Vortices, Gauged Sigma Model, and Kirwan Map

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1. Sigma Model and Holomorphic Curves
2. Symplectic Vortex Equation
3. Quantization of the Kirwan Map
4. Open Quantum Kirwan Map
I. Sigma Model and Holomorphic Curves
• For many reasons people are interested in studying the space of smooth maps between two manifolds with respect to an energy functional.

• The simplest example is that of geodesics in a Riemannian manifold, which are minimizers of the energy functional

\[ E(x) = \frac{1}{2} \int_a^b |x'(t)|^2 dt, \quad x : [a, b] \to X. \]

• \textbf{σ-model}: the theory of maps \( u : D \to X \) with Lagrangians

\[ E(u) \approx \frac{1}{2} \int_D |\nabla u|^2 dvol_X + \cdots. \]

• If we include “superpartners” to the maps \( u \), then theories with supersymmetries become extremely simple and beautiful.
• Critical dimension: conformal invariance.
• Propagation of strings.
• $\mathcal{N} = (2, 2)$: $X$ is Kähler, instantons are **holomorphic curves**.
• **Gromov–Witten Invariants** (correlation functions)
  \[ \langle \alpha_1, \ldots, \alpha_k \rangle^X_{\Sigma} \in F, \quad \alpha_1, \ldots, \alpha_k \in H^*(X; F). \]
• **Quantum cohomology**
  \[ \langle \alpha \ast \beta, \gamma \rangle := \langle \alpha, \beta, \gamma \rangle^X_{S^2}, \quad \forall \alpha, \beta, \gamma \in H^*(X; F). \]
• More generally, a **cohomological field theory** (CohFT)
  \[ \langle \alpha_1, \ldots, \alpha_n; Z \rangle^X_{g,n}, \quad \forall \alpha_i \in H^*(X), \quad Z \in H^*(\overline{\mathcal{M}}_{g,n}). \]
A sequence of holomorphic curves may bubble off holomorphic spheres. Domain can also degenerate: a sphere with four markings can degenerate into a stable curve with two components.

Two different degenerations can be connected in $\overline{M}_{0,4}$. The associativity of quantum product essentially follows from this picture.

More general splitting axioms of CohFT also follows from this kind of relations in $\overline{M}_{g,n}$.
II. The Symplectic Vortex Equation
• Let $X$ be Kähler/symplectic. Let $K$ be a compact Lie group acting on $X$. A **moment map** is a smooth equivariant mapping $\mu : X \rightarrow \mathfrak{k}^*$ satisfying

$$d(\mu \cdot \xi) = \omega(X_\xi, \cdot), \quad \forall \xi \in \mathfrak{k}.$$

• If $0 \in \mathfrak{k}^*$ is a regular value of $\mu$, then $\bar{X} := \mu^{-1}(0)/K$ is an orbifold or manifold, called the **symplectic reduction**.

• Many examples come from GIT (geometric invariant theory).

**Example**

$X = \mathbb{C}^2$, $K = S^1$ acts diagonally. Then a moment map is

$$\mu(z_1, z_2) = \frac{i}{\pi} \left(|z_1|^2 + |z_2|^2 - c\right).$$

For $c > 0$, $\bar{X} \simeq \mathbb{P}^1$. (All toric manifolds are obtained in similar ways.)
• Given \((X, K, \mu)\) and Riemann surface \(\Sigma\), variables of the symplectic vortex equation are gauge fields \(A \in \Omega^1(\Sigma, \mathfrak{k})\) and matter fields \(u : \Sigma \to X\).

• The **Yang–Mills–Higgs** functional reads

\[
\mathcal{YHM}(A, u) = \frac{1}{2} \int_\Sigma \left( |d_A u|^2 + |F_A|^2 + |\mu(u)|^2 \right) dvol_\Sigma.
\]

• If \(A = \phi ds + \psi dt\), then

\[
d_A u = ds \otimes (\partial_s u + X_\phi(u)) + dt \otimes (\partial_t u + X_\psi(u)).
\]

• The equation reads

\[
\overline{\partial}_A u = 0, \quad \ast F_A + \mu(u) = 0.
\]

• In general, given a nontrivial principal \(K\)-bundle \(P \to \Sigma\), \(A\) is a connection on \(P\) and \(u\) is a section of \(P \times_K X\).
For vector bundles, there is the so-called Hitchin–Kobayashi correspondence (Narasimhan–Seshadri, Donaldson, Uhlenbeck–Yau).

- Bradlow: HK correspondence for stable pairs (generalized by Banfield).

