\mathbb{Z}^g$; and it is easy to see that it is analytically trivial if and only if $s\in\mathcal{L}(\Omega)$. Indeed by definition this factor of automorphy is analytically trivial if and only if there is a nowhere vanishing holomorphic function $\phi(t)$ in $\mathbb{C}^g$ such that $\phi(t+\mu+Z\nu)=\sigma_s(\mu+Z\nu)\phi(t)$ for all $\mu,\nu\in\mathbb{Z}^g$. Any such function can be written as $\phi(t)=\exp 2\pi i h(t)$ for some holomorphic function $h(t)$ on $\mathbb{C}^g$, and in terms of this function the flat factor of automorphy σ_s is analytically trivial if and only if

$$h(t+\mu+Z\nu)=h(t)-{}^t\nu\, s-m(\mu,\nu)\quad\text{for some } m(\mu,\nu)\in\mathbb{Z}. \tag{G.30}$$

If there is such a function $h(t)$ then $\partial_j h(t)$ is invariant under translations through the lattice vectors in $\mathcal{L}(\Omega)$ for any index $1\le j\le g$, so $\partial_j h(t)$ is a bounded holomorphic function in $\mathbb{C}^g$ hence it must be constant; the function $h(t)$ consequently must be a linear function $h(t)={}^t\alpha t$ for some $\alpha\in\mathbb{C}^g$. In that case (G.30) reduces to the condition that ${}^t\alpha(\mu+Z\nu)=-{}^t\nu s-m(\mu,\nu)$ for all $\mu,\nu\in\mathbb{Z}^{2g}$. In particular for $\nu=0$ it follows that ${}^t\alpha\cdot\nu\in\mathbb{Z}$ for all $\mu\in\mathbb{Z}^{2g}$, hence that $\alpha\in\mathbb{Z}^{2g}$, and therefore ${}^t\nu s\in\mathcal{L}(\Omega)$ for all $\nu\in\mathbb{Z}^{2g}$ so that $s\in\mathcal{L}(\Omega)$. Conversely if $s\in\mathcal{L}(\Omega)$ so that $s=m+Zn$ for $m,n\in\mathbb{Z}^{2g}$, and if $h(t)=-{}^tnt$ then $h(t+\mu+Z\nu)-h(t)-{}^t\nu(m+Zn)=-{}^tn(\mu+Z\nu)-{}^t\nu(m+Zn)=-{}^tn\mu+{}^t\nu m\in\mathbb{Z}$ so $\sigma_s(\mu+Z\nu)$ is analytically trivial. In general, if $s\in\frac{1}{r}\mathbb{Z}^g$ for an integer r it is clear from (G.29) that $\sigma_s(\mu+Z\nu)^r=1$ for all μ,ν, hence that $\Theta(t+\mu+Z\nu;s,Z)^r=\xi_Z(\mu+Z\nu)^r\Theta(t;s,Z)^r$; thus the r-th power of the translate $\theta(t;s,Z)$ for any such a value s is also a relatively automorphic function for the r-power of the theta factor of automorphy, so

$$\Theta(t;s,Z)^r\in\Gamma(M,\Xi_Z^r)\quad\text{whenever}\quad s\in\tfrac{1}{r}\mathbb{Z}^g. \tag{G.31}$$

Classically translates of the theta function were handled by writing a vector $s\in\mathbb{C}^{2g}$ as the sum

$$s=Z\alpha+\beta\quad\text{for some}\quad \alpha,\,\beta\in\mathbb{C}^g, \tag{G.32}$$

since the columms of the matrix $\Omega=\begin{pmatrix}\mathrm{I} & Z\end{pmatrix}$ are a basis for the vector space $\mathbb{C}^{2g}$. The translate $\Theta(t;s,Z)=\Theta(s+t;Z)$ of the theta series through a vector s of the form (G.32) was expressed[2] in terms of a *theta series with characteristic* $[\alpha|\beta]$, defined by

$$\Theta[\alpha|\beta](t;Z)=\sum_{n\in\mathbb{Z}^g}\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t(n+\alpha)Z(n+\alpha)+{}^t(n+\alpha)(t+\beta)\Big). \tag{G.33}$$

[2]The classical notatation is $\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(t;Z)$, which has been used quite persistently despite its rather ungainly form; in the present book a modified notation is used, for purely aesthetic reasons.

Since

$$\tfrac{1}{2}\,{}^t(n+\alpha)Z(n+\alpha)+{}^t(n+\alpha)(t+\beta)=\tfrac{1}{2}\,{}^tnZn+{}^tn(t+Z\alpha+\beta)+\tfrac{1}{2}\,{}^t\alpha Z\alpha+{}^t\alpha(t+\beta)$$

the series (G.33) can be written

$$\text{(G.34)}\qquad \Theta[\alpha|\beta](t;Z)=\Theta(t+Z\alpha+\beta;Z)\cdot\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t\alpha Z\alpha+\,{}^t\alpha(t+\beta)\Big),$$

so the theta function with characteristic $[\alpha,\beta]$ is the product of the exponential of a simple linear function and the translate $\Theta(t;s,Z)$ of the theta function $\Theta(t;Z)$ through the vector $s=Z\alpha+\beta$. It then follows from (G.28) that for any $\mu,\nu\in\mathbb{Z}^g$