For example, given a holomorphic line bundle $L \to \Sigma$ and holomorphic sections $\phi, \psi \in H^0(\Sigma, L)$, there is a unique metric $H$ on $L$ such that

$$\ast F_H + \frac{i}{\pi} \left( |\phi|^2 + |\psi|^2 - c \right) = 0.$$  

It is equivalent to a vortex $(A, u)$ for $X = \mathbb{C}^2$ and $K = S^1$, where $A$ is the Chern connection of $(L, H)$ and $u = (\phi, \psi)$.

**Theorem (General case, Mundet)**

*Given a Kähler manifold $X$ acted by a reductive group $G = K^\mathbb{C}$. A gauge class of vortices over $\Sigma$ is equivalent to a holomorphic $G$-bundle $P \to \Sigma$ with a holomorphic section $u : \Sigma \to P \times_G X$ such that $(P, u)$ satisfies certain stability condition. (The stability condition depends on the metric on $\Sigma$ and $\mu$.)*
• **Gauged GW** Over compact surfaces, formally counting vortices defines a correlation function (Hamiltonian GW invariants)
\[
\langle \alpha_1, \ldots, \alpha_k \rangle^X_\Sigma \in F, \; \alpha_1, \ldots, \alpha_k \in H^*_K(\Sigma, F).
\]

1. **Mundet 2003**: HGW for \( K = S^1 \) and \( X \) compact symplectic.
2. **Cieliebak–Gaio–Mundet–Salamon 2002**: general \( K \) with \( X \) aspherical.
3. **CohFT over \( H^*_K(X) \)**: unfinished project of Mundet–Tian for \( K = S^1 \).

• **Gauged Ham.Floer** (Xu 2016) For general \( K \) and aspherical \( X \), using vortex equation on the cylinder perturbed by a \( K \)-invariant Hamiltonian, one can define a Floer homology over integers.

• **Gauged Lag.Floer** (Frauenfelder 2003) For certain (non-invariant) Lagrangian in \( X \) that descends to \( \bar{L} \subset \bar{X} \) which is the fixed point set of an anti-symplectic involution.
• From physical perspective, 2D $\mathcal{N} = (2, 2)$ SUSY allows more general Lagrangians which contain the so-called “F-term”.

• Geometrically, it means we may have a holomorphic potential function $\mathcal{W} : X \to \mathbb{C}$ which is $G$-invariant. The field theory (GLSM) based on such targets was invented by Witten 1993.

• When $G$ is trivial or a finite abelian group, this gives an orbifold Landau–Ginzburg model. For example, for $\mathcal{W} : \mathbb{C}^n \to \mathbb{C}$ quasihomogeneous polynomial, there is the FJRW (Fan–Jarvis–Ruan and Witten) theory.

• There are many works on rigorizing GLSM. Besides the algebraic geometry approaches, Tian and I are developing a symplectic approach of GLSM, based on the analysis of gauged Witten equation

$$\bar{\partial}_A u + \nabla \mathcal{W}(u) = 0,$$
$$\ast F_A + \mu(u) = 0.$$
III. Quantum Kirwan Map
Quantum Kirwan Map

- There is a classical map (surjective over \( \mathbb{Q} \))
  \[(\text{Kirwan Map}) : \kappa : H^*_K(X) \to H^*(\bar{X}).\]
- It is natural to ask the following.

**Question**

Does \( \kappa \) intertwine the theory upstairs with the theory downstairs? For example

\[
\langle \alpha_1, \ldots, \alpha_k \rangle^X_{\text{HGW}} \overset{?}{=} \langle \kappa(\alpha_1), \ldots, \kappa(\alpha_n) \rangle^\bar{X}_{\text{GW}}.
\]

- The question should be answered by using the adiabatic limit:
  \[
  \overline{\partial}_A u = 0, \quad \ast F_A + \epsilon^{-2} \mu(u) = 0.
  \]
- This is the equation of motion for \( \| d_A u \|_{L^2}^2 + \epsilon^2 \| F_A \|_{L^2}^2 + \epsilon^{-2} \| \mu(u) \|_{L^2}^2 \).
- As \( \epsilon \to 0 \), \( \| \mu(u) \|_{L^2} \) is small, \( u \) approximates a holomorphic curve in \( \bar{X} \).
Bubbling in the Adiabatic Limit and Affine Vortices

- As $\epsilon \to 0$, bubbling classified by Gaio–Salamon 2005:
  1. Energy blows up rate $\gg \epsilon^{-2}$ $\Rightarrow$ holomorphic spheres in $X$ bubble off.
  2. Energy blows up rate $\ll \epsilon^{-2}$ $\Rightarrow$ holomorphic spheres in $\bar{X}$ bubble off.
  3. Energy blows up rate $\approx \epsilon^{-2}$ $\Rightarrow$ affine vortices bubbles off.

- An affine vortex is a solution to the vortex equation over $\mathbb{C}$.
- Not conformal invariant, but only translation invariant.

Example
A vortex in $\mathbb{C}^2$ over $\Sigma$ is equivalent to 2 sections $\phi, \psi \in H^0(\Sigma, L)$. It defines a map $[\phi, \psi] : \Sigma \to \mathbb{P}^1$ if $\phi, \psi$ have no common zero.

By the HK correspondence, for any $\epsilon$, there is an vortex $(A_{\phi,\psi}^\epsilon, u_{\phi,\psi}^\epsilon)$ on $\Sigma$.

Xu 2015: If in a sequence a pair of zeroes of $\phi, \psi$ come together in a rate $\ll \epsilon$, then a sphere in $\mathbb{P}^1$ bubbles off; if the rate $\approx \epsilon$ or $\gg \epsilon$, then an affine vortex bubbles off.
Properties and Classifications

1. Gaio–Salamon 2005, Ziltener 2009: Affine vortices have well-defined limits at infinity as a $K$-orbit in $\mu^{-1}(0)$.

2. Ziltener 2009: Optimal energy decays at $\infty$ (slightly worse than holomorphic maps $\nu : \mathbb{C} \to \mathcal{X}$).


1. Taubes 1980: $X = \mathbb{C}$, $K = S^1$. Affine vortices $\approx$ polynomials.

2. Xu 2015: $X = \mathbb{C}^n$, $K = S^1$. Affine vortices $\approx n$ polynomials.

3. Venugopalan–Woodward 2016: HK correspondence for affine vortices, when $X$ is either a compact Kähler manifold or a linear space, acted by a reductive $G = K^\mathbb{C}$. 
• If there is no affine vortices, or if they “do not contribute,” then we can prove a statement like (Gaio–Salamon 2005, under certain conditions)
\[
\langle \alpha_1, \ldots, \alpha_k \rangle_{HG}^X = \langle \kappa(\alpha_1), \ldots, \kappa(\alpha_k) \rangle_{\bar{X}}^\bar{X}.
\]

• In general Salamon conjectured that using affine vortices one can define a quantization of \( \kappa \), called the quantun Kirwan map
\[
\kappa^Q = \kappa + \text{higher order term} : H_k^*(X; F) \to H^*(\bar{X}; F)
\]
which indeed intertwines the two theories.

• **Difficulty**: Hard analysis (redevelop everything parallel to pseudoholomorphic curves) unfinished project of Ziltener; Venugopalan–Xu 2016: *local model for moduli space of affine vortices*.

• **Algebraic case**: when \( X \) is projective, proved by Woodward 2015.
• Let $V$ be a vector space over $F$. A **CohFT algebra** on $V$ is a collection of compositions $m_n : V^\otimes k \otimes H^*(\bar{M}_{0,n}) \to V$ satisfying certain splitting axioms.

• Quantum cohomology of $\bar{X}$: on $\bar{H} = H^*(\bar{X}; F)$, $\bar{m}_n$ defined by genus zero GW invariants of $\bar{X}$.

• Equivariant quantum cohomology of $X$: on $H := H^*_K(X; F)$, $m_n$ defined by genus zero equivariant GW invariants. This is defined via the moduli space of holomorphic spheres in $X$ modulo $K$-action.

• **Nguyen–Woodward–Ziltener 2014**: $\kappa^Q$ is a collection of maps

$$\kappa_n : H^\otimes n \otimes H^*(\bar{M}_{n,1}; F) \to \bar{H}$$

which is a morphism of CohFT algebras. Namely, satisfying another type of splitting axioms.
• The axiom of morphism of CohFT algebra is dictated by the moduli space $\mathcal{M}_{n,1}$ of configurations of $n$ points in $\mathbb{C}$ modulo translation.

• Points coming together form spherical components.

• The algebraic conclusion of the degeneration picture is exactly a morphism of CohFT algebras.
• There are similar type of degenerations on the level of solutions.
• Besides sphere bubbling caused by energy concentration, a sequence of affine vortices can also degenerate in the following way.

\[
\text{Nonzero energy} \quad \text{growing size} \quad \text{growing distance}
\]

\[
\text{growing size} \quad \{ \text{nonzero energy} \} \quad \text{growing distance} \quad \{ \text{nonzero energy} \} \quad \text{growing size}
\]
IV. Open Quantum Kirwan Map
• Counting holomorphic disks with boundaries in a Lagrangian submanifold $\bar{L} \subset \bar{X}$ defines the so-called **Fukaya algebra** $Fuk(\bar{L})$.