(G.35)

$$\begin{aligned}\Theta[\alpha|\beta](t+\mu+Z\nu;Z)&=\Theta(t+\mu+Z\nu+Z\alpha+\beta;Z)\cdot\\&\qquad\cdot\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t\alpha Z\alpha+\,{}^t\alpha(t+\mu+Z\nu+\beta)\Big)\\&=\Theta(t+Z\alpha+\beta;Z)\exp -2\pi i\big(\tfrac{1}{2}\,{}^t\nu Z\nu+\,{}^t\nu(t+Z\alpha+\beta)\big)\cdot\\&\qquad\cdot\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t\alpha Z\alpha+\,{}^t\alpha(t+\mu+Z\nu+\beta)\Big)\\&=\Theta[\alpha|\beta](t;Z)\exp -2\pi i\Big(\tfrac{1}{2}\,{}^t\nu Z\nu-\,{}^t\mu\alpha+\,{}^t\nu(t+\beta)\Big),\end{aligned}$$

which can be rewritten

$$\text{(G.36)}\quad \Theta[\alpha|\beta](t+\mu+Z\nu;Z)=\sigma_{[\alpha|\beta]}(\mu+Z\nu)\cdot\Xi_Z(\mu+Z\nu,t)\cdot\Theta[\alpha|\beta](t;Z)$$

for the flat factor of automorphy $\sigma_{[\alpha|\beta]}\in\operatorname{Hom}(\mathcal{L}(\Omega),\mathbb{C}^*)$ for which

$$\text{(G.37)}\qquad \sigma_{[\alpha|\beta]}(\mu+Z\nu)=\exp 2\pi i\,\big({}^t\mu\alpha-\,{}^t\nu\beta\big)\,.$$

The theta function with characteristic $[\alpha,\beta]$ consequently is a relatively automorphic function for the product of the flat factor of automorphy $\sigma_{[\alpha|\beta]}(\mu+Z\nu)$ and the theta factor of automorphy $\Xi_Z(\mu+Z\nu,t)$.

Theta functions with rational characteristics are particularly interesting. For example, if α and β are half-integer vectors then $Z\alpha+\beta$ is a half-period for the period matrix $\Omega=\begin{pmatrix}\mathrm{I} & Z\end{pmatrix}$, while if $\alpha=0$ and β is a half-integer vector then $Z\alpha+\beta$ is a real half-period. In general a theta function $\Theta[\alpha|\beta](t;Z)$ with half-integral characteristic is either an odd or an even function of t. Of course $\Theta(t;Z)=\Theta[0|0](t;Z)$ is an even function, as noted in (G.3), and it follows from the definition (G.33) upon replacing the index of summation n by m where $n+\alpha=-(m+\alpha)$, so m ranges through $\mathbb{Z}^g$ as n does, that

$$\begin{aligned}\Theta[\alpha|\beta](-t;Z)&=\sum_{n\in\mathbb{Z}^g}\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t(n+\alpha)Z(n+\alpha)+\,{}^t(n+\alpha)(-t+\beta)\Big)\\&=\sum_{m\in\mathbb{Z}^g}\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t(m+\alpha)Z(m+\alpha)-\,{}^t(m+\alpha)(-t+\beta)\Big)\\&=\sum_{m\in\mathbb{Z}^g}\exp 2\pi i\Big(\tfrac{1}{2}\,{}^t(m+\alpha)Z(m+\alpha)+\,{}^t(m+\alpha)(t+\beta)-2\,{}^t\alpha\beta\Big)\end{aligned}$$

since $2\beta \in Z^g$ hence $\exp 4\pi i\, {}^t m\beta = 1$; consequently

$$\Theta[\alpha|\beta](-t;Z) = \exp -4\pi i\, {}^t\alpha\beta \cdot \Theta[\alpha|\beta](t;Z). \tag{G.38}$$

The half-integral characteristic $[\alpha|\beta]$ is said to be *even* if $4\,{}^t\alpha\beta \in \mathbb{Z}$ is an even integer and *odd* if $4\,{}^t\alpha\beta \in \mathbb{Z}$ is an odd integer, so that

$$\exp -4\pi i[\alpha|\beta] = \begin{cases} 1 & \text{if } [\alpha|\beta] \text{ is even,} \\ -1 & \text{if } [\alpha|\beta] \text{ is odd .} \end{cases} \tag{G.39}$$

Consequently the theta function $\Theta[\alpha|\beta](t;Z)$ with an odd half-integral characteristic is an odd function so has a zero of odd order at the origin; and by (G.34) the zero of the function $\Theta[\alpha|\beta](t;Z)$ at the origin has the same order as that of the zero of the theta function $\Theta(t;Z)$ at the point $Z\alpha+\beta$, so the theta function $\Theta(t;Z)$ has a zero of odd order at any half-integral point $Z\alpha+\beta$ for which $[\alpha|\beta]$ is an odd characteristic. Similarly the theta function $\Theta[\alpha|\beta](t;Z)$ with an even half-integral characteristic is an even function so is either nonzero or has a zero of even order at the origin; and by (G.34) the zero of the function $\Theta[\alpha|\beta](t;Z)$ at the origin has the same order as that of the zero of the theta function $\Theta(t;Z)$ at the point $Z\alpha+\beta$, so the theta function $\Theta(t;Z)$ is either nonzero or has a zero of even order at any half-integral point $Z\alpha+\beta$ for which $[\alpha|\beta]$ is an even characteristic. There are $2^{g-1}(2^g+1)$ even half-periods and $2^{g-1}(2^g-1)$ odd half-periods, which is essentially demonstrated in Corollary **??** in Chapter **??**.