• It is an $A_\infty$ algebra, i.e., a cochain group $C(\bar{L})$ of $\bar{L}$ with compositions

\[ \bar{m}_0 : F \to C(\bar{L}), \quad \bar{m}_1 : C(\bar{L}) \to C(\bar{L}), \quad \bar{m}_2 : C(\bar{L}) \otimes C(\bar{L}) \to C(\bar{L}), \ldots \]

In the best case, $\bar{m}_0 = 0$, $\bar{m}_1$ is a differential, $\bar{m}_2$ is associative, and $\bar{m}_k = 0$ for higher $k$. This notion is a real analog of CohFT algebra.

• (Fukaya–Oh–Ohta–Ono) Consider the moduli space of holomorphic disks with $k + 1$ marked points. Then for $a_1, \ldots, a_k \in C(\bar{L})$, define

\[ \bar{m}_k(a_1, \ldots, a_k) = (ev_\infty)_* \left[ (ev_1 \times \cdots \times ev_k)^*(a_1, \ldots, a_k) \cap [M_{k+1}(\bar{L})]^{\text{vir}} \right]. \]

• $M_{k+1}(\bar{L})$ behaves like a manifold with corners. The virtual fundamental chain is defined in a very sophisticated way using Kuranishi structures.
• An important property of $A_\infty$ algebra is \textbf{unobstructedness}. It is called weakly unobstructed if there exists some $b \in \mathcal{C}(\bar{L})$ such that

$$\sum_k \bar{m}_k(b, \ldots, b) \equiv 0 \mod F \cdot \text{PD}[\bar{L}].$$

• On the solution set $\mathcal{MC}(\bar{L})$, define the \textbf{potential function}

$$\bar{W}(b) = \sum_k \bar{m}_k(b, \ldots, b)/\text{PD}[\bar{L}] \in F.$$

• Detect nontrivial Floer cohomology; mirror Landau–Ginzburg model.

• FOOO proved for the toric case $\bar{W}$ is (implicitly) related to the Givental–Hori–Vafa potential. For Fano toric manifolds, they coincide.

• More explicit relation was proved by Chan–Lau–Leung–Tseng for semi-Fano toric manifolds.
• If $\tilde{X} = X//K$, then $\tilde{L} \subset \tilde{X}$ can be lifted to $L \subset \mu^{-1}(0) \subset X$.

• (Woodward 2011) Consider $K$-orbits of holomorphic disks in $X$. The evaluations at boundary markings are still in $L/K \simeq \tilde{L}$.

Example
$X = \mathbb{C}^2$, $K = S^1$, $L = S^1 \times S^1$. A holomorphic disk $u : \mathbb{D}^2 \to \mathbb{C}^2$

$$u(z) = \left( e^{i\theta_1} \frac{z - \alpha}{1 - \alpha_1 z}, e^{i\theta_2} \frac{z - \beta}{1 - \alpha_2 z} \right)$$

descends to a disk in $\mathbb{P}^1$ if $\alpha_1 \neq \alpha_2$. (Upstairs is more regular!)

• Using these object (quasidisks) one can define another $A_\infty$ algebra $Fuk^K(L)$ on the same cochain group $C(\tilde{L})$.

• For the toric case, Woodward verified that the quasimap potential function coincides with the Givental–Hori–Vafa potential.
• Woodward 2011: potential function upstairs and downstairs should be related via affine vortices over $\mathbb{H}$.

**Theorem (Woodward–Xu)**

(For $X, L$ satisfying certain conditions, including the case that $\tilde{X}$ is a rational semi-positive toric manifolds) there is $c \in QH^*(\tilde{X})$ and an $A_\infty$ morphism from $Fuk^K(L)$ to $Fuk^c(\tilde{L})$. Namely, there are multilinear maps $\phi_n : C(\tilde{L})^n \to C(\tilde{L})$ satisfying axioms of $A_\infty$ morphisms.
Remarks

Remark

1. Transversality is treated by Cieliebak–Mohnke’s stabilizing divisor technique. Charest–Woodward firstly developed this version of $\text{Fuk}(\bar{L})$.
2. Gluing affine vortices, Xu 2016
3. Similar results are obtained by Fukaya via the natural Lagrangian correspondence $\mathcal{X} \xrightarrow{\text{Lag}} \bar{\mathcal{X}}$.

Corollary

1. The map $\tilde{\phi} : b \mapsto \sum_k \phi_k(b, \ldots, b)$ maps $\mathcal{MC}^K(L)$ into $\mathcal{MC}^c(\bar{L})$ and
   \[ \bar{W}^c(\tilde{\phi}(b)) = W^K(b). \]
   So unobstructedness of $\text{Fuk}^K(L)$ implies unobstructedness of $\text{Fuk}^c(\bar{L})$.
3. Provides a simple way to identify unobstructed and Floer nontrivial Lagrangians in GIT quotients.
Thanks